Problem 7.3.21

Consider the numerical integration rule $\int_{-1}^{1} f(x)\ dx \approx Af\left(-\sqrt{\frac{3}{5}}\right) + Bf(0) + Cf\left(\sqrt{\frac{3}{5}}\right)$.

(a) The following is the linear system that must be solved to determine A, B, and C:

$$\int_{-1}^{1} dx = 2 = A + B + C$$
$$\int_{-1}^{1} x\ dx = 0 = -\sqrt{\frac{3}{5}}A + \sqrt{\frac{3}{5}}C$$
$$\int_{-1}^{1} x^2\ dx = \frac{2}{3} = \frac{3}{5}A + \frac{3}{5}C$$

Solving this system in MATLAB with ”linsolve” gives

$$A = 0.5555556 = \frac{5}{9}$$
$$B = 0.8888889 = \frac{8}{9}$$
$$C = 0.5555556 = \frac{5}{9}$$

(b) The following are the integrals that must be evaluated to find A, B, and C using Newton-Cotes:

$$\int_{-1}^{1} \frac{(x-0)(x-\sqrt{\frac{3}{5}})}{(-\sqrt{\frac{3}{5}}-0)(-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}})}\ dx = 0.5555556 = \frac{5}{9}$$
$$\int_{-1}^{1} \frac{(x+\sqrt{\frac{3}{5}})(x-\sqrt{\frac{3}{5}})}{(-\sqrt{\frac{3}{5}})(\sqrt{\frac{3}{5}})}\ dx = 0.8888889 = \frac{8}{9}$$
$$\int_{-1}^{1} \frac{(x+\sqrt{\frac{3}{5}})(x-0)}{\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}})(\sqrt{\frac{3}{5}}-0)}\ dx = 0.5555556 = \frac{5}{9}$$

Problem 7.3.22

Let $\int_{-1}^{1} f(x)\ dx \approx \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$.

(a) We wish to solve $\int_{0}^{\pi/2} x\ dx$ using the above formula. Using the change of intervals formula, our given Gaussian quadrature becomes $\frac{\pi}{4}\left(\frac{5}{9}(\frac{\pi}{4}\cdot(-\sqrt{\frac{3}{5}}) + \frac{\pi}{4}) + \frac{8}{9}(\frac{\pi}{4}) + \frac{5}{9}(\frac{\pi}{4}\cdot\sqrt{\frac{3}{5}} + \frac{\pi}{4})\right) = 1.23370$.

(b) We wish to solve $\int_{0}^{4} \frac{\sin t}{t}\ dt$ using the above formula. Using the change of interval formula, our given Gaussian quadrature becomes $2\left(\frac{5}{9}\frac{\sin(2\cdot(-\sqrt{\frac{3}{5}})+2)}{2\cdot(-\sqrt{\frac{3}{5}})} + \frac{8}{9}\frac{\sin 2}{2} + \frac{5}{9}\frac{\sin(2\cdot(\sqrt{\frac{3}{5}})+2)}{2\cdot(\sqrt{\frac{3}{5}})}\right) = 1.75802$.

Problem 7.4.4

Recall Simpson's rule, which states $\int_{a}^{b} f(x)\ dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$. The second column of the Romberg array takes the form of $R(n,1) = R(n,0) + \frac{1}{4^m-1}[R(n,0) - R(0,0)] =$

Problem 7.4.5

We wish to show by induction that $I - R(n, m-1) = a_1h^{2m} + a_2h^{2m+2} + a_3h^{2m+4} + ...$ Let our base case be when $m = 1$, then we have $I - R(n, 0) = a_1h^2 + a_2h^4 + ...$, so the base case holds. Now for our inductive step, suppose for $m \in \mathbb{N}$ that $I - R(n, m-1) = a_1h^{2m} + a_2h^{2m+2} + a_3h^{2m+4} + ... \forall n \in \mathbb{N}$. Recall that $h_n = 2h_{n-1}$.

$$\begin{aligned}
\Rightarrow I - \frac{4^m R(n, m-1) - R(n-1, m-1)}{4^m - 1} &= \frac{4^m(I - R(n, m-1))}{4^m - 1} - \frac{I - R(n-1, m-1)}{4^m - 1} \\
&= \frac{4^m(a_1h_{n+1}^{2m} + a_2h_{n+1}^{2m+2} + ...)}{4^m - 1} - \frac{a_1h_n^{2m} + a_2h_n^{2m} + ...}{4^m - 1} \\
&= \frac{4^m(a_1(\frac{h_n}{2})^{2m} + a_2(\frac{h_n}{2})^{2m+2} + ...) - (a_1h_n^{2m} + a_2h_n^{2m+2} + ...)}{4^m - 1} \\
&= b_1h_{n+1}^{2m+2} + b_2h_{n+1}^{2m+4} + ...
\end{aligned}$$

as desired.

Problem 7.4.6

We wish to apply the Romberg algorithm to find $R(2, 2)$ for the following integrals. For this, I wrote a MATLAB code called "romberg.m".

(a) $\displaystyle\int_1^3 \frac{dx}{x}$. We now construct the Romberg array:

$n\backslash m$	0	1	2
0	1.333	0	0
1	1.667	1.1111	0
2	1.1167	1.1000	1.0993

Thus, $R(2, 2) = 1.0993$.

(b) $\displaystyle\int_0^{\pi/2} (\frac{x}{\pi})^2\, dx$ in terms of π. As before, we construct the Romberg array using MATLAB:

$n\backslash m$	0	1	2
0	0.1936	0	0
1	0.1473	0.1309	0
2	0.1350	0.1309	0.1309

In terms of π, this is approximately

$n\backslash m$	0	1	2
0	$\frac{\pi}{16}$	0	0
1	$\frac{3\pi}{64}$	$\frac{\pi}{24}$	0
2	$\frac{11\pi}{256}$	$\frac{\pi}{24}$	$\frac{\pi}{24}$

Thus, $R(2, 2) = \frac{\pi}{24}$.