Sect 6.2 (Cont...)

Recall: In the previous class we introduced the ideas of orthonormal sets, orthonormal bases, and orthogonal matrices

Def

A matrix U in $m \times n$ is orthogonal if its columns form an orthonormal set.

It followed ...

Thm

An $m \times n$ matrix U is orthogonal if and only if $U^T U = I$ $(n \times n)$

Notes

- If U is square $\overset{n \times n}{}$ in the Thm (above), then $U^T U = I \Rightarrow U^T = U^{-1}$, ie, the unique matrix such that $U^{-1} U = U U^{-1} = I$

  ↳ also implies that the rows of U are an orth. set.

Thm

Let U be an $m \times n$ orthogonal matrix, and let $\underline{x}$ and $\underline{y}$ be vectors in $\mathbb{R}^n$, then

a) $\| U\underline{x} \| = \| \underline{x} \|$ (norm invariant)

b.) $(U\underline{x}) \cdot (U\underline{y}) = \underline{x} \cdot \underline{y}$

c.) $(U\underline{x}) \cdot (U\underline{y}) = 0$ if and only if $\underline{x} \cdot \underline{y} = 0$

Pf

a) $$\|U\underline{x}\|^2 = (U\underline{x})\cdot(U\underline{x})$$
$$= (U\underline{x})^T(U\underline{x})$$
$$= \underline{x}^T U^T U \underline{x}$$
$$= \underline{x}^T(I\underline{x})$$
$$= \underline{x}^T\underline{x}$$
$$= \|\underline{x}\|^2$$

So $\|U\underline{x}\|^2 = \|\underline{x}\|^2 \iff \|U\underline{x}\| = \|\underline{x}\|$

b.-c.) Similarly, we see

$$(U\underline{x})\cdot(U\underline{y}) = (U\underline{x})^T(U\underline{y})$$
$$= \underline{x}^T U^T U \underline{y}$$
$$= \underline{x}^T(I\underline{y})$$
$$= \underline{x}^T\underline{y}$$
$$= \underline{x}\cdot\underline{y}$$

If $(U\underline{x})\cdot(U\underline{y}) = 0 \iff \underline{x}\cdot\underline{y} = 0$ for c.)

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Notes

- This means orthogonal matrices do not change the length of vectors under transformation, even when the dimension changes!

Ex

Let $\underline{U} = \begin{vmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{vmatrix} \in \mathbb{R}^{3\times 2}$ and $\underline{x} = \begin{vmatrix} \sqrt{2} \\ 3 \end{vmatrix}$

Show that $\|U\underline{x}\| = \|\underline{x}\|$ given that U has orthonormal columns.

First, $U\underline{x} = \begin{vmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{vmatrix} \begin{vmatrix} \sqrt{2} \\ 3 \end{vmatrix} = \begin{vmatrix} \sqrt{2}/\sqrt{2} + 2 \\ \sqrt{2}/\sqrt{2} - 2 \\ 0 + 1 \end{vmatrix} = \begin{vmatrix} 3 \\ -1 \\ 1 \end{vmatrix}$

$$\|Ux\| = \sqrt{3^2 + (-1)^2 + (1)^2} = \sqrt{11}$$

$$\|\underline{x}\| = \sqrt{(\sqrt{2})^2 + (3)^2} = \sqrt{11} \checkmark$$

Ex

Last class, we showed $\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ were orthogonal where

$$\underline{v}_1 = \begin{vmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{vmatrix}$$

The matrix $U = \begin{vmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{vmatrix} \in \mathbb{R}^{3\times 3}$, so $U^T = U^{-1}$ and the rows of U must also be orthogonal. Lets show this.

$$U = \begin{vmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{vmatrix}$$

For rows $\underline{u}_i$,

$$\underline{u}_1 \cdot \underline{u}_2 = \begin{vmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \end{vmatrix} \begin{vmatrix} 1/\sqrt{11} \\ 2/\sqrt{6} \\ -4/\sqrt{66} \end{vmatrix} = 3/11 - 2/6 + 4/66 = 18/66 - 22/66 + 4/66 = 0$$

$$\underline{u}_1 \cdot \underline{u}_3 = \begin{vmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \end{vmatrix} \begin{vmatrix} 1/\sqrt{11} \\ 1/\sqrt{6} \\ 7/\sqrt{66} \end{vmatrix} = 3/11 - 1/6 - 7/66 = 18/66 - 11/66 - 7/66 = 0$$

$$\underline{u}_2 \cdot \underline{u}_3 = \begin{vmatrix} 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \end{vmatrix} \begin{vmatrix} 1/\sqrt{11} \\ 1/\sqrt{6} \\ 7/\sqrt{66} \end{vmatrix} = 1/11 + 2/6 - 28/66 = 6/66 + 22/66 - 28/66 = 0$$

This shows the rows are orthogonal. Check yourself that they are unit vectors, $\underline{u}_1^T \underline{u}_1 = \underline{u}_2^T \underline{u}_2 = \underline{u}_3^T \underline{u}_3 = 1$

Sect 6.3: Orthogonal Projections

Consider an orthonormal basis $\{\underline{u}_1 \cdots \underline{u}_n\}$ for $\mathbb{R}^n$. Then, any vector $\underline{y}$ in $\mathbb{R}^n$ can be written as

$$\underline{y} = c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_n \underline{u}_n$$

for weights $c_1 \cdots c_n$. Consider splitting $\underline{y}$ into two pieces

$$\underline{y} = \underbrace{(c_1 \underline{u}_1 + \cdots + c_j \underline{u}_j)} + \underbrace{(c_{j+1} \underline{u}_{j+1} + \cdots + c_n \underline{u}_n)}$$

$$\underline{y} = \underline{z}_1 + \underline{z}_2$$

where $\underline{z}_1$ in $W = \text{span}\{\underline{u}_1 \cdots \underline{u}_j\}$

z_2 in $W^\perp = \text{span}\{\underline{u}_{j+1} \cdots \underline{u}_n\}$

↳ j is arbitrary

↳ can split the bases u_j in non-numerical order

We can see that $\underline{z}_1$ and $\underline{z}_2$ are orthogonal
This nice, but what if we only one set, i.e, we don't have an orthonormal basis for all of $\mathbb{R}^n$. What if we only have a basis for $W = \text{span}\{\underline{u}_1 \cdots \underline{u}_j\}$

__Thm__

Let W be a subspace of $\mathbb{R}^n$. Then, each $\underline{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\underline{y} = \underline{\hat{y}} + \underline{z}$$

where $\hat{y}$ is in W and $\underline{z}$ is in $W^\perp$.
If $\{\underline{u}_1 \cdots \underline{u}_p\}$ is a orthogonal basis for W, then

$$\hat{y} = \left(\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 + \cdots + \left(\frac{\underline{y} \cdot \underline{u}_p}{\underline{u}_p \cdot \underline{u}_p}\right)\underline{u}_p$$

and $\underline{z} = \underline{y} - \underline{\hat{y}}$

Notes

- If $\{\underline{u}_1 \cdots \underline{u}_p\}$ is an orthonormal basis, then $\underline{u}_1 \cdot \underline{u}_1 = \cdots = \underline{u}_p \cdot \underline{u}_p = 1$ and

$$\hat{\underline{y}} = (\underline{y} \cdot \underline{u}_1)\underline{u}_1 + (\underline{y} \cdot \underline{u}_2)\underline{u}_2 + \cdots + (\underline{y} \cdot \underline{u}_p)\underline{u}_p$$

- $\hat{\underline{y}}$ is the orthogonal projection of $\underline{y}$ onto the subspace W, written $\hat{\underline{y}} = \text{proj}_W(\underline{y})$

Ex

Let $\underline{u}_1 = \begin{vmatrix} 2 \\ 5 \\ -1 \end{vmatrix}$, $\underline{u}_2 = \begin{vmatrix} -2 \\ 1 \\ 1 \end{vmatrix}$ and $\underline{y} = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}$

and $W = \text{span}\{\underline{u}_1, \underline{u}_2\}$. Find $\hat{\underline{y}} = \text{proj}_W(\underline{y})$ and write $\underline{y} = \hat{\underline{y}} + \underline{z}$ where $\underline{z}$ is in $W^\perp$

$$\hat{\underline{y}} = \left(\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 + \left(\frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\right)\underline{u}_2 = \frac{9}{30}\begin{vmatrix} 2 \\ 5 \\ -1 \end{vmatrix} + \frac{3}{6}\begin{vmatrix} -2 \\ 1 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 18/30 \\ 45/30 \\ -9/30 \end{vmatrix} + \begin{vmatrix} -30/30 \\ 15/30 \\ 15/30 \end{vmatrix} = \begin{vmatrix} -2/5 \\ 2 \\ 1/5 \end{vmatrix}$$

Subtracting to get $\underline{z}$

$$\underline{z} = \underline{y} - \hat{\underline{y}} = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} - \begin{vmatrix} -2/5 \\ 2 \\ 1/5 \end{vmatrix} = \begin{vmatrix} 7/5 \\ 0 \\ 14/5 \end{vmatrix}$$

Check that $\underline{z}$ is orthogonal to $\hat{\underline{y}}$ (do this yourself!)

Thm

Let W be a subspace of $\mathbb{R}^n$, let $\underline{y}$ be a vector in $\mathbb{R}^n$, and let $\underline{\hat{y}} = \text{proj}_W(\underline{y})$, then $\underline{\hat{y}}$ is the closest point in W to $\underline{y}$ in the sense

$$\| \underline{y} - \underline{\hat{y}} \| < \| \underline{y} - \underline{v} \|$$

for all $\underline{v}$ in W where $\underline{v} \neq \underline{\hat{y}}$.

Pf: next class...