Sect 6.1: Inner Products, Length, and Orthogonality

Goals: Extend $\mathbb{R}^2$ & $\mathbb{R}^3$ concepts for length, distance, and perpendicularity to $\mathbb{R}^n$. Then, we look at the implications.

- How long are vectors?
- How "far" is one vector from another?
- What is the angle between two vectors?

All three of these question are answered using the inner product.

Def For $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^n$, the <u>inner product</u> of $\underline{u}$ and $\underline{v}$ is given by

$$\underline{u}^T\underline{v} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{1\times n} \left.\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right\} n\times 1$$

$$= u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{k=1}^{n} u_k v_k$$

Observations

- $\underline{u}^T\underline{v}$ is a scalar

  $\rightarrow$ 2 vectors give one number as output

- we often refer to the inner product on vectors as a <u>dot product</u>, written

$$\underline{u}^T \underline{v} = \underline{u} \cdot \underline{v}$$

<u>Ex</u>

For $\underline{u} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$, then

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = 3(0) + 1(2) + (-1)(5)$$

$$\underline{u} \cdot \underline{v} = 0 + 2 - 5 = -3$$

<u>Properties of inner products</u> (Thm)

Let $\underline{u}$, $\underline{v}$, and $\underline{w}$ be vector in $\mathbb{R}^n$, and let c be a scalar in $\mathbb{R}$, then

a.) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ (order doesn't matter)

<u>Pf.</u> for $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then we see

$$\underline{u} \cdot \underline{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

$$= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \underline{v} \cdot \underline{u} \checkmark$$

b.) $(\underline{u}+\underline{v})\cdot\underline{w} = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w}$

Pf. $(\underline{u}+\underline{v})\cdot\underline{w} = \begin{vmatrix} u_1+v_1 & u_2+v_2 & \cdots & u_n+v_n \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{vmatrix}$

$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n$

$= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + \cdots + u_nw_n + v_nw_n$

$= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n)$

$= \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w}$

c.) $(c\underline{u})\cdot\underline{v} = c(\underline{u}\cdot\underline{v}) = \underline{u}(c\underline{v})$

Pf

$(c\underline{u})\cdot\underline{v} = \begin{vmatrix} cu_1 & cu_2 & \cdots & cu_n \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{vmatrix}$

$= (cu_1)v_1 + (cu_2)v_2 + \cdots + (cu_n)v_n$

$= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = c(\underline{u}\cdot\underline{v})$

$= u_1(cv_1) + u_2(cv_2) + \cdots + u_n(cv_n) = \underline{u}\cdot(c\underline{v})$

d.) $\underline{u}\cdot\underline{u} \geq 0$ and $\underline{u}\cdot\underline{u} = 0$ if and only if $\underline{u} = \underline{0}$

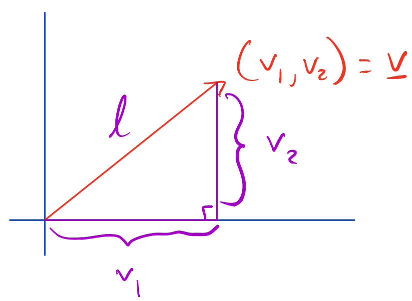
Pf. $\underline{u}\cdot\underline{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n$

$= u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$

because $u_k^2 \geq 0$ for all $k = 1 \cdots n$. Also we see $\underline{u} \cdot \underline{u} = 0$ if and only if $u_k^2 = 0$ for all $k = 1 \cdots n \Rightarrow u_k = 0$ for all $k = 1 \cdots n \Rightarrow \underline{u} = \underline{0}$

Length of a Vector in $\mathbb{R}^n$

In 2D, the length of a vector is something we see in geometry



$$\ell = \sqrt{v_1^2 + v_2^2}$$

= hypotenuse of right tri.

We extend this idea analogously to $\mathbb{R}^n$

Def

The norm (or length) of a vector $\underline{v}$ in $\mathbb{R}^n$ is the non-negative scalar (≥ 0)

$$\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$$

$$= \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$$

$$= \sqrt{\sum_{k=1}^{n} v_k^2}$$

Ex Let $\underline{u} = \begin{vmatrix} 3 \\ 4 \\ 0 \end{vmatrix}$, then $\|\underline{u}\| = \sqrt{3^2 + 4^2 + 0^2} = 5$

Notes

- we also call this the Euclidean norm (there are others out there)

- $\|c\underline{v}\| = |c|\,\|\underline{v}\|$ for scalar $c \in \mathbb{R}$, $\underline{v} \in \mathbb{R}^n$

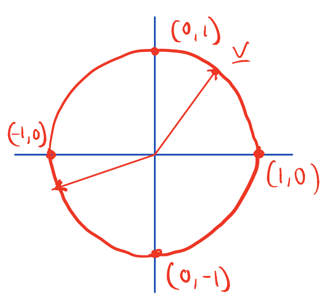
<u>Pf.</u> $$\|c\underline{v}\| = \sqrt{(c\underline{v})\cdot(c\underline{v})}$$
$$= \sqrt{c^2(\underline{v}\cdot\underline{v})}$$
$$= \sqrt{c^2}\sqrt{\underline{v}\cdot\underline{v}}$$
$$= |c|\,\|\underline{v}\|$$

- We often look at the square of the norm

$$\|\underline{v}\| = \sqrt{\underline{v}\cdot\underline{v}} \iff \|\underline{v}\|^2 = \underline{v}\cdot\underline{v}$$

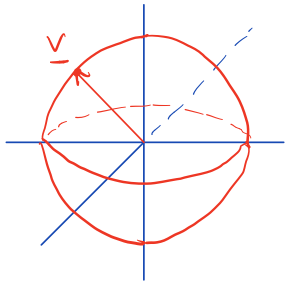
Def

We call any vector $\underline{v}$ with length (norm) $\|\underline{v}\| = 1$ a <u>unit vector</u>. In $\mathbb{R}^2$, that's all points on circle of radius 1



$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2} = 1$$

In $\mathbb{R}^3$, this is the surface of the unit sphere



$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1$$

Thought exercise: extend this idea in $\mathbb{R}^n$

- We can always scale any vector so that it becomes a unit vector. This is called <u>normalizing</u> a vector.

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|}$$

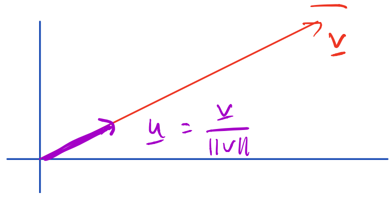
So, we see that

$$\|\underline{u}\|^2 = \left(\sqrt{\underline{u} \cdot \underline{u}}\right)^2$$

$$= \underline{u} \cdot \underline{u} = \frac{\underline{v}}{\|\underline{v}\|} \cdot \frac{\underline{v}}{\|\underline{v}\|}$$

$$= \frac{1}{\|\underline{v}\|^2} \left(\underline{v} \cdot \underline{v}\right) = \frac{1}{\|\underline{v}\|^2} \left(\|\underline{v}\|^2\right) = 1$$

Note that normalizing a vector scales the length but does <u>not</u> change the direction



Ex

Find a unit vector $\underline{u}$ for $\underline{v} = \begin{vmatrix} 1 \\ -2 \\ 2 \\ 0 \end{vmatrix}$

$\underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|}$ so we need $\|\underline{v}\|$

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$$

$$= \sqrt{1^2 + (-2)^2 + 2^2 + 0^2}$$

$$= \sqrt{9} = 3$$

always take + square root b/c length is ≥ 0

Then, $\underline{u} = \dfrac{1}{3} \cdot \underline{v} = \begin{vmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{vmatrix}$.

Exercise: Check $\|\underline{u}\| = 1$