## Sect 6.3: (Cont...)

**Thm:**

Let $W$ be a subspace of $\mathbb{R}^n$, let $\underline{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\underline{y}} = \text{proj}_W(\underline{y})$, then $\hat{\underline{y}}$ is the closest point in $W$ to $\underline{y}$ in the sense that

$$\|\underline{y} - \hat{\underline{y}}\| < \|\underline{y} - \underline{v}\|$$

for all $\underline{v}$ in $W$ such that $\underline{v} \neq \hat{\underline{y}}$

**Pf.**

Let $\underline{v}$ in $W$ with $\underline{v} \neq \hat{\underline{y}}$, then $\hat{\underline{y}} - \underline{v}$ is in $W$ because $W$ closed under addition. We also know that $\underline{y} - \hat{\underline{y}}$ is orthogonal to $W$ and therefore orthogonal to $\hat{\underline{y}} - \underline{v}$. Then we see

$$\underline{y} - \underline{v} = \underline{y} \underbrace{- \hat{\underline{y}} + \hat{\underline{y}}}_{\text{add } 0} - \underline{v}$$

$$\underline{y} - \underline{v} = \underbrace{(\underline{y} - \hat{\underline{y}})}_{\in W^\perp} + \underbrace{(\hat{\underline{y}} - \underline{v})}_{\in W}$$

So, by the Pythagorean Thm.

$$\|\underline{y} - \underline{v}\|^2 = \|\underline{y} - \hat{\underline{y}}\|^2 + \|\hat{\underline{y}} - \underline{v}\|^2$$

$$> \|\underline{y} - \hat{\underline{y}}\|^2 \quad \text{because } \|\hat{\underline{y}} - \underline{v}\|^2 > 0$$

We can the define the distance of a vector $\underline{y}$ in $\mathbb{R}^n$ to a subspace $W$ as the distance from $\underline{y}$ to the nearest point $W$, i.e.,

$$\| \underline{y} - \hat{\underline{y}} \| \quad \text{where} \quad \hat{\underline{y}} = \text{proj}_W(\underline{y})$$

## Ex

Find the distance from $\underline{y}$ to $W = \text{span}\{\underline{u}_1, \underline{u}_2\}$ for

$$\underline{y} = \begin{vmatrix} -1 \\ -5 \\ 10 \end{vmatrix}, \quad \underline{u}_1 = \begin{vmatrix} 5 \\ -2 \\ 1 \end{vmatrix}, \quad \underline{u}_2 = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$$

First, we compute $\hat{\underline{y}} = \text{proj}_W(\underline{y})$

$$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2$$

$$= \frac{15}{30} \begin{vmatrix} 5 \\ -2 \\ 1 \end{vmatrix} + \frac{-21}{6} \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 5 \\ -2 \\ 1 \end{vmatrix} + \frac{-7}{2} \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} = \begin{vmatrix} -1 \\ -8 \\ 4 \end{vmatrix}$$

Then $\underline{y} - \hat{\underline{y}} = \begin{vmatrix} -1 \\ -5 \\ 10 \end{vmatrix} - \begin{vmatrix} -1 \\ -8 \\ 4 \end{vmatrix} = \begin{vmatrix} 0 \\ 3 \\ 6 \end{vmatrix}$

And $\| \underline{y} - \hat{\underline{y}} \| = \sqrt{0^2 + 3^2 + 6^2} = \sqrt{45}$ is the distance of $\underline{y}$ to the subspace $W$

Lastly, we want to relate the formula

$$\hat{y} = \mathrm{proj}_W(\underline{y}) = (y \cdot \underline{u}_1)\underline{u}_1 + \cdots + (\underline{y} \cdot \underline{u}_p)\underline{u}_p$$

for an orthonormal basis $\{\underline{u}_1 \cdots \underline{u}_p\}$ of $W$ and $\underline{y}$ in $\mathbb{R}^n$ to the orthogonal matrix $U = \begin{vmatrix} \underline{u}_1 & \cdots & \underline{u}_p \end{vmatrix}$

First, we observe that

$$U^T \underline{y} = \begin{vmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_p^T \end{vmatrix} \underline{y} = \begin{vmatrix} \underline{u}_1^T \underline{y} \\ \underline{u}_2^T \underline{y} \\ \vdots \\ \underline{u}_p^T \underline{y} \end{vmatrix} = \begin{vmatrix} \underline{u}_1 \cdot \underline{y} \\ \underline{u}_2 \cdot \underline{y} \\ \vdots \\ \underline{u}_p \cdot \underline{y} \end{vmatrix} = \begin{vmatrix} \underline{y} \cdot \underline{u}_1 \\ \underline{y} \cdot \underline{u}_2 \\ \vdots \\ \underline{y} \cdot \underline{u}_p \end{vmatrix}$$

Multiplying by $U$ we get

$$U(U^T\underline{y}) = \begin{vmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_p \end{vmatrix} \begin{vmatrix} \underline{y} \cdot \underline{u}_1 \\ \underline{y} \cdot \underline{u}_2 \\ \vdots \\ \underline{y} \cdot \underline{u}_p \end{vmatrix}$$

$$= (\underline{y} \cdot \underline{u}_1)\underline{u}_1 + (\underline{y} \cdot \underline{u}_2)\underline{u}_2 + \cdots + (\underline{y} \cdot \underline{u}_p)\underline{u}_p$$

$$= \mathrm{proj}_W(\underline{y})$$

$$= \hat{y}$$

Thus, we've shown that $\hat{y} = \mathrm{proj}_W(\underline{y}) = UU^T\underline{y}$

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## Sect. 6.4: Gram-Schmidt Process

**Goal:** We've shown that orthogonal vectors/sets have lots of nice properties. But, how do we get an orthogonal set from a non-orthogonal set?

**Strategy:** A series of orthogonal projections.

**Thm** (Gram-Schmidt Process)

Given a (non-orthogonal) basis $\{\underline{x}_1 \cdots \underline{x}_p\}$ for a nonzero subspace $W$ of $\mathbb{R}^n$, define

$$\underline{v}_1 = \underline{x}_1$$

portion of $\underline{x}_2$ in the $\underline{v}_1$ direction

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1$$

portion of $\underline{x}_3$ in the $\underline{v}_1$ direction

$$\underline{v}_3 = \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \frac{\underline{x}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

portion of $x_3$ in the $\underline{v}_2$ direction

$$\vdots$$

$$\underline{v}_p = \underline{x}_p - \frac{\underline{x}_p \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_2} \underline{v}_1 - \cdots - \frac{\underline{x}_p \cdot \underline{v}_{p-1}}{\underline{v}_{p-1} \cdot \underline{v}_{p-1}} \underline{v}_{p-1}$$

The resulting set $\{\underline{v}_1 \cdots \underline{v}_p\}$ is an orthogonal basis for $W$. We have

$$\text{span}\{\underline{v}_1 \cdots \underline{v}_p\} = \text{span}\{\underline{x}_1 \cdots x_p\} = W$$

and $\text{span}\{\underline{v}_1 \cdots \underline{v}_k\} = \text{span}\{\underline{x}_1 \cdots \underline{x}_k\}$ for $1 \le k \le p$

## Ex

Let $\underline{x}_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$, $\underline{x}_2 = \begin{vmatrix} 0 \\ 1 \\ 1 \\ 1 \end{vmatrix}$, $\underline{x}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 1 \end{vmatrix}$ and $W = \text{span}\{\underline{x}_1, \underline{x}_2 \underline{x}_3\}$

Find an orthogonal basis for $W = \text{span}\{\underline{x}_1 \underline{x}_2 \underline{x}_3\}$

$$\underline{v}_1 = \underline{x}_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$$

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 = \begin{vmatrix} 0 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \frac{3}{4} \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{vmatrix}$$

$$\underline{v}_3 = \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} v_1 - \frac{\underline{x}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

$$= \begin{vmatrix} 0 \\ 0 \\ 1 \\ 1 \end{vmatrix} - \frac{2}{4} \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \frac{\left(\frac{1}{2}\right)}{\left(\frac{12}{16}\right)} \begin{vmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{vmatrix} = \begin{vmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{vmatrix}$$

$\frac{2}{3}$

**Check:** $\underline{v}_1 \cdot \underline{v}_3 = \underline{v}_1 \cdot \underline{v}_2 = \underline{v}_2 \cdot \underline{v}_3 = 0$ (orthogonal)

## Notes:

- we can modify the Gram-Schmidt Process to produce orthonormal bases by normalizing the basis vectors at each step

## Ex (above cont...)

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{2} \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{vmatrix}$$

$$\underline{u}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\left(\frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)^{1/2}} \left| \begin{matrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{matrix} \right| = \left| \begin{matrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{matrix} \right|$$

$$\underline{u}_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|} = \frac{1}{\left(4/9 + 1/9 + 1/9\right)^{1/2}} \left| \begin{matrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{matrix} \right| = \left| \begin{matrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{matrix} \right|$$

<u>Check:</u> $\|\underline{u}_1\| = \|\underline{u}_2\| = \|\underline{u}_3\| = 1$ (unit vectors)

$\underline{u}_1 \cdot \underline{u}_2 = \underline{u}_1 \cdot \underline{u}_3 = \underline{u}_2 \cdot \underline{u}_3 = 0$ (orthogonal

- Note that the formula for the kth Gram-Schmidt vector gets longer as k grows. The process gets really costly as the of basis vectors gets large