MATH 6250 - Surfaces

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Functions

```
In[1]:= norm[j_] := FullSimplify[Sqrt[TrigReduce[j[[1]]^2 + j[[2]]^2 + j[[3]]^2]]]

In[2]:= Surface[x_, u_, v_] := Module[{Ip, IIp, E, F, G, area, l, m, n,
          Sp, K, H, k1, k2, Vec, uvec1, uvec2, ugraph, vgraph, vvec1, vvec2},
         E = FullSimplify[Dot[D[x[u, v], u], D[x[u, v], u]]];
         F = FullSimplify[Dot[D[x[u, v], u], D[x[u, v], v]]];
         G = FullSimplify[Dot[D[x[u, v], v], D[x[u, v], v]]];
         Ip = {{E, F}, {F, G}};
         (*area = Integrate[Sqrt[E G-F^2],u,v];*)
         l = FullSimplify[Dot[D[D[x[u, v], u], u], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
             norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         m = FullSimplify[Dot[D[D[x[u, v], u], v], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
             norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         n = FullSimplify[Dot[D[D[x[u, v], v], v], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
             norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         IIp = {{l, m}, {m, n}};
         Sp = FullSimplify[Inverse[Ip].IIp];
         K = FullSimplify[Det[Sp]];
         H = FullSimplify[Tr[Sp] / 2];
         k1 = FullSimplify[ H + Sqrt[H^2 - K]];
         k2 = FullSimplify[H - Sqrt[H^2 - K]];
         Vec = Eigenvectors[Sp];
         uvec1 = ReplaceAll[Vec[[1, 1]], u → u[v]];
         uvec2 = ReplaceAll[Vec[[1, 2]], u → u[v]];
         (*usolution = DSolve[u'[v]==uvec1/uvec2,u[v],v];*)
         vvec1 = ReplaceAll[Vec[[2, 1]], v → v[u]];
         vvec2 = ReplaceAll[Vec[[2, 2]], v → v[u]];
         (*vsolution = Simplify[DSolve[v'[u]==vvec2/vvec1,v[u],u]];*)
         ugraph = Show[Table[Module[{usol},
             usol = DSolve[{u'[v] == uvec1 / uvec2, u[0] == i}, u[v], v];
             ParametricPlot3D[x[(u[v] /. usol), v], {v, -4, 4}]], {i, -4, 4, 0.5}],
           PlotRange → {{-4, 4}, {-4, 4}, {-4, 4}}];
         vgraph = Show[Table[Module[{vsol},
             vsol = DSolve[{v'[u] == vvec2 / vvec1, v[0] == j}, v[u], u];
             ParametricPlot3D[x[u, (v[u] /. vsol)], {u, -4, 4}]], {j, -4, 4, 0.5}],
           PlotRange → {{-4, 4}, {-4, 4}, {-4, 4}}];
         CellPrint[{
```

```
    Cell[TextData[{"Ip = ", Cell[BoxData[ToBoxes[MatrixForm[Ip]]]]}], "Text"],
    (*Cell[
     TextData[{"Surface area = ", Cell[BoxData[ToBoxes[area]]]}],"Text"],*)
    Cell[TextData[{"IIp = ", Cell[BoxData[ToBoxes[MatrixForm[IIp]]]]}], "Text"],
    Cell[TextData[{"Sp = ", Cell[BoxData[ToBoxes[MatrixForm[Sp]]]]}], "Text"],
    Cell[TextData[{"K = ", Cell[BoxData[ToBoxes[K]]]}], "Text"],
    Cell[TextData[{"H = ", Cell[BoxData[ToBoxes[H]]]}], "Text"],
    Cell[
     TextData[{"Vec = ", Cell[BoxData[ToBoxes[MatrixForm[Vec]]]]}], "Text"],
    Cell[BoxData[ToBoxes[Show[ugraph, vgraph]]], "Output"]
    (*Cell[BoxData[
      ToBoxes[Plot[Table[u[v]//.{usolution[[1,1]],C[1]→CC},{CC,-10,10,0.25}],
        {v,-4,4},AspectRatio→Automatic,PlotRange→{{-4,4},{-4,4}}]]],"Output"],
    Cell[BoxData[ToBoxes[Plot[Table[v[u]//.{vsolution[[1,1]],C[1]→CC},
          {CC,-10,10,0.25}],{u,-4,4},AspectRatio→Automatic,
         PlotRange→{{-4,4},{-4,4}}]]],"Output"]'*)
   }]
  (*{MatrixForm[Vec],Show[ugraph],
   Show[vgraph], DSolve[{u'[v]==uvec1/uvec2,u[0]==i},u[v],v],
   DSolve[{v'[u]==vvec2/vvec1,v[0]==j},v[u],u],Show[ugraph,vgraph]}*)
 ];
```

Test

```
x[u_, v_] := {u, v, u v};
```

```
Surface[x, u, v]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

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$$\mathrm{Ip} = \begin{pmatrix} 1+v^2 & u\,v \\ u\,v & 1+u^2 \end{pmatrix}$$

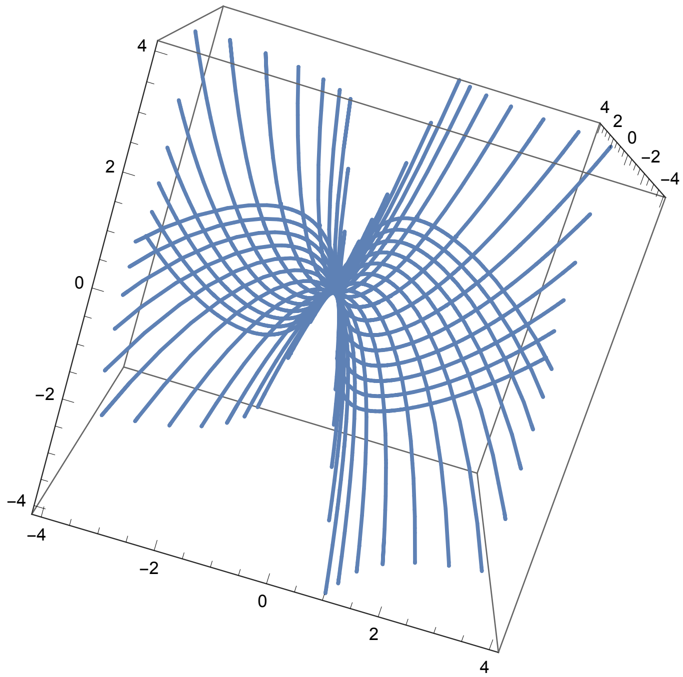
$$\mathrm{IIp} = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2+v^2}} \\ \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix}$$

$$\mathrm{Sp} = \begin{pmatrix} -\frac{u\,v}{(1+u^2+v^2)^{3/2}} & \frac{1+u^2}{(1+u^2+v^2)^{3/2}} \\ \frac{1+v^2}{(1+u^2+v^2)^{3/2}} & -\frac{u\,v}{(1+u^2+v^2)^{3/2}} \end{pmatrix}$$

$$K = -\frac{1}{(1+u^2+v^2)^2}$$

$$H = -\frac{u\,v}{\left(1+u^2+v^2\right)^{3/2}}$$

$$\text{Vec} = \begin{pmatrix} -\frac{\sqrt{\left(1+u^2\right)\left(1+v^2\right)}}{1+v^2} & 1 \\ \frac{\sqrt{\left(1+u^2\right)\left(1+v^2\right)}}{1+v^2} & 1 \end{pmatrix}$$




```
x[u_, v_] := {(3 + Cos[u]) Cos[v], (3 + Cos[u]) Sin[v], Sin[u]};
Surface[x, u, v]
```

$$\text{Ip} = \begin{pmatrix} 1 & 0 \\ 0 & (3+\text{Cos}[u])^2 \end{pmatrix}$$

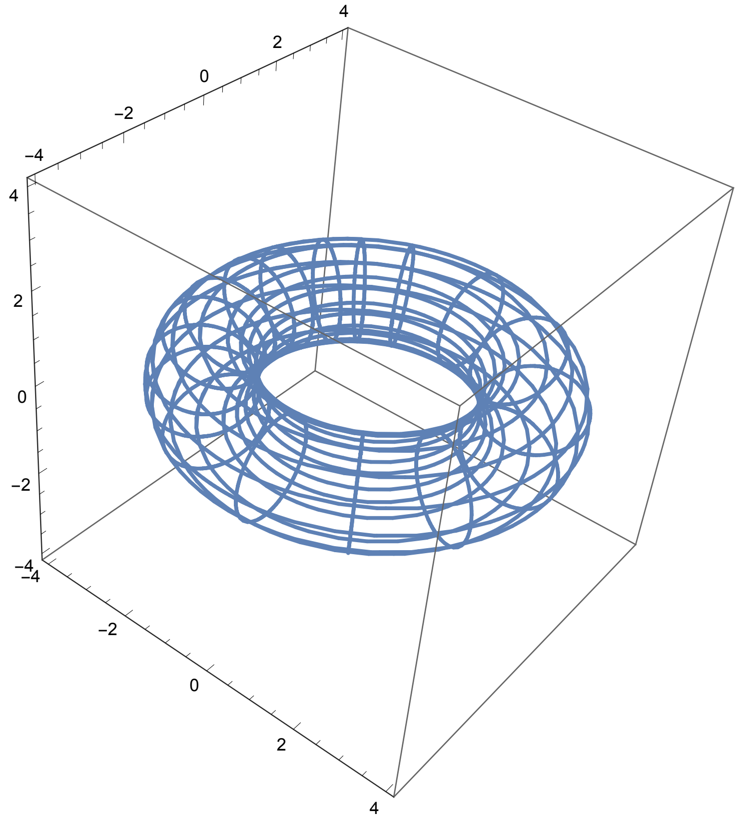
$$\text{IIp} = \begin{pmatrix} \frac{3+\text{Cos}[u]}{\sqrt{(3+\text{Cos}[u])^2}} & 0 \\ 0 & \text{Cos}[u]\sqrt{(3+\text{Cos}[u])^2} \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} \frac{3+\text{Cos}[u]}{\sqrt{(3+\text{Cos}[u])^2}} & 0 \\ 0 & \frac{\text{Cos}[u]}{\sqrt{(3+\text{Cos}[u])^2}} \end{pmatrix}$$

$$K = \frac{1}{1+3\,\text{Sec}[u]}$$

$$H = \frac{3+2\,\text{Cos}[u]}{2\sqrt{(3+\text{Cos}[u])^2}}$$

$$\text{Vec} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$




```
x[u_, v_] := {u Cos[v], u Sin[v], v};
Surface[x, u, v]
```

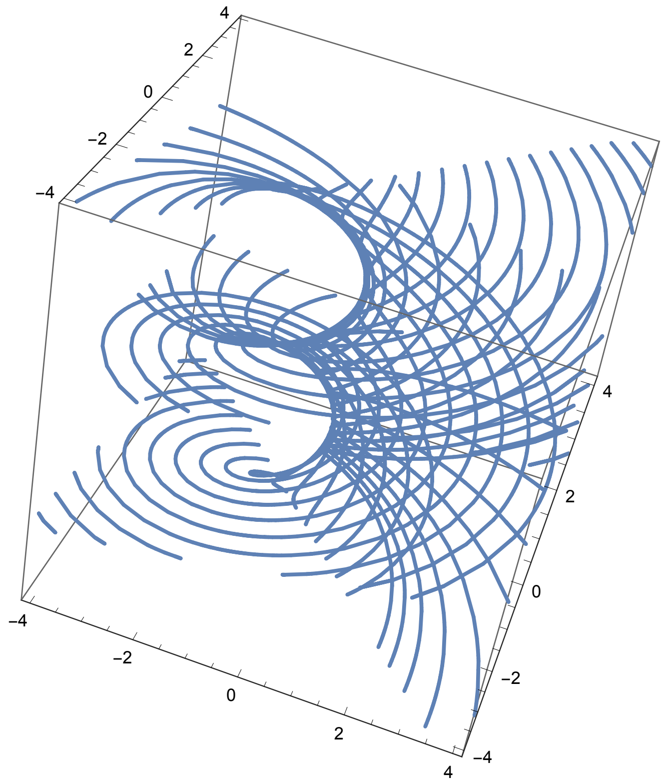
$$Ip = \begin{pmatrix} 1 & 0 \\ 0 & 1 + u^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{\sqrt{1+u^2}} & 0 \end{pmatrix}$$

$$Sp = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{(1+u^2)^{3/2}} & 0 \end{pmatrix}$$

$$K = -\frac{1}{(1+u^2)^2}$$

$$H = 0$$




```
x[u_, v_] := 4 {Cosh[u] Cos[v], Cosh[u] Sin[v], u};
```

```
Surface[x, u, v]
```

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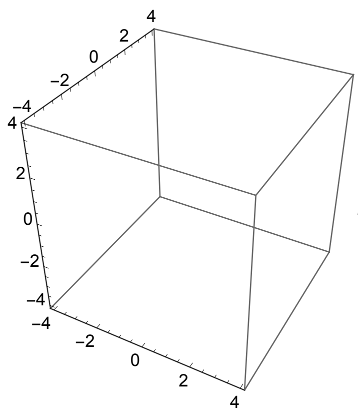
$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right.$



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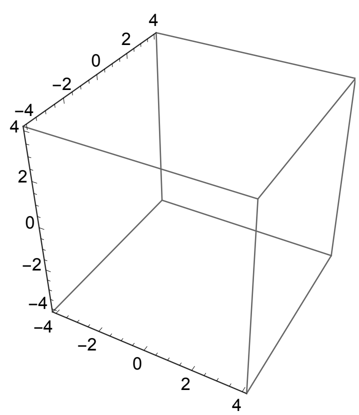

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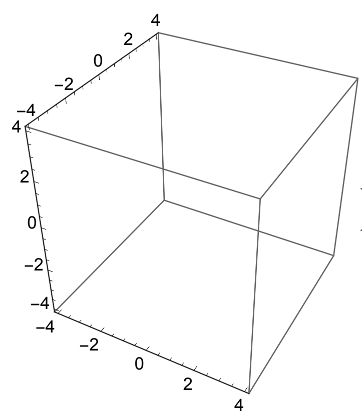


}

```
x[u_, v_] := 2 {Sin[u] Cos[v], Sin[u] Sin[v], Cos[u]};
```

```
Surface[x, u, v]
```

$$\text{Ip} = \begin{pmatrix} 4 & 0 \\ 0 & 4\,\text{Sin}[u]^2 \end{pmatrix}$$

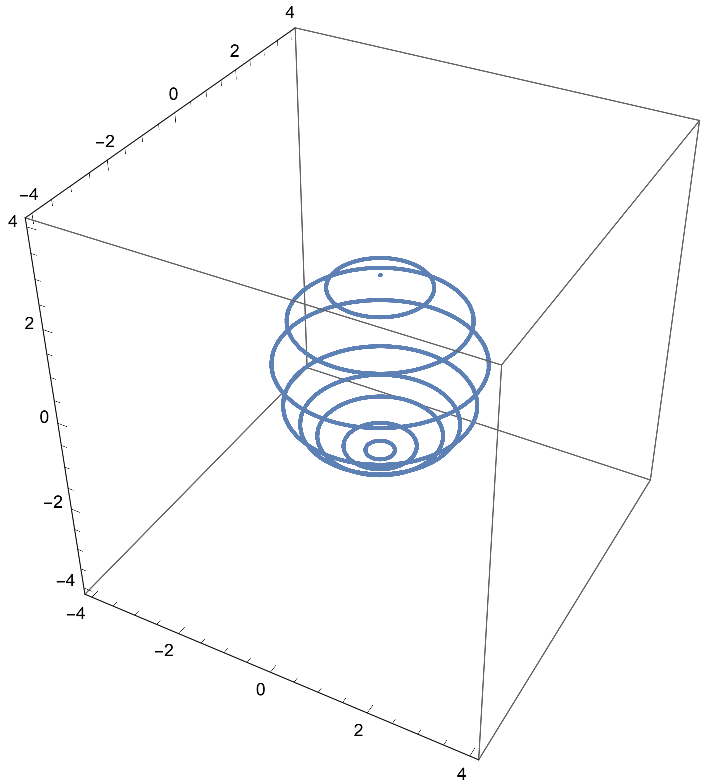
$$\text{IIp} = \begin{pmatrix} -2\,\text{Csc}[u]\,\sqrt{\text{Sin}[u]^2} & 0 \\ 0 & -2\,\text{Sin}[u]\,\sqrt{\text{Sin}[u]^2} \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} -\frac{1}{2}\,\text{Csc}[u]\,\sqrt{\text{Sin}[u]^2} & 0 \\ 0 & -\frac{1}{2}\,\text{Csc}[u]\,\sqrt{\text{Sin}[u]^2} \end{pmatrix}$$

$K = \frac{1}{4}$

$H = -\frac{1}{2} \text{Csc}[u] \sqrt{\text{Sin}[u]^2}$

$\text{Vec} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



Homework 5

Problem 3

A

```
q[u_, v_] := a {Cos[v] Sin[u], Sin[u] Sin[v], Cos[u]};
Surface[q, u, v]
```

$\text{Ip} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \text{Sin}[u]^2 \end{pmatrix}$

$\text{IIp} = \begin{pmatrix} -\frac{a^3 \text{Sin}[u]}{\sqrt{a^4 \text{Sin}[u]^2}} & 0 \\ 0 & -\frac{\text{Sin}[u] \sqrt{a^4 \text{Sin}[u]^2}}{a} \end{pmatrix}$

$$Sp = \begin{pmatrix} -\frac{a\,\mathrm{Sin}[u]}{\sqrt{a^4\,\mathrm{Sin}[u]^2}} & 0 \\ 0 & -\frac{a\,\mathrm{Sin}[u]}{\sqrt{a^4\,\mathrm{Sin}[u]^2}} \end{pmatrix}$$

$$K = \frac{1}{a^2}$$

$$H = -\frac{a\,\mathrm{Sin}[u]}{\sqrt{a^4\,\mathrm{Sin}[u]^2}}$$

B

```
q[u_, v_] := {(a + b Cos[u]) Cos[v], (a + b Cos[u]) Sin[v], b Sin[u]}
Surface[q, u, v]
```

$$Ip = \begin{pmatrix} b^2 & 0 \\ 0 & (a + b\,\mathrm{Cos}[u])^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} \frac{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}{a+b\,\mathrm{Cos}[u]} & 0 \\ 0 & \frac{\mathrm{Cos}[u]\,\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}{b} \end{pmatrix}$$

$$Sp = \begin{pmatrix} \frac{a+b\,\mathrm{Cos}[u]}{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}} & 0 \\ 0 & \frac{b\,\mathrm{Cos}[u]}{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}} \end{pmatrix}$$

$$K = \frac{1}{b^2 + a\,b\,\mathrm{Sec}[u]}$$

$$H = \frac{a + 2\,b\,\mathrm{Cos}[u]}{2\,\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}$$

C

```
q[u_, v_] := {u Cos[v], u Sin[v], v};
Surface[q, u, v]
```

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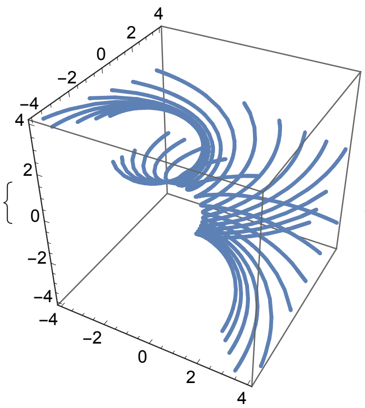
{



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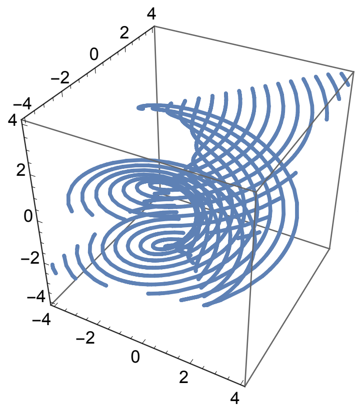
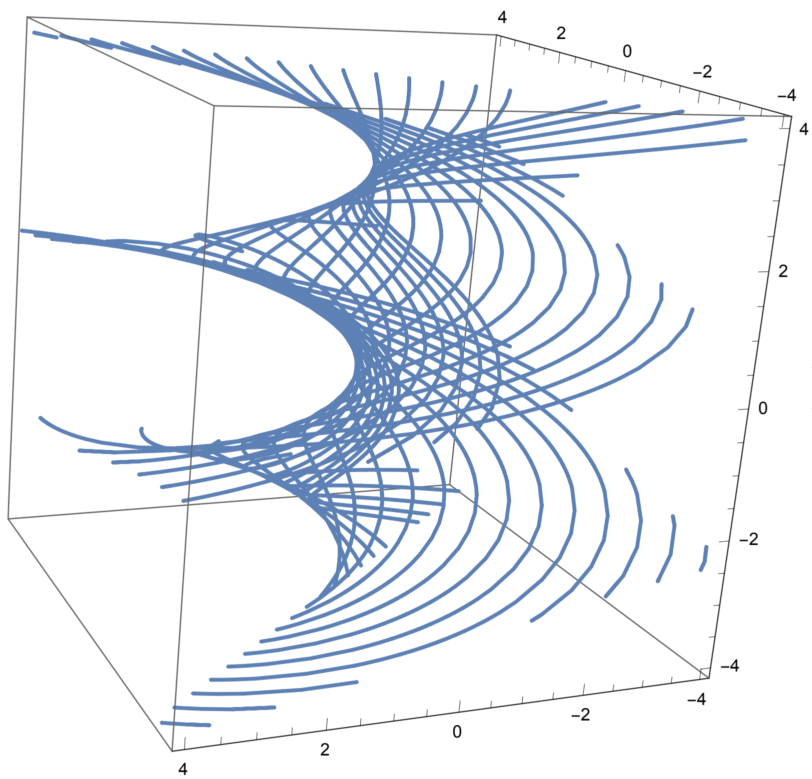


, {{u[v] → Sinh[v + ArcSinh[i]]}},

{{v[u] → j - ArcSinh[u]}},



}

D

In[3]:=
```
q[u_, v_] := {Cosh[u] Cos[v], Cosh[u] Sin[v], u}
```

In[4]:= **Surface[q, u, v]**

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Power::infy : Infinite expression $\frac{1}{0}$ encountered . >>

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$$\text{Ip} = \begin{pmatrix} \text{Cosh}[u]^2 & 0 \\ 0 & \text{Cosh}[u]^2 \end{pmatrix}$$

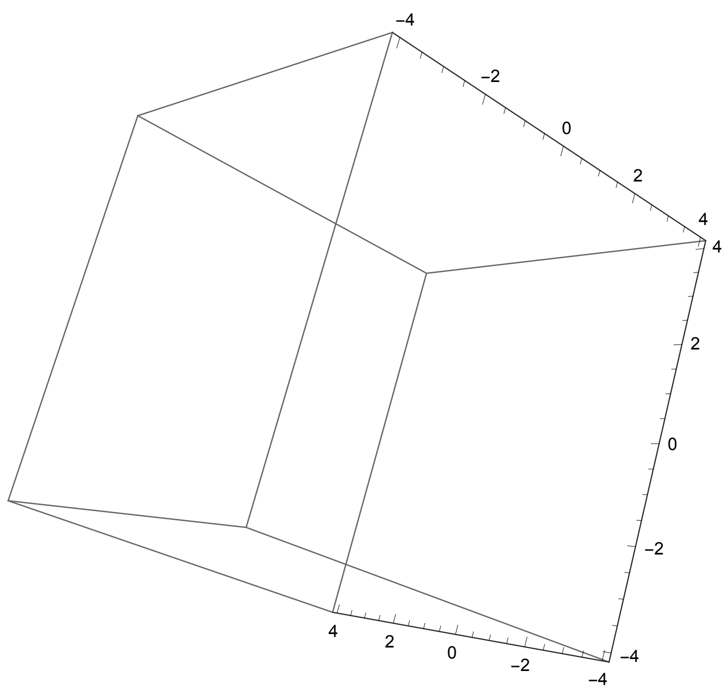
$$\text{IIp} = \begin{pmatrix} -\sqrt{\text{Cosh}[u]^4}\ \text{Sech}[u]^2 & 0 \\ 0 & \sqrt{\text{Cosh}[u]^4}\ \text{Sech}[u]^2 \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} -\frac{1}{\sqrt{\text{Cosh}[u]^4}} & 0 \\ 0 & \frac{1}{\sqrt{\text{Cosh}[u]^4}} \end{pmatrix}$$

$$K = -\text{Sech}[u]^4$$

$$H = 0$$

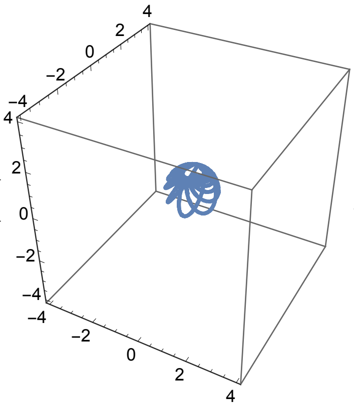
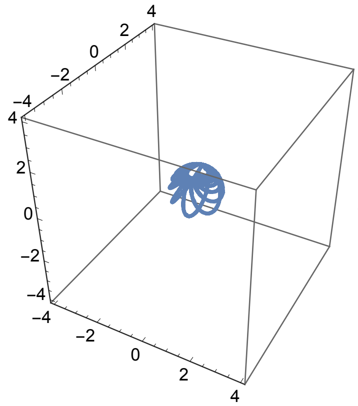
$$\text{Vec} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

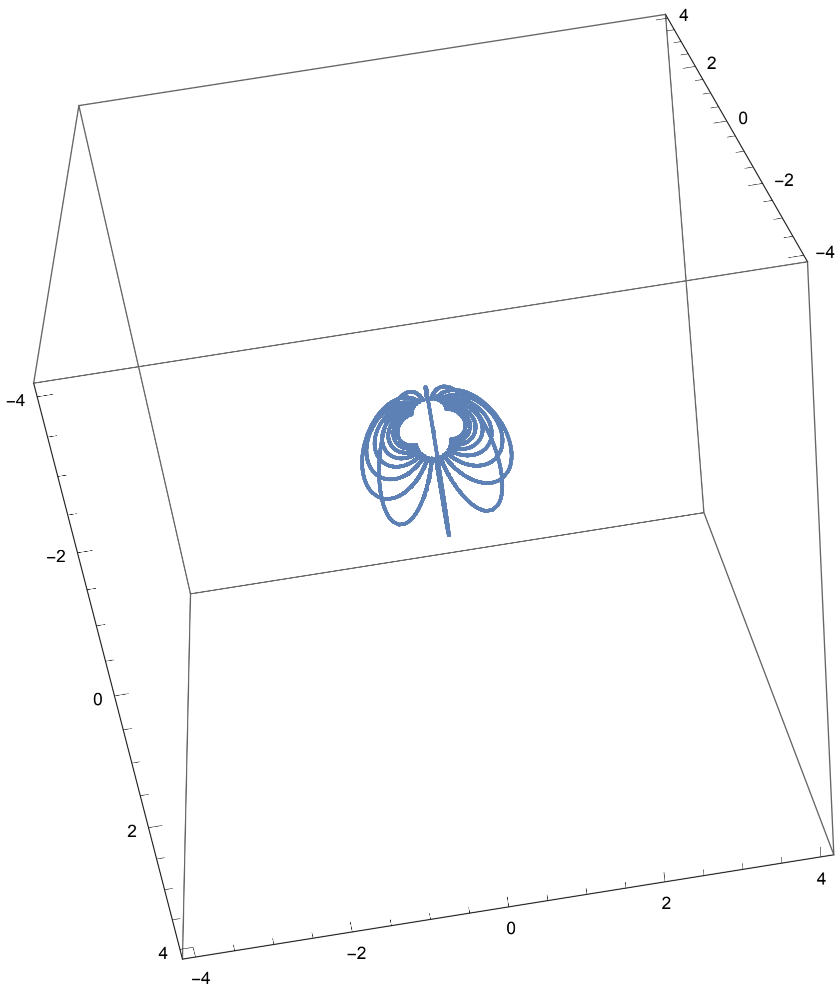


Problem 8

$$q[u\_, v\_] := \left\{\frac{2\,u}{u^{\wedge}2 + v^{\wedge}2 + 1}, \frac{2\,v}{u^{\wedge}2 + v^{\wedge}2 + 1}, \frac{u^{\wedge}2 + v^{\wedge}2 - 1}{u^{\wedge}2 + v^{\wedge}2 + 1}\right\}$$

```
Surface[q, u, v]
```

{  ,  , {{u[v] → i}},

{{v[u] → j}},  }

Problem 16

```
q[u_, v_] := a {Cosh[u] Cos[v], Cosh[u] Sin[v], u};
Surface[q, u, v]
```

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Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

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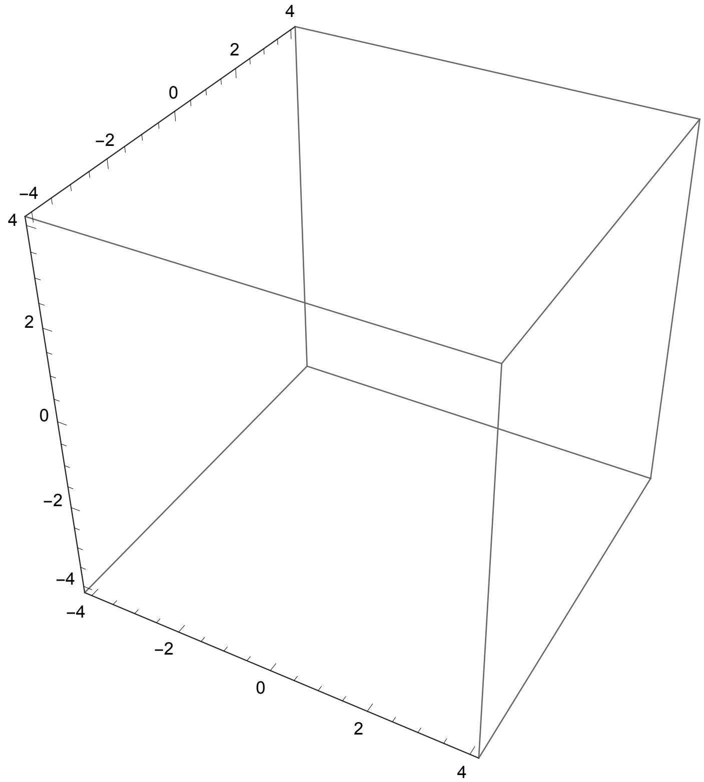
$$\text{Ip} = \begin{pmatrix} a^2 \,\text{Cosh}[u]^2 & 0 \\ 0 & a^2 \,\text{Cosh}[u]^2 \end{pmatrix}$$

$$\text{IIp} = \begin{pmatrix} -\frac{a^3 \,\text{Cosh}[u]^2}{\sqrt{a^4 \,\text{Cosh}[u]^4}} & 0 \\ 0 & \frac{a^3 \,\text{Cosh}[u]^2}{\sqrt{a^4 \,\text{Cosh}[u]^4}} \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} -\frac{a}{\sqrt{a^4 \,\text{Cosh}[u]^4}} & 0 \\ 0 & \frac{a}{\sqrt{a^4 \,\text{Cosh}[u]^4}} \end{pmatrix}$$

$$K = -\frac{\text{Sech}[u]^4}{a^2}$$

$$H = 0$$



Integrate$\left[a^2 \operatorname{Cosh}[u]^2, \{u, -1/a, 1/a\}, \{v, 0, 2\,\mathrm{Pi}\}\right]$

$$a\,\pi\left(2 + a\,\operatorname{Sinh}\left[\frac{2}{a}\right]\right)$$

```
Plot[{{t Cosh[1/t]}, {Sqrt[3 - t^2]}, {-Sqrt[3 - t^2]}},
 {t, -5, 5}, AspectRatio → Automatic, PlotRange → {{-5, 5}, {-5, 5}}]
```