**Problem 7.1.1**

Let us begin with $f(x) = \sum_{i=0}^{n} f(x_i)l_i(x) + \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)w(x)$ where $w(x) = \prod_{i=0}^{n}(x-x_i)$. Thus, we find that $f'(x) = \sum_{i=0}^{n} f(x_i)l_i'(x) + \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)w'(x) + \frac{1}{(n+1)!}w(x)\frac{d}{dx}f^{(n+1)}(\xi_x)$. Suppose we let $x_a$ be a node for $f$, then we know $w(x_a) = 0$ since it is a product with the term $(x_a - x_a)$ multiplied somewhere inside it. Our result becomes $f'(x_a) = \sum_{i=0}^{n} f(x_i)l_i'(x_a) + \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)w'(x_a)$. Now we seek to find $w'(x_a)$. Since $w(x) = \prod_{i=0}^{n}(x-x_i)$, we find that $w'(x) = \sum_{i=0}^{n}\prod_{\substack{j=0\\ j\neq i}}^{n}(x-x_j)$, which is know as the derivative for a product. By plugging in $x_a$, we find $w'(x_a) = \prod_{\substack{j=0\\ j\neq a}}^{n}(x_a-x_j)$, where the summation disappears as every other term becomes zero. Plugging this result back into the original derivative, we find $f'(x_a) = \sum_{i=0}^{n} f(x_i)l_i'(x_a) + \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\prod_{\substack{j=0\\ j\neq a}}^{n}(x_a-x_j)$ as desired.

**Problem 7.1.6**

Taking some Taylor expansions, we find

$$\begin{aligned}
f(x+2h) &= f(x) + f'(x)(2h) + \frac{f''(x)}{2}(2h)^2 + \frac{f'''(x)}{6}(2h)^3 + \frac{f^{(4)}(x)}{4!}(2h)^4 + \frac{f^{(5)}(\xi_1)}{5!}(2h)^5\\
f(x-2h) &= f(x) + f'(x)(-2h) + \frac{f''(x)}{2}(-2h)^2 + \frac{f'''(x)}{6}(-2h)^3 + \frac{f^{(4)}(x)}{4!}(-2h)^4 + \frac{f^{(5)}(\xi_2)}{5!}(-2h)^5\\
f(x+h) &= f(x) + f'(x)(h) + \frac{f''(x)}{2}(h)^2 + \frac{f'''(x)}{6}(h)^3 + \frac{f^{(4)}(x)}{4!}(h)^4 + \frac{f^{(5)}(\xi_3)}{5!}(h)^5\\
f(x-h) &= f(x) + f'(x)(-h) + \frac{f''(x)}{2}(-h)^2 + \frac{f'''(x)}{6}(-h)^3 + \frac{f^{(4)}(x)}{4!}(-h)^4 + \frac{f^{(5)}(\xi_4)}{5!}(-h)^5
\end{aligned}$$

Thus, we find that $-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h) \approx 12hf'(x)$, so $f'(x) \approx \frac{1}{12h}[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)]$. Likewise we find that $-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h) \approx 12h^2f''(x)$, so $f''(x) \approx \frac{1}{12h^2}[-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)]$ as desired. As seen from the Taylor expansions, each of these has an error term of $O(h^5)$. Since we are dividing by an $12h$, our error term becomes $O(h^4)$ as desired.

**Problem 7.1.7**