from V_i to V_l. Now on the one hand, if there is an edge $\{V_l, V_j\}$, then $a_{il}^{(m)} a_{lj} = a_{il}^{(m)}$ is the number of walks of length $m + 1$ from V_i to V_j of the form

$$V_i \to \cdots \to V_l \to V_j$$

On the other hand, if $\{V_l, V_j\}$ is not an edge, then there are no walks of length $m + 1$ of this form from V_i to V_j and

$$a_{il}^{(m)} a_{lj} = a_{il}^{(m)} \cdot 0 = 0$$

It follows that the total number of walks of length $m + 1$ from V_i to V_j is given by

$$a_{i1}^{(m)} a_{1j} + a_{i2}^{(m)} a_{2j} + \cdots + a_{in}^{(m)} a_{nj}$$

But this is just the (i, j) entry of A^{m+1}. ■

EXAMPLE 7 To determine the number of walks of length 3 between any two vertices of the graph in Figure 1.4.2, we need only compute

$$A^3 = \begin{Bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 & 4 \\ 0 & 4 & 4 & 4 & 2 \end{Bmatrix}$$

Thus, the number of walks of length 3 from V_3 to V_5 is $a_{35}^{(3)} = 4$. Note that the matrix A^3 is symmetric. This reflects the fact that there are the same number of walks of length 3 from V_i to V_j as there are from V_j to V_i. ■

SECTION 1.4 EXERCISES

1. Explain why each of the following algebraic rules will not work in general when the real numbers a and b are replaced by $n \times n$ matrices A and B.

(a) $(a + b)^2 = a^2 + 2ab + b^2$

(b) $(a + b)(a - b) = a^2 - b^2$

2. Will the rules in Exercise 1 work if a is replaced by an $n \times n$ matrix A and b is replaced by the $n \times n$ identity matrix I?

3. Find nonzero 2×2 matrices A and B such that $AB = O$.

4. Find nonzero matrices A, B, and C such that

$$AC = BC \qquad \text{and} \qquad A \neq B$$

5. The matrix

$$A = \begin{Bmatrix} 1 & -1 \\ 1 & -1 \end{Bmatrix}$$

has the property that $A^2 = O$. Is it possible for a nonzero symmetric 2×2 matrix to have this property? Prove your answer.

6. Prove the associative law of multiplication for 2×2 matrices; that is, let

$$A = \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix}, \quad B = \begin{Bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Bmatrix},$$

$$C = \begin{Bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{Bmatrix}$$

and show that

$$(AB)C = A(BC)$$

7. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3 . What will A^n turn out to be?

8. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3. What will A^{2n} and A^{2n+1} turn out to be?

9. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Show that $A^n = O$ for $n \geq 4$.

10. Let A and B be symmetric $n \times n$ matrices. For each of the following, determine whether the given matrix must be symmetric or could be nonsymmetric:

(a) $C = A + B$ **(b)** $D = A^2$
(c) $E = AB$ **(d)** $F = ABA$
(e) $G = AB + BA$ **(f)** $H = AB - BA$

11. Let C be a nonsymmetric $n \times n$ matrix. For each of the following, determine whether the given matrix must necessarily be symmetric or could possibly be nonsymmetric:

(a) $A = C + C^T$ **(b)** $B = C - C^T$
(c) $D = C^T C$ **(d)** $E = C^T C - CC^T$
(e) $F = (I + C)(I + C^T)$
(f) $G = (I + C)(I - C^T)$

12. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

13. Use the result from Exercise 12 to find the inverse of each of the following matrices:

(a) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

14. Let A and B are $n \times n$ matrices. Show that if

$$AB = A \quad \text{and} \quad B \neq I$$

then A must be singular.

15. Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

16. Prove that if A is nonsingular then A^T is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T$$

Hint: $(AB)^T = B^T A^T$.

17. Let A be an $n \times n$ matrix and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Show that if $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then the matrix A must be singular.

18. Let A be a nonsingular $n \times n$ matrix. Use mathematical induction to prove that A^m is nonsingular and

$$(A^m)^{-1} = (A^{-1})^m$$

for $m = 1, 2, 3, \ldots$.

19. Let A be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

20. Let A be an $n \times n$ matrix. Show that if $A^{k+1} = O$, then $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

21. Given

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

show that R is nonsingular and $R^{-1} = R^T$.

22. An $n \times n$ matrix A is said to be an *involution* if $A^2 = I$. Show that if G is any matrix of the form

$$G = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

then G is an involution.

23. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ (i.e., $\mathbf{u}^T\mathbf{u} = 1$) and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is an involution.

24. A matrix A is said to be *idempotent* if $A^2 = A$. Show that each of the following matrices are idempotent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$

5. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

6. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that

$$E_3E_2E_1A = U$$

where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1}E_2^{-1}E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

7. Let

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$$

(a) Express A^{-1} as a product of elementary matrices.

(b) Express A as a product of elementary matrices.

8. Compute the LU factorization of each of the following matrices.

(a) $\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Verify that

$$A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$.

(i) $\mathbf{b} = (1, 1, 1)^T$ **(ii)** $\mathbf{b} = (1, 2, 3)^T$

(iii) $\mathbf{b} = (-2, 1, 0)^T$

10. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & 0 \\ 9 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(f)** $\begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

11. Given

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

compute A^{-1} and use it to:

(a) Find a 2×2 matrix X such that $AX = B$.

(b) Find a 2×2 matrix Y such that $YA = B$.

12. Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

Solve each of the following matrix equations.

(a) $AX + B = C$ **(b)** $XA + B = C$

(c) $AX + B = X$ **(d)** $XA + C = X$

13. Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

14. Let U and R be $n \times n$ upper triangular matrices and set $T = UR$. Show that T is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

15. Let A be a 3×3 matrix and suppose that

$$2\mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

How many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain. Is A nonsingular? Explain.

16. Let A be a 3×3 matrix and suppose that

$$\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Is A nonsingular? Explain your answers.

Proof The proof is by induction on n. Clearly, the result holds if $n = 1$, since a 1×1 matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that A is a $(k + 1) \times (k + 1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T) \tag{9}$$

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore,

$$\det(A^T) = \det(A)$$ ■

Theorem 2.1.3 *If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A.*

Proof In view of Theorem 2.1.2, it suffices to prove the theorem for lower triangular matrices. The result follows easily using the cofactor expansion and induction on n. The details are left for the reader (see Exercise 8 at the end of the section). ■

Theorem 2.1.4 *Let A be an $n \times n$ matrix.*

(i) *If A has a row or column consisting entirely of zeros, then* $\det(A) = 0$.
(ii) *If A has two identical rows or two identical columns, then* $\det(A) = 0$.

Both of these results can be easily proved with the use of the cofactor expansion. The proofs are left for the reader (see Exercises 9 and 10). ■

In the next section we look at the effect of row operations on the value of the determinant. This will allow us to make use of Theorem 2.1.3 to derive a more efficient method for computing the value of a determinant.

SECTION 2.1 EXERCISES

1. Let

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

(a) Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22}, and A_{23}.

(c) Use your answers from part (b) to compute $\det(A)$.

2. Use determinants to determine whether the following 2×2 matrices are nonsingular:

(a) $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -6 \\ 2 & 4 \end{pmatrix}$

3. Evaluate the following determinants:

(a) $\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix}$ **(b)** $\begin{vmatrix} 5 & -2 \\ -8 & 4 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix}$ **(d)** $\begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{vmatrix}$ **(f)** $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}$

(g) $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

(h) $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix}$

4. Evaluate the following determinants by inspection:

(a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ **(d)** $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix}$

5. Evaluate the following determinant. Write your answer as a polynomial in x:

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

6. Find all values of λ for which the following determinant will equal 0:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

7. Let A be a 3×3 matrix with $a_{11} = 0$ and $a_{21} \neq 0$. Show that A is row equivalent to I if and only if

$$-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

8. Write out the details of the proof of Theorem 2.1.3.

9. Prove that if a row or a column of an $n \times n$ matrix A consists entirely of zeros, then $\det(A) = 0$.

10. Use mathematical induction to prove that if A is an $(n+1) \times (n+1)$ matrix with two identical rows, then $\det(A) = 0$.

11. Let A and B be 2×2 matrices.

(a) Does $\det(A + B) = \det(A) + \det(B)$?

(b) Does $\det(AB) = \det(A)\det(B)$?

(c) Does $\det(AB) = \det(BA)$?

Justify your answers.

12. Let A and B be 2×2 matrices and let

$$C = \begin{Bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{Bmatrix}, \quad D = \begin{Bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{Bmatrix},$$

$$E = \begin{Bmatrix} 0 & \alpha \\ \beta & 0 \end{Bmatrix}$$

(a) Show that $\det(A + B) = \det(A) + \det(B) + \det(C) + \det(D)$.

(b) Show that if $B = EA$, then $\det(A + B) = \det(A) + \det(B)$.

13. Let A be a symmetric tridiagonal matrix (i.e., A is symmetric and $a_{ij} = 0$ whenever $|i - j| > 1$). Let B be the matrix formed from A by deleting the first two rows and columns. Show that

$$\det(A) = a_{11}\det(M_{11}) - a_{12}^2\det(B)$$

2.2 Properties of Determinants

In this section we consider the effects of row operations on the determinant of a matrix. Once these effects have been established, we will prove that a matrix A is singular if and only if its determinant is zero, and we will develop a method for evaluating determinants by using row operations. Also, we will establish an important theorem about the determinant of the product of two matrices. We begin with the following lemma:

Lemma 2.2.1 *Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then*

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$