$$\begin{cases} -u'' = f \\ u(0)=1, \ u(1)=2 \end{cases}$$

finite difference BC

$$A\vec{u} = \vec{f}, \quad \vec{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

blurring:

$$\underbrace{g(x)}_{\text{obtain}} = \underbrace{u(x)}_{\text{original}} + \boxed{\int \underbrace{K(x-y)}_{\substack{\text{convolution} \\ \text{kernel}}} u(y)\,dy}$$

$$\vec{u}(x) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}$$

Toeplitz

$$\vec{g} = A\vec{u}, \quad A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} k_0 & k_1 & \cdots & k_{N-1} \\ k_1 & k_0 & \cdots & \vdots \\ \vdots & & \ddots & \\ k_{N-1} & \cdots & k_1 & k_0 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

Two important issues:

- efficiency;
- stability: sensitivity to perturbations

Review part of linear algebra.

vector norms: $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}^+$

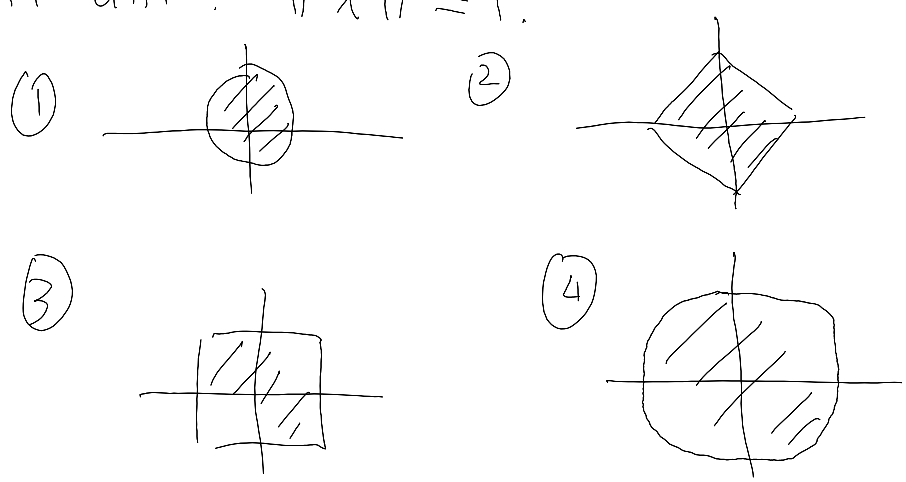
e.g. ①. $\|\vec{x}\|_2^2 = \vec{x}\cdot\vec{x} = \vec{x}^*\vec{x} = \langle \vec{x}, \vec{x}\rangle = \vec{x}^T\vec{x}$

②. l^1-norm: $\|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i|$

③. l^∞-norm: $\|\vec{x}\|_\infty = \max_{1\le i\le m} |x_i|$

④ l^p-norm: $\|\vec{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}}$ $\quad 1 \le p < \infty$

unit disk. $\|\vec{x}\| \le 1$.



What about $p \in [0, 1)$.

⑤. Weighted norm:

$\|\vec{x}\|_W = \|W\vec{x}\|_2$ $\qquad$ Given $W = \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{bmatrix}$

$w_i > 0$

matrix norms: $\qquad A \in \mathbb{R}^{m\times n}$

two ways to define:

①. view the matrix as a vector in $\mathbb{R}^d$, $d = m\times n$.

① view the matrix as a vector in $\mathbb{R}^{d}$, $d = m \times n$.

$$\|A\|_F = \left(\sum_{i,j} |x_{ij}|^2\right)^{\frac{1}{2}}$$

② Induced norm (preferred way).

$$A : \mathbb{R}^n \to \mathbb{R}^m.$$

$\forall \vec{x} \in \mathbb{R}^n$, $A\vec{x} \in \mathbb{R}^m$

$$\|A\|_{(m,n)} = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}} \qquad \begin{matrix}\sup \\ \max\end{matrix}$$

where $\| \; \|_{(n)}$ is a vector norm in $\mathbb{R}^n$.

$$= \sup_{\substack{\vec{x} \in \mathbb{R}^n \\ \|\vec{x}\|_{(n)} = 1}} \|A\vec{x}\|_{(m)}$$

$A = [\vec{a}_1 | \cdots | \vec{a}_n]$

e.g. $\|A\|_1 = \max_{1 \leq j \leq n} \|\vec{a}_j\|_1$

$\|A\|_\infty = \max_{1 \leq i \leq m} \|\vec{a}_i^*\|_1$, $\vec{a}_i^*$ is ith row of A.

What's the induced 2-norm?

$$\|A\|_2 = ?$$

unitary transformation:

Q unitary matrix. $QQ^* = I$

$Q^*Q = I$

$\|QA\|_2 = \|A\|_2$.

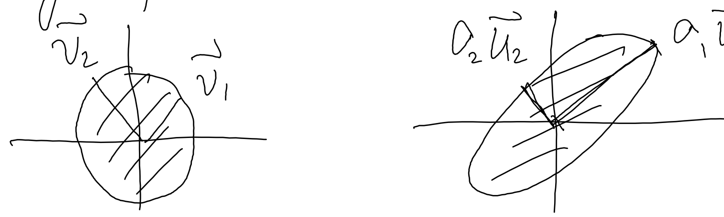
$\|QA\|_F = \|A\|_F$

Singular value decomposition (SVD)

$A \in \mathbb{R}^{2\times 2}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$

image of unit disk (norm-2)



$\|\vec{u}_1\| = \|\vec{u}_2\| = 1$

$$A\underbrace{[\vec{v}_1, \vec{v}]}_{V} = [\sigma_1 \vec{u}_1, \sigma_2 \vec{u}_2]$$

$$= \underbrace{[\vec{u}_1, \vec{u}_2]}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma}$$

$$AV = U\Sigma \iff A = U\Sigma V^*$$