Sect 4.1: Vector Spaces & Subspaces

Goal: Create the theoretical framework that underlies vector operations and transformations.

Def A vector space is a nonempty set V of vectors with 2 defined operations, addition and scalar multiplication, and subject 10 axioms (rule). These axioms must hold for all $\underline{u}, \underline{v}, \underline{w} \in V$ and scalars $c, d \in \mathbb{R}$

vector addition

1.) for $\underline{u}, \underline{v} \in V$, then $\underline{u}+\underline{v} \in V$
2.) $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
3.) $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$
4.) there exists an additive identity, the zero vector $\underline{0}$, such that $\underline{u}+\underline{0}=\underline{u}$
5.) there exists an additive inverse, $-\underline{u}$, such that $\underline{u}+(-\underline{u}) = \underline{0}$

scalar multiplication

6.) for $\underline{u} \in V$ and scalar $c \in \mathbb{R}$, $c\underline{u} \in V$
7.) $c(\underline{u}+\underline{v}) = c\underline{u}+c\underline{v}$
8.) $(c+d)\underline{u} = c\underline{u}+d\underline{u}$ for c, d scalars
9.) $c(d\underline{u}) = (cd)\underline{u}$
10.) there exists a scalar multiplicative identity $c=1$, such that $c \cdot \underline{u} = 1 \cdot \underline{u} = \underline{u}$

- All 10 axioms must hold for a vector space.

Ex

The spaces $\mathbb{R}^n$ we've used so far are all vector spaces

Ex

Let $\mathbb{P}^n$ be the set of all polynomials of degree less than or equal to n, written

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

for scalars $a_0 \cdots a_n$. $\mathbb{P}^n$ is a vector space.

Let's show this for $\mathbb{P}^2$

1.) let $u(x) = a_0 + a_1 x + a_2 x^2 \in \mathbb{P}^2$

$v(x) = b_0 + b_1 x + b_2 x^2 \in \mathbb{P}^2$

then $u(x) + v(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in \mathbb{P}^2$

2.) $u(x) + v(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

$= (b_0 + a_0) + (b_1 + a_1)x + (a_2 + b_2)x^2$

$= b_0 + b_1 x + b_2 x^2 + a_0 + a_1 x + a_2 x^2$

$= v(x) + u(x)$

3.) $\left(u(x) + v(x)\right) + w(x)$ for $w(x) = c_0 + c_1 x + c_2 x^2$

$= \left((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\right) + c_0 + c_1 x + c_2 x^2$

$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + (a_2 + b_2 + c_2)x^2$

$= a_0 + a_1 x + a_2 x^2 + \left((b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2\right)$

$= u(x) + \left(v(x) + w(x)\right)$

4.) let $\underline{0} = 0 + 0x + 0x^2$, then

$$u(x) + \underline{0} = (a_0 + 0) + (a_1 + 0)x + (a_2 + 0)x^2$$
$$= a_0 + a_1 x + a_2 x^2$$
$$= u(x)$$

5.) for $u(x) = a_0 + a_1 x + a_2 x^2$, let $-u(x) = -a_0 - a_1 x - a_2 x^2$
then $u(x) + (-u(x)) = 0 + 0x + 0x^2 = \underline{0}$

6.) Let $c \in \mathbb{R}$, then

$$cu(x) = c\left(a_0 + a_1 x + a_2 x^2\right)$$
$$= ca_0 + ca_1 x + ca_2 x^2 \in \mathbb{P}^2$$

7.) for $c \in \mathbb{R}$, $u(x), v(x) \in \mathbb{P}^2$, then

$$c\left(u(x) + v(x)\right) = c\left((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\right)$$
$$= c(a_0 + b_0) + c(a_1 + b_1)x + c(a_2 + b_2)x^2$$
$$= c\left(a_0 + a_1 x + a_2 x^2\right) + c\left(b_0 + b_1 x + b_2 x^2\right)$$
$$= cu(x) + cv(x)$$

8.) $c, d \in \mathbb{R}$, $u(x) \in \mathbb{P}^2$ then

$$(c + d)\, u(x) = (c + d)\left(a_0 + a_1 x + a_2 x^2\right)$$
$$= c\left(a_0 + a_1 x + a_2 x^2\right) + d\left(a_0 + a_1 x + a_2 x^2\right)$$
$$= c\, u(x) + d\, u(x)$$

9.) for $c, d \in \mathbb{R}$ and $u(x) \in \mathbb{P}^2$

$$c(d\,u(x)) = c(da_0 + da_1 x + da_2 x^2)$$
$$= cda_0 + cda_1 x + cda_2 x^2$$
$$= (cd)(a_0 + a_1 x + a_2 x^2)$$
$$= (cd)\,u(x)$$

10.) for $c = 1$, $u(x) \in \mathbb{P}^2$, then

$$1 \cdot u(x) = 1 \cdot (a_0 + a_1 x + a_2 x^2)$$
$$= a_0 + a_1 x + a_2 x^2$$
$$= u(x) \quad \#$$

$\Longrightarrow \mathbb{P}^2$ is a vector space.

<u>Exercise</u>: Show this for a general $\mathbb{P}^n$ (proof is nearly identical)

<u>Ex</u> Let V be the set of all real-valued functions defined on a fixed domain $\mathbb{D}$, $f: \mathbb{D} \to \mathbb{R}$, then V is a vector space.

For example, the functions $f(x) = 3$ and $g(x) = \sin(x)$ on the domain $\mathbb{D} = \mathbb{R}$ are elements of V, and

$$f(x) + g(x) = 3 + \sin(x)$$

is also defined on $\mathbb{D} = \mathbb{R}$, so their sum is in V. (property 1)

Def

A <u>subspace</u> H of a vector space V is a subset that satisfies 3 properties

1.) $\underline{0} \in V$ is in H, $\underline{0} \in H$
2.) H is closed under vector addition, i.e, for $\underline{u}, \underline{v} \in H$ then $\underline{u} + \underline{v} \in H$
3.) H is closed under scalar multiplication, i.e. for $\underline{u} \in H$ and scalar $c \in \mathbb{R}$, $c\underline{u} \in H$

↳ All the other axioms of the vector space are automatically satisfied if our subspace satisfies these 3

<u>Ex</u> The set $H = \{\underline{0}\}$ containing only the zero vector is the <u>zero subspace</u>.

<u>Ex</u> The subspace H of polynomials of degree less than or equal to n, $H = \mathbb{P}^n$, is a subspace of the vector space of real-valued functions on the Domain $\mathbb{D} = \mathbb{R}$ introduced above.

Remarks

- To show a subset is a vector space or a subspace, you must show that all the appropriate properties hold.

• To show a subset is not a subspace, show that one or more properties do not hold.

<u>Ex</u> Show that H, the set of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ for scalar s is not a subspace of $\mathbb{R}^2$

Let $\underline{u} = (3s_1, 2+5s_1) = \begin{bmatrix} 3s_1 \\ 2+5s_1 \end{bmatrix}$

$\underline{v} = (3s_2, 2+5s_2) = \begin{bmatrix} 3s_2 \\ 2+5s_2 \end{bmatrix}$

then $\underline{u}+\underline{v} = \begin{bmatrix} 3s_1 \\ 2+5s_1 \end{bmatrix} + \begin{bmatrix} 3s_2 \\ 2+5s_2 \end{bmatrix} = \begin{bmatrix} 3(s_1+s_2) \\ 4+5(s_1+s_2) \end{bmatrix}$

$= (3(s_1+s_2), 4+5(s_1+s_2))$

$\Longrightarrow$ not of the form $(3s, 2+5s)$, so $\underline{u}+\underline{v}$ not in H, so the subset not closed under vector addition and therefore not a subspace of $\mathbb{R}^2$.

<u>Exercise:</u> Show that H above is also not closed under scalar multiplication.

<u>Theorem</u>

If $\underline{v}_1 \cdots \underline{v}_p$ are p vectors in a vector space V, then span$\{\underline{v}_1 \cdots \underline{v}_p\}$ is a subspace V.

We call span$\{\underline{v}_1 \cdots \underline{v}_p\}$ the subspace spanned (or generated) by the set $\{\underline{v}_1 \cdots \underline{v}_p\}$

__Ex__

Let H be set of vectors of the form $(a-3b,\ b-a,\ a,\ b)$ for scalars $a, b \in \mathbb{R}$. Show that H is a subspace of $\mathbb{R}^4$

First, write in vector notation

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ b \\ 0 \\ b \end{bmatrix} = a \underset{\underline{v}_1}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + b \underset{\underline{v}_2}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

This shows $H = \operatorname{span}\{\underline{v}_1\ \underline{v}_2\}$, so by the theorem we know H is a subspace of $\mathbb{R}^4$

__Ex__ For what value of h will $\underline{y}$ be in $\operatorname{span}\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \quad \underline{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Solve $\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right] \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

$y \in \operatorname{span}\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ if the equation is consistent, and the system is consistent only if $h-5=0 \Rightarrow h=5$