1. *Section 1.1 Problem 2*
   Let $\alpha(t) = (a\cos t, a\sin t, bt)$. Then, $\alpha'(t) = (-a\sin t, a\cos t, b)$. From this, it is found that $||\alpha'(t)|| = \sqrt{(-a\sin^2 t)^2 + (a\cos^2 t + b^2}$, which can be reduced through trig identities to $||\alpha'(t)|| = \sqrt{a^2+b^2}$. Now to parametrize this, the formula $s(t) = \int_a^t ||\alpha'(t)||dt$. This produces $s(t) = \int_0^t \sqrt{a^2+b^2}dt$ which results in $s(t) = \sqrt{a^2+b^2}t$. Therefore, $t(s) = \frac{1}{\sqrt{a^2+b^2}}s$. The parametrization of this is $\beta(s) = \alpha(t(s)) = (a\cos\frac{1}{\sqrt{a^2+b^2}}s, a\sin\frac{1}{\sqrt{a^2+b^2}}s, \frac{b}{\sqrt{a^2+b^2}}s)$.

2. *Section 1.1 Problem 4*
   Let $\alpha(x) = (x, f(x)$. Finding this graph's derivative gives $\alpha'(x) = (1, f'(x))$. Then its magnitude is $||\alpha'(x)|| = \sqrt{1+(f'(x))^2}$. Using the definition of arclength, it is clear that length $= \int_a^b \sqrt{1+(f'(x))^2}dx$.

3. *Section 1.1 Problem 5*

   (a) Let $\alpha(t) = (t, \cosh t)$ for $0 \le t \le b$. $\alpha'(t) = (1, \sinh t)$, and $||\alpha'(t)|| = \sqrt{1+\sinh^2 t}$. Notice that $1 + \sinh^2 t = \cosh^2 2$. Now $s(t) = \int_0^b \sqrt{1+\sinh^2 t}dt = \int_0^b \sqrt{\cosh^2 t}dt$. Taking the integral gives $\sinh b - \sinh 0 = \sinh b$.

   (b) Let $y = \sinh t$. Taking the definition of hyperbolic sine, it can be shown that $2y = e^t - e^{-t}$. Rearranging this and multiplying both sides by e^t gives $(e^t)^2 - 2y(e^t) - 1 = 0$. By solving for e^b with the quadratic equation, it can be seen that

   $$e^t = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2y \pm \sqrt{(-2y)^2+4}}{2} = \frac{2y \pm \sqrt{4(y^2+1)}}{2} = \frac{2y \pm 2\sqrt{y^2+1}}{2} = y \pm \sqrt{y^2+1}$$

   After this, by taking the natural log the following is obtained, noting that e^t cannot be negative: $t = \ln(y + \sqrt{y^2+1})$. Therefore the inverse of $\sinh t$ is $\ln(t + \sqrt{t^2+1})$, and the reparametrized catenary is $\beta(s) = (\ln(s + \sqrt{s^2+1}), \cosh(\ln(s + \sqrt{s^2+1})))$.

4. *Section 1.1 Problem 8*
   Let $P, Q \in \mathbb{R}^3$ and $\alpha : [a,b] \to \mathbb{R}^3$. Note that $\alpha(a) = P$ and $\alpha(b) = Q$. Finally, let $\mathbf{v} = Q - P$. Using the definition of the length, we can state for the partition $\mathcal{P} = \{a, b\}$:

   $$\ell(\alpha, \mathcal{P}) = \sum_{i=1}^{k} ||\alpha(t_i) - \alpha(t_{i-1})||$$

   Using the proof provided on page 8 of our textbook, it is seen that:

   $$\ell(\alpha, \mathcal{P}) = \sum_{i=1}^{k} ||\alpha(t_i) - \alpha(t_{i-1})|| = \sum_{i=1}^{k} \left|\left| \int_{t_{i-1}}^{t_i} \alpha'(t)dt \right|\right| \le \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} ||\alpha'(t)||dt = \int_a^b ||\alpha'(t)||dt$$

   From this, it is apparent that $\sum_{i=1}^{k} ||\alpha(t_i) - \alpha(t_{i-1})|| \le length(\alpha)$. Since $\mathcal{P} = \{a, b\}$, it can bee seen that $||\alpha(a) - \alpha(b)|| = ||\alpha(b) - \alpha(a)|| = ||Q - P|| = ||\mathbf{v}|| \le length(\alpha)$.

5. *Section 1.1 Problem 9*
   Begin by noting that for this system, all forces must be equal. Let $\gamma(t) = (x, f(x))$. Given two points $\mathbf{T}(x+h)$ and $-\mathbf{T}(x)$, we can find their respective tangent lines: $T_0(1, f'(x+h))$ and $T_0(-1, f'(x))$, where T_0 is the magnitude of the tension. Define the gravity acting on the system as $g\delta \int_x^{x+h} \sqrt{1+(f'(x))^2}dx$. Based on this step up and the fact that all forces must be equal, it can be seen that $T_0 f'(x+h) - T_0 f'(x) - g\delta \int_x^{x+h} \sqrt{1+(f'(x))^2}dx = 0$. By taking the limit of both sides as $h \to 0$ and using the

definition of a limit definition of a derivative, it can be seen that:

$$\lim_{h\to 0} \frac{T_0(f'(x+h)-f'(x)) - g\delta \int_x^{x+h} \sqrt{1+(f'(x))^2}dx}{h} = T_0 f''(x) - g\delta\sqrt{1+(f'(x))^2} = 0$$

Through rearrangement, it is easy to see that $f''(x) = \frac{g\delta}{T_0}\sqrt{1+(f'(x))^2}$.

Now, let $f''(x) = \frac{df'(x)}{dx}$. By separation of variables, this allows $\displaystyle\int \frac{df'(x)}{\sqrt{1+(f'(x))^2}} = \int \frac{g\delta}{T_0}dx$. Let $f'(x) = \sinh v$ and $df'(x) = \cosh v dv$. Then the equation becomes $\displaystyle\int \frac{\cosh v}{\sqrt{1+\sinh^2 v}}dv = \int \frac{\cosh v}{\sqrt{\cosh^2 v}}dv = \int \frac{\cosh v}{\cosh v}dv = \int 1 dv = \frac{g\delta}{T_o}x$. Thus, $v = \frac{g\delta}{T_0}x$ and $\sinh^{-1} f'(x) = \frac{g\delta}{T_0}x$. This can also be stated as $f'(x) = \sinh \frac{g\delta}{T_o}x$. Integrating both sides gives $f(x) = \frac{T_0}{g\delta}\cosh\frac{g\delta}{T_0}x + c$. Since $C = \frac{T_0}{g\delta}$, this gives the desired result of $f(x) = C\cosh\frac{x}{C} + c$.

6. *Section 1.1 Problem 10*
Begin by letting $\gamma = (x(s), y(s))$ be the arclength parametrization of the curve s.t. a square rolls smoothly (ie. the center of the square does not vary with height). For this to be true, the we must find the vector $\overrightarrow{OC}$, which is shown in the figure below. Therefore, let $\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QC}$. As

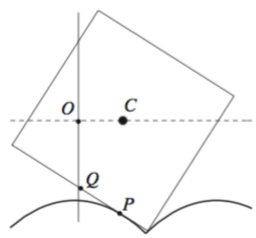

Figure 1: *Figure 1.13 from Shifrin's textbook*

mentioned above, $\overrightarrow{OC} = (x(s), 0)$. The vector $\overrightarrow{OP}$ can be seen to equal $\gamma(s)$, and $\overrightarrow{QP} = s\gamma'(s)$ because it represents the distance travelled by arclength. Note that this second quanitity needs to be reversed to solve the problem in question, so $\overrightarrow{PQ} = -s\gamma'(s)$. Next, Since $\gamma(s)$ is arclength parametrized and $\overrightarrow{QC}$ is length 1, it is true that $<\overrightarrow{QC}, \gamma'(s)>= 0$ since $\overrightarrow{QC}$ is orthogonal to $\gamma'(s)$. Likewise, since $\overrightarrow{QC}$ is counterclockwise from $\gamma'(s)$, we can define $\overrightarrow{QC} = (-y(s), x(s))$. By combining all these terms and looking at the y portions only, we can see that $0 = y(s) - sy'(s) + x'(s)$. Let us differentiate this to see that $0 = y'(s) - y'(s) - sy''(s) + x''(s)$, which when rearranged gives $sy''(s) = x''(s)$. Since $\gamma(s)$ is arclength parametrized, we can say that $<(x''(s), y''(s)), (x'(s), y'(s))>= 0$ and therefore $x'(s)x''(s) = -y'(s)y''(s)$. Using this fact with the result we obtained from differentiation, it is seen that $sx'(s)y''(s) = -y'(s)y''(s)$. Rearranging this gives $s = -\frac{y'(s)}{x'(s)}$. This can be further altered to show $s = -\frac{dy}{dx}$ By utilizing the equation found in problem 4, we can state that $\frac{ds}{dx} = \sqrt{1+s^2}$ or $\frac{dx}{ds} = \frac{1}{\sqrt{1+s^2}}$. Multiplying both sides by $\frac{dy}{dx}$, the above can be rewritten to $\frac{dy}{ds} = \frac{-s}{\sqrt{1+s^2}}$. Replace $s = \sinh x$ and solve the differential equation. This yields $y = \displaystyle\int \frac{-\sinh x\cosh x}{\sqrt{1+\sinh^2 x}}dx$. Using hyperbolic trig identities gives $y = \displaystyle\int -\sinh x ds$ or $y = -\cosh x + c$. Therefore, the road should be designed by $f(x) = -\cosh x + c$.

7. *Section 1.1 Problem 14*
   Let $\alpha(t)$ be a smooth parametrized plane curve such that $\alpha : [a, b]$. Let $||\alpha(s) - \alpha(t)||$ depend only on $|s - t|$, or $||\alpha(s) - \alpha(t)|| = C|s - t|$. Begin by dividing both sides by $|s - t|$, or $\frac{||\alpha(s)-\alpha(t)||}{|s-t|} = C$. By taking the limit of both sides, the following is obtained:

$$\lim_{s \to t} \frac{||\alpha(s) - \alpha(t)||}{|s - t|} = \lim_{s - t} C$$

   . In other words, $||\alpha'(t)|| = C$, which implies that α is a subset of a line. From this, rearrange the equaiton above to give $\sqrt{x'(t)^2 + y'(t)^2} = C$ or $x'(t)^2 + y'(t)^2 = C^2$. Taking this equation's derivative gives $2x'x'' + 2y'y'' = 0$, which after dividing the 2, shows $< (x', y'), (x'', y'') >= 0$, where the independent variable is removed for ease of typing. It can thus be seen that $|\alpha'| = |\alpha''| = C$. Since α is smooth, it can be concluded that $|\alpha| = |\alpha'| = |\alpha''| = C$, or that all derivatives of α are orthogonal to each other. Therefore, we can say that $|\alpha| = C$ or $x(t)^2 + y(t)^2 = C^2$, which is a subset of a circle.

Work was collaborated on with Hollis Neel.