The procedure we have just described will work in general whenever the number of sample points is even. If, say, $N = 2m$, and we permute the columns of F_{2m} so that the odd columns are first, then the reordered Fourier matrix $F_{2m}P_{2m}$ can be partitioned into $m \times m$ blocks

$$F_{2m}P_{2m} = \begin{Bmatrix} F_m & D_m F_m \\ F_m & -D_m F_m \end{Bmatrix}$$

where D_m is a diagonal matrix whose (j, j) entry is ω_{2m}^{j-1}. The discrete Fourier transform can then be computed in terms of two transforms of length m. Furthermore, if m is even, then each length m transform can be computed in terms of two transforms of length $\frac{m}{2}$, and so on.

If, initially, N is a power of 2, say, $N = 2^k$, then we can apply this procedure recursively through k levels of recursion. The amount of arithmetic required to compute the FFT is proportional to $Nk = N \log_2 N$. In fact, the actual amount of arithmetic operations required for the FFT is approximately $5N \log_2 N$. How dramatic of a speedup is this? If we consider, for example, the case where $N = 2^{20} = 1{,}048{,}576$, then the DFT algorithm requires $8N^2 = 8 \cdot 2^{40}$ operations, that is, approximately 8.8 trillion operations. On the other hand, the FFT algorithm requires only $100N = 100 \cdot 2^{20}$, or approximately 100 million, operations. The ratio of these two operations counts is

$$r = \frac{8N^2}{5N \log_2 N} = 0.08 \cdot 1{,}048{,}576 = 83{,}886$$

In this case, the FFT algorithm is approximately 84,000 times faster than the DFT algorithm.

SECTION 5.5 EXERCISES

1. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1, 0)^T, (0, 1)^T\}$

(b) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$

(c) $\{(1, -1)^T, (1, 1)^T\}$

(d) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

2. Let

$$\mathbf{u}_1 = \begin{Bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{Bmatrix}, \quad \mathbf{u}_2 = \begin{Bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{Bmatrix}, \quad \mathbf{u}_3 = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{Bmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1, 1, 1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ using Theorem 5.5.2 and use Parseval's formula to compute $\|\mathbf{x}\|$.

3. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ of Exercise 2. Let $\mathbf{x} = (1, 2, 2)^T$. Find the projection $\mathbf{p}$ of $\mathbf{x}$ onto S. Show that $(\mathbf{p} - \mathbf{x}) \perp \mathbf{u}_2$ and $(\mathbf{p} - \mathbf{x}) \perp \mathbf{u}_3$.

4. Let θ be a fixed real number and let

$$\mathbf{x}_1 = \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix}$$

(a) Show that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

(b) Given a vector $\mathbf{y}$ in $\mathbb{R}^2$, write it as a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

(c) Verify that

$$c_1^2 + c_2^2 = \|\mathbf{y}\|^2 = y_1^2 + y_2^2$$

5. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$ and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$. If $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, determine the value of $|\mathbf{u}^T\mathbf{u}_2|$.

6. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V and let

$$\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 \qquad \text{and} \qquad \mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$$

Determine the value of each of the following:

(a) $\langle \mathbf{u}, \mathbf{v}\rangle$ **(b)** $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$

(c) The angle θ between $\mathbf{u}$ and $\mathbf{v}$

7. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\langle \mathbf{u}_1, \mathbf{x}\rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, c_3?

8. The functions $\cos x$ and $\sin x$ form an orthonormal set in $C[-\pi, \pi]$. If

$$f(x) = 3\cos x + 2\sin x \quad \text{and} \quad g(x) = \cos x - \sin x$$

use Corollary 5.5.3 to determine the value of

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

9. The set

$$S = \left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \cos 3x, \cos 4x\right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined by (2).

(a) Use trigonometric identities to write the function $\sin^4 x$ as a linear combination of elements of S.

(b) Use part (a) and Theorem 5.5.2 to find the values of the following integrals:

(a) $\int_{-\pi}^{\pi} \sin^4 x \cos x\,dx$ **(b)** $\int_{-\pi}^{\pi} \sin^4 x \cos 2x\,dx$

(c) $\int_{-\pi}^{\pi} \sin^4 x \cos 3x\,dx$ **(d)** $\int_{-\pi}^{\pi} \sin^4 x \cos 4x\,dx$

10. Write out the Fourier matrix F_8. Show that F_8P_8 can be partitioned into block form:

$$\begin{Bmatrix} F_4 & D_4F_4 \\ F_4 & -D_4F_4 \end{Bmatrix}$$

11. Prove that the transpose of an orthogonal matrix is an orthogonal matrix.

12. If Q is an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then how does the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ compare with the angle between $\mathbf{x}$ and $\mathbf{y}$? Prove your answer.

13. Let Q be an $n \times n$ orthogonal matrix. Use mathematical induction to prove each of the following.

(a) $(Q^m)^{-1} = (Q^T)^m = (Q^m)^T$ for any positive integer m.

(b) $\|Q^m\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$.

14. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

15. Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $|d| = 1$.

16. Show that the product of two orthogonal matrices is also an orthogonal matrix. Is the product of two permutation matrices a permutation matrix? Explain.

17. How many $n \times n$ permutation matrices are there?

18. Show that if P is a symmetric permutation matrix then $P^{2k} = I$ and $P^{2k+1} = P$.

19. Show that if U is an $n \times n$ orthogonal matrix then

$$\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T = I$$

20. Use mathematical induction to show that if $Q \in \mathbb{R}^{n\times n}$ is both upper triangular and orthogonal, then $\mathbf{q}_j = \pm\mathbf{e}_j$, $j = 1, \ldots, n$.

21. Let

$$A = \begin{Bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$$

(a) Show that the column vectors of A form an orthonormal set in $\mathbb{R}^4$.

(b) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$ for each of the following choices of $\mathbf{b}$.

(a) $\mathbf{b} = (4, 0, 0, 0)^T$ **(b)** $\mathbf{b} = (1, 2, 3, 4)^T$

(c) $\mathbf{b} = (1, 1, 2, 2)^T$

22. Let A be the matrix given in Exercise 21.

(a) Find the projection matrix P that projects vectors in $\mathbb{R}^4$ onto $R(A)$.

(b) For each of your solutions $\mathbf{x}$ to Exercise 21(b), compute $A\mathbf{x}$ and compare it with $P\mathbf{b}$.

23. Let A be the matrix given in Exercise 21.

(a) Find an orthonormal basis for $N(A^T)$.

(b) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $N(A^T)$.

24. Let A be an $m \times n$ matrix, let P be the projection matrix that projects vectors in $\mathbb{R}^m$ onto $R(A)$, and let Q be the projection matrix that projects vectors in $\mathbb{R}^n$ onto $R(A^T)$. Show that to $A\mathbf{x} = \mathbf{b}$, however, in this case one should not compute $\mathbf{c} = Q^T\mathbf{b}$ directly. Instead, as each column vector $\mathbf{q}_k$ is determined, one modifies the right hand side vector obtaining a modified vector $\mathbf{b}_k$ and then sets $c_k = \mathbf{q}_k^T\mathbf{b}_k$. An algorithm for solving least squares problems using the modified Gram–Schmidt QR factorization is given in Section 7 of Chapter 7.

SECTION 5.6 EXERCISES

1. For each of the following, use the Gram–Schmidt process to find an orthonormal basis for $R(A)$.

(a) $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}$

2. Factor each of the matrices in Exercise 1 into a product QR, where Q is an orthogonal matrix and R is upper triangular.

3. Given the basis $\{(1, 2, -2)^T, (4, 3, 2)^T, (1, 2, 1)^T\}$ for $\mathbb{R}^3$, use the Gram–Schmidt process to obtain an orthonormal basis.

4. Consider the vector space $C[-1, 1]$ with inner product defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

5. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.

(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$

6. Repeat Exercise 5 using

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 20 \\ 10 \end{pmatrix}$$

7. Given $\mathbf{x}_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram–Schmidt process.]

8. Use the Gram–Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (4, 2, 2, 1)^T$, $\mathbf{x}_2 = (2, 0, 0, 2)^T$, and $\mathbf{x}_3 = (1, 1, -1, 1)^T$.

9. Repeat Exercise 8 using the modified Gram–Schmidt process and compare answers.

10. Let A be an $m \times 2$ matrix. Show that if both the classical Gram–Schmidt process and the modified Gram–Schmidt process are applied to the column vectors of A, then both algorithms will produce the exact same QR factorization, even when the computations are carried out in finite-precision arithmetic (i.e., show that both algorithms will perform the exact same arithmetic computations).

11. Let A be an $m \times 3$ matrix. Let QR be the QR factorization obtained when the classical Gram–Schmidt process is applied to the column vectors of A, and let $\tilde{Q}\tilde{R}$ be the factorization obtained when the modified Gram–Schmidt process is used. Show that if all computations were carried out using exact arithmetic then we would have

$$\tilde{Q} = Q \quad \text{and} \quad \tilde{R} = R$$

and show that when the computations are done in finite-precision arithmetic, $\tilde{r}_{23}$ will not necessarily be equal to r_{23} and consequently $\tilde{r}_{33}$ and $\tilde{\mathbf{q}}_3$ will not necessarily be the same as r_{33} and $\mathbf{q}_3$.

12. What will happen if the Gram–Schmidt process is applied to a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, but $\mathbf{v}_3 \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Will the process fail? If so, how? Explain.

13. Let A be an $m \times n$ matrix of rank n and let $\mathbf{b} \in \mathbb{R}^m$. Show that if Q and R are the matrices derived from applying the Gram–Schmidt process to the column vectors of A and

$$\mathbf{p} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n$$

is the projection of $\mathbf{b}$ onto $R(A)$, then

(a) $\mathbf{c} = Q^T\mathbf{b}$ **(b)** $\mathbf{p} = QQ^T\mathbf{b}$

(c) $QQ^T = A(A^TA)^{-1}A^T$