Sect. 2.8: Subspaces of $\mathbb{R}^n$

Goal: Develop theory and vocabulary to describe the properties of sets of vectors

Reason: the solutions to lin. systems of equations are determined by the columns of the matrix $A = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] \in \mathbb{R}^{m \times n}$

Def

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that satisfies the following 3 properties:

a.) the zero vector is in H, $\underline{0} \in H$

b.) for $\underline{u}, \underline{v} \in H$, their sum $\underline{u} + \underline{v}$ is also in H

c.) for $\underline{u} \in H$ and scalar $c \in \mathbb{R}$, $c\underline{u}$ is also in H

↳ b.c. we say subspaces are closed under addition and scalar multiplication.

Ex

$H = \text{span}\{\underline{v}_1, \underline{v}_2\}$ for $\underline{v}_1 \, \underline{v}_2 \in \mathbb{R}^n$. Is this a subspace?

To-do: check all 3 properties.

a.) is $\underline{0} \in \text{span}\{\underline{v}_1, \underline{v}_2\}$, i.e, can we find $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 = \underline{0}$$

Yes! $c_1 = 0$, $c_2 = 0$, then $0 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2 = \underline{0}$
so $\underline{0} \in H$ ✓

b.) is $\underline{u} + \underline{v} \in H$ for $\underline{u}, \underline{v} \in H$.
by definition, if $\underline{u}, \underline{v} \in H$ then we can write

$$\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2 \quad \text{for} \quad a_1, a_2 \in \mathbb{R}$$
$$\underline{v} = b_1 \underline{v}_1 + b_2 \underline{v}_2 \quad\quad b_1, b_2 \in \mathbb{R}$$

then
$$\begin{aligned} \underline{u} + \underline{v} &= a_1 \underline{v}_1 + a_2 \underline{v}_2 + b_1 \underline{v}_1 + b_2 \underline{v}_2 \\ &= \underbrace{(a_1 + b_1)}\underline{v}_1 + \underbrace{(a_2 + b_2)}\underline{v}_2 \\ &= c_1 \underline{v}_1 + c_2 \underline{v}_2 \end{aligned}$$

$\Rightarrow$ $\underline{u} + \underline{v}$ is a lin. comb. of $\underline{v}_1, \underline{v}_2$, so
by definition $\underline{u} + \underline{v} \in \text{span}\{\underline{v}_1\ \underline{v}_2\} = H$ ✓

c.) is $c\underline{u} \in H$ for $\underline{u} \in H$, $c \in \mathbb{R}$
let $\underline{u} = a_1 \underline{v}_1 + a_2 \underline{v}_2$ and $c \in \mathbb{R}$,
the we get

$$\begin{aligned} c\underline{u} &= c(a_1 \underline{v}_1 + a_2 \underline{v}_2) \\ &= c a_1 \underline{v}_1 + c a_2 \underline{v}_2 \\ &= c_1 \underline{v}_1 + c_2 \underline{v}_2 \in \text{span}\{\underline{v}_1 \underline{v}_2\} = H \end{aligned}$$ ✓

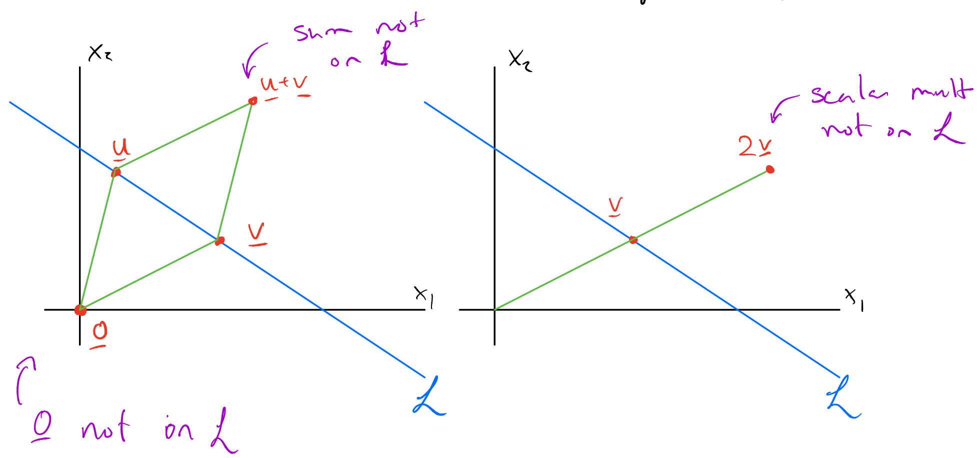
We showed all 3 properties hold, so $\text{span}\{\underline{v}_1 \underline{v}_2\} = H$
is a subspace of $\mathbb{R}^n$ #

↳ Similar proof shows span$\{\underline{v}_1 \cdots \underline{v}_p\}$ is a subspace of $\mathbb{R}^n$

<u>Ex</u> (not a subspace)

Any line L that doesn't pass through the origin is not a subspace because it does not contain the zero vector and it's not closed under addition or scalar multiplication

$\mathbb{R}^2$



<u>Column Space & Null Space of a Matrix</u>

We are interested in subspaces associated with columns of matrices:

<u>Def</u>

The <u>column space</u> of a matrix A, denoted Col A, is the subspace spanned by the columns of A. That is

$$\text{Col } A = \text{span}\{\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n\}$$

↳ Col A is the set of all linear combinations of the columns of A.

↳ $A\underline{x} = \underline{b}$ only has a solution if $\underline{b}$ is linear combination of the columns of A, i.e., $\underline{b} \in \text{Col } A$

<u>Ex</u>

Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Is $\underline{b} \in \text{Col } A$?

$\underline{b} \in \text{Col } A \iff A\underline{x} = \underline{b}$ is consistent, so we row reduce

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system $A\underline{x} = \underline{b}$ is consistent, so $\underline{b} \in \text{Col } A$

<u>Def.</u>

The <u>null space</u> of a matrix A, denoted Nul A, is the subspace of all solutions $\underline{x}$ of the homogeneous equation $A\underline{x} = \underline{0}$

<u>Pf.</u> Need to verify all 3 properties of a subspace

a.) for $\underline{x} = \underline{0}$, we that $A\underline{x} = A\underline{0} = \underline{0}$,

so $\underline{x} = 0$ is a solution to the homogeneous equation, i.e., $\underline{0} \in \text{Nul } A$ ✓

b.) let $\underline{u}, \underline{v} \in \text{Nul } A$, that means by definition $A\underline{u} = \underline{0}$ and $A\underline{v} = \underline{0}$.

we check that $\underline{u} + \underline{v} \in \text{Nul } A$ by

$$A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v}$$
$$= \underline{0} + \underline{0}$$
$$= \underline{0}$$

so, $\underline{u}+\underline{v} \in \text{Nul } A$ ✓

c.) let $\underline{u} \in \text{Nul } A$ and $c \in \mathbb{R}$, then we see

$$A(c\underline{u}) = c(A\underline{u}) = c \cdot \underline{0} = \underline{0}$$

So $c\underline{u} \in \text{Nul } A$ ✓

We satisfy all 3 properties, so $\text{Nul } A$ is a subspace of $\mathbb{R}^n$.

<u>Def</u>

A <u>basis</u> for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H than spans all of H

↳ smallest # of vectors needed to produce any other vector in the space

↳ the columns of any invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$

$\hookrightarrow$ the columns of the identity are commonly used as a basis for $\mathbb{R}^n$

Ex

Find a basis for Nul A for

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

First, Solve $A\underline{x} = \underline{0}$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so x_2, x_4, x_5 are free variables, and a general solution is

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_3}$$

The vectors $\underline{v}_1\ \underline{v}_2\ \underline{v}_3$ are a basis for Nul A.

Ex Find a basis for Col A where

$$A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \underline{a}_4 & \underline{a}_5 \end{bmatrix}$$

we see $\underline{a}_3 = -3\underline{a}_1 + 2\underline{a}_2$

$\underline{a}_4 = 5\underline{a}_1 - \underline{a}_2$

so for any vector $\underline{v} \in \text{Col } A$, we get

$$\underline{v} = c_1 \underline{a}_1 + c_2 \underline{a}_2 + c_3 \underline{a}_3 + c_4 \underline{a}_4 + c_5 \underline{a}_5$$

$$\underline{v} = c_1 \underline{a}_1 + c_2 \underline{a}_2 + c_3 (-3\underline{a}_1 + 2\underline{a}_2) + c_4 (5\underline{a}_1 - \underline{a}_2) + c_5 \underline{a}_5$$

↳ we only need $\underline{a}_1$ $\underline{a}_2$ & $\underline{a}_5$

$\underline{v}$ is a linear combination of $\underline{a}_1$ $\underline{a}_2$ $\underline{a}_5$. These vectors are linearly independent, so they form a basis for Col A.

<u>Theorem</u> The pivot columns of A form a basis for Col A.

↳ can always row reduce, find pivot columns to get a basis.