Sect 5.3 (Cont...)

Def.

An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix, i.e., there exist invertible P and diagonal matrix D such that

$$A = P^{-1} D P$$

Thm (Diagonalization Theorem)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

The matrix $P = \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_p \end{vmatrix}$ (invertible) is formed by the eigenvectors of A.

and $D = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix}$ has the corresponding eigenvalues.

Pf. $\Rightarrow$ (diagonalizable $\Rightarrow$ e-vectors, e-values)

Let's assume $A = PDP^{-1}$ is diagonalizable.
We first that

$$A = PDP^{-1}$$
$$AP = PD(P^{-1}P)$$
$$AP = PD$$

Next, we notice that for $P = \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{vmatrix}$, then we have

$$AP = A \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{vmatrix}$$
$$= \begin{vmatrix} A\underline{v}_1 & \cdots & A\underline{v}_n \end{vmatrix}$$

We also have

$$PD = \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{vmatrix} = \begin{vmatrix} \lambda_1 \underline{v}_1 & \cdots & \lambda_n \underline{v}_n \end{vmatrix}$$

Equate the columns of AD and PD, for each column we get

$$A\underline{v}_j = \lambda_j \underline{v}_j \quad \text{for } j = 1:n$$

This shows by definition that the columns of P are eigenvectors of A corresponding to the diagonal entries of D.

$\Leftarrow$ (e-values, e-vectors $\Rightarrow$ diagonalizable)

We start with n eigenpairs $\lambda_j \ \underline{v}_j$ for $j = 1:n$ with the relationship

$$A\underline{v}_j = \lambda_j \underline{v}_j$$

Also, we assume $\{ \underline{v}_1 \cdots \underline{v}_n \}$ are lin. ind.
Set $P = \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{vmatrix}$ and define the matrix D such that

$$D = \begin{vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{vmatrix}$$

Then we see that $AP = PD$. However, by assumption $\underline{v}_1 \cdots \underline{v}_n$ are lin. ind. This means that P is invertible, so

$$AP = PD \Rightarrow P^{-1}AP = D$$

i.e, A is diagonalizable. #

Notes:

- We say $\{\underline{v}_1 \cdots \underline{v}_n\}$ are an <u>eigenvector basis</u> for $\mathbb{R}^n$
- Not every matrix has n linearly independent eigenvectors, i.e, not all square matrices are diagonalizable!

<u>Diagonalizing a Matrix</u>

How can we diagonalize a matrix?

$$A = \begin{vmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{vmatrix}$$

1.) Find the e-values of A

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\big((-5-\lambda)(1-\lambda)+9\big) - 3\big(-3(1-3)+9\big) + 3\big(-9-3(-5-\lambda)\big)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - 3(3\lambda+6) + 3(3\lambda+6)$$

$$= -\lambda^3 - 4\lambda^2 + \lambda^2 - 4\lambda + 4\lambda + 4 - 9\lambda - 18 + 9\lambda + 18$$

$$= -\lambda^3 - \lambda^2 + 4$$

Setting $=0$ to find the roots, we get

$$0 = -\lambda^3 - \lambda^2 + 4$$

$$0 = -(\lambda-1)(\lambda+2)^2 \implies \text{eigenvalues } \lambda = 1,\ \lambda = -2$$

2.) Find the eigenvectors corresponding to the e-values

For $\lambda = 1$, solve $(A-\lambda I)x = 0$. This gives

$$\left|\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right| \sim \left|\begin{array}{cccc} 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right| \sim \left|\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right|$$

$$\sim \left|\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right| \sim \left|\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

$$\underline{x} = x_3 \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix} \implies \underline{v}_1 = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$$

← single e-vector for $\lambda = 1$, multiplicity 1

For $\lambda = -2$, solve $(A - \lambda I)\underline{x} = \underline{0}$

$$\left|\begin{array}{cccc} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right| \sim \left|\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

$$\underline{x} = x_2 \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix} + x_3 \begin{vmatrix} -1 \\ 0 \\ 1 \end{vmatrix} \Rightarrow \underline{v}_2 = \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix}, \quad \underline{v}_3 = \begin{vmatrix} -1 \\ 0 \\ 1 \end{vmatrix}$$

2 e-vectors for $\lambda = -2$
multiplicity 2

So, our diagonalization is

$$D = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$P = \begin{vmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

3.) Let's check that $AP = PD \Rightarrow P^{-1}AP = D$

$$AP = \begin{vmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{vmatrix}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

=

$$PD = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

Check P^{-1} exists, i.e, $\det(P) = 0$

$\det(P) = 1(1-0) + 1(-1-0) - 1(0-1) = 1 \neq 0$ ✓

Q: When is a matrix diagonalizable?

Conditions for Diagonalizability

Thm

An $n \times n$ matrix A with n distinct eigenvalues is diagonalizable.

Pf.

If $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \neq 0$ are n distinct e-values of A, then the e-vectors $\{\underline{v}_1 \cdots \underline{v}_n\}$ are linearly independent. So, by the diagonalization theorem, A is diagonalizable

The situation gets more complicated when λ_j are not all distinct

↳ some λ_j have multiplicity > 1 in the characteristic equation

↳ if mult. > 1 for some λ_j, the matrix can still be diagonalizable

(see example above where $\lambda = -2$ had mult. 2 & the matrix was still diagonalizable).

Theorem

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1 \cdots \lambda_p$ $(p < n)$

a) For $k = 1 \cdots p$, the dimension of the e-space for λ_k is less than or equal to the multiplicity of the e-value λ_k

Ex $\det(A - \lambda I) = (\lambda-4)^4 (\lambda-2)^2 (\lambda-3) = 0$

a) says that dimension of e-space corresponding to $\lambda = 4$ is ≤ 4

b) A is diagonalizable if and only if the sum of the dimensions of the e-spaces equals n. This happens if $\lambda_1 \cdots \lambda_n$ are distinct (prev. theorem) or if the dimensions of the e-spaces for each λ_k equals the multiplicity of λ_k

Ex

Prev. diagonalization example, $\lambda_2 = -2$ had multiplicity 2 in the characteristic equation, and its e-space had dimension 2. So, the matrix was diagonalizable.

c.) If A is diagonalizable and $\mathcal{B}_k$ for $k=1:p$ are bases for the e-space of each λ_k, then the collection $\mathcal{B}_1 \cdots \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.