SECTION 2.2 EXERCISES

1. Evaluate each of the following determinants by inspection.

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

2. Let

$$A = \begin{Bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{Bmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.

(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

3. For each of the following, compute the determinant and state whether the matrix is singular or nonsingular:

(a) $\begin{Bmatrix} 3 & 1 \\ 6 & 2 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 3 & 1 \\ 4 & 2 \end{Bmatrix}$

(c) $\begin{Bmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{Bmatrix}$ **(d)** $\begin{Bmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{Bmatrix}$

(e) $\begin{Bmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{Bmatrix}$

(f) $\begin{Bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{Bmatrix}$

4. Find all possible choices of c that would make the following matrix singular:

$$\begin{Bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{Bmatrix}$$

5. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

(a) $\det(AB)$ **(b)** $\det(3A)$

(c) $\det(2AB)$ **(d)** $\det(A^{-1}B)$

8. Show that if E is an elementary matrix, then E^T is an elementary matrix of the same type as E.

9. Let E_1, E_2, and E_3 be 3×3 elementary matrices of types I, II, and III, respectively, and let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from I by multiplying its second row by 3. Find the values of each of the following:

(a) $\det(E_1A)$ **(b)** $\det(E_2A)$

(c) $\det(E_3A)$ **(d)** $\det(AE_1)$

(e) $\det(E_1^2)$ **(f)** $\det(E_1E_2E_3)$

10. Let A and B be row equivalent matrices, and suppose that B can be obtained from A by using only row operations I and III. How do the values of $\det(A)$ and $\det(B)$ compare? How will the values compare if B can be obtained from A using only row operation III? Explain your answers.

11. Let A be an $n \times n$ matrix. Is it possible for $A^2 + I = O$ in the case where n is odd? Answer the same question in the case where n is even.

12. Consider the 3×3 Vandermonde matrix

$$V = \begin{Bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{Bmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. *Hint*: Make use of row operation III.

(b) What conditions must the scalars x_1, x_2, and x_3 satisfy in order for V to be nonsingular?

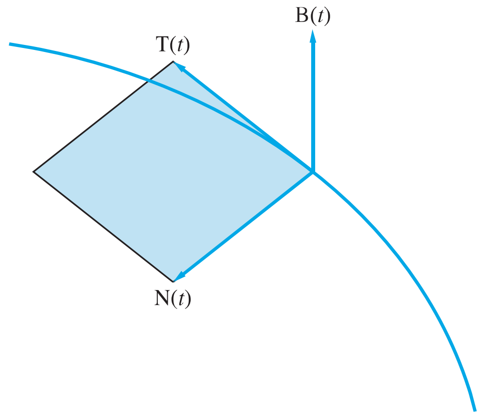

Figure 2.3.3.

The vector $\mathbf{B}(t)$ defined by (5) is called the *binormal* vector (see Figure 2.3.3).

SECTION 2.3 EXERCISES

1. For each of the following, compute (i) det(A), (ii) adj A, and (iii) A^{-1}:

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2. Use Cramer's rule to solve each of the following systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 - x_2 &= 1 \end{aligned}$ **(b)** $\begin{aligned} 2x_1 + 3x_2 &= 2 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$

(c) $\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ 4x_1 + 5x_2 + x_3 &= 8 \\ -2x_1 - x_2 + 4x_3 &= 2 \end{aligned}$

(d) $\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned}$

(e) $\begin{aligned} x_1 + x_2 \qquad\qquad &= 0 \\ x_2 + x_3 - 2x_4 &= 1 \\ x_1 \quad + 2x_3 + x_4 &= 0 \\ x_1 + x_2 \quad + x_4 &= 0 \end{aligned}$

3. Given

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

determine the $(2, 3)$ entry of A^{-1} by computing a quotient of two determinants.

4. Let A be the matrix in Exercise 3. Compute the third column of A^{-1} by using Cramer's rule to solve $A\mathbf{x} = \mathbf{e}_3$.

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(a) Compute the determinant of A. Is A nonsingular?

(b) Compute adj A and the product A adj A.

6. If A is singular, what can you say about the product A adj A?

7. Let B_j denote the matrix obtained by replacing the jth column of the identity matrix with a vector $\mathbf{b} = (b_1, \ldots, b_n)^T$. Use Cramer's rule to show that

$$b_j = \det(B_j) \qquad \text{for} \quad j = 1, \ldots, n$$

8. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that

$$\det(\operatorname{adj} A) = (\det(A))^{n-1}$$

9. Let A be a 4×4 matrix. If

$$\operatorname{adj} A = \begin{Bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{Bmatrix}$$

(a) calculate the value of $\det(\operatorname{adj} A)$. What should the value of $\det(A)$ be? *Hint*: Use the result from Exercise 8.

(b) find A.

10. Show that if A is nonsingular, then $\operatorname{adj} A$ is nonsingular and

$$(\operatorname{adj} A)^{-1} = \det(A^{-1})A = \operatorname{adj} A^{-1}$$

11. Show that if A is singular, then $\operatorname{adj} A$ is also singular.

12. Show that if $\det(A) = 1$, then

$$\operatorname{adj}(\operatorname{adj} A) = A$$

13. Suppose that Q is a matrix with the property $Q^{-1} = Q^T$. Show that

$$q_{ij} = \frac{Q_{ij}}{\det(Q)}$$

14. In coding a message, a blank space was represented by 0, an A by 1, a B by 2, a C by 3, and so on. The message was transformed using the matrix

$$A = \begin{Bmatrix} -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{Bmatrix}$$

and sent as

$$-19, 19, 25, -21, 0, 18, -18, 15, 3, 10,$$
$$-8, 3, -2, 20, -7, 12$$

What was the message?

15. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Show each of the following:

(a) $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ **(b)** $\mathbf{y} \times \mathbf{x} = -(\mathbf{x} \times \mathbf{y})$

(c) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$

(d) $\mathbf{z}^T(\mathbf{x} \times \mathbf{y}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

16. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$ and define the skew-symmetric matrix A_x by

$$A_x = \begin{Bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{Bmatrix}$$

(a) Show that $\mathbf{x} \times \mathbf{y} = A_x\mathbf{y}$.

(b) Show that $\mathbf{y} \times \mathbf{x} = A_x^T\mathbf{y}$.

Chapter Two Exercises

MATLAB EXERCISES

The first four exercises that follow involve integer matrices and illustrate some of the properties of determinants that were covered in this chapter. The last two exercises illustrate some of the differences that may arise when we work with determinants in floating-point arithmetic.

In theory, the value of the determinant should tell us whether the matrix is nonsingular. However, if the matrix is singular and its determinant is computed using finite-precision arithmetic, then, because of roundoff errors, the computed value of the determinant may not equal zero. A computed value near zero does not necessarily mean that the matrix is singular or even close to being singular. Furthermore, a matrix may be nearly singular and have a determinant that is not even close to zero (see Exercise 6).

1. Generate random 5×5 matrices with integer entries by setting

$$A = \mathtt{round}(10 * \mathtt{rand}(5))$$

and

$$B = \mathtt{round}(20 * \mathtt{rand}(5)) - 10$$

where c_1 and c_2 are arbitrary scalars. The solution set S corresponds to a plane in 3-space that does not pass through the origin. See Figure 3.2.3.

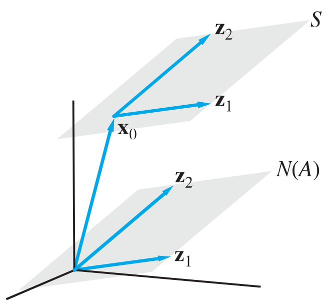

Figure 3.2.3.

SECTION 3.2 EXERCISES

1. Determine whether the following sets form subspaces of $\mathbb{R}^2$:

(a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$

(b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$

(c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$

(d) $\{(x_1, x_2)^T \mid |x_1| = |x_2|\}$

(e) $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

(c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$

(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(g) The set of all singular 2×2 matrices

4. Determine the null space of each of the following matrices:

(a) $\begin{Bmatrix} 2 & 1 \\ 3 & 2 \end{Bmatrix}$

(b) $\begin{Bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{Bmatrix}$

(c) $\begin{Bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{Bmatrix}$

(d) $\begin{Bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{Bmatrix}$

5. Determine whether the following are subspaces of P_4 (be careful!):

(a) The set of polynomials in P_4 of even degree

(b) The set of all polynomials of degree 3

(c) The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$

(d) The set of all polynomials in P_4 having at least one real root

6. Determine whether the following are subspaces of $C[-1, 1]$:

(a) The set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$

(b) The set of odd functions in $C[-1, 1]$

(c) The set of continuous nondecreasing functions on $[-1, 1]$

(d) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$

(e) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ or $f(1) = 0$

7. Show that $C^n[a, b]$ is a subspace of $C[a, b]$.

8. Let A be a fixed vector in $\mathbb{R}^{n\times n}$ and let S be the set of all matrices that commute with A, that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n\times n}$.

9. In each of the following determine the subspace of $\mathbb{R}^{2\times 2}$ consisting of all matrices that commute with the given matrix:

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

10. Let A be a particular vector in $\mathbb{R}^{2\times 2}$. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) $S_1 = \{B \in \mathbb{R}^{2\times 2} \mid BA = O\}$

(b) $S_2 = \{B \in \mathbb{R}^{2\times 2} \mid AB \neq BA\}$

(c) $S_3 = \{B \in \mathbb{R}^{2\times 2} \mid AB + B = O\}$

11. Determine whether the following are spanning sets for $\mathbb{R}^2$:

(a) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ **(b)** $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

12. Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers.

(a) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$

(b) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$

(c) $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

(d) $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$

(e) $\{(1,1,3)^T, (0,2,1)^T\}$

13. Given

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}$$

(a) Is $\mathbf{x} \in \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

14. Let A be a 4×3 matrix and let $\mathbf{b} \in \mathbb{R}^4$. How many possible solutions could the system $A\mathbf{x} = \mathbf{b}$ have if $N(A) = \{\mathbf{0}\}$? Answer the same question in the case $N(A) \neq \{\mathbf{0}\}$. Explain your answers.

15. Let A be a 4×3 matrix and let

$$\mathbf{c} = 2\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$$

(a) If $N(A) = \{\mathbf{0}\}$, what can you conclude about the solutions to the linear system $A\mathbf{x} = \mathbf{c}$?

(b) If $N(A) \neq \{\mathbf{0}\}$, how many solutions will the system $A\mathbf{x} = \mathbf{c}$ have? Explain.

16. Let $\mathbf{x}_1$ be a particular solution to a system $A\mathbf{x} = \mathbf{b}$ and let $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ be a spanning set for $N(A)$. If

$$Z = \begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \end{pmatrix},$$

show that $\mathbf{y}$ will be a solution to $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{y} = \mathbf{x}_1 + Z\mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^3$.

17. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

(a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

(b) If we delete one of the vectors, say, $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

18. In $\mathbb{R}^{2\times 2}$, let

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$.

19. Which of the sets that follow are spanning sets for P_3? Justify your answers.

(a) $\{1, x^2, x^2 - 2\}$ **(b)** $\{2, x^2, x, 2x + 3\}$

(c) $\{x + 2, x + 1, x^2 - 1\}$ **(d)** $\{x + 2, x^2 - 1\}$

20. Let S be the vector space of infinite sequences defined in Exercise 15 of Section 3.1. Let S_0 be the set of $\{a_n\}$ with the property that $a_n \to 0$ as $n \to \infty$. Show that S_0 is a subspace of S.

21. Prove that if S is a subspace of $\mathbb{R}^1$, then either $S = \{\mathbf{0}\}$ or $S = \mathbb{R}^1$.

22. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent.

(a) $N(A) = \{\mathbf{0}\}$. **(b)** A is nonsingular.

(c) For each $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

23. Let U and V be subspaces of a vector space W. Prove that their intersection $U \cap V$ is also a subspace of W.

24. Let S be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$ and let T be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_2$. Is $S \cup T$ a subspace of $\mathbb{R}^2$? Explain.