Step 1
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right)$$

Step 2
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right)$$

Step 3
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{array}\right)$$

Figure 1.1.2.

SECTION 1.1 EXERCISES

1. Use back substitution to solve each of the following systems of equations:

(a) $$\begin{aligned} x_1 - 3x_2 &= 2 \\ 2x_2 &= 6 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 2x_2 + x_3 &= 5 \\ 3x_3 &= 9 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 5 \\ 3x_2 + x_3 - 2x_4 &= 1 \\ -x_3 + 2x_4 &= -1 \\ 4x_4 &= 4 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ 2x_2 + x_3 - 2x_4 + x_5 &= 1 \\ 4x_3 + x_4 - 2x_5 &= 1 \\ x_4 - 3x_5 &= 0 \\ 2x_5 &= 2 \end{aligned}$$

2. Write out the coefficient matrix for each of the systems in Exercise 1.

3. In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

(a) $$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - x_2 &= 2 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 - 4x_2 &= 4 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 - x_2 &= 3 \\ -4x_1 + 2x_2 &= -6 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

4. Write an augmented matrix for each of the systems in Exercise 3.

5. Write out the system of equations that corresponds to each of the following augmented matrices:

(a) $$\left\{\begin{array}{cc|c} 3 & 2 & 8 \\ 1 & 5 & 7 \end{array}\right\}$$

(b) $$\left\{\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 2 & 3 & -4 & 0 \end{array}\right\}$$

(c) $$\left\{\begin{array}{ccc|c} 2 & 1 & 4 & -1 \\ 4 & -2 & 3 & 4 \\ 5 & 2 & 6 & -1 \end{array}\right\}$$

(d) $$\left\{\begin{array}{cccc|c} 4 & -3 & 1 & 2 & 4 \\ 3 & 1 & -5 & 6 & 5 \\ 1 & 1 & 2 & 4 & 8 \\ 5 & 1 & 3 & -2 & 7 \end{array}\right\}$$

6. Solve each of the following systems.

(a) $$\begin{aligned} x_1 - 2x_2 &= 5 \\ 3x_1 + x_2 &= 1 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + x_2 &= 8 \\ 4x_1 - 3x_2 &= 6 \end{aligned}$$

(c) $$\begin{aligned} 4x_1 + 3x_2 &= 4 \\ \tfrac{2}{3}x_1 + 4x_2 &= 3 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

(e) $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}$$

(f) $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

(g) $$\begin{aligned} \tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + 2x_3 &= -1 \\ x_1 + 2x_2 + \tfrac{3}{2}x_3 &= \tfrac{3}{2} \\ \tfrac{1}{2}x_1 + 2x_2 + \tfrac{12}{5}x_3 &= \tfrac{1}{10} \end{aligned}$$

(h) $$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 \qquad + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

7. The two systems

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &= 5 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 + x_2 &= -1 \\ 4x_1 + 3x_2 &= 1 \end{aligned}$$

have the same coefficient matrix but different right-hand sides. Solve both systems simultaneously by eliminating the first entry in the second row of the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 4 & 3 & 5 & 1 \end{array}\right)$$

and then performing back substitutions for each of the columns corresponding to the right-hand sides.

8. Solve the two systems

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= 9 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 - 2x_3 &= 9 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= -2 \end{aligned}$$

by doing elimination on a 3×5 augmented matrix and then performing two back substitutions.

9. Given a system of the form

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

where m_1, m_2, b_1, and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation of parts (a) and (b).

10. Consider a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constants. Explain why a system of this form must be consistent.

11. Give a geometrical interpretation of a linear equation in three unknowns. Give a geometrical description of the possible solution sets for a 3×3 linear system.

1.2 Row Echelon Form

In Section 1.1 we learned a method for reducing an $n \times n$ linear system to strict triangular form. However, this method will fail if, at any stage of the reduction process, all the possible choices for a pivot element in a given column are 0.

EXAMPLE 1 Consider the system represented by the augmented matrix

$$\begin{array}{cl} \left(\begin{array}{ccccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) & \begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \\ \\ \end{array} \end{array}$$

If row operation III is used to eliminate the nonzero entries in the last four rows of the first column, the resulting matrix will be

$$\begin{array}{cl} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) & \begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \\ \\ \end{array} \end{array}$$

At this stage, the reduction to strict triangular form breaks down. All four possible choices for the pivot element in the second column are 0. How do we proceed from

Finally, using the third row of the table, we get

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_3$$

These equations can be rewritten as a homogeneous system:

$$\begin{aligned} -\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &= 0 \\ \tfrac{1}{4}x_1 - \tfrac{2}{3}x_2 + \tfrac{1}{4}x_3 &= 0 \\ \tfrac{1}{4}x_1 + \tfrac{1}{3}x_2 - \tfrac{3}{4}x_3 &= 0 \end{aligned}$$

The reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is one free variable: x_3. Setting $x_3 = 3$, we obtain the solution $(5, 3, 3)$, and the general solution consists of all multiples of $(5, 3, 3)$. It follows that the variables x_1, x_2, and x_3 should be assigned values in the ratio

$$x_1 : x_2 : x_3 = 5 : 3 : 3$$

This simple system is an example of the closed Leontief input-output model. Leontief's models are fundamental to our understanding of economic systems. Modern applications would involve thousands of industries and lead to very large linear systems. The Leontief models will be studied in greater detail later in Section 6.8 of the book.

SECTION 1.2 EXERCISES

1. Which of the matrices that follow are in row echelon form? Which are in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix}$ **(h)** $\begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. The augmented matrices that follow are in row echelon form. For each case, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

(a) $\left[\begin{array}{rr|r} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$ **(b)** $\left[\begin{array}{rr|r} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{rrr|r} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{rrr|r} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(e) $\left[\begin{array}{rrr|r} 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(f) $\left[\begin{array}{rrr|r} 1 & -1 & 3 & 8 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

3. The augmented matrices that follow are in reduced row echelon form. In each case, find the solution set to the corresponding linear system.

(a) $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$ (b) $\left[\begin{array}{rrr|r} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(c) $\left[\begin{array}{rrr|r} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$

(e) $\left[\begin{array}{rrrr|r} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{rrr|r} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

4. For each of the systems in Exercise 3, make a list of the lead variables and a second list of the free variables.

5. For each of the systems of equations that follow, use Gaussian elimination to obtain an equivalent system whose coefficient matrix is in row echelon form. Indicate whether the system is consistent. If the system is consistent and involves no free variables, use back substitution to find the unique solution. If the system is consistent and there are free variables, transform it to reduced row echelon form and find all solutions.

(a) $\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - x_2 &= 9 \end{aligned}$ (b) $\begin{aligned} 2x_1 - 3x_2 &= 5 \\ -4x_1 + 6x_2 &= 8 \end{aligned}$

(c) $\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$ (d) $\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 2x_3 &= 1 \\ 11x_1 + 2x_2 + x_3 &= 14 \end{aligned}$

(e) $\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$

(f) $\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= 1 \\ 7x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$

(g) $\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 - x_3 - x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + x_4 &= 5 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 4 \end{aligned}$ (h) $\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 1 \\ -5x_1 + 8x_2 &= 4 \end{aligned}$

(i) $\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$

(j) $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned}$

(k) $\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ x_1 - 5x_2 + x_4 &= 5 \end{aligned}$

(l) $\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 2 \\ x_1 + 4x_2 - 2x_3 &= 1 \\ 5x_1 - 8x_2 + 2x_3 &= 5 \end{aligned}$

6. Use Gauss–Jordan reduction to solve each of the following systems.

(a) $\begin{aligned} x_1 + x_2 &= -1 \\ 4x_1 - 3x_2 &= 3 \end{aligned}$

(b) $\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ 3x_1 + x_2 + 2x_3 - x_4 &= -1 \end{aligned}$

(c) $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$

(d) $\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$

7. Give a geometric explanation of why a homogeneous linear system consisting of two equations in three unknowns must have infinitely many solutions. What are the possible numbers of solutions of a nonhomogeneous 2×3 linear system? Give a geometric explanation of your answer.

8. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right)$$

For what values of a will the system have a unique solution?

9. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right)$$

(a) Is it possible for the system to be inconsistent? Explain.

(b) For what values of β will the system have infinitely many solutions?

10. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right)$$

(a) For what values of a and b will the system have infinitely many solutions?

(b) For what values of a and b will the system be inconsistent?

11. Given the linear systems

(i) $\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 7x_2 &= 8 \end{aligned}$ **(ii)** $\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 7x_2 &= 7 \end{aligned}$

solve both systems by incorporating the right-hand sides into a 2×2 matrix B and computing the reduced row echelon form of

$$(A|B) = \left(\begin{array}{rr|rr} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right)$$

12. Given the linear systems

(i)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

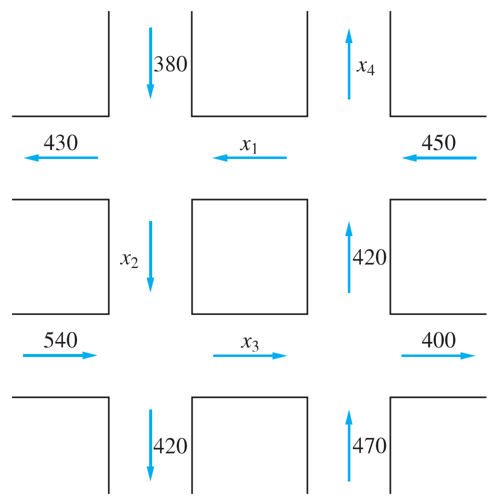
(ii)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ -x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 &= -2 \end{aligned}$$

solve both systems by computing the row echelon form of an augmented matrix $(A|B)$ and performing back substitution twice.

13. Given a homogeneous system of linear equations, if the system is overdetermined, what are the possibilities as to the number of solutions? Explain.

14. Given a nonhomogeneous system of linear equations, if the system is underdetermined, what are the possibilities as to the number of solutions? Explain.

15. Determine the values of x_1, x_2, x_3, x_4 for the following traffic flow diagram.



16. Consider the traffic flow diagram that follows, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are fixed positive integers. Set up a linear system in the unknowns x_1, x_2, x_3, x_4 and show that the system will be consistent if and only if

$$a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$$

What can you conclude about the number of automobiles entering and leaving the traffic network?

For Web searches, a more sophisticated algorithm is necessary for ranking the pages that contain all of the key search words. In Chapter 6 we will study a special type of matrix model for assigning probabilities in certain random processes. This type of model is referred to as a *Markov process* or a *Markov chain*. In Section 6.3 we will see how to use Markov chains to model Web surfing and obtain rankings of Web pages.

References

1. Berry, Michael W., and Murray Browne, *Understanding Search Engines: Mathematical Modeling and Text Retrieval*, SIAM, Philadelphia, 1999.
2. Langville, Amy N., and Carl D. Meyer, *Google's PageRank and Beyond: The Science of Search Engine Rankings*, Princeton University Press, 2012.

SECTION 1.3 EXERCISES

1. If

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix}$$

compute

(a) $2A$ **(b)** $A + B$

(c) $2A - 3B$ **(d)** $(2A)^T - (3B)^T$

(e) AB **(f)** BA

(g) $A^T B^T$ **(h)** $(BA)^T$

2. For each of the pairs of matrices that follow, determine whether it is possible to multiply the first matrix times the second. If it is possible, perform the multiplication.

(a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 5 \end{bmatrix}$

3. For which of the pairs in Exercise 2 is it possible to multiply the second matrix times the first, and what would the dimension of the product matrix be?

4. Write each of the following systems of equations as a matrix equation:

(a) $\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 2x_1 - 3x_2 &= 5 \end{aligned}$

(b) $\begin{aligned} x_1 + x_2 \phantom{{}- x_3} &= 5 \\ 2x_1 + x_2 - x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 7 \end{aligned}$

(c) $\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$

5. If

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{bmatrix}$$

verify that

(a) $5A = 3A + 2A$ **(b)** $6A = 3(2A)$

(c) $(A^T)^T = A$

6. If

$$A = \begin{bmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{bmatrix}$$

verify that

(a) $A + B = B + A$

(b) $3(A + B) = 3A + 3B$

(c) $(A + B)^T = A^T + B^T$

7. If

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$$

verify that

(a) $3(AB) = (3A)B = A(3B)$,

(b) $(AB)^T = B^T A^T$

8. If

$$A = \begin{Bmatrix} 2 & 4 \\ 1 & 3 \end{Bmatrix}, \quad B = \begin{Bmatrix} -2 & 1 \\ 0 & 4 \end{Bmatrix}, \quad C = \begin{Bmatrix} 3 & 1 \\ 2 & 1 \end{Bmatrix}$$

verify that

(a) $(A + B) + C = A + (B + C)$

(b) $(AB)C = A(BC)$

(c) $A(B + C) = AB + AC$

(d) $(A + B)C = AC + BC$

9. Let

$$A = \begin{Bmatrix} 1 & 2 \\ 1 & -2 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}, \quad \mathbf{c} = \begin{Bmatrix} -3 \\ -2 \end{Bmatrix}$$

(a) Write $\mathbf{b}$ as a linear combination of the column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

(b) Use the result from part (a) to determine a solution of the linear system $A\mathbf{x} = \mathbf{b}$. Does the system have any other solutions? Explain.

(c) Write $\mathbf{c}$ as a linear combination of the column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

10. For each of the choices of A and $\mathbf{b}$ that follow, determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent by examining how $\mathbf{b}$ relates to the column vectors of A. Explain your answers in each case.

(a) $A = \begin{Bmatrix} 2 & 1 \\ -2 & -1 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$

(b) $A = \begin{Bmatrix} 1 & 4 \\ 2 & 3 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$

(c) $A = \begin{Bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

11. Let A be a 5×3 matrix. If

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_2 + \mathbf{a}_3$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

12. Let A be a 3×4 matrix. If

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4$$

then what can you conclude about the number of solutions to the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

13. Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose augmented matrix $(A|\mathbf{b})$ has reduced row echelon form

$$\left\{\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right\}$$

(a) Find all solutions to the system.

(b) If

$$\mathbf{a}_1 = \begin{Bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{Bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{Bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{Bmatrix}$$

determine $\mathbf{b}$.

14. Suppose in the search and screen example in Application 2 the committee decides that research is actually 1.5 times as important as teaching and 3 times as important as professional activities. The committee still rates teaching twice as important as professional activities. Determine a new weight vector $\mathbf{w}$ that reflects these revised priorities. Determine also a new rating vector $\mathbf{r}$. Will the new weights have any effect on the overall rankings of the candidates?

15. Let A be an $m \times n$ matrix. Explain why the matrix multiplications $A^T A$ and AA^T are possible.

16. A matrix A is said to be *skew symmetric* if $A^T = -A$. Show that if a matrix is skew symmetric, then its diagonal entries must all be 0.

17. In Application 3, suppose that we are searching the database of seven linear algebra books for the search words *elementary, matrix, algebra*. Form a search vector $\mathbf{x}$, and then compute a vector $\mathbf{y}$ that represents the results of the search. Explain the significance of the entries of the vector $\mathbf{y}$.

18. Let A be a 2×2 matrix with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$. Show that A can be factored into a product of the form

$$\begin{Bmatrix} 1 & 0 \\ \alpha & 1 \end{Bmatrix} \begin{Bmatrix} a_{11} & a_{12} \\ 0 & b \end{Bmatrix}$$

What is the value of b?

1.4 Matrix Algebra

The algebraic rules used for real numbers may or may not work when matrices are used. For example, if a and b are real numbers, then

$$a + b = b + a \quad \text{and} \quad ab = ba$$