PHY250: Harmonis Oscillations

Anabela R. Turlione

Digipen

Fall 2021

Oscillations

The Harmonic Oscillator

Simple Harmonic Motion

Many objects vibrate or oscillate:

- An object at the end of a spring.
- A tuning fork
- The electric and Magnetic fields in the electromagnetic radiation.
- The atoms of a solid vibrate about their relatively fixed positions.
- etc...

Simple Harmonic Motion

Consider a system under the influence of a force proportional to the displacement

Simple Harmonic Motion

Consider a system under the influence of a force proportional to the displacement

$$\vec{F} = -k\vec{\Delta x}$$

Simple Harmonic Motion

Consider a system under the influence of a force proportional to the displacement

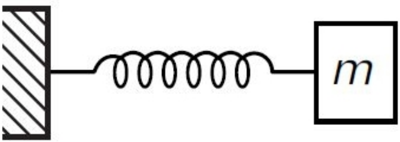
$$\vec{F} = -k\vec{\Delta x} \quad \leftarrow Hook's\ Law \tag{1}$$

Simple Harmonic Motion

Consider a system under the influence of a force proportional to the displacement

$$\vec{F} = -k\vec{\Delta x} \quad \leftarrow Hook's\ Law \tag{1}$$

This is the case of a mass on a spring:



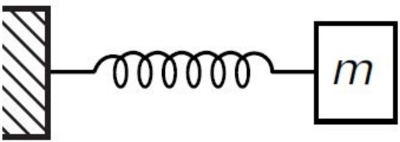
higher the k

Simple Harmonic Motion

Consider a system under the influence of a force proportional to the displacement

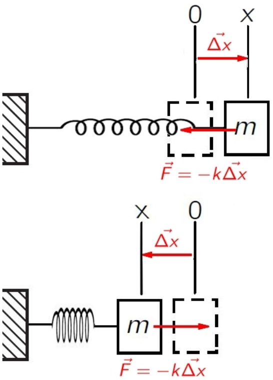
$$\vec{F} = -k\vec{\Delta x} \quad \leftarrow \textit{Hook's Law} \tag{1}$$

This is the case of a mass on a spring:

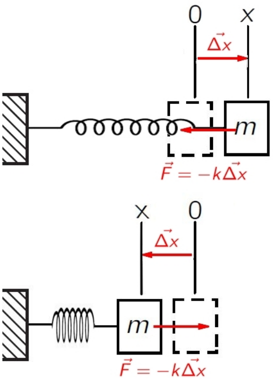


higher the $k \rightarrow$ the stiffer the spring.

Simple Harmonic Motion



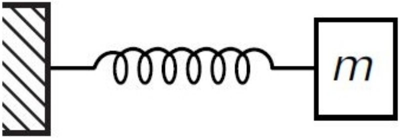
Simple Harmonic Motion



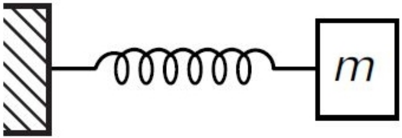
The force is a *Restoring Force.*

Simple Harmonic Motion

We are going to consider:



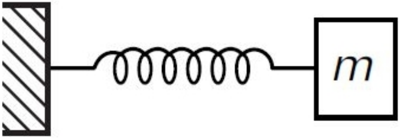
Simple Harmonic Motion



We are going to consider:

- Massless spring.

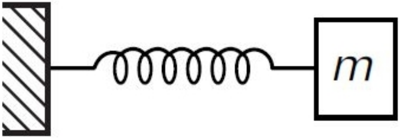
Simple Harmonic Motion



We are going to consider:

- Massless spring.
- The material of the spring in the elastic regime.

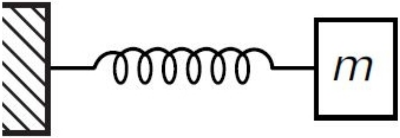
Simple Harmonic Motion



We are going to consider:

- Massless spring.
- The material of the spring in the elastic regime.
- There are no friction or drag forces.

Simple Harmonic Motion



We are going to consider:

- Massless spring.
- The material of the spring in the elastic regime.
- There are no friction or drag forces.
- The coils are not close to touching.

Simple Harmonic Motion

What is the position x as a function of the time?

Simple Harmonic Motion

What is the position x as a function of the time?

In this case, the force is not constant, then we can not use the results for a constant acceleration motion

Simple Harmonic Motion

What is the position x as a function of the time?

In this case, the force is not constant, then we can not use the results for a constant acceleration motion

$$x \neq \frac{1}{2}at^2 + v_o t + x_o$$

To find the equations of motion for x, we use the second Newton's law:

Simple Harmonic Motion

What is the position x as a function of the time?

In this case, the force is not constant, then we can not use the results for a constant acceleration motion

$$x \neq \frac{1}{2}at^2 + v_o t + x_o$$

To find the equations of motion for x, we use the second Newton's law:

$$\begin{aligned} \sum F &= ma \\ -kx &= m\frac{d^2x}{dt^2} \end{aligned}$$

Simple Harmonic Motion

we rearrange this to obtain,

Simple Harmonic Motion

we rearrange this to obtain,

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \tag{2}$$

Simple Harmonic Motion

we rearrange this to obtain,

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \qquad (2)$$

Equation of motion for the simple harmonic oscillator.

Simple Harmonic Motion

we rearrange this to obtain,

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \tag{2}$$

Equation of motion for the simple harmonic oscillator.

How do we solve it?

Simple Harmonic Motion

we rearrange this to obtain,

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \tag{2}$$

Equation of motion for the simple harmonic oscillator.

How do we solve it?

Trial solution:

Simple Harmonic Motion

we rearrange this to obtain,

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \tag{2}$$

Equation of motion for the simple harmonic oscillator.

How do we solve it?

Trial solution:

$$x(t) = A cos(\omega t + \phi) \tag{3}$$

Simple Harmonic Motion

If we put the trial solution 3 into the equation 2, we obtain:

Simple Harmonic Motion

If we put the trial solution 3 into the equation 2, we obtain:

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \tag{4}$$

- Greater the mass m

Simple Harmonic Motion

If we put the trial solution 3 into the equation 2, we obtain:

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \tag{4}$$

- Greater the mass $m \rightarrow$ the lower the frequency ω.

Simple Harmonic Motion

If we put the trial solution 3 into the equation 2, we obtain:

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \tag{4}$$

- Greater the mass $m \rightarrow$ the lower the frequency ω.
- Stiffer the spring k

Simple Harmonic Motion

If we put the trial solution 3 into the equation 2, we obtain:

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \tag{4}$$

- Greater the mass $m \rightarrow$ the lower the frequency ω.
- Stiffer the spring $k \rightarrow$ the higher ω.

Simple Harmonic Motion

$x(t)$:periodic function

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t+T),$$

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t+T), T = period \tag{5}$$

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t+T), T = period \tag{5}$$

$$\rightarrow Acos(\omega t + \phi) = Acos(\omega t + \omega T + \phi) \tag{6}$$

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t+T), T = period \tag{5}$$

$$\rightarrow Acos(\omega t + \phi) = Acos(\omega t + \omega T + \phi) \tag{6}$$

$$\rightarrow \omega T = 2\pi$$

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t + T), T = period \tag{5}$$

$$\rightarrow Acos(\omega t + \phi) = Acos(\omega t + \omega T + \phi) \tag{6}$$

$$\rightarrow \omega T = 2\pi \rightarrow \ \omega = \frac{2\pi}{T}$$

Simple Harmonic Motion

$x(t)$:periodic function

$$\rightarrow x(t) = x(t+T), T = period \tag{5}$$

$$\rightarrow Acos(\omega t + \phi) = Acos(\omega t + \omega T + \phi) \tag{6}$$

$$\rightarrow \omega T = 2\pi \rightarrow \boxed{\omega = \frac{2\pi}{T}} \tag{7}$$

Simple Harmonic Motion

$\boxed{T}$

Simple Harmonic Motion

$$\boxed{T} \rightarrow period, \quad (units : s) \tag{8}$$

Simple Harmonic Motion

$$\boxed{T} \rightarrow period, \quad (units : s) \tag{8}$$

$$\boxed{\omega = \frac{2\pi}{T}}$$

Simple Harmonic Motion

$$\boxed{T} \rightarrow period, \quad (units : s) \tag{8}$$

$$\boxed{\omega = \frac{2\pi}{T}} \rightarrow angular\ \ frequency, \quad (units : rad \cdot s^{-1}) \tag{9}$$

Simple Harmonic Motion

$$\boxed{T} \rightarrow period, \quad (units : s) \tag{8}$$

$$\boxed{\omega = \frac{2\pi}{T}} \rightarrow angular\ \ frequency, \quad (units : rad \cdot s^{-1}) \tag{9}$$

$$\boxed{f = \frac{\omega}{2\pi}}$$

Simple Harmonic Motion

$$\boxed{T} \rightarrow period, \quad (units: s) \tag{8}$$

$$\boxed{\omega = \frac{2\pi}{T}} \rightarrow angular\ \ frequency, \quad (units: rad \cdot s^{-1}) \tag{9}$$

$$\boxed{f = \frac{\omega}{2\pi}} \rightarrow frequency, \quad (units: s^{-1} = Hz) \tag{10}$$

Simple Harmonic Motion

$$x(t) = A cos(\omega t + \phi)$$

Simple Harmonic Motion

$$x(t) = A cos(\omega t + \phi)$$

what are A and ϕ?

Simple Harmonic Motion

$$x(t) = Acos(\omega t + \phi)$$

what are A and ϕ?

$$x(t=0) \quad = \quad x_0$$

Simple Harmonic Motion

$$x(t) = Acos(\omega t + \phi)$$

what are A and ϕ?

$$\begin{aligned} x(t=0) &= x_0 \\ v(t=0) &= v_0 \end{aligned}$$

$$\rightarrow Acos(\phi) = x_0$$

Simple Harmonic Motion

$$x(t) = Acos(\omega t + \phi)$$

what are A and ϕ?

$$\begin{aligned} x(t=0) &= x_0 \\ v(t=0) &= v_0 \end{aligned}$$

$$\begin{aligned} \rightarrow Acos(\phi) &= x_0 \\ \rightarrow -A\omega sin(\phi) &= v_0 \end{aligned}$$

Simple Harmonic Motion

$$x(t) = Acos(\omega t + \phi)$$

what are A and ϕ?

$$\begin{aligned} x(t=0) &= x_0 \\ v(t=0) &= v_0 \end{aligned}$$

$$\begin{aligned} \rightarrow Acos(\phi) &= x_0 \\ \rightarrow -A\omega sin(\phi) &= v_0 \end{aligned}$$

$$\left.\begin{aligned} &\phi = tan^{-1}\left(\frac{v_0}{\omega x_0}\right) \\ &A = \frac{x_0}{cos(\phi)} \end{aligned}\right\}$$

Simple Harmonic Motion

$$x(t) = A cos(\omega t + \phi)$$

what are A and ϕ?

$$\begin{aligned} x(t=0) &= x_0 \\ v(t=0) &= v_0 \end{aligned}$$

$$\begin{aligned} \rightarrow A cos(\phi) &= x_0 \\ \rightarrow -A\omega sin(\phi) &= v_0 \end{aligned}$$

$$\left.\begin{aligned} \phi &= tan^{-1}\left(\frac{v_0}{\omega x_0}\right) \\ A &= \frac{x_0}{cos(\phi)} \end{aligned}\right\} \textit{depend on the initial velocity and position}$$

Simple Harmonic Motion

For example, if $x(t = 0) = x_0$ and $v(t = 0) = 0$,

Simple Harmonic Motion

For example, if $x(t=0) = x_0$ and $v(t=0) = 0$,

$$v_0 = -A\omega sin(\phi) = 0 \to \phi = n\pi, \; n \; integer$$

$$x_0 = Acos(n\pi) \to A(-1)^n = x_0$$

Simple Harmonic Motion

For example, if $x(t=0) = x_0$ and $v(t=0) = 0$,

$$v_0 = -A\omega sin(\phi) = 0 \rightarrow \phi = n\pi, \; n \; integer$$

$$x_0 = Acos(n\pi) \rightarrow A(-1)^n = x_0$$

if $x_0 > 0$, we can take $A > 0$ and $n = 0$

Simple Harmonic Motion

For example, if $x(t=0) = x_0$ and $v(t=0) = 0$,

$$v_0 = -A\omega sin(\phi) = 0 \rightarrow \phi \quad = \quad n\pi, \; n \; integer$$

$$x_0 = Acos(n\pi) \rightarrow A(-1)^n \quad = \quad x_0$$

if $x_0 > 0$, we can take $A > 0$ and $n = 0$

Then,

$$A = x_0$$

Simple Harmonic Motion

For example, if $x(t=0)=x_0$ and $v(t=0)=0$,

$$v_0 = -A\omega sin(\phi) = 0 \rightarrow \phi = n\pi, \; n \; integer$$

$$x_0 = Acos(n\pi) \rightarrow A(-1)^n = x_0$$

if $x_0 > 0$, we can take $A > 0$ and $n = 0$

Then,

$$A = x_0$$

$$\boxed{x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)}$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$A cos(\phi)=0$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$A cos(\phi)=0 \rightarrow \phi = \boxed{\frac{(2n+1)}{2}\pi}, \quad n \ integer$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$Acos(\phi)=0 \rightarrow \phi = \boxed{\frac{(2n+1)}{2}\pi}, \;\; n \; integer$$

$$\rightarrow v_0 =$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$Acos(\phi)=0 \rightarrow \phi = \boxed{\frac{(2n+1)}{2}\pi}, \ \ n \ integer$$

$$\rightarrow v_0 = \ -A\omega sin\left(\frac{2n+1}{2}\pi\right)$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$Acos(\phi)=0 \rightarrow \phi = \boxed{\frac{(2n+1)}{2}\pi}, \quad n \ integer$$

$$\rightarrow v_0 = -A\omega sin\left(\frac{2n+1}{2}\pi\right) = \boxed{-A\omega(-1)^n}$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$A cos(\phi) = 0 \rightarrow \phi = \boxed{\frac{(2n+1)}{2}\pi}, \quad n\ integer$$

$$\rightarrow v_0 = -A\omega sin\left(\frac{2n+1}{2}\pi\right) = \boxed{-A\omega(-1)^n}$$

if $v_0 < 0$, we can take $A > 0$ and $n = 0$. Then,

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$A cos(\phi)=0 \to \phi = \boxed{\frac{(2n+1)}{2}\pi}, \quad n \ integer$$

$$\to v_0 = \ - A\omega sin\left(\frac{2n+1}{2}\pi\right) = \boxed{-A\omega(-1)^n}$$

if $v_0 < 0$, we can take $A > 0$ and $n = 0$. Then,

$$A = -\frac{v_0}{\omega}$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$Acos(\phi)=0\rightarrow\phi=\boxed{\frac{(2n+1)}{2}\pi},\ \ n\ integer$$

$$\rightarrow v_0=\ -A\omega sin\left(\frac{2n+1}{2}\pi\right)=\boxed{-A\omega(-1)^n}$$

if $v_0<0$, we can take $A>0$ and $n=0$. Then,

$$A=-\frac{v_0}{\omega}$$

$$x(t)=A\cos\left(\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right)$$

Simple Harmonic Motion

if $x(t=0)=0$ and $v(t=0)=v_0$, then

$$Acos(\phi)=0 \to \phi = \boxed{\frac{(2n+1)}{2}\pi}, \quad n \ integer$$

$$\to v_0 = -A\omega sin\left(\frac{2n+1}{2}\pi\right) = \boxed{-A\omega(-1)^n}$$

if $v_0<0$, we can take $A>0$ and $n=0$. Then,

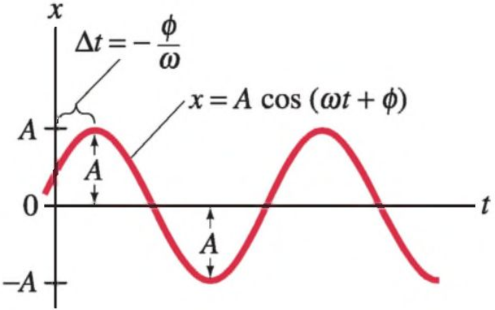
$$A=-\frac{v_0}{\omega}$$

$$x(t)=A\cos\left(\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right) = \boxed{Asin\left(\sqrt{\frac{k}{m}}t\right)}$$

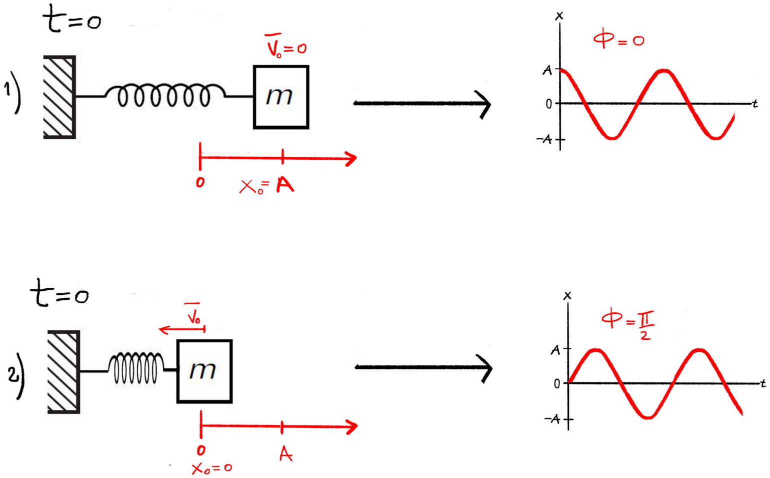
Summarizing...

For a Simple Harmonic Oscillator:

$$x(t) = Acos(\omega t + \phi)$$

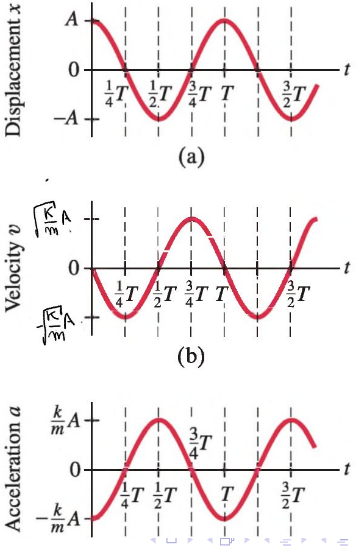


Summarizing...



Summarizing...

$\phi = 0 \rightarrow$



Simple Harmonic Motion

We can write the solution in a different form:

Simple Harmonic Motion

We can write the solution in a different form:

$$x(t) = A cos(\omega t + \phi)$$

Simple Harmonic Motion

We can write the solution in a different form:

$$x(t) = Acos(\omega t + \phi) = Acos(\phi)cos(\omega t) - Asin(\phi)sin(\omega t)$$

Simple Harmonic Motion

We can write the solution in a different form:

$$x(t) = Acos(\omega t + \phi) = Acos(\phi)cos(\omega t) - Asin(\phi)sin(\omega t)$$

Rename the constants:

Simple Harmonic Motion

We can write the solution in a different form:

$$x(t) = Acos(\omega t + \phi) = Acos(\phi)cos(\omega t) - Asin(\phi)sin(\omega t)$$

Rename the constants:

$$\begin{aligned} Acos(\phi) &= A' \\ Asin(\phi) &= B' \end{aligned}$$

Simple Harmonic Motion

We can write the solution in a different form:

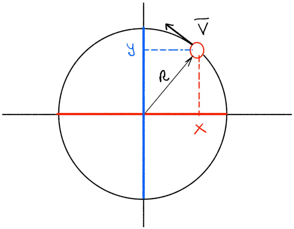
$$x(t) = Acos(\omega t + \phi) = Acos(\phi)cos(\omega t) - Asin(\phi)sin(\omega t)$$

Rename the constants:

$$\begin{aligned} Acos(\phi) &= A' \\ Asin(\phi) &= B' \end{aligned}$$

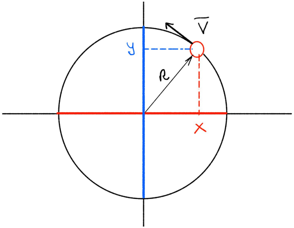
$$\rightarrow x(t) = A'cos(\omega t) + B'sin(\omega t)$$

Simple Harmonic Motion Related to Uniform Circular Motion



$$\vec{F} = -m\omega^2 R\hat{r} \rightarrow \begin{aligned} F_x &= -m\omega^2 Rcos(\omega t + \phi) \\ F_y &= -m\omega^2 Rsin(\omega t + \phi) \end{aligned}$$

Simple Harmonic Motion Related to Uniform Circular Motion



$$\vec{F} = -m\omega^2 R\hat{r} \rightarrow \begin{aligned} F_x &= -m\omega^2 Rcos(\omega t + \phi) \\ F_y &= -m\omega^2 Rsin(\omega t + \phi) \end{aligned}$$

$$\begin{aligned} x = Rcos(\omega t + \phi) \rightarrow F_x &= -m\omega^2 x \\ y = Rsin(\omega t + \phi) \rightarrow F_y &= -m\omega^2 y \end{aligned}$$

Simple Harmonic Motion Related to Uniform Circular Motion

We can analyze oscillatory motion in a simpler way if we imagine it to be a projection of something going in a circle.

If we do this, we will be able to analyze our one-dimensional oscillator with circular motions, which is a lot easier than having to solve a differential equation. The trick in doing this is to use complex numbers.

Generalization

The Simple Harmonic Oscillator equation is a linear differential equation,

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + a_1\frac{dx}{dt} + a_o x = 0$$

with $n = 2$

The trial solution for this kind of equations is $x(t) = Ae^{\alpha t}$.

If we replace this solution in the case of the Harmonic Oscillator, we obtain the equivalent equation

$$\alpha^2 + \frac{k}{m} = 0 \rightarrow \alpha = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

Generalization

The, the general solution of the equation is,

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{11}$$

Using the identity, $e^{i\omega t} = cos(\omega t) + isin(\omega t)$ Rearranging it, we can obtain again the solution in terms of *cos* and *sin*:

$$x(t) = A' cos(\omega t) + B' sin(\omega t) \tag{12}$$

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \rightarrow U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \rightarrow U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

where we set $U(x = 0) = 0$.

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \rightarrow U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

where we set $U(x=0) = 0$.

Then the total mechanical energy of a Simple Harmonic Oscillator is,

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \to U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

where we set $U(x=0)=0$.

Then the total mechanical energy of a Simple Harmonic Oscillator is,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{14}$$

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \rightarrow U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

where we set $U(x=0) = 0$.

Then the total mechanical energy of a Simple Harmonic Oscillator is,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{14}$$

If we use the expressions for x and v,

Energy in the Simple Harmonic Oscillator

The potential of a particle that has a Simple Harmonic motion is,

$$F = -kx \rightarrow U = -\int F dx = \frac{1}{2}kx^2 \tag{13}$$

where we set $U(x = 0) = 0$.

Then the total mechanical energy of a Simple Harmonic Oscillator is,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{14}$$

If we use the expressions for x and v,

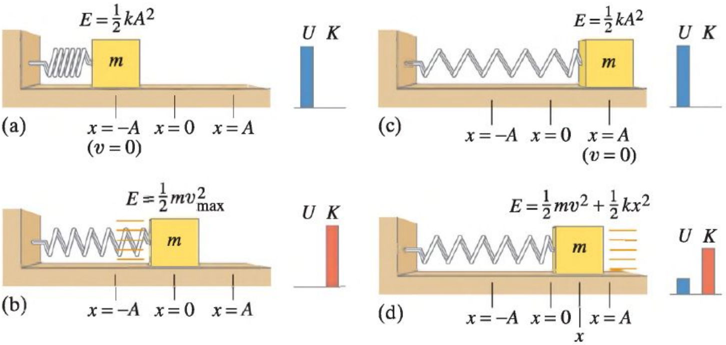
$$E = \frac{1}{2}m[A\omega sin(\omega t + \phi)]^2 + \frac{1}{2}k[Acos(\omega t + \phi)]^2 \tag{15}$$

Energy in the Simple Harmonic Oscillator

$$E = \frac{1}{2}kA^2[sin(\omega t + \phi)^2 + cos(\omega t + \phi)]^2 = \frac{1}{2}kA^2 \qquad (16)$$

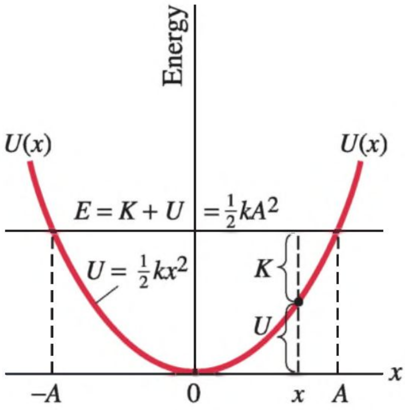
Energy in the Simple Harmonic Oscillator

$$E = \frac{1}{2}kA^2[sin(\omega t + \phi)^2 + cos(\omega t + \phi)]^2 = \frac{1}{2}kA^2 \qquad (16)$$



Energy in the Simple Harmonic Oscillator

Potential Energy Graph

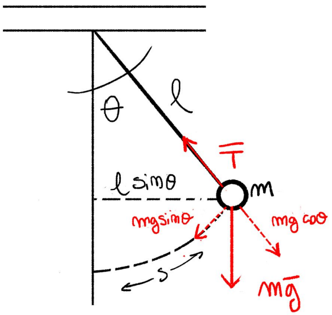


The Simple Pendulum

For small displacements, the motion of a symple pendulum is essentially simple harmonic.

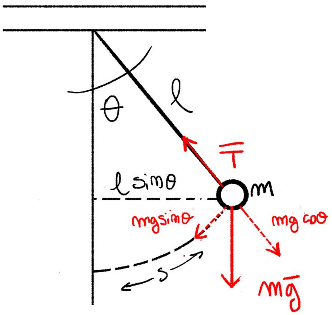
The Simple Pendulum

For small displacements, the motion of a symple pendulum is essentially simple harmonic.



The Simple Pendulum

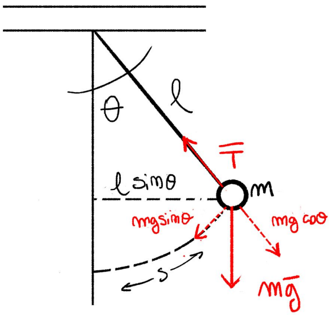
For small displacements, the motion of a symple pendulum is essentially simple harmonic.



Ecuation of motion?

The Simple Pendulum

For small displacements, the motion of a symple pendulum is essentially simple harmonic.



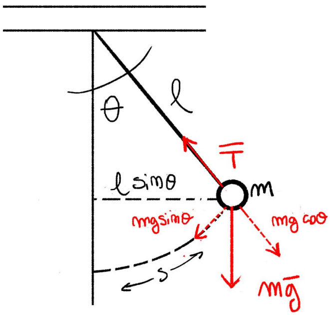
Ecuation of motion?

$$T - mgcos\theta = ma_c \quad (17)$$

$$-mgsin\theta = ma_T \quad (18)$$

The Simple Pendulum

For small displacements, the motion of a symple pendulum is essentially simple harmonic.



Ecuation of motion?

$$
\begin{aligned}
T - mgcos\theta &= ma_c \quad (17)\\
-mgsin\theta &= ma_T \quad (18)
\end{aligned}
$$

We are going to use the equation 18:

$$a_T + gsin\theta = 0 \quad (19)$$

The Simple Pendulum

Then we have to solve $a_T + gsin\theta = 0$, for θ.

The Simple Pendulum

Then we have to solve $a_T + gsin\theta = 0$, for θ.

Note that the acceleration is related with θ:

The Simple Pendulum

Then we have to solve $a_T + gsin\theta = 0$, for θ.

Note that the acceleration is related with θ:

$$
\begin{aligned}
a_T &= \ell\alpha \\
\alpha &= \frac{d^2\theta}{dt^2}
\end{aligned}
$$

The Simple Pendulum

Then we have to solve $a_T + gsin\theta = 0$, for θ.

Note that the acceleration is related with θ:

$$
\begin{aligned}
a_T &= \ell\alpha \\
\alpha &= \frac{d^2\theta}{dt^2}
\end{aligned}
$$

Then, we can write the equation for θ as,

The Simple Pendulum

Then we have to solve $a_T + gsin\theta = 0$, for θ.

Note that the acceleration is related with θ:

$$\begin{aligned} a_T &= \ell\alpha \\ \alpha &= \frac{d^2\theta}{dt^2} \end{aligned}$$

Then, we can write the equation for θ as,

$$\ell\frac{d^2\theta}{dt^2} + gsin\theta = 0 \qquad (20)$$

The Simple Pendulum

The equation 20 is not exactly the equation of a simple harmonic oscillator, but we can make the following approximation for small angles:

$$sin\theta \sim \theta$$

where θ is measured in radians.

The Simple Pendulum

Note that:

$$\begin{aligned} sin\theta &= \sum \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} \\ &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \end{aligned}$$

if $\theta \leq 15^\circ \rightarrow sin\theta \sim \theta$

$$\begin{aligned} sin(15^\circ) &= 0.2588190451 \\ 15^\circ &= 0.26179938779 \ rad \end{aligned}$$

The Simple Pendulum

Then, for small angles,

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The solution of this equation is:

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The solution of this equation is:

$$\theta(t) = \theta_0 cos(\omega t) \tag{22}$$

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The solution of this equation is:

$$\theta(t) = \theta_0 cos(\omega t) \tag{22}$$

were

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The solution of this equation is:

$$\theta(t) = \theta_0 cos(\omega t) \tag{22}$$

were

$$\omega = \sqrt{\frac{g}{\ell}} \tag{23}$$

The Simple Pendulum

Then, for small angles,

$$\ell \frac{d^2\theta}{dt^2} + g\theta = 0 \tag{21}$$

The solution of this equation is:

$$\theta(t) = \theta_0 cos(\omega t) \tag{22}$$

were

$$\omega = \sqrt{\frac{g}{\ell}} \tag{23}$$

and we consider that at $t = 0$, the initial angle is θ_0 and the velocity is $v_0 = 0$.

The Simple Pendulum

The period of the motion is,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \tag{24}$$

The Simple Pendulum

The period of the motion is,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \tag{24}$$

We can measure g using a pendulum!

The Simple Pendulum

The period of the motion is,

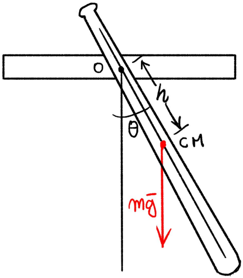
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \tag{24}$$

We can measure g using a pendulum!

Question: If a simple pendulum is taken from sea level to the top of a high mountain and started at the same angle of 5°, it would oscillate at the top of the mountain (a) slightly slower, (b) slightly faster, (c) at exactly the same frequency, (d) none of these.

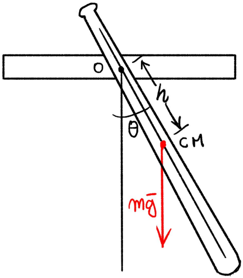
The Physical Pendulum

The term physical pendulum refers to any real extended object which oscillates back and forth.



The Physical Pendulum

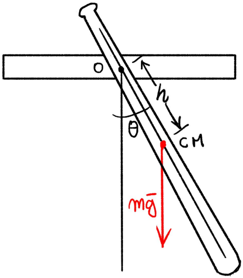
The term physical pendulum refers to any real extended object which oscillates back and forth.



$$torque \rightarrow N = -hmg\ sin\theta$$

The Physical Pendulum

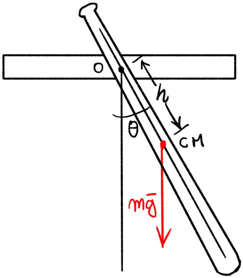
The term physical pendulum refers to any real extended object which oscillates back and forth.



$$
\begin{aligned}
torque \rightarrow N &= -hmg \; sin\theta \\
\rightarrow I\alpha &= -hmg \; sin\theta
\end{aligned}
$$

The Physical Pendulum

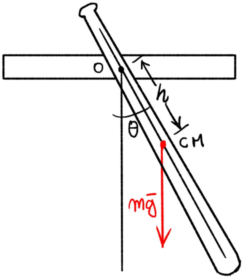
The term physical pendulum refers to any real extended object which oscillates back and forth.



$$\begin{aligned} torque \rightarrow N &= -hmg\ sin\theta \\ \rightarrow I\alpha &= -hmg\ sin\theta \\ \rightarrow I\frac{d^2\theta}{dt^2} &= -hmg\ sin\theta \end{aligned}$$

The Physical Pendulum

The term physical pendulum refers to any real extended object which oscillates back and forth.

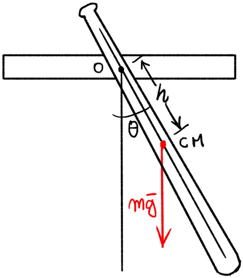


$$
\begin{aligned}
torque \rightarrow N &= -hmg\ sin\theta \\
\rightarrow I\alpha &= -hmg\ sin\theta \\
\rightarrow I\frac{d^2\theta}{dt^2} &= -hmg\ sin\theta
\end{aligned}
$$

The, the equation for small angles is,

The Physical Pendulum

The term physical pendulum refers to any real extended object which oscillates back and forth.



$$
\begin{aligned}
torque \rightarrow N &= -hmg\ sin\theta \\
\rightarrow I\alpha &= -hmg\ sin\theta \\
\rightarrow I\frac{d^2\theta}{dt^2} &= -hmg\ sin\theta
\end{aligned}
$$

The, the equation for small angles is,

$$\frac{d^2\theta}{dt^2} + \frac{hmg}{I}\theta = 0 \tag{25}$$

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

where,

$$\omega = \sqrt{\frac{hmg}{I}} \tag{26}$$

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

where,

$$\omega = \sqrt{\frac{hmg}{I}} \tag{26}$$

where the velocity at $t = 0$ is $v_0 = 0$ and the angle is $\theta(t = 0) = \theta_0$.

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

where,

$$\omega = \sqrt{\frac{hmg}{I}} \tag{26}$$

where the velocity at $t = 0$ is $v_0 = 0$ and the angle is $\theta(t = 0) = \theta_0$. The period of the oscillations is,

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

where,

$$\omega = \sqrt{\frac{hmg}{I}} \tag{26}$$

where the velocity at $t = 0$ is $v_0 = 0$ and the angle is $\theta(t = 0) = \theta_0$. The period of the oscillations is,

$$T = 2\pi\sqrt{\frac{I}{hmg}} \tag{27}$$

The Physical Pendulum

The solution of the equation 25 is:

$$\theta(t) = \theta_0 cos(\omega t)$$

where,

$$\omega = \sqrt{\frac{hmg}{I}} \tag{26}$$

where the velocity at $t = 0$ is $v_0 = 0$ and the angle is $\theta(t = 0) = \theta_0$. The period of the oscillations is,

$$T = 2\pi\sqrt{\frac{I}{hmg}} \tag{27}$$

We can measure the inertia moment using a pendulum motion.

Summarizing...

A restoring force gives place to an simple harmonic motion.

$$F = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Summarizing...

A restoring force gives place to an simple harmonic motion.

$$F = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution is...

Summarizing...

A restoring force gives place to an simple harmonic motion.

$$F = -kx \to \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution is...

$$x(t) = Acos(\omega t + \phi) = A'cos(\omega t) + B'sin(\omega t)$$

Summarizing...

A restoring force gives place to an simple harmonic motion.

$$F = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution is...

$$x(t) = Acos(\omega t + \phi) = A'cos(\omega t) + B'sin(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

Summarizing...

A restoring force gives place to an simple harmonic motion.

$$F = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution is...

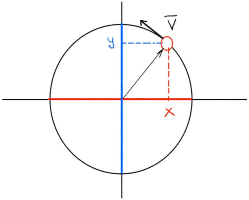
$$x(t) = Acos(\omega t + \phi) = A'cos(\omega t) + B'sin(\omega t)$$

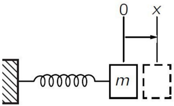
$$\omega = \sqrt{\frac{k}{m}}$$

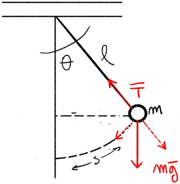
and the other constant depends on the intitial conditions.

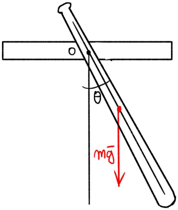
Summarizing...

All the following systems have simple harmonic motion..









Damped Harmonic Motion

Damped Harmonic Motion

Drag force due to the viscosity of the air:

Damped Harmonic Motion

Drag force due to the viscosity of the air:

$$\boxed{F_d = -bv}$$

Damped Harmonic Motion

Drag force due to the viscosity of the air:

$$\boxed{F_d = -bv}$$

Then,

$$ma = -kx - bv$$

Damped Harmonic Motion

Drag force due to the viscosity of the air:

$$\boxed{F_d = -bv}$$

Then,

$$ma = -kx - bv$$

$$\rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0} \tag{28}$$

Damped Harmonic Motion

The drag force depends on v,

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Trial solution:

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Trial solution:

$$x(t) = Ae^{-\gamma t} cos\omega' t$$

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Trial solution:

$$x(t) = Ae^{-\gamma t} cos\omega' t$$

What are γ and ω'?

Damped Harmonic Motion

The drag force depends on v, $\rightarrow$ it is not conservative $\rightarrow$ the energy is lost $\rightarrow$ the amplitude decreases

Trial solution:

$$x(t) = Ae^{-\gamma t} cos\omega' t$$

What are γ and ω'?

To find them, we have to introduce the solution in the equation (28)...

Damped Harmonic Motion

We find...

$$\gamma = \frac{b}{2m} \tag{29}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{30}$$

Damped Harmonic Motion

We find...

$$\gamma = \frac{b}{2m} \tag{29}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{30}$$

We can add a phase constant, in this case we considered $\phi = 0$, then

$$v_0 = 0, \;\; A = x_0$$

Damped Harmonic Motion

The frequency is,

Damped Harmonic Motion

The frequency is,

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{31}$$

Damped Harmonic Motion

The frequency is,

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (31)$$

Damped Harmonic Motion

The frequency is,

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{31}$$

So, the frequency is lower, and the period longer.

Damped Harmonic Motion

The frequency is,

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{31}$$

So, the frequency is lower, and the period longer.

Limit for b:

Damped Harmonic Motion

The frequency is,

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{31}$$

So, the frequency is lower, and the period longer.

Limit for b:

$$b^2 < 4mk \tag{32}$$

Damped Harmonic Motion

When $b^2 > 4mk$, the solution is:

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{33}$$

Damped Harmonic Motion

When $b^2 > 4mk$, the solution is:

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{33}$$

a_1 *and* a_2

Damped Harmonic Motion

When $b^2 > 4mk$, the solution is:

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{33}$$

a_1 *and* a_2 $\rightarrow$ *depend on* m, k, *and* b

Damped Harmonic Motion

When $b^2 > 4mk$, the solution is:

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{33}$$

a_1 *and* a_2 $\rightarrow$ *depend on m, k, and b*

C_1 *and* C_2

Damped Harmonic Motion

When $b^2 > 4mk$, the solution is:

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t} \tag{33}$$

a_1 *and* a_2 $\rightarrow$ *depend on* m, k, *and* b

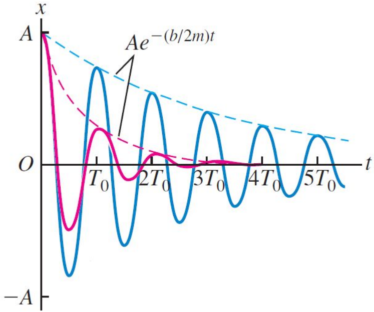
C_1 *and* C_2 $\rightarrow$ *depend on the intitial conditions*

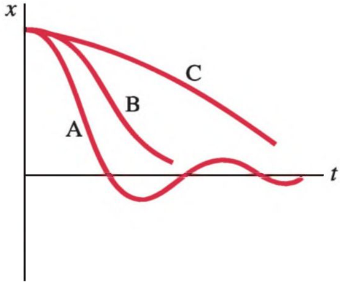
Damped Harmonic Motion

Then, the solutions look like this...

Damped Harmonic Motion

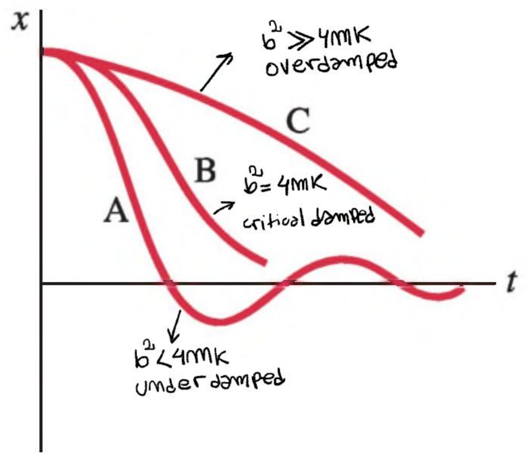
Then, the solutions look like this...





Damped Harmonic Motion

Then, the solutions look like this...



Damped Harmonic Motion

Energy in damped Oscillations

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt}$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kx \frac{dx}{dt}$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kx \frac{dx}{dt}$$

$$\rightarrow \frac{dE}{dt}$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

$$\rightarrow \frac{dE}{dt} = v_x(ma_x + kx)$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kx \frac{dx}{dt}$$

$$\rightarrow \frac{dE}{dt} = v_x(ma_x + kx)$$

$$\rightarrow \frac{dE}{dt}$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

$$\rightarrow \frac{dE}{dt} = v_x(ma_x + kx)$$

$$\rightarrow \frac{dE}{dt} = v_x(-bv_x)$$

Damped Harmonic Motion

Energy in damped Oscillations

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

$$\rightarrow \frac{dE}{dt} = v_x(ma_x + kx)$$

$$\rightarrow \frac{dE}{dt} = v_x(-bv_x) = -bv_x^2$$

Example 1

A simple pendulum has a length of ℓ. It is set swinging with small-amplitude oscillations. After a time Δt, the amplitude is only 50% of what it was initially, (a) What is the value of γ for the motion? (b) By what factor does the difference between the frequencies, $f - f'$, differ from f, the undamped frequency?

Test your understanding...

1. Is the acceleration of a simple harmonic oscillator ever zero? If so, where?

Test your understanding...

1. Is the acceleration of a simple harmonic oscillator ever zero? If so, where?
2. For a simple harmonic oscillator, when (if ever) are the displacement and velocity vectors in the same direction? When are the displacement and acceleration vectors in the same direction?

Test your understanding...

1. Is the acceleration of a simple harmonic oscillator ever zero? If so, where?
2. For a simple harmonic oscillator, when (if ever) are the displacement and velocity vectors in the same direction? When are the displacement and acceleration vectors in the same direction?
3. Two equal masses are attached to separate identical springs next to one another. One mass is pulled so its spring stretches 20 *cm* and the other is pulled so its spring stretches only 10 *cm*. The masses are released simultaneously. Which mass reaches the equilibrium point first?

Test your understanding...

1. Is the acceleration of a simple harmonic oscillator ever zero? If so, where?
2. For a simple harmonic oscillator, when (if ever) are the displacement and velocity vectors in the same direction? When are the displacement and acceleration vectors in the same direction?
3. Two equal masses are attached to separate identical springs next to one another. One mass is pulled so its spring stretches 20 cm and the other is pulled so its spring stretches only 10 cm. The masses are released simultaneously. Which mass reaches the equilibrium point first?
4. A thin uniform rod of mass m is suspended from one end and oscillates with a frequency f . If a small sphere of mass $2m$ is attached to the other end, does the frequency increase or decrease? Explain.

Forced oscillations; Resonance

Adding an external force

Forced oscillations; Resonance

Adding an external force

$$F_{ext} = F_0 cos \omega t \tag{34}$$

Forced oscillations; Resonance

Adding an external force

$$F_{ext} = F_0 cos\omega t \tag{34}$$

$\omega_0 \neq \sqrt{\frac{k}{m}}$ (natural frequency of the spring).

If the frequency of the force is near the natural frequency of the spring, the amplitude of the motion can become very large. This effect is known as **resonance** and the natural frequency of the system is f_0, the **resonant frequency**.

Forced oscillations; Resonance

Equation of motion:

$$ma = -kx - bv + F_0 cos\omega t \tag{35}$$

Forced oscillations; Resonance

Equation of motion:

$$ma = -kx - bv + F_0 cos\omega t \quad (35)$$

$$\rightarrow \frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 cos\omega t \quad (36)$$

Forced oscillations; Resonance

Equation of motion:

$$ma = -kx - bv + F_0 cos\omega t \tag{35}$$

$$\rightarrow \frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 cos\omega t \tag{36}$$

The trial solution is this time,

Forced oscillations; Resonance

Equation of motion:

$$ma = -kx - bv + F_0 cos\omega t \tag{35}$$

$$\rightarrow \frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 cos\omega t \tag{36}$$

The trial solution is this time,

$$x(t) = A_0 sin(\omega t + \phi_0) \tag{37}$$

Forced oscillations; Resonance

We can find A_0 and ϕ_0 by direct substitution,

Forced oscillations; Resonance

We can find A_0 and ϕ_0 by direct substitution,

$$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}}$$

Forced oscillations; Resonance

We can find A_0 and ϕ_0 by direct substitution,

$$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}}$$

and

$$\phi_0 = tan^{-1}\frac{\omega_0^2 - \omega^2}{\omega(b/m)}$$

Forced oscillations; Resonance

We can find A_0 and ϕ_0 by direct substitution,

$$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}}$$

and

$$\phi_0 = tan^{-1}\frac{\omega_0^2 - \omega^2}{\omega(b/m)}$$