# Sect 4.7: Change of Bases

For a given vector space, the can be many different bases

↳ All we need is a linearly independent, spanning set. There can be an infinite number of these.

↳ The columns of any $n \times n$ invertible matrix are a basis for $\mathbb{R}^n$

Ex

Take $\mathbb{R}^2$, here some bases

$$\mathcal{B}_1 = \left\{ \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \end{vmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \begin{vmatrix} 1 \\ -1 \end{vmatrix} \right\}, \quad \mathcal{B}_3 = \left\{ \begin{vmatrix} 4 \\ 2 \end{vmatrix}, \begin{vmatrix} 5 \\ 3 \end{vmatrix} \right\}$$

We also showed in Sect. 2.9 that can express any vector $\underline{x} \in V$ in terms of a given basis

↳ For a basis $\mathcal{B} = \{ \underline{b}_1 \cdots \underline{b}_n \}$, find weights $c_1 \cdots c_n$ such that

$$\underline{x} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \cdots + c_n \underline{b}_n$$

We called $[\underline{x}]_{\mathcal{B}} = \begin{vmatrix} c_1 \\ \vdots \\ c_n \end{vmatrix}$ the coordinates of $x$ relative to the basis $\mathcal{B}$. These coordinates exist if the system $\left| \underline{b}_1 \cdots \underline{b}_n \;\vdots\; \underline{x} \right|$ is consistent.

Ex

Find the coordinates of $\underline{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$. The coordinates $[\underline{x}]_{\mathcal{B}}$ are the solution to the equation

$$\underbrace{\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[x]_{\mathcal{B}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\underline{x}}$$

Solve by

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

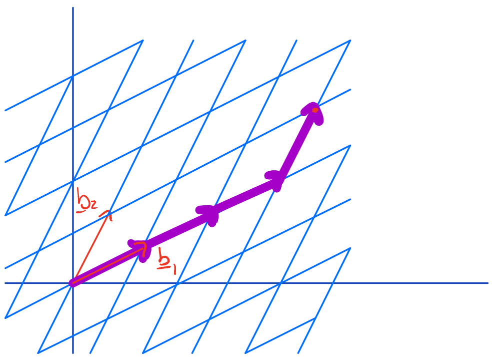
$$= \frac{1}{-6+4} \begin{bmatrix} -2 & +1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

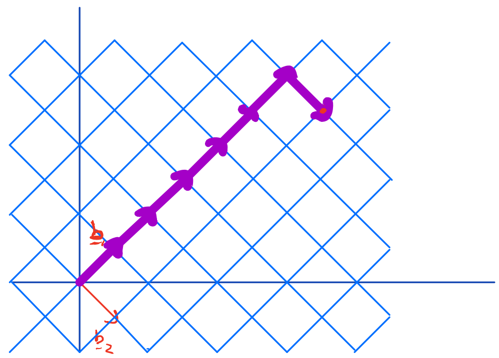
$$= \frac{1}{-2} \begin{bmatrix} -5 \\ -11 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 11/2 \end{bmatrix}$$

The coordinates of $\underline{x}$ relative to $\mathcal{B}$ are $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ 11/2 \end{bmatrix}$

## Geometric Interpretation

Writing the coordinates of a vector relative to two bases is equivalent to writing $\underline{x}$ as a sum of different sets of vectors




$$[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad [x]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

for $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

So, we express the same point in $\mathbb{R}^2$ is terms of linear combinations of vectors from two different bases.

## Question:

Given two bases $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1 \cdots \underline{c}_n\}$ and the coordinates of $\underline{x}$ relative to both bases, $[\underline{x}]_{\mathcal{B}}$ and $[x]_{\mathcal{C}}$, can we find a map from one to the other

$$[\underline{x}]_{\mathcal{B}} \xrightarrow{?} [\underline{x}]_{\mathcal{C}}$$

Answer: We can find the map, let's see how...

## Theorem

Let $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1 \cdots \underline{c}_n\}$ be bases for a vector space $V$. There exists a unique matrix, denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\underline{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\underline{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the coordinates of the vectors in $\mathcal{B}$ relative to $\mathcal{C}$. That is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[ [\underline{b}_1]_{\mathcal{C}} \ [\underline{b}_2]_{\mathcal{C}} \ \cdots \ [\underline{b}_n]_{\mathcal{C}} \right]$$

## Notes

- ↳ We need to solve for $[\underline{b}_j]_{\mathcal{C}}$ for every vector in $\mathcal{B}$. That is $n$ problems!
- ↳ If the vectors in $\mathcal{C}$ form a square, $n \times n$ matrix, $C = [\underline{c}_1 \cdots \underline{c}_n]$, then we can find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ by

$$\begin{aligned} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} &= \left[ [\underline{b}_1]_{\mathcal{C}} \ [\underline{b}_2]_{\mathcal{C}} \ \cdots \ [\underline{b}_n]_{\mathcal{C}} \right] \\ &= \left[ C^{-1}\underline{b}_1 \ C^{-1}\underline{b}_2 \ \cdots \ C^{-1}\underline{b}_n \right] \\ &= C^{-1} \left[ \underline{b}_1 \ \underline{b}_2 \ \cdots \ \underline{b}_n \right] = C^{-1}B \end{aligned}$$

where $B = \begin{vmatrix} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_n \end{vmatrix}$. So, find $C^{-1}$ and multiply $C^{-1}B$ to get $\underset{C \leftarrow B}{P}$

↳ The matrix $\underset{C \leftarrow B}{P}$ converts $|x|_B \longrightarrow |x|_C$

To convert the opposite direction

$$|x|_B \longleftarrow |x|_C$$

we take $\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1}$, the inverse.

This is given by

$$\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1} = \left(C^{-1}B\right)^{-1} = B^{-1}C$$

Ex

Given $B = \left\{ \begin{vmatrix} -9 \\ 1 \end{vmatrix}, \begin{vmatrix} -5 \\ -1 \end{vmatrix} \right\}$ and $C = \left\{ \begin{vmatrix} 1 \\ -4 \end{vmatrix}, \begin{vmatrix} 3 \\ -5 \end{vmatrix} \right\}$ and $\underline{x} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$, find $|\underline{x}|_B$, $|\underline{x}|_C$, $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$.

To find $|\underline{x}|_B$, we solve

$$\underset{B}{\begin{vmatrix} -9 & -5 \\ 1 & -1 \end{vmatrix}} \underset{[\underline{x}]_B}{\begin{vmatrix} u_1 \\ u_2 \end{vmatrix}} = \underset{\underline{x}}{\begin{vmatrix} 1 \\ 1 \end{vmatrix}}$$

We see $B^{-1} = \begin{vmatrix} -9 & -5 \\ 1 & -1 \end{vmatrix}^{-1} = \dfrac{1}{9+5} \begin{vmatrix} -1 & -5 \\ 1 & -9 \end{vmatrix}$

This gives $|\underline{x}|_B = B^{-1}\underline{x} = \dfrac{1}{14} \begin{vmatrix} -1 & -5 \\ 1 & -9 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \dfrac{1}{14} \begin{vmatrix} 4 \\ -10 \end{vmatrix}$

We find $[\underline{x}]_{\mathcal{C}}$ by solving

$$\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C \qquad [\underline{x}]_{\mathcal{C}} = \underline{x}$$

We get $C^{-1} = \dfrac{1}{-5+12} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix}$

and $\quad [\underline{x}]_{\mathcal{C}} = C^{-1}\underline{x} = \dfrac{1}{7} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, we calculate

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = C^{-1}B = \frac{1}{7} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -9 & -3 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 42 & 28 \\ -35 & -21 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = B^{-1}C = \frac{1}{14} \begin{bmatrix} -1 & 5 \\ -1 & -9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} -21 & -28 \\ 35 & 42 \end{bmatrix} = \begin{bmatrix} -3/2 & -2 \\ 5/2 & 3 \end{bmatrix}$$

Check $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\underline{x}]_{\mathcal{B}} = [\underline{x}]_{\mathcal{C}}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\underline{x}]_{\mathcal{C}} = [\underline{x}]_{\mathcal{B}}$

Also, check $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1}$.

## Alternative way to compute $\underset{C\leftarrow B}{P}$

If computing $C^{-1}$ & $B^{-1}$ is too hard, we can $\underset{C\leftarrow B}{P}$ and $\underset{B\leftarrow C}{P}$ using row reduction

1.) Form augmented matrix $[C \mid B]$

2.) Row reduce the augmented matrix so that $C$ goes to the identity, this gives

$$\left[ C \mid B \right] \sim \left[ I \mid \underset{C\leftarrow B}{P} \right]$$

Similarly, row reducing $\left[ B \mid C \right] \sim \left[ I \mid \underset{B\leftarrow C}{P} \right]$

Ex

Using $\mathcal{B} = \left\{ \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$

$$\left[ C \mid B \right] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad I \qquad\quad \underset{C\leftarrow B}{P}$$

We get $\underset{C\leftarrow B}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ identical to the example before.

↳ This method is typically easier for large matrices.