$$\mathbf{r} = 0.40\left[0.40\begin{bmatrix}0.35\\0.20\\0.25\\0.20\end{bmatrix} + 0.60\begin{bmatrix}0.3289\\0.1739\\0.2188\\0.2784\end{bmatrix}\right] + 0.40\begin{bmatrix}0.21\\0.29\\0.33\\0.17\end{bmatrix} + 0.20\begin{bmatrix}0.23\\0.28\\0.28\\0.21\end{bmatrix}$$

$$= 0.40\begin{bmatrix}0.3373\\0.1843\\0.2313\\0.2470\end{bmatrix} + 0.40\begin{bmatrix}0.21\\0.29\\0.33\\0.17\end{bmatrix} + 0.20\begin{bmatrix}0.23\\0.28\\0.28\\0.21\end{bmatrix} = \begin{bmatrix}0.2649\\0.2457\\0.2805\\0.2088\end{bmatrix}$$

The candidate with the highest rating is O'Leary. Gauss comes in second. Ipsen and Taussky are third and fourth, respectively. ■

SECTION 5.3 EXERCISES

1. Find the least squares solution of each of the following systems:

(a) $\begin{aligned} x_1 + x_2 &= 3\\ 2x_1 - 3x_2 &= 1\\ 0x_1 + 0x_2 &= 2\end{aligned}$ **(b)** $\begin{aligned} -x_1 + x_2 &= 10\\ 2x_1 + x_2 &= 5\\ x_1 - 2x_2 &= 20\end{aligned}$

(c) $\begin{aligned} x_1 + x_2 + x_3 &= 4\\ -x_1 + x_2 + x_3 &= 0\\ -x_2 + x_3 &= 1\\ x_1 \qquad + x_3 &= 2\end{aligned}$

2. For each of your solutions $\hat{\mathbf{x}}$ in Exercise 1:

(a) determine the projection $\mathbf{p} = A\hat{\mathbf{x}}$.

(b) calculate the residual $r(\hat{\mathbf{x}})$.

(c) verify that $r(\hat{\mathbf{x}}) \in N(A^T)$.

3. For each of the following systems $A\mathbf{x} = \mathbf{b}$, find all least squares solutions:

(a) $A = \begin{bmatrix}1 & 2\\2 & 4\\-1 & -2\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}3\\2\\1\end{bmatrix}$

(b) $A = \begin{bmatrix}1 & 1 & 3\\-1 & 3 & 1\\1 & 2 & 4\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}-2\\0\\8\end{bmatrix}$

4. For each of the systems in Exercise 3, determine the projection $\mathbf{p}$ of $\mathbf{b}$ onto $R(A)$ and verify that $\mathbf{b} - \mathbf{p}$ is orthogonal to each of the column vectors of A.

5. (a) Find the best least squares fit by a linear function to the data

x	-1	0	1	2
y	0	1	3	9

(b) Plot your linear function from part (a) along with the data on a coordinate system.

6. Find the best least squares fit to the data in Exercise 5 by a quadratic polynomial. Plot the points $x = -1, 0, 1, 2$ for your function and sketch the graph.

7. Given a collection of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n)^T$$
$$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$$

and let $y = c_0 + c_1 x$ be the linear function that gives the best least squares fit to the points. Show that if $\bar{x} = 0$, then

$$c_0 = \bar{y} \qquad \text{and} \qquad c_1 = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}}$$

8. The point $(\bar{x}, \bar{y})$ is the *center of mass* for the collection of points in Exercise 7. Show that the least squares line must pass through the center of mass. [*Hint*: Use a change of variables $z = x - \bar{x}$ to translate the problem so that the new independent variable has mean 0.]

9. Let A be an $m \times n$ matrix of rank n and let $P = A(A^TA)^{-1}A^T$.

(a) Show that $P\mathbf{b} = \mathbf{b}$ for every $\mathbf{b} \in R(A)$. Explain this property in terms of projections.

(b) If $\mathbf{b} \in R(A)^\perp$, show that $P\mathbf{b} = \mathbf{0}$.

(c) Give a geometric illustration of parts (a) and (b) if $R(A)$ is a plane through the origin in $\mathbb{R}^3$.

10. Let A be an 8×5 matrix of rank 3, and let $\mathbf{b}$ be a nonzero vector in $N(A^T)$.

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(b) How many least squares solutions will the system $A\mathbf{x} = \mathbf{b}$ have? Explain.

11. Let $P = A(A^TA)^{-1}A^T$, where A is an $m \times n$ matrix of rank n.

(a) Show that $P^2 = P$.

(b) Prove that $P^k = P$ for $k = 1, 2, \ldots$.

(c) Show that P is symmetric. [*Hint*: If B is nonsingular, then $(B^{-1})^T = (B^T)^{-1}$.]

12. Show that if

$$\begin{pmatrix} A & I \\ O & A^T \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

then $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ and $\mathbf{r}$ is the residual vector.

13. Let $A \in \mathbb{R}^{m \times n}$ and let $\hat{\mathbf{x}}$ be a solution of the least squares problem $A\mathbf{x} = \mathbf{b}$. Show that a vector $\mathbf{y} \in \mathbb{R}^n$ will also be a solution if and only if $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$, for some vector $\mathbf{z} \in N(A)$. [*Hint*: $N(A^TA) = N(A)$.]

14. Find the equation of the circle that gives the best least squares circle fit to the points $(-1, -2)$, $(0, 2.4)$, $(1.1, -4)$, and $(2.4, -1.6)$.

15. Suppose that in the search procedure described in Example 4, the search committee made the following judgments in evaluating the teaching credentials of the candidates:

(i) Gauss and Taussky have equal teaching credentials.

(ii) O'Leary's teaching credentials should be given 1.25 times the weight of Ipsen's credentials and 1.75 times the weight given to the credentials of both Gauss and Taussky.

(iii) Ipsen's teaching credentials should be given 1.25 times the weight given to the credentials of both Gauss and Taussky.

(a) Use the method given in Application 4 to determine a weight vector for rating the teaching credentials of the candidates.

(b) Use the weight vector from part (a) to obtain overall ratings of the candidates.

5.4 Inner Product Spaces

Scalar products are useful not only in $\mathbb{R}^n$, but in a wide variety of contexts. To generalize this concept to other vector spaces, we introduce the following definition.

Definition and Examples

Definition

An **inner product** on a vector space V is an operation on V that assigns, to each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V, a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ satisfying the following conditions:

I. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$.

II. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in V.

III. $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V and all scalars α and β.

A vector space V with an inner product is called an **inner product space**.

The Vector Space $\mathbb{R}^n$

The standard inner product for $\mathbb{R}^n$ is the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$$

Given a vector $\mathbf{w}$ with positive entries, we could also define an inner product on $\mathbb{R}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i \tag{1}$$

The entries w_i are referred to as *weights*.

SECTION 5.4 EXERCISES

1. Let $\mathbf{x} = (-1, -1, 1, 1)^T$ and $\mathbf{y} = (1, 1, 5, -3)^T$. Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{x}+\mathbf{y}\|_2$ and verify that the Pythagorean law holds.

2. Let $\mathbf{x} = (1, 1, 1, 1)^T$ and $\mathbf{y} = (8, 2, 2, 0)^T$.

(a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

(c) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

(d) Compute $\|\mathbf{x} - \mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, $\|\mathbf{x}\|_2$ and verify that the Pythagorean law is satisfied.

3. Use equation (1) with weight vector $\mathbf{w} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^T$ to define an inner product for $\mathbb{R}^3$, and let $\mathbf{x} = (1, 1, 1)^T$ and $\mathbf{y} = (-5, 1, 3)^T$.

(a) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal with respect to this inner product.

(b) Compute the values of $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ with respect to this inner product.

4. Given

$$A = \begin{Bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{Bmatrix}$$

determine the value of each of the following.

(a) $\langle A, B \rangle$ **(b)** $\|A\|_F$

(c) $\|B\|_F$ **(d)** $\|A + B\|_F$

5. Show that equation (2) defines an inner product on $\mathbb{R}^{m\times n}$.

6. Show that the inner product defined by equation (3) satisfies the last two conditions of the definition of an inner product.

7. In $C[0, 1]$, with inner product defined by (3), compute

(a) $\langle e^x, e^{-x} \rangle$ **(b)** $\langle x, \sin \pi x \rangle$ **(c)** $\langle x^2, x^3 \rangle$

8. In $C[0, 1]$, with inner product defined by (3), consider the vectors 1 and x.

(a) Find the angle θ between 1 and x.

(b) Determine the vector projection $\mathbf{p}$ of 1 onto x and verify that $1 - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

(c) Compute $\|1 - \mathbf{p}\|$, $\|\mathbf{p}\|$, $\|1\|$ and verify that the Pythagorean law holds.

9. In $C[-\pi, \pi]$ with inner product defined by (6), show that $\cos mx$ and $\sin nx$ are orthogonal and that both are unit vectors. Determine the distance between the two vectors.

10. Show that the functions x and x^2 are orthogonal in P_5 with inner product defined by (5), where $x_i = (i - 3)/2$ for $i = 1, \ldots, 5$.

11. In P_5 with inner product as in Exercise 10 and norm defined by

$$\|p\| = \sqrt{\langle p, p \rangle} = \left\{ \sum_{i=1}^{5} [p(x_i)]^2 \right\}^{1/2}$$

compute

(a) $\|x\|$ **(b)** $\|x^2\|$

(c) the distance between x and x^2

12. If V is an inner product space, show that

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

satisfies the first two conditions in the definition of a norm.

13. Show that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

defines a norm on $\mathbb{R}^n$.

14. Show that

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$$

defines a norm on $\mathbb{R}^n$.

15. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for each of the following vectors in $\mathbb{R}^3$.

(a) $\mathbf{x} = (-3, 4, 0)^T$ **(b)** $\mathbf{x} = (-1, -1, 2)^T$

(c) $\mathbf{x} = (1, 1, 1)^T$

16. Let $\mathbf{x} = (5, 2, 4)^T$ and $\mathbf{y} = (3, 3, 2)^T$. Compute $\|\mathbf{x} - \mathbf{y}\|_1$, $\|\mathbf{x} - \mathbf{y}\|_2$, and $\|\mathbf{x} - \mathbf{y}\|_\infty$. Under which norm are the two vectors closest together? Under which norm are they farthest apart?

17. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in an inner product space. Show that if $\mathbf{x} \perp \mathbf{y}$ then the distance between $\mathbf{x}$ and $\mathbf{y}$ is

$$\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^{1/2}$$

18. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space that satisfy the Pythagorean law

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal.

19. In $\mathbb{R}^n$ with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$$

derive a formula for the distance between two vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$.