For a good portion of this homework, when work is not shown, *Mathematica* was utilized. The code for it can be found at the end.

1. *Section 1.2 Problem 1*

   (a) $\kappa = 1$

   (b) $\kappa = \sqrt{\frac{1}{(1+s^2)^2}}$

   (c) $\kappa = \frac{1}{2}\sqrt{\frac{1}{2-2s^2}}$

2. *Section 1.2 Problem 3*

   For this portion, all results will be found in the *Mathematica* code at the end due to the results being too painful to type in LaTeX.

3. *Section 1.2 Problem 4*

   We wish to show that $\kappa = \frac{|f''|}{(1+(f')^2)^{3/2}}$ for a plane curve. Begin by noting that $\kappa = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3}$. Also note that since $f(x)$ is a plane curve, $\tau = 0$. Let $\alpha(x) = (x, f(x), 0)$, where the z compoenent is zero because of a lack of torsion. Now, it can easily be shown that $\alpha'(x) = (1, f'(x), 0)$ and $\alpha''(x) = (0, f''(x), 0)$. Since the numerator of $\kappa = ||\alpha' \times \alpha''||$, we can find the cross product of the given function's first and second derivatives to be $|f''(x)|$. Likewise, it is known that $|\alpha'(x)| = (1 + (f'(x))^2)^{1/2}$. By cubing this, it is seen that $|\alpha'(x)|^3 = (1 + (f(x))^2)^{3/2}$.
   $\therefore \kappa = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3} = \frac{|f''|}{(1+(f')^2)^{3/2}}$ □

4. *Section 1.2 Problem 11*

   We wish to show that
   $$\tau(s(t)) = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2}$$
   To begin, note that the triple product may be rewritten to $\alpha''' \cdot (\alpha' \times \alpha'')$. From class, we know that $(\alpha' \times \alpha'') = \kappa v^3 \mathbf{B}$. Likewise, $\alpha'' = v'\mathbf{T} + \kappa v^2 \mathbf{N}$.
   We begin by calculating $\frac{d}{dt}\alpha''(t)$.
   $$\begin{aligned}
   \alpha''' &= v''\mathbf{T} + v'v^2\kappa\mathbf{N} + 2vv'\kappa\mathbf{N} + v^3\kappa'\mathbf{N} + v^3\kappa\mathbf{N}' \\
   &= v''\mathbf{T} + v'v^2\kappa\mathbf{N} + 2vv'\kappa\mathbf{N} + v^3\kappa'\mathbf{N} + v^3\kappa(\tau\mathbf{B} - \kappa\mathbf{T}) \\
   &= v''\mathbf{T} + v'v^2\kappa\mathbf{N} + 2vv'\kappa\mathbf{N} + v^3\kappa'\mathbf{N} + v^3\kappa\tau\mathbf{B} - v^3\kappa^2\mathbf{T}
   \end{aligned}$$
   Now, note that $\mathbf{B} \cdot \mathbf{T}, \mathbf{B} \cdot \mathbf{N} = 0$ and $\mathbf{B} \cdot \mathbf{B} = |\mathbf{B}|^2$. So, $\alpha''' \cdot (\alpha' \times \alpha'') = \alpha''' \cdot \kappa v^3 \mathbf{B}$.
   $\implies \alpha''' \cdot (\alpha' \times \alpha'') = \tau\kappa v^3\mathbf{B} \cdot v^3\kappa\mathbf{B} = |\kappa v^3 \mathbf{B}|^2\tau$ because $\mathbf{B}$, $\mathbf{T}$, and $\mathbf{N}$ are orthogonal. By dividing by $|\alpha' \times \alpha''|^2$, this becomes τ!
   $$\therefore \frac{\alpha''' \cdot (\alpha' \times \alpha'')}{|\alpha' \times \alpha''|^2} = \tau$$
   □

5. *Section 1.2 Problem 15*

   (a) Begin by assuming that $\beta = \alpha + \lambda\mathbf{T} + \eta\mathbf{N}$, where η is an arbitrary variable. We wish to show that $\eta = \mu$. μ is defined as $|\beta - \alpha|$, as given in the problem. Rearrange the above equation and dot both sides by $\mathbf{N}$ to get
   $$(\beta - \alpha) \cdot \mathbf{N} = \lambda\mathbf{TN} + \eta\mathbf{NN}.$$

Note that $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{N} \cdot \mathbf{N} = 1$. Thus, the above can be rewritten to $(\beta - \alpha) \cdot \mathbf{N} = \eta$. By taking the absolute value of tboth sides, we can see that

$$|(\beta - \alpha)||\mathbf{N}| = |\beta - \alpha| = |\eta| = \eta.$$

Thus, $\eta = \mu$, so $\mathbf{N}$ has the coefficent μ.

Now, we must show that $\lambda = 0$. Because $\alpha(s)$ and $\beta(s)$ are parallel, it is true that $\beta' + \alpha' = 0$. By moving α to the left side and taking the derivative of $\beta = \alpha + \lambda\mathbf{T} + \mu\mathbf{N}$, it can be seen that

$$\beta' - \alpha' = \lambda'\mathbf{T} + \lambda\kappa\mathbf{N} - \mu\kappa\mathbf{T}$$

By noting that $\beta' = T_\beta$ and that $\alpha = -\beta$, this can be rewritten to

$$2\mathbf{T}_\beta = (\lambda' - \mu\kappa)\mathbf{T}_\beta + (\lambda\kappa)\mathbf{N}_\beta.$$

As can be seen by this equation, for it to be true, $\lambda = 0$. Therefore, $\beta(s) = \alpha(s) + \mu\mathbf{N}_\beta$, and the chord μ is normal to the curve at both points by definiton of the vector $\mathbf{N}$. $\square$

(b) From part (a), we know that $\alpha' = -\beta'$. We want to show that $\frac{1}{\kappa_\alpha} + \frac{1}{\kappa_\beta} = \mu$. As proved above, $\beta = \alpha + \mu\mathbf{N}_\alpha$ and $\alpha = \beta + \mu\mathbf{N}_\beta$. Differentiating both sides gives:

$$\begin{aligned} \mathbf{T}_\beta &= \mathbf{T}_\alpha + \mu(-\kappa_\alpha\mathbf{T}_\alpha) & \qquad \mathbf{T}_\alpha &= \mathbf{T}_\beta + \mu(-\kappa_\beta\mathbf{T}_\beta) \\ -\mathbf{T}_\alpha &= \mathbf{T}_\alpha + \mu(-\kappa_\alpha\mathbf{T}_\alpha) & -\mathbf{T}_\beta &= \mathbf{T}_\beta + \mu(-\kappa_\beta\mathbf{T}_\beta) \\ -2\mathbf{T}_\alpha &= -\mu\kappa_\alpha\mathbf{T}_\alpha & -2\mathbf{T}_\beta &= -\mu\kappa_\beta\mathbf{T}_\beta \\ \frac{1}{\kappa_\alpha} &= \frac{\mu}{2} & \frac{1}{\kappa_\beta} &= \frac{\mu}{2} \end{aligned}$$

Adding these equations gives $\frac{\mu}{2} + \frac{\mu}{2} = \frac{1}{\kappa_\alpha} + \frac{1}{\kappa_\beta} = \mu$. $\square$

6. *Section 1.2 Problem 20*

(a) Let α and β have the same normal line. At t, $\beta(t)$ is some distance along the normal line from $\alpha(t)$. This gives us that $\beta(s) = \alpha(s) + r(s)\mathbf{N}_\alpha$. We wish to show that $r(s)$ is a constant. Begin by differentiating to obtain

$$\begin{aligned} \mathbf{T}_\beta &= \mathbf{T}_\alpha + r'\mathbf{N}_\alpha - r\kappa\mathbf{T}_\alpha + r\tau\mathbf{B}_\alpha \\ &= (1 - r\kappa)\mathbf{T}_\alpha + r'\mathbf{N}_\alpha + r\tau\mathbf{B}_\alpha \end{aligned}$$

Since $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ are on the same line, any vector orthogonal to one must be orthogonal to the other. Thus, we dot the above expression by $\mathbf{N}_\alpha$ to obtain

$$\mathbf{T}_\beta \cdot \mathbf{N}_\alpha = (1 - r\kappa)\mathbf{T}_\alpha \cdot \mathbf{N}_\alpha + r'\mathbf{N}_\alpha \cdot \mathbf{N}_\alpha + r\tau\mathbf{B}_\alpha \cdot \mathbf{N}_\alpha.$$

$\implies 0 = r'$, so thus r(s) is a constant.

(b) We wish to show that the angle between $\mathbf{T}_\beta$ and $\mathbf{T}_\alpha$ is constant. To do this, begin with the statement found in (a) to say $\beta = \alpha + r\mathbf{N}$, where β is not necessarily arclength parametrized. By differentiating, we find $\beta' = \mathbf{T}_\alpha + r(-\kappa\mathbf{T}_\alpha + \tau\mathbf{B}_\alpha)$ or

$$v_\beta\mathbf{T}_\beta = (1 - r\kappa_\alpha)\mathbf{T}_\alpha + (r\tau_\alpha)\mathbf{B}_\alpha,$$

where $\mathbf{T}_\beta$ is the unit tangent for β. By dividing through by v_β we obtain

$$\mathbf{T}_\beta = \frac{(1 - r\kappa_\alpha)}{v_\beta}\mathbf{T}_\alpha + \frac{(r\tau_\alpha)}{v_\beta}\mathbf{B}_\alpha.$$

Now, replace the two fractions with f and g, respectively. Then, by differentiation the following is obtained:

$$v_\beta \kappa_\beta \mathbf{N}_\beta = f'\mathbf{T}_\alpha + f(\kappa_\alpha \mathbf{N}_\alpha) + g'\mathbf{B}_\alpha + g(-\tau_\alpha \mathbf{N}_\alpha)$$

Notice that the left-hand side has no $\mathbf{T}_\alpha$ or $\mathbf{B}_\alpha$ terms, so therefore $f', g' = 0$.

To use this fact, we will now compute $\langle \mathbf{T}_\beta, \mathbf{T}_\alpha \rangle$. Using the first derivative found above, it is easy to see that

$$\mathbf{T}_\beta \cdot \mathbf{T}_\alpha = \frac{1 - r\kappa_\alpha}{v_\beta}.$$

Since the dot product can be defined as $|\mathbf{T}_\beta||\mathbf{T}_\alpha| \cos\theta$, where $|\mathbf{T}_\beta|, |\mathbf{T}_\alpha| = 1$, this becomes

$$\frac{1 - r\kappa_\alpha}{v_\beta} = \cos\theta \quad \text{or} \quad \cos^{-1}\left(\frac{1 - r\kappa_\alpha}{v_\beta}\right) = \theta$$

But since, from above, $f' = 0$, we can say that θ is a constant. □

(c) From (b), we know that

$$1 - r\kappa_\alpha = v_\beta C_1 \quad \text{and} \quad r\tau_\alpha = v_\beta C_2$$

where C_1, C_2 are constants. By rearranging the second equation, we obtain $-\frac{C_1}{C_2} r\tau_\alpha = -v_\beta C_1$. When these two equations are added together, it results in

$$1 = r\kappa + c\tau$$

as desired. □

(d) Begin with the fact that $1 = r\kappa + c\tau$, which was found in (c). Taking this equation's derivative gives $0 = r\kappa' + c\tau'$, since r,c are constants their derivatives are zero. By dividing by the constant r, this becomes $0 = \kappa' + j\tau'$, where j is the new constant. From this, it is apparent that $\kappa = \tau + m$, where m is an arbitrary constant. Plugging this result back into the original equation gives $1 = r(\tau + m) + c\tau$. Collecting terms, we find that

$$\frac{1 - rm}{r + c} = \tau.$$

This proves that τ is a constant.

We will now use this fact to show that infinitely many β result in a circular helix. Let $\beta_i = \alpha + r_i \mathbf{N}_\alpha$ and $\beta_j = \alpha + r_j \mathbf{N}_\alpha$. These are both Bertrand mates to α. Thus, it is true that $1 = r_i\kappa + c_i\tau$ and $1 = r_j\kappa + c_j\tau$. We can state that these are equivalent and rearrange them to show

$$\begin{aligned} r_i\kappa + c_i\tau &= r_j\kappa + c_j\tau \\ (r_i - r_j)\kappa + (c_i - c_j)\tau &= 0 \\ (c_i - c_j)\tau &= (r_j - r_i)\kappa \\ \implies \frac{\tau}{\kappa} &= \frac{r_j - r_i}{c_i - c_j} \end{aligned}$$

Therefore, since τ/κ is a constant, α is a generalized helix by Proposition 2.5 of the textbook. But from what was shown above, τ is a constant. Thus, for τ/κ to be constant with τ already constant, κ must also be constant. But a circular helix is defined as having constant τ and κ, so α must be a circular helix. □

7. *Section 1.2 Problem 22*

   Since $\alpha = c\int_a^t (\mathbf{Y} \times \mathbf{Y}')du$, we can say that $\alpha' = c(\mathbf{Y} \times \mathbf{Y}')$. Also $\alpha'' = c[(\mathbf{Y}' \times \mathbf{Y}') \times (\mathbf{Y} \times \mathbf{Y}'')] = c(\mathbf{Y} \times \mathbf{Y}'')$. Finally $\alpha''' = c[(\mathbf{Y}' \times \mathbf{Y}'') \times (\mathbf{Y} \times \mathbf{Y}''')] = c(\mathbf{Y}' \times \mathbf{Y}'')$, since $\mathbf{Y}$ is $\mathcal{C}^2$, $\mathbf{Y}''' = 0$. From Problem 11, we know that

   $$\tau(s(t)) = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2}.$$

   I was unable to complete this problem, but I believe that my setup is correct and will ultimately show that $\tau = \frac{1}{c}$.

Work was collaborated on with Hollis Neel.