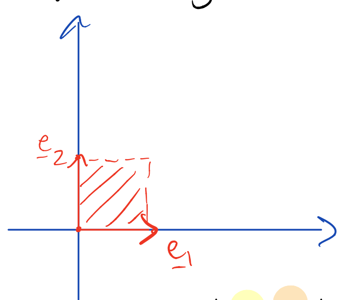
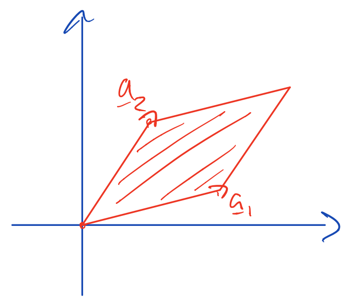
Sect 3.3 (Cont...)

Today, we look at determinants as area & volume.

In $\mathbb{R}^2$, we can view the columns of a square matrix A as two vectors determining a parallelogram

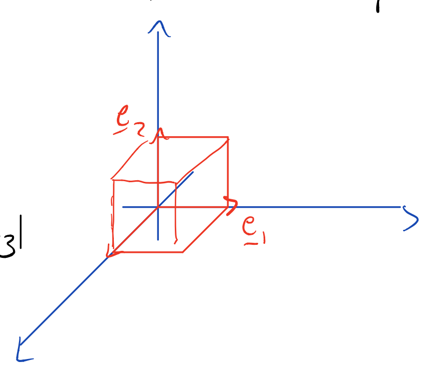




$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad A = |\underline{a}_1 \ \underline{a}_2|$$

In $\mathbb{R}^3$, the 3 vectors determine a parallelepiped

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\underline{e}_1 \ \underline{e}_2 \ \underline{e}_3|$$



In this geometric interpretation of the columns of a matrix, determinants are useful.

Theorem

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is given by

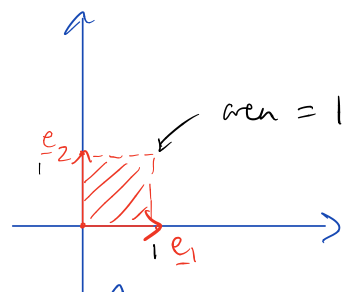
$$\text{Area} = |\det A|$$

If A is 3×3, $|\det A|$ is the volume of the parallelepiped determined by the columns of A.

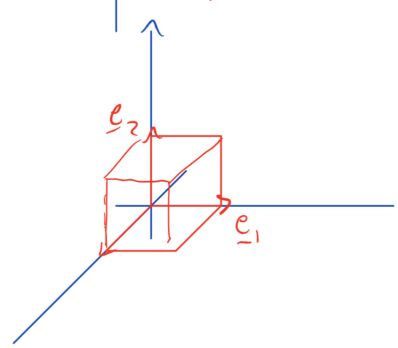
$$\text{Vol} = |\det A|$$

Ex

$$I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \Rightarrow |\det I_2| = 1$$



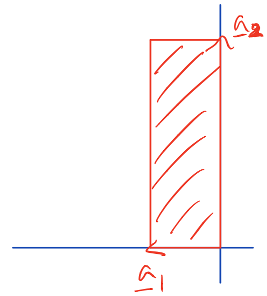
$$I_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow |\det I_3| = 1$$



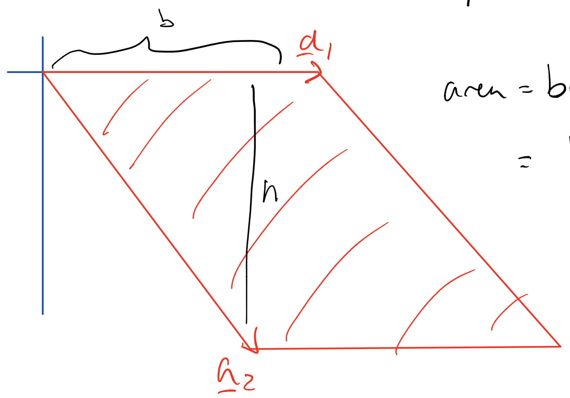
So, the identity matrix can be viewed as the unit rectangle or unit rectangular solid with area/volume in $\mathbb{R}^2/\mathbb{R}^3$.

$$A = \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} \Rightarrow |\det A| = |(-1\cdot 3 - 0)| = 3$$

we see $\text{area} = b\cdot h = 3\cdot 1 = |\det A|$



$$A = \begin{vmatrix} 4 & 3 \\ 0 & -4 \end{vmatrix} \Rightarrow |\det A| = |-4\cdot 4 - 0\cdot 3| = 16$$



$\text{area} = b\cdot h = 4\cdot 4 = 16$ ✓

$= |\det A| = 16$ ✓

Ex

What is the volume of the parallelepiped determined by the columns of A. for

$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\text{Vol} = |\det A| = \left| (-1)^{1+1}\cdot(-1)\cdot(0\cdot 1 - 1\cdot 1) + (-1)^{1+2}\cdot 0\cdot \det A_{12} + (-1)^{1+3}\cdot 0\cdot \det A_{13} \right|$$

$$= |1\cdot(-1)\cdot(0-1)| = 1$$

Effect on Area/Volume in Transformations

We can also see how linear transformations defined by a square matrix A have a geometric interpretation in terms of $\det A$

Theorem

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. transformation defined by the matrix A, then if P be a parallelogram in $\mathbb{R}^2$, then

$$\text{area}(T(P)) = |\det A| \cdot \text{area}(P)$$

change in area

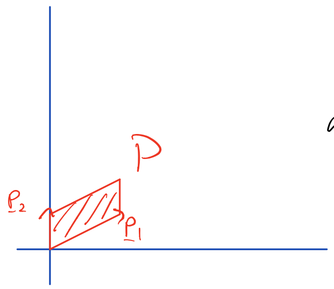
If T is determined by $A \in \mathbb{R}^{3\times 3}$ and P is a parallelepiped, then

$$\text{vol}(T(P)) = |\det A| \cdot \text{vol}(P)$$

change in volume

Ex

Consider the parallelogram defined by $P = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}$ and let T be lin. transformation determined by $A = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$. The what is the volume of $T(P)$?

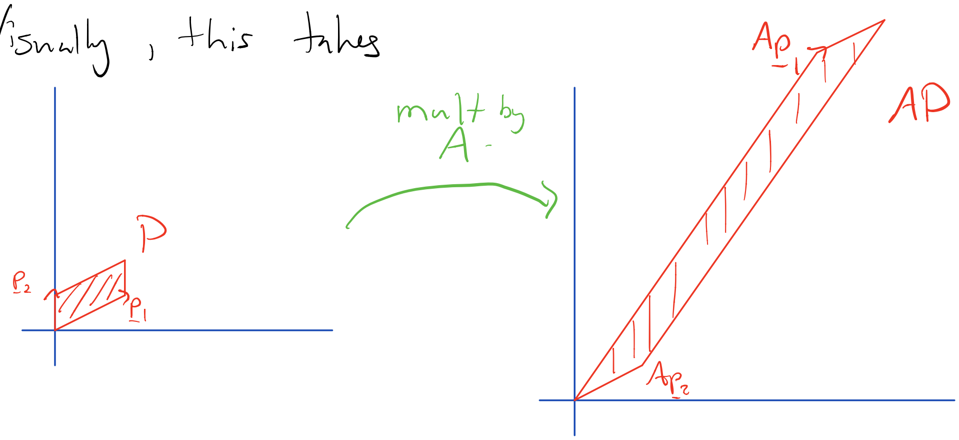


$$\text{area } P = |\det P| = |(2\cdot 1 - 0\cdot 1)| = 2$$

Transforming $T(P)$ is equivalent to multiplying P by A; so we get

$$T(P) = AP = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 1 \\ 10 & 2 \end{vmatrix}$$

Visually, this takes



$$\begin{aligned} \text{area}(AP) &= |\det A| \cdot \text{area } P \\ &= |\det A| \cdot |\det P| \\ &= |(3\cdot 2 - 4\cdot 1)| \cdot 2 \\ &= 2\cdot 2 = 4 \end{aligned}$$

This is the same as

$$\text{area}(AP) = |\det AP| = \left|\det \begin{vmatrix} 7 & 1 \\ 10 & 2 \end{vmatrix}\right| = |14 - 10| = 4$$

$\hookrightarrow$ This has important interpretations in engineering and medicine

- strain on a tissue
- deformation of a material

<u>Ex</u> Consider the parallelepiped S defined by $\underline{b}_1 = \begin{vmatrix} 1 \\ 3 \\ 0 \end{vmatrix}$, $\underline{b}_2 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ and $\underline{b}_3 = \begin{vmatrix} 2 \\ 5 \\ 0 \end{vmatrix}$, and let T be a lin. transformation defined by $A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix}$

Compute the volume of $T(S)$

$$\hookrightarrow \text{vol}(S) = \left| \det \left| \underline{b}_1\ \underline{b}_2\ \underline{b}_3 \right| \right|$$

$$= \left| \det \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 5 \\ 0 & 1 & 0 \end{vmatrix} \right|$$

$$= \left| (-1)^{3+1} \cdot 0 \cdot \det A_{31} + (-1)^{3+2} \cdot 1 \cdot \det A_{32} + (-1)^{3+3} \cdot 0 \cdot \det A_{33} \right|$$

$$= \left| -1 \cdot 1 \cdot \det \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \right| = \left| -1 \cdot (5-6) \right| = 1$$

So, $\text{vol}(S) = 1$. From the theorem, the volume of $T(S)$ is given by

$$\text{vol}\, T(S) = |\det A| \cdot \text{vol}(S)$$

$$= |\det A|$$

$$= \left| \det \left(\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} \right) \right|$$

$$= |1 \cdot (-1) \cdot 2| = 2$$

So, $\operatorname{vol} T(S) = 2$. Again, note that this is identical to

$$\operatorname{vol} AB = \left| \det\left(A \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 5 \\ 0 & 1 & 0 \end{vmatrix}\right) \right| = \left| \det\left(\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 5 \\ 0 & 1 & 0 \end{vmatrix} \right) \right|$$

$$= \left| \det\left(\begin{vmatrix} 1 & 2 & 2 \\ -3 & 0 & -5 \\ \multicolumn{3}{c}{\text{[redacted]}} \end{vmatrix} \right) \right|$$

$$= \left| (-1)^{3+1} \cdot 0 \cdot \det A_{31} + (-1)^{3+2} \cdot 2 \cdot \det A_{32} + (-1)^{3+3} \cdot 0 \cdot \det A_{33} \right|$$

$$= \left| -1 \cdot 2 \cdot \det \begin{vmatrix} 1 & 2 \\ -3 & -5 \end{vmatrix} \right| = |-1 \cdot 2 \cdot 1| = 2$$

Same answer, but this is harder because it required matrix multiplication.