Ore's Theorem

CS 111 Presentation

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Youssef Koreatam Michael Wong Gabriel Serrano

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Questions

Introduction: Background Knowledge

To understand **Ore's theorem**, we must briefly recapitulate what it means for a graph to be **Hamiltonian**.

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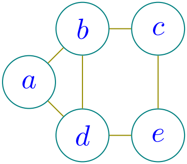
Example:



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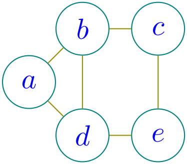
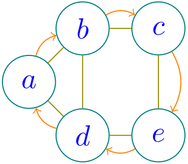




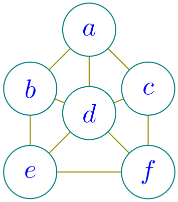
$T = \{a, b, c, e, d, a\}$

Introduction: Ore's Theorem Statement

Example I

First Case

$G_1(V, E)$ **satisfies** Ore's theorem Ω and **has** a Hamiltonian cycle T.



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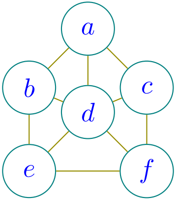

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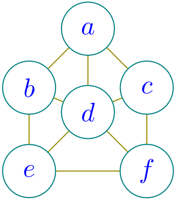

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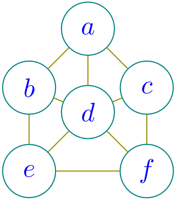

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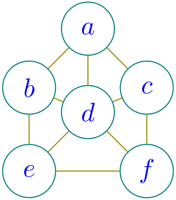

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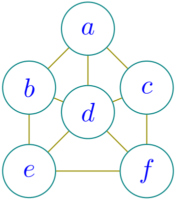

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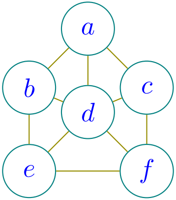

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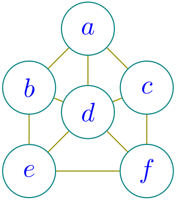

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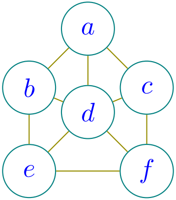

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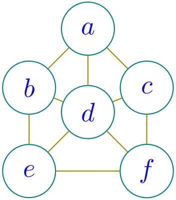

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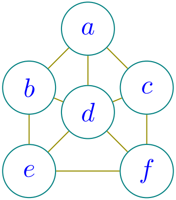

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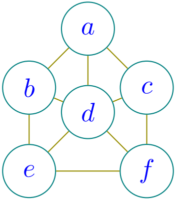

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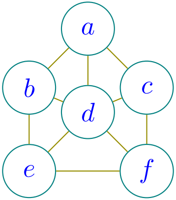

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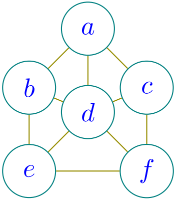

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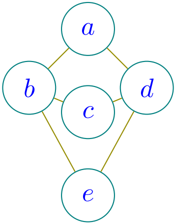
This graph **satisfies** Ω and **has**
a Hamiltonian cycle $T = \{d, a, b, e, f, c, d\}$.

Example II

Second Case

$G_2(V, E)$ **does not satisfy** Ω and **does not have** a Hamiltonian cycle T.



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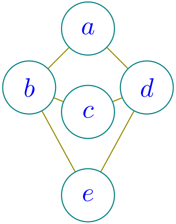

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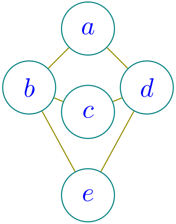

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$$\deg(a) + \deg(c) = 4$$

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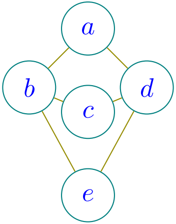

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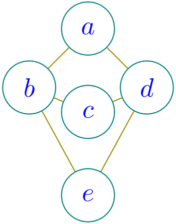

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$$\deg(a) + \deg(c) = 4 \ngeq n$$
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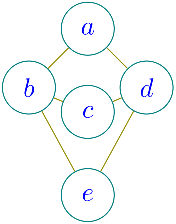

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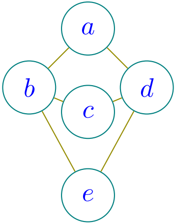


$$n := |V| = 5 \geq 3 \quad \vdash \quad \Omega_1$$

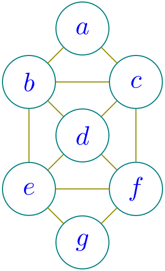
$$\begin{gathered}\deg(a) + \deg(c) = 4 \not\geq n \\ \deg(a) + \deg(e) = 4 \not\geq n \\ \deg(b) + \deg(d) = 6 \geq n \\ \deg(c) + \deg(e) = 4 \not\geq n \\ \nvdash \Omega_2\end{gathered}$$

This graph **does not satisfy** Ω and **does not have** a Hamiltonian cycle.

Example III

Third Case

$G_3(V, E)$ **does not satisfy** Ω and **has** a Hamiltonian cycle T.



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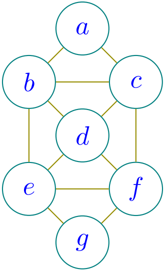

$$n := |V| = 7 \geq 3 \quad \vdash \quad \Omega_1$$

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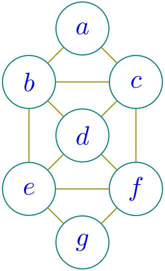

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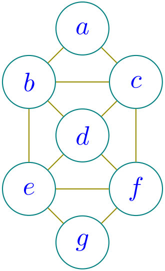

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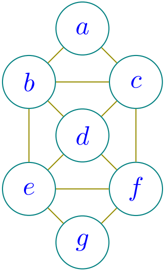

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This graph **does not satisfy** Ω but still **has** a Hamiltonian cycle $T = \{a, b, d, e, g, f, c, a\}$.

Applications

Hamiltonian cycles have a variety of real-world applications.

Network routing:

- finding the most effective and efficient round-trip path between points A to B
- improving network traffic by reducing delays from client to server
- optimizing network runtime

Picture a data packet moving through a network of routers to reach a server, aiming to find a path that visits each router exactly once, and thereby optimizing data transmission efficiency.

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Navigation systems:

- finding a path that travelers can take to both reach every destination and reduce estimated time of arrival
- development of algorithms to prevent travelers from revisiting destinations and inefficient paths

In some navigation systems, the primary goal is to find an efficient *round-trip* cycle that visits each destination once, ensuring smooth transport without revisiting locations.

Conclusion: Summary

Conclusion: Questions