Sect 3.3: Cramer's Rule, Volume, and Linear Transformations

Goal: Continue exploring uses & interpretations of the determinant.

Notation

For an $n \times n$ matrix A and a vector $\underline{b} \in \mathbb{R}^n$, let $A_i(\underline{b})$ be the $n \times n$ matrix formed by replacing the ith column of A with $\underline{b}$

Ex

$$A = \begin{vmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{vmatrix} \longrightarrow A_i(\underline{b}) = \begin{vmatrix} \underline{a}_1 & \cdots & \underline{a}_{i-1} & \underline{b} & \underline{a}_{i+1} & \cdots & \underline{a}_n \end{vmatrix}$$

replace ith col. with $\underline{b}$

$$A = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & -2 \end{vmatrix} \quad \underline{b} = \begin{vmatrix} 5 \\ 6 \\ 7 \end{vmatrix} \longrightarrow A_2(\underline{b}) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & -2 \\ 3 & 7 & -2 \end{vmatrix}$$

Theorem (Gabriel Cramer)

Let A be an invertible $n \times n$ matrix, then for any $\underline{b} \in \mathbb{R}^n$, the unique solution $\underline{x}$ of $A\underline{x} = \underline{b}$ has entries $x_i = 1 \cdots n$ given by

$$x_i = \frac{\det A_i(\underline{b})}{\det A} \quad \text{for } i = 1 \cdots n$$

↳ allows us to compute a single, specific entry in the solution $\underline{x}$.

$\hookrightarrow$ in practice, its usually too expensive to use for the complete system.

<u>Ex</u> Find the 3rd entry of the solution $A\underline{x} = \underline{b}$ where

$$A = \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} \quad \& \quad \underline{b} = \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix}$$

Lets find det A to see if A is invertible

$$\det A = (-1)^{1+1}(1)\det\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2}(-2)\det\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix}$$

$$+ (-1)^{1+3}(4)\det\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1\cdot 1\cdot(0\cdot 2 - (1\cdot(-1))) + (-1)\cdot(-2)\cdot(3\cdot 2 - 0\cdot(-1)) + 1\cdot 4\cdot(3\cdot 1 - 0\cdot 0)$$

$$= 1\cdot(2) + 2(6) + 4(3) = 26 \neq 0$$

$\Rightarrow$ A is invertible, so compute $\det A_3(\underline{b})$

$$A_3(\underline{b}) = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\det A_3(\underline{b}) = (-1)^{1+1}\cdot 1\cdot\det\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2}\cdot(-2)\det\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$+ (-1)^{1+3}\cdot 0\cdot\det\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1\cdot 1\cdot(0\cdot 2 - 1\cdot 1) + (-1)\cdot(-2)(3\cdot 2 - 0\cdot 1) + 0$$

$$= 1\cdot(-1) + 2(6) + 0 = 11$$

So by Cramer's Rule

$$x_3 = \frac{\det A_3(\underline{b})}{\det A} = \frac{11}{26}$$

↳ effective, but lots of work.

We can also use Cramer's Rule to compute a formula for A^{-1}

↳ more expensive than row-reducing on the augmented matrix $\left[A \mid I \right] \sim \left[I \mid A^{-1} \right]$

Cramer's Rule for finding A^{-1}

First, we note the jth column of A^{-1} is the vector $\underline{x}$ such that

$$A\underline{x} = \underline{e}_j \quad \text{where} \quad \underline{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ jth entry}$$

$\underline{x}$ is jth column of A^{-1} because $\underline{x} = A^{-1}\underline{e}_j$ = jth col of A^{-1}

Cramer's Rule gives the ith entry in $\underline{x}$, so it gives us $[A^{-1}]_{ij}$ by

$$\left[A^{-1}_{ij} \right] = \frac{\det A_i(\underline{e}_j)}{\det A}$$

If we do this for all rows $i = 1 \cdots n$ and for all columns e_j for $j = 1 \cdots n$, we can find all the entries of A^{-1}

↳ hugely expensive, it requires computing n^2 determinants

Looking closer,

$$\det A_i(\underline{e}_j) = \left| \underline{a}_1 \cdots \underline{a}_{i-1} \; \underline{e}_j \; \underline{a}_{i+1} \cdots \underline{a}_n \right|$$

Cofactor expansion along the ith column of $A_i(\underline{e}_j)$ we get

$$\det A_i(\underline{e}_j) = (-1)^{i+j} \det A_{ij} = C_{ij}$$

we call C_{ij} the ijth <u>cofactor</u> of A. The matrix A^{-1} computed using Cramer's Rule for every entry is

$$A^{-1} = \frac{1}{\det A} \begin{vmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{vmatrix}^T = \frac{1}{\det A} \operatorname{adj} A$$

where adj A is the transpose of matrix of cofactors, called the <u>adjoint</u>.

↳ We've seen this for 2x2 matrices

$$A^{-1} = \frac{1}{ad-bc} \begin{vmatrix} a & -b \\ -c & d \end{vmatrix}$$

Our formula came from this rule!

<u>Theorem</u>

Let A be $n \times n$ and invertible then $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$

Ex Compute A^{-1} for $A = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix}$ using the adjoint formula.

First, check A^{-1} exists using $\det A$

$$\det A = (-1)^{1+1} \cdot 2 \cdot \det \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} + (-1)^{1+2} \cdot 1 \det \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$+ (-1)^{1+3} \cdot 3 \cdot \det \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot (-1 \cdot -2 - 4 \cdot 1) - 1 \cdot 1 \cdot (1 \cdot -2 - 1) + 1 \cdot 3 \cdot (1 \cdot 4 - 1 \cdot -1)$$

$$= -4 + 3 + 15 = 14 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

Now, we compute all the cofactors

$$C_{11} = (-1)^{1+1} \det \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2$$

$$C_{12} = (-1)^{1+2} \det \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

$$C_{13} = (-1)^{1+3} \det \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = (-1)^{2+1} \det \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14$$

$$C_{22} = (-1)^{2+2} \det \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$$

$$C_{23} = (-1)^{2+3} \det \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = (-1)^{3+1} \det \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$$

$$C_{32} = (-1)^{3+2} \det \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{33} = (-1)^{3+3} \det \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

Putting all these into the adjoint matrix, we get

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{14} \begin{vmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & 3 \end{vmatrix}$$

Confirm $A^{-1}A = I$ to check

$$A^{-1}A = \frac{1}{14} \begin{vmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & 3 \end{vmatrix} \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= \frac{1}{14} \begin{vmatrix} -4+14+4 & -2-14+16 & -6+14-8 \\ 6-7+1 & 3+7+4 & 9-7-2 \\ 10-7-3 & 5+7-12 & 15-7+6 \end{vmatrix}$$

$$= \frac{1}{14} \begin{vmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I \checkmark$$

$\Rightarrow$ our inverse is correct.

↳ moral of the story: the formula works, but it is way more work than row-reduction so do $\left| A \mid I \right| \sim \left| I \mid A^{-1} \right|$