It follows from (4) that

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} s_{ij}x_j\right)\mathbf{v}_j = \mathbf{0}$$

By the linear independence of the $\mathbf{v}_i$'s, it follows that

$$\sum_{j=1}^{n} s_{ij}x_j = 0 \qquad i = 1, \ldots, n$$

or, equivalently,

$$S\mathbf{x} = \mathbf{0}$$

Since S is nonsingular, $\mathbf{x}$ must equal $\mathbf{0}$. Therefore, $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent and hence they form a basis for V. The matrix S is the transition matrix corresponding to the change from the ordered basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ to $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

In many applied problems, it is important to use the right type of basis for the particular application. In Chapter 5, we will see that the key to solving least squares problems is to switch to a special type of basis called an *orthonormal* basis. In Chapter 6, we will consider a number of applications involving the *eigenvalues* and *eigenvectors* associated with an $n \times n$ matrix A. The key to solving these types of problems is to switch to a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

SECTION 3.5 EXERCISES

1. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(a) $\mathbf{u}_1 = (1, 1)^T, \quad \mathbf{u}_2 = (-1, 1)^T$

(b) $\mathbf{u}_1 = (1, 2)^T, \quad \mathbf{u}_2 = (2, 5)^T$

(c) $\mathbf{u}_1 = (0, 1)^T, \quad \mathbf{u}_2 = (1, 0)^T$

2. For each of the ordered bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ in Exercise 1, find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Exercise 1, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

4. Let $E = [(5, 3)^T, (3, 2)^T]$ and let $\mathbf{x} = (1, 1)^T$, $\mathbf{y} = (1, -1)^T$, and $\mathbf{z} = (10, 7)^T$. Determine the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

5. Let $\mathbf{u}_1 = (1, 1, 1)^T$, $\mathbf{u}_2 = (1, 2, 2)^T$, and $\mathbf{u}_3 = (2, 3, 4)^T$.

(a) Find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) Find the coordinates of each of the following vectors with respect to the ordered basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(i) $(3, 2, 5)^T$ **(ii)** $(1, 1, 2)^T$ **(iii)** $(2, 3, 2)^T$

6. Let $\mathbf{v}_1 = (4, 6, 7)^T$, $\mathbf{v}_2 = (0, 1, 1)^T$, and $\mathbf{v}_3 = (0, 1, 2)^T$, and let $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ be the vectors given in Exercise 5.

(a) Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) If $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

7. Given

$$\mathbf{v}_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad \mathbf{v}_2 = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}, \quad S = \begin{Bmatrix} 3 & 5 \\ 1 & -2 \end{Bmatrix}$$

find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S will be the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

8. Given

$$\mathbf{v}_1 = \begin{Bmatrix} 2 \\ 6 \end{Bmatrix}, \quad \mathbf{v}_2 = \begin{Bmatrix} 1 \\ 4 \end{Bmatrix}, \quad S = \begin{Bmatrix} 4 & 1 \\ 2 & 1 \end{Bmatrix}$$

find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

9. Let $[x, 1]$ and $[2x - 1, 2x + 1]$ be ordered bases for P_2.

(a) Find the transition matrix representing the change in coordinates from $[2x - 1, 2x + 1]$ to $[x, 1]$.

(b) Find the transition matrix representing the change in coordinates from $[x, 1]$ to $[2x - 1, 2x + 1]$.

10. Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis

$$[1, 1 + x, 1 + x + x^2]$$

11. Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $F = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two ordered bases for $\mathbb{R}^n$, and set

$$U = (\mathbf{u}_1, \ldots, \mathbf{u}_n), \qquad V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

Show that the transition matrix from E to F can be determined by calculating the reduced row echelon form of $(V|U)$.

3.6 Row Space and Column Space

If A is an $m \times n$ matrix, each row of A is an n-tuple of real numbers and hence can be considered as a vector in $\mathbb{R}^{1\times n}$. The m vectors corresponding to the rows of A will be referred to as the *row vectors* of A. Similarly, each column of A can be considered as a vector in $\mathbb{R}^m$, and we can associate n *column vectors* with the matrix A.

Definition

If A is an $m \times n$ matrix, the subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the **row space** of A. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A.

EXAMPLE 1 Let

$$A = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{Bmatrix}$$

The row space of A is the set of all 3-tuples of the form

$$\alpha(1, 0, 0) + \beta(0, 1, 0) = (\alpha, \beta, 0)$$

The column space of A is the set of all vectors of the form

$$\alpha \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \beta \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \gamma \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix}$$

Thus the row space of A is a two-dimensional subspace of $\mathbb{R}^{1\times 3}$, and the column space of A is $\mathbb{R}^2$. ■

Theorem 3.6.1 *Two row equivalent matrices have the same row space.*

Proof If B is row equivalent to A, then B can be formed from A by a finite sequence of row operations. Thus the row vectors of B must be linear combinations of the row vectors of A. Consequently, the row space of B must be a subspace of the row space of A. Since A is row equivalent to B, by the same reasoning, the row space of A is a subspace of the row space of B. ■

EXAMPLE 5 Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

Solution

The subspace $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is the same as the column space of the matrix

$$X = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

The row echelon form of X is

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns $\mathbf{x}_1, \mathbf{x}_2$ of X will form a basis for the column space of X. Thus, $\dim \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = 2$. ■

SECTION 3.6 EXERCISES

1. For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

2. In each of the following, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the given vectors.

(a) $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$$

(a) Compute the reduced row echelon form U of A. Which column vectors of U correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the lead variables.

(b) Which column vectors of A correspond to the lead variables of U? These column vectors form a basis for the column space of A. Write each of the remaining column vectors of A as a linear combination of these basis vectors.

4. For each of the following choices of A and $\mathbf{b}$, determine whether $\mathbf{b}$ is in the column space of A and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent:

(a) $A = \begin{Bmatrix} 1 & 2 \\ 2 & 4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 4 \\ 8 \end{Bmatrix}$

(b) $A = \begin{Bmatrix} 3 & 6 \\ 1 & 2 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

(c) $A = \begin{Bmatrix} 2 & 1 \\ 3 & 4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 4 \\ 6 \end{Bmatrix}$

(d) $A = \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

(e) $A = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 2 \\ 5 \\ 2 \end{Bmatrix}$

(f) $A = \begin{Bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 5 \\ 10 \\ 5 \end{Bmatrix}$

5. For each consistent system in Exercise 4, determine whether there will be one or infinitely many solutions by examining the column vectors of the coefficient matrix A.

6. How many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space of A and the column vectors of A are linearly dependent? Explain.

7. Let A be a $6 \times n$ matrix of rank r and let $\mathbf{b}$ be a vector in $\mathbb{R}^6$. For each choice of r and n that follows, indicate the possibilities as to the number of solutions one could have for the linear system $A\mathbf{x} = \mathbf{b}$. Explain your answers.

(a) $n = 7, r = 5$ **(b)** $n = 7, r = 6$

(c) $n = 5, r = 5$ **(d)** $n = 5, r = 4$

8. Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.

(a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

(b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

9. Let A and B be 6×5 matrices. If $\dim N(A) = 2$, what is the rank of A? If the rank of B is 4, what is the dimension of $N(B)$?

10. Let A be an $m \times n$ matrix whose rank is equal to n. If $A\mathbf{c} = A\mathbf{d}$, does this imply that $\mathbf{c}$ must be equal to $\mathbf{d}$? What if the rank of A is less than n? Explain your answers.

11. Let A be an $m \times n$ matrix. Prove that

$$\operatorname{rank}(A) \leq \min(m, n)$$

12. Let A and B be row equivalent matrices.

(a) Show that the dimension of the column space of A equals the dimension of the column space of B.

(b) Are the column spaces of the two matrices necessarily the same? Justify your answer.

13. Let A be a 4×3 matrix and suppose that the vectors

$$\mathbf{z}_1 = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}, \quad \mathbf{z}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

form a basis for $N(A)$. If $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, find all solutions of the system $A\mathbf{x} = \mathbf{b}$.

14. Let A be a 4×4 matrix with reduced row echelon form given by

$$U = \begin{Bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

If

$$\mathbf{a}_1 = \begin{Bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{Bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{Bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{Bmatrix}$$

find $\mathbf{a}_3$ and $\mathbf{a}_4$.

15. Let A be a 4×5 matrix and let U be the reduced row echelon form of A. If

$$\mathbf{a}_1 = \begin{Bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{Bmatrix}, \quad \mathbf{a}_2 = \begin{Bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{Bmatrix},$$

$$U = \begin{Bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{Bmatrix}$$

(a) find a basis for $N(A)$.

(b) given that $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{Bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{Bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{Bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{Bmatrix}$$

(i) find all solutions to the system.

(ii) determine the remaining column vectors of A.

16. Let A be a 5×8 matrix with rank equal to 5 and let $\mathbf{b}$ be any vector in $\mathbb{R}^5$. Explain why the system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

EXAMPLE 13 Let $D: P_3 \to P_3$ be the differentiation operator, defined by

$$D(p(x)) = p'(x)$$

The kernel of D consists of all polynomials of degree 0. Thus, $\ker(D) = P_1$. The derivative of any polynomial in P_3 will be a polynomial of degree 1 or less. Conversely, any polynomial in P_2 will have antiderivatives in P_3, so each polynomial in P_2 will be the image of polynomials in P_3 under the operator D. It then follows that $D(P_3) = P_2$. ■

SECTION 4.1 EXERCISES

1. Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each linear transformation accomplishes.

(a) $L(\mathbf{x}) = (-x_1, x_2)^T$ **(b)** $L(\mathbf{x}) = -\mathbf{x}$

(c) $L(\mathbf{x}) = (x_2, x_1)^T$ **(d)** $L(\mathbf{x}) = \frac{1}{2}\mathbf{x}$

(e) $L(\mathbf{x}) = x_2\mathbf{e}_2$

2. Let L be the linear operator on $\mathbb{R}^2$ defined by

$$L(\mathbf{x}) = (x_1 \cos\alpha - x_2 \sin\alpha,\ x_1 \sin\alpha + x_2 \cos\alpha)^T$$

Express x_1, x_2, and $L(\mathbf{x})$ in terms of polar coordinates. Describe geometrically the effect of the linear transformation.

3. Let $\mathbf{a}$ be a fixed nonzero vector in $\mathbb{R}^2$. A mapping of the form

$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$

is called a *translation*. Show that a translation is not a linear operator. Illustrate geometrically the effect of a translation.

4. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((1, 2)^T) = (-2, 3)^T$$

and

$$L((1, -1)^T) = (5, 2)^T$$

find the value of $L((7, 5)^T)$.

5. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$.

(a) $L(\mathbf{x}) = (x_2, x_3)^T$ **(b)** $L(\mathbf{x}) = (0, 0)^T$

(c) $L(\mathbf{x}) = (1 + x_1, x_2)^T$

(d) $L(\mathbf{x}) = (x_3, x_1 + x_2)^T$

6. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$.

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

(c) $L(\mathbf{x}) = (x_1, 0, 0)^T$

(d) $L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)^T$

7. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$.

(a) $L(A) = 2A$ **(b)** $L(A) = A^T$

(c) $L(A) = A + I$ **(d)** $L(A) = A - A^T$

8. Let C be a fixed $n \times n$ matrix. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$:

(a) $L(A) = CA + AC$ **(b)** $L(A) = C^2A$

(c) $L(A) = A^2C$

9. Determine whether the following are linear transformations from P_2 to P_3.

(a) $L(p(x)) = xp(x)$

(b) $L(p(x)) = x^2 + p(x)$

(c) $L(p(x)) = p(x) + xp(x) + x^2p'(x)$

10. For each $f \in C[0, 1]$, define $L(f) = F$, where

$$F(x) = \int_0^x f(t)\,dt \qquad 0 \le x \le 1$$

Show that L is a linear operator on $C[0, 1]$ and then find $L(e^x)$ and $L(x^2)$.

11. Determine whether the following are linear transformations from $C[0, 1]$ into $\mathbb{R}^1$:

(a) $L(f) = f(0)$ **(b)** $L(f) = |f(0)|$

(c) $L(f) = [f(0) + f(1)]/2$

(d) $L(f) = \left\{\int_0^1 [f(x)]^2\,dx\right\}^{1/2}$

12. Use mathematical induction to prove that if L is a linear transformation from V to W, then

$$\begin{aligned} &L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) \\ &\quad = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_n L(\mathbf{v}_n) \end{aligned}$$

13. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let L_1 and L_2 be two linear transformations mapping V into a vector space W. Show that if

$$L_1(\mathbf{v}_i) = L_2(\mathbf{v}_i)$$

for each $i = 1, \ldots, n$, then $L_1 = L_2$ [i.e., show that $L_1(\mathbf{v}) = L_2(\mathbf{v})$ for all $\mathbf{v} \in V$].

14. Let L be a linear operator on $\mathbb{R}^1$ and let $a = L(1)$. Show that $L(x) = ax$ for all $x \in \mathbb{R}^1$.

15. Let L be a linear operator on a vector space V. Define L^n, $n \geq 1$, recursively by

$$L^1 = L$$
$$L^{k+1}(\mathbf{v}) = L(L^k(\mathbf{v})) \qquad \text{for all } \mathbf{v} \in V$$

Show that L^n is a linear operator on V for each $n \geq 1$.

16. Let $L_1\colon U \to V$ and $L_2\colon V \to W$ be linear transformations, and let $L = L_2 \circ L_1$ be the mapping defined by

$$L(\mathbf{u}) = L_2(L_1(\mathbf{u}))$$

for each $\mathbf{u} \in U$. Show that L is a linear transformation mapping U into W.

17. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$ **(b)** $L(\mathbf{x}) = (x_1, x_2, 0)^T$

(c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

18. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. For each linear operator L in Exercise 17, find $L(S)$.

19. Find the kernel and range of each of the following linear operators on P_3:

(a) $L(p(x)) = xp'(x)$ **(b)** $L(p(x)) = p(x) - p'(x)$

(c) $L(p(x)) = p(0)x + p(1)$

20. Let $L\colon V \to W$ be a linear transformation, and let T be a subspace of W. The *inverse image* of T, denoted $L^{-1}(T)$, is defined by

$$L^{-1}(T) = \{\mathbf{v} \in V | L(\mathbf{v}) \in T\}$$

Show that $L^{-1}(T)$ is a subspace of V.

21. A linear transformation $L\colon V \to W$ is said to be *one-to-one* if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e., no two distinct vectors $\mathbf{v}_1, \mathbf{v}_2$ in V get mapped into the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}_V\}$.

22. A linear transformation $L\colon V \to W$ is said to map V *onto* W if $L(V) = W$. Show that the linear transformation L defined by

$$L(\mathbf{x}) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$$

maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

23. Which of the operators defined in Exercise 17 are one-to-one? Which map $\mathbb{R}^3$ onto $\mathbb{R}^3$?

24. Let A be a 2×2 matrix, and let L_A be the linear operator defined by

$$L_A(\mathbf{x}) = A\mathbf{x}$$

Show that

(a) L_A maps $\mathbb{R}^2$ onto the column space of A.

(b) if A is nonsingular, then L_A maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

25. Let D be the differentiation operator on P_3, and let

$$S = \{p \in P_3 \,|\, p(0) = 0\}$$

Show that

(a) D maps P_3 onto the subspace P_2, but $D\colon P_3 \to P_2$ is not one-to-one.

(b) $D\colon S \to P_3$ is one-to-one but not onto.

4.2 Matrix Representations of Linear Transformations

In Section 4.1, it was shown that each $m \times n$ matrix A defines a linear transformation L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$, where

$$L_A(\mathbf{x}) = A\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^n$. In this section, we will see that, for each linear transformation L mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that

$$L(\mathbf{x}) = A\mathbf{x}$$