"Thanks to Gödal's incompleteness theorems, everything you thought you knew about math is wrong. So basically, you're all getting a degree in lies."
-Random abstract algebra professor from the internet

1. *Section 2.1 Problem 1*

   Let S be the unit sphere centered at 0 with a north pole N as (0,0,1). Likewise, let P be any point other than N on S and let P* be a point on the equator that intersects the line formed by connecting P and N, denoted by (u,v,0). Thus, we can describe P by a multiple of the line connecting N and P*, $(0, 0, 1) + \lambda(u, v, -1)$ since P* can be inside or outside of the sphere depending on which hemisphere P is on. This can be rewritten to be $(\lambda u, \lambda v, 1 - \lambda)$. Since S is a unit sphere, it must be true that $x^2 + y^2 + z^2 = 1$. By plugging our parametrization of P into this, we obtain

   $$(\lambda u)^2 + (\lambda v)^2 + (1 - \lambda)^2 = 1$$

   By factoring and rearranging, this shows that

   $$\lambda = \frac{2}{u^2 + v^2 + 1}$$

   By plugging this into the parametrization above, this gives

   $$P = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right),$$

   as desired. □

2. *Section 2.1 Problem 2*

   Let $\alpha(t) = x(u(t), v(t))$, where $a \le t \le b$. From curve theory, we know that

   $$length(\alpha(t)) = \int_a^b |\alpha'(t)|dt.$$

   Since α and its derivative have regular parametrization, we can find $|\alpha'(t)| = \sqrt{(\alpha'(t))^2} = \sqrt{\alpha' * \alpha'} = \sqrt{I_p(\alpha', \alpha')}$. Thus, we can say

   $$length(\alpha(t)) = \int_a^b \sqrt{I_{\alpha(t)}(\alpha'(t), \alpha'(t))}dt,$$

   as desired. Note that $\alpha'(t) = u'(t)x_u + v'(t)x_v$. Then, actually writing out $I_p(\alpha'(t), \alpha'(t))$, we find that

   $$I_p(\alpha'(t), \alpha'(t)) = (u')^2 x_u \cdot x_u + 2u'v'x_u \cdot x_v + (v')^2 x_v \cdot x_v = E(u')^2 + 2Fu'v' + G(v')^2.$$

   Finally, let $\alpha \subset M$ and $\alpha* \subset M*$, where M and M* are isometric surfaces. Since they are isometric, $I_p = I_p*$, and thus by the above formula $length(\alpha) = length(\alpha*)$. □

3. *Section 2.1 Problem 3*

   These problems were solved in *Mathematica*.

4. *Section 2.1 Problem 5*

   Let all normal lines pass through the origin. Thus, any position x on the surface can be described by $x = a\vec{n}$, where $\vec{n}$ is the unit normal vector. By differentiating this equation, we obtain

   $$x_u = a_u\vec{n} + a\vec{n}_u \quad \text{and} \quad x_v = a_v\vec{n} + a\vec{n}_v.$$

   Dotting these results with $\vec{n}$, it is easy to see that $a_u, a_v = 0$ and thus a is a constant. This means that x can be described by unit normals facing different directions all multiplied by the same a. Since a is a constant, the equation shows that every x is a away from the origin, implying that it is a (part of at least) sphere. □

5. *Section 2.1 Problem 8*

For our parametrization to be conformal, the angels measured in the uv-plane must agree with the corresponding angles in T_pM for all P. More so, as mentioned in the text (and proven in another problem that was not assigned), this statement is equivalent to the conditions $E = G, F = 0$. Thus, it is sufficient to show these conditions are met. Using *Mathematica*, it is found that

$$E = G = \frac{4}{(1+u^2+v^2)^2} \quad \text{and} \quad F = 0.$$

as desired. Thus, the parametrization found in (1) is conformal. □

6. *Section 2.1 Problem 12*

 (a) Let $x(u, v) = \alpha(u) + v\beta(u)$. Then, $x_u = \alpha'(u) + v\beta'(u)$ and $x_v = \beta(u)$. Since M is a surface and thus every point has a neighborhood that is regularly parametrized, x_u, x_v form a basis spanning a plane, and thus $x_u \cdot x_v = 0$. Doing this actual computation results in $0 = \alpha'(u) \cdot \beta'(u)$, as desired.

 (b) Since α', β, β' are linearly dependent, it must be true that $a\alpha' + b\beta + c\beta' = 0$. From before, we know that $\alpha' \cdot \beta = 0$. By dotting the above equation with β, we find that $a\alpha' \cdot \beta + b|\beta| + c\beta \cdot \beta' = 0$, but $\alpha \cdot \alpha, \beta \cdot \beta' = 0$, so $b|\beta| = 0$. We know that $|\beta| = 1$ from the problem, so $b = 0$. Thus $a\alpha' + c\beta' = 0$. From the problem, $\alpha' \neq 0$, so we know that $a\alpha' = -c\beta'$, and thus $\beta' = -\frac{a}{c}\alpha' = \lambda\alpha'$ as desired.

 i. Let $\lambda(u) = 0$. Then, $\beta' = 0$ by the above function. So, $x(u, v) = \alpha + v\beta$, where β is constant. Furthermore, $\beta = 1$ since $|\beta| = 1$. Thus $x(u, v) = \alpha + v$, which is the book's definition for a cylinder.

 ii. Let λ be a nonzero constant. Thus, $\beta' = \lambda\alpha'$. Further, $\beta = c\alpha$, where c is some constant resulting from the integration potentially altering λ. Them, $x(u, v) = \alpha + vc\alpha = (1 + vc)\alpha$. Likewise, $x_u = (1 + vc)\alpha'$ and $x_v = c\alpha$. Crossing these, we find $x_u \times x_v = (1 + vc)\alpha' \times c\alpha$ which is nonzero so long as $v \neq -c$. Thus, the cross product fails (=0) only at one point, which is the vertex, and the surface is a cone.

 iii. Let $\lambda, \lambda' \neq 0$ anywhere on M. $\beta' = \lambda\alpha$ and $\beta = \eta\alpha$, where η has all the integration junk. Then, $x_u = \alpha' + v\eta'\alpha + v\eta\alpha'$ and $x_v = \eta\alpha$. Therefore, $x_u \times x_v = \alpha' \times \eta\alpha$, which is away from the directrix, showing that M is a tangent developable.

7. *Section 2.1 Problem 13*

8. *Section 2.1 Problem 16*

 (a) Using *Mathematica*, the surface area was computer to be $a\pi(2 + a\sinh(\frac{2}{a}))$.

 (b) For $R_0 > \sqrt{3}$, this is true based off the graph $f(t) = t\cosh\frac{1}{t}$.

9. *Section 2.1 Problem 17*

In the question, it says to find a plane that is tangent to the torus twice. The only region where this is possible is the hole inside the torus, where a plane can be tangent to the opposite points of the inside circle. By tilting the plane at a certain angle, where it still holds true that it is tangent at two points, the plane can cut through the torus in such a way to produce two circles on the newly created surface, which is the third family of circles.