Problem 6.1.1

For these problems, we will use Lagrange interpolation, where $l_i = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$.

a. From this, we find $l_0 = \frac{x-7}{10}$ and $l_1 = \frac{x-3}{10}$, so our interpolation becomes

$$p_1(x) = 5 * \frac{x-7}{10} - \frac{x-3}{10}.$$

b. Using the table, we find $l_0 = \frac{(x-1)(x-2)}{30} = \frac{x^2-3x+2}{30}$, $l_1 = \frac{x^2-9x+14}{6}$, $\frac{x^2-8x+7}{-5}$. So we find

$$p_2(x) = 146 * \frac{x^2-3x+2}{30} + 2 * \frac{x^2-4x+14}{6} - \frac{x^2-8x+7}{5}.$$

c. We find $l_0 = \frac{(x-7)(x-1)(x-2)}{(3-7)(3-1)(3-2)}, l_1 = \frac{(x-3)(x-1)(x-2)}{(7-3)(7-1)(7-2)}, l_2 = \frac{(x-3)(x-7)(x-2)}{(1-3)(1-7)(1-2)}$, and $l_3 = \frac{(x-3)(x-7)(x-1)}{(2-1)(2-7)(2-3)}$. Then

$$p_3(x) = 10l_0 + 146l_1 + 2l_2 + l_3.$$

d. This problem will have the same l_i for $0 \leq i \leq 3$, but with different y_i values. So,

$$p_3(x) = 12l_0 + 146l_1 + 2l_2 + l_3.$$

e. Since most of the y-values are zero, we find that

$$p_5(x) = \frac{(x-1.5)(x-2.7)(x-3.1)(x+6.6)(x-11.0)}{(-2.1-1.5)(-2.1-2.7)(-2.1-3.1)(-2.1+6.6)(-2.1-11.0)}.$$

Problem 6.1.2

From Lagrange, we know $p(x) = \sum_{k=0}^{n} y_k l_k(x)$. From the problem, we know $\exists L : f \mapsto p \Rightarrow Lf = p(x)$ and $Lf = \sum_{k=0}^{n} y_k l_k$. But, $y_k = f(x_k)$, so we have $Lf = \sum_{k=0}^{n} f(x_k) l_k$ as desired. Now let's apply L to $(af + bg)$. We then find

$$\begin{aligned}
L(af+bg) &= \sum_{i=0}^{n} (af(x_i) + bg(x_i)) l_i \\
&= \sum_{i=0}^{n} af(x_i)l_i + bg(x_i)l_i \\
&= a\sum_{i=0}^{n} f(x_i)l_i + b\sum_{i=0}^{n} g(x_i)l_i \\
&= aLf + bLg.
\end{aligned}$$

Problem 6.1.3

Let $Gf = \sum_{i=0}^{n} f(x_i) l_i^2$. From the text, l_i is a polynomial of degree i. Thus, there is a term $f(x_n)l_n^2$, where l_n is a polynomial of degree n. Since l_n is squared, its leading term of ax^n will become a^2x^{2n} for some nonzero a. Thus, Gf is a polynomail of degree at most 2n. Each $l_i = \prod_{j=0, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$. Thus, $l_i(x_j) = 0$ if $i \neq j$ and $l_i(x_i) = 1$ if $i = j$. So for any given x_i, $Gf(x_i) = f(x_i)$, so it interpolates f at the nodes. Regardless of what x_i we choose, $(l_i(x_i))^2$ will be positive by the squaring. Thus, only $f(x_i)$ determines the sign, so Gf

is nonnegative so long as f is too.

Problem 6.1.4

Let q be an arbitrary polynomial of degree at most n, and let $p = Lq$, so $p(x) = Lq = \sum_{i=0}^{n} q(x_i)l_i$. Thus, $p(x)$ is a polynomial of degree at most n, and $p(x_i) - q(x_i) = 0$ for $0 \leq i \leq n$. This implies that $p - q$ is a polynomail of degree at most n with n+1 distinct roots, so $p - q \equiv 0$. Thus, $p = q$ and $Lq = q$.

Problem 6.1.5

Recall that $Lf = \sum_{i=0}^{n} f(x_i)l_i$. Suppose $f(x_i) = 1$ for $0 \leq i \leq n$. Then $q(x) = Lf = \sum_{i=0}^{n} f(x_i)l_i = \sum_{i=0}^{n} l_i$. Since $q(x)$ interpolates $f(x)$, we know $q(x_i) - 1 = 0$. This implies $q(x) - 1$, a polynomial of degree at most n, has n+1 zeros. Therefore, $q(x) - 1 \equiv 0$ for all x and $q(x) = 1$, so $\sum_{i=0}^{n} l_i = 1$.

Problem 6.1.9

Suppose g interpolates f at $x_0, x_1, x_2, ..., x_{n-1}$ and suppose h interpolates f at $x_1, x_2, x_3, ..., x_n$. Let $f(x_i) = g(x_i) + \frac{x_0 - x_i}{x_n - x_0}[g(x_i) - h(x_i)]$. Plugging in x_0, we find $f(x_0) = g(x_0) + 0$, and since $g(x_0)$ interpolates $f(x_0)$, this is true. For $1 \leq i \leq n-1$, plugging in x_i results in $f(x_i) = g(x_i) + 0$, since $g(x_i) = h(x_i)$ for the given range, so again this interpolates f. For x_n, we have $f(x_n) = g(x_n) + \frac{x_0 - x_n}{x_n - x_0}[g(x_n) - h(x_n)] = g(x_n) - [g(x_n) - h(x_n)] = h(x_n)$, which interpolates $f(x_n)$, so the given formula interpolates f at all the nodes.

Problem 6.1.10

Let $p(x) = y_0l_0 + y_1l_1 + ... + y_nl_n = \displaystyle\sum_{i=0}^{n} y_il_i$, where $l_i = \displaystyle\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$. The coefficient for x_n will thus be computed with $\displaystyle\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$. Notice for the numerator in the product, we will obtain a polynomial of degree n who is monic. We may thus break this up and focus on $\displaystyle\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} x^n$. Thus we see $\displaystyle\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1}$ is x^n's coefficient.

Problem 6.1.11

From 6.1.10, we found $\displaystyle\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1}$ to be the coefficient for x^n. For any polynomial q of degree $\neq n - 1$, this coefficient must be zero, or it would not be of degree at most n-1. Since $q(x_i) = y_i$, $\displaystyle\sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1} = \sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1} = 0$, as desired.