Sensing weak anharmonicities through Vacuum Induced Coherences

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November 5, 2022

Abstract

A scheme has been proposed to sense weak anharmonic perturbations in a dissipatively coupled anti parity-time symmetric systems. High sensitivity to anharmonicities is obtained between the coherences between the two-modes constituting the system which are induced by a common vacuum. This vacuum induced coherence generates a pole on the real axis, after including the anharmonic term under certain conditions we see that the total response of the system is not divergent unlike the linear response but is instead highly sensitive to the perturbative parameters. The specific system considered here is that of a weakly anharmonic Yttrium Iron Garnet sphere interacting with a cavity indirectly through a waveguide. A small change in the anharmonicity leads to a substantial change in the induced spin current.

1 Introduction

First we try to understand the general schematics of the system. We begin with $\mathcal{PT}$ symmetries in the following section which are incredibly useful in describing open quantum systems which is more relevant from an experimental point of view since we cannot have an ideal closed quantum system and also make measurements on it. Next we understand how exceptional points in a system can be similar to diabolical points which explain the standard degeneracies in a Hermitian Hamiltonian except that exceptional points do not introduce degeneracies with different eigenstates rather just are points where the eigenvalues coalesce to give parallel eigenstates.

We then introduce the idea of vacuum induced coherences which are particularly relevant to open quantum systems where the probabilities are not always conserved. After understanding the basic underlying principles, we move on to the general two mode system where we first consider the linear response and then include the anharmonic part to study the behaviour in more detail. We then see how this can be applied to the case of Yttrium Iron Garnet sphere (YIG) interacting with the cavity where the changes in the anharmonic parameters and/or the drive powers provide substantial changes in the induced spin current.

2 Basics of $\mathcal{PT}$ symmetry

The purpose of this section is to introduce the idea of $\mathcal{PT}$ symmetry and to understand the importance of using $\mathcal{PT}$ symmetric systems to describe open quantum systems.

2.1 Open, Closed and $\mathcal{PT}$-symmetric systems

A system is said to be closed or isolated if it is not in contact with its environment i.e. the dynamics of the system is not influenced by its environment. Such systems are typically described by Hermitian Hamiltonians, which also takes into account the fact that probability and energy of the system is conserved, since the eigenvalues of a Hermitian matrix are always real, there is no imaginary part in them which would account for decay. *Closed systems* are idealized systems as any measurement that needs to be made requires contact with its environment.

All physical realistic systems are open systems which are subject to external influences by its environment because probability and/or energy flows from (to) the environment to (from) the system and this will always happen in a realistic scenario since we cannot completely prevent interactions between the system and the environment. Even attempting to measure any parameter of the system would mean that it is interacting with the environment in some way. Such a system cannot be in equilibrium at the time of interaction or measurement.

We can consider the observer and the system of interest to be part of the same system thus the total probability of this entire system is conserved but this is very difficult to formulate mathematically as we cannot describe the interactions and measurements of the experimenter in an accurate manner.

An alternative way to study the open sub-systems independently (that is without considering the environment and the system of interest to be one big system) would be to construct a non-isolated system that has no net flux of probability by adding an identical copy of the original sub-system except that it has the opposite probability flux of the original sub-system. This would ensure that the combined set of systems have no net probability flux i.e. there is no gain/loss for the composite system. The loss(gain) in one sub-system will be compensated by the gain(loss) in the other sub-system. It is not necessary that this compensation has to be in a way that couples the systems.

The composite loss-gain system exhibits a type of symmetry called $\mathcal{PT}$ symmetry. Here $\mathcal{P}$ represents the generic parity (space-reflection) operator and $\mathcal{T}$ represents the time-reversal operator effectively switching the decay effects in the system to amplification and vice versa. $\mathcal{P}$ interchanges the gain and loss components of a system and $\mathcal{T}$ changes a system with gain into a system with loss and vice versa which is basically a time reversal of one of the processes.

2.2 Types of $\mathcal{PT}$ symmetries

Although the composite system has no net probability flux, we cannot say that it is always in equilibrium since the probability of each sub-system changes with time. However, if we can physically couple the two sub-systems such that the probability gained in one sub-system flows into the sub-system with loss it is possible that the composite system is in a *dynamic equilibrium.* A system that would be in a dynamic equilibrium is said to be an *unbroken $\mathcal{PT}$-symmetric* system and the one not in a dynamic equilibrium is said to be a *broken $\mathcal{PT}$-symmetric* system.

From these definitions we can see that unbroken $\mathcal{PT}$-symmetric systems come closest to the definition of a *closed* sub-system since it is in equilibrium, but it is not a closed system because *it is* in contact with its environment and interacting with it. Similarly a broken $\mathcal{PT}$-symmetric system resembles an open system since it is not in equilibrium and also is interacting with its environment. But even this does not allow us to conclude that it can be generalized as an open system since the net probability flux vanishes and the system has $\mathcal{PT}$ symmetry, unlike most open systems. This clearly tells us that $\mathcal{PT}$-symmetric systems are a *special class of systems* that are intermediate between *open* and *closed systems.*

The coupling parameter generally decides the nature of the $\mathcal{PT}$ symmetry. The transition from a *broken $\mathcal{PT}$-symmetric* to an *unbroken $\mathcal{PT}$-symmetric* system occurs abruptly at a critical value of this coupling parameter. This transition is called the *$\mathcal{PT}$ phase transition*

2.3 Operators $\mathcal{P}, \mathcal{T}$

Here the definitions of $\mathcal{P}$ and $\mathcal{T}$ operators are provided. $\mathcal{P}$ is the parity operator, interchanges the 2 sub-systems, which is given by the linear matrix operator

$$\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Notice that this is in agreement with the fact that applying the parity operator twice must leave the system invariant which is seen in $\mathcal{P}^2 = 1$ and $\mathcal{P} = \mathcal{P}^{-1}$.

$\mathcal{T}$ is a time reversal operator, we cannot directly define it such that t can be replaced by *-t*. Instead we look at the time-dependent Schrodinger equation $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ and also note that the general time dependent factor associated with the wave functions go as $e^{itf(x)}$ (where x is some parameter). From this we can tell that the time reversal operator must provide the complex conjugate of the operator/function that it acts upon. Thus to reverse the sign of t one must reverse the sign of i.

Unlike the parity operator $\mathcal{P}, \mathcal{T}$ does not have a matrix representation as it is also an antilinear operator, however, it is a reflection operator since $\mathcal{T}^2 = 1$. The operators $\mathcal{P}$ and $\mathcal{T}$ are independent of each other in a physical sense so they must commute: $[\mathcal{P}, \mathcal{T}] = 0$.

3 Exceptional points

3.1 Physical significance of exceptional points

Usually when a perturbation is added, we observe the phenomenon of energy level repulsion that is associated with the external strength parameter. Consider for instance the atomic spectra plotted versus an external magnetic field, clearly we see that originally degenerate levels (if any) become non-degenerate and separate from each other or in other words, repel from each other's energy levels. When such plots are extended into the complex plane of the external parameter, presence of complex branch points which connect the two repelling energy levels become significant.

It has been shown that for a real strength parameter where the Hamiltonian is Hermitian, the branch points always occur at complex values of the parameter, which would mean that the Hamiltonian extended into the corresponding complex plane is no longer Hermitian. Due to this the properties of degenerate eigenstates in hermitian Hamiltonians where we get non-identical or rather non-parallel degenerate eigenstates are no longer valid. Such points are termed as *exceptional points* (EPs).

Mathematically speaking, EPs generally occur in eigenvalue problems that depend on a parameter. Generally, variation of the parameter by extending it into the complex plane leads to situations where the eigenvalues coincide. Such eigenvalues are not referred to as degenerate states since it is conceptually different from the usual degeneracies obtained in hermitian Hamiltonians which have two distinct eigenstates whereas in this case we only get *one* eigenstate instead of two.

Now in the case of regular Hermitian Hamiltonians where there is no gain or loss, the degeneracies are such that the eigenstates are mutually orthogonal to each other. Such points where the eigenvalues coincide are called diabolical points for Hermitian Hamiltonians. Refer to Appendix A for a detailed derivation that finds the exceptional points for a 2-level system in terms of the perturbative parameter.

3.2 Application to $\mathcal{PT}-$ Hamiltonians

Hamiltonians that are $\mathcal{PT}$-symmetric can have a real spectrum even if they might be non-Hermitian. If the eigenstates are also $\mathcal{PT}$-symmetric i.e. $\mathcal{PT}\,|\psi_E\rangle = const\,|\psi_E\rangle$, then the eigenvalues are real and are complex if it is not PT-symmetric. The transition point(s), the value of the parameters at which the $\mathcal{PT}$-symmetry is broken, are also the values where the *exceptional points* of a system appear. For Hermitian Hamiltonians, EPs occur at complex values, since the PT-symmetric Hamiltonians can in general be non-Hermitian, the EPs can also occur for real parameter values.

4 Vacuum Induced Coherence

Let us consider a 3-level system in a V configuration where there is no coherent field applied. Now we will see how vacuum field can induce coherences in the absence of an external field. It happens through the process of spontaneous emission. According to the master equation, we have

$$\dot{\rho_{aa}} = -2\gamma_{ca}\rho_{aa}, \quad \dot{\rho_{bb}} = -2\gamma_{cb}\rho_{bb}$$

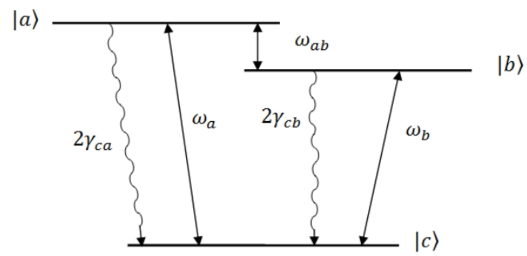

Figure 1: V system where we assume that the 2 excited energy levels are of nearly the same energy

As seen from these equations, both the spontaneous emissions are independent of each other. But this is only because we assumed that the energy levels are non-degenerate and and unevenly spaced. So when we had the relation that $\omega_{ij} + \omega_{kl} = 0$, we said that 4 indices have to be related in such a way that at maximum only 2 of them are unique. But in this V system we assume degenerate levels, thus we have more freedom for the above relation $\omega_{ij} + \omega_{kl} = 0$. So from here we obtain the new set of equations,

$$\dot{\rho_{aa}} = -2\gamma_{ca}\rho_{aa} - \sqrt{\gamma_{ca}\gamma_{cb}}cos\theta(\rho_{ab}e^{i\omega_{ab}t} + \rho_{ba}e^{-i\omega_{ab}t})$$
$$\dot{\rho_{bb}} = -2\gamma_{cb}\rho_{bb} - \sqrt{\gamma_{ca}\gamma_{cb}}cos\theta(\rho_{ab}e^{i\omega_{ab}t} + \rho_{ba}e^{-i\omega_{ab}t})$$
$$\dot{\rho_{ab}} = -(\gamma_{ca} + \gamma_{cb})\rho_{ab} - \sqrt{\gamma_{ca}\gamma_{cb}}cos\theta(\rho_{aa} + \rho_{bb})e^{-i\omega_{ab}t})$$

where $cos\theta = (\vec{d^*_{ac}}.\vec{d_{bc}})/(|d_{ac}||d_{bc}|)$. Thus the populations are coupled to coherences and vice versa which stem from the fact that the spontaneous emissions are not independent of each other. From this we also see that the coherence term ρ_{ab} in the steady state is non-zero. This is known as vacuum induced coherence (VIC).

As mentioned earlier, if ω_{ab} was not small or in other words if the energy levels were non-degenerate, the time dependent terms in the above set of differential equations would average out to 0 and no interferences occur, leading back to the standard independent decay equations. The interferences are most significant when the levels are degenerate and when the transition dipole moments are almost parallel to each other ($cos\theta \rightarrow 1$. The steady state solutions depend on the initial conditions, if $\rho_{aa}(0) = 1$, then in steady state,

$$\rho_{aa} = \rho_{bb} = 1/4; \quad \rho_{ab} = -1/4$$

The population is coherently trapped in the excited superposition state as $\rho_{ab} \neq 0$. This can also be viewed as the quenching of the linewidth of the spontaneous emission which thus leads to an infinite lifetime or in other words, trapping.

5 Two Mode system

5.1 Describing the system

Consider the general model for a two-mode anharmonic system. The Hamiltonian is given by,

$$H/\hbar = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a^\dagger b + b^\dagger a) + U(b^{\dagger 2}b^2) + i\Omega(b^\dagger e^{-i\omega_d t} - be^{i\omega_d t})$$

where ω_a and ω_b are the respective natural frequencies of the 2 uncoupled modes a and b (the diagonal elements in the Hamiltonian), g is the coherent Hermitian coupling between them (the off-diagonal element in the Hamiltonian). U is the perturbative parameter where the anharmonic perturbation (Kerr nonlinearity for cavities) is specific to mode b, which is driven by a laser of frequency ω_d. Ω represents the Rabi frequency for the 2 modes. Here dissipation into the environment has not been taken into account. It is known that dissipation can happen through two ways, one where each subsystem is independently coupled with its own local heat bath and interacts with it and the other is where there is a common heat reservoir that interacts with both the systems.

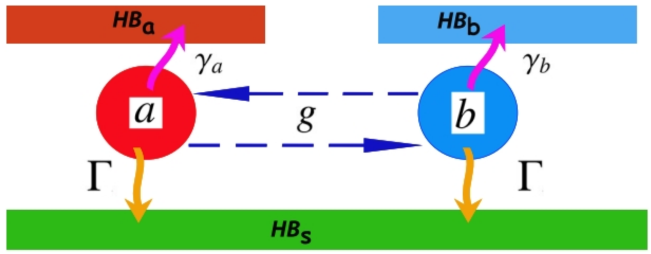

Figure 2: A general two-mode system with both of them dissipating with the local and the common heat baths with the rates $\gamma_{a(b)}$ and Γ respectively. The parameter g represents the coupling between them.

5.2 Dynamics of this system

The master equation for this system in terms of the density matrix ρ is given by,

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}[H,\rho] + \gamma_a \mathcal{L}(a)\rho + \gamma_b \mathcal{L}(b)\rho + 2\Gamma\mathcal{L}(c)\rho,$$

where Γ introduces the coherences, the Liouvillian superoperator $\mathcal{L}$ is defined as $\mathcal{L}(\sigma)\rho = 2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma$. Assuming both the modes, a and b, are coupled with the common heat bath in an identical manner, we will define $c = (a+b)/\sqrt{(2)}$. The master equation is valid under the condition that the phase difference of light propogation from one mode to another is a multiple of 2π.

The mean value equations for a and b are then given by,

$$\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = -i\mathscr{H}\begin{pmatrix}a\\ b\end{pmatrix} - 2iU(b^\dagger b)\mathscr{R}\begin{pmatrix}a\\ b\end{pmatrix} + \Omega e^{-i\omega_d t}\begin{pmatrix}0\\ 1\end{pmatrix}$$

,

where $\mathscr{H} = \begin{pmatrix}\omega_a - i(\gamma_a+\Gamma) & g - i\Gamma\\ g - i\Gamma & \omega_b - i(\gamma_b+\Gamma)\end{pmatrix}$ and $\mathscr{R} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. To deal with the nonlinear term, mean-field approximation has been applied. A transformation of the vector $(a,b) \rightarrow e^{i\omega_d t}(a,b)$ removes the time dependence in the last term and also transforms $\mathcal{H}$ to $\mathcal{H} - \omega_d I$ where I is the 2x2 identity matrix. So we are now in the laser's frame of reference.

To study the linear dynamics before considering the non-linear part let us take U = 0. Define $\delta = \omega_a - \omega_b$, mode detunings as $\Delta_i = \omega_i - \omega_d$ where $i = a, b$, in this case, the eigenvalues of $\mathcal{H}$ will be $\lambda_\pm = (\omega_a+\omega_b)/2 - i(\gamma_0+\Gamma) \pm \sqrt{(\delta/2 - i\gamma_{ab})^2 + (g - i\Gamma)^2}$ where $\gamma_{ab} = (\gamma_a - \gamma_b)/2$. As long as the Im($\lambda_\pm$) ¡ 0, the system will be stable as the amplitude will be decaying and not growing. The steady state solutions for the mean values of the amplitudes, under the condition that it's a stable system, Im($\lambda_\pm$) ¡ 0 which ensures the amplitude does not get amplified, will be

$$\frac{\langle a\rangle}{\Gamma + ig} = \frac{\langle b\rangle}{i\Delta_a - \gamma_a - \Gamma} = \frac{\Omega}{(\omega_d - \lambda_+)(\omega_d - \lambda_-)}$$

From this we can see that the linear response diverges if any one of the eigenvalues approaches the real axis.
The generalized 2x2 matrix $\mathscr{H}$ includes both the types of systems, coupled ($\Gamma = 0$) and dissipative ($g = 0$). But the two individual cases are the ones which exhibit some form of symmetries, coherently coupled ones exhibit PT-symmetry ($\mathcal{PT}^\dagger\mathcal{H}\mathcal{PT} = \mathcal{H}$) and dissipatively coupled systems exhibit anti-PT symmetry ($\mathcal{PT}^\dagger\mathcal{H}\mathcal{PT} = -\mathcal{H}$). So for the current system, choosing $\gamma_{ab} = g = 0$, $\Delta_a = -\Delta_b = \delta/2$, and $\Gamma \neq 0$. Both the modes are dissipative even though they are oppositely detuned. Switching to the rotating frame of the laser as the reference frame, the reformulated Hamiltonian is

$$\mathscr{H}^{(d)}_{aPT} = \begin{pmatrix}\delta/2 - i(\gamma_0+\Gamma) & -i\Gamma\\ -i\Gamma & -\delta/2 - i(\gamma_0+\Gamma)\end{pmatrix}$$

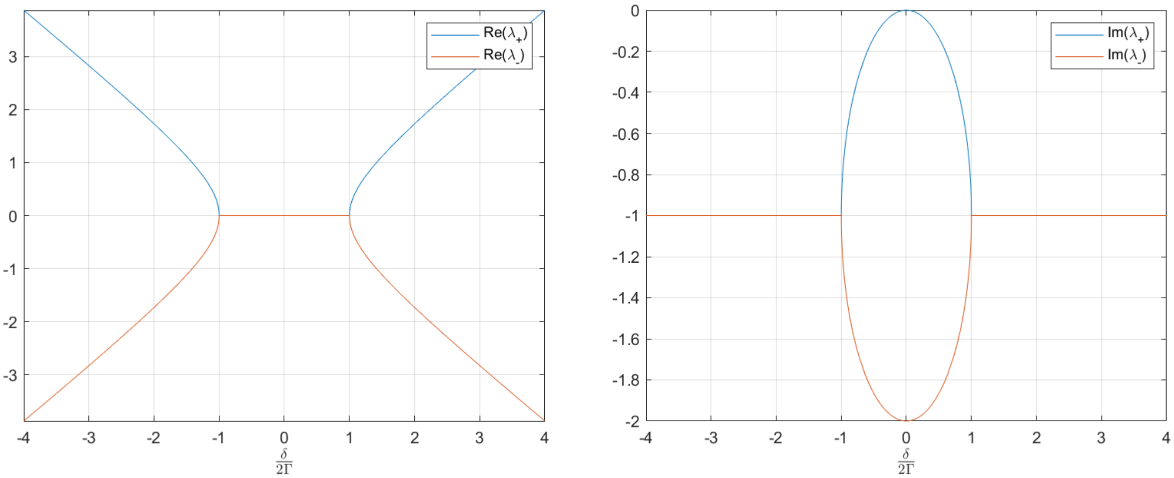

Figure 3: a) Eigenfrequencies for the given system. The EPs occur when $\delta/2\Gamma = 1$. b)Linewidths of the corresponding eigenfrequencies. The vaccum induced coherence linewidth supression occurs at $\delta = 0$

The shifted eigenvalues obtained are

$$\lambda_{\pm} = -i(\gamma_0 + \Gamma) \pm \sqrt{\delta^2/4 - \Gamma^2}$$

The eigenvalues of this matrix are plotted in the above figures. The case where the eigenvalues are imaginary corresponds to the case of broken anti-PT symmetry. As long as the stability criterion as mentioned earlier, $\mathrm{Im}(\lambda_{\pm})$ ¡ 0 is fulfilled, the responses are inversely related to $det(\mathscr{H}^{(d)}_{aPT}) = (\omega_d - \lambda_+)(\omega_d - \lambda_-) = -(\delta^2/4 + \gamma_0(2\Gamma + \gamma_0))$.

The broken anti-PT phase brings in real singularities when $\gamma_0 \to 0$ and $\delta = 0$, as seen above where the imaginary part of the eigenvalue (λ_+) tends to 0, let us name this point X. $\gamma_0 = 0$ would imply that none of the modes suffer spontaneous losses to its independent heat reservoirs or they although it does interact with the common reservoir since Γ need not be 0 too. Note that since we already take $\gamma_{ab} = 0$, the system is already not symmetric wrt PT operator and thus $\gamma_a = \gamma_b = 0$. Thus this point X distinguishes an especially long lived response of that mode which would lead to a buildup in in the steady-state amplitude of that mode.

It can be seen that the point X, the VIC (vacuum induced coherence) induced linewidth supression in the anti-PT symmetric case is functionally analogous to an exceptional point in PT-symmetric systems. The PT-symmetric configuration for coherently coupled systems can be obtained when we consider the dissipation to the common reservoir, the detunings, and the overall loss of the combined system to be negligible i.e. $\Delta = \Gamma = \gamma_0 = 0$. $\gamma_0 = 0$ implies $\gamma_a = -\gamma_b = \gamma_{ab}$ which is characteristic of PT-symmetric systems, that is a loss in one sub-system is compensated by gain in the other subsystem. It also happens that $\mod \gamma_{ab} = g$, which defines the EP introduces real singularities which again reduce the linewidth to 0.

Now we need to consider the non-linear part in the Hamiltonian and then re-analyze the case where the linewidth tends to 0 (or lifetime tends to ∞). So solving the original equation which had the non-linear term and doing it in the rotating frame of reference of the laser with $g = 0$ and $\gamma_a = \gamma_b = \gamma_0$, the steady state relations of the wavefunction amplitudes will be given by,

$$\dot{a} = -(i\delta/2 + \gamma_0 + \Gamma)a - \Gamma b$$
$$\dot{b} = -\Gamma - (-i\delta/2 + \gamma_0 + \Gamma)b - 2iU|b|^2 b + \Omega$$

In the steady state, $\dot{a} = \dot{b} = 0$

$$-(i\delta/2 + \gamma_0 + \Gamma)a - \Gamma b = 0$$

$$-\Gamma - (-i\delta/2 + \gamma_0 + \Gamma)b - 2iU|b|^2 b + \Omega = 0$$

Now we define a new parameter, $\gamma = \gamma_0 + \Gamma$ and we solve for b. The intensity $x = |b|^2$ will then be given by,

$$\frac{\alpha^2}{\gamma^2 + (\delta/2)^2} x - \frac{2U\alpha\delta}{\gamma^2 + (\delta/2)^2} x^2 + 4U^2 x^3 = I$$

where $\alpha = \Gamma - \gamma^2 - (\delta/2)^2$ and $I = \Omega^2$. We operate at low enough drive powers so that the presence of bistable responses are not significant. Near the point X, where $\delta \to 0$ and $\gamma \to 0$, α becomes small and higher orders of α can be ignored for a given Rabi frequency Ω. From this approximation we directly get a relation between the intensity x and the parameter U which is,

$$x = (I/4U^2)^{1/3}$$

For two mode systems where the exceptional points are singularities, the splitting of the coalescent eigenenergies around the EP varies as $\epsilon^{1/N}$ where ϵ is the perturbative parameter and N is the number of values that coalesce at the given EP, in this case $N = 2$. However in the setup considered in the paper, the sensitivity of the intensity of mode b, x, wrt U is given by $|dx/dU| \propto U^{-5/3}$ (refer an anaolgy in Appendix B).

6 Magnon Photon systems

6.1 What are magnons?

Magnons can be thought of as a unit of energy that corresponds to a unit change in magnetic field strength in a magnetic substance, more rigorously reducing the total spin along the magnetization direction by $\hbar$ and the magnetization by $\gamma\hbar$ where γ is the gyromagnetic ratio. For example, all the individual atomic spins in an iron sample tend to align themselves in the same direction to reinforce each other's field, now suppose one atom is flipped then the total magnetic energy of the system is decreased. For this to have happened some external energy needs to be supplied. This energy that decreases the magnetic strength of a group of atoms constitutes a magnon.

Instead of thinking about it as reversing the spin of one atom in a group of atoms, we can think of it as decreasing the spin strength of all atoms in the group each by a lesser amount to produce the same effect in the bulk. This partial reversal spreads through the entire solid as a wave of discrete energy transferal. This wave is called a spin wave as magnetic fields in atoms are generated from the spins of unpaired electrons. Thus, a magnon is a quantized spin wave. So the decrease in magnetic strength that can be associated with many things for instance, increase in temperature corresponds to the presence of a larger number of magnons.

Thus this also gives us a picture that magnons can be thought of as a quasi-particle. Since they are quantized, they carry a fixed amount of energy and lattice momentum and are spin-1, thus also bosonic in nature.

6.2 In the case of a YIG sphere

This idea of variation of the intensity wrt the perturbative parameter U can be applied to the specific case of Kerr nonlinearity in a Yttrium Iron Garnet sphere (YIG) sample.

The original paper considers the integrated apparatus comprising a microwave cavity and a YIG sphere both interfacing with a one-dimensional waveguide. As seen in the above figure, the cavity and the YIG sphere do not have any spatial overlap thus there cannot be any direct coupling between them although since they can interact with the waveguide which happens to be a form of dissipation, they can thus said to be having an indirect coupling (a dissipative coupling) between them. It is this dissipative coupling which instills the VIC into the system.

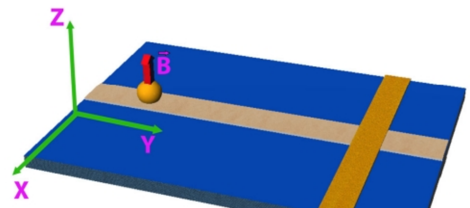

Figure 4: This is the given experimental setup, where the yellow sphere represents the YIG sphere and the microwave cavity runs transverse to the waveguide which interacts with YIG sphere through the transmission line. A static magnetic field has been applied in the z-direction

7 Cavity-magnonic setup

7.1 Description

The Hamiltonian in this case is the same as the one that we earlier used to describe a general 2-mode system except that b will be replaced by the magnonic operator m. With the anti-PT symmetric choices $\Delta_a = -\Delta_m = \delta/2, \gamma_a = \gamma_b = \gamma_0$ and defining $\gamma_0 + \Gamma = \gamma$, we once again solve for the steady state equations like earlier. Using similar definition of variables like $x = |m|^2$ which now represents the spin-current response.

Now we again consider the spontaneous losses to the independent heat baths to be 0, $\gamma_0 = 0$ and set $\Gamma = \gamma = 2\pi$ x 10MHz. $\alpha = -\delta^2/4$, and in the given limit where the detuning is very small (close to resonance), we can approximate the spin-current response using the same expression $x = (I/4U^2)^{1/3}$ in the region $\delta/2\pi < 1$MHz which is the range for us to clearly observe the variation wrt U. The drive power D_p is related to the Rabi frequency as $\Omega = \gamma_e(5\pi\rho d D_p/3c)^{1/2}$, γ_e is the gyromagnetic ratio, ρ is the denisty of Fe^{3+} ion density, $d = 1$mm is the diameter of the YIG sample and c is the speed of light.

7.2 Sensing the anharmonicity

Even for low powers, $D_p = 1\mu W$, there is a huge increase in the induced spin-current around the point $\delta = 0$. We know that at the point $\delta = 0$, the imaginary part of the eigenvalue will be 0 if we only consider the linear response which leads to a divergent response. This real singularity can be averted if we include the nonlinear term. This would still preserve the high sensitivity of the variation with the parameter U. From the plots below, we see that for various values of Γ the quenching of the spin current at higher values of δ are different.

When $\Gamma < \gamma$, a considerable decline in the spin-induced current is observed which can be compensated by increasing the drive-power from $1\mu W$ to $1mW$. Doing that brings back the sensitivities to similar levels as was the case earlier. The importance of having 0 coherent coupling between the 2 modes can be seen here as well, if $g \neq 0$ then the real singularity would have been a complex one, thus the linewidth of the state would've been finite, which would mean that there will be no divergence in the linear response and an even lesser sharp response of the resonance when we include the nonlinear perturbation and hence decreased sensitivity. A tenfold would increase in U would barely increase the peak intensity of the spin-induced current unlike the purely dissipative mode where it would increase the peak intensity by a factor of almost 5 even though all other parameters are kept the same. This is thus clearly a method which is extremely useful in the case of zero coherently coupled systems since the sensitivity is high, the paper also cites other references where such systems have been engineered experimentally in recent years.

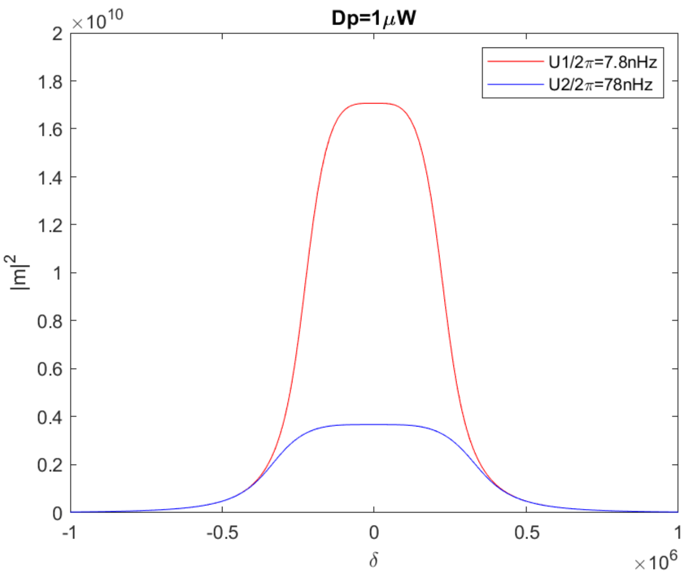

Figure 5: Spin currents plotted for different nonlinearities for a given drive power. Here we see that 1 10 fold increase in the anharmonicity corresponds to a significant increase in the induced current which aids in the sensitivity

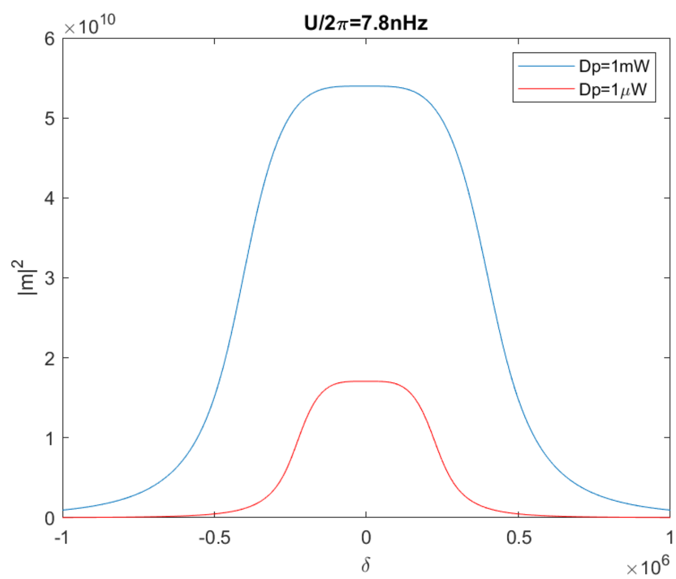

Figure 6: Same nonlinearity but different drive powers. It is clearly seen that increaesing the drive power enhances the sensing for a weak anharmonicity.

8 Summary

Hence we have analyzed a way to sense anharmonicities, more specifically Kerr non-linearity of the mode. We exploit the fact that one of the eigenvalues has a very long lifetime evident from its negligible linewidth. This is possible because of the effective coupling between the two modes in the system, which in this case are the cavity and the magnons, and an indirect interaction between them in the form of a common reservoir. Thus even though there is no direct spatial coupling between these 2 modes, the common reservoir provides the indirect coupling. This along with the fact that the spontaneous emissions from the modes with the immediate independent reservoirs is negligible ensures that VIC strongly dominates and thus increase sensitivity as a result of it. Similar ideas can be generalized to any two-mode system as well.

9 Appendix

9.1 APPENDIX A: EXCEPTIONAL POINTS IN A 2 LEVEL SYSTEM

Consider the Hamiltonian, the eigenvalues will be

$$H(\lambda) = H_0 + \lambda V$$
$$H(\lambda) = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} + \lambda \begin{pmatrix} \epsilon_1 & \delta_1 \\ \delta_2 & \epsilon_2 \end{pmatrix}$$
$$det(H - \mu I) = 0$$

where μ is an eigenvalue,

$$\begin{vmatrix} \omega_1 + \lambda\epsilon_1 - \mu & \lambda\epsilon_1 \\ \lambda\epsilon_2 & \omega_2 + \lambda\epsilon_2 - \mu \end{vmatrix} = 0$$
$$(\omega_1 + \lambda\epsilon_1 - \mu)(\omega_2 + \lambda\epsilon_2 - \mu) - \lambda^2\delta_1\delta_2 = 0$$
$$\mu^2 - \mu(\omega_1 + \omega_2 + \lambda(\epsilon_1 + \epsilon_2)) + (\omega_1 + \lambda\epsilon_1)(\omega_2 + \lambda\epsilon_2) - \lambda^2\delta_1\delta_2 = 0$$

The discriminant of this equation must be zero for it to have identical roots. For this we may adjust the parameter λ to get the eigenvalues to coincide to get the exceptional point as stated earlier.

$$(\omega_1 + \omega_2 + \lambda(\epsilon_1 + \epsilon_2))^2 = 4((\omega_1 + \lambda\epsilon_1)(\omega_2 + \lambda\epsilon_2) - \lambda^2\delta_1\delta_2)$$
$$\lambda^2((\epsilon_1 + \epsilon_2)^2 + 4\delta_1\delta_2 - 4\epsilon_1\epsilon_2) + \lambda(2(\omega_1 + \omega_2)(\epsilon_1 + \epsilon_2) - 4(\omega_1\epsilon_2 + \omega_2\epsilon_1)) + (\omega_1 + \omega_2)^2 - 4\omega_1\omega_2 = 0$$
$$\lambda^2((\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2) + \lambda(2(\omega_1 - \omega_2)(\epsilon_1 - \epsilon_2)) + (\omega_1 - \omega_2)^2 = 0$$

Solving the quadratic equation in λ we get,

$$\lambda = \frac{-2(\omega_1 - \omega_2)(\epsilon_1 - \epsilon_2) \pm \sqrt{4(\omega_1 - \omega_2)^2(\epsilon_1 - \epsilon_2)^2 - 4(\omega_1 - \omega_2)^2((\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2)}}{2((\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2)}$$
$$\lambda = \frac{-(\omega_1 - \omega_2)(\epsilon_1 - \epsilon_2) \pm \sqrt{(\omega_1 - \omega_2)^2(-4\delta_1\delta_2)}}{(\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2}$$
$$\lambda = (\omega_1 - \omega_2)\frac{-(\epsilon_1 - \epsilon_2) \pm 2i\sqrt{\delta_1\delta_2}}{(\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2}$$

From this, we see that the eigenstates coincide at specific values of λ given by,

$$\lambda_1 = \frac{-i(\omega_1 - \omega_2)}{i(\epsilon_1 - \epsilon_2) + 2\sqrt{\delta_1\delta_2}} \quad \lambda_2 = \frac{-i(\omega_1 - \omega_2)}{i(\epsilon_1 - \epsilon_2) - 2\sqrt{\delta_1\delta_2}}$$

The value of the energies at these values of λ are given by,

$$E_{1,2}(\lambda) = \frac{1}{2}(\omega_1 + \omega_2 + \lambda(\epsilon_1 + \epsilon_2) \pm \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2}\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)})$$
$$E(\lambda_{1,2}) = \frac{\epsilon_1\omega_2 - \epsilon_2\omega_1 \pm i\sqrt{\delta_1\delta_2}(\omega_1 + \omega_2)}{\epsilon_1 - \epsilon_2 \mp 2i\sqrt{\delta_1\delta_2}}$$

9.2 APPENDIX B: SENSITIVITY OF PARAMETER ESTIMATION NEAR THE EXCEPTIONAL POINT OF A NON-HERMITIAN SYSTEM

Here we see how the sensitivity to eigenmode splitting around the exceptional point occurs wrt the perturbative parameter. Consider the 2-level system Hamiltonian,

$$H = (\omega_a - i\gamma_a/2)a^{dagger}a + (\omega_b - i\gamma_b/2)b^{dagger}b + g(a^{dagger}b + b^{dagger}a)$$

where ω_a, ω_b are the resonant frequencies of the two levels, units are such that $\hbar = 1$, a and b are the 2 modes of the system and g is the coupling parameter, γ_a and γ_b are the decay rates for the two modes.

Writing the Hamiltonian in matrix form and finding the eigenvalues, define the following quantities

$$\omega_0 = (\omega_a + \omega_b)/2, \delta = \omega_a - \omega_b, \gamma_0 = (\gamma_a + \gamma_b)/2, \gamma_{ab} = (\gamma_a - \gamma_b)/2$$

Consider the detuning δ to be the perturbative parameter which can be varied slightly by introducing a small disturbance in one of the modes which would alter the resonant frequency which would in turn change the detuning which can be sensed. The energy eigenvalues will be obtained as

$$\lambda_{\pm} = \omega_0 - i\gamma_0/2 \pm \sqrt{g^2 + (\delta/2 - i\gamma_{ab})^2}.$$

The EP point occurs at $g = \gamma_{ab}/2$ where the eigenvalues are degenerate and coalesce. Around the EP, the energy difference is approximately given by

$$2\sqrt{|\gamma_{ab}|(g - |\gamma_{ab}|/2) - i\gamma_{ab}\delta/2}$$

The susceptibility of the energy splitting diverges as

$$\frac{d\chi}{d\delta} = \frac{-i\gamma_{ab}/2}{\sqrt{|\gamma_{ab}|(g - |\gamma_{ab}|/2 - i\gamma_{ab}\delta/2}}$$

Thus we see that the dependence of the splitting goes as $\delta^{1/2}$, for the case where N modes coalesce we can say that it would vary as $\delta^{1/N}$ where δ is the perturbative parameter.

10 References

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