Sect 4.5: Dimension of a Basis

Goal: In the previous lectures, we have introduced the idea of a basis, a linearly independent set of vectors spanning a subspace.

↳ a basis is the <u>smallest</u> spanning set for a subspace.

↳ a basis is the <u>largest</u> linearly independent set of vectors for a subspace.

Def:

The number of vectors, p, in a basis is known as a subspace's <u>dimension</u>.

Theorem

If $\mathcal{B} = \{ \underline{b}_1 \dots \underline{b}_p \}$ is a basis for the subspace H, then any set of more than p vectors is linearly dependent.

↳ if # vectors is greater than the dimension of the subspace, p, then the set is linearly dependent.

↳ every linearly independent set has $\leq p$ vectors.

Theorem

If $\mathcal{B}$ is a basis for H, and $\mathcal{B}$ has p vectors, then every basis for H has exactly p vectors.

↳ if we find one basis, all the others must be the same size.

Ex

$$\underline{e}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad \underline{e}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \implies \mathcal{B}_1 = \{\underline{e}_1\ \underline{e}_2\} \text{ is a basis for } \mathbb{R}^2$$

$$\underline{b}_1 = \begin{vmatrix} 1 \\ 2 \end{vmatrix} \quad \underline{b}_2 = \begin{vmatrix} -1 \\ 0 \end{vmatrix} \implies \mathcal{B}_2 = \{\underline{b}_1\ \underline{b}_2\} \text{ is a basis for } \mathbb{R}^2$$

or, the columns $\underline{a}_1\ \underline{a}_2$ of any invertible 2x2 matrix $A = |\underline{a}_1\ \underline{a}_2|$ form a basis for $\mathbb{R}^2$

$\implies$ All the bases for $\mathbb{R}^2$ have dimension 2.

Categories of Dimension

1.) $H = \{\underline{0}\}$ - the zero subspace is defined as having dimension $\dim H = 0$

2.) A subspace H with basis $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_p\}$ for some integer p is called a finite-dimensional basis. with $\dim H = p$.

Ex $\mathbb{R}^2, \mathbb{R}^3, \cdots \mathbb{R}^n$, and all the subspaces we've seen in this class.

3.) A subspace H not spanned by a finite set of vectors is <u>infinite-dimensional</u>

<u>Ex</u> The vector of all polynomials of any degree, denoted $\mathbb{P}$, is infinite-dimensional

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

(can keep going $x^{1000}, x^{10000}, x^{100000} \cdots \to \infty$)

The infinite-dimensional standard basis for $\mathbb{P}$ is

$$\mathcal{B} = \{1, x, x^2, x^3, \cdots\}$$

<u>Ex</u>

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} \text{ for } a,b,c,d \in \mathbb{R} \right\}$$

First, we separate to set

$$\begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_2} + c\underbrace{\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}}_{\underline{v}_3} + d\underbrace{\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}}_{\underline{v}_4}$$

We know $\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\ \underline{v}_4\}$ span H, but we need a lin. ind. spanning set.

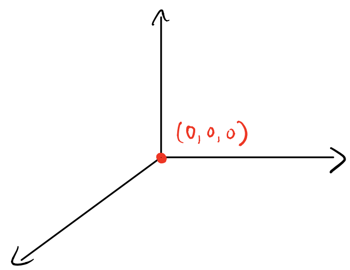
- add $\underline{v}_1 \neq \underline{0}$
- add $\underline{v}_2$ because $\underline{v}_2$ not a multiple of $\underline{v}_1$

- throw out $\underline{v}_3$ because $\underline{v}_3 = -2\underline{v}_2$
- add $\underline{v}_4$ because it is not in span $\{\underline{v}_1\ \underline{v}_2\}$

$\Rightarrow$ So, $\{\underline{v}_1\ \underline{v}_2\ \underline{v}_4\}$ is a linearly independent, spanning set, and therefore a basis.

<u>Ex</u> Different dimensional subspaces for $\mathbb{R}^3$

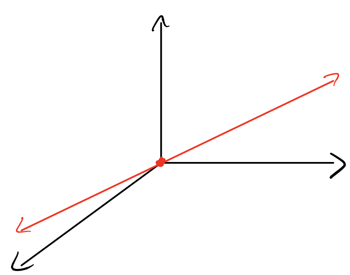
0-dimensional



Only the zero-vector

$H = \{\underline{0}\}$

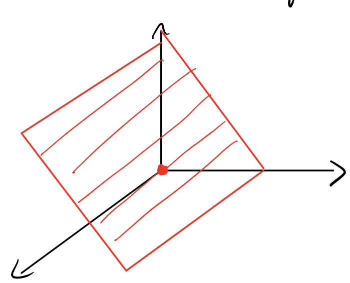
1-dimensional subspace



Any line through the origin, the span of a single vector

$H = \text{span}\{\underline{v}_1\}$

2-dimensional subspace



Any plane through the origin, span of two vectors

$H = \text{span}\{\underline{v}_1\ \underline{v}_2\}$

3-dimensional subspace All of $\mathbb{R}^3$, $H = \{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$

Theorem

Let H be a subspace of finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

↳ dim of subspace $\leq$ dim of vector space

↳ we can always add vectors to get a basis.

Theorem

Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set with exactly p vectors in V is a basis. Also, any set with exactly p vectors that spans the space is a basis.

↳ if we know $\dim V = p$, then we only need to check one criteria or the othe for a set of p vectors to be a basis.

Dimension of Nul A and Col A

Facts:

- The dimension of Nul A is the number of free variables in the system $A\underline{x} = \underline{0}$
- The dimension of Col A is the number of pivot columns in A.

Ex Find the dimensions for Nul A and Col A of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

We row reduce the augmented system

$$\left[A \;\middle|\; \underline{0} \right] = \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2 pivot columns $\Rightarrow \dim(\text{Col } A) = 2$

3 free variables $\Rightarrow \dim(\text{Nul } A) = 3$