Problem 6.2.3

Let $f \in C^n[a,b]$. Suppose $x_0 \in (a,b)$ and $x_i \to x_0$ for $1 \leq k \leq n$. By Theorem 4 of the section, we know $f[x_0, x_1, ..., x_n] = \frac{1}{n!} f^{(n)}(\xi)$, where $\xi \in (a,b)$. As each x_i converges to x_0, it must be that $\xi \to x_0$ as well. Thus we find that $f[x_0, x_1, ..., x_n]$ will converge to $\frac{1}{n!} f^{(n)}(x_0)$, as desired.

Problem 6.2.4

Suppose f is a polynomial of degree k. Consider for any $n > k$ $f[x_0, x_1, ..., x_n]$. By definition, $f[x_0, x_1, ..., x_n] = c_n$, where c_n is the coefficient for x^n. As f is only a polynomial of degree k, any term of degree greater than k must be zero, or else f would be of that degree. Thus, $f[x_0, x_1, ..., x_n] = 0$.

Problem 6.2.6

We proceed to prove this by induction. To start, we look at the base case:

$$\begin{aligned}(\alpha f + \beta g)[x_0] &= (\alpha f + \beta g)(x_0) \\ &= \alpha f(x_0) + \beta g(x_0) \\ &= \alpha f[x_0] + \beta g[x_0]\end{aligned}$$

Thus, the base case holds. Now for the inductive step, suppose $(\alpha f + \beta g)[x_0, x_1, ..., x_{n-1}] = \alpha f[x_0, x_1, ..., x_{n-1}] + \beta g[x_0, x_1, ..., x_{n-1}]$. So $f[x_0, x_1, ..., x_n] = \dfrac{f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$ and in turn we have $\alpha f[x_0, x_1, ..., x_n] = \alpha \dfrac{f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$. Likewise we have similar results for $g(x)$. So we find that their sum is $\dfrac{\alpha(f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]) + \beta(g[x_1, x_2, ..., x_n] - g[x_0, x_1, ..., x_{n-1}])}{x_n - x_0}$. Notice that by our inductive hypothesis, we may rewrite our numerator as $\dfrac{(\alpha f + \beta g)[x_1, x_2, ..., x_n]}{x_n - x_0} - \dfrac{(\alpha f + \beta g)[x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$. From this, it is clear to see that $(\alpha f + \beta g)[x_0, x_1, ..., x_n] = \alpha f[x_0, x_1, ..., x_n] + \beta g[x_0, x_1, ..., x_n]$, as desired.

Problem 6.2.7

We know that $f[x_i, x_{i+1}] = \dfrac{f[x_i] - f[x_{i+1}]}{x_i - x_{i+1}}$. Recall that $f[x_i, x_{i+1}] = f'(x_i)$. Thus we find that

$$\begin{aligned}(fg)' &= (fg)[x_i, x_{i+1}] \\ &= \frac{(fg)[x_i] - (fg)[x_{i+1}]}{x_i - x_{i+1}} \\ &= \frac{f(x_i)g(x_i) - f(x_{i+1})g(x_{i+1})}{x_i - x_{i+1}} \\ &= \frac{f(x_i)g(x_i) - f(x_{i+1})g(x_i) + f(x_{i+1})g(x_i) - f(x_{i+1})g(x_{i+1})}{x_i - x_{i+1}} \\ &= \frac{(f(x_i) - f(x_{i+1}))g(x_i)}{x_i - x_{i+1}} + \frac{f(x_{i+1})(g(x_i) - g(x_{i+1}))}{x_i - x_{i+1}} \\ &= f[x_i, x_{i+1}]g[x_i] + f[x_{i+1}]g[x_i, x_{i+1}] \\ &= f'g + fg'\end{aligned}$$

Problem 6.2.19

Let $u(x)$ interpolate f at $x_0, x_1, ..., x_{n-1}$ and let $v(x)$ interpolate f at $x_1, x_2, ..., x_n$. Suppose $g(x) = \frac{[(x_n - x_0)u(x) + (x - x_0)v(x)]}{x_n - x_0}$. We wish to show that $g(x)$ interpolates f for $x_0, x_1, ..., x_n$. Let $x = x_0$. Then we have $g(x_0) = \frac{(x_n - x_0)u(x_0) + 0}{x_n - x_0} = u(x_0)$. Since $u(x)$ interpolates f at x_0, $g(x)$ does too. Now let $x = x_1$. Then $g(x_1) = \frac{(x_n - x_1)u(x_1) + (x_1 - x_0)v(x_1)}{x_n - x_0} = \frac{x_n u(x_1) - x_1 u(x_1) + x_1 v(x_1) - x_0 v(x_1)}{x_n - x_0}$. Since both $u(x)$ and $v(x)$ interpolate f at x_1, $u(x_1) = v(x_1)$. So then $g(x_1) = \frac{x_n u(x_1) - x_0 u(x_1)}{x_n - x_0} = u(x_1)$, so again $g(x)$ interpolates f. This will be true for any x_i where $1 \leq i \leq n-1$. Now let $x = x_n$. Thus $g(x_n) = \frac{0 + (x_n - x_0)v(x_n)}{x_n - x_0} = v(x_n)$. Since $v(x)$ interpolates f at x_n, $g(x)$ does too and in turn $g(x)$ interpolates f for $x_0, x_1, ..., x_n$ as desired.

Problem 6.2.20

Notice that by construction, any a_i is linear, so in turn any b_i is quadratic, as it multiplies a_i terms by x. Then c_i multiplies b_i by x, so it must be cubic.

Problem 6.2.21

This may be generalized as $p_{i,i}(x) = y_i$ and $p_{i,j}(x) = \frac{(x_j - x)p_{i,j-1}(x) + (x - x_i)p_{i+1,j}(x)}{x_j - x_i}$.

Problem 6.2.22

The divided difference table is as follows:

0	51	-48	23	$-\frac{16}{7}$
1	3	-2	7	
2	1	40		
7	201			

Thus, the Newton interpolating polynomial will be: $51 - 48x + 23x(x-1) - \frac{16}{7}x(x-1)(x-2)$.

Problem 6.2.23

With the original polynomial, we find $p(3) = -38$. Since we want a polynomial $q(x)$ such that $q(3) = 10$, we let $q(x) = p(x) + x(x+1)(x-1)(x-2)$ to get the desired result.

Problem 6.2.24

The divided difference table is as follows:

4	63	26	6	1
2	11	2	5	
0	7	7		
3	28			

Thus, the Newton interpolating polynomial will be: $p(x) = 63 + 26(x-4) + 6(x-4)(x-2) + (x-4)(x-2)x$.

Problem 6.3.1

The divided difference table is as follows:

0	2	-9	3	7	5
0	2	-6	10	17	
1	-4	4	44		
1	-4	48			
2	44				

Thus, the extended Newton interpolating polynomial will be: $p(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$.

Problem 6.3.2

In the previous problem, we determined $p(x)$ as a quartic polynomial. Now we wish to find some $q(x)$ that satisfies $q(3) = 2$ and interpolates all of the previous points for $p(x)$. Notice that $p(3) = 308$, so we seek some $g(x)$ to append to our previous polynomial such that $g(3) = -306$ and $q(3) = p(3) + g(3) = 2$. We let $g(x) = -\frac{17}{2}x^2(x-1)^2(x-2)$, so $g(3) = -306$. Thus we find our quintic polynomial to be $q(x) = p(x) - \frac{17}{2}x^2(x-1)^2(x-2)$.

Problem 6.3.3

For Hermite interpolation, when given $p(x_i) = c_{i0}$ and $p'(x_i) = c_{i1}$, we have $p(x) = \sum_{i=0}^{n} c_{i0}A_i(x) + \sum_{i=0}^{n} c_{i1}B_i(x)$, where $A_i(x) = [1 - 2(x - x_i)l_i'(x_i)]l_i^2(x)$ and $B_i(x) = (x - x_i)l_i^2(x)$ for $0 \leq i \leq n$. Thus, given $p(x_i) = y_i$ and $p'(x_i) = 0$, we find $p(x) = \sum_{i=0}^{n} y_i[1 - 2(x - x_i)l_i'(x_i)]l_i^2(x)$, where $l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$.