Sect. 4.2: Null Space and Column Space

Goal: Last class, we introduced vector spaces and subspaces. Today, we want to talk about two specific subspaces in the context of linear systems.

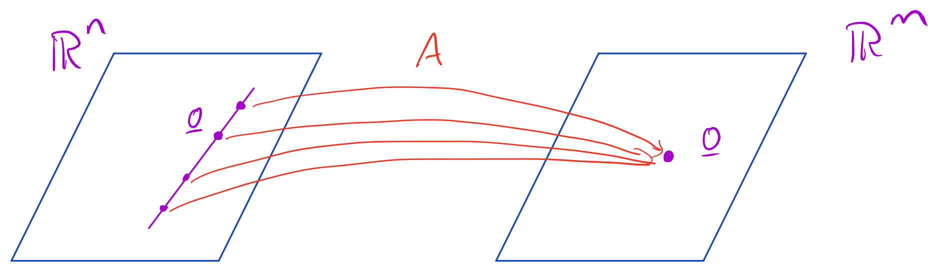
Def.

The null space of an $m \times n$ matrix A, denoted $\text{Nul } A$, is the set of all vectors that satisfy the homogeneous equation $A\underline{x} = 0$.
In set notation,

$$\text{Nul } A = \{ \underline{x} \mid \underline{x} \in \mathbb{R}^n \text{ and } A\underline{x} = \underline{0} \}$$

such that

- " set of $\underline{x}$ such that $\underline{x}$ is in $\mathbb{R}^n$ and $A\underline{x} = \underline{0}$ "
- As a picture



the set of all vectors $\underline{x} \in \mathbb{R}^m$ that map to $\underline{0}$ under $A\underline{x}$

- To test if $\underline{v} \in \mathbb{R}^n$ is in the null space of A, simply check if $A\underline{v} = \underline{0}$

Ex

Is $\underline{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ in Nul A for $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

Check $A\underline{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\underline{u} \in$ Nul A.

Theorem

The null space of $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^n$.

Pf. (Check the 3 requirements for a subspace)

$\underline{0}$ vector : Check $\underline{0} \in \mathbb{R}^n$ is in Nul A

$\underline{0} \in$ Nul A if $A\underline{0} = \underline{0}$. We can see this easily

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

closed under addition : For $\underline{u}, \underline{v} \in$ Nul A, check that $\underline{u} + \underline{v} \in$ Nul A.

We know $\underline{u} \in$ Nul $A \Rightarrow A\underline{u} = 0$

$\underline{v} \in$ Nul $A \Rightarrow A\underline{v} = 0$

So we check $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v}$

$= \underline{0} + \underline{0} = \underline{0}$

We have $A(\underline{u}+\underline{v}) = 0 \Rightarrow \underline{u}+\underline{v} \in \text{Nul}\, A$ ✓

closed under scalar mult.

For $\underline{u} \in \text{Nul}\, A$ and scalar $c \in \mathbb{R}$,
we check that $c\underline{u} \in \text{Nul}\, A$

We know $\underline{u} \in \text{Nul}\, A \Rightarrow A\underline{u} = \underline{0}$

So we check
$$
\begin{aligned}
A(c\underline{u}) &= c(A\underline{u}) \\
&= c(\underline{0}) \\
&= \underline{0}
\end{aligned}
$$

$A(c\underline{u}) = \underline{0} \Rightarrow c\underline{u} \in \text{Nul}\, A$ ✓

We've showed that Nul A satisfies the 3 criteria to be subspace.

We find an explicit expression for Nul A in the following way...

Ex Find a spanning set for Nul A where

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

1st, row reduce $A\underline{x} = \underline{0}$

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next, we write the solution to $A\underline{x} = \underline{0}$ in parametric vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_3}$$

Then $\text{Nul}\, A = \text{span}\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$

Column Space of Matrix

Def The column space of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $\text{Col}\, A$, is the set of all linear combinations of the columns of A, ie, for $A = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n]$ we have

$$\text{Col}\, A = \text{span}\{\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n\}$$

Theorem

The column space of $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^m$.

Pf Check 3 criteria for subspace

$\underline{0} \in \text{Col}\, A$ We see that for weights

$x_1 = x_2 = \cdots = x_n = 0$, then

$$A\underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n = \underline{0}$$

$$\Rightarrow \underline{0} \in \text{Col}\, A \checkmark$$

closed under addition

Given $\underline{b}_1, \underline{b}_2 \in \text{Col } A$, we need to check that $\underline{b}_1 + \underline{b}_2 \in \text{Col } A$.

$\underline{b}_1 \in \text{Col } A \Rightarrow$ there exist $\underline{x}_1$ such that

$$A\underline{x}_1 = \underline{b}_1$$

$\underline{b}_2 \in \text{Col } A \Rightarrow$ there exist $\underline{x}_2$ such that

$$A\underline{x}_2 = \underline{b}_2$$

We then see that

$$A(\underline{x}_1 + x_2) = A\underline{x}_1 + A\underline{x}_2$$

$$= \underline{b}_1 + \underline{b}_2$$

So there exists $\underline{x} = \underline{x}_1 + x_2$ such that $A\underline{x} = \underline{b}_1 + \underline{b}_2$

$\Rightarrow \underline{b}_1 + \underline{b}_2 \in \text{Col } A$. ✓

closed under scalar mult.

For $\underline{b} \in \text{Col } A$ and some scalar $c \in \mathbb{R}$, check that $c\underline{b} \in \text{Col } A$

$\underline{b} \in \text{Col } A$ means there exists $\underline{x}$ such that

$$A\underline{x} = \underline{b}$$

We see that for $c\underline{x}$, we set

$$A(c\underline{x}) = c(A\underline{x})$$

$$= c\,\underline{b}$$

So there exist $\underline{v} = c\underline{x}$ such that $A\underline{v} = c\underline{b}$ ✓

All 3 criteria are satisfied $\Rightarrow$ so Col A is a subspace of $\mathbb{R}^m$.

<u>Ex</u> Find A such Col $A = W$ where

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} \text{ for } a,b \in \mathbb{R} \right\}$$

First, we see $W = \left\{ a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ for } a,b \in \mathbb{R} \right\}$

$$= \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

So take $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$, then we have that

$$\text{Col } A = \text{span}\{\underline{a}_1, \underline{a}_2\} = \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} = W$$

<u>Comparing & Contrasting Nul A and Col A</u>

Let A be an $m \times n$ matrix

Nul A	Col A
1.) Nul A is a subspace of $\mathbb{R}^n$	1.) Col A is a subspace of $\mathbb{R}^m$
2.) To check if $\underline{x} \in$ Nul A, see if $A\underline{x} = \underline{0}$	2.) To check if $\underline{b} \in$ Col A, we need $A\underline{x} = \underline{b}$ is consistent

3) Every vector $\underline{v} \in \text{Nul } A$ satisfies $A\underline{v} = \underline{0}$

4) $\text{Nul } A = \{\underline{0}\}$ if and only if $A\underline{x} = \underline{0}$ only has the trivial solution

5.) $\text{Nul } A = \{\underline{0}\}$ if and only if $\underline{x} \longmapsto A\underline{x}$ is a one-to-one map

3.) Vectors in $\text{Col } A$ are all linear combinations of the cols. of A.

4) $\text{Col } A = \mathbb{R}^m$ if and only if $A\underline{x} = \underline{b}$ is consistent for all $\underline{b} \in \mathbb{R}^m$

5.) $\text{Col } A = \mathbb{R}^m$ if and only if $\underline{x} \longmapsto A\underline{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$

Ex

Let $A = \begin{vmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{vmatrix}$

If $\text{Col } A$ is a subspace of $\mathbb{R}^k$, what is k?

$\underline{a}_1\ \underline{a}_2\ \underline{a}_3\ \underline{a}_4 \in \mathbb{R}^3 \Rightarrow \text{Col } A$ is a subspace of $\mathbb{R}^3$

If $\text{Nul } A$ is a subspace of $\mathbb{R}^\ell$, what is ℓ?

for $A\underline{x}$ to be defined, we need $x \in \mathbb{R}^4$, so $\text{Nul } A$ is a subspace of $\mathbb{R}^4$.

Find $\underline{v} \in \text{Col } A$. Easy, and column or combination of columns in A will do.

$$\underline{v} = \begin{vmatrix} 2 \\ -2 \\ 3 \end{vmatrix} \in \text{Col } A$$

Find $\underline{v} \in \text{Nul } A$. Harder, we solve $A\underline{v} = \underline{0}$

$$\left| A \,\vdots\, 0 \right| = \left|\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right| \sim \left|\begin{array}{ccccc} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right|$$

$x_1 = -9x_3$

$x_2 = 5x_3$

$x_3 = \text{free}$

$x_4 = 0$

$$x_3 = 1 \Rightarrow \underline{v}_1 = \begin{vmatrix} -9 \\ 5 \\ 1 \\ 0 \end{vmatrix} \in \text{Nul } A$$

$$x_3 = -2 \Rightarrow \underline{v}_2 = \begin{vmatrix} 18 \\ -10 \\ -2 \\ 0 \end{vmatrix} \in \text{Nul } A$$

Check that $A\underline{v}_1 = A\underline{v}_2 = \underline{0} \Rightarrow \underline{v}_1, \underline{v}_2 \in \text{Nul } A$

Is $\underline{u} = \begin{vmatrix} 3 \\ -2 \\ -1 \\ 0 \end{vmatrix}$ in Nul A?

Check $$A\underline{u} = \begin{vmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{vmatrix} \begin{vmatrix} 3 \\ -2 \\ -1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ -3 \\ 3 \end{vmatrix} \neq 0$$

so $\underline{u}$ not in Nul A.

Is $\underline{v} = \begin{vmatrix} 3 \\ -1 \\ 3 \end{vmatrix}$ in Col A?

Solve $$\left|\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right| \sim \left|\begin{array}{ccccc} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{array}\right|$$

The system is consistent, so $\underline{v} \in \text{Col } A$.