Problem 4.3.1

For these problems, I wrote a code in MATLAB called "gaussianelim.m".

(a) For the first part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1.5 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} 1.25 \\ -0.75 \\ 0.5 \end{bmatrix}$$

For the second part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix} \quad x = \begin{bmatrix} 1.25 \\ -0.75 \\ 0.5 \end{bmatrix}$$

(b) For the first part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} \frac{3}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{bmatrix}$$

For the second part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{11}{4} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{4} \end{bmatrix} \quad x = \begin{bmatrix} \frac{3}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{bmatrix}$$

(c) For the first part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} -1 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

For the second part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

(d) For the first part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

For the second part, I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -1 & -\frac{1}{6} & 1 & 0 \\ 2 & \frac{1}{3} & -\frac{2}{13} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 2 & 1 \\ 0 & 0 & \frac{13}{3} & -\frac{83}{6} \\ 0 & 0 & 0 & -\frac{6}{13} \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

(e) For the first part I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & -\frac{16}{9} & 1 & 0 \\ 2 & -\frac{1}{3} & \frac{6}{31} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & -\frac{62}{3} & -\frac{25}{3} \\ 0 & 0 & 0 & -\frac{12}{31} \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

For the second part I found

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{10} & 1 & 0 \\ 4 & -\frac{4}{15} & -\frac{19}{9} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{6}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 4.3.11

Let A be tridiagonal and let $c_0 = 0$ and $a_n = 0$. Suppose $|d_i| > |a_i| + |c_{i-1}|$. By definition $d_i \neq 0$. Let $|d_i'| = \left|d_i - \frac{a_{i-1}}{d_{i-1}} c_{i-1}\right|$. Thus, we wish to show that $|d_i'| > 0$. We find that $|d_i'| = \left|d_i - \frac{a_{i-1}}{d_{i-1}} c_{i-1}\right| \geq |d_i| - \left|\frac{a_{i-1}}{d_{i-1}}\right| |c_{i-1}| > |a_i| + |c_{i-1}| - \left|\frac{a_{i-1}}{d_{i-1}}\right| |c_{i-1}|$ since our matrix is columnwise dominant. Thus we have $|d_i'| > |a_i| + |c_{i-1}| \left(1 - \left|\frac{a_{i-1}}{d_{i-1}}\right|\right)$. Now, we will show that $\left|\frac{a_{i-1}}{d_{i-1}}\right| < 1$. Since $|d_i| > |a_i| + |c_{i-1}| > |a_i|$, we know that $|d_{i-1}| > |a_{i-1}|$, so $\left|\frac{a_{i-1}}{d_{i-1}}\right| < 1$. Thus, we find that $|d_i'| > |a_i| + |c_{i-1}| \left(1 - \left|\frac{a_{i-1}}{d_{i-1}}\right|\right) > |a_i| > 0$, so $a_i \neq 0$. Hence $|d_i'| > |a_{i-1}| > 0 \Rightarrow |d_i'| \neq 0$ and we are done.

Problem 4.3.21

Work for this problem was done using my MATLAB code "gaussianelim.m".

(a) $x = \begin{bmatrix} 0.81475769 \\ -2.3569839 \\ -0.64607822 \end{bmatrix}$

(b) $x = \begin{bmatrix} 0.81475769 \\ -2.3569839 \\ -0.64607822 \end{bmatrix}$

Problem 4.3.29

Using my MATLAB code "gaussianelim.m", I found the determinant to be -6.

Problem 4.4.1

Let $x, y \in V$, where V is a vector space.

(a) Suppose $x \neq 0$. Then, $||x||_\infty = \max_{1\leq i\leq n} |x_i|$. Since x is nonzero, at least one of its elements must be nonzero. Letting this nonzero element be at position i, we find that $|x_i| > 0$. If this is the only nonzero element, then we are done as it is the max. Otherwise, there may exist another nonzero element greater than $|x_i|$, which then will be the max. Regardless, we find that $||x||_\infty > 0$. Now let $\lambda \in \mathbb{R}$. Then, λx will have elements of the form λx_i. Thus, $||\lambda x||_\infty = \max_{1\leq i\leq n} |\lambda x_i| = |\lambda| \max_{1\leq i\leq n} |x_i| = |\lambda|||x||_\infty$, as desired. Finally, $||x+y||_\infty = \max_{1\leq i\leq n} |x_i + y_i|$. By the normal triangle inequality, we find $\max_{1\leq i\leq n} |x_i+y_i| \leq \max_{1\leq i\leq n} (|x_i| + |y_i|) = \max_{1\leq i\leq n} |x_i| + \max_{1\leq i\leq n} |y_i| = ||x||_\infty + ||y||_\infty$, as desired.

(b) Suppose $x \neq 0$. Then $||x||_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Since x is nonzero, at least one of its elements must be nonzero, which we shall call x_i. Then $x_i^2 > 0$. Since $||x||_2$ is a summation of n terms like this, that may or may not also be nonzero, we can see that $||x||_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} > 0$ as desired. Now let $\lambda \in \mathbb{R}$. Then, λx will have elements of the form λx_i. Thus, $||\lambda x||_2 = \left(\sum_{i=1}^n (\lambda x_i)^2\right)^{1/2} = \left(\lambda^2 \sum_{i=1}^n x_i^2\right)^{1/2} = |\lambda| \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, as desired. Finally, $||x+y||_2 = \left(\sum_{i=1}^n (x_i+y_i)^2\right)^{1/2} = \left(\sum_{i=1}^n x_i^2 + 2x_iy_i + y_i^2\right)^{1/2} \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2} + \left(\sum_{i=1}^n y_i^2\right)^{1/2}$, as desired.

(c) Suppose $x \neq 0$. Then $||x||_1 = \sum_{i=1}^n |x_i|$. Since x is nonzero, at least one of its elements must be nonzero. Letting this nonzero element be at position i, we find that $|x_i| > 0$. This will be true for any other nonzero element of x, so we find that $||x||_1 = \sum_{i=1}^n |x_i| > 0$. Now let $\lambda \in \mathbb{R}$. Then, λx will have elements of the form λx_i. Thus, we find that $||\lambda x||_1 = \sum_{i=1}^n |\lambda x_i| = \sum_{i=1}^n |\lambda||x_i| = |\lambda| \sum_{i=1}^n |x_i| = \lambda||x||_1$ as desired. Finally, $||x+y||_1 = \sum_{i=1}^n |x_i+y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = ||x||_1 + ||y||_1$, as desired.

Problem 4.4.2

Let $x \in \mathbb{R}^n$. Then, $||x||_2^2 = \sum_{i=1}^n x_i^2 \leq \left(\sum_{i=1}^n x_i^2 + 2 * \sum_{i,j,i\neq j} |x_i||x_j|\right) = ||x||_1^2$. Likewise, since $||x||_\infty$ only takes the largest element, and does not sum all the elements, we find that $||x||_\infty = \max_{1\leq i\leq n} |x_i| = |x_j| = \left(x_j^2\right)^{1/2} \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Thus we find that $||x||_\infty \leq ||x||_2 \leq ||x||_1$. Notice that the equalities will hold for any vector with a single nonzero element.

Problem 4.4.3

Notice that $||x||_1 = \sum_{i=1}^n |x_i| \leq \sum_{i=1}^n \max_{1\leq i\leq n} |x_i| = n * \max_{1\leq i\leq n} = n||x||_\infty$. Likewise, $||x||_2^2 = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n \max_{1\leq i\leq n} |x_i|^2 = n * \max_{1\leq i\leq n} |x_i|^2 = n||x||_\infty^2$, so $||x||_2 \leq \sqrt{n}||x||_\infty$.

Problem 4.4.4

Let $x \in \mathbb{R}$. Then, we know $f(x) \leq \sup f(x)$ and $g(x) \leq \sup g(x)$. Thus, $f(x) + g(x) \leq \sup f(x) + \sup g(x)$. Thus $\sup f(x) + \sup g(x)$ is some upper bound for $f(x) + g(x)$, though it is not necessarily the least upper bound. So $\sup\,[f(x) + g(x)] \leq \sup f(x) + \sup g(x)$ as desired.

Problem 4.4.7

(a) No. For example take $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Then, $\max\{|x_2|, |x_3|\} = 0$, but the vector is nonzero.

(b) No, as for some $\lambda \in \mathbb{R}$ it follows that $||\lambda x|| = \sum_{i=1}^{n} |\lambda x_i|^3 = |\lambda|^3 \sum_{i=1}^{n} |x_i|^3 \neq |\lambda|||x||$.

(c) No, take for example $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. Then $||x + y|| = 4$ but $||x|| + ||y|| = 2$ so $||x + y|| \not\leq ||x|| + ||y||$.

(d) Yes, this satisfies all the postulates.

(e) Yes, this satisfies all the postulates.

```
Here is my code:
---------------------------------------------------------------------------
type gaussianelim.m


function x = gaussianelim(A,b,p,q)
%This is an algorithm made by Alexander Winkles to solve problems of the
%form Ax = b using Gaussian elimination in various settings.
%
%A  : the matrix
%b  : the vector
%p  : the permutation of the matrix, where (1,2,...,n) is the given matrix
%q  : indication which way the system is to be solved
%       q == 1 - Gaussian elimination (currently not working)
%       q == 2 - LU factorization
%       q == 3 - Scaled row pivoting

if size(A,1) ~= size(A,2)
    fprintf('This matrix is not square!')
else
    n = size(A,1);
    A1 = A;
%%% Gaussian elimination %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    if q == 1


        %Performs the Gaussian elimination to create an upper triangular
        %matrix A
```

```
        for k=1:n-1
            for i=k+1:n
                z = A(p(i),k)/A(p(k),k);
                A(p(i),k) = 0;
                for j = k+1:n
                    A(p(i),j) = A(p(i),j) - z*A(p(k),j);
                end;
            end;
        end;

        %Solves the system Ax = b with new matrix A

        for i = 1:n %(error somewhere here)
            sum = 0;
            for j = 1:i-1
                sum = sum + A(p(i),j)*z(j);
            end;
            z(i) = b(p(i)) - sum;
        end;
        disp(z)

        for i=n:-1:1
            sum = 0;
            for j=i+1:n
                sum = sum + A(p(i),j)*x(j);
            end;
            x(i) = (z(i) - sum)/A(p(i),i);
        end;

    end;

%%% LU factorization %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    if q == 2
        U = zeros(n,n);
        L = zeros(n,n);
        for i=1:n
            L(i,i) = 1;
```

```
    for j=i:n
        sum = 0;
        for k=1:i-1
            sum = sum + L(i,k)*U(k,j);
        end;

        U(i,j) = A(i,j) - sum;
    end;
    for j=i+1:n
        sum = 0;
        for k=1:i-1
            sum = sum + L(j,k)*U(k,i);
        end;

        L(j,i) = (A(j,i) - sum)/U(i,i);
    end;
end;

%Solves the system using LU results

x = zeros(n,1);
for i=1:n
    sum = 0;
    for j=1:i-1
        sum = sum + L(i,j)*z(j);
    end;
    z(i) = b(p(i)) - sum;
end;
for i = n:-1:1
    sum = 0;
    for j=i+1:n
        sum = sum + U(i,j)*x(j);
    end;
    x(i) = (z(i) - sum)/U(i,i);
end;
fprintf('\nThe L matrix is:\n')
disp(L)
fprintf('\nThe U matrix is:\n')
```

```
        disp(U)
    end;

%%% Scaled row pivoting %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    if q == 3
        % factorization phase
        for i=1:n
            p(i) = i;
            s = max(abs(A'));
        end;
        for k=1:n-1
            r = abs(A(p(k),k)/s(p(k)));
            kp = k;
            for i = (k+1):n
                t = abs(A(p(i),k)/s(p(i)));
                if t > r, r = t; kp = i; end;
            end;
            l = p(kp); p(kp) = p(k); p(k) = l;
            for i = (k+1):n
                A(p(i),k) = A(p(i),k)/A(p(k),k);
                for j = (k+1):n
                    A(p(i),j) = A(p(i),j)-A(p(i),k)*A(p(k),j);
                end;
            end;
        end;

        y = zeros(n,1);
        y(1) = b(p(1));
        for i = 2:n
            y(i) = b(p(i));
            for j = 1:(i-1)
                y(i) = y(i)-A(p(i),j)*y(j);
            end;
        end;

        x = zeros(n,1);
        x(n) = y(n)/A(p(n),n);
```

```
        for i = (n-1):-1:1
            x(i) = y(i);
            for j = (i+1):n
                x(i) = x(i) - A(p(i),j)*x(j);
            end;
            x(i) = x(i)/A(p(i),i);
        end;

        P = zeros(n,n);
        for i=1:length(p)
            for j=1:length(p)
                if j == p(i)
                    P(i,j) = 1;
                end;
            end;
        end;

        K = P*A1;

        [L,U] = lufactor(K,1);

        fprintf('\nThe L matrix is:\n')
        disp(L)
        fprintf('\nThe U matrix is:\n')
        disp(U)

    end;

end;
--------------------------------------------------------------------------


Problem 1:


part A:
A = [-1 1 -4; 2 2 0; 3 3 2];
a = [0 1 1/2];
gaussianelim(A,a,[1 2 3], 2)
```

```
The L matrix is:
    1.0000         0         0
   -2.0000    1.0000         0
   -3.0000    1.5000    1.0000



The U matrix is:
    -1     1    -4
     0     4    -8
     0     0     2



ans =


    1.2500
   -0.7500
   -0.5000

gaussianelim(A,a,[1 2 3], 3)


The L matrix is:
    1.0000         0         0
   -0.5000    1.0000         0
    1.5000         0    1.0000



The U matrix is:
     2     2     0
     0     2    -4
     0     0     2



ans =


    1.2500
   -0.7500
   -0.5000
```

```
part B:

B = [1 6 0; 2 1 0; 0 2 1];

b = [3 1 1];

format rat

gaussianelim(B,b,[1 2 3], 2)

The L matrix is:
       1              0              0
       2              1              0
       0             -2/11           1

The U matrix is:
       1              6              0
       0            -11              0
       0              0              1

ans =

       3/11
       5/11
       1/11

gaussianelim(B,b,[1 2 3], 3)

The L matrix is:
       1              0              0
       0              1              0
       1/2            11/4           1

The U matrix is:
       2              1              0
       0              2              1
       0              0            -11/4
```

```
ans =


       3/11
       5/11
       1/11


part C:
C = [-1 1 0 -3; 1 0 3 1; 0 1 -1 -1; 3 0 1 2];
c = [4 0 3 1];
gaussianelim(C,c,[1 2 3 4], 2)

The L matrix is:
       1              0              0              0
      -1              1              0              0
       0              1              1              0
      -3              3              2              1



The U matrix is:
      -1              1              0             -3
       0              1              3             -2
       0              0             -4              1
       0              0              0             -3



ans =


       1
       2
       0
      -1

gaussianelim(C,c,[1 2 3 4], 3)

The L matrix is:
       1              0              0              0
       0              1              0              0
       1/3            0              1              0
```

```
      -1/3           1             1/2           1



The U matrix is:
       3             0             1             2
       0             1            -1            -1
       0             0             8/3           1/3
       0             0             0            -3/2



ans =


       1
       2
      -1/48038396025285288
      -1


part D:
D = [6 -2 2 4; 12 -8 4 10; 3 -13 3 3; -6 4 2 -18];
d = [0 -10 -39 -16];
gaussianelim(D,d,[1 2 3 4], 2)

The L matrix is:
       1             0             0             0
       2             1             0             0
       1/2           3             1             0
      -1            -1/2           2             1



The U matrix is:
       6            -2             2             4
       0            -4             0             2
       0             0             2            -5
       0             0             0            -3



ans =
```

```
     1
     3
    -2
     1

gaussianelim(D,d,[1 2 3 4], 3)

The L matrix is:
     1           0           0           0
     1/2         1           0           0
    -1          -1/6         1           0
     2           1/3        -2/13        1



The U matrix is:
     6          -2           2           4
     0         -12           2           1
     0           0           13/3       -83/6
     0           0           0           -6/13



ans =

     1
     3
    -2
     1

part E:
E = [1 0 2 1; 4 -9 2 1; 8 16 6 5; 2 3 2 1];
e = [2 14 -3 0];
gaussianelim(E,e,[1 2 3 4], 2)

The L matrix is:
     1           0           0           0
     4           1           0           0
     8         -16/9         1           0
     2          -1/3         6/31        1
```

```
The U matrix is:
      1            0            2            1
      0           -9           -6           -3
      0            0          -62/3        -25/3
      0            0            0          -12/31



ans =


      1
     -1
      1/1939049839562297
      1

gaussianelim(E,e,[1 2 3 4], 3)

The L matrix is:
      1            0            0            0
      2            1            0            0
      1/2          1/10         1            0
      4           -4/15        -19/9         1



The U matrix is:
      2            3            2            1
      0          -15           -2           -1
      0            0            6/5          3/5
      0            0            0            2



ans =


      1
     -1
      0
      1
```

```
Problem 21:

format long
K = [0.2641 0.1735 0.8642; 0.9411 0.0175 0.1463; -0.8641 -0.4243 0.0711];
k = [-0.7521 0.6310 0.2501];
format long
gaussianelim(K,k,[1 2 3 4], 2)

The L matrix is:
   1.000000000000000                   0                   0
   3.563422945853843   1.000000000000000                   0
  -3.271866717152593  -0.238648271671779   1.000000000000000



The U matrix is:
   0.264100000000000   0.173500000000000   0.864200000000000
                   0  -0.600753881105642  -2.933210109806891
                   0                   0   2.198641693807667



ans =

   0.814757689669357
  -2.356983928355609
  -0.646078216005530

gaussianelim(K,k,[1 2 3], 3)

The L matrix is:
   1.000000000000000                   0                   0
  -0.918180852194241   1.000000000000000                   0
   0.280629051110403  -0.412973651527701   1.000000000000000



The U matrix is:
   0.941100000000000   0.017500000000000   0.146300000000000
                   0  -0.408231835086601   0.205429858676017
```

```
                  0                  0   0.907981088692803


ans =

   0.814757689669357
  -2.356983928355609
  -0.646078216005530

Problem 29:

gaussianelim(Q,[1 1 1 1],[1 2 3 4],3)
     0     0     0     1
     0     1     0     0
     0     0     1     0
     1     0     0     0


The L matrix is:
   1.000000000000000                  0                  0                  0
                   0   1.000000000000000                  0                  0
   0.500000000000000   1.000000000000000   1.000000000000000                  0
                   0  -1.000000000000000                  0   1.000000000000000


The U matrix is:
   2.000000000000000                  0   1.000000000000000                  0
                   0   1.000000000000000                  0   1.000000000000000
                   0                  0   1.500000000000000  -1.000000000000000
                   0                  0                  0   2.000000000000000


ans =

   0.333333333333333
                   0
   0.333333333333333
   1.000000000000000
```

```
det(Q)

ans =

    -6

diary off
```