Sect 2.9: Dimension & Rank

Goal: Continue building on our understanding abount sets of vectors in $\mathbb{R}^n$

We ended last class by introducing the idea of a basis.

Def A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans all of H.

Key Idea: Let $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_p\}$ be a basis of p vectors for the subspace H. The vectors of $\mathcal{B}$ span H, i.e. any vector $x \in H$ can be written as a lin. comb. of vectors in $\mathcal{B}$

$$x = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \cdots + c_p \underline{b}_p$$

for $c_1 \cdots c_p \in \mathbb{R}$ scalars.

↳ The scalars $c_1 \cdots c_p$ are unique, i.e., there is one unique to express $\underline{x} \in H$ in terms of $\mathcal{B}$.

Pf. Assume there are other scalars $d_1 \cdots d_p$ such that

$$\underline{x} = d_1 \underline{b}_1 + d_2 \underline{b}_2 + \cdots + d_p \underline{b}_p$$

Then we have

$$\underline{0} = \underline{x} - \underline{x}$$
$$= \left(c_1 \underline{b}_1 + \cdots + c_p \underline{b}_p\right) - \left(d_1 \underline{b}_1 + \cdots + d_p \underline{b}_p\right)$$
$$= (c_1 - d_1)\underline{b}_1 + (c_2 - d_2)\underline{b}_2 + \cdots + (c_p - d_p)\underline{b}_p$$

$$\Rightarrow (c_j - d_j) = 0 \quad \text{for all } j = 1 \cdots p$$

$$\Rightarrow c_j = d_j \quad \text{for all } j = 1 \cdots p$$

So, we have a contradictions $\Rightarrow$ c are unique.

Def. Suppose the set $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_p\}$ is a basis for a subspace H. For each $\underline{x}$ in H, the coordinates of $\underline{x}$ relative to $\mathcal{B}$ are the weights $c_1 \cdots c_p$ such that

$$\underline{x} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \cdots + c_p \underline{b}_p$$

We call the vector

$$\left[\underline{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

the coordinate vector of $\underline{x}$ relative to $\mathcal{B}$.

Remark

- You've already been doing this with the standard basis without knowing it.

Consider the standard basis in $\mathbb{R}^3$

$$\underline{e}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad \underline{e}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad \underline{e}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

We can express any point $(5, 2, -1)$ in $\mathbb{R}^3$ in terms of the standard basis

$$5 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + 2 \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} - 1 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 5 \\ 2 \\ -1 \end{vmatrix} = \underline{x}$$

Our vector $\underline{x}$ is really <u>the coordinate vector of $\underline{x}$ relative to the standard basis.</u>

<u>Ex</u> How can we find $c_1 \cdots c_p$ for a different basis?

Let $\underline{v}_1 = \begin{vmatrix} 3 \\ 6 \\ 2 \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} -1 \\ 0 \\ 1 \end{vmatrix}, \quad \underline{x} = \begin{vmatrix} 3 \\ 12 \\ 7 \end{vmatrix}$, and $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$

be a basis for a subspace H. Can we find $[\underline{x}]_{\mathcal{B}}$. We need c_1 & c_2 such that

$$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2$$

$$\Rightarrow \begin{vmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} = \begin{vmatrix} 3 \\ 12 \\ 7 \end{vmatrix}$$

Solving, we get

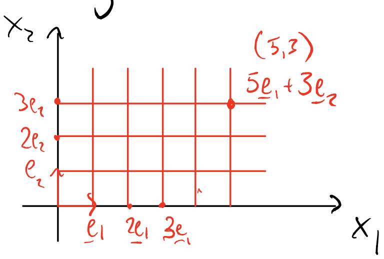
$$\begin{vmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{vmatrix} \sim \cdots \sim \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array}$$

So $[\underline{x}]_{\mathcal{B}} = \begin{vmatrix} 2 \\ 3 \end{vmatrix}$

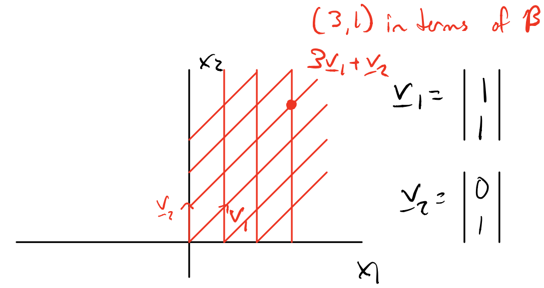
__Note:__ If the system is inconsistent, $\underline{x}$ is not in H and cannot express in terms of the basis $\mathcal{B}$.

Geometric Interpretation ($\mathbb{R}^2$)

We can view the vectors of a basis as "tiling" the subspace.



$\underline{e}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $\quad \underline{e}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$



$\underline{v}_1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $\quad \underline{v}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

Dimension & Rank

__Def.__ The __dimension__ of a nonzero subspace H, denoted $\dim H$, is the number of vectors in any basis of H

↳ The dimension of $H = \{\underline{0}\}$ is defined as zero.

__Ex__ $\mathbb{R}^n$ has dimension n. For example, the n vectors of the standard basis form a basis for $\mathbb{R}^n$.

Def. The <u>rank</u> of a matrix, denote rank A, is the dimension of its column space, col A.

↳ since the pivot columns of A form a basis for Col A, the rank of A is the number of pivot columns.

Ex Find rank A for $A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot columns $\Rightarrow$ rank $A = 3$

Rank-Nullity Theorem

If A has n columns, then

$$\text{rank } A + \dim(\text{Nul } A) = n$$

Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of p elements of H is a basis of H. Also, any set of p elements that spans H is a basis of H.

↳ Finding a basis just means finding p lin. ind. vectors in H.

↳ Foreshadowing: we can generate these basis vectors.

Invertible Matrix Theorem (Continued)

Let A be an $n \times n$ matrix, the following are equivalent.

(a.)-(l) previously in Sect. 2.3

m.) the columns of A form a basis for $\mathbb{R}^n$

n.) $\text{Col } A = \mathbb{R}^n$

o.) $\dim(\text{Col } A) = n$

q.) $\text{Nul } A = \{\underline{0}\}$

r.) $\dim(\text{Nul } A) = 0$

Ex Determine the dimension of the subspace H of $\mathbb{R}^3$ spanned by

$$\underline{v}_1 = \begin{vmatrix} 2 \\ -8 \\ 6 \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} 3 \\ -7 \\ -1 \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} -1 \\ 6 \\ -7 \end{vmatrix}$$

Row reduce

$$\begin{vmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{vmatrix} \sim \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{vmatrix} \sim \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

2 pivot columns $\Rightarrow$ $\dim H = 2$

$A = |\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3|$ then $\dim(\text{Col } A) = \text{rank } A = 2$

$\dim(\text{Nul } A) = 3 - \text{rank } A = 1$