Sect. 4.9: Application to Markov Chain

Today, we'll go back and cover a specific application related to probability and modeling.

Questions

- How does a system evolve over time?
- Does a given system reach an equilibrium?
- What is that equilibrium?

Def

A vector $\underline{x}$ with non-negative entries that sum to 1 is a <u>probability vector</u>.

Ex

$$\underline{x} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} 0.05 \\ 0.65 \\ 0.3 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Def

A <u>stocastic matrix</u> is a square matrix whose columns are porbababability vectors

Ex

$$P = \begin{bmatrix} 0.6 & 1.0 \\ 0.4 & 0.0 \end{bmatrix} \quad P = \begin{bmatrix} 0.6 & 0.1 & 0.0 \\ 0.3 & 0.2 & 0.8 \\ 0.1 & 0.7 & 0.2 \end{bmatrix}$$

Fact:

- The product of a stocastic matrix and a probability vector is a probability vector, i.e., stocastic matrices map probability vectors to probability vectors

↳ Can prove this (we don't)

Ex $P = \begin{vmatrix} 0.6 & 1.0 \\ 0.4 & 0.0 \end{vmatrix}$ $\underline{x} = \begin{vmatrix} 0.1 \\ 0.9 \end{vmatrix}$

$$P\underline{x} = \begin{vmatrix} 0.6 & 1.0 \\ 0.4 & 0.0 \end{vmatrix} \begin{vmatrix} 0.1 \\ 0.9 \end{vmatrix} = \begin{vmatrix} 0.06 + 0.90 \\ 0.04 + 0.00 \end{vmatrix} = \begin{vmatrix} 0.96 \\ 0.04 \end{vmatrix}$$

Def

A Markov chain is a sequence of probability vectors $x_0, x_1, x_2, \cdots$ with stocastic matrix P such that

$$\underline{x}_1 = P\underline{x}_0 \quad , \quad \underline{x}_2 = P\underline{x}_1 \quad , \quad \underline{x}_3 = P\underline{x}_2$$

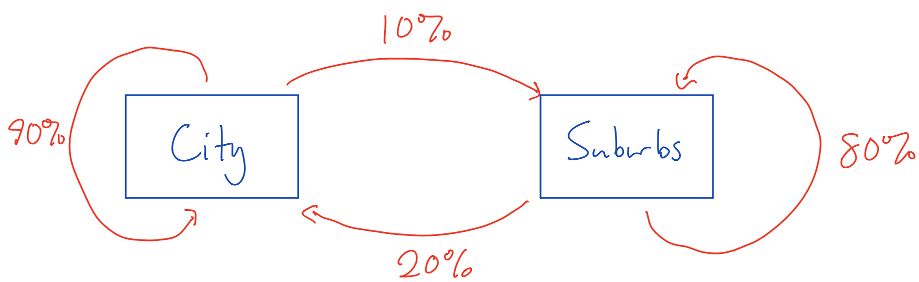
That is,

$$\underline{x}_{k+1} = P\underline{x}_k$$

We often call $\underline{x}_k$ the state vector.

Ex

Imagine a simple population model of people moving from the city to the suburbs and vice versa. Every year, 10% of city dwellers move to the suburbs and 90% stay. 20% of suburbanites move to the city and 80% stay. This year (time 0), 60% of the population lives in the city and 40% in the suburbs.



What percentage of residents will live in each place in 3 years?

- initial state: $\underline{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \begin{array}{l} \leftarrow \text{\% in city} \\ \leftarrow \text{\% in suburbs} \end{array}$

- stocastic matrix: $P = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$

The evolution of the system is given by the Markov chain

year 1: $\underline{x}_1 = P\underline{x}_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.62 \\ 0.38 \end{bmatrix}$

year 2: $\underline{x}_2 = P\underline{x}_1 = \begin{vmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{vmatrix} \begin{vmatrix} 0.62 \\ 0.38 \end{vmatrix} = \begin{vmatrix} 0.634 \\ 0.366 \end{vmatrix}$

year 3: $\underline{x}_3 = P\underline{x}_2 = \begin{vmatrix} 0.6438 \\ 0.3562 \end{vmatrix}$

We observe that every year a greater portion of people live in the city and a smaller portion in the country, but the rate of change slows down.

↳ What happens in the long term? 100 years? 1000 years?

$$\underline{x}_{10} = \begin{vmatrix} 0.6648\ldots \\ 0.3352\ldots \end{vmatrix} \qquad \underline{x}_{100} \approx \begin{vmatrix} 0.6667 \\ 0.3333 \end{vmatrix}$$

It seems that the Markov chain $\underline{x}_0, \underline{x}_1, \underline{x}_2, \ldots$ converges a fixed vector $\underline{x} = \begin{vmatrix} 2/3 \\ 1/3 \end{vmatrix}$.

↳ This evolution of Markov chains to fixed vectors is used to model populations, disease, dynamic systems, trends, ...

Def

If P is a stocastic matrix, then a <u>steady-state vector</u> (or <u>equilibrium vector</u>) for P is a probability vector $\underline{q}$ such that:

$$P\underline{q} = \underline{q}$$ ← apply P doesn't change the state

Notes

- $\underline{q}$ is the eigenvector corresponding to $\lambda = 1$, which is always the largest eigenvalue of a stocastic matrix

- to find the steady-state vector $\underline{q}$, we can solve $P\underline{q} = \underline{q} \Rightarrow (P - I)\underline{q} = \underline{0}$

Ex Find the steady-state vector for

$$P = \begin{vmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{vmatrix}$$

$$(P - I)\underline{q} = \underline{0} \Rightarrow \left|\begin{array}{cc|c} -0.1 & 0.2 & 0 \\ 0.1 & -0.2 & 0 \end{array}\right| \sim \left|\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right|$$

Thus we set $\underline{q} = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$. Making $\underline{q}$ a prob. vector. by $\underline{q} = \begin{vmatrix} 2/3 \\ 1/3 \end{vmatrix}$, the steady-state vector we observed experimentally on the computer above.

Fact

- Not every stocastic matrix has a steady-state vector.

Ex

$$P = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad X_0 = \begin{vmatrix} a \\ b \end{vmatrix} \quad \text{where } a + b = 1,\ a, b \geq 0$$

Markov chain

$$\underline{x}_1 = Px_0 = \begin{vmatrix} b \\ a \end{vmatrix}, \quad \underline{x}_2 = P\underline{x}_1 = \begin{vmatrix} a \\ b \end{vmatrix}, \quad \underline{x}_3 = P\underline{x}_2 = \begin{vmatrix} b \\ a \end{vmatrix}, \ \ldots$$

So, the system oscillates without reaching a steady state.

Def

A stochastic matrix is regular if there exists some integer $k \geq 0$ such that $P^k = \underbrace{P * P * \cdots * P}_{k \text{ times}}$ contains strictly positive entries.

↳ i.e. P^k has no 0's

Thm

If P is an $n \times n$, regular stocastic matrix, then P has a unique steady-state vector $\underline{q}$. Furthermore, for any initial state $\underline{x}_0$, the Markov chain $\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots$ converges to $\underline{q}$ as $k \to \infty$