Sect 6.1 (Cont...)

Recall: We were looking at the ideas of length, distance and perpendicularity in $\mathbb{R}^n$

↳ All based on the inner product / dot product

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

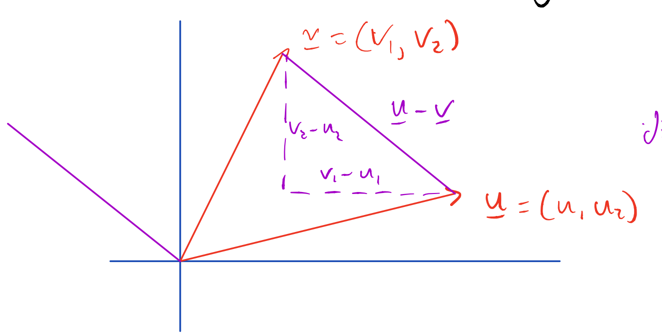
$$= \sum_{k=1}^{n} u_k v_k$$

We defined the length / norm of a vector by

$$\| \underline{u} \| = \sqrt{\underline{u} \cdot \underline{u}}$$

Today, we want to start by looking at ways to define the distance from one vector to another in $\mathbb{R}^n$.

To start, a drawing in 2D



$$\text{dist}(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \|$$

Notes on distance

- $\text{dist}(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\| = \|\underline{v}-\underline{u}\| = \text{dist}(\underline{v},\underline{u})$
- $\text{dist}(\underline{u},\underline{v}) \geq 0$, just like length/norm
- evaluated in $\mathbb{R}^n$, we get

$$\text{dist}(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\| = \sqrt{(\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})}$$

$$= \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

<u>Ex</u> Find $\text{dist}(\underline{u},\underline{v})$ for $\underline{u} = \begin{vmatrix} 5 \\ 2 \\ -1 \\ 3 \end{vmatrix}$, $\underline{v} = \begin{vmatrix} 3 \\ -1 \\ 2 \\ 0 \end{vmatrix}$

$$\text{dist}(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\| = \sqrt{(5-3)^2 + (2+1)^2 + (-1-2)^2 + (3-0)^2}$$

$$= \sqrt{4+9+9+9}$$

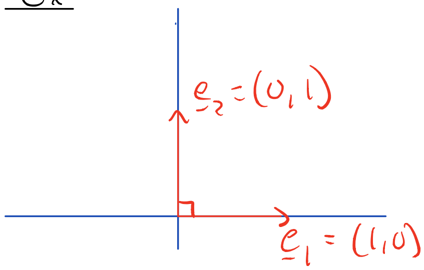
$$= \sqrt{31}$$

<u>Orthogonality</u>

Orthogonal is the extension of "perpendicular" to n-dimensions.

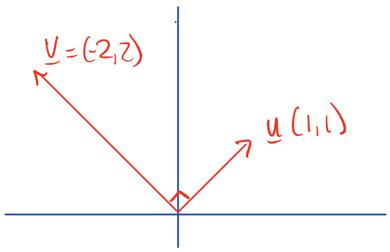
<u>Def</u> Two vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^n$ are <u>orthogonal</u> if $\underline{u}\cdot\underline{v} = 0$. The zero vector, $\underline{0}$, is orthogonal to every $\underline{u}$ in $\mathbb{R}^n$

Ex



$$\underline{e}_1 \cdot \underline{e}_2 = \begin{vmatrix} 1 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0$$

$\Rightarrow$ $\underline{e}_1, \underline{e}_2$ are orthogonal



$$\underline{u} \cdot \underline{v} = \begin{vmatrix} 1 & 1 \end{vmatrix} \begin{vmatrix} -2 \\ 2 \end{vmatrix} = 1 \cdot (-2) + 1 \cdot 2 = 0$$

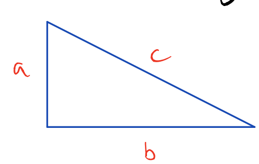
$\Rightarrow$ $\underline{u}, \underline{v}$ are orthogonal

Also, in physics, the normal vector to a plane or surface is orthogonal to the plane or surface.

Thm (Pythagorean Theorem)

Two vectors $\underline{u}$ and $\underline{v}$ are orthogonal if and only if $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$

$\hookrightarrow$ compare this to the 2D version for a right triangle



$$c^2 = a^2 + b^2$$

$\underline{\text{Pf}}$ $\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$

$$= \underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$= \|\underline{u}\|^2 + \underbrace{2\underline{u}\cdot\underline{v}}_{=0} + \|\underline{v}\|^2$$

$= 0$ b/c $\underline{u}\cdot\underline{v} = 0$ by orthogonality

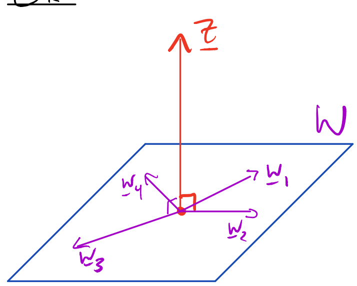
$$\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 \quad \checkmark$$

Orthogonal Complements

We can extend the idea of orthogonality to sets and subspaces

- a vector $\underline{z}$ is orthogonal to a subspace W if $\underline{z}\cdot\underline{w} = 0$ for every vector $\underline{w}$ in W.

$\underline{\text{Ex}}$



For the plane W (a subspace), every vector $\underline{w}$ on the plane is orthogonal to $\underline{z}$

- the set of all vectors $\underline{z}$ orthogonal to a subspace W, it is called the orthogonal complement of W, denoted $W^{\perp}$

Ex (algebraic)

Let $W = \left\{ \underline{w} = \begin{vmatrix} a \\ b \\ 0 \\ 0 \end{vmatrix} \text{ for } a, b \in \mathbb{R} \right\}$

$W^{\perp} = \left\{ \underline{z} = \begin{vmatrix} 0 \\ 0 \\ c \\ d \end{vmatrix} \text{ for } c, d \in \mathbb{R} \right\}$

Note that for any $\underline{w} \in W$ and $\underline{z} \in W^{\perp}$, we get

$$\underline{w} \cdot \underline{z} = \begin{vmatrix} a & b & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ c \\ d \end{vmatrix} = a \cdot 0 + b \cdot 0 + 0 \cdot c + 0 \cdot d = 0$$

Observations

- A vector $\underline{x}$ is in $W^{\perp}$ if and only if $\underline{x}$ is orthogonal to every vector in a set that spans W.

↳ consider a basis $\mathcal{B} = \{ \underline{v}_1 \cdots \underline{v}_p \}$ for a subspace W. Since $\mathcal{B}$ is a basis, $\underline{v}_1 \cdots \underline{v}_p$ span W. If $\underline{x} \cdot \underline{v}_k = 0$ for all $k = 1 \cdots p$, then $\underline{x}$ is in $W^{\perp}$, i.e., it is orthogonal to W

Pf Let $\underline{x} \in W^{\perp}$ and let $\{ \underline{v}_1 \cdots \underline{v}_p \}$ be a set such that $W = \text{span}\{ \underline{v}_1, \cdots \underline{v}_p \}$

Then, for $\underline{w}$ in W, we can write

$$\underline{w} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_p \underline{v}_p$$

for some weights $a_1 \cdots a_p$. Then we see

$$\begin{aligned} \underline{x} \cdot \underline{w} &= \underline{x} \cdot (a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_p \underline{v}_p) \\ &= \underline{x} \cdot (a_1 \underline{v}_1) + \underline{x} \cdot (a_2 \underline{v}_2) + \cdots + \underline{x} \cdot (a_p \underline{v}_p) \\ &= a_1 (\underline{x} \cdot \underline{v}_1) + a_2 (\underline{x} \cdot \underline{v}_2) + \cdots + a_p (\underline{x} \cdot \underline{v}_p) \\ &= 0 \end{aligned}$$

if and only if $\underline{x} \cdot \underline{v}_k = 0$ for all $k = 1 \cdots p$

- The orthogonal complement, $W^\perp$, is a subspace of $\mathbb{R}^n$.

$\underline{\text{Pf}}$ 1.) Let $\underline{w} \in W$, then $\underline{0} \cdot \underline{w} = \underline{0}$, so by definition $\underline{0} \in W^\perp$

2.) (closed under add.) Let $\underline{u} \in W^\perp$ and $\underline{v} \in W^\perp$ we need to show $\underline{u} + \underline{v} \in W^\perp$

Let $\underline{w}$ be any vector in W, we get that

$$(\underline{u} + \underline{v}) \cdot \underline{w} = \underbrace{\underline{u} \cdot \underline{w}}_{\substack{=0 \text{ b/c} \\ \underline{u} \in W^\perp}} + \underbrace{\underline{v} \cdot \underline{w}}_{\substack{=0 \text{ b/c} \\ \underline{v} \in W^\perp}} = 0$$

So, $(\underline{u} + \underline{v}) \cdot \underline{w} = 0 \implies \underline{u} + \underline{v}$ is in $W^\perp$

3.) (closed under scalar mult.)

Let $\underline{u} \in W^{\perp}$, let $c \in \mathbb{R}$ be a scalar and consider $\underline{w}$ any vector in W, then we get

$$(c\underline{u}) \cdot \underline{w} = c\,\underbrace{(\underline{u} \cdot \underline{w})}_{=0 \text{ b/c } \underline{u} \in W^{\perp}} = c \cdot 0 = 0$$

So, $c\underline{u}$ is orthogonal to all $\underline{w}$ in W, so $c\underline{u}$ is in $W^{\perp}$.

$W^{\perp}$ contains $\underline{0}$, and is closed under vector addition and scalar mult., so it is a subspace.