Answers

MA 304 EXAM 2 ——-spring 2019

REMARKS There are 8 problems. Problems 1-4 are each worth 12 points while problems 5-8 are each worth 13 points. Show all *relevant* work. **NO CALCULATORS, NO CELL PHONES.**

1.

a. Let S be the subspace of P_3 (quadratics) consisting of all polynomials p(x) such that p(0) = 0, and let T be the subspace of all polynomials q(x) such that q(1) = 0. Find bases for T and $S \cap T$.

$p(x) = a + bx + cx^2$

$p(0) = 0$ so $a = 0$

$q(1) = 0 = a + b + c$

so $c = -a - b$

$q(x) = a(1 - x^2) + b(x - x^2)$

$T = sp\{1 - x, x - x^2\}$

$S \cap T = sp\{x - x^2\}$

b. Given

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$$

find vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ so that $\mathbf{S}$ will be the transition matrix from $[\mathbf{v_1}, \mathbf{v_2}]$ to $[\mathbf{u_1}, \mathbf{u_2}]$.

$\{v_1, v_2\} \xrightarrow{S} \{u_1, u_2\}$

$V \searrow \{e_1, e_2\} \nearrow U^{-1}$

$S = U^{-1}V$

$U = VS^{-1} = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix}$

$u_1 \quad u_2$

2.

a. How many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space of A and the column vectors of A are linearly dependent? Explain!

b in column space $\Rightarrow \geq 1$ soln.

linear dep. of col. $\Rightarrow$ ——→ infinitely many

b. Let $\mathbf{A}$ be an $m \times n$ matrix with rank equal to n. Show that if $\mathbf{x} \neq 0$ and $\mathbf{Ax} = \mathbf{y}$, then $\mathbf{y} \neq 0$.

$A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = y$

$\vec{x} \neq 0$ and linear independence of $\{\vec{a}_j\} \Rightarrow \vec{y} \neq 0$

3. Find the kernel and range of each operator on P_3.

a. Lp(x) = $xp'(x)$

$L1 = 0$
$Lx = x$
$Lx^2 = 2x^2$

ker L = a{1}

Range = sp{x, x^2}

b. Lp(x) = p(0)x + p(1)

$L1 = x+1$ $\quad L(a+bx+cx^2) = 0$
$Lx = 1$ $\quad \Rightarrow a = 0$
$Lx^2 = 1$ $\quad b = -c$

kernel: sp{x - x^2}

range: sp{1, x}

4.

a. If $L(x_1, x_2, x_3) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)$ is a linear operator on R^3 find a matrix A such that L(**x**) = A**x** for every **x** in R^3.

$L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\3\\2\end{pmatrix}$

$L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$

$$A = \begin{array}{ccc} Le_1 & Le_2 & Le_3 \\ \end{array}\begin{pmatrix}0 & 0 & 2\\ 3 & 1 & 0\\ 2 & 0 & -1\end{pmatrix}$$

b. Find the standard matrix representation for the linear operator L that reflects each vector **x** in R^2 about the x-axis and then rotates it 90° in the counterclockwise direction.

reflection: $(x, y) \rightarrow (x, -y)$

so $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ -y\end{pmatrix}$

rotation by 90° $\begin{pmatrix}\cos 90 & -\sin 90 & 0\\ \sin 90 & \cos 90\end{pmatrix}$

$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

5.

a. Let L be the operator on P_3 defined by $L(p(x)) = xp'(x) + p''(x)$.

Find the matrix A representing L with respect to $[1, x, 1+x^2]$

$L(1) = 0$

$L(x) = x$

$L(1+x^2) = 2(1+x^2)$

$$A = \begin{array}{c|ccc} & 1 & x & 1+x^2 \\ \hline 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 \\ 1+x^2 & 0 & 0 & 2 \end{array}$$

b. Show that if A and B are similar matrices, then det(A) = det(B).

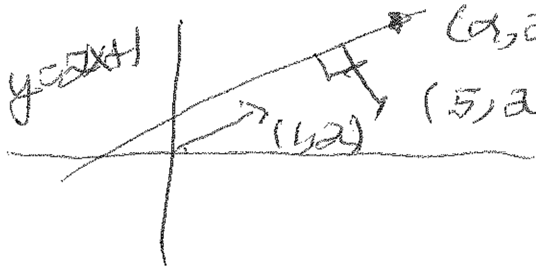
$A = S^{-1}BS$

$\det A = \det S^{-1} \det B \det S = \det B$

since $\det S^{-1} \det S = 1$

6.

a. Find the point on the line y = 2x + 1 that is closest to the point (5, 2).



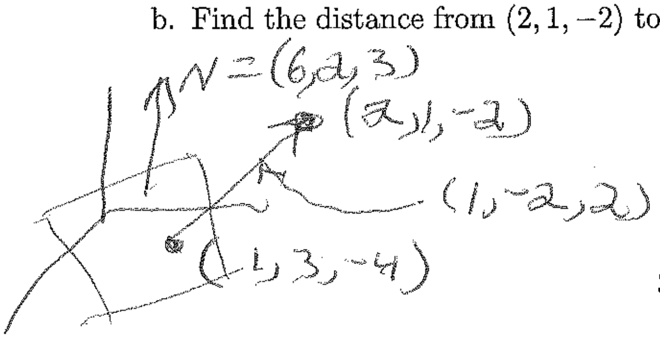
$(5-\alpha, 1-2\alpha)\cdot(1,2) = 0$

$7 = 5\alpha$

$\alpha = 7/5$

$(7/5, 19/5)$

b. Find the distance from $(2, 1, -2)$ to the plane $6(x-1) + 2(y-3) + 3(z+4) = 0$.



$d = \dfrac{|(1,-2,2)\cdot(6,2,3)|}{\sqrt{49}}$

$= 8/7$

$x^2 - 2x + 1$
$x - 1$
$x^2 - 4$

7.

a. Find the orthogonal complement of the subspace of R^3 spanned by $(1, 2, 1)$ and $(1, -1, 2)$.

$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5/3 \\ 0 & 1 & -1/3 \end{pmatrix}$

$x = -\frac{5}{3} z$
$y = \frac{1}{3} z$
$z = z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

b. Is it possible for a matrix to have the vector (3, 1, 2) in its row space and $(2, 1, 1)^T$ in its null space? Explain.

$(3,1,2)$ is in $R(A^T) = N(A)^\perp$
but $(3,1,2) \not\perp (2,1,1)$

NO

8.

a. Find the best least squares fit to the data $(-1, 0), (0, 1), (1, 3), (2, 9)$ by a quadratic po;ynomial.

$p(x) = a + bx + cx^2$
$p(-1) = a - b + c = 0$
$p(0) = a \qquad = 1$
$p(1) = a + b + c = 3$
$p(2) = a + 2b + 4c = 9$

$\underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cong \underbrace{\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}}_{b}$

$A^T A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^T b$

b. Let A be an 8×5 matrix of rank 3, and let $\mathbf{b}$ be a nonzero vector in $N(A^T)$. How many least squares solutions will the system $A\mathbf{x} = \mathbf{b}$ have? Explain!

$b \in N(A^T)$
so if $Ax \cong b$
$A^T A x = A^T b = 0$
$A^T A_{5\times 5}$ will have rank 3
so

infinitely many solns
2-d null space