Problem 4.5.7

Let $1 = ||A|| > ||B||$. Suppose $A - B$ was not invertible. Then, it will be singular, so $\exists x$ such that $||x|| = 1$ and $(A - B)x = 0$, which may be rewritten as $Ax = Bx$. From this, we find $1 = ||x|| = ||Ax|| \leq ||A||\ ||x|| = ||B||\ ||x|| = ||B|| < 1$. Thus, we have found a contradiction, so $A - B$ is invertible.

Problem 4.5.8

Suppose $||A|| < 1$. By Theorem 1, if we rewrite $I + A$ as $I - (-A)$, then $(I - (-A))^{-1} = \sum_{k=0}^{\infty}(-A)^k = I - A + A^2 - ...$ as desired.

Problem 4.5.9

Suppose

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Then, $||BA - I|| = ||I|| = 1$, but $||AB - I|| < 1$, so we have found an example where the statement is false.

Problem 4.6.1

Suppose A is diagonally dominant and let $Q = D$ as in the Jacobi method. Since

$$Q = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

it follows that

$$Q^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}.$$

Thus, $Q^{-1}A$ will have diagonal elements of 1, and will retain its diagonal dominance. Then, $I - Q^{-1}A$ will have diagonal elements of 0. Since $Q^{-1}A$ was diagonally dominant with diagonals of 0, we know for each row $(1 \leq i \leq n)$ that $\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < 1$. Thus, $||I - Q^{-1}A||_\infty = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < 1$. Since this is defined to be the maximal norm, all other norms are less than it. Thus, $\inf_{||\cdot||} ||I - Q^{-1}A|| < 1$ and by Theorem 4 of the section $\rho(A) < 1$.

Problem 4.6.2

Suppose A is unit row diagonally dominant, or

$$a_{ii} = 1 > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

Then,

$$I - A = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{bmatrix}$$

By Theorem 4.4.1, for any norm $||\cdot||$ on R^n, it follows that $||A|| = \sup_{||u||=1}\{||Au|| : u \in \mathbb{R}^n\}$. Without loss of generality, let us choose $u = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}} \end{bmatrix}^T$. Then, $(I-A)u = \begin{bmatrix} \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{1j}| & \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{2j}| & \dots & \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{nj}| \end{bmatrix}^T$. Notice that each of these elements is less than 1, so $||I-A||_\infty = \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{n} |a_{ij}| < 1$, so by Theorem 4.6.1, the Richardson iteration is successful.

Problem 4.6.7

Let $Q = D - C_L$. Suppose A is diagonally dominant. Suppose λ is an eigenvalue of $I - Q^{-1}A$ and x is its associated eigenvector. Without loss of generality, assume $||x||_\infty = 1$. So we have $[I - Q^{-1}A]x = \lambda x$. Since Q is the lower triangle of A, $-\sum\limits_{j=i+1}^{n} a_{ij}x_j = \lambda \sum\limits_{j=1}^{i} a_{ij}x_j$ for $1 \leq i \leq n$. This can be transposed to $\lambda a_{ii}x_i = -\lambda \sum\limits_{j=1}^{i-1} a_{ij}x_j - \sum\limits_{j=i+1}^{n} a_{ij}x_j$. Select i such that $|x_i| = 1 \geq |x_j| \forall j$. Then we find $|\lambda||a_{ii}| \leq |\lambda| \sum\limits_{j=1}^{i-1} |a_{ij}| + \sum\limits_{j=i+1}^{n} |a_{ij}|$. From this, we find $|\lambda| \leq \{\sum\limits_{j=i+1}^{n} |a_{ij}|\}\{|a_{ii}| - \sum\limits_{j=1}^{i-1} |a_{ij}|\}^{-1} < 1$. Since A is diagonally dominant, it follows that $||I - Q^{-1}A||_\infty < 1$.

Problem 4.6.9

Suppose the ith equation of $Ax = b$ is divided by a_{ii}. We then have a new system $A'x = b'$, where the diagonal elements of the new matrix A' are all 1, and $D' = I$. Recall that the inverse of a diagonal matrix D is

$$D^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}.$$

We know the Jacobi method is $Dx^{(k)} = (C_L + C_U)x^{(k-1)} + b$, or $x^{(k)} = D^{-1}(C_L + C_U)x^{(k-1)} + D^{-1}b$. Notice then that $D^{-1}b = b'$. Likewise, $D^{-1}(C_L + C_U) = (C'_L + C'_U) = (I - A')$, as $A' = D' - C'_L - C'_U$. Thus, we may rewrite the Jacobi method as $x^{(k)} = (I - A')x^{(k-1)} + b'$, which is the Richardson method applied to the new system $A'x = b'$.

Problem 4.6.28

Let

$$\mathcal{J} = I - D^{-1}A$$
$$\mathcal{G} = I - (D - C_L)^{-1}A$$

Let $x^{(k)} = Gx^{(k-1)} + c$ be an arbitrary linear iterative process where G is the iterative matrix. We wish to show that the above matrices are the iterative matrices for the Jacobi and Gauss-Seidel methods, respectively.

Plugging in $\mathcal{J}$, we find $x^{(k)} = (I - D^{-1}A)x^{(k-1)} + c = (I - D^{-1}[D - C_L - C_U])x^{(k-1)} + c = D^{-1}(C_L + C_U)x^{(k-1)} + c \Rightarrow Dx^{(k)} = (C_L + C_U)x^{(k-1)} + b$, where $b = Dc$. Thus, this is the iterative matrix for the Jacobi method. Plugging in $\mathcal{G}$, we find $x^{(k)} = [I - (D - C_L)^{-1}A]x^{(k-1)} + c = [I - (D - C_L)^{-1}(D - C_L - C_U)]x^{(k-1)} + c = [I - I + (D - C_L)^{-1}C_U]x^{(k-1)} + c \Rightarrow (D - C_L)x^{(k)} = C_U x^{(k-1)} + b$, where $b = (D - C_L)c$. Thus, this is the iterative matrix for the Gauss-Seidel method.

The splitting matrices an iterative matrices given in the text are correct, and can be derived from chapter equations 13 and 15.

Problem 4.6.29

Using MATLAB, I found

$$I - Q^{-1}A = \begin{bmatrix} 0 & 1/2 & & & & \\ 0 & 1/4 & 1/2 & & & \\ 0 & 1/8 & 1/4 & 1/2 & & \\ 0 & \vdots & \vdots & \vdots & \ddots & \\ 0 & 1/2^n & 1/2^{n-1} & 1/2^{n-2} & \dots & 1/4 \end{bmatrix}.$$

Problem 4.7.2

By Lemma 1 of the section, we know $q(x) =< x, Ax > -2 < x, b >$. Taking the derivative, we find that $\frac{dq(x)}{dx} = 2Ax - 2b$. Thus, the critical point is $x = A^{-1}b$. To check that this is a minimum, $\frac{d}{dx}\left(\frac{dq(x)}{dx}\right) = 2A$. Since A is positive definite, $2A \Rightarrow$ it is a minimum, and $q(x)_{min} = \langle A^{-1}b, b\rangle - 2\langle A^{-1}b, b\rangle = -\langle b, A^{-1}b\rangle$.

Problem 4.7.4

Suppose A is positive definite and let b be a fixed vector. Let $r = b - Ax$ be the residual vector and let $e = A^{-1}b - x$ be the error vector. Then, $\langle r, e\rangle = \langle b - Ax, A^{-1}b - x\rangle = \langle b, A^{-1}b - x\rangle - \langle Ax, A^{-1}b - x\rangle = \langle b, A^{-1}b\rangle - \langle b, x\rangle - \langle Ax, A^{-1}b\rangle + \langle Ax, x\rangle$. Notice that if $Ax = b$, this becomes 0. Otherwise, as A is positive definite, its inverse is likewise positive definite. Thus, $\langle b, A^{-1}b\rangle > 0$ and $\langle Ax, x\rangle > 0$, and thus $\langle r, e\rangle > 0$.

Problem 4.7.7

By equation 4 of the section, we know $q(x + tv) = q(x) - \frac{\langle v, b - Ax^2\rangle}{\langle v, Av\rangle}$, or $q(x^{(k+1)}) = q(x^{(k)}) - \frac{\langle v, b - A(x^{(k)})^2\rangle}{\langle v, Av\rangle}$. In steepest descent, we are choosing $v = b - Ax^{(k)} = r$, we can write the following: $q(x^{(k+1)}) = q(x^{(k)}) - \frac{\langle r^{(k)}, b - A(x^{(k)})^2\rangle}{\langle r^{(k)}, Ar^{(k)}\rangle}$. Then, $\langle r^{(k)}, b - A(x^{(k)})^2\rangle = \sum_{i=1}^{n} r_i^{(k)}(b_i - Ax_i^{(k)} \Rightarrow \sum_{i=1}^{n}\langle r_i^{(k)}, r_i^{(k)}\rangle = ||r^{(k)}||^2 \Rightarrow q(x^{(k+1)}) = q(x^{(k)}) - \frac{||r^{(k)}||^2}{\langle r^{(k)}, Ar^{(k)}\rangle}$.