Answers

MA 304 EXAM 2 ——-spring 2018

REMARKS There are 8 problems. Problems 1-4 are each worth 13 points while problems 5-8 are each worth 12 points. Show all *relevant* work. **NO CALCULATORS.**

1. Let S be the subspace of C[a, b] spanned by e^x, xe^x, and x^2e^x. Let D be the differentiation operator of S. Find the matrix representing D with respect to $[e^x, xe^x, x^2e^x]$.

$De^x = e^x$
$Dxe^x = xe^x + e^x$
$Dx^2e^x = x^2e^x + 2xe^x$

$$D_A = \begin{array}{ccc} De^x & Dxe^x & Dx^2e^x \end{array}$$
$$D_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Find the kernel and range of each of the following linear operators on P_3 which has a basis of $\{1, x, x^2\}$.

a. $L(p(x)) = xp'(x)$.

$L(1) = 0$
$L(x) = x$
$L(x^2) = 2x^2$

$R(L) = sp\{x, x^2\}$
$N(L) = sp\{1\}$

b. $L(p(x)) = p(x) - p'(x)$

$L(1) = 1$
$L(x) = x - 1$
$L(x^2) = x^2 - 2x$

$L_A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
$\det L_A = 1$

$R(L) = sp\{1, x, x^2\}$
$N(L) = \{0\}$

c. $L(p(x)) = p(0)x + p(1)$.

$L(1) = x + 1$
$L(x) = 0 + 1$
$L(x^2) = 0 + 1$

$R(L) = sp\{1, x\}$
$N(L) = sp\{x^2 - x\}$

3. Let $L : R^2 \to R^2$ be a linear operator. If

$$L((1,2)^T) = (-2,3)^T$$

$a\binom{1}{2}+b\binom{1}{-1}=\binom{7}{5}$ so $\begin{pmatrix}1&1\\2&-1\end{pmatrix}\binom{a}{b}=\binom{7}{5}$

and

$\binom{a}{b} = -\frac{1}{3}\begin{pmatrix}-1&-1\\-2&1\end{pmatrix}\binom{7}{5}$

$$L((1,-1)^T) = (5,2)^T$$

$= \frac{1}{3}\begin{pmatrix}1&1\\2&-1\end{pmatrix}\binom{7}{5}$

$= \frac{1}{3}\binom{12}{9} = \binom{4}{3}$

find the value of $L((7,5)^T)$.

$L\binom{7}{5} = 4L\binom{1}{2}+3L\binom{1}{-1}$

$= 4\binom{-2}{3}+3\binom{5}{2} = \binom{7}{18}$

$\binom{7}{18}$

4.

a. Find the standard matrix representation of the linear operator L which reflects each vector $\mathbf{x}$ about the line $x_2 = x_1$ and then projects it onto the x_1-axis.

$\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}$

b. Let L be the linear transformation mapping P_2 into R^2 defined by

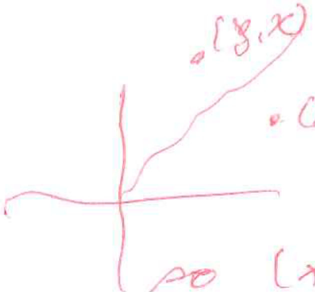
$$L(p(x)) = \begin{bmatrix}\int_0^1 p(x)dx \\ p(0)\end{bmatrix}$$

Find a matrix A such that

$$\begin{array}{cc} L(1) & L(x) \end{array}$$
$$\begin{pmatrix}1 & 1/2\\ 1 & 0\end{pmatrix}$$

$$L(\alpha+\beta x)) = A\binom{\alpha}{\beta}$$

$$\begin{array}{cc} L(1) & L(x) \end{array}$$
$$\begin{pmatrix}1 & 1/2\\ 1 & 0\end{pmatrix}$$



$\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}\binom{x}{y} = \binom{y}{0}$

$= \begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}$

so $(x,y) \to (y,x)$

$\begin{pmatrix}1&0\\0&0\end{pmatrix}\binom{y}{x} = \binom{y}{0}$

$(y,0)$

$L_M = \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$

5. Let V be the subspace of C[a, b] spanned by $1, e^x, e^{-x}$, and let D be the differentiation operator on V.

a. Find the transition matrix S representing the change of coordinates from the ordered basis $[1, e^x, e^{-x}]$ to the ordered basis $[1, \cosh x, \sinh x]$ where $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

bad - good

good to bad

$$V = \begin{array}{c|ccc} & 1 & \cosh x & \sinh x \\ 1 & 1 & 0 & 0 \\ e^x & 0 & 1/2 & 1/2 \\ e^{-x} & 0 & 1/2 & -1/2 \end{array} \qquad V^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

b. Find the matrix A representing D with respect to the ordered basis $[1, \cosh x, \sinh x]$.

$D(1) = 0$

$D(\cosh) = \sinh$

$D \sinh = \cosh$

$$A = \begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \cosh & 0 & 0 & 1 \\ \sinh & 0 & 1 & 0 \end{array}$$

6.

a. Find the nearest point to the plane $6(x - 1) + 2(y - 3) + 3(z + 4) = 0$ for the point $(2, 1, -2)$.

$\Rightarrow 6x + 2y + 3z = 0$,

2 points in plane: $(1, 0, -2)$, $(1, -3, 0)$

nearest point $= (a, b, c)$ so

$[(2, 1, -2) - (a, b, c)] \cdot (1, 0, -2) = 0$

$[$ same $] \cdot (1, -3, 0) = 0$

$6a + 2b + 3c = 0$

$a \quad -2c = -6$

$a - 3b \quad = 0$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{50}{49} \\ \frac{33}{49} \\ \frac{-122}{49} \end{pmatrix}$$

b. Find the distance from $(2, 1, -2)$ to the plane $6(x - 1) + 2(y - 3) + 3(z + 4) = 0$.

$\Rightarrow 6x + 2y + 3z = 0$

$$d = \left| \frac{(2, 1, -2) \cdot (6, 2, 3)}{7} \right|$$

8/7

7.

a. Is it possible for a matrix to have the vector (3, 1, 2) in its row space and $(2, 1, 1)^T$ in its null space? Explain.

NO! $N(A) \perp R(A^T)$

$(2,1,1) \in N(A)$

$(3,1,2) \in R(A^T)$

but $(2,1,1)\cdot(3,1,2) \neq 0$

b. If A is an $m \times n$ matrix of rank r, what are the dimensions of N(A) and $N(A^T)$? Explain.

$A: R^n \to R^m$

$\dim N(A) + \underbrace{\dim R(A)}_{r} = n$

$\dim R(A^T) = \dim R(A)$ $\quad A^T: R^m \to R^n$

$\dim N(A) = n - r$

$\dim N(A^T) = m - r$

8. Find the best least squares fit by quadratic polynomials to the data $(-1, 0), (0, 1), (1, 3), (2, 9)$. (**Write the answer in matrix form: Do not invert any matrix**).

$p(x) = a + bx + cx^2$

$$\begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \end{array} \quad \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}}_{b}$$

$$\boxed{A^T A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^T \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}}$$