Answers

MA 304 EXAM 1 ——-spring 2019

REMARKS There are 8 problems. Problems 1-4 are each worth 12 points while problems 5-8 are each worth 13 points. Show all *relevant* work. **NO CALCULATORS.**

1. Given the following matrix equation

$$w + 2x - 3y + z = 1$$
$$-w - x + 4y - z = 6$$
$$-2w - 4x + 7y - z = 1$$

a. Put the above system into reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 2 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right)$$

b. Write an expression for **all** solutions to the matrix equation and write it as an equation of some plane, i.e., $(w, x, y, z)^T =$

$$w = 7 - 6z$$
$$x = 4 + z$$
$$y = 3 - z$$
$$z = z$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -6 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

a. Write **b** as a linear combination of the columns of **A**.

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

b. Use the results from a. to determine a solution of the linear system $\mathbf{Ax} = \mathbf{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

c. Let A be a 5×3 matrix. If a_1, a_2, a_3 represent the columns of A and

$$b = a_1 + a_2 = a_2 + a_3$$

then what can you conclude about the number of solutions of the linear system $Ax = b$? Explain!

∞ many

$$A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = b$$

so 2 solns $\Rightarrow$ ∞ many

3. Consider the 3×3 Vandermonde matrix:

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^3 \end{pmatrix} 2$$

8 a. Show that $det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

3rd row expansion $\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} - x_3 \begin{vmatrix} 1 & x_1^2 \\ 1 & x_2^2 \end{vmatrix} + x_3^2 \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$

$= x_1 x_2 (x_2 - x_1) - x_3 (x_2 + x_1)(x_2 - x_1) + x_3^2 (x_2 - x_1)$

$= (x_2 - x_1)[(x_3 - x_2)(x_3 - x_1)]$

4 b. What conditions must the scalars x_1, x_2, x_3 satisfy for V to be nonsingular?

x_1, x_2, x_3 all destinct

4. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

Find A^{-1} by computing the reduced row echelon form of A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 4 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 3/4 & 1/4 & -3/4 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

5.

a. Compute the LU factorization of

$$A = \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 2 & 4 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$\to \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}}_{U}$$

b. Let

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Find a 2×2 matrix X which solves AX + B = X.

$$\frac{1}{5}\begin{pmatrix} -8 & -10 \\ 9 & 10 \end{pmatrix}$$

$AX - X = -B$

$(A - I)X = -B$

$X = -(A - I)^{-1} B = -\left[\frac{1}{5}\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right] =$

$\det A = 0$ $\qquad 2(2)-2(2)=0$

6.

a. Is the following set a spanning set for R^3. Justify your answer.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{pmatrix}$$

$$\{(2,1,-2),(3,2,-2),(2,2,0)\}$$

$$\rightarrow \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ No rows not L.I.

b. Is the following a spanning set for P_3 (quadratic polynomials).

$x^2 \quad x \quad 1$

$$\{x+2, x+1, x^2-1\}$$

$$B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad |B| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

B invertible so columns L.I.

YES

7.

a. Use Cramer's rule to solve the following system

$$2x + 3y = 2$$
$$3x + 2y = 5$$

$$x = \frac{\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}} = \frac{-11}{-5}$$

$$\frac{\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}}{-5} = \frac{4}{-5}$$

$$\underline{x = \frac{11}{5}, \; y = -\frac{4}{5}}$$

b. Evaluate the determinant of the following matrix. Write you answer as a polynomial in x.

$$A = \begin{pmatrix} a+x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{pmatrix}$$

$(a-x)(x^2) - (-bx-c)$ $\qquad \underline{-x^3 + ax^2 + bx + c}$

8.

a. Let $\mathbf{x_1}, \mathbf{x_2}$ and $\mathbf{x_3}$ be linearly independent vectors in R^n and let

$$\mathbf{y_1} = \mathbf{x_2} + \mathbf{x_1}, \quad \mathbf{y_2} = \mathbf{x_3} + \mathbf{x_2}, \quad \mathbf{y_3} = \mathbf{x_3} + \mathbf{x_1}$$

Are $\mathbf{y_1}, \mathbf{y_2}$ and $\mathbf{y_3}$ linearly independent? Prove your answer.

$a y_1 + b y_2 + c y_3 = 0$

$a(x_2 + x_1) + b(x_3 + x_2) + c(x_3 + x_1) = 0$

$(b+c)x_3 + (a+b)x_2 + (a+c)x_1 = 0$

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2 \neq 0$ so L.I.

b. Let $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_k}$ be linearly independent vector in R^n, and let A be a nonsingular $n \times n$ matrix. Define $\mathbf{y_j} = \mathbf{A}\mathbf{x_j}$ for $j = 1, \cdots, k$. Show that $\mathbf{y_1}, \mathbf{y_2}, \cdots, \mathbf{y_k}$ are linearly independent.

$a_1 y_1 + \cdots + a_k y_k = 0$

so $a_1 A x_1 + \cdots + a_k A x_k = 0$

so $A(a_1 x_1 + \cdots + a_k x_k) = 0$

implies $a_1 x_1 + \cdots + a_k x_k = 0$ since A^{-1} exists

implies $a_1 = a_2 = \cdots = a_k = 0$

since $\{x_1, \cdots, x_k\}$ is L.I.