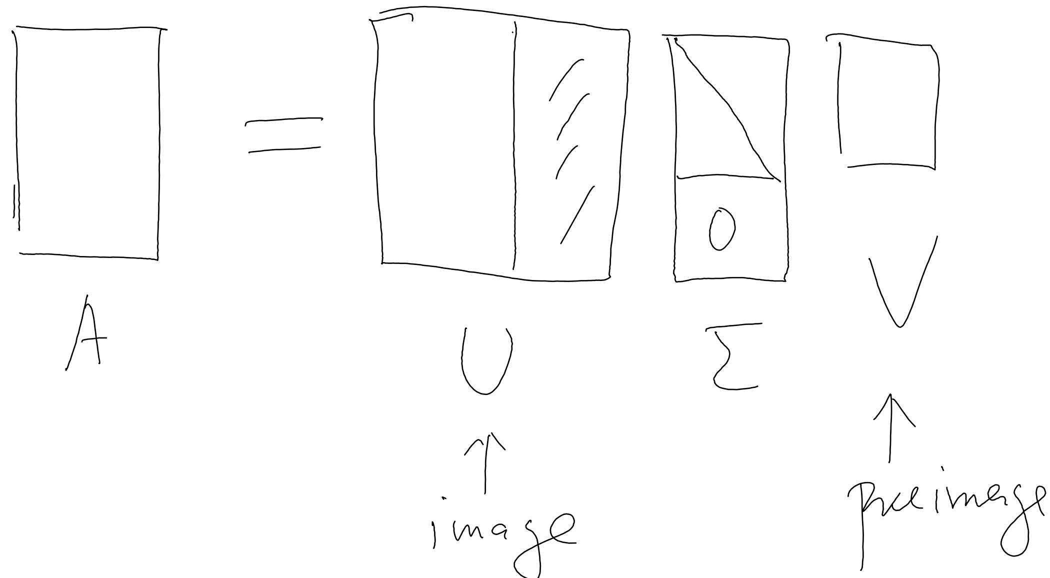
2017-08-28

Monday, August 28, 2017 3:06 PM

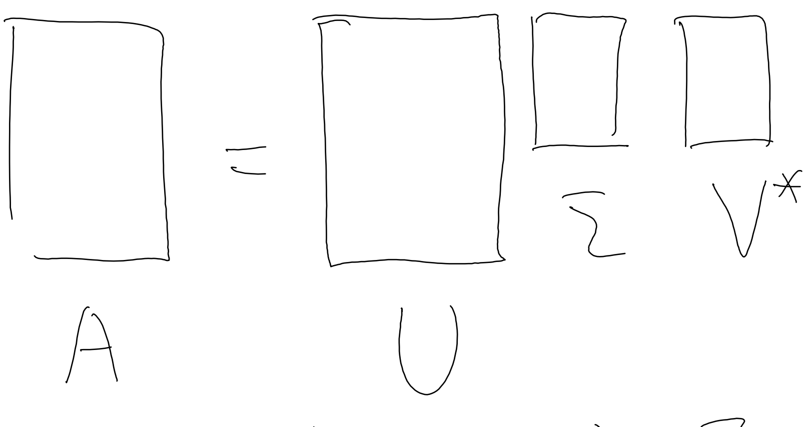
$A = U\Sigma V^*$ $\quad A \in \mathbb{R}^{m\times n}$

$U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, $V \in \mathbb{R}^{n\times n}$

$m \geq n$,



Reduced SVD:



Thm: (Existence and Uniqueness). Every matrix $A \in \mathbb{C}^{m\times n}$ has a SVD. The singular values are uniquely determined. If A is square and σ_j are distinct, the left and right singular vectors $\{\vec{u}_i\}$ and $\{\vec{v}_i\}$ are uniquely determined up to a complex sign.

Properties:

(1). $\text{Rank}(A) = \#$ of nonzero singular values.

$$A^*A = V\Sigma^*\cancel{U^*U}\Sigma V^*$$
$$= V\Sigma^*\Sigma V^* = V\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{bmatrix}V^*$$

(2) $\|A\|_2 = \sigma_1 \qquad (\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r)$

$\|A\|_F = \left(\sum_{i=1}^{r} \sigma_i^2\right)^{\frac{1}{2}}$ HW.

(3) $|\det(A)| = \prod_{i=1}^{m} \sigma_i$ if $A \in \mathbb{R}^{m\times m}$

Gaussian Elimination.

$$L_3(L_2(L_1A))$$

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \xrightarrow{L_1} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\times} & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{L_1} \begin{bmatrix} 0 & \times & \times & \times \\ 0 & \boxed{\times} & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \boxed{\times} & \times \end{bmatrix}$$

$$\xrightarrow{L_3} \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$$

$$\vec{x}_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ x_{(k+1)k} \\ \vdots \\ x_{m,k} \end{bmatrix} \xrightarrow{L_k} \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$-l_{jk} = -\frac{x_{jk}}{x_{kk}} \qquad m \geq j \geq (k+1)$$

$$L_k = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{(k+1)k} & \ddots & \\ & & \vdots & & \\ & & -l_{m,k} & & 1 \end{bmatrix}$$

$$L_k = I - \vec{l}_k \vec{e}_k^{\,*} \qquad \text{where}$$

$$\vec{l}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{bmatrix}, \quad \vec{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k+1$$

$$(\underline{I} - \vec{l}_k \vec{e}_k^*)(\underline{I} + \vec{l}_k \vec{e}_k^*)$$

$$= \underline{I} - \vec{l}_k \underbrace{\vec{e}_k^* \vec{l}_k}_{0} \vec{e}_k^* \qquad \vec{e}_k^* \vec{l}_k = 0$$

$$= \underline{I}$$

$$L_k^{-1} = \underline{I} + \vec{l}_k \vec{e}_k^*$$

$$L_k^{-1} L_{k+1}^{-1} = (\underline{I} + \vec{l}_k \vec{e}_k^*)(\underline{I} + \vec{l}_{k+1} \vec{e}_{k+1}^*)$$

$$= \underline{I} + \vec{l}_k \vec{e}_k^* + \vec{l}_{k+1} \vec{e}_{k+1}^* + \vec{l}_k \underbrace{\vec{e}_k^* \vec{l}_{k+1}}_{0} \vec{e}_{k+1}^*$$

$$= \underline{I} + \vec{l}_k \vec{e}_k^* + \vec{l}_{k+1} \vec{e}_{k+1}^*$$

$$= \left[\vec{e}_1 \,\middle|\, \cdots \,\middle|\, \vec{e}_k + \vec{l}_k \,\middle|\, \vec{e}_{k+1} + \vec{l}_k \,\middle|\, \cdots \,\middle|\, \vec{e}_m \right]$$

$$L \cdots L_2 L_1 A = U \leftarrow \text{upper}$$

$$L_{m-1} \cdots L_2 L_1 A = U \leftarrow \text{upper triangular}$$

$$A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}}_{L} U$$

$$A = LU$$

LU factorization.

$$L = \begin{bmatrix} 1 & & & 0 \\ l_{21} & \ddots & & \\ \vdots & & & \\ l_{m1} & l_{m2} & \cdots & 1 \end{bmatrix}$$