Problem 6.4.3

Let $S_i(x) = \frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + C(x-t_i) + D(t_{i+1}-x)$ where $h_i \equiv t_{i+1} - t_i$. Now let $t_{i+1} \equiv t_i + h_i$. We find that

$$\begin{aligned} S_i(x) &= \frac{z_i}{6h_i}(t_i + h_i - x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + C(x-t_i) + D(t_i + h_i - x) \\ &= -\frac{z_i}{6h_i}(x - t_i - h_i)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + C(x-t_i) - D(x - t_i - h_i) \end{aligned}$$

as required.

Problem 6.4.7

Let

$$f(x) = \begin{cases} a(x-2)^2 + b(x-3)^3 & x \in (-\infty, 1] \\ c(x-2)^2 & x \in [1,3] \\ d(x-2)^2 + e(x-3)^3 & x \in [3, \infty) \end{cases}$$

Define $f_1(x)$ to be the function defined on $(-\infty, 1]$, define $f_2(x)$ to be the function over $[1,3]$ and define $f_3(x)$ to be the function defined on $[3,\infty)$. Then we find the following derivatives:

$$\begin{aligned} &f_1(x) = a(x-2)^2 + b(x-3)^3 && f_2(x) = c(x-2)^2 && f_3(x) = d(x-2)^2 + e(x-3)^3 \\ &f_1'(x) = 2a(x-2) + 3b(x-1)^2 && f_2'(x) = 2c(x-2) && f_3'(x) = 2d(x-2) + 3e(x-3)^2 \\ &f_1''(x) = 2a + 6b(x-1) && f_2''(x) = 2c && f_3''(x) = 2d + 6e(x-3) \end{aligned}$$

For cubic splines, we know they are C^2 so their derivatives must be continuous at the given knots. From this, we find that $a = c = d$. Any nonzero values for b, e will make this system a cubic spline. If they were zero, then this would simply be a polynomial, and not piecewise. Now, using the parameters given from the problem, placing $x = 1$ into the system finds $a = c = d = 7$. Setting $x = 0$, we find $7(-2)^2 + b(-1)^3 = 26 \Rightarrow b = 2$. Setting $x = 4$, we find $7(2)^2 + e(1)^3 = 25 \Rightarrow e = -3$.

Problem 6.4.11

Let

$$f(x) = \begin{cases} 3 + x - 9x^2 & x \in [0,1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [1,2] \end{cases}$$

Using a similar scheme for defininining functions from Problem 7, we find the following derivatives:

$$\begin{aligned} f_1(x) &= 3 + x - 9x^2 & f_2(x) &= a + b(x-1) + c(x-1)^2 + d(x-1)^3 \\ f_1'(x) &= 1 - 18x & f_2'(x) &= b + 2c(x-1) + 3d(x-1)^2 \\ f_1''(x) &= -18 & f_2''(x) &= 2c + 6d(x-1) \end{aligned}$$

From this, letting $x = 1$ we find that $a = -5, b = -17, c = -9$. Now letting $f''(2) = 2(-9) + 6d(2-1) = 0 \Rightarrow d = 3$.

Problem 6.4.12

Let

$$f(x) = \begin{cases} x^3 + x & x \leq 0 \\ x^3 - x & x \geq 0 \end{cases}$$

From this we find the following derivatives:

$$\begin{aligned} f_1(x) &= x^3 + x & f_2(x) &= x^3 - x \\ f_1'(x) &= 3x^2 + 1 & f_2'(x) &= 3x^2 - 1 \\ f_1''(x) &= 6x & f_2''(x) &= 6x \end{aligned}$$

This is not a cubic spline, as $f_1'(0) \neq f_2'(0)$, so its first derivative is not continuous. Thus, $\lim\limits_{x \uparrow 0} 6x = 0$ and $\lim\limits_{x \downarrow 0} 6x = 0$, so $\lim\limits_{x \uparrow 0} f''(x) = \lim\limits_{x \downarrow 0} f''(x)$.

Problem 6.4.13

Let

$$f(x) = \begin{cases} 1 + x - x^3 & x \in [0, 1] \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & x \in [1, 2] \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3 & x \in [2, 3] \end{cases}$$

From this we find the following derivatives:

$$\begin{aligned} f_1(x) &= 1 + x - x^3 & f_2(x) &= 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & f_3(x) &= 4(x-2) + 9(x-2)^2 - 3(x-2)^3 \\ f_1'(x) &= 1 - 3x^2 & f_2'(x) &= -2 - 6(x-1) + 12(x-1)^2 & f_3'(x) &= 4 + 18(x-2) - 9(x-2)^2 \\ f_1''(x) &= -6x & f_2''(x) &= -6 + 24(x-1) & f_3''(x) &= 18 - 18(x-2) \end{aligned}$$

Upon examining these derivatives, we find that this spline does interpolate the given data. All the knots give the approrpiate answers, and the derivatives are continuous at the knots. Additionally, $f''(0) = f''(3) = 0$, confirming it is natural.