Sect 6.2 : Orthogonal Sets

Recall : In the previous section, we introduced orthogonality of vectors and the orthogonal complement of a subspace.

Def A set of vectors $\{\underline{u}_1 \cdots \underline{u}_p\}$ in $\mathbb{R}^n$ is an <u>orthogonal set</u> if each pair of distinct vectors in the set is orthogonal. That, $\underline{u}_i \cdot \underline{u}_j = 0$ for $i \neq j$

<u>Ex</u>

Show that $\{\underline{u}_1\ \underline{u}_2\ \underline{u}_3\}$ is an orthogonal set.

$$\underline{u}_1 = \begin{vmatrix} 3 \\ 1 \\ 1 \end{vmatrix}, \quad \underline{u}_2 = \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix}, \quad \underline{u}_3 = \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix}$$

Need to show $\underline{u}_i \cdot \underline{u}_j = 0$ for $i \neq j$

$$\underline{u}_1 \cdot \underline{u}_2 = \begin{vmatrix} 3 & 1 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix} = -3 + 2 + 1 = 0$$

$$\underline{u}_1 \cdot \underline{u}_3 = \begin{vmatrix} 3 & 1 & 1 \end{vmatrix} \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix} = -3/2 - 2 + 7/2 = 0$$

$$\underline{u}_2 \cdot \underline{u}_3 = \begin{vmatrix} -1 & 2 & 1 \end{vmatrix} \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix} = 1/2 - 4 + 7/2 = 0$$

Each distinct pair is orthogonal, so we have an orthogonal set.

Thm

If $S = \{\underline{u}_1 \cdots \underline{u}_p\}$ is an orthogonal set of nonzero in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace $W = \text{span}\{\underline{u}_1 \cdots \underline{u}_p\}$

Pf

We note that if $\{\underline{u}_1 \cdots \underline{u}_p\}$ are linear independent then the equation

$$\underline{0} = c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p \qquad (*)$$

has only the trivial solution $(c_1 = c_2 = \cdots = c_p = 0)$

To show $c_1 \cdots c_p = 0$, take the dot product of $(*)$ with each $\underline{u}_1 \cdots \underline{u}_p$

$$\underline{u}_1 \cdot \underline{0} = \underline{u}_1 \cdot (c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p)$$

$$0 = c_1 \underbrace{(\underline{u}_1 \cdot \underline{u}_1)}_{\neq 0} + c_2 \underbrace{(\underline{u}_1 \cdot \underline{u}_2)}_{= 0 \text{ by orth.}} + \cdots + c_p \underbrace{(\underline{u}_1 \cdot \underline{u}_p)}_{= 0 \text{ by orth.}}$$

$$0 = c_1 (\underline{u}_1 \cdot \underline{u}_1)$$

$$0 = c_1$$

If we do this for all $\underline{u}_j$ for $j = 1 \cdots p$

$$0 = \underline{u}_j \cdot \underline{0} = \underline{u}_j \cdot (c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p) = c_j (\underline{u}_j \cdot \underline{u}_j)$$

So, we get that $c_j = 0$ for all $j = 1 \cdots p$, this shows that (*) has only the trivial solution and $S = \{ \underline{u}_1 \cdots \underline{u}_p \}$ must be linearly independent.

<u>Def</u>

An <u>orthogonal basis</u> for a subspace W of $\mathbb{R}^n$ is a basis W that is also an orthogonal set.

<u>Thm</u>

Let $\{ \underline{u}_1 \cdots \underline{u}_p \}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then, for every $\underline{y}$ in W, we can write

$$\underline{y} = c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p$$

and the weights $c_j = \dfrac{\underline{y} \cdot \underline{u}_j}{\underline{u}_j \cdot \underline{u}_j}$ for all $j = 1 \cdots p$

$\hookrightarrow$ Previously, we had to solve a linear system to get the coordinates of $\underline{y}$ relative to a basis.

<u>Pf</u> Since $\{ \underline{u}_1 \cdots \underline{u}_p \}$ is a basis, we know we can write

$$y = c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p \qquad (*)$$

for weights $c_1 \cdots c_p$. We then take the

Dot product of (*) with $\underline{u}_j$

$$\underline{u}_j \cdot \underline{y} = \underline{u}_j \cdot \left(c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_p \underline{u}_p\right)$$

$$\underline{u}_j \cdot \underline{y} = \underbrace{c_1 (\underline{u}_j \cdot \underline{u}_1)}_{=0 \text{ by orth.}} + \cdots + c_j (\underline{u}_j \cdot \underline{u}_j) + \cdots + \underbrace{c_p (\underline{u}_j \cdot \underline{u}_p)}_{=0 \text{ by orth.}}$$

$$\underline{u}_j \cdot \underline{y} = c_j (\underline{u}_j \cdot \underline{u}_j)$$

$$c_j = \frac{\underline{y} \cdot \underline{u}_j}{\underline{u}_j \cdot \underline{u}_j}$$. This works for all $j = 1 \cdots p$

#

Ex

Write $\underline{y} = \begin{vmatrix} 6 \\ 1 \\ -8 \end{vmatrix}$ as a linear combination of

$S = \{\underline{u}_1\ \underline{u}_2\ \underline{u}_3\}$ for $\underline{u}_1 = \begin{vmatrix} 3 \\ 1 \\ 1 \end{vmatrix}$, $\underline{u}_2 = \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix}$, $\underline{u}_3 = \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix}$

First, we compute all the dot products

$$\underline{y} \cdot \underline{u}_1 = \begin{vmatrix} 6 & 1 & -8 \end{vmatrix} \begin{vmatrix} 3 \\ 1 \\ 1 \end{vmatrix} = 18 + 1 - 8 = 11$$

$$\underline{y} \cdot \underline{u}_2 = \begin{vmatrix} 6 & 1 & -8 \end{vmatrix} \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix} = -6 + 2 - 8 = -12$$

$$\underline{y} \cdot \underline{u}_3 = \begin{vmatrix} 6 & 1 & -8 \end{vmatrix} \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix} = -3 - 2 - 28 = -33$$

$$\underline{u}_1 \cdot \underline{u} = \begin{vmatrix} 3 & 1 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 1 \\ 1 \end{vmatrix} = 9 + 1 + 1 = 11$$

$$\underline{u}_2 \cdot \underline{u}_2 = \begin{vmatrix} -1 & 2 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix} = 1 + 4 + 1 = 6$$

$$\underline{u}_3 \cdot \underline{u}_3 = \begin{vmatrix} -1/2 & -2 & 7/2 \end{vmatrix} \begin{vmatrix} -1/2 \\ -2 \\ 7/2 \end{vmatrix} = 1/4 + 4 + \frac{49}{4} = \frac{66}{4} = 33/2$$

So $c_1 = \dfrac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} = \dfrac{11}{11} = 1$

$$c_2 = \frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\underline{y} \cdot \underline{u}_3}{\underline{u}_3 \cdot \underline{u}_3} = \frac{-33}{(33/2)} = -2$$

Which gives $\underline{y} = \underline{u}_1 - 2\underline{u}_2 - 2\underline{u}_3$ (check this answer)

Orthogonal Projection

Consider $\underline{y}$ in $\mathbb{R}^n$. Given a vector $\underline{u}$, can we split $\underline{y}$ into the sum of two pieces

$$\underline{y} = \hat{\underline{y}} + \underline{z}$$

where $\hat{\underline{y}} = \alpha \underline{u}$ is the orthogonal projection of $\underline{y}$ in the direction $\underline{u}$ and $\underline{z}$ is orthogonal $\underline{y}$?

We call $\underline{z}$ the complement of $\underline{y}$ orthogonal to $\underline{u}$.

We can do this. To show its possible, we first note

$$\underline{y} = \hat{\underline{y}} + \underline{z} = \alpha \underline{u} + \underline{z} \Rightarrow \underline{z} = \underline{y} - \alpha \underline{u}$$

We then need $\hat{\underline{y}} \cdot \underline{z} = 0$ (orthogonal)

$$0 = \hat{\underline{y}} \cdot \underline{z}$$

$$0 = (\alpha \underline{u}) \cdot z$$

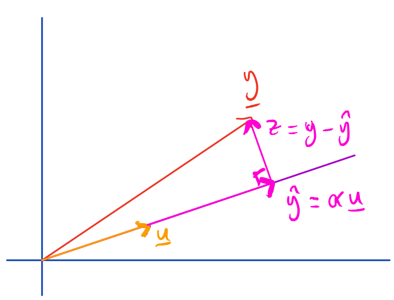
$$0 = (\alpha \underline{u}) \cdot (\underline{y} - \alpha \underline{u})$$

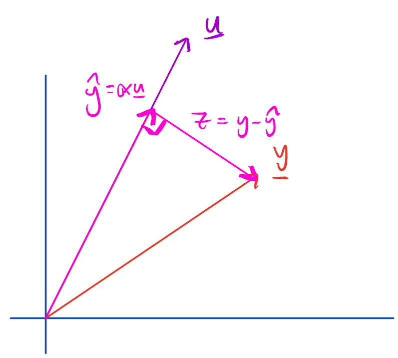
$$0 = \underline{u} \cdot (\underline{y} - \alpha \underline{u})$$

$$0 = \underline{u} \cdot \underline{y} - \alpha \underline{u} \cdot \underline{u}$$

$$\alpha = \frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}}$$

Geometrically,





So, we're breaking $\underline{y}$ up into pieces corresponding to $\underline{u}$ and a vector orthogonal to it.

Ex Let $\underline{y} = \begin{vmatrix} 7 \\ 6 \end{vmatrix}$ and $\underline{u} = \begin{vmatrix} 4 \\ 2 \end{vmatrix}$, find the orthogonal projection of $\underline{y}$ onto $\underline{u}$

$$\underline{y} \cdot \underline{u} = \begin{vmatrix} 7 & 6 \end{vmatrix} \begin{vmatrix} 4 \\ 2 \end{vmatrix} = 40$$

$$\underline{u} \cdot \underline{u} = \begin{vmatrix} 4 & 2 \end{vmatrix} \begin{vmatrix} 4 \\ 2 \end{vmatrix} = 20$$

So, we set $\alpha = \dfrac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} = \dfrac{40}{20} = 2 \Rightarrow \hat{\underline{y}} = \begin{vmatrix} 8 \\ 4 \end{vmatrix} = 2\underline{u}$

The complement of $\underline{y}$ orthogonal to $\underline{u}$ is then

$$\underline{z} = \underline{y} - \hat{\underline{y}} = \begin{vmatrix} 7 \\ 6 \end{vmatrix} - \begin{vmatrix} 8 \\ 4 \end{vmatrix} = \begin{vmatrix} -1 \\ 2 \end{vmatrix}$$

Note $\hat{\underline{y}} \cdot \underline{z} = \begin{vmatrix} 8 & 4 \end{vmatrix} \begin{vmatrix} -1 \\ 2 \end{vmatrix} = -8 + 8 = 0 \Rightarrow$ they're orthogonal