Recharacterization of Photons in Plasma Medium to Mimic an Ideal Bose Gas

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Abstract

The properties of a Bose Einstein condensate are discussed and we see the role of chemical potential for the same. The thermodynamical properties of a plasma-photon system have also been studied using a statistical physics approach. Photons in a plasma medium acquire an effective mass due to the presence of a finite chemical potential, with all these modified properties of a photon in a different medium, it can also be proved that the Planck's law of black body radiation is also modified where we observe a point of criticality that was previously absent for a photon without an effective mass. We also see how this very critical point is relevant in the context of Bose Einstein Condensates (BECs) which has also been experimentally realized in recent years.

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Acknowledgement

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Motivation

We have always known of photons as massless particles. In this paper, we see photons in a new light. That is, here we generate an effective photon mass and see that they exhibit the same properties as that of an ideal Bose gas. Since we normally don't associate photons with matter, the fact that they show some similarity to particles with mass is surprising and fascinating. This paper follows whatever we have been studying in the PH-301 course and builds upon the fundamentals that are under the scope of the syllabus.

1 Introduction

We have already studied the statistics of classical ideal gases and quantum ideal gases. We begin this report with a brief review of quantum ideal gases and observe a peculiar property of quantum ideal gases when we consider the specific case of Bose Einstein statistics. Bose Einstein condensation is a quantum phenomenon in Bose gases in which a large number of Bosons simultaneously occupy the ground state.

Bose Einstein Condensate (BEC) was first theoretically predicted in 1925 by Bose and discoevered experimentally towards the end of the 20th century. It seems to be a very natural thing to happen, multiple bosons all in the ground state simultaneously which is possible because all of them are bosons which does not need to follow Pauli's exclusion principle like fermions. What really makes this a quantum phenomenon, or rather something specific to BE statistics, is that the temperatures need not be too low for a significant occupancy in the ground state. We discuss the details of the same by including mathematical arguments in the section that covers BEC.

Einstein's work which was based on Bose's initial work on photon gases showed that only massive particles (particles with mass) that follow BE statistics, bosons, can undergo this condensation procedure. This meant that photons, which are massless (rest mass is 0) cannot form a BEC. However, by changing the experimental conditions, like having a medium that is made up of plasma, we see that photons can have an effective mass which allows them to form a BEC.

We investigate this generation of the effective mass of photons by considering the Bosonic grand canonical potential which incorporates the effective mass of the photon in a plasma medium. In this paper, we have discussed the corresponding massive quasiparticle, bulk plasmon-polaritons. The effective mass of photons also modifies the Planck's black body radiation which is also discussed in this paper.

2 Quantum Ideal Gases

When we consider an ideal gas made of classical particles the most important assumption is that the particles do not interact among themselves or in other words, the interaction potential is 0. The same treatment can be applied to the case of quantum particles as well which would result in a quantum ideal gas.

Consider an ideal gas enclosed in a volume V at an equilibrium temperature T. Let us denote each energy state by r, the energy of a particle in state r by ϵ_r, the number of particles in state r by n_r. As mentioned earlier, the interaction energy between particles in an ideal gas is negligible so the total energy of the system can be written as a summation of all the individual energies of each particle in its particular energy state which is thus given by,

$$E_R = n_1\epsilon_1 + n_2\epsilon_2 + ...n_n\epsilon_n = \sum_r n_r\epsilon_r$$

the summations are over all possible states r of all particles. The number of particles are also constant thus we also obtain the following constraint

$$\sum_r n_r = N$$

We now calculate the partition function

$$Z = \sum_R exp(-\beta E_r) = \sum_R exp(-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...))$$

The summation is done over all the possible states a particle in the ideal gas can take under the same constraint that the number of particles have to be constant.

When we are talking about identical quantum particles, the symmetry of the wave function under exchange of particles comes into the picture. There are two possible cases under this:

1. Particles with half integral spin (Fermi-Dirac Statistics)
2. Particles with integral spin (Bose-Einstein Statistics)

2.1 Fermi-Dirac Statistics

This is applicable when the total spin angular momentum is half integral (1/2, 3/2, 5/2, ..). In this case, quantum mechanics dictates that the wave function must be anti-symmetric, i.e., under exchange of any two particles, the wave function becomes the negative of itself.

$$\Psi(...Q_i....Q_j....) = -\Psi(...Q_j....Q_i....)$$

Due to this fundamental rule, there cannot be more than 1 particle in a single state (as the sum of all possible wave functions gives 0).
Therefore n_s can only take the values 0 or 1, and the partition function is given by,

$$\ln Z = \alpha N + \sum_r \ln\left(1 + e^{-\alpha - \beta\epsilon_r}\right)$$

2.2 Bose-Einstein Statistics

This is the case where the total spin angular momentum is integral (1, 2, 3, ...). In such a case, under exchange of two particles, the wave function remains the same, i.e., symmetric wave function.

$$\Psi(...Q_i....Q_j....) = \Psi(...Q_j....Q_i....)$$

As interchange of particles does not lead to a new state, there is no limit on the number of particles in a single state.
Therefore n_s can only take values between 0 to N, then the partition function will be given by,

$$\ln Z = \alpha N - \sum_r \ln\left(1 - e^{-\alpha - \beta\epsilon_r}\right)$$

2.3 Photon Statistics

A simpler case in Bose-Einstein Statistics comes up when we are dealing with photons. Photons stored in a container of volume V are readily absorbed and emitted by it's walls. As the walls can absorb 1 photon of energy E, and emit 2 photons of energy E/2, the number of particles can increase while energy of the complete system is conserved. Due to this, there is no bound on the number of particles removing the constraint on N, and the partition function can be summed up from 0 to ∞.

The partition function is now given by,

$$Z = \sum_{n_1, n_2, ...} e^{-\beta n_1 \epsilon_1} e^{-\beta n_2 \epsilon_2} ...$$

$$Z = \left(\sum_{n_1} e^{-\beta n_1 \epsilon_1}\right)\left(\sum_{n_2} e^{-\beta n_2 \epsilon_2}\right)\left(\sum_{n_3} e^{-\beta n_3 \epsilon_3}\right)....$$

Using summation series for Geometric Progression, we get the partition function as

$$Z = \left(\frac{1}{1 - e^{-\beta\epsilon_1}}\right)\left(\frac{1}{1 - e^{-\beta\epsilon_2}}\right)\left(\frac{1}{1 - e^{-\beta\epsilon_3}}\right).....$$

3 Plasma Medium

Plasma is another state of matter. Plasma is a quasi-neutral electrically conducting medium that consists of unbound positive and negative charged particles. These charged particles/ions are what sets them apart from the other states of matter. Although they don't have fixed volume and shape, they differ from ionized gas in various ways such as interaction of the particle, electrical conductivity etc. It is the most abundant form of matter in the universe that's makes up the sun and other stars. Plasma can be generated in artificial conditions by heating a gas to high temperatures (6000K-10000K) and subjecting it to a strong electromagnetic field. So far, we don't have any material that can withstand such high temperatures without melting. Hence, they must intrinsically heated (increase KE of the electrons directly).

3.1 Why Plasma?

The mass of the positive ions is generally much larger than those of negative ions, since they are generally electrons. We consider the positive ions to be stationary therefore, any small displacement in group of electrons with respect to the position of positive ions leads to the oscillation of electrons as a whole about a fixed equilibrium state. This is analogous to that of a ball bobbing up and down in water, where the restoring force is gravity. Instead of gravity, here electromagnetic interactions provide the restoring force.

Photons are packets of energy (light particles) that are generally considered massless. Due to this "masslessness", photons can be generated with infinitesimal energy. As the photon number is not conserved, we say that N becomes unbounded and gives rise to photon statistics. In this paper, we inject photons into a plasma medium to generate an effective photon mass. This effective mass arises due to the change in dispersion relation of photon due to collective oscillation of charged particles in the plasma medium. Because of the newly introduced mass, a chemical potential is induced in photos, due to which, photons now cannot be generated with infinitesimally small energy. Hence, the number of photons is conserved and are expected to follow Bose-Einstein Statistics.

3.2 Model for the Photon-Plasma System

In this paper, we consider a grand canonical ensemble framework of a plasmon-polariton gas in order to describe the photon gas in homogeneous isotropic plasma medium. We describe the system by a Hamiltonian considering a system of N_ch charges interacting with a quantized monochromatic EM field,

$$H = \frac{1}{2m}\sum_{i=1}^{N_ch}\left(p_i - \frac{e}{c}A\right)^2 + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where $p_i's$ m are the momenta and the mass of the charges which have unit charge e and the summation is over all N_ch charged particles. The linearly polarized EM mode is represented by the vector potential A. The terms $a^\dagger$ and a are the creation and annihilation operators of the mode with angular frequency ω. In case of a single particle, this Hamiltonian can be diagonalized by using displacement (D) and a Bogoliubov (C) transformation. This solution can be generalised for a system of more particles($N_{ch}e > 1$). In this scenario, the system can be considered as a plasma where the Coloumb interactions are dampened by Debye screening - a free electron gas. The homogeneity of the plasma is achieved by taking the limit $p_i \to 0$ for all i. This gives a uniformly distributed N_{ch}e net charge in the box (volume V) according to the Heisenberg's uncertainty principle. Applying the above described operations on the above Hamiltonian results in an effective Hamiltonian of a free harmonic quantum oscillator describing a plasmon-polariton system.

$$\mathscr{H} = \hbar\Omega(bb^\dagger + \frac{1}{2}),$$

where $b^\dagger$ and b are the annihilation and creation operators of the plasmon-polariton particles with angular frequency Ω, which is defined as $\Omega = \sqrt{\omega^2 + \omega_p^2}$, where $\omega_p = \sqrt{4\pi e^2 n_p/m}$ defines the plasma frequency of the medium with plasma density $n_p = N_{ch}/V$. Likewise, the "plasma energy" can be obtained as $\epsilon_p = \hbar\omega_p$. By comparing this with the dispersion relation of a relativistic particle, this is considered as the "rest energy" of the photon. Thus, from this rest energy we create an "effective mass" $m = \epsilon_p/\hbar\omega$ on the photon as discussed above.

The grand potential is now defined as

$$\Phi = \frac{1}{\beta}\sum_k \ln(1 - e^{\beta(\epsilon_k - \mu)})$$

and hence the system now follows the properties of a Bose Gas and Bose-Einstein statistics. Here $\beta = 1/k_B T$, r indicates a one particle state and μ is the chemical potential that arises due to conservation of the number of quasi-particles. This chemical potential is what makes the plasmon-polariton system different from that of a free photon gas where photons are massless and can be created with an infinitesimal amount of energy, i.e., $\mu = 0$.

4 Thermodynamic Properties

Throughout this discussion, we assume that the plasma medium and photons are in thermal equilibrium and that the plasma medium is homogeneous and isotropic. We get the number states per unit volume as

$$dN(p) = \frac{8\pi\nu^2}{c^3}d\nu$$

from Section 6.2, Using $\nu = \sqrt{\epsilon^2 - \epsilon_p^2}/h$ and $\nu d\nu = \epsilon d\epsilon/h^2$, we get the energy density of states as

$$\rho(\epsilon)d\epsilon = \frac{8\pi V}{c^3 h^3}\epsilon\sqrt{\epsilon^2 - \epsilon_p^2}d\epsilon$$

Taking the thermodynamical limit, the summation over every state is replaced by the integral over phase space ($\sum_k \equiv \int\ d^3x\, d^3p$). Thus, the grand potential becomes

$$\Phi = \int_{\epsilon_p}^{\infty} \ln(1 - e^{\beta(\epsilon - \mu)})\, d^3x\, d^3p = \int_{\epsilon_p}^{\infty} \ln(1 - e^{\beta(\epsilon - \mu)})\rho(\epsilon)d\epsilon$$

$$\Phi = \frac{1}{\beta}\int_{\epsilon_p}^{\infty} \frac{8\pi V}{c^3 h^3}\ln(1 - e^{\beta(\epsilon - \mu)})\epsilon\sqrt{\epsilon^2 - \epsilon_p^2}d\epsilon$$

We now introduce the following dimensionless quantities to simplify the calculation: $\epsilon' = \epsilon/\epsilon_p$, $x = \beta\epsilon_p$, $\mu_p = \mu/\epsilon_p$, and $z = e^{-\mu_p}$ The integral becomes,

$$\Phi = \frac{8\pi V x^3}{\beta^4 c^3 h^3}\int_1^{\infty} \epsilon'\sqrt{\epsilon'^2 - 1}\ln(1 - e^{-x\epsilon'}z)$$

From this grand potential(see Appendix A), we can derive other thermodynamical properties of the system by taking partial derivative wrt to some parameter.

$$\overline{N} = (\partial_\mu \Phi)_{V,T},\ \ \overline{E} = (\partial_\beta \beta\Phi)_{\beta\mu},\ \ S = -(\partial_T \Phi)_{\mu,V},\ \ P = \overline{E}/3V$$

The average photon number in plasma is,

$$\overline{N} = \frac{V}{\lambda_T^3}\int_1^{\infty} \frac{x^3\epsilon'\sqrt{\epsilon'^2 - 1}}{e^{x(\epsilon' - 1)}z^{-1} - 1}d\epsilon'$$

In the thermodynamic limit ($N \to \infty, V \to \infty$, while N/V is kept constant) we see that it is more natural to consider the particle density instead of the total number of particles, we also see that it can be expressed in pretty much the same expresison as above.

$$\overline{n} = \frac{1}{\lambda_T^3}\int_1^{\infty} \frac{x^3\epsilon'\sqrt{\epsilon'^2 - 1}}{e^{x(\epsilon' - 1)}z^{-1} - 1}d\epsilon'$$

The above integral is a monotonically increasing function of z and has a maxima when $z = 1$, when the system is found to be critical i.e. $\overline{n_c} = \overline{n}\ z = 1$. This equation is true only for *thermal states* as the density of states tells us that the population of the ground state is 0, and above the critical threshold, the particles accumulate in the ground state with energy ϵ_p by forming a BEC.

Similarly from the final form of the grand potential function, and the partial derivatives, we get the relations as

$$\overline{N} = \frac{V}{\lambda_T^3} \sum_{j=1}^{\infty} \frac{(e^x z)^j}{j} K_2(jx)$$

$$\overline{E} = \frac{V}{\lambda_T^3} \epsilon_p x \sum_{j=1}^{\infty} \frac{(e^x z)^j}{j^2} (jxK_1(jx) + 3K_2(jx))$$

$$S = \frac{V}{\lambda_T^3} k_B x^2 \sum_{j=1}^{\infty} \frac{(e^x z)^j}{j} (jxK_1(jx) + (4 - j\log(z) + x))K_2(jx)$$

$$P = \frac{1}{3\lambda_T^3} \epsilon_p x \sum_{j=1}^{\infty} \frac{(e^x z)^j}{j^2} (jxK_1(jx) + 3K_2(jx))$$

5 Bose Einstein Condensate

Bose Einstein Condensate (BEC) is a state of matter in which separate atoms or even subatomic particles, cooled to extremely low temperatures, coalesce into a *single quantum mechanical entity* i.e. it can be represented by a *single wave-function* even on large scales (close to macroscopic ones). This form of matter was first predicted by Albert Einstein in 1924 inspired by the calculations of Satyendra Nath Bose.

As the name suggests, this state of matter can be attained only in the case of Bosons. This is a very natural consequence since only multiple Bosonic particles can occupy the same set of quantum numbers as other Bosonic particles which thus allows the entire set of atoms/particles to be represented by *one* wave-function.

5.1 Two state system - Canonical ensemble

Consider a two-level system with N particles with the energy levels $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon$. Let us study this system in the canonical ensemble using Bose-Einstein statistics.

First considering the simple case where $N = 1$, the partition function will be written as,

$$Z = \sum_r e^{-\beta E_k} = 1 + e^{-\beta\epsilon}$$

The expected fraction of particles in the ground state is,

$$\frac{\langle N_{ground} \rangle}{N} = 1.P_{ground} + 0.P_{excited} = \frac{1}{Z_1} = \frac{1}{1 + e^{-\beta E}}$$

All these relations are general irrespective of the statistics because we only consider one particle. Now let us consider the general case of N particles. With Bose-Einstein statistics, the case where we have m particles in the ground state and $N - m$ in the excited state constitutes one state. m can vary from 0 to N, so we have $N + 1$ states in total.

$$Z = \sum_{k=0}^{N} e^{-k\beta\epsilon} = \frac{1 - e^{-(N+1)\beta\epsilon}}{1 - e^{-\beta\epsilon}}$$

Thus the expected number in the ground state is,

$$\begin{aligned}\langle N^{BE}_{ground}\rangle &= \frac{1}{Z_N}[N.1 + (N-1).e^{-\beta\epsilon} + ... + 0.e^{-N\beta\epsilon}] \\ &= \frac{1}{e^{\beta\epsilon - 1}} + \frac{N+1}{1-e^{-(N+1)\beta\epsilon}}\end{aligned}$$

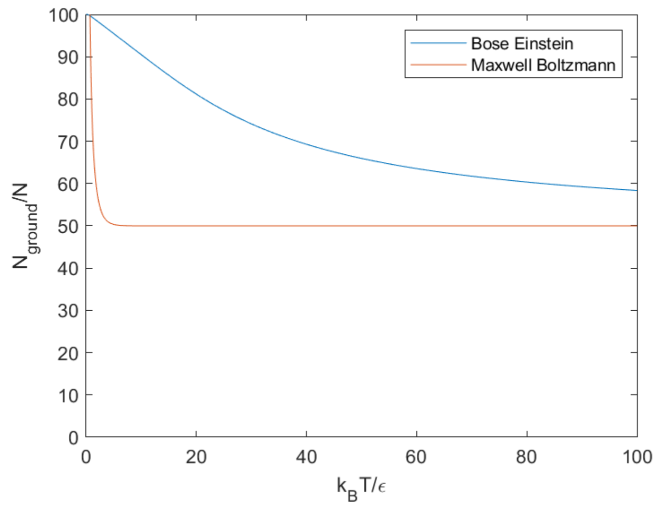


Figure 1: The above plot has been simulated numerically for the case where the total number of particles is 100. We see how we can have a much higher fraction of particles in the ground state even for higher values of k_BT/ϵ which is the most important takeaway from the Bose Einstein statistics approach.

The most important takeaway from here is that the temperature can be significantly higher and we can still have a sizeable fraction of particles in the ground state. This is a unique feature of Bose-Einstein statistics which is essentially what that helps us achieve BEC.

Thus, Bose-Einstein condensation is a phase transition whereby the ground state becomes highly occupied. Now let us expand equation (!) in 2 limits, one where $N >> k_BT/\epsilon$ and $N << k_BT$. Considering the first limit, we get the following,

$$\frac{\langle N_{ground}\rangle}{N} = 1 - \frac{k_BT}{N\epsilon} + ...$$

This equation can be used in the high-temperature limit where the classical behaviour is approached as $T \to \infty$. In the second limit,

$$\frac{\langle N_{ground}\rangle}{N} = \frac{1}{2} + \frac{N\epsilon}{12k_BT} + ...$$

The second limit tells us how the growth tends to 1 as $T \to 0$. The comparison of these approximations have been plotted below,

The most important point to be considered here is that even for very large values of temperature, the ground state occupancy is still very high. To have an intuitive idea of it we can consider 10 million particles, with the temperature more than a thousand times the excited state's energy, the occupancy of the ground state is still close to 90% and only 10% in the excited state.

5.2 Grand Canonical ensemble

In general not every physical system can be accurately described using a 2-level system we can have an infinite number of states where it becomes difficult to calculate $\langle N_{graound}\rangle$. We thus need to resort

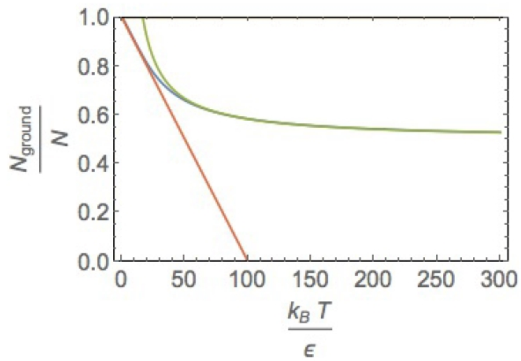


Figure 2: The crossover point is roughly where both the approximations break down. The crossover is roughly at $k_BT = N\epsilon$. This is also the order of the critical temperature for this two-state model indicating the onset of Bose-Einstein condensate

to using a grand canonical ensemble

For a grand canonical ensemble, the partition function will be given by,

$$\mathcal{Z} = \prod_r \frac{1}{1 - e^{-\beta(\epsilon_r - \mu)}}$$

The number of accessible states will then be,

$$\Omega = k_B T ln \mathcal{Z}$$

$$\Omega = k_B T \sum_r ln[1 - e^{-\beta(\epsilon_r - \mu)}]$$

The expectation value of the number of particles in the state i will then be given by,

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_r - \mu) - 1}}$$

Clearly the average number of particles for a given state must always be a non-negative number, if we set the ground state's energy as 0 (for reference), we see that the chemical potential μ must always be negative ($\epsilon_0 - \mu > 0$ and thus $\mu < 0$). When the chemical potential then approaches the ground state energy while still being negative, the occupancy of the ground state will *diverge*. This phenomenon is called *Bose-Einstein Condensation.*

5.3 Occupation of ground state

We have seen that the ground state occupation is very high or in other words, occupied *macroscopically.* Now, consider the density of states expression,

$$D(E)\ \sqrt{E} \qquad \lim_{E \to 0} D(E) = 0$$

Thus, when we find $\int D(E)dE$, we will get the population of the ground state to be 0 whereas we just showed that the ground state is macroscopically occupied. This problem can be solved by considering a special form of the Bosonic grand canonical potential.

$$\beta\Omega(T, V, z) = \sum_r ln[1 - e^{-\beta(\epsilon_r - \mu)})]$$

$$\beta\Omega(T, V, z) = (2s)\frac{V}{(2\pi)^3} 4\pi \int_0^\infty k^2 ln[1 - ze^{-\beta\epsilon(k)}]dk + (2s)ln(1 - z)$$

where the last term is the contribution from the ground state energy. What we have essentially done in these 2 steps is that we have replaced the summation over discrete number of states into an integral over a *continuum* of all the possible states and the ground state's contribution.

In the end, the Bose-Einstein condensate is characterized by divergences in occupation numbers. The ground state is the only state (all the particles occupy a single quantum state) that is occupied macroscopically.

6 Photon Condensate

6.1 General Realization

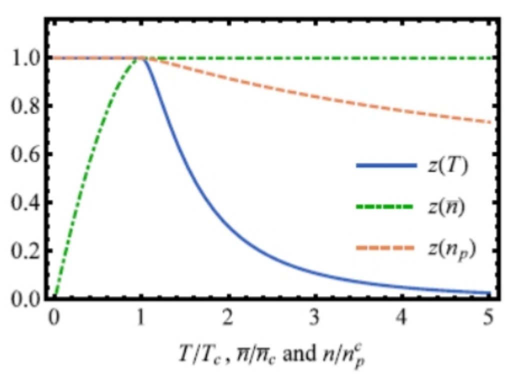
Bose Einstein condensate of photons has been demonstrated by Klaers, Schmitt, Vewinger, and Weitz in a specific optical microcavity. Dye molecules filled into the cavity repeatedly absorb and re-emit photons thus providing a thermalization mechanism needed to the phase transition. If the number of density of the grand-canonical gas exceeds a critical value, the excess photons occupy the ground state of the resonator macroscopically, exactly how it would be like in a BEC. The design of the cavity makes the photon gas harmonically trapped in an effectively two-dimensional resonator.

The main idea in this setup is that we make sure that the photons are confined in the cavity long enough for them to reach equilibrium. Unlike atoms, photons when cooled in a cavity, they diminish in number as they get absorbed by the walls. By confining laser light within the cavity by placing two concave mirrors, we can possibly create the conditions for light required to thermally obtain an equilibrium gas of conserved particles.

6.2 Photon BEC in our medium

As we saw earlier, $\overline{n}$ depends on z and as z increases, it reaches a critical point above which photons start to occupy the ground state. The same phenomenon happens for an ideal Bose gas in 3 dimensions when reaching the critical temperature or particle density. Besides the photon (plasmon-polariton) density and temperature, the plasma density can also drive the current system to its criticality. The chemical potential also depends on all parameters and a critical values exist for each of them $\mu_p = 1$ or $z = 1$.

Using these dependence of z on T, $\bar{(}n)$, n_p, we can increase($\bar{(}n)$, n_p) or decrease(T) the parameters to push the photon system towards criticality and into the BEC region. By doing this, the dependence Thermodynamical propeties also and they start to deviate from the photon gas regime into the BEC regime.



7 Blackbody Radiation

7.1 Introduction

Objects with some Temperature tend to emit radiations which results in a outward flow of energy. But for the system to be in equilibrium, the energy emitted should be replenished. The first step towards understanding this radiation being in equilibrium was made by Kirchoff using Cavity Radiation.

Blackbody Radiation refers to the electromagnetic radiation surrounding a body in thermal equilibrium with its surroundings, emitted by an idealized non-reflective body (a black body). The thermal radiation spontaneously emitted by many ordinary objects can be approximated as black-body radiation. This radiation depends only on the system's properties, and not on the radiation absorbed by the body. Each blackbody has a characteristic, continuous frequency spectrum called the Planck's spectrum.

The energy density of radiation per unit frequency is described by the Planck's formula.

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

For deriving the Planck's Formula, let us consider a cube of length L. So, the wavelengths that can be fit into the box (in x-direction) are those for which

$$\frac{l\lambda_x}{2} = L$$

where l takes positive integral values. Similarly, for the y-direction and z-direction,

$$\frac{m\lambda_y}{2} = L \quad \frac{n\lambda_z}{2} = L$$

The expression for the waves which fit in the box in the x-direction is

$$A(x) = A_o \sin(k_x x)$$

Now, $k_x = 2\pi/\lambda_x$, substituting it we get,

$$k_x = \frac{\pi l}{L}$$

Similarly, for the y-direction and z-direction,

$$k_y = \frac{\pi m}{L} \quad k_z = \frac{\pi n}{L}$$

For the 3-dimensional Wave vector,

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

$$k^2 = \frac{\pi^2}{L^2}(l^2 + m^2 + n^2) = \frac{\pi^2 p^2}{L^2}$$

Now for counting the number of modes available, we assume a sphere shell of radius p with an differential extension dp, then we find that,

$$dN(p) = N(p)dp = \frac{1}{8}4\pi p^2 dp$$

The 1/8 factor comes as we consider only the positive values of l,m and n. Since $k = \pi p/L$ and $dk = \pi dp/L$, we find

$$dN(p) = \frac{L^3}{2\pi^2}k^2 dk$$

But, $L^3 = V$ is the volume of the box and $k = 2\pi\nu/c$. Therefore, we can rewrite the expression

$$dN(p) = \frac{V}{2\pi^2}k^2dk = \frac{V}{2\pi^2}\frac{8\pi^3\nu^2}{c^3}d\nu = \frac{4\pi\nu^2V}{c^3}d\nu$$

We multiply this result by 2 to accommodate the two allowed independent modes of EM waves. Thus, we get the number of states per unit volume as

$$dN(p) = \frac{8\pi\nu^2}{c^3}d\nu$$

We now use the Boltzmann distribution to find the expected occupancy of the modes in Thermal Equilibrium. The probability that a mode has Energy $E\nu = nh\nu$, where we have n photons in this mode.

$$p(n) = \frac{e^{\frac{-E_n}{kT}}}{\sum_{n=0}^{\infty} e^{\frac{-E_n}{kT}}}$$

therefore, average energy

$$\overline{E_\nu} = \sum_{n=0}^{\infty} E_n p(n) = \frac{\sum_{n=0}^{\infty} E_n e^{\frac{-E_n}{kT}}}{\sum_{n=0}^{\infty} e^{\frac{-E_n}{kT}}}$$

$$\overline{E_\nu} = \frac{\sum_{n=0}^{\infty} nh\nu e^{\frac{-nh\nu}{kT}}}{\sum_{n=0}^{\infty} e^{\frac{-nh\nu}{kT}}}$$

Using the summation series, (where $x = e^{\frac{-h\nu}{kT}}$)

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ...$$

The mean energy mode is,

$$\overline{E_\nu} = \frac{h\nu e^{\frac{-h\nu}{kT}}}{1 - e^{\frac{-h\nu}{kT}}} = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

Using these equations, the energy density of radiation in this frequency range is

$$u(\nu)d\nu = dN(p)\overline{E_\nu} = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1}d\nu$$

This is the Planck's distribution function.

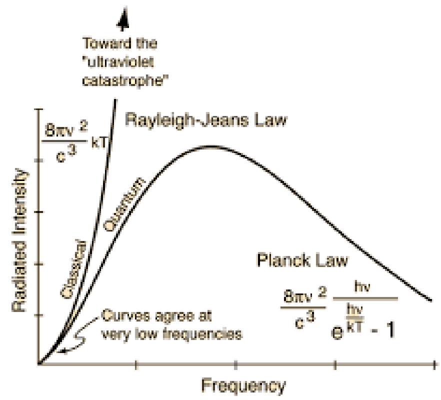
At the thermodynamic Limit T>>1, $h\nu/kT << 1$, then

$$e^{\frac{h\nu}{kT}} = 1 + \frac{h\nu}{kT} + ...$$

Therefore, the Planck's distribution function becomes,

$$u(\nu)d\nu = dN(p)\overline{E_\nu} = \frac{8\pi h\nu^3}{c^3}\frac{1}{\frac{h\nu}{kT}}d\nu = \frac{8\pi\nu^2 kT}{c^3}d\nu$$

$$\overline{E} = kT$$



7.2 Modified BlackBody Radiation

As the system behaves differently depending on the oscillation frequency as free photon gas($\omega_p << kT/\hbar$) or heavy Bose-gas ($\omega_p >> kT/\hbar$), we can study the change in properties of the thermal radiation in the presence of the medium.

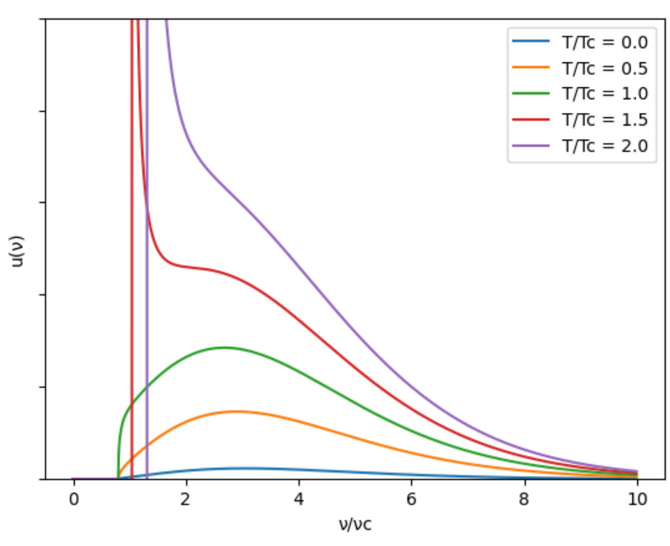
We can get the Energy distribution for the system discussed in this paper from Section 5, therefore we get the modified Planck's distribution as

$$u(\nu) = \frac{8\pi h\nu^2}{c^3}\frac{\sqrt{\nu^2 - \nu_p^2}}{e^{\beta h(\nu-\nu_p)}z^{-1} - 1}; \quad \nu_p = \frac{\omega}{2\pi}$$

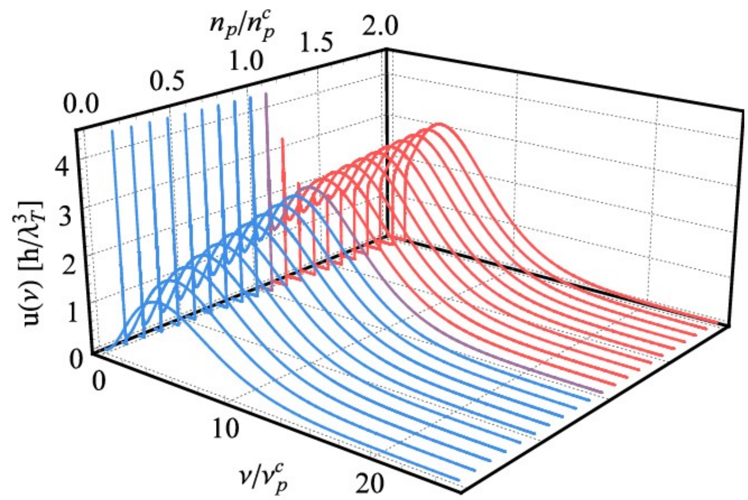
The total energy E is given by,

$$E = V\int_{\nu_p}^{\infty} u(\nu)\, d\nu$$

Below the oscillation frequency(ν_p), the system acts as a free photon gas and there is no radiation. At the oscillation frequency, the system start radiating as it starts behaving like Bose gas. Beyond criticality(when z = 1), a sharp peak starts appearing at ν_p as the particles start to accumulate in a coherent BEC phase. Although a singularity exists at this point, since its normalizable the total energy of the system remains $\overline{E} = \overline{N}\epsilon_p$ as all particles join the condensate.
We can see from the expression of $u(\nu)$, that it depends on T(through β). Taking T/T_c as a parameter, we can plot $u(\nu)$ wrt ν/ν_p.



From the above plot we can see that as temperature goes below T_c the photon gas undergoes BEC and a sharp peak appears near ν_p
Similarly for other parameters such as n_p(plasma density), we can see the criticality remains.



8 Summary

We saw how introducing photons into the plasma medium generates an effective non-zero mass for photons which allows it to form a BEC. This has been demonstrated in recent experiments by using an optical microcavity. We also analyzed how photons in this plasmon-polariton medium mimic the thermodynamic properties of an ideal Bose gas and also derived the expressions for the same. The introduction of effective mass also modifies the Planck's law and how the potential to form BEC plays a role in the modification of the law. The critical points at which the photon gas transitions into a BEC are studied.

These non-Planckian radiations can be observed and used to study the Cosmic Microwave Background radiations at low frequencies. We saw that the distribution develops a sharp condensation peak at the plasma frequency in the BEC state corresponding to coherent radiation which can be detected and perhaps could be used as a new type of source for coherent radiation. Thus, the statistical properties of photon gas in plasma medium were studied analytically and numerically.

9 Appendix

9.1 Appendix A: Grand Potential

We solve the integral for the grand potential as mentioned earlier,

$$\Phi = A\int_1^\infty \epsilon\sqrt{\epsilon^2-1}ln[1-e^{-x(\epsilon-1)}z]$$

Taking the summation series for $\ln[1-e^{-x(\epsilon-1)}z]$

$$\Phi = -A\int_1^\infty d\epsilon\epsilon\sqrt{\epsilon^2-1}\sum_{j=1}^\infty \frac{(e^x z)^j}{j}e^{-jx\epsilon}$$

$$= -A\sum_{j=1}^\infty \frac{(e^x z)^j}{j}\int_1^\infty d\epsilon\epsilon\sqrt{\epsilon^2-1}e^{-jx\epsilon}$$

We now evaluate $\int_1^\infty d\epsilon\epsilon\sqrt{\epsilon^2-1}e^{-jx\epsilon}$ by using laplace transform,

$$\mathcal{L}[\epsilon\sqrt{\epsilon^2-1}](jx) = \int_1^\infty d\epsilon\epsilon\sqrt{\epsilon^2-1}e^{-jx\epsilon}$$

$$\mathcal{L}[\epsilon\sqrt{\epsilon^2-1}](jx) = \int_0^\infty dycosh(y)sinh^2(y)e^{-jxcosh(y)}$$

$$= \int_0^\infty dycosh^3(y)e^{-jxcosh(y)} - \int_0^\infty dycosh(y)e^{-jxcosh(y)}$$

$$= \frac{1}{4}\int_0^\infty dy(cosh(3y)+3cosh(y))e^{-jxcosh(y)} - \int_0^\infty dycosh(y)e^{-jxcosh(y)}$$

$$= \frac{1}{4}(K_3(jx)+3K_1(jx)) - K_1(jx) = \frac{K_2(jx)}{jx}$$

Here the we use the Macdonald function, whose integral form is given by

$$K_s(x) = \int_0^\infty dtcosh(st)e^{-xcosh(t)},$$

and its recursion relation

$$\frac{2s}{x}K_s(x) = e^{(s-1)\pi\iota}K_{s-1}(x) - e^{(s+1)\pi\iota}K_{s+1}(x)$$

Thus we get the final form of the grand potential as,

$$\Phi = -A\sum_{j=1}^{\infty}\frac{(e^x z)^j}{j}\frac{K_2(jx)}{jx}; \quad A = \frac{8\pi V x^3}{\beta^4 c^3 h^3}$$

9.2 Appendix B: Thermodynamical Properties at the extremes

At the limiting cases, we can simplify the grand potential and other properties to approximate their behaviour to study them.

When $x << 1$ (and s>0), the Macdonald function $K_s(y) = 2^{s-1}\Gamma(s)y^{-s} + \mathcal{O}(y^2)$, hence the grand potential is

$$\Phi = -\frac{V}{\lambda_T^3}\frac{2}{\beta}\sum_{j=1}^{\infty}\frac{(e^x z)^j}{j^4} = -2\frac{V}{\lambda_T^3}\frac{1}{\beta}Li_4(e^{\ln z+x})$$

$$\overline{N} = \frac{V}{\lambda_T^3}\frac{2}{\beta}\frac{\partial}{\partial\mu}\Big(Li_4(e^x z)\Big)_{V,T} = \frac{2V}{\lambda_T^3}Li_3(e^x z)$$

$$\overline{E} = -2V\frac{\partial}{\partial\beta}\left(\frac{Li_4(e^x z)}{\lambda_T^3}\right)_{\beta\mu} = \frac{6V}{\lambda_T^3}\frac{1}{\beta}Li_4(e^x z)$$

$$S = 2V\frac{\partial}{\partial T}\left(\frac{1}{\lambda_T^3\beta}Li_4(e^x z)\right)_{\mu,V} = 8k_B\frac{V}{\lambda_T^3}Li_4(e^x z) - 2k_B\frac{V}{\lambda_T^3}(lnz+x)Li_3(e^x z)$$

$$P = \frac{\overline{E}}{3V} = \frac{2}{\lambda_T^3}\frac{1}{\beta}Li_4(e^x z)$$

When $x >> 1$ (and s>0), the Macdonald function $K_s(y) = e^{-y}\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{y}} + O(1/y^{3/2})$, hence the grand potential is

$$\Phi = -\frac{V}{\lambda_T^3}\sqrt{\frac{\pi}{2}}\frac{x^{3/2}}{\beta}\sum_{j=1}^{\infty}\frac{(z)^j}{j^{5/2}} = -\frac{V}{\lambda_T^3}\sqrt{\frac{\pi}{2}}\frac{x^{3/2}}{\beta}Li_{5/2}(z)$$

$$\overline{N} = \frac{V}{\lambda_T^3}\frac{2}{\beta}\frac{\partial}{\partial\mu}\bigg(Li_4(e^x z)\bigg)_{V,T} = \sqrt{\frac{\pi}{2}}\frac{V}{\lambda_T^3}x^{3/2}Li_{3/2}(z)$$

$$\overline{E} = -2V\frac{\partial}{\partial\beta}\left(\frac{Li_4(e^x z)}{\lambda_T^3}\right)_{\beta\mu} = \sqrt{\frac{\pi}{2}}\frac{V}{\lambda_T^3}x^{3/2}\epsilon_p Li_{3/2}(z) + \frac{3}{2}\sqrt{\frac{\pi}{2}}\frac{V}{\lambda_T^3}\frac{x^{3/2}}{\beta}Li_{5/2}(z)$$

$$S = -\left(\frac{\partial\Phi}{\partial T}\right)_{\mu,V} = \sqrt{\frac{\pi}{2}}\frac{V}{\lambda_T^3}x^{3/2}k_B\left(\frac{5}{2}Li_{5/2}(z) - Li_{3/2}(z)ln(z)\right)$$

$$P = \frac{1}{3}\sqrt{\frac{\pi}{2}}\frac{1}{\lambda_T^3}x^{3/2}\epsilon_p Li_{3/2}(z) + \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\lambda_T^3}\frac{x^{3/2}}{\beta}Li_{5/2}(z)$$

where the definition of polylogarithm is used, $Li_s(y) = \sum_{j=1}^{\infty} y^j/j^s$

9.3 Appendix C: Polariton

Polaritons are quasi-particles that are made up of photons which are strongly coupled to an electric dipole or a magnetic dipole carrying an excitation. Examples of such a polariton include an electron-hole pair in a semiconductor, which forms an exciton polariton. Another example includes oscillating electrons at the surface of a metal which creates a surface-plasmon polariton. The polariton is a bosonic quasi-particle. We can say that the polariton picture is valid whenever the weak coupling limit is an invalid approximation. A major feature of polaritons is a strong dependency of the propagation speed of light through the crystal on the frequency of the photon.

10 References

- Statistical Theory of Photon Gas
- Quantum gases
- Ideal Bose gas
- Polariton - 1
- Polariton - 2
- Bose Einstein Condensation of photons in optical microcavities
- Bose Einstein Condensate introduction
- Fundamentals of statistical mechanics and thermal physics - Frederick Reif
- Black Body Radiation
- Black Body Radiation - Wikipedia
- Plasma 1
- Plasma - Wikipedia