EXAMPLE 9 Show that the vectors $1, x, x^2$, and x^3 are linearly independent in $C((-\infty, \infty))$.

Solution

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since $W[1, x, x^2, x^3] \not\equiv 0$, the vectors are linearly independent. ■

SECTION 3.3 EXERCISES

1. Determine whether the following vectors are linearly independent in $\mathbb{R}^2$:

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

3. For each of the sets of vectors in Exercise 2, describe geometrically the span of the given vectors.

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

5. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in a vector space V.

(a) If we add a vector $\mathbf{x}_{k+1}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

(b) If we delete a vector, say, $\mathbf{x}_k$, from the collection, will we still have a linearly independent collection of vectors? Explain.

6. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1$$

Are $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

7. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \quad \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1$$

Are $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

8. Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$ **(b)** $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$ **(d)** $x + 2, x^2 - 1$

9. For each of the following, show that the given vectors are linearly independent in $C[0, 1]$:

(a) $\cos \pi x, \ \sin \pi x$ **(b)** $x^{3/2}, \ x^{5/2}$

(c) $1, \ e^x + e^{-x}, \ e^x - e^{-x}$ **(d)** $e^x, \ e^{-x}, \ e^{2x}$

10. Determine whether the vectors $\cos x, 1$, and $\sin^2(x/2)$ are linearly independent in $C[-\pi, \pi]$.

11. Consider the vectors $\cos(x+\alpha)$ and $\sin x$ in $C[-\pi, \pi]$. For what values of α will the two vectors be linearly dependent? Give a graphical interpretation of your answer.

12. Given the functions $2x$ and $|x|$, show that
(a) these two vectors are linearly independent in $C[-1, 1]$.
(b) the vectors are linearly dependent in $C[0, 1]$.

13. Prove that any finite set of vectors that contains the zero vector must be linearly dependent.

14. Let $\mathbf{v}_1$, and $\mathbf{v}_2$ be two vectors in a vector space V. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of the vectors is a scalar multiple of the other.

15. Prove that any nonempty subset of a linearly independent set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is also linearly independent.

16. Let A be an $m \times n$ matrix. Show that if A has linearly independent column vectors, then $N(A) = \{\mathbf{0}\}$.
[*Hint*: For any $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$.]

17. Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be a nonsingular $n \times n$ matrix. Define $\mathbf{y}_i = A\mathbf{x}_i$ for $i = 1, \dots, k$. Show that $\mathbf{y}_1, \dots, \mathbf{y}_k$ are linearly independent.

18. Let A be a 3×3 matrix and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be vectors in $\mathbb{R}^3$. Show that if the vectors

$$\mathbf{y}_1 = A\mathbf{x}_1, \quad \mathbf{y}_2 = A\mathbf{x}_2, \quad \mathbf{y}_3 = A\mathbf{x}_3$$

are linearly independent, then the matrix A must be nonsingular and the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ must be linearly independent.

19. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a spanning set for the vector space V, and let $\mathbf{v}$ be any other vector in V. Show that $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.

20. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be linearly independent vectors in a vector space V. Show that $\mathbf{v}_2, \dots, \mathbf{v}_n$ cannot span V.

3.4 Basis and Dimension

In Section 3.3, we showed that a spanning set for a vector space is minimal if its elements are linearly independent. The elements of a minimal spanning set form the basic building blocks for the whole vector space, and consequently, we say that they form a *basis* for the vector space.

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a **basis** for a vector space V if and only if

(i) $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.
(ii) $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V.

EXAMPLE 1 The *standard basis* for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; however, there are many bases that we could choose for $\mathbb{R}^3$. For example,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are both bases for $\mathbb{R}^3$. We will see shortly that any basis for $\mathbb{R}^3$ must have exactly three elements. ■

The standard way to represent a polynomial in P_n is in terms of the functions $1, x, x^2, \ldots, x^{n-1}$, and consequently, the standard basis for P_n is $\{1, x, x^2, \ldots, x^{n-1}\}$.

Although these standard bases appear to be the simplest and most natural to use, they are not the most appropriate bases for many applied problems. (See, for example, the least squares problems in Chapter 5 or the eigenvalue applications in Chapter 6.) Indeed, the key to solving many applied problems is to switch from one of the standard bases to a basis that is in some sense natural for the particular application. Once the application is solved in terms of the new basis, it is a simple matter to switch back and represent the solution in terms of the standard basis. In the next section, we will learn how to switch from one basis to another.

SECTION 3.4 EXERCISES

1. In Exercise 1 of Section 3.3, indicate whether the given vectors form a basis for $\mathbb{R}^2$.

2. In Exercise 2 of Section 3.3, indicate whether the given vectors form a basis for $\mathbb{R}^3$.

3. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

4. Given the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -6 \\ 4 \\ -8 \end{bmatrix}$$

what is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

5. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

(d) Give a geometric description of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

6. In Exercise 2 of Section 3.2, some of the sets formed subspaces of $\mathbb{R}^3$. In each of these cases, find a basis for the subspace and determine its dimension.

7. Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a + b, a - b + 2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

8. Given $\mathbf{x}_1 = (1,\ 1,\ 1)^T$ and $\mathbf{x}_2 = (3,\ -1,\ 4)^T$:

(a) Do $\mathbf{x}_1$ and $\mathbf{x}_2$ span $\mathbb{R}^3$? Explain.

(b) Let $\mathbf{x}_3$ be a third vector in $\mathbb{R}^3$ and set $X = (\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)$. What condition(s) would X have to satisfy in order for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ to form a basis for $\mathbb{R}^3$?

(c) Find a third vector $\mathbf{x}_3$ that will extend the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ to a basis for $\mathbb{R}^3$.

9. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be linearly independent vectors in $\mathbb{R}^3$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^2$.

(a) Describe geometrically Span$(\mathbf{a}_1, \mathbf{a}_2)$.

(b) If $A = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{b} = A\mathbf{x}$, then what is the dimension of Span$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b})$? Explain.

10. The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix},$$

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

11. Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis for S.

12. In Exercise 3 of Section 3.2, some of the sets formed subspaces of $\mathbb{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

13. In $C[-\pi, \pi]$, find the dimension of the subspace spanned by $1, \cos 2x, \cos^2 x$.

14. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x - 1, x^2 + 1$

(b) $x, x-1, x^2+1, x^2-1$

(c) $x^2, x^2-x-1, x+1$ (d) $2x, x-2$

15. Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(0)=0$, and let T be the subspace of all polynomials $q(x)$ such that $q(1)=0$. Find bases for

(a) S (b) T (c) $S \cap T$

16. In $\mathbb{R}^4$, let U be the subspace of all vectors of the form $(u_1, u_2, 0, 0)^T$, and let V be the subspace of all vectors of the form $(0, v_2, v_3, 0)^T$. What are the dimensions of U, V, $U \cap V$, $U+V$? Find a basis for each of these four subspaces. (See Exercises 23 and 25 of Section 3.2.)

17. Is it possible to find a pair of two-dimensional subspaces U and V of $\mathbb{R}^3$ whose intersection is $\{\mathbf{0}\}$? Prove your answer. Give a geometrical interpretation of your conclusion. *Hint*: Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be bases for U and V, respectively. Show that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are linearly dependent.

18. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then

$$\dim(U+V) = \dim U + \dim V$$

3.5 Change of Basis

Many applied problems can be simplified by changing from one coordinate system to another. Changing coordinate systems in a vector space is essentially the same as changing from one basis to another. For example, in describing the motion of a particle in the plane at a particular time, it is often convenient to use a basis for $\mathbb{R}^2$ consisting of a unit tangent vector $\mathbf{t}$ and a unit normal vector $\mathbf{n}$ instead of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

In this section, we discuss the problem of switching from one coordinate system to another. We will show that this can be accomplished by multiplying a given coordinate vector $\mathbf{x}$ by a nonsingular matrix S. The product $\mathbf{y} = S\mathbf{x}$ will be the coordinate vector for the new coordinate system.

Changing Coordinates in $\boldsymbol{R}^2$

The standard basis for $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$. Any vector $\mathbf{x}$ in $\mathbb{R}^2$ can be expressed as a linear combination

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

The scalars x_1 and x_2 can be thought of as the *coordinates* of $\mathbf{x}$ with respect to the standard basis. Actually, for any basis $\{\mathbf{y}, \mathbf{z}\}$ for $\mathbb{R}^2$, it follows from Theorem 3.3.2 that a given vector $\mathbf{x}$ can be represented uniquely as a linear combination

$$\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$$

The scalars α and β are the coordinates of $\mathbf{x}$ with respect to the basis $\{\mathbf{y}, \mathbf{z}\}$. Let us order the basis elements so that $\mathbf{y}$ is considered the first basis vector and $\mathbf{z}$ is considered the second, and denote the ordered basis by $[\mathbf{y}, \mathbf{z}]$. We can then refer to the vector $(\alpha, \beta)^T$ as the *coordinate vector* of $\mathbf{x}$ with respect to $[\mathbf{y}, \mathbf{z}]$. Note that, if we reverse the order of the basis vectors and take $[\mathbf{z}, \mathbf{y}]$, then we must also reorder the coordinate vector. The coordinate vector of $\mathbf{x}$ with respect to $[\mathbf{z}, \mathbf{y}]$ will be $(\beta, \alpha)^T$. When we refer to a basis using subscripts, such as $\{\mathbf{u}_1, \mathbf{u}_2\}$, the subscripts assign an ordering to the basis vectors.