Sect 5.1: Eigenvalues & Eigenvectors

Eigenvectors and eigenvectors are special vectors and scalars assiatul with a square matrix A. Chapter 5 introduces them and explores their significance

Applications

- energy states of multi-body systems in physics
- Page Rank Algorithm ("Google") is driven by eigenvalues.
- Machine learning
- Resonant frequencies.

Def.

An eigenvector of an $n \times n$ matrix A is a non-zero vector $\underline{x}$ such that $A\underline{x} = \lambda \underline{x}$ for some scalar $\lambda \in \mathbb{R}$.

A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\underline{x}$ of the equation $A\underline{x} = \lambda \underline{x}$. Again, the solution(s) $\underline{x}$ are called eigenvectors corresponding to the eigenvalue λ.

Observations

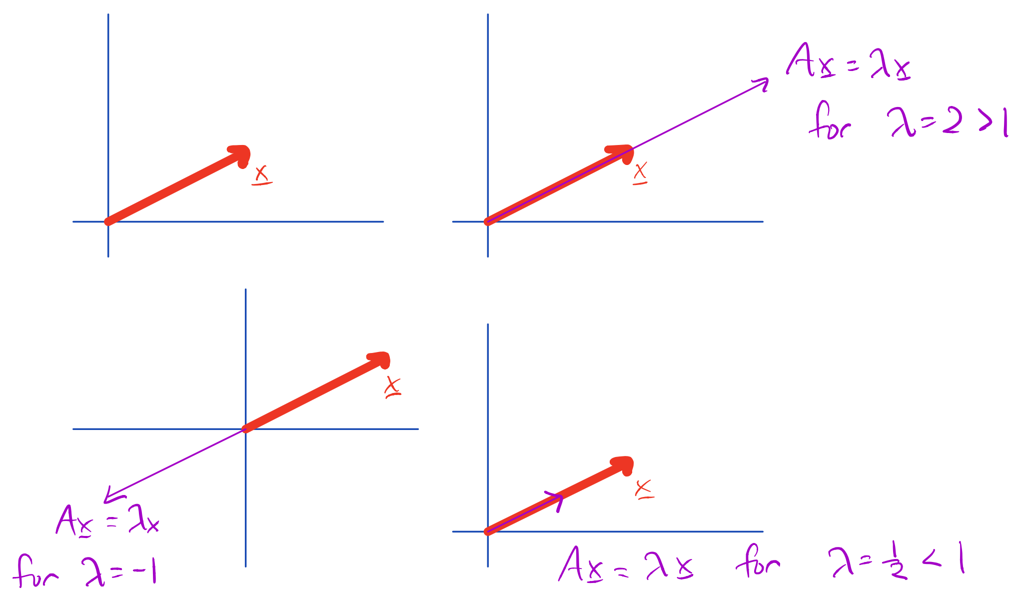
- "Eigen" comes from the German word "own". This is because eigenvectors and eigenvalues are characteristic of or belong to the matrix A.

  $\rightarrow$ Hilbert & Helmholtz.

- We'll abbreviate in the notes: "e-vector" & "e-value"

- e-vector(s) and an e-value are paired, so we often refer to an eigenpair ("e-pair") $\underline{x}$ and λ.

- The matrix A scales or dilates e-vectors by a factor λ, but it does not change their direction. The transformed vector $A\underline{x} = \lambda\underline{x}$ lies on the same line.



- if $\underline{x}$ is an e-vector with corresponding e-value λ, then the vector $c\underline{x}$ is also an eigenvector for any scalar $c \in \mathbb{R}$
  - ↳ "e-vectors are unique up to a constant"

$$A\underline{x} = \lambda \underline{x} \Rightarrow A(c\underline{x}) = \lambda(c\underline{x})$$
$$c(A\underline{x}) = c(\lambda \underline{x})$$
$$\tfrac{1}{c}c(A\underline{x}) = \tfrac{1}{c}c(\lambda \underline{x})$$
$$A\underline{x} = \lambda \underline{x}$$

<u>Ex</u> (check something is an e-vector)

Let $A = \begin{vmatrix} 1 & 6 \\ 5 & 2 \end{vmatrix}$, $\underline{u} = \begin{vmatrix} 6 \\ -5 \end{vmatrix}$, and $\underline{v} = \begin{vmatrix} 3 \\ -2 \end{vmatrix}$.

Are $\underline{u}$ and $\underline{v}$ e-vectors of A?

Evaluate $A\underline{u}$ and $A\underline{v}$

$$A\underline{u} = \begin{vmatrix} 1 & 6 \\ 5 & 2 \end{vmatrix} \begin{vmatrix} 6 \\ -5 \end{vmatrix} = \begin{vmatrix} -24 \\ 20 \end{vmatrix} = -4 \begin{vmatrix} 6 \\ -5 \end{vmatrix} = \lambda \underline{u}$$

So $\underline{u}$ is an e-vector with e-value $\lambda = -4$

$$A\underline{v} = \begin{vmatrix} 1 & 6 \\ 5 & 2 \end{vmatrix} \begin{vmatrix} 3 \\ -2 \end{vmatrix} = \begin{vmatrix} -9 \\ 11 \end{vmatrix} \neq \lambda \begin{vmatrix} 3 \\ -2 \end{vmatrix}$$

So $\underline{v}$ is not an e-vector

Ex (check something is an e-value)

Check that $\lambda = 7$ is an e-value of $A = \begin{vmatrix} 1 & 6 \\ 5 & 2 \end{vmatrix}$

We need to show $A\underline{x} = \lambda \underline{x}$ for some x.
This means we need a solution to

$$A\underline{x} = 7\underline{x}$$
$$A\underline{x} - 7\underline{x} = \underline{0}$$
$$(A - 7\underline{I})\underline{x} = \underline{0}$$

← always insert an identity matrix here

We've seen this! This is the homogeneous equation for the matrix $(A - 7I)$. So, we set up an augmented system.

First, evaluate $A - 7I$

$$A - 7\underline{I} = \begin{vmatrix} 1 & 6 \\ 5 & 2 \end{vmatrix} - 7 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} -6 & 6 \\ 5 & -5 \end{vmatrix}$$

Solve the homogeneous equations

$$\left|\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right| \sim \begin{vmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = x_2 \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

So, the e-vector is $\underline{x} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ (Remember, e-vectors are unique up to a constant, so choosing any $x_2 \in \mathbb{R}$ would work, but we chose $x_2 = 1$.)

<u>Remarks</u>

- Row reduction works to determine e-vectors, but e-values. We'll learn how to determine e-values in Sect. 5.2.

- Above, the e-vector was the solution to $(A - \lambda I)\underline{x} = \underline{0}$.

  In general, any solution to $(A - \lambda I)\underline{x} = \underline{0}$ for a fixed λ is an e-vector associated with the e-value λ. There may be more than one. In fact, the set of e-vectors corresponding to the e-value λ is the null space of $(A - \lambda I)$

  ↳ This means the set of e-vectors corresponding to an e-value is a subspace of $\mathbb{R}^n$.

<u>Def.</u>

The <u>eigenspace</u> of A corresponding to the e-value λ is the null space of $(A - \lambda I)$

$\underline{Ex}$ Let $A = \begin{vmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{vmatrix}$ and $\lambda = 2$ an e-value, then find a basis for the corresponding e-space.

Need to find a basis for the null space of $A - 2I$. To do this, solve $(A-2I)\underline{x} = \underline{0}$

$$A - 2I = \begin{vmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{vmatrix} - \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{vmatrix}$$

Now, solve $(A-2I)\underline{x} = \underline{0}$

$$\begin{vmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{vmatrix} \sim \begin{vmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

In parametric vector form, the solution is

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_2 \underbrace{\begin{vmatrix} 1/2 \\ 1 \\ 0 \end{vmatrix}}_{\underline{v}_1} + x_3 \underbrace{\begin{vmatrix} -3 \\ 0 \\ 1 \end{vmatrix}}_{\underline{v}_2}$$

Then, a basis for the null space of $(A-2I)$, and by definition a basis for the e-space of A corresponding to $\lambda = 2$, is $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$

Notes

- Any linear combination of vectors in the e-space is an e-vector

Ex $\underline{u} = 2\underline{v}_1 - \underline{v}_2$

$$= 2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

Then $A\underline{u} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 16-2-6 \\ 8+2-6 \\ 8-2-8 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -2 \end{bmatrix} = \underset{\lambda}{2}\underset{\underline{u}}{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}}$