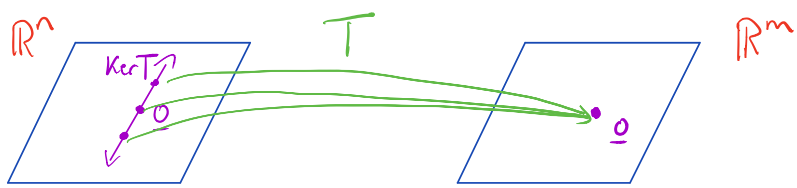
Sect 4.2 (Cont...)

Recall, we were talking about Nul A & Col A, two subspaces associated with a matrix A.

We know that matrix multiplication defines a linear transformation.

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{by} \quad \underline{x} \in \mathbb{R}^n \longmapsto A\underline{x} \in \mathbb{R}^m$$

Def The <u>null space</u> of a linear transformation is also called the <u>kernel</u>, the set of all $\underline{u} \in \mathbb{R}^n$ that map to $\underline{0}$, i.e., $T(\underline{u}) = \underline{0}$



The "column space" of a linear transformation is the <u>range</u>, the set of vectors $\underline{w} \in \mathbb{R}^m$ such that there exist $\underline{v} \in \mathbb{R}^n$ such that $T(\underline{v}) = \underline{w}$

Sect 4.3: Linear Independent Sets & Base

Goal: Find efficient ways to express subspaces, especially Nul A and Col A

<u>Def</u> A set of vectors $\{\underline{v}_1 \cdots \underline{v}_p\}$ in a vector space V is <u>linearly independent</u> if the equation

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_p \underline{v}_p = \underline{0}$$

$$\left(\text{or} \quad \left| \underline{v}_1 \cdots \underline{v}_p \,\middle|\, \underline{0} \right| \right)$$

has only the trivial solution, $c_1 = c_2 = \cdots = c_p = 0$

If there is a nontrivial solution, then the set is <u>linearly dependent</u>.

<u>Ex</u>

$$\underline{v}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} 3 \\ 0 \\ 1 \end{vmatrix}$$

$\{\underline{v}_1\, \underline{v}_2\, \underline{v}_3\}$ is linearly independent because

$$\left| \underline{v}_1\ \underline{v}_2\ \underline{v}_3 \,\middle|\, \underline{0} \right| = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \Rightarrow \underline{c} = \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \underline{0}$$

<u>Ex</u>

$$\underline{v}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad \underline{v}_2 = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} 3 \\ 1 \\ 0 \end{vmatrix}$$

$\{\underline{v}_1\, \underline{v}_2\, \underline{v}_3\}$ is linearly dependent because

$$\left| \underline{v}_1\ \underline{v}_2\ \underline{v}_3 \,\middle|\, \underline{0} \right| = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \Rightarrow \underline{c} = c_3 \begin{vmatrix} -1 \\ -1 \\ 1 \end{vmatrix} \neq \underline{0}$$

For $c_3 = 1$, we get

$$-\underline{v}_1 - \underline{v}_2 + \underline{v}_3 = \underline{0} \quad \Rightarrow \quad \underline{v}_3 = \underline{v}_1 + \underline{v}_2$$

Theorem

An indexed set $\{\underline{v}_1 \cdots \underline{v}_p\}$ for $p \geq 2$ and $\underline{v}_1 = \underline{0}$ is linearly dependent if and only if some $\underline{v}_j$ $(j>1)$ is a linear combination of $\underline{v}_1 \cdots \underline{v}_{j-1}$

↳ That is, there exist weights $c_1 \cdots c_{j-1}$ (not all zero) such that

$$\underline{v}_j = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_{j-1} \underline{v}_{j-1}$$

Def

Let H be a subspace of a vector space V. An indexed set $\mathcal{B} = \{\underline{b}_1 \cdots \underline{b}_p\}$ in V is a basis for H

1) $\mathcal{B}$ is linearly independent

2) the subspace spanned $\mathcal{B}$ coincides with H

$$H = \text{span}\{\underline{b}_1 \cdots \underline{b}_p\}$$

Note:

- if $H = V$, then $\mathcal{B}$ spans V (every vector space is a subspace of itself)

Ex

The columns of an $n \times n$, invertible matrix $A = |\underline{a}_1 \cdots \underline{a}_n|$ are a basis for $\mathbb{R}^n$

→ Theorem (IMT) from Sect 2.9.

Ex

The standard basis vectors

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ are a basis for $\mathbb{R}^n$.

Ex $S = \{1, x, x^2, \dots, x^n\}$ is the standard basis for $\mathbb{P}^n$, the vector space of polynomials of degree $\leq n$

$\Rightarrow$ any polynomial of degree $\leq n$ can be written as

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n x^n = 0$$

for basis S and weights $a_0\ a_1 \cdots a_n$

Ex

Are $\underline{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ a basis $\mathbb{R}^3$?

$\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ are linearly independent and span $\mathbb{R}^3$ (a basis) if $A = [\underline{v}_1\ \underline{v}_2\ \underline{v}_3]$ is invertible. We know A is invertible if $\det A \neq 0$ (can also row reduce)

$$\det \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & 7 & 5 \end{vmatrix} = (-1)^{1+1} \cdot 3 \cdot \det \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \det \begin{vmatrix} -4 & -2 \\ 7 & 5 \end{vmatrix}$$

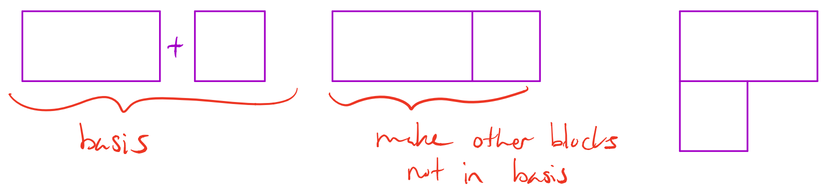
$$+ (-1)^{3+1} \cdot 6 \cdot \det \begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3\cdot(5-7) + 0 + 6\cdot(-4+2) \neq 0$$

So A is invertible $\Rightarrow \{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ a basis for $\mathbb{R}^3$

Interpretations

- The vectors in a basis are the "building blocks" to produce every other vector in the subspace.



In some sense, a basis contains the fundamental pieces

- A basis is the <u>smallest</u> spanning set for a subspace
- A basis is the <u>largest</u> set of linearly independent vectors in a subspace.

Theorem

Let $S = \{\underline{v}_1 \cdots \underline{v}_p\}$ be a set in a vectorspace V and let $H = \text{span}\{\underline{v}_1 \cdots \underline{v}_p\}$

a.) If one of the vectors $\underline{v}_k$ is a linear combination of the other vectors in S, then the set formed by removing $\underline{v}_k$ from S, $\{\underline{v}_1 \cdots \underline{v}_{k-1}, \underline{v}_{k+1} \cdots \underline{v}_p\}$ still spans H.

b.) If $H \neq \{\underline{0}\}$, some subset of S is a basis for H.

This gives a strategy for finding bases.

- Find a large, linearly dependent spanning set
- Throw out vectors until the set spans but is linearly independent
- The remaining linearly independent, spanning set is basis!

Bases for Nul A & Col A

We've already seen how to get a basis for Nul A

Finding a basis for Nul A

- solve homogeneous equation $A\underline{x} = \underline{0}$
- write answer in parametric vector form
- the vectors from par. vector form are a lin. ind., spanning set for Nul A, i.e., a basis

Ex

$A = \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}$, find a basis for Nul A.

$$\begin{bmatrix} 1 & 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 3 & -1 & -7 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & -1 & -1 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 13 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \underline{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

so $\underline{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for Nul A

To find a basis for Col A, we use the following...

<u>Theorem</u>

The pivot columns of a matrix A form a basis for Col A.

<u>Ex</u> The same matrix...

$$A = \begin{array}{cccc} & & & \\ \left[\begin{matrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{matrix}\right] \\ \underline{a}_1 \;\; \underline{a}_2 \;\;\;\;\;\; \underline{a}_4 \end{array} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so $\mathcal{B} = \{\underline{a}_1, \underline{a}_2, \underline{a}_4\}$ are a basis for Col A