# Ore's Theorem

## CS 111 Presentation

Youssef Adam  Thomas Kang
Youssef Koreatam  Michael Wong  Gabriel Serrano

Department of Computer Science, University of California, Riverside

March 19, 2024

# Introduction: Questions

Below are three questions that you may be able to answer by the end of this presentation.

- What are the two criteria a graph must meet in order to satisfy Ore's theorem and therefore be a Hamiltonian graph?
- If a graph $G$ does not meet both criteria to satisfy Ore's theorem, is $G$ Hamiltonian?
- What is one example of a real-world application of Hamiltonian cycles, and how does this tie in with Ore's theorem?

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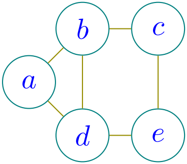

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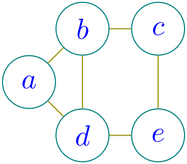


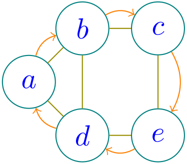


$T = \{a, b, c, e, d, a\}$

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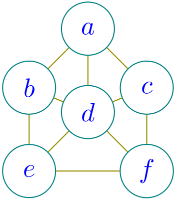
- $\Omega$ be a condition (Ore's theorem) with two subconditions such that:
  - $\Omega_1$: $|V| \geq 3$,
  - $\Omega_2$: the sum of the degrees of all nonadjacent pairs $\geq |V|$.

If $G$ satisfies $\Omega$, then we may conclude that $G$ is a Hamiltonian graph (i.e., $G$ has a Hamiltonian cycle $T$).

# Example I

*First Case*

$G_1(V, E)$ **satisfies** Ore's theorem $\Omega$ and **has** a Hamiltonian cycle $T$.



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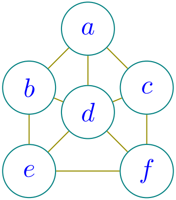


$$n := |V| = 6 \geq 3 \quad \vdash \quad \Omega_1$$

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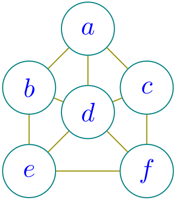


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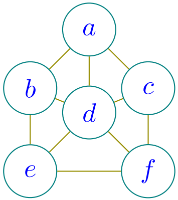


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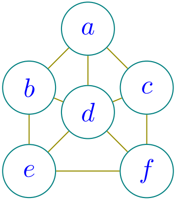


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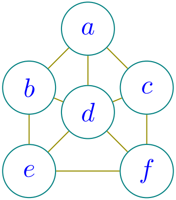


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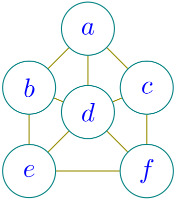


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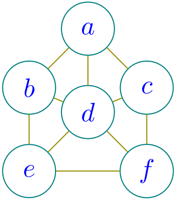


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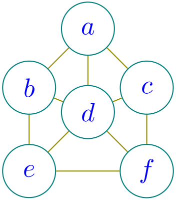


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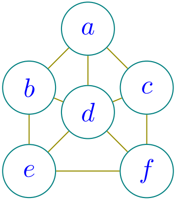


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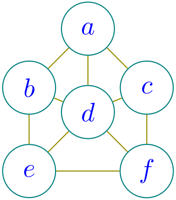


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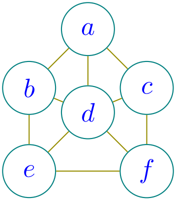


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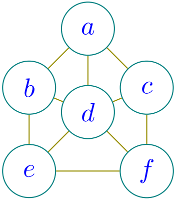


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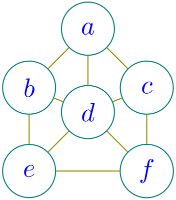


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This graph **satisfies** $\Omega$ and **has**
a Hamiltonian cycle $T = \{d, a, b, e, f, c, d\}$.

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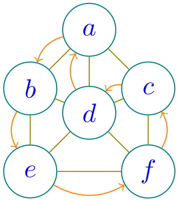


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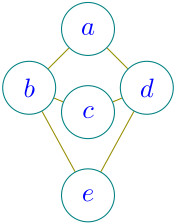

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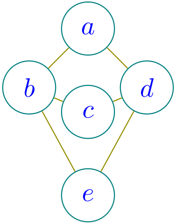


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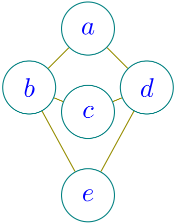


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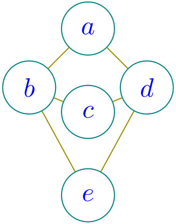


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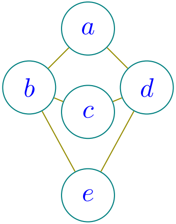


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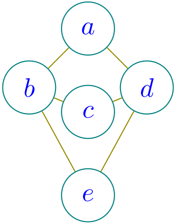


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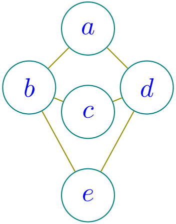


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This graph **does not satisfy** $\Omega$ and **does not have** a Hamiltonian cycle.

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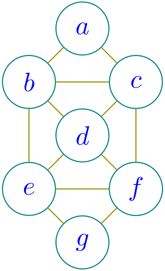
*Third Case*

$G_3(V, E)$ **does not satisfy** $\Omega$ and **has** a Hamiltonian cycle $T$.



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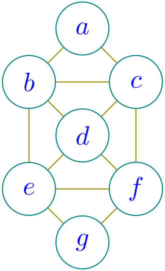


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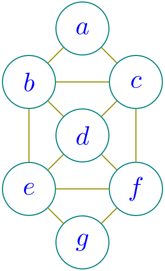


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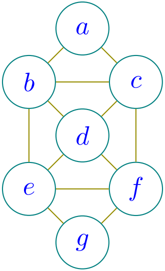


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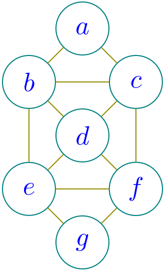


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This graph **does not satisfy** $\Omega$ but still **has** a Hamiltonian cycle $T = \{a, b, d, e, g, f, c, a\}$.

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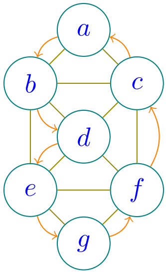


$$n := |V| = 7 \geq 3 \quad \vdash \quad \Omega_1$$

$$\begin{array}{ccc}
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\end{array}$$

$$\not\vdash \ \Omega_2$$

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# Applications

**Hamiltonian cycles** have a variety of real-world applications.

Network routing:

- finding the most effective and efficient round-trip path between points A to B
- improving network traffic by reducing delays from client to server
- optimizing network runtime

Picture a data packet moving through a network of routers to reach a server, aiming to find a path that visits each router exactly once, and thereby optimizing data transmission efficiency.

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**Hamiltonian cycles** have a variety of real-world applications.

Navigation systems:

- finding a path that travelers can take to both reach every destination and reduce estimated time of arrival
- development of algorithms to prevent travelers from revisiting destinations and inefficient paths

In some navigation systems, the primary goal is to find an efficient *round-trip* cycle that visits each destination once, ensuring smooth transport without revisiting locations.

# Conclusion: Summary

Ore's theorem has many practical, mathematical applications. This theorem can be applied to anything that deals with systems that can be represented by vertices and graph, such as:

- transportation systems
- neural networks
- sewage pipes
- electrical lines
- social networks

Any individual can apply this theorem to solve problems involving such matters. Ore's theorem is a revolutionary theorem that has enabled humanity to progress expeditiously in the field of mathematics and technological innovations.

# Conclusion: Questions

Below are the questions displayed at the beginning of the presentation.

- What are the two criteria a graph must meet in order to satisfy Ore's theorem and therefore be a Hamiltonian graph?
- If a graph $G$ does not meet both criteria to satisfy Ore's theorem, is $G$ Hamiltonian?
- What is one example of a real-world application of Hamiltonian cycles, and how does this tie in with Ore's theorem?