Sect 5.3: (cont...)

Def.

An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix, i.e.

$$P^{-1} A P = D$$

↳ $P = \begin{vmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{vmatrix}$ for eigenvectors of A

↳ $D = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix}$ for eigenvalues of A

Thm

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1 \cdots \lambda_p$, then

a.) for $k = 1 \cdots p$, the dimension of the eigenspace corresponding to λ_k is $\leq$ to the multiplicity of the eigenvalue λ_k

b.) A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n. This happens if $p = n$ (i.e. n distinct eigenvalues) or if the dimension of the eigenspace equals the multiplicity for all λ_k, $k = 1 \cdots p$.

c.) if A is diagonalizable and $\mathcal{B}_k$ are bases for the eigenspaces for λ_k, then the collection of $\mathcal{B}_1 \dots \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$

<u>Fact:</u> Two matrices can have the eigenvalues, but one is diagonalizable and the other is not.

<u>Ex</u>

Consider $A = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

For A, we have $\lambda = 2$ with multiplicity 2. Let's find the eigenspace associated with $\lambda = 2$

Solve $(A - 2I)\underline{x} = \underline{0}$. This gives

$$\begin{vmatrix} 2-2 & 1 \\ 0 & 2-2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

So, the solution is

$$\begin{cases} 0 \cdot x_1 + 1 \cdot x_2 = 0 \\ 0 \cdot x_1 + 0 \cdot x_2 = 0 \end{cases} \implies x_2 = 0, \ x_1 = \text{free}$$

$$\underline{x} = \begin{vmatrix} x_1 \\ 0 \end{vmatrix} = x_1 \underbrace{\begin{vmatrix} 1 \\ 0 \end{vmatrix}}_{\underline{v}_1}$$

We see that even though $\lambda = 2$ has mult. 2, the eigenspace has a basis of dimension 1 given by $\mathcal{B}_{\lambda=2} = \{ \underline{v}_1 \}$. Therefore, A is not diagonalizable

Now, consider B. B also has $\lambda = 2$ as an eigenvalue with multiplicity 2. To find a basis for the corresponding eigen space, we solve $(A - 2I)\underline{x} = \underline{0}$

$$\begin{vmatrix} 2-2 & 0 \\ 0 & 2-2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{array}{l} x_1 = \text{free} \\ x_2 = \text{free} \end{array}$$

So $$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = x_1 \underbrace{\begin{vmatrix} 1 \\ 0 \end{vmatrix}}_{\underline{v}_1} + x_2 \underbrace{\begin{vmatrix} 0 \\ 1 \end{vmatrix}}_{\underline{v}_2}$$

So the basis for the eigenspace associated with $\lambda = 2$ is $\mathcal{B} = \{ \underline{v}_1, \underline{v}_2 \}$. $\dim(\mathcal{B}) = 2$ is equal the multiplicity of $\lambda = 2$, so our matrix B is diagonalizable

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Ex

Diagonalize $A = \begin{vmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{vmatrix}$

A is triangular, so the char. eqn. is

$$\det(A - \lambda I) = (5-\lambda)^2(-3-\lambda)^2 = 0$$

$$\Rightarrow \lambda = 5, -3 \quad (\text{both with mult. } 2)$$

Find a basis for e-space corresponding to $\lambda = 5$ by solving $(A - 5I)\underline{x} = \underline{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -8 & 0 & 0 \\ 0 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 16 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{x} = x_3 \underbrace{\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_1} + x_4 \underbrace{\begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_2}$$

$\mathcal{B}_{\lambda=5} = \{\underline{v}_1, \underline{v}_2\}$ has dimension 2 for $\lambda = 5$ with multiplicity 2.

Find a basis for e-space corresponding to $\lambda = -3$ by solving $(A + 3I)\underline{x} = \underline{0}$

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{array}$$

$$\underline{x} = \begin{vmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{vmatrix} = x_3 \underbrace{\begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}}_{\underline{v}_3} + x_4 \underbrace{\begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}}_{\underline{v}_4}$$

$\mathcal{B}_{\lambda=-3} = \{\underline{v}_3\ \underline{v}_4\}$ has dimension 2 for $\lambda=-3$ with mult. 2

The sum of the dimensions of the eigenspaces is

$$\dim \mathcal{B}_{\lambda=5} + \dim \mathcal{B}_{\lambda=-3} = 2 + 2 = n \checkmark$$

The matrix A is diagonalizable and

$$P = \begin{vmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{v}_4 \end{vmatrix} \quad D = \begin{vmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} \text{ with } P^{-1}AP = D$$