Sect 3.1: Introduction to Determinants

Recall from Ch. 2, we had a formula for computing the inverse of a 2×2 matrix

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

if $\det A = ad - bc \neq 0$. We want to introduce determinants for general $n\times n$ matrices.

Def.

For $n \geq 2$, the <u>determinant</u> of an $n\times n$ matrix $A = [a_{ij}]$ is defined recursively by

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Typically, we do this across the first row, $i=1$

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

↳ we call $(-1)^{i+j} \det A_{ij} = C_{ij}$ the ijth <u>cofactor</u>. We compute $\det A$ by <u>cofactor expansion</u>.

↳ For A $n\times n$, the matrix A_{ij} is the $(n-1)\times(n-1)$ submatrix obtained by eliminating the ith row and jth col. of A

$$A = \begin{vmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & 2 & 0 \end{vmatrix} \implies A_{11} = \begin{vmatrix} 0 & 4 & -1 \\ 1 & 0 & 7 \\ 4 & 2 & 0 \end{vmatrix}$$

$$A_{23} = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & 7 \\ 0 & 4 & 0 \end{vmatrix}$$

<u>Ex</u> Compute the determinant of $A = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \leftarrow$

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

$$= 1 \cdot 1 \cdot \det \begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} + (-1) \cdot 5 \det \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} + 1 \cdot 0 \cdot \det \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$$

$$= 1 \cdot 1 \cdot \left(4 \cdot 0 - (-1)(-2) \right) + (-1) \cdot 5 \cdot \left(2 \cdot 0 - (-1) \cdot 0 \right) + 1 \cdot 0 \cdot \left(2 \cdot (-2) - 0 \cdot 4 \right)$$

$$= 1 \cdot (0 - 2) - 5(0 - 0) + 0 \cdot (-4 - 0)$$

$$= -2$$

$\nwarrow$ det A is always a scalar.

<u>Facts</u>

- det A is always a scalar
- det A for $A \in \mathbb{R}^{2\times 2}$ is always $ad - bc$
- we can compute the cofactor expansion formula along any row or column.

Ex Compute det A for

$$A = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$\det A = \sum_{i=1}^{3} (-1)^{i+3} a_{i3} \det A_{i3}$$

$$= (-1)^{1+3} a_{13} \det A_{13} + (-1)^{2+3} \cdot a_{23} \det A_{23} + (-1)^{3+3} \cdot a_{33} \det A_{33}$$

$$= 1 \cdot 0 \cdot \det A_{13} + (-1) \cdot (-1) \det A_{23} + 1 \cdot 0 \cdot \det A_{33}$$

$$= (-1) \cdot (-1) \det A_{23} = 1 \cdot \det \begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} = 1 \cdot (-2) - 0 \cdot 5 = -2$$

So, $\det A = -2$ ✓. This agrees with the previous example.

- the determinant of any triangular matrix is the product of the diagonal.

Ex compute det A for

$$A = \begin{vmatrix} 1 & 3 & -2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

Cofactor expansion along the 1st column gives

$$\det A = \sum_{i=1}^{4} (-1)^{i+1} a_{i1} \det A_{i1}$$

$$= (-1)^2 \cdot a_{11} \det A_{11} + (-1)^3 \cdot a_{21} \det A_{21}$$

$$+ (-1)^4 \cdot a_{31} \det A_{31} + (-1)^5 a_{41} \det A_{41}$$

$$= 1 \cdot 1 \cdot \det \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} + (-1)^3 \cdot 0 \cdot \det A_{21}$$

$$+ (-1)^4 \cdot 0 \cdot \det A_{31} + (-1)^5 \cdot 0 \cdot \det A_{41}$$

$$= 1 \cdot \det \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= 1 \cdot \left((-1)^{1+1} \cdot 2 \cdot \det \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} + (-1)^{2+1} \cdot 0 \cdot \det A_{21} + (-1) \cdot 0 \cdot \det (A_{31}) \right)$$

$$= 1 \cdot \left(1 \cdot 2 \cdot (-1 \cdot 4 - 0 \cdot 0) + 0 + 0 \right)$$

$$= 1 \cdot 2 \cdot (-4) = -8. = \det A$$

But, the product of the diagonal is

$$1 \cdot 2 \cdot (-1) \cdot 4 = 8 = \det A.$$

<u>Strategy</u>: Always look for at the matrix and choose the "easiest" column/row for cofactor expansion

↳ Choose columns/rows with lots of 0's, save work.

Ex $A = \begin{vmatrix} 2 & 3 & 0 & -1 \\ 0 & -1 & 0 & 3 \\ 1 & 4 & 5 & 0 \\ 0 & 3 & 0 & 1 \end{vmatrix}$ Expand along column 3

$$\det A = (-1)^{1+3} \cdot 0 \cdot \det A_{13} + (-1)^{2+3} \cdot 0 \cdot \det A_{23} + (-1)^{3+3} \cdot 5 \cdot \det A_{33} + (-1)^{4+3} \cdot 0 \cdot \det A_{43}$$

$$= 1 \cdot 5 \cdot \det \begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$ Expand along column 1

$$= 1 \cdot 5 \cdot \left((-1)^{1+1} \cdot 2 \cdot \det \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix} + (-1)^{2+1} \cdot 0 \cdot \det A_{21} + (-1)^{3+1} \cdot 0 \cdot \det A_{31} \right)$$

$$= 5 \cdot \left(1 \cdot 2 \cdot (-1 \cdot 1 - 3 \cdot 3) \right) = 5 \cdot 2 \cdot (-10) = -100 \checkmark$$

Sect 3.2: Properties of Determinants

Let's learn about properties of determinants. What do they tell us about the matrix A?

First, let's explore how elementary row operations affect $\det A$, i.e., how does $\det A$ relate to row equivalent matrices.

a.) If a multiple of a row of A is added to another row to produce a matrix B, then

$$\det A = \det B$$

b.) If we swap two rows, then $\det B = -\det A$

c.) If one row of A is scaled by k to produce B, then $\det B = k \cdot \det A$.

Ex Find $\det A$ for $A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

$$A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \sim \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = B$$

add mult. of r_1

swap r_2, r_3

$\det B = 1 \cdot 3 \cdot (-5) = -15$ and $\det A = -\det B = 15$ because we swapped rows once.

So, we avoided cofactor expansion by row reducing to a triangular matrix.