Sect 4.6: Rank

Goal: We want to look at the properties and relationships between the rows and columns of a matrix.

Begin by recalling the transpose of a matrix

$$\underbrace{A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}}_{m \times n} \implies \underbrace{A^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{vmatrix}}_{n \times m}$$

Ex

$$B = \begin{vmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 5 \end{vmatrix} \implies B^T = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 0 & 5 \end{vmatrix}$$

3×2 $\qquad$ 2×3

Recently, we have been looking at Col A, the set of linear combinations of the columns of A. One idea is to look at the linear combinations of the rows of A, (i.e., the columns of A^T)

Def

The <u>row space</u> of a matrix A is the set of all linear combinations of the rows of A, denoted $\text{Row } A$.

↳ equivalent to the span of the rows of A

Ex

$$A = \begin{vmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{vmatrix} \in \mathbb{R}^{4\times 5}$$

$$\left.\begin{aligned} \underline{r}_1 &= \begin{vmatrix} -2 & -5 & 8 & 0 & -17 \end{vmatrix} \\ \underline{r}_2 &= \begin{vmatrix} 1 & 3 & -5 & 1 & 5 \end{vmatrix} \\ \underline{r}_3 &= \begin{vmatrix} 3 & 11 & -19 & 7 & 1 \end{vmatrix} \\ \underline{r}_4 &= \begin{vmatrix} 1 & 7 & -13 & 5 & -3 \end{vmatrix} \end{aligned}\right\} \text{rows of } A$$

$$\text{Row } A = \text{span}\{\underline{r}_1\ \underline{r}_2\ \underline{r}_3\ \underline{r}_4\}$$

$$= \left\{ \underline{x} \;\middle|\; \begin{array}{l} \underline{x} = c_1\underline{r}_1 + c_2\underline{r}_2 + c_3\underline{r}_3 + c_4\underline{r}_3 \\ \quad \text{for weights } c_1 \cdots c_4 \in \mathbb{R} \end{array} \right\}$$

can write $\underline{r}_j$ as row vectors or column vectors as needed, just be clear

Facts

- $\text{Row } A = \text{Col } A^T$ and $\text{Col } A = \text{Row } A^T$

- <u>Theorem</u>: If two matrices A and B are row equivalent, then $\text{Row } A = \text{Row } B$. If B is in echelon form, then the nonzero rows of B form a basis for the row space of both A and B.

  ↳ slightly different than $\text{Col } A$, where we had to use the columns of the original matrix A, not the reduced B.

<u>Ex</u> Find $\text{Row } A$, $\text{Col } A$, and $\text{Nul } A$ for

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Start by row reducing to find $\text{Row } A$ & $\text{Col } A$

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\text{Col } A = \text{span}\{\underline{a}_1, \underline{a}_2, \underline{a}_4\}$

$$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$$\text{Row } A = \text{span}\{\underline{r}_1 \ \underline{r}_2 \ \underline{r}_3\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 20 \end{bmatrix} \right\}$$

make sure to use the rows of the row-reduced matrix, not the original A

To get Nul A, we must solve $A\underline{x} = \underline{0}$, the homogeneous system

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 & 0 \\ 1 & 3 & -5 & 1 & 5 & 0 \\ 3 & 11 & -19 & 7 & 1 & 0 \\ 1 & 7 & -13 & 5 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 & 0 \\ 0 & 1 & -2 & 2 & -7 & 0 \\ 0 & 0 & 0 & -4 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 0 & 10 & 0 \\ 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} \Rightarrow \underline{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Nul } A = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- To check if $\underline{w}$ is in Row A, see if $A^T\underline{x} = \underline{w}$ is consistent, so solve for $\underline{x}$.

Next, we want introduce some theorems which establish important relationships between the subspaces & their dimensions. We also introduce the idea of rank.

Def

The rank of A is the dimension of the column space of A

↳ # pivot columns in the echelon form of A

↳ # basis vectors for Col A

↳ $\text{rank } A = \dim(\text{Col } A) = \dim(\text{Row} A)$

The nullity of A is the dimension of the null space of A

↳ # basis vectors for Nul A

↳ # free variables in $A\underline{x} = \underline{0}$

Rank Theorem (Rank-Nullity Theorem)

The dimensions of Col A and Row A for an $m \times n$ matrix A are equal, and equal rank A.

This dimension, rank A, satisfies the following equation:

cols in A

$$\text{equivalent}\begin{cases} \text{rank } A + \text{nullity } A = n \\ \dim(\text{Col } A) + \dim(\text{Nul } A) = n \end{cases}$$

Ex

For $A \in \mathbb{R}^{7 \times 9}$ and $\dim(\text{Nul } A) = 2$, what is rank A?

We have # columns $= n = 9$, so the relation between rank & nullity satisfies

$$\begin{aligned} \text{rank } A + \text{nullity } A &= n \\ \text{rank } A + 2 &= 9 \\ \text{rank } A &= 7 \end{aligned}$$

Ex

Can $A \in \mathbb{R}^{6 \times 9}$ have $\dim(\text{Nul } A) = 2$?

Again, # cols. in $A = n = 9$, so the rank-nullity relation is

$$\text{rank } A + \text{nullity } A = n$$

$$\text{rank } A + 2 = 9$$

$$\text{rank } A = 7$$

However, the columns of A are in $\mathbb{R}^6$, so $\dim(\text{Col } A)$ cannot exceed 6
(can't have more than 6 basis vectors for Col A because it is a subspace of $\mathbb{R}^6$)
So, $\text{rank } A \leq 6$ cannot equal 7, so the dimension of Nul A cannot be 2.

Note

- In general, $\text{rank}(A) = \dim(\text{Col } A) = \dim(\text{Nul } A)$ for $A \in \mathbb{R}^{m \times n}$ cannot exceed $\min\{m, n\}$

Lastly, we want to extend the invertible matrix theorem from Ch. 2,3.

Theorem (IMT Cont...)

Let $A \in \mathbb{R}^{n \times n}$ a square matrix, then the following statements are equivalent to A being an invertible matrix.

m.) the columns of A are a basis for $\mathbb{R}^n$

n.) $\text{Col } A = \mathbb{R}^n$ and $\text{Row } A = \mathbb{R}^n$

o.) $\text{rank } A = \dim(\text{Col } A) = n$

p.) $\text{Nul } A = \{\underline{0}\}$, the zero subspace

9.) nullity $A = \dim(\text{Nul } A) = 0$