It is easily seen that the eigenvalues of T are $\lambda_1 = 2$ and $\lambda_2 = 3$. If we set $A = S^{-1}TS$, then the eigenvalues of A should be the same as those of T.

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}$$

We leave it to the reader to verify that the eigenvalues of this matrix are $\lambda_1 = 2$ and $\lambda_2 = 3$. ■

SECTION 6.1 EXERCISES

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ **(h)** $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

(i) $\begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ **(j)** $\begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

(k) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ **(l)** $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

2. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

3. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

5. Let A and B be $n \times n$ matrices. Show that if none of the eigenvalues of A are equal to 1, then the matrix equation

$$XA + B = X$$

will have a unique solution.

6. Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that, for $m \geq 1$, λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m.

7. Let A be an $n \times n$ matrix and let $B = I - 2A + A^2$.

(a) Show that if $\mathbf{x}$ is an eigenvector of A belonging to an eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of B belonging to an eigenvalue μ of B. How are λ and μ related?

(b) Show that if $\lambda = 1$ is an eigenvalue of A, then the matrix B will be singular.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

9. An $n \times n$ matrix is said to be *nilpotent* if $A^k = O$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are 0.

10. Let A be an $n \times n$ matrix and let $B = A - \alpha I$ for some scalar α. How do the eigenvalues of A and B compare? Explain.

11. Let A be an $n \times n$ matrix and let $B = A + I$. Is it possible for A and B to be similar? Explain.

12. Show that A and A^T have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

13. Show that the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

will have complex eigenvalues if θ is not a multiple of π. Give a geometric interpretation of this result.

14. Let A be a 2×2 matrix. If $\operatorname{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

15. Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that

$$\lambda_j = a_{jj} + \sum_{i \neq j} (a_{ii} - \lambda_i) \quad \text{for} \quad j = 1, \ldots, n$$

16. Let A be a 2×2 matrix and let $p(\lambda) = \lambda^2 + b\lambda + c$ be the characteristic polynomial of A. Show that $b = -\operatorname{tr}(A)$ and $c = \det(A)$.

17. Let λ be a nonzero eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Show that $A^m\mathbf{x}$ is also an eigenvector belonging to λ for $m = 1, 2, \ldots$.

18. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. If $A - \lambda I$ has rank k, what is the dimension of the eigenspace corresponding to λ? Explain.

19. Let A be an $n \times n$ matrix. Show that a vector $\mathbf{x}$ in either $\mathbb{R}^n$ or $\mathbb{C}^n$ is an eigenvector belonging to A if and only if the subspace S spanned by $\mathbf{x}$ and $A\mathbf{x}$ has dimension 1.

20. Let $\alpha = a + bi$ and $\beta = c + di$ be complex scalars and let A and B be matrices with complex entries.

(a) Show that

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta} \qquad \text{and} \qquad \overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$$

(b) Show that the (i, j) entries of $\overline{AB}$ and $\overline{A}\,\overline{B}$ are equal and hence that

$$\overline{AB} = \overline{A}\,\overline{B}$$

21. Let Q be an orthogonal matrix.

(a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.

(b) Show that $|\det(Q)| = 1$.

22. Let Q be an orthogonal matrix with an eigenvalue $\lambda_1 = 1$ and let $\mathbf{x}$ be an eigenvector belonging to λ_1. Show that $\mathbf{x}$ is also an eigenvector of Q^T.

23. Let Q be a 3×3 orthogonal matrix whose determinant is equal to 1.

(a) If the eigenvalues of Q are all real and if they are ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, determine the values of all possible triples of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$.

(b) In the case that the eigenvalues λ_2 and λ_3 are complex, what are the possible values for λ_1? Explain.

(c) Explain why $\lambda = 1$ must be an eigenvalue of Q.

24. Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be eigenvectors of an $n \times n$ matrix A and let S be the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$. Show that S is *invariant* under A (i.e., show that $A\mathbf{x} \in S$ whenever $\mathbf{x} \in S$).

25. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Show that if B is any matrix that commutes with A, then the eigenspace $N(A - \lambda I)$ is invariant under B.

26. Let $B = S^{-1}AS$ and let $\mathbf{x}$ be an eigenvector of B belonging to an eigenvalue λ. Show that $S\mathbf{x}$ is an eigenvector of A belonging to λ.

27. Let A be an $n \times n$ matrix with an eigenvalue λ and let $\mathbf{x}$ be an eigenvector belonging to λ. Let S be a nonsingular $n \times n$ matrix and let α be a scalar. Show that if

$$B = \alpha I - SAS^{-1}, \quad \mathbf{y} = S\mathbf{x}$$

then $\mathbf{y}$ is an eigenvector of B. Determine the eigenvalue of B corresponding to $\mathbf{y}$?

28. Show that if two $n \times n$ matrices A and B have a common eigenvector $\mathbf{x}$ (but not necessarily a common eigenvalue), then $\mathbf{x}$ will also be an eigenvector of any matrix of the form $C = \alpha A + \beta B$.

29. Let A be an $n \times n$ matrix and let λ be a nonzero eigenvalue of A. Show that if $\mathbf{x}$ is an eigenvector belonging to λ, then $\mathbf{x}$ is in the column space of A. Hence the eigenspace corresponding to λ is a subspace of the column space of A.

30. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and let A be a linear combination of the rank 1 matrices $\mathbf{u}_1\mathbf{u}_1^T, \mathbf{u}_2\mathbf{u}_2^T, \ldots, \mathbf{u}_n\mathbf{u}_n^T$. If

$$A = c_1\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + c_n\mathbf{u}_n\mathbf{u}_n^T$$

show that A is a symmetric matrix with eigenvalues $c_1, c_2, \ldots, c_n$ and that $\mathbf{u}_i$ is an eigenvector belonging to c_i for each i.

31. Let A be a matrix whose columns all add up to a fixed constant δ. Show that δ is an eigenvalue of A.

32. Let λ_1 and λ_2 be distinct eigenvalues of A. Let $\mathbf{x}$ be an eigenvector of A belonging to λ_1 and let $\mathbf{y}$ be an eigenvector of A^T belonging to λ_2. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

33. Let A and B be $n \times n$ matrices. Show that

(a) If λ is a nonzero eigenvalue of AB, then it is also an eigenvalue of BA.

(b) If $\lambda = 0$ is an eigenvalue of AB, then $\lambda = 0$ is also an eigenvalue of BA.

34. Prove that there do not exist $n \times n$ matrices A and B such that

$$AB - BA = I$$

[*Hint*: See Exercises 10 and 33.]

35. Let $p(\lambda) = (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0)$ be a polynomial of degree $n \geq 1$, and let

$$C = \begin{Bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{Bmatrix}$$

(a) Show that if λ_i is a root of $p(\lambda) = 0$, then λ_i is an eigenvalue of C with eigenvector $\mathbf{x} = (\lambda_i^{n-1}, \lambda_i^{n-2}, \ldots, \lambda_i, 1)^T$.

(b) Use part (a) to show that if $p(\lambda)$ has n distinct roots then $p(\lambda)$ is the characteristic polynomial of C.

The matrix C is called the *companion matrix* of $p(\lambda)$.

At time $t = 0$, we have

$$x_1(0) = x_2(0) = 0 \quad \text{and} \quad x_1'(0) = x_2'(0) = 2$$

It follows that

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 - c_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} c_2 + \sqrt{3}c_4 &= 2 \\ c_2 - \sqrt{3}c_4 &= 2 \end{aligned}$$

and hence

$$c_1 = c_3 = c_4 = 0 \quad \text{and} \quad c_2 = 2$$

Therefore, the solution to the initial value problem is simply

$$\mathbf{X}(t) = \begin{Bmatrix} 2\sin t \\ 2\sin t \end{Bmatrix}$$

The masses will oscillate with frequency 1 and amplitude 2.

APPLICATION 3 Vibrations of a Building

For another example of a physical system, we consider the vibrations of a building. If the building has k stories, we can represent the horizontal deflections of the stories at time t by the vector function $\mathbf{Y}(t) = (y_1(t), y_2(t), \ldots, y_k(t))^T$. The motion of a building can be modeled by a second-order system of differential equations of the form

$$M\mathbf{Y}''(t) = K\mathbf{Y}(t)$$

The *mass matrix* M is a diagonal matrix whose entries correspond to the concentrated weights at each story. The entries of the *stiffness matrix* K are determined by the spring constants of the supporting structures. Solutions of the equation are of the form $\mathbf{Y}(t) = e^{i\sigma t}\mathbf{x}$, where $\mathbf{x}$ is an eigenvector of $A = M^{-1}K$ belonging to an eigenvalue λ and σ is a square root of λ.

SECTION 6.2 EXERCISES

1. Find the general solution of each of the following systems:

(a) $\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}$

(b) $\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= -y_1 - 3y_2 \end{aligned}$

(c) $\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}$

(d) $\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= y_1 + y_2 \end{aligned}$

(e) $\begin{aligned} y_1' &= 3y_1 - 2y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$

(f) $\begin{aligned} y_1' &= y_1 + y_3 \\ y_2' &= 2y_2 + 6y_3 \\ y_3' &= y_2 + 3y_3 \end{aligned}$

2. Solve each of the following initial value problems:

(a) $\begin{aligned} y_1' &= -y_1 + 2y_2 \\ y_2' &= 2y_1 - y_2 \end{aligned}$
$y_1(0) = 3, y_2(0) = 1$

(b) $\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= 2y_1 + y_2 \end{aligned}$
$y_1(0) = 1, y_2(0) = -2$

(c) $\begin{aligned} y_1' &= 2y_1 - 6y_3 \\ y_2' &= y_1 - 3y_3 \\ y_3' &= y_2 - 2y_3 \end{aligned}$
$y_1(0) = y_2(0) = y_3(0) = 2$

(d) $y_1' = y_1 + 2y_3$
$y_2' = y_2 - y_3$
$y_3' = y_1 + y_2 + y_3$
$y_1(0) = y_2(0) = 1, y_3(0) = 4$

3. Given

$$\mathbf{Y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{x}_n$$

is the solution to the initial value problem:

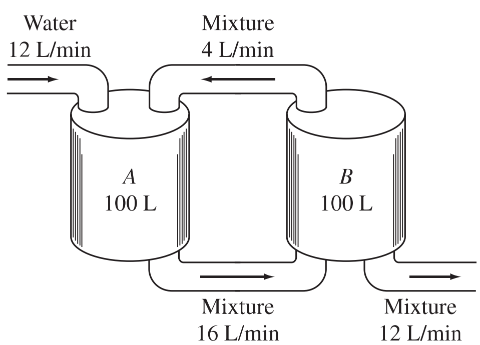
$$\mathbf{Y}' = A\mathbf{Y}, \qquad \mathbf{Y}(0) = \mathbf{Y}_0$$

(a) show that

$$\mathbf{Y}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

(b) let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)^T$. Assuming that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent, show that $\mathbf{c} = X^{-1}\mathbf{Y}_0$.

4. Two tanks each contain 100 liters of a mixture. Initially, the mixture in tank A contains 40 grams of salt while tank B contains 20 grams of salt. Liquid is pumped in and out of the tanks as shown in the accompanying figure. Determine the amount of salt in each tank at time t.



5. Find the general solution of each of the following systems:

(a) $y_1'' = -2y_2$
$y_2'' = y_1 + 3y_2$

(b) $y_1'' = 2y_1 + y_2'$
$y_2'' = 2y_2 + y_1'$

6. Solve the initial value problem

$$y_1'' = -2y_2 + y_1' + 2y_2'$$
$$y_2'' = 2y_1 + 2y_1' - y_2'$$

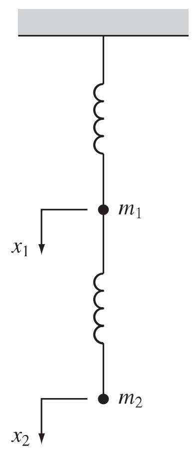
$$y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = -3,\ y_2'(0) = 2$$

7. In Application 2, assume that the solutions are of the form $x_1 = a_1 \sin \sigma t$, $x_2 = a_2 \sin \sigma t$. Substitute these expressions into the system and solve for the frequency σ and the amplitudes a_1 and a_2.

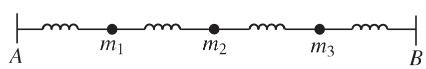
8. Solve the the problem in Application 2, using the initial conditions

$$x_1(0) = x_2(0) = 1,\ x_1'(0) = 4, \text{ and } x_2'(0) = 2$$

9. Two masses are connected by springs as shown in the accompanying diagram. Both springs have the same spring constant, and the end of the first spring is fixed. If x_1 and x_2 represent the displacements from the equilibrium position, derive a system of second-order differential equations that describes the motion of the system.



10. Three masses are connected by a series of springs between two fixed points as shown in the accompanying figure. Assume that the springs all have the same spring constant, and let $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the displacements of the respective masses at time t.



(a) Derive a system of second-order differential equations that describes the motion of this system.

(b) Solve the system if $m_1 = m_3 = \frac{1}{3}$, $m_2 = \frac{1}{4}$, $k = 1$, and

$$x_1(0) = x_2(0) = x_3(0) = 1$$
$$x_1'(0) = x_2'(0) = x_3'(0) = 0$$

11. Transform the nth-order equation

$$y^{(n)} = a_0 y + a_1 y' + \cdots + a_{n-1}y^{(n-1)}$$

into a system of first-order equations by setting $y_1 = y$ and $y_j = y_{j-1}'$ for $j = 2, \ldots, n$. Determine the characteristic polynomial of the coefficient matrix of this system.