PHY250

Anabela R. Turlione

Digipen

Fall 2020

General information of the course

Class Schedule: Tuesday/Thursday 15:30 – 17:25

Class room: Ada Byron

Professor: Anabela Turlione

Contact: anabela.turlione@digipen.edu - 1029

Class web page: PHY250 at distance.digipen.edu

Office hours: by appointment

General information of the course

Course Objectives

1. Solve particle collisions
2. Describe the motion of rigid bodies
3. Main characteristics of fluids
4. Oscillations: Physical pendula and elactic bodies
5. Creation and propagation of mechanical wawes. Waves equation
6. Sound
7. Light

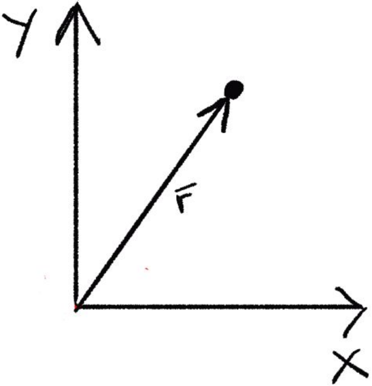
General information of the course

Grading Policy

1. HOMEWORKS 30 %
2. MIDTERM 30 %
3. FINAL EXAM 40 %

Kinematics

Mathematical description of the motion, without taking into account what causes it.



$$position \rightarrow \vec{r} = \vec{r}(t)$$

$$velocity \rightarrow \vec{v} = \frac{d\vec{r}}{dt}$$

$$acceleration \rightarrow \vec{a} = \frac{d\vec{v}}{dt}$$

Kinematics

Cartesian Coordinates

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1}$$

$$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \tag{2}$$

$$\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k} \tag{3}$$

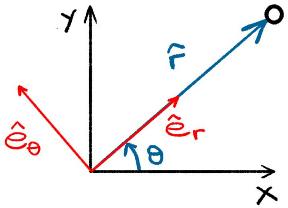
Kinematics

Unitary vectors

The unitary vectors, or versors, are vectors of length 1. They indicate directions in the space.

$\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unitary vectors, in Cartesian coordinates.

Kinematics



Circular Coordinates

- $\hat{e}_r$: direction in which the radial distance increases
- $\hat{e}_\theta$: direction in which the angle increases, counterclockwise from the positive x-axis

Kinematics

If we describe the motion in terms of $\hat{e}_r$ and $\hat{e}_\theta$,

$$\vec{r} = r\hat{e}_r \tag{4}$$

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$$\vec{v} = \frac{d(r\hat{e}_r)}{dt} \tag{5}$$

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using the relations : $\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}\hat{e}_\theta, \quad \frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r$

Kinematics

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$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta \tag{6}$$

Kinematics

If we define the angular velocity as $\omega = \frac{d\theta}{dt}$, then

$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\omega\hat{e}_\theta \tag{7}$$

If the radious of the motion is constant, then, we recover the result for the circular motion,

$$v_\theta = r\omega \tag{8}$$

For the acceleration, we obtain,

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{e}_r + \left(2\frac{dr}{dt}\omega + r\alpha\right)\hat{e}_\theta \tag{9}$$

where we have defined the angular acceleration as $\alpha = \frac{d\omega}{dt}$

Kinematics

If the radious of the motion is cosntant,

$$\vec{a} = \left(-r\omega^2\right)\hat{e}_r + \left(+r\alpha\right)\hat{e}_\theta \tag{10}$$

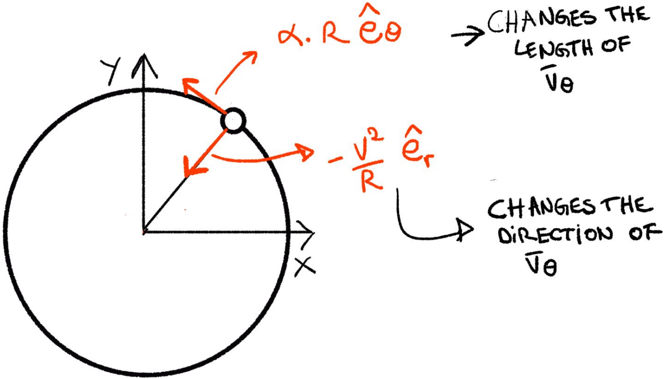
or it can be expressed as,

$$\vec{a} = -\left(\frac{v^2}{r}\right)\hat{e}_r + \left(+r\alpha\right)\hat{e}_\theta \tag{11}$$

Where the first term is the well known expression for the centripetal acceleration.

$$\vec{a_c} = -\left(\frac{v^2}{r}\right)\hat{e}_r \tag{12}$$

Kinematics


$\alpha . R \hat{e}_\theta$
CHANGES THE LENGTH OF $\vec{V}_\theta$
$-\frac{V^2}{R} \hat{e}_r$
CHANGES THE DIRECTION OF $\vec{V}_\theta$
Y
X

Kinematics

To summarize, we can make a paralelism between the expresions corresponding to spacial and angular coordintes

spatial coordinates	angular coordinates
$\vec{r}$	θ
$\vec{v} = \frac{d\vec{r}}{dt}$	$\omega = \frac{d\theta}{dt}$
$\vec{a} = \frac{d^2\vec{r}}{dt^2}$	$\alpha = \frac{d^2\theta}{dt^2}$
$\vec{F} = m\vec{a}$	$\tau = I\alpha$

Dynamics

What causes the the motion?

- Newton's Laws
- Conservation Theorems

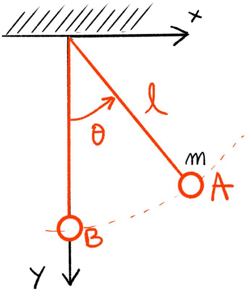
Dynamics

Newton's laws:

1. Every object continues in its state of rest, or on uniform velocity in a straight line, as long as no net force acts on it.
2. The acceleration of an object is directly proportional to the force acting on it, and is inversely proportional to the object's mass. The direction of the acceleration is in the direction of the net force acting on the object $\rightarrow \sum \vec{F} = m\vec{a}$
3. Whenever one object exerts a force on a second object, the second exerts an equal force in the opposite direction on the first. $\rightarrow \vec{F}_{12} = -\vec{F}_{21}$

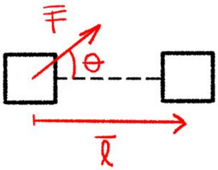
Dynamics

Example: Conider the simple pendulum of the figure. What is the tension of the rod in the points A (corresponding to the extreme) and B?



Work and energy

Work done by a constant force:



$$w = \vec{F} \cdot \vec{l} \qquad (13)$$

$$w = Fl\ cos\theta \qquad (14)$$

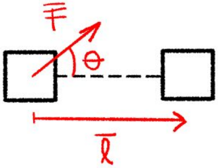
The units in SI are $[w] = N \cdot m = J$, and in cgs, $[w] = dyne \cdot cm = 1erg$.

Then. . .

1. No displacement $\rightarrow$ No Work

Work and energy

Work done by a constant force:



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$$w = Fl\ cos\theta \qquad (14)$$

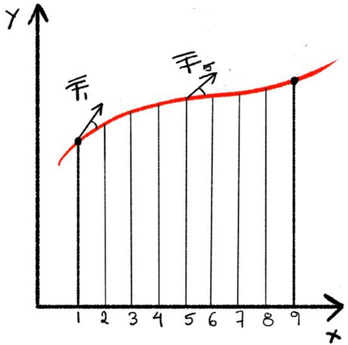
The units in SI are $[w] = N \cdot m = J$, and in cgs, $[w] = dyne \cdot cm = 1erg$.

Then...

1. No displacement $\rightarrow$ No Work
2. $\theta = 90 \rightarrow$ No Work

Work and energy

Work done by a non-constant force:



$$\Delta w_{12} = \vec{F}_1 \cdot \vec{l}_1 = F_1 \Delta l_1 \ cos\theta_1$$

$$\vdots$$

$$\Delta w_{i\ i+1} = \vec{F}_i \cdot \vec{l}_i = F_i \Delta l_i \ cos\theta_i$$

Work and energy

the total work made from the point 1 to 9 is:

$$\Delta w_{19} = \sum_{i=1}^{8} \Delta w_{i\ i+1} \tag{15}$$

If $\Delta l \rightarrow 0$, then

$$\Delta w = \int_1^9 \vec{F} \cdot \vec{dl} = \int_1^9 F cos\theta dl \tag{16}$$

So how do we solve the integral?

$$\int_1^9 \vec{F} \cdot \vec{dl} = \int_1^9 (F_x, F_y, F_z) \cdot (dx, dy, dz) = \int_1^9 (F_x\ dx + F_y\ dy + F_z\ dz) \tag{17}$$

Work and energy

ENERGY $\rightarrow$ this is one of the most important concepts in science. There is no a general definition, but we can define different types.

Kinetic Energy:

A moving object can do work on another object it strikes. So, it has the ability to do work- It has energy. If we define the Kinetic Energy as,

$$E_k = \frac{1}{2}mv^2 \tag{18}$$

We can show that,

$$W = \Delta E_k \tag{19}$$

Work and energy

Potential Energy-Conservative Forces:

A force is conservative if the net work done by the force on an object moving around any closed path is zero. In this case, the force is a **Conservative force** and the work only depends on the initial and final points.

Work and energy

Gravitational Potential

The gravitational potential is the work that must be done by a force to lift an object at constant velocity, against the gravity. Then, near the earth surface,

$$\vec{F}_{ext} = -\vec{F}_G = mg\hat{\jmath} \tag{20}$$

$$W_{ext} = \int_1^2 \vec{F}_{ext} \cdot d\vec{l} = mg \int_1^2 \hat{\jmath} \cdot (dx\hat{\imath} + dy\hat{\jmath}) = mg \int_1^2 dy \tag{21}$$

$$\rightarrow W_{ext} = mg\Delta y \tag{22}$$

Work and Energy

And the work made by the gravitational force is,

$$W_G = -mg\Delta y \tag{23}$$

If we define the gravitational potential energy near the surphace of the earth as,

$$U_G = mgy + C \tag{24}$$

where, C is a constant, then

$$W_G = -\Delta U \tag{25}$$

Work and Energy

General expression of the Potential Energy

$$\Delta U = -W = \int_1^2 \vec{F} \cdot d\vec{l} \tag{26}$$

Then in 1D,

$$U(x) = -\int F(x)dx + C \tag{27}$$

and

$$F(x) = -\frac{dU}{dx} \tag{28}$$

Work and Energy

Example: Spring - Hooke Law

$$\vec{F}_s = -kx\hat{\imath} \tag{29}$$

$$\Delta U = -\int_1^2 \vec{F}_s \cdot d\vec{l} = -\int_0^x (-kx)dx \tag{30}$$

then,

$$\Delta U = \frac{1}{2}kx^2 \tag{31}$$

$$\rightarrow U = \frac{1}{2}kx^2 + C \tag{32}$$

Work and Enery

Mechanical energy

Consider a conservative system, then,

$$\Delta E_k = W \tag{33}$$

and

$$\Delta U = -W \tag{34}$$

Then,

$$\Delta E_k + \Delta U = 0 \rightarrow E_{k2} + U_2 = E_{k1} + U_1 \tag{35}$$

Work and Enery

So there is a quantity that is constant for the system, it is conserved. We call this constant the **Mechanical Energy** of the system,

$$E = E_k + U \tag{36}$$

And, in general,

$$\Delta E_k + \Delta U + [\mathit{change\ in\ all\ other\ forms\ of\ energy}] = 0 \tag{37}$$

This is the law of conservation of energy, its validity rests on experimental observations.

Work and Enery

Example: Total Energy for the gravitational potential Newton's

Law of Universal Gravitation:

$$\vec{F} = -G\frac{Mm}{r^2}\hat{r} \tag{38}$$

$$\Delta U = -\int_1^2 \vec{F}\cdot d\vec{l} = GMm\int_1^2 \frac{\hat{r}\cdot d\vec{l}}{r^2} \tag{39}$$

$$= GMm(-\frac{1}{r})\Big|_1^2 = -GMm\left(\frac{1}{r_2}-\frac{1}{r_1}\right) \tag{40}$$

Work and Energy

$$\rightarrow U(r) = -G\frac{Mm}{r} + C \tag{41}$$

Then, the total Energy is

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} + C \tag{42}$$

Work and Energy

Power

The power, P, is the rate at which work i done.

$$P = \frac{dW}{dt} \tag{43}$$

$$average\ power \rightarrow P = \frac{W}{\Delta t} \tag{44}$$

The units of the power in SI are $[P] = N \cdot ms^{-1} = Js^{-1}$, and in cgs, $[w] = dyne \cdot cm^{-1} = erg\ s^{-1}$.

Linear Momentum

- In the previous section we studied one of several conservation laws in physcis: The conservation of Energy
- In this section we discuss Linear Momentum and its conservation.
- The conservation of Linear Momentum and of Energy is usefull when dealing with a system of two or more objects that interacts which each other (collisions).

Linear Momentum

We define the linear momentum as,

$$\vec{p} = m\vec{v} \tag{45}$$

Then, derivating each sides the equation, we obtain

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{46}$$

But, we know from the Newton's second Law that

$$\sum \vec{F} = m\vec{a} \tag{47}$$

Linear Momentum

Then,

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \tag{48}$$

- This is a more general exprssion of the Newton's 2nd Law, because includes the case of variable mass.
- The more momentum an object has, the harder is to stop it, and the greatter effect it will have on another object.
- When the sum of all the external forces acting in the system is zero, **the Linear Momentum is conserved**, that is

$$\frac{d\vec{p}}{dt} = 0 \rightarrow p = constant \tag{49}$$

Linear Momentum

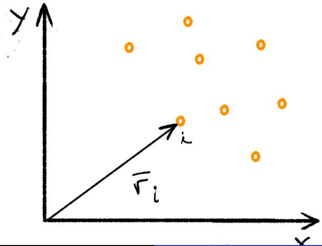
Conservation Theorem I

The total Linear Momentum of a particle is conserved when the total force on it is zero.

System of particles

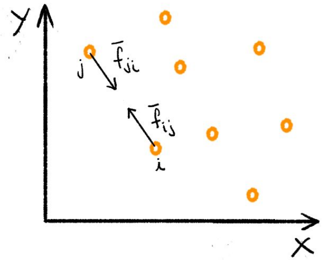
Thus far, we have solved problems in term of singleparticles. In this section we extend the discusion to describe a system of particles, these particles may form

- a loose aggregate (a pile of rocks or a volume of gas).
- a Rigid Body



System of particles

Newton's third Law plays a prominent role in the dynamics of a system pf particles.



- $\vec{f}_{ij}$: is the force on i due to j
- $\vec{f}_{ij} = -\vec{f}_{ij}$
- The forces are colinear

The total force acting on the particle i is $\vec{F}_i$. It has two terms,

- the term due to expernal forces: $\vec{F}_i^e$
- the term due to internal forces: $\vec{f}_i = \sum_{ij} \vec{f}_{ij}$, $i \neq j$

$$\vec{F}_i = \vec{F}_i^e + \vec{f}_i \tag{50}$$

We define the total linear momentum of the system as the sum of the linear momentum of all the particles,

$$\vec{P} = \sum \vec{p}_i = \sum m\vec{v}_i \tag{51}$$

System of particles

Then, if we calculate the time derivatives, we obtain,

$$\frac{d\vec{P}}{dt} = \sum m \frac{d\vec{v}_i}{dt} = \sum \vec{F}_i$$

Where the last term is the net force acting on the system,

$$\vec{F}_{net} = \sum \vec{F}_i$$

Then, from the last two equations,

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \tag{52}$$

This is the Newton's second Law for a system of particles.

System of particles

The net force sum can be splited in two terms,

$$\vec{F}_{net} = \sum \vec{F} = \sum_i \vec{F}_i^e + \sum_{ij} \vec{f}_{ij}, \;\; i \neq j \tag{53}$$

Let's note that because of the 3rd Law of Newton, all the internal forces acting between the particles are canceled.

$$\sum_{ij} \vec{f}_{ij} = \vec{f}_{12} + \vec{f}_{13} + \cdots + \vec{f}_{21} + \vec{f}_{23} + \cdots + \vec{f}_{31} + \vec{f}_{32} + \cdots$$

$$\vec{f}_{12} + \vec{f}_{13} + \cdots - \vec{f}_{12} + \vec{f}_{23} + \cdots - \vec{f}_{13} - \vec{f}_{23} + \cdots = 0 \tag{54}$$

System of particles

Then, for a system of particles,

$$\vec{F}_{net} = \sum_i \vec{F}_i^e \tag{55}$$

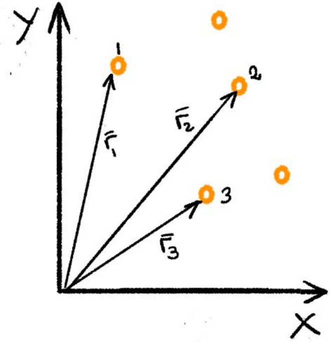
And

$$\sum_i \vec{F}_i^e = \frac{d\vec{P}}{dt} \tag{56}$$

When all the external forces acting on the system are zero, **the Linear Momentum of the system is conserved**.

$$\vec{F}_{net} = 0 \rightarrow \vec{P} = constant \tag{57}$$

Center of Mass

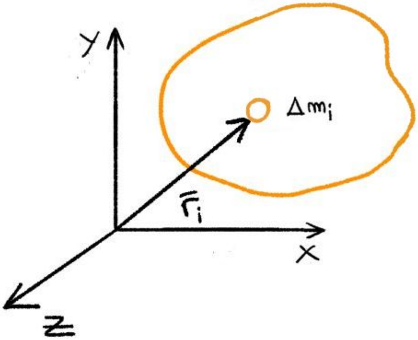


The total mass of the system is: $M = \sum_i m_i$

We define the **Center of Mass** of the system as,

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i \tag{58}$$

Center of Mass



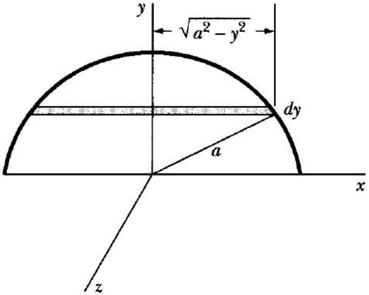
For a continuous distribution of mass,

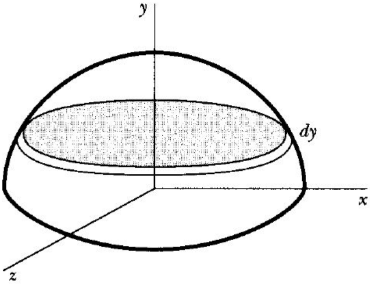
$$\vec{R} = \frac{1}{M} \lim_{\Delta m_i \to 0} \sum_i \Delta m_i \vec{r}_i \quad (59)$$

$$\vec{R} = \frac{1}{M} \int \vec{r} dm \quad (60)$$

Center of Mass

Example: Find the center of mass of a solid hemisphere of constant density.





Center of Mass

We can find a relation between the total Linear Momentum of the system and the center of mass,

$$\vec{P} = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) \quad (61)$$

but, the expression inside the parentesis is,

$$\sum_i m_i \vec{r}_i = M\vec{R} \quad (62)$$

Then,

$$\vec{P} = \frac{d}{dt}(M\vec{R}) = M\vec{V} \quad (63)$$

Center of Mass

We can do the same with the expression for the net force:

$$\vec{F}_{net} = \sum_i m_i \frac{d^2 \vec{r}_i}{dt^2} = \frac{d^2}{dt^2}\left(\sum_i m_i \vec{r}_i\right) \tag{64}$$

Then,

$$\vec{F}_{net} = \frac{d^2}{dt^2}(M\vec{R}) = M\vec{A} \tag{65}$$

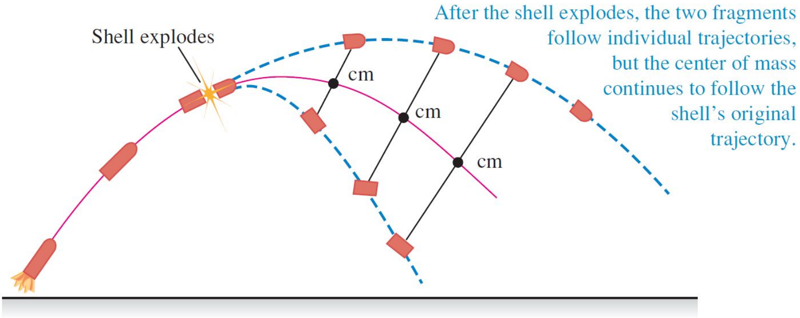
Center of Mass

We have obtained two important results,

- The Linear Momentum of the system is the same as if a single particle of mass M were located at the position of the center of mass and moving in the manner the center of mass moves.
- The total Linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass.

Center of Mass

Example:



Center of Mass

Energy of a system of particles

The total kinetic energy of the system is,

$$E_k = \sum_i \frac{1}{2} m_i v_i^2 \tag{66}$$

If the forces are conservative, the total potential energy is,

$$U = \sum_i U_i + \sum_{i \neq j} U*_{ij} \tag{67}$$

Center of Mass

If the forces are conservative, then the total energy of the system is conserved,

$$E = E_k + U = \sum_i \frac{1}{2} m_i v_i^2 + \sum_i U_i + \sum_{i \neq j} U*_{ij} \tag{68}$$

total energy = kinetic energy + ext potential energy + int potential energy

Impulse and collisions

In this section we apply the conservation laws to describe the interaction of two particles.

Interaction $\rightarrow$ the motion of one particle respect to another is governed by the force that describe the interaction.

- Contact interactions $\rightarrow$ Collisions
- No contact (electromagnetic force, gravity) $\rightarrow$ Scattering

Impulse and Collisions

Consider first the collision between two balls in one dimension, we assume:

1. The net external force acting on the system is zero.
2. all the forces during the collision are the forces of each ball exerts on the other.
3. Then, although the moment of each one the balls changes, the total momentum of the system remains constant.

Impulse and Collisions

If no net external forces act on the system, the total Momentum of the system is conserved.

$$momentum\ before = momentum\ after \tag{69}$$

$$m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v'}_A + m_B\vec{v'}_B \tag{70}$$

Impulse and Collisions

Elastic Collision

- No heat is produced in the collision

If there is not energy losing though heat emissions, then, the total energy of the system of the system is conserved,

$$E = E_A + E_B = constant \tag{71}$$

Impulse and Collisions

If there are no external forces, the initial energy of each one of the particles are, respectively

$$E_A = \frac{1}{2}mv_A^2, \;\; E_B = \frac{1}{2}mv_B^2$$

and,

$$E'_A = \frac{1}{2}mv'^2_A, \;\; E'_B = \frac{1}{2}mv'^2_B$$

During the brief moment of contact, the energy is stored in the form of elastic potential energy.

Impulse and Collisions

Then, the total kinetic energy before and after the collision is conserved,

$$E_A + E_B = E'_A + E'_B \tag{72}$$

$$\rightarrow \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = \frac{1}{2}mv'^2_A + \frac{1}{2}mv'^2_B \tag{73}$$

During the brief moment of contact, the energy is stored in the form of elastic potential energy.

Impulse and Collisions

We now apply both conservation laws:

$$m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v'}_A + m_B\vec{v'}_B$$

and

$$\frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = \frac{1}{2}m{v'}_A^2 + \frac{1}{2}m{v'}_B^2$$

Impulse and Collisions

We have two equations that we can solve for two unknowns. If we know the masses and the initial velocities, we can obtain v'_A and v'_B,

$$\vec{v}'_B - \vec{v}'_A = \vec{v}_A - \vec{v}_B$$

The relative speed of the two objects after the collision has the same magnitude (but opposite direction) as before the collision, no matter what the masses are.

Impulse and Collisions

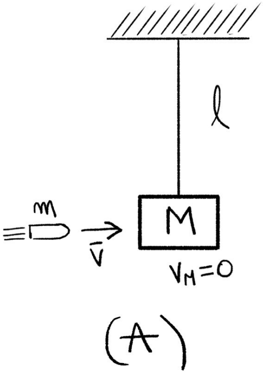
Inelastic Collisions

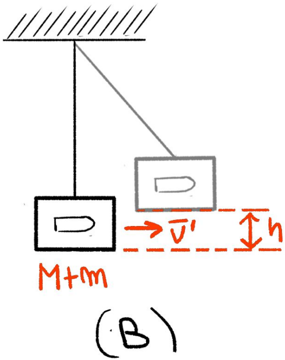
- The kinetic energy is not conserved,

$$E = E_A + E_B = E'_A + E'_B + \ \textit{thermal energy} \tag{74}$$

Impulse and Collisions

Example: Ballistic pendulum
(**Solved on whiteboard**)





Impulse and Collisions

collisions in 2D or 3D

Conservation of Momentum can also be appliyed to collisions in 2 or 3 dimensions.
In this case the equation for the conservation of the momentum is vectorial, then

$$\vec{P} = \vec{p}_A + \vec{p}_B = \vec{p}'_A + \vec{p}'_B = \vec{P}' \tag{75}$$

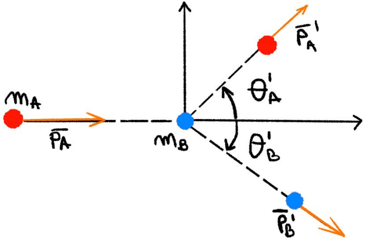
where $\vec{P}$ is the initial total momentum and $\vec{P}'$ is the final.
Then we have 2 equations and 4 unknowns:

$$\rightarrow p_{Ax} + p_{Bx} = p'_{Ax} + p'_{Bx} \tag{76}$$

$$\rightarrow p_{Ay} + p_{By} = p'_{Ay} + p'_{By} \tag{77}$$

Impulse and Collisions

Consider for example a non-head-on collision in which a moving object (called the "projectile") strikes a second object initially at rest (the "target").



$$m_A v_{Ax} = m_A v'_{Ax} + m_B v'_{Bx}$$
$$0 = m_A v'_{Ay} + m_B v'_{By}$$

$$m_A v_A = m_A v'_A cos(\theta'_A) + m_B v'_B cos(\theta'_B)$$
$$0 = m_A v'_A sin(\theta'_A) + m_B v'_B sin(\theta'_B)$$

Impulse and Collisions

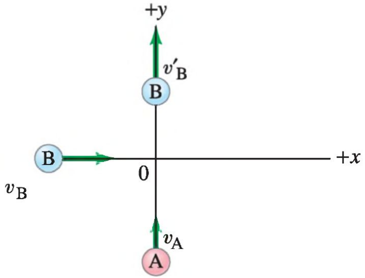
We need more information to fully solve the problem.

- How is the collision? is the kinetic energy conserved? $\rightarrow$ one more eq.
- Some information about the final state of the system.

Impulse and Collisions

Example: Two billiard balls of equal mass move at right angles and

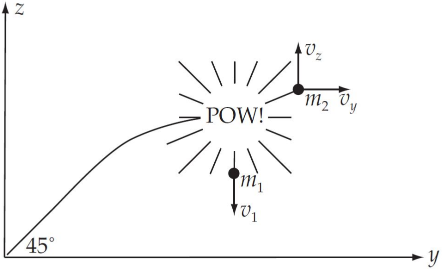
meet at the origin of an xy coordinate system. After the collision (assumed elastic), the second ball is moving along the positive y axis . What is the final direction of ball A, and what are the speeds of the two balls?



Impulse and Collisions

Example:

A projectile is fired at an angle of 45º with initial kinetic energy E_0. At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down. What are the velocities of m_1 and m_2?



Impulse and Collisions

Impulse resolution of collisions

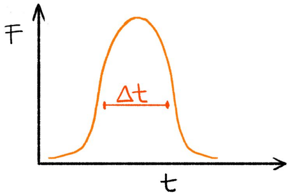
If we don't know how is the collision force, Conservation Theorems are not enough to solve the problem, we need some information of the final state.

We can somehow, model the collision through the definition of the restitution coefficient and the impulse.

Impulse and collisions

Collisions $\rightarrow$ time involved

In a collision, the force between objects jumps from 0 to a high value in a very short time, and then, to 0 again.



In general, we don't know $F(t)$, but we can know the change on linear momentum, $\Delta\vec{p}$ produced by the collision.

Impulse and Collisions

We can find a relation between the collision force and the variation of the linear momentum,

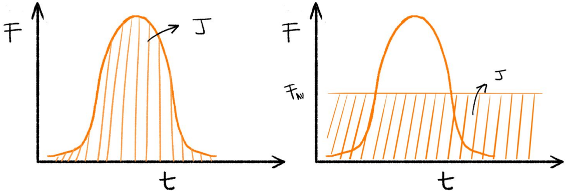
$$\vec{F} = \frac{d\vec{p}}{dt} \rightarrow \vec{F}dt = d\vec{p} \tag{78}$$

Then, we define the impulse as

$$\vec{J} = \int \vec{F}dt = \Delta\vec{p} \tag{79}$$

Impulse and Collisions

The impulse is the area under the $F(t)$ curve



The impulse can also be calculated as

$$\vec{J} = \vec{F}_{avg}\Delta t \quad (80)$$

where, $\vec{F}_{avg}$, is the time-average force.

Impulse and Collisions

In a collision, the impulse is related with the contact force,

$$\vec{J}_A = \int \vec{f}_{AB}dt = \Delta\vec{p}_A \quad (81)$$

$$\vec{J}_B = \int \vec{f}_{BA}dt = \Delta\vec{p}_B \quad (82)$$

But we know from the 3rd Newton's Law that,

$$\vec{f}_{AB} = -\vec{f}_{BA}$$

This add one more equation to our system. Then,

$$\vec{J}_A = -\vec{J}_B$$

Impulse and Collisions

Now we can relate the final velocities with the initial velocities,

$$\begin{aligned} \vec{J}_A &= m_A(\vec{v'}_A - \vec{v}_A) = \vec{J} \\ \vec{J}_B &= m_A(\vec{v'}_B - \vec{v}_B) = -\vec{J} \end{aligned}$$

Then,

$$\begin{aligned} \vec{V}'_A &= \vec{v}_A + \frac{\vec{J}}{m_A} \\ \vec{V}'_B &= \vec{v}_B - \frac{\vec{J}}{m_B} \end{aligned}$$

Impulse and Collisions

To obtain the last relation that we need, we introduce the definition of **restitution** coefficient:

$$e = -\frac{(\vec{v}'_A - \vec{v}'_B) \cdot \hat{n}}{(\vec{v}_A - \vec{v}_B) \cdot \hat{n}} \tag{83}$$

where $\hat{n}$ is the direction vector that points in the direction of $\vec{J}$

Impulse and Collisions

- $|e| = 1 \;\rightarrow$ the collision is completely elastic (the relative velocity is the same after the collision)
- $0 < |e| < 1 \;\rightarrow$ the relative velocity decreases
- $|e| = 0 \;\rightarrow$ the collision is completely inelastic (the two bodies get stick)

Impulse and Collisions

Then, we can now solve the following system for J

$$\begin{aligned}
\vec{V}_A' &= \vec{v}_A - \frac{J}{m_A}\hat{n} \\
\vec{V}_B' &= \vec{v}_B + \frac{J}{m_B}\hat{n} \\
e &= -\frac{(\vec{v}_A' - \vec{v}_B')\cdot\hat{n}}{(\vec{v}_A - \vec{v}_B)\cdot\hat{n}}
\end{aligned}$$

$$\rightarrow J = -\frac{(1+e)\vec{v}_r\cdot\hat{n}}{m_A^{-1} + m_B^{-1}} \tag{84}$$

where, $\vec{v}_r = \vec{v}_A - \vec{v}_B$, is the relative velocity.

In this section we will discuss the kinematics of rotational motion and then its dynamics (involving torque), as well as rotational kinetic energy and angular momentum (the rotational analog of linear momentum). $\rightarrow$ Describe Rigid Rotational Motion

Rigid Bodies

Collection of particles whose:

- relatives distances are constrained to remain absolutely fixed (microscopic displacement neglected).
- macroscopic displacement can take place (elastic deformations), but we are going to neglect them.
- If the Body has a continues distribution of mass $\rightarrow$ all the sums are replaced by integrals

Angular Momentum and Torque

The angular momentum of a single particle is,

$$\vec{L} = \vec{r}x\vec{p} \tag{85}$$

and the **torque** is,

$$\vec{N} = \vec{r}x\vec{F} \tag{86}$$

$$\rightarrow \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt}x\vec{p} + \vec{r}x\frac{d\vec{p}}{dt} \tag{87}$$

$$\rightarrow \frac{d\vec{L}}{dt} = \vec{N} \tag{88}$$

Angular Momentum and Torque

Conservation Theorem II:

The Angular Momentum of a particle subject to no toque is conserved.

Angular Momentum and Torque

Angular Momentum for a system of particles:

$$\vec{L} = \sum_i \vec{r_i} x \vec{p_i} = \sum_i \vec{r_i} \ x \ m_i \vec{v_i} \tag{89}$$

and, we can write:

$$\vec{r_i} = \vec{R} + \vec{r'}_i \rightarrow \vec{v_i} = \vec{V} + \vec{v'}_i \tag{90}$$

where $\vec{r'}_i$ is the position of the particle i respect to the center of mass. Then,

$$\vec{L} = \vec{R} x \vec{P} + \sum_i \vec{r'}_i x \vec{p'}_i \tag{91}$$

Angular Momentum and Torque

The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.

Angular Momentum and Torque

It can be shown that,

$$\frac{d\vec{L}}{dt} = \sum_i \vec{N}_i^{(e)} = \vec{N}^{(e)} \tag{92}$$

No external torque $\rightarrow$ No angular momentum variation.

Inertia Momentum

Rotation and translation of a system as a rigid body:

$$\vec{r_i} = \vec{R} + \vec{r'}_i \tag{93}$$

$$\rightarrow \vec{v_i} = \vec{V} + \vec{v'}_i = \vec{V} + \vec{\omega} x \vec{r'}_i \tag{94}$$

Then, the kinetic energy of the system can be expressed as,

$$T = \sum_i \frac{1}{2} m_i (\vec{V} + \vec{\omega} x \vec{r'}_i)^2 \tag{95}$$

$$\rightarrow T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_i m_i (\vec{\omega} x \vec{r'}_i)^2 \tag{96}$$

Inertia Momentum

Planar motion $\rightarrow \vec{\omega} \perp \vec{r}$, then

$$\sum_i m_i(\vec{\omega} x \vec{r'}_i)^2 = \sum_i m_i \omega^2 r'^2_i = \omega^2 \sum_i m_i r'^2_i \tag{97}$$

We define the Inertia Momentum as,

$$I = \sum_i m_i r'^2_i \tag{98}$$

Then,

$$\rightarrow T = \frac{1}{2} M V^2 + \frac{1}{2} \omega^2 I \tag{99}$$

Inertia Momentum

If we now calculate the Angular Momentum respect to some point fixed to the body,

$$\vec{L} = \sum_i m_i \vec{r'}_i x (\vec{\omega} x \vec{r'}_i) \tag{100}$$

If the motion is planar,

$$\vec{L} = \sum_i m_i r_i'^2 \vec{\omega} = \vec{\omega} \sum_i m_i {r'}_i^2 \tag{101}$$

$$\rightarrow \vec{L} = I\vec{\omega} \quad and \quad \vec{N} = I\vec{\alpha} \tag{102}$$

Inertia Momentum

If the rotation is not planar, then in general $\vec{\omega}$ is not parallel to $\vec{r'}_i$, then the expression for the linear momentum, torque and the kinetic energy take more complex expressions:

$$T = \frac{1}{2}MV^2 + \vec{\omega} \cdot (\mathsf{I}\ \vec{\omega}) \tag{103}$$

$$\vec{L} = \mathsf{I}\ \vec{\omega} \tag{104}$$

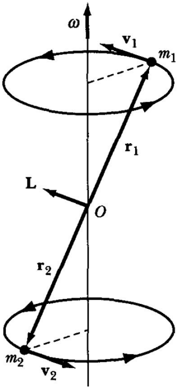
$$\vec{N} = \mathsf{I}\ \vec{\alpha} \tag{105}$$

$$\tag{106}$$

Where, I is the Inertia Tensor, and $\vec{L}$ is not parallel to ω, at least the body is rotating around a principal inertial axis.

Inertia Momentum

Example of Non-Planar Motion

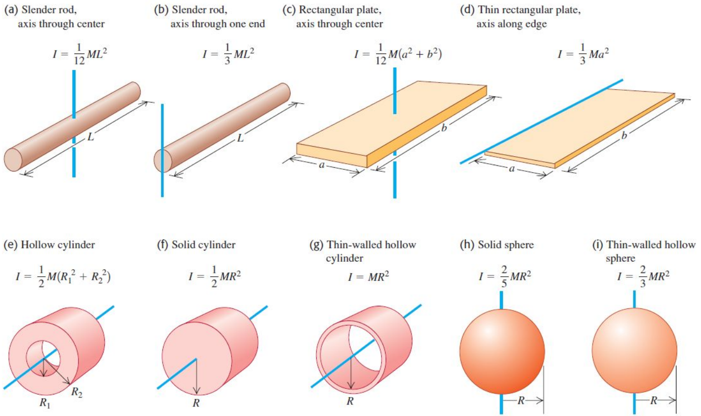


Inertia Momentum

Inertia Momentum of a continues mass distribution:

$$\sum_i m_i r_i'^2 \rightarrow \int_V r^2 dm \tag{107}$$

Inertia Momentum



Inertia Momentum

The potential energy for a system of particles is,

$$U = \sum_i U_i + \sum_{i \neq j} U*_{ij} \tag{108}$$

The second term is constant for a rigid body , then it is not relevant. Thus,

$$U = \sum_i U_i \tag{109}$$

and for a continues distribution of mass,

$$U = \int dU \tag{110}$$

Inertia Momentum

Gravitational Potential Energy

$$U = \sum_i U_i = \sum_i m_i g y_i = g \sum_i m_i y_i = gMY_{cm} \tag{111}$$

Or, for a continues body,

$$U = \int dU = \int gydm = g \int ydm = gMY_{cm} \tag{112}$$

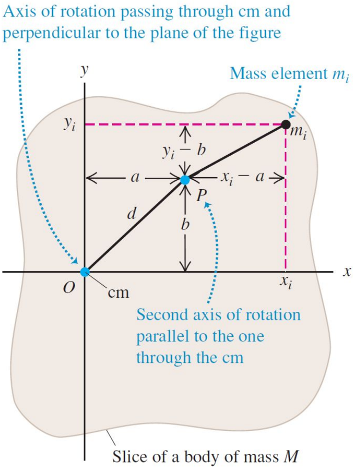
Inertia Momentum

Parallel-Axis Theorem

There is not only one Moment of Inertia for a rigid body, it depends of the axis of rotation. There is a simple relationship between the moment of inertia I_{cm} of a body of mass M about an axis through its center of mass and the moment of inertia I_p about any other axis parallel to the original but displaced from it by a distance d.

$$I_p = I_{cm} + Md^2 \quad (paralel - axis\ \ theorem) \tag{113}$$

Inertia Momentum



Proof

Moment of inertia about the CM,

$$I_{cm} = \sum_i m_i(x_i^2 + y_i^2) \qquad (114)$$

Moment of inertia about P,

$$I_p = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2] \quad (115)$$

Inertia Momentum

Expanding the squared terms and regruping,

$$I_p = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

The second and third terms are x_{cm} and y_{cm} in the center of mass system, so they are 0. Then,

$$I_p = I_{cm} + d^2 M$$

Inertia Momentum

To summarize ...

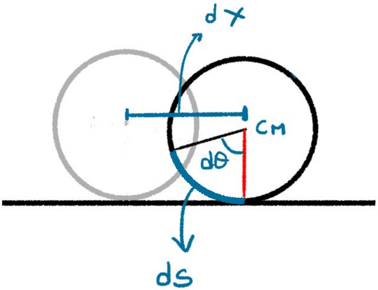
- Force $\rightarrow$ displacement of the cm $\vec{F} = M\vec{a}$
- Torque $\rightarrow$ rotation around an axis $\vec{N} = I\vec{\alpha}$

In a Rigid Bodie we have to take into consideration,

- Force magnitude
- Force direction
- Where it is appliyed

Inertia Momentum

Rolling without slipping



$$dx = ds = Rd\theta$$

$$\rightarrow \frac{dx}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$\rightarrow v_{cm} = R\frac{d\theta}{dt} = R\omega \qquad (116)$$

Inertia Momentum

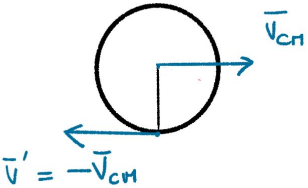
Then, the speed of the contact point respect to the center of mass is,

$$v' = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

and the speed respect to the floor, is

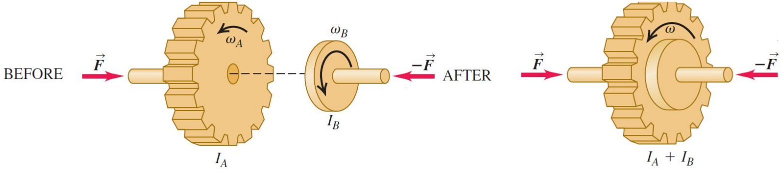
$$v = 0$$

The friction acting on the body is static $\rightarrow$ does not make work.



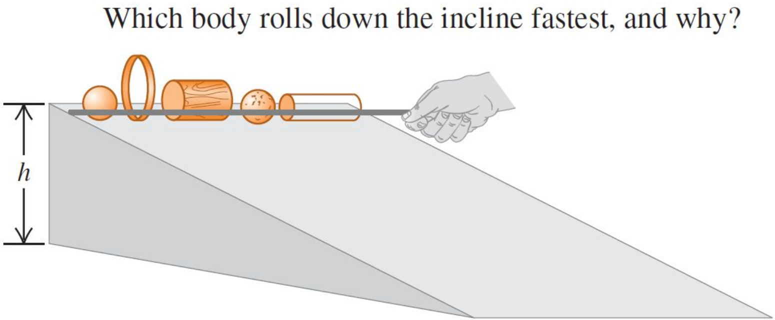
Inertia Momentum

Example: Consider an engine flywheel (A) and a clutch plate (B) attached to a transmission shaft. Their moments of inertia are and initially I_A and I_B, they are rotating with constant angular speeds ω_A and ω_B, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed ω. Derive an expression for ω.



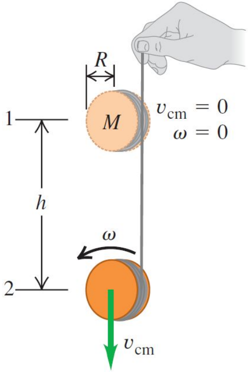
Inertia Momentum

Example:



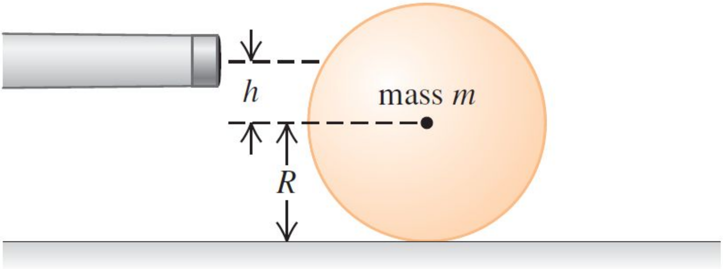
Inertia Momentum

Example: Calculate the speed of a primitive yo-yo.



Inertia Momentum

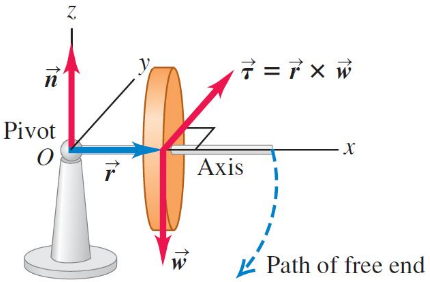
Example: A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball. The hit lasts for a short time Δt. For what value of h will the ball roll without slipping?

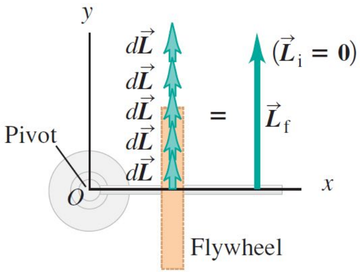


Inertia Momentum

Giroscope

Consider a toy gyroscope that's supported at one end:



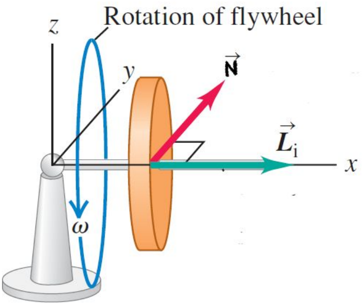


If the flywheel is not spinning, it falls.

Inertia Momentum

Giroscope

If the flywheel is spinning around a symmetry axis, then, there is an initial angular momentum along the axis



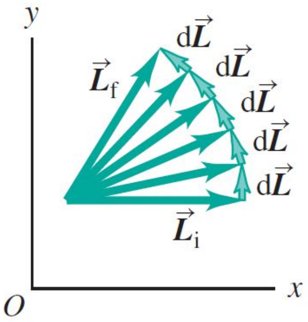
$$\frac{d\vec{L}}{dt} = \vec{N} \rightarrow d\vec{L} = \vec{N}dt$$

and

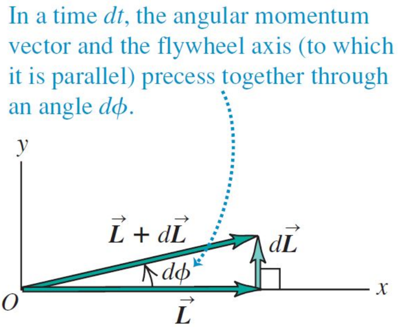
$$\vec{N} \perp \vec{L}$$

Inertia Momentum

$\vec{L}$ only changes the direction $\rightarrow$ the flywheel does not fall.
The rotation axis rotates around the z-axis counterclockwise $\rightarrow$ **Precession Motion**



Inertia Momentum



Precession Angular Speed

$$\frac{d\phi}{dt} = \Omega$$

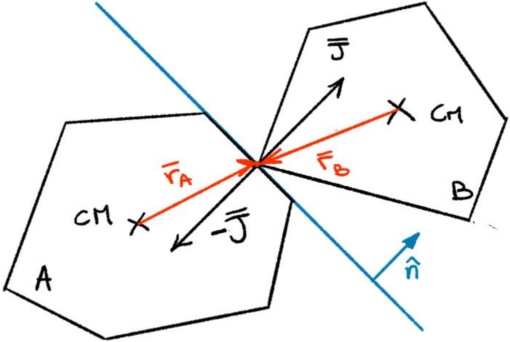
$$d\phi = \frac{dL}{L}$$

$$\rightarrow \frac{d\phi}{dt} = \frac{dL}{dt}\frac{1}{L} = \frac{rMg}{I\omega}$$

Impulse and Collisions

Collisions of Rigid Bodies

Linear displacement + Rotation



Impulse and Collisions

Linear displacement of the center of mass:

$$\vec{V_{cm}}'_A = \vec{v_{cm}}_A - \frac{J}{m_A}\hat{n}$$

$$\vec{V_{cm}}'_B = \vec{v_{cm}}_B + \frac{J}{m_B}\hat{n}$$

Impulse and Collisions

Rotation around the center of mass:

If the collision force produces a torque $\neq 0$ on the impact point respect to the center of mass, then, we also have rotation after the collision:

$$\frac{d\vec{L}}{dt} = \vec{N} = \vec{r} x \vec{F} \rightarrow \Delta\vec{L} = \int (\vec{r} x \vec{F}) dt = \vec{r} x \vec{J}$$

$$I \Delta\vec{\omega} = \vec{r} x \vec{J}$$

Impulse and Collisions

Then, we can relate the final angular velocities with the initial, through

$$\vec{\omega}'_A = \vec{\omega}_A - \frac{J}{I_A}(\vec{r}_A x \hat{n}) \quad (117)$$

$$\vec{\omega}'_B = \vec{\omega}_B + \frac{J}{I_B}(\vec{r}_B x \hat{n}) \quad (118)$$

Impulse and Collisions

We define now the restitution coefficient as,

$$e = -\frac{(\vec{v}'_A - \vec{v}'_B) \cdot \hat{n}}{(\vec{v}_A - \vec{v}_B) \cdot \hat{n}} \tag{119}$$

where the velocities are this time, the velocities of the contact points:

$$\begin{aligned} \vec{v_A} &= \vec{v}_{cmA} + \vec{\omega}_A x \vec{r}_A \\ \vec{v_B} &= \vec{v}_{cmB} + \vec{\omega}_B x \vec{r}_B \end{aligned}$$

Impulse and Collisions

Replacing all the final velocities and angular velocities in the expresion for e, we obtain an expression for the collision impulse in terms of e and the initial velocities,

$$\rightarrow J = \frac{-(1+e)\vec{v}_r \cdot \hat{n}}{m_A^{-1} + m_B^{-1} + [I_1^{-1}(\vec{r}_1 x \hat{n}) x \vec{r}_1 + I_2^{-1}(\vec{r}_2 x \hat{n}) x \vec{r}_2] \cdot \hat{n}} \tag{120}$$

Equilibrium conditions

We will see now how forces and torques act within structures and determine whether the structures can sustain those forces without deformation.

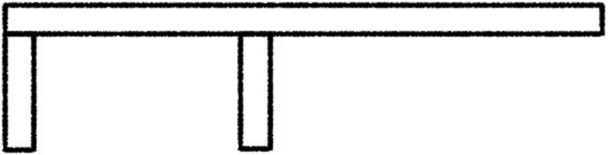
Equilibrium conditions

- First condition: $\sum \vec{F} = 0$
- Second condition: $\sum \vec{N} = 0$

Equilibrium

Example

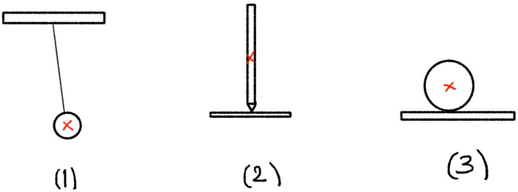
Forces on a diving board



Stability and Balance

How an object in equilibrium reacts to a slightly displacement?

- Return to its original position $\rightarrow$ stable equilibrium
- Moves farther its original position $\rightarrow$ unstable equilibrium
- remains in its new position $\rightarrow$ neutral equilibrium



Stability and Balance

Definition: Center of gravity

Is the point which the force of gravity does not produces torque. If the gravity field is uniform, it coincides with the center of mass.

In general, if the **center of gravity** (CG) is bellow the point of support, the equilibrium will be stable.

An object whose **center of gravity** is above its base of support, will be stable if a vertical line projected downward from the CG falls within the base of support.

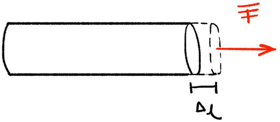
Stability and Balance

Example: A person continually shifts the body so that its CG is over the feet.

In general, the larger the base and the lower the CG $\rightarrow$ more stable equilibrium.

Elasticity, Stress and Strain

Hooke's Law



Consider that an external force produces a macroscopic deformation of the body:

$$\vec{F} = k\Delta l \tag{121}$$

This valid if, Δl is small compared with the length of the body, and is valid on almost any solid material from iron to bone.

Elasticity, Stress and Strain

Young's Modulus

Δl depends on,

- F
- The material
- its dimensions

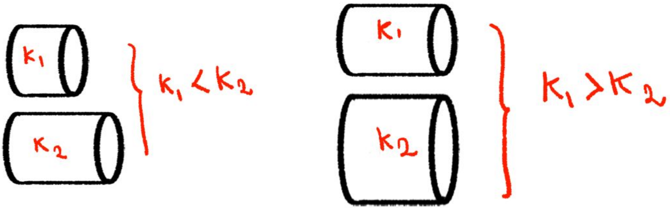
$$\Delta l = \frac{1}{E}\frac{F}{A}l_0 \tag{122}$$

Where E is the Young's modulus and depends on the material. The units of E are N/m^2

Elasticity, Stress and Strain

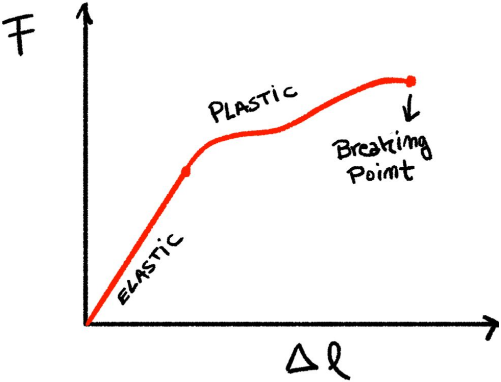
Then,

$$k = \frac{EA}{l_0} \tag{123}$$



Elasticity, Stress and Strain

Applied force vs. elongation for a typical metal under tension.



Elasticity, Stress and Strain

Stress and Strain

$$stress = \frac{force}{area} = \frac{F}{A} \tag{124}$$

$$strain = \frac{change\ in\ length}{original\ length} = \frac{\Delta l}{l_0} \tag{125}$$

Then,

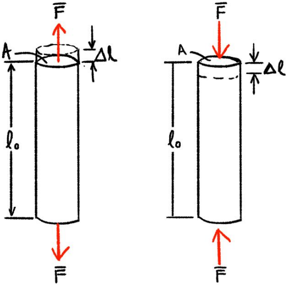
$$\frac{F}{A} = E\,\frac{\Delta l}{l_0} \tag{126}$$

The strain is proportional to the stress in the elastic region.

Elasticity, Stress and Strain

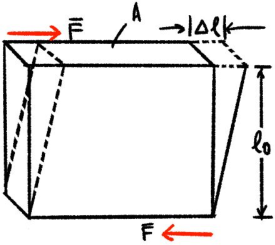
External forces applied to and object give rise to internal forces, or stress, within the material itself.

Tension And compression



Elasticity, Stress and Strain

Shear stress: it changes shape, not dimensions



$$\Delta l = \frac{1}{G}\frac{F}{A} l_0 \quad (127)$$

Where G is the shear modulus.

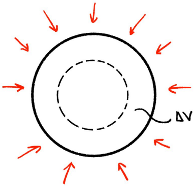
Elasticity, Stress and Strain


(a)


(b)

Elasticity, Stress and Strain

Consider now that there are inward forces acting on the object from all sides.



The change in volume is proportional to the change in presure.

$$\frac{\Delta V}{V_0} = -\frac{1}{B}\Delta P \tag{128}$$

Where B is the bulk modulus, and P is the presure,

Elasticity, Stress and Strain

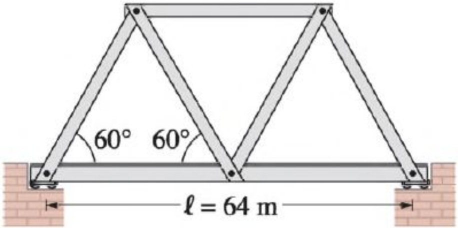
Elastic moduli

TABLE 12–1 Elastic Moduli

Material	Young's Modulus, E (N/m^2)	Shear Modulus, G (N/m^2)	Bulk Modulus, B (N/m^2)
Solids			
Iron, cast	100×10^9	40×10^9	90×10^9
Steel	200×10^9	80×10^9	140×10^9
Brass	100×10^9	35×10^9	80×10^9
Aluminum	70×10^9	25×10^9	70×10^9
Concrete	20×10^9		
Brick	14×10^9		
Marble	50×10^9		70×10^9
Granite	45×10^9		45×10^9
Wood (pine) (parallel to grain)	10×10^9		
(perpendicular to grain)	1×10^9		
Nylon	5×10^9		
Bone (limb)	15×10^9	80×10^9	
Liquids			
Water			2.0×10^9
Alcohol (ethyl)			1.0×10^9
Mercury			2.5×10^9
Gases†			

Example

Determine the tension or compression in each of the struts of the truss bridge shown in the figure. Use the method of joints, which involves (1) drawing a free-body diagram of the truss as a whole, and (2) drawing a free-body diagram for each of the pins (joints), one by one, and setting $\sum \vec{F} = 0$ for each pin. Ignore the mass of the struts. Assume all triangles are equilateral.



Example

Suspension bridge. Determine the shape of the cable between the two towers of a suspension bridge, assuming the weight of the roadway is supported uniformly along its length. Ignore the weight of the cable.