Sect 3.2: Properties of Determinants

Recall the cofactor expansion formula for computing the determinant of an $n \times n$ matrix A.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

or

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Goal: Learn what $\det A$ tells us about the matrix A.

First, let's see what elementary row operations do to the determinant, i.e., how $\det A$ relates to the determinant of row-equivalent matrices.

Let A be square, then

a.) if a multiple of one row of A is added to another row to produce the matrix B, then

$$\det B = \det A$$

b) if two rows of A are swapped to produce B, then

$$\det B = -\det A$$

c) if one row of A is multiplied by a scalar k to produce B, then

$$\det B = k \cdot \det A$$

Ex Find $\det A$ for $A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

$$A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \sim \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = B$$

add mult of r_1 to r_2, r_3

row swap r_2 & r_3
$-\det A$

$\det B = 1 \cdot 3 \cdot (-5) = -15$, and $\det A = -\det B = 15$

So, we avoid cofactor expansion. Instead, we have to track how the row ops change $\det B$ compared to $\det A$.

Theorem

A square, $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Ex Is A invertible? $A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix}$

$$A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix} \underset{\text{add } 2r_1 \text{ to } r_3}{\sim} \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} \underset{\text{add } -r_2 \text{ to } r_3}{\sim} \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 9 \end{vmatrix} = B$$

So, $\det A = \det B$, but $\det B = 0$ using cofactor expansion along the 3rd row, and $\det A = 0 \Rightarrow A$ is not invertible.

Remark

- recall that $A\underline{x} = \underline{b}$ has a unique solution for every $\underline{b}$ if A is $n \times n$ and invertible. given by $\underline{x} = A^{-1}\underline{b}$

Theorem

If A is $n \times n$, then $\det A^T = \det A$

↳ We know $\det A$ can be computed via cofactor expansion along any or column. But, cofactor expansion along a row of A is cofactor expansion along a column of A^T, so the the determinants must agree.

Theorem

If A and B are $n \times n$, then $\det AB = \det A \cdot \det B$

Ex Verify the theorem for $A = \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix}$, $B = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}$

First, $AB = \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 25 & 20 \\ 14 & 13 \end{vmatrix}$

$\det AB = ad - bc = 25 \cdot 13 - 20 \cdot 14 = 45$

$\det A = 6 \cdot 2 - 3 \cdot 1 = 9$

$\det B = 4 \cdot 2 - 3 \cdot 1 = 5$

$\det A \cdot \det B = 45$

Let's do some examples to use the properties of the determinant ...

Ex

Use determinants to determine if $\underline{v}_1\ \underline{v}_2\ \underline{v}_3$ are linearly independent.

$$\underline{v}_1 = \begin{vmatrix} 4 \\ 6 \\ -7 \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} -7 \\ 0 \\ 2 \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} -3 \\ -5 \\ 6 \end{vmatrix}$$

We know $A = \begin{vmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{vmatrix}$ is invertible if and only if $\underline{v}_1\ \underline{v}_2\ \underline{v}_3$ are lin. ind., but A is invertible if and only if $\det A \neq 0$

$$A = \begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix},$$

$$\det A = (-1)^{1+1} 4 \cdot \det A_{11} + (-1)^{1+2}(-7) \det A_{12} + (-1)^{1+3}(-3) \det A_{13}$$

$$\det A = 1 \cdot 4 \cdot \det \begin{vmatrix} 0 & -5 \\ 2 & 6 \end{vmatrix} - 1 \cdot (-7) \det \begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + 1 \cdot (-3) \det \begin{vmatrix} 6 & 0 \\ -7 & 2 \end{vmatrix}$$

$$= 4 \cdot \left(0 - (-10) \right) + 7 \cdot (36 - 35) - 3 \cdot (12 - 0)$$

$$= 40 + 7 - 36 = 11 \neq 0$$

So $\underline{v}_1\, \underline{v}_2\, \underline{v}_3$ are lin. independent b/c $\det A \neq 0$

$\Rightarrow$ A invertible $\Rightarrow$ columns of A are lin. ind.