MATH 6250 - Surfaces

Alexander Winkles

Functions

```
In[26]:= norm[j_] := FullSimplify[√(TrigReduce[j[[1]]^2 + j[[2]]^2 + j[[3]]^2])]

In[25]:= Surface[x_, u_, v_] := Module[{Ip, IIp, E, F, G, area, l, m, n,
          Sp, K, H, k1, k2, Vec, uvec1, uvec2, ugraph, vgraph, vvec1, vvec2},
         E = FullSimplify[Dot[D[x[u, v], u], D[x[u, v], u]]];
         F = FullSimplify[Dot[D[x[u, v], u], D[x[u, v], v]]];
         G = FullSimplify[Dot[D[x[u, v], v], D[x[u, v], v]]];
         Ip = (E  F
               F  G);
         (*area = Integrate[Sqrt[E G-F^2],u,v];*)
         l = FullSimplify[Dot[D[D[x[u, v], u], u], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
              norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         m = FullSimplify[Dot[D[D[x[u, v], u], v], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
              norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         n = FullSimplify[Dot[D[D[x[u, v], v], v], (Cross[D[x[u, v], u], D[x[u, v], v]]) /
              norm[Cross[D[x[u, v], u], D[x[u, v], v]]]]];
         IIp = (l  m
                m  n);
         Sp = FullSimplify[Inverse[Ip].IIp];
         K = FullSimplify[Det[Sp]];
         H = FullSimplify[Tr[Sp] / 2];
         k1 = FullSimplify[ H + Sqrt[H^2 - K]];
         k2 = FullSimplify[H - Sqrt[H^2 - K]];
         Vec = Eigenvectors[Sp];
         uvec1 = ReplaceAll[Vec[[1, 1]], u → u[v]];
         uvec2 =  ReplaceAll[Vec[[1, 2]], u → u[v]];
         (*usolution = DSolve[u'[v]==uvec1/uvec2,u[v],v];*)
         vvec1 = ReplaceAll[Vec[[2, 1]], v → v[u]];
         vvec2 =  ReplaceAll[Vec[[2, 2]], v → v[u]];
         (*vsolution = Simplify[DSolve[v'[u]==vvec2/vvec1,v[u],u]];*)
         ugraph = Show[Table[Module[{usol},
              usol = DSolve[{u'[v] == uvec1 / uvec2, u[0] == i}, u[v], v];
              ParametricPlot3D[x[(u[v] /. usol), v], {v, -4, 4}]], {i, -4, 4, 0.5}],
           PlotRange → {{-4, 4}, {-4, 4}, {-4, 4}}];
         vgraph = Show[Table[Module[{vsol},
              vsol = DSolve[{v'[u] == vvec2 / vvec1, v[0] == j}, v[u], u];
```

```
      ParametricPlot3D[x[(v[u] /. vsol), u], {u, -4, 4}]], {j, -4, 4, 0.5}],
    PlotRange → {{-4, 4}, {-4, 4}, {-4, 4}}];
  CellPrint[{
    Cell[TextData[{"Ip = ", Cell[BoxData[ToBoxes[MatrixForm[Ip]]]]}], "Text"],
    (*Cell[
     TextData[{"Surface area = ", Cell[BoxData[ToBoxes[area]]]}],"Text"],*)
    Cell[TextData[{"IIp = ", Cell[BoxData[ToBoxes[MatrixForm[IIp]]]]}], "Text"],
    Cell[TextData[{"Sp = ", Cell[BoxData[ToBoxes[MatrixForm[Sp]]]]}], "Text"],
    Cell[TextData[{"K = ", Cell[BoxData[ToBoxes[K]]]}], "Text"],
    Cell[TextData[{"H = ", Cell[BoxData[ToBoxes[H]]]}], "Text"],
    Cell[BoxData[ToBoxes[Show[ugraph, vgraph]]], "Output"]
    (*Cell[BoxData[ToBoxes[
       Plot[Table[u[v]//.{usolution[[1,1]],C[1]→CC},{CC,-10,10,0.25}],{v,-4,4},
        Aspec\.14tRatio→Automatic,PlotRange→{{-4,4},{-4,4}}]]],"Output"],
    Cell[BoxData[ToBoxes[Plot[Table[v[u]//.{vsolution[[1,1]],C[1]→CC},
          {CC,-10,10,0.25}],{u,-4,4},AspectRatio→Automatic,
         PlotRange→{{-4,4},{-4,4}}]]],"Output"]'*)
   }]
  (*Show[ugraph,vgraph]*)
 ];
```

Test

In[13]:=
```
x[u_, v_] := {u, v, u v};
```

In[14]:=
```
Surface[x, u, v]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

General::stop : Further output of Solve::ifun will be suppressed during this calculation. ≫

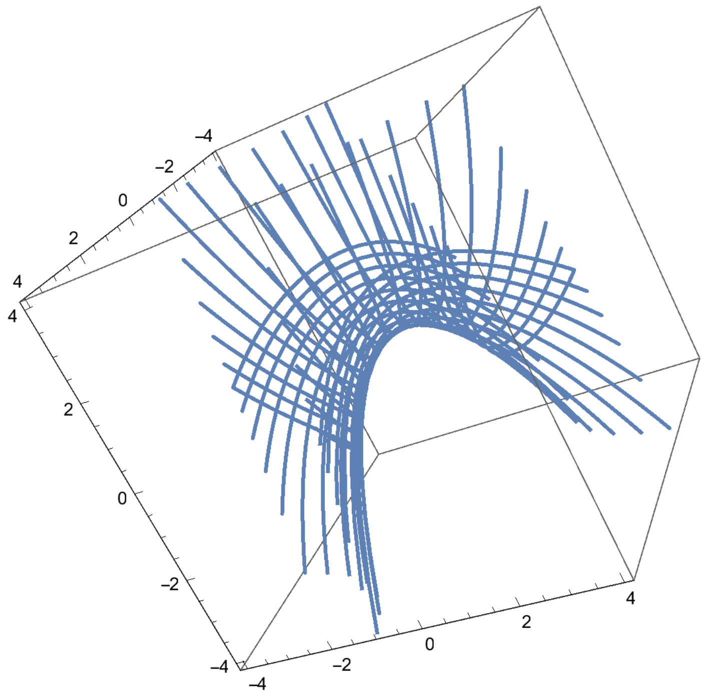
$$\text{Ip} = \begin{pmatrix} 1+v^2 & u\,v \\ u\,v & 1+u^2 \end{pmatrix}$$

$$\text{IIp} = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2+v^2}} \\ \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} -\frac{u\,v}{(1+u^2+v^2)^{3/2}} & \frac{1+u^2}{(1+u^2+v^2)^{3/2}} \\ \frac{1+v^2}{(1+u^2+v^2)^{3/2}} & -\frac{u\,v}{(1+u^2+v^2)^{3/2}} \end{pmatrix}$$

$$K = -\frac{1}{(1+u^2+v^2)^2}$$

$$H = -\frac{u\,v}{\left(1+u^2+v^2\right)^{3/2}}$$



In[27]:=
```
x[u_, v_] := {(2 + Cos[u]) Cos[v], (2 + Cos[u]) Sin[v], Sin[u]};
```

In[28]:=
```
Surface[x, u, v]
```

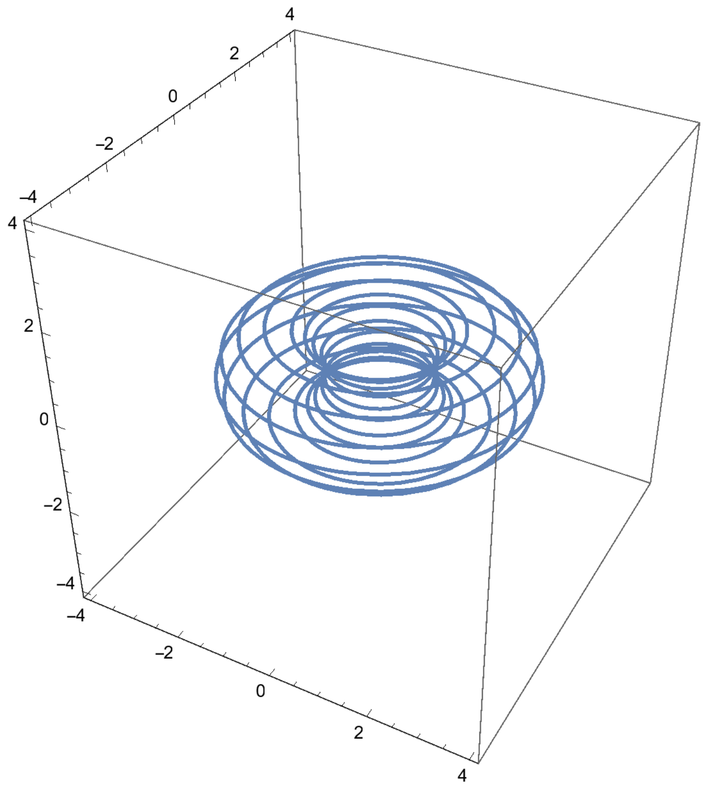
$$Ip = \begin{pmatrix} 1 & 0 \\ 0 & (2+\mathrm{Cos}[u])^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} \frac{2+\mathrm{Cos}[u]}{\sqrt{(2+\mathrm{Cos}[u])^2}} & 0 \\ 0 & \mathrm{Cos}[u]\sqrt{(2+\mathrm{Cos}[u])^2} \end{pmatrix}$$

$$Sp = \begin{pmatrix} \frac{2+\mathrm{Cos}[u]}{\sqrt{(2+\mathrm{Cos}[u])^2}} & 0 \\ 0 & \frac{\mathrm{Cos}[u]}{\sqrt{(2+\mathrm{Cos}[u])^2}} \end{pmatrix}$$

$$K = \frac{1}{1+2\,\mathrm{Sec}[u]}$$

$$H = \frac{1+\mathrm{Cos}[u]}{\sqrt{(2+\mathrm{Cos}[u])^2}}$$



In[19]:=
```
x[u_, v_] := {u Cos[v], u Sin[v], u};
```

In[20]:=
```
Surface[x, u, v]
```

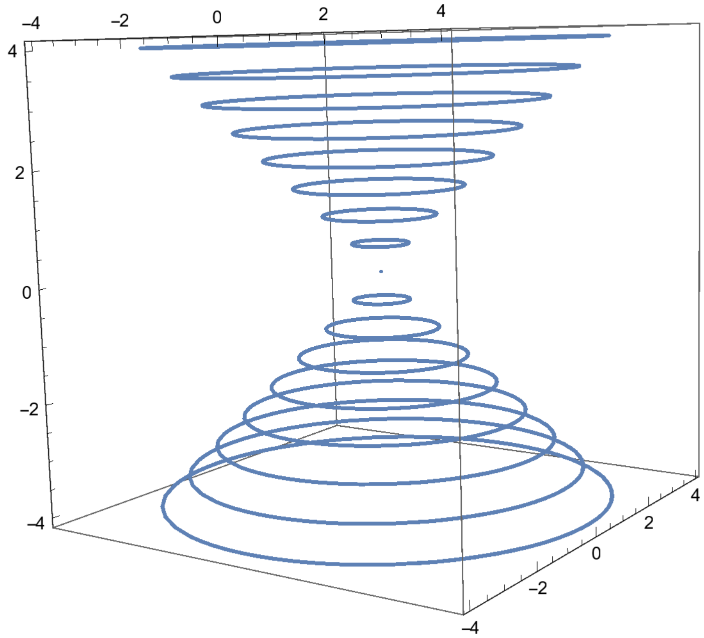
$$Ip = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sqrt{u^2}}{\sqrt{2}} \end{pmatrix}$$

$$Sp = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}\sqrt{u^2}} \end{pmatrix}$$

$$K = 0$$

$$H = \frac{1}{2\sqrt{2}\sqrt{u^2}}$$



Homework 5

Problem 3

A

```
q[u_, v_] := a {Cos[v] Sin[u], Sin[u] Sin[v], Cos[u]};
Surface[q, u, v]
```

$$Ip = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\ \mathrm{Sin}[u]^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} -\frac{a^3\ \mathrm{Sin}[u]}{\sqrt{a^4\ \mathrm{Sin}[u]^2}} & 0 \\ 0 & -\frac{\mathrm{Sin}[u]\ \sqrt{a^4\ \mathrm{Sin}[u]^2}}{a} \end{pmatrix}$$

$$Sp = \begin{pmatrix} -\frac{a\ \mathrm{Sin}[u]}{\sqrt{a^4\ \mathrm{Sin}[u]^2}} & 0 \\ 0 & -\frac{a\ \mathrm{Sin}[u]}{\sqrt{a^4\ \mathrm{Sin}[u]^2}} \end{pmatrix}$$

$$K = \frac{1}{a^2}$$

$$H = -\frac{a\ \mathrm{Sin}[u]}{\sqrt{a^4\ \mathrm{Sin}[u]^2}}$$

B

```
q[u_, v_] := {(a + b Cos[u]) Cos[v], (a + b Cos[u]) Sin[v], b Sin[u]}
Surface[q, u, v]
```

$$\mathrm{Ip} = \begin{pmatrix} b^2 & 0 \\ 0 & (a + b\,\mathrm{Cos}[u])^2 \end{pmatrix}$$

$$\mathrm{IIp} = \begin{pmatrix} \frac{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}{a+b\,\mathrm{Cos}[u]} & 0 \\ 0 & \frac{\mathrm{Cos}[u]\,\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}{b} \end{pmatrix}$$

$$\mathrm{Sp} = \begin{pmatrix} \frac{a+b\,\mathrm{Cos}[u]}{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}} & 0 \\ 0 & \frac{b\,\mathrm{Cos}[u]}{\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}} \end{pmatrix}$$

$$K = \frac{1}{b^2+a\,b\,\mathrm{Sec}[u]}$$

$$H = \frac{a+2\,b\,\mathrm{Cos}[u]}{2\,\sqrt{b^2\,(a+b\,\mathrm{Cos}[u])^2}}$$

C

```
q[u_, v_] := {u Cos[v], u Sin[v], b v};
Surface[q, u, v]
```

$$\mathrm{Ip} = \begin{pmatrix} 1 & 0 \\ 0 & b^2 + u^2 \end{pmatrix}$$

$$\mathrm{IIp} = \begin{pmatrix} 0 & -\frac{b}{\sqrt{b^2+u^2}} \\ -\frac{b}{\sqrt{b^2+u^2}} & 0 \end{pmatrix}$$

$$\mathrm{Sp} = \begin{pmatrix} 0 & -\frac{b}{\sqrt{b^2+u^2}} \\ -\frac{b}{(b^2+u^2)^{3/2}} & 0 \end{pmatrix}$$

$$K = -\frac{b^2}{(b^2+u^2)^2}$$

$$H = 0$$

D

```
q[u_, v_] := a {Cosh[u] Cos[v], Cosh[u] Sin[v], u}
Surface[q, u, v]
```

$$\mathrm{Ip} = \begin{pmatrix} a^2\,\mathrm{Cosh}[u]^2 & 0 \\ 0 & a^2\,\mathrm{Cosh}[u]^2 \end{pmatrix}$$

$$\text{IIp} = \begin{pmatrix} -\frac{a^3\,\text{Cosh}[u]^2}{\sqrt{a^4\,\text{Cosh}[u]^4}} & 0 \\ 0 & \frac{a^3\,\text{Cosh}[u]^2}{\sqrt{a^4\,\text{Cosh}[u]^4}} \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} -\frac{a}{\sqrt{a^4\,\text{Cosh}[u]^4}} & 0 \\ 0 & \frac{a}{\sqrt{a^4\,\text{Cosh}[u]^4}} \end{pmatrix}$$

$$K = -\frac{\text{Sech}[u]^4}{a^2}$$

$$H = 0$$

Problem 8

```
q[u_, v_] := {2 u/(u^2 + v^2 + 1), 2 v/(u^2 + v^2 + 1), (u^2 + v^2 - 1)/(u^2 + v^2 + 1)}
```

$$q[u\_,\ v\_] := \left\{\frac{2\,u}{u\,^\wedge 2 + v\,^\wedge 2 + 1},\ \frac{2\,v}{u\,^\wedge 2 + v\,^\wedge 2 + 1},\ \frac{u\,^\wedge 2 + v\,^\wedge 2 - 1}{u\,^\wedge 2 + v\,^\wedge 2 + 1}\right\}$$

```
Surface[q, u, v]
```

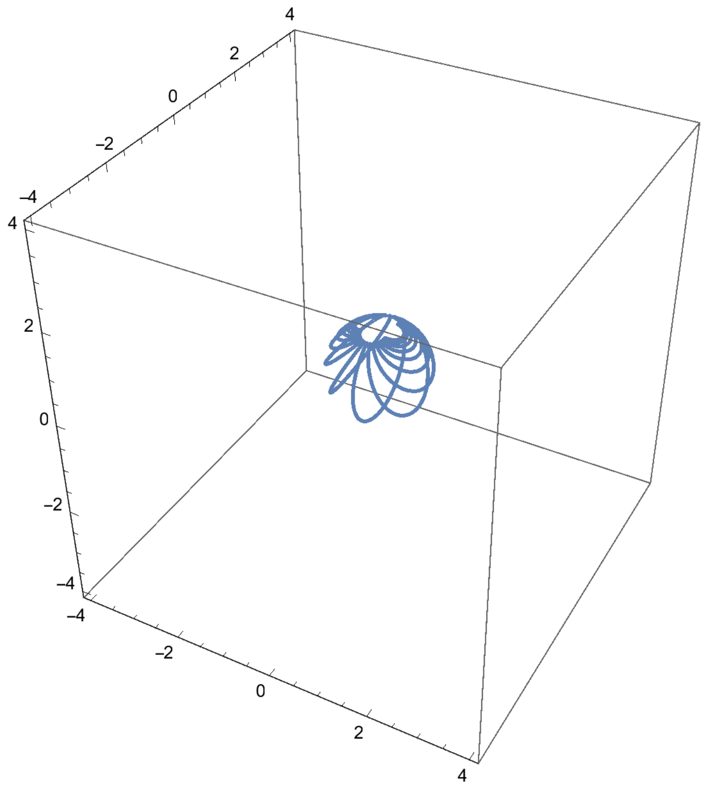
$$\text{Ip} = \begin{pmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{pmatrix}$$

$$\text{IIp} = \begin{pmatrix} 4\sqrt{\frac{1}{(1+u^2+v^2)^4}} & 0 \\ 0 & 4\sqrt{\frac{1}{(1+u^2+v^2)^4}} \end{pmatrix}$$

$$\text{Sp} = \begin{pmatrix} \sqrt{\frac{1}{(1+u^2+v^2)^4}}\,(1+u^2+v^2)^2 & 0 \\ 0 & \sqrt{\frac{1}{(1+u^2+v^2)^4}}\,(1+u^2+v^2)^2 \end{pmatrix}$$

$$K = 1$$

$$H = \sqrt{\frac{1}{(1+u^2+v^2)^4}}\,(1+u^2+v^2)^2$$



Problem 16

In[9]:=
```
q[u_, v_] := a {Cosh[u] Cos[v], Cosh[u] Sin[v], u};
```

In[10]:=
```
Surface[q, u, v]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

General::stop : Further output of Power::infy will be suppressed during this calculation. >>

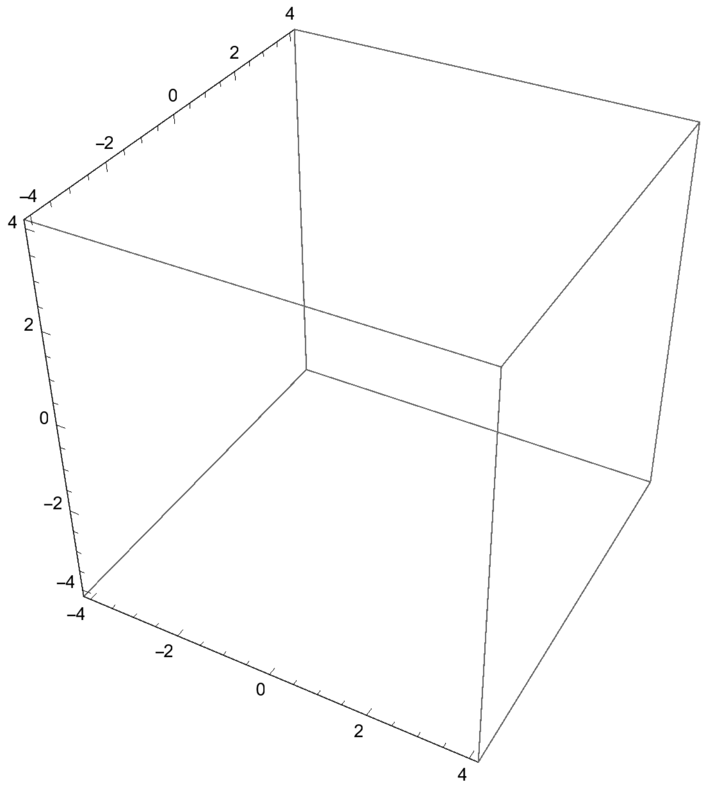
$$Ip = \begin{pmatrix} a^2 \operatorname{Cosh}[u]^2 & 0 \\ 0 & a^2 \operatorname{Cosh}[u]^2 \end{pmatrix}$$

$$IIp = \begin{pmatrix} -\frac{a^3 \operatorname{Cosh}[u]^2}{\sqrt{a^4 \operatorname{Cosh}[u]^4}} & 0 \\ 0 & \frac{a^3 \operatorname{Cosh}[u]^2}{\sqrt{a^4 \operatorname{Cosh}[u]^4}} \end{pmatrix}$$

$$Sp = \begin{pmatrix} -\frac{a}{\sqrt{a^4 \operatorname{Cosh}[u]^4}} & 0 \\ 0 & \frac{a}{\sqrt{a^4 \operatorname{Cosh}[u]^4}} \end{pmatrix}$$

$$K = -\frac{\operatorname{Sech}[u]^4}{a^2}$$

H = 0




```
Integrate[a^2 Cosh[u]^2, {u, -1 / a, 1 / a}, {v, 0, 2 Pi}]
```

$a\,\pi\left(2 + a\,\mathrm{Sinh}\left[\frac{2}{a}\right]\right)$

In[21]:=
```
Plot[{{t Cosh[1/t]}, {Sqrt[3 - t^2]}, {-Sqrt[3 - t^2]}},
  {t, -5, 5}, AspectRatio → Automatic, PlotRange → {{-5, 5}, {-5, 5}}]
```

Out[21]=