Sect. 5.2 (Cont...)

Recall: We were finding the roots of the characteristic equation, $\det(A - \lambda I) = 0$. These roots are the eigenvalues of the matrix A.

Def

We define the (algebraic) <u>multiplicity</u> of an evalue λ is its multiplicity (power) as a root of the characteristic equation.

Ex

$$A = \begin{vmatrix} 2 & 1 & 0 & 5 & 0 & 1 \\ 0 & 2 & -4 & 0 & -1 & 0 \\ 0 & 0 & 2 & 2 & -0 & -3 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \in \mathbb{R}^{6\times 6}$$

The characteristic equation of A is

$$\det(A - \lambda I) = (2-\lambda)^3 (3-\lambda)^2 (-1-\lambda) = 0$$

The root $\lambda = 2$ has multiplicity 3, so $\lambda = 2$ has algebraic multiplicity 3.

↳ $\lambda = 3$ has multiplicity 2

↳ $\lambda = -1$ has multiplicity 1

Ex

Find the e-values and e-vectors of $A = \begin{vmatrix} -4 & 2 \\ 6 & 7 \end{vmatrix}$

1.) Find e-values

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{vmatrix} -4-\lambda & 2 \\ 6 & 7-\lambda \end{vmatrix} \right) = 0$$

$$(-4-\lambda)(7-\lambda) - 12 = 0$$

$$\lambda^2 - 7\lambda + 4\lambda - 28 - 12 = 0$$

$$\lambda^2 - 3\lambda - 40 = 0$$

$$(\lambda - 8)(\lambda + 5) = 0 \Rightarrow \text{evalues}$$

$$\lambda = 8, \ \lambda -5$$

2.) Find e-vectors

For $\lambda = 8$, solve $(A - 8I)\underline{x} = \underline{0}$

$$\left| \begin{array}{cc|c} -4-8 & 2 & 0 \\ 6 & 7-8 & 0 \end{array} \right| = \begin{vmatrix} -12 & 2 & 0 \\ 6 & -1 & 0 \end{vmatrix} \sim \begin{vmatrix} -12 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\sim \begin{vmatrix} 1 & -1/6 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = x_2 \underset{\underline{v}_1}{\begin{vmatrix} 1/6 \\ 1 \end{vmatrix}} \Rightarrow \text{e-vector } \underline{v}_1 \text{ corresponding to } \lambda = 8$$

For $\lambda = -5$, solve $(A + 5I)\underline{x} = \underline{0}$

$$\begin{vmatrix} -4+5 & 2 & 0 \\ 6 & 7+5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 6 & 12 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

So

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = x_2 \underbrace{\begin{vmatrix} -2 \\ 1 \end{vmatrix}}_{\underline{v}_2} \Rightarrow \text{e-vector } \underline{v}_2 \text{ corresponding to } \lambda = -5$$

Similarity

Def. Two $n \times n$ matrices A and B are similar if there exists a invertible matrix P such that

$$P^{-1} A P = B$$

Notes

- $$P^{-1} A P = B$$
$$\underbrace{(P P^{-1})}_{I} A P = P B$$
$$A P = P B$$
$$A \underbrace{(P P^{-1})}_{I} = P B P^{-1}$$
$$A = P B P^{-1}$$
$$A = Q^{-1} B Q \quad \text{where } P = Q^{-1}$$

This shows A similar to $B \Leftrightarrow B$ similar to A

Theorem

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial, and hence the same e-values.

Pf. $B = P^{-1}AP$ by similarity, we want to show $\det(B - \lambda I) = \det(A - \lambda I)$

$$B - \lambda I = P^{-1}AP - \lambda I \quad \text{(sub in } B = P^{-1}AP\text{)}$$

$$= P^{-1}AP - \lambda(P^{-1}P) \quad \text{(sub } I = P^{-1}P\text{)}$$

$$= P^{-1}(AP - \lambda P) \quad \text{(factor } P^{-1} \text{ left)}$$

$$= P^{-1}(A - \lambda I)P \quad \text{(factor } P \text{ right)}$$

Taking the determinant on both sides

$$\det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

We know (from HW!) $\det(P^{-1}) = \frac{1}{\det(P)}$,

so $\det(P) \cdot \det(P^{-1}) = \det(P) \cdot \frac{1}{\det(P)} = 1$

This gives

$$\det(B - \lambda I) = 1 \cdot \det(A - \lambda I)$$

$$= \det(A - \lambda I)$$

So, the characteristic polynomials of A and B are the same!

Warning

- Similarity $\Rightarrow$ same e-values
- Not the same as same evalues $\not\Rightarrow$ similarity

  Two matrices can have the same evalues but not be similar.
- Similarity is not the same as row equivalence.

Ex.

Show that if $A = QR$ where Q is invertible, then show A is similar to $B = RQ$

We need to show $B = P^{-1}AP$ for some invertible matrix P.

We see

$$B = RQ$$

$$B = IRQ \quad \text{(mult. by identity)}$$

$$B = (Q^{-1}Q)RQ \quad (\text{sub. in } I = Q^{-1}Q)$$

$$B = Q^{-1}(\underbrace{QR}_{A})Q$$

$$B = Q^{-1}AQ = P^{-1}AQ$$

Thus, for $P = Q$, we have A is similar to B.

Sect. 5.3: Diagonalization

Some types of matrices are easy to multiply. We particularly like diagonal matrices.

Ex

$$D = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$D^2 = D \cdot D = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & b^2 \end{vmatrix}$$

$$D^3 = D \cdot D^2 = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \begin{vmatrix} a^2 & 0 \\ 0 & b^2 \end{vmatrix} = \begin{vmatrix} a^3 & 0 \\ 0 & b^3 \end{vmatrix}$$

$$D^k = D \cdot D^{k-1} = \begin{vmatrix} a^k & 0 \\ 0 & b^k \end{vmatrix} \quad \text{for } k \geq 1$$

The ease of multiplication extends if a general matrix A is similar to a diagonal matrix D

$$A = P^{-1} D P$$

$$A^2 = A \cdot A = (P^{-1} D P)(P^{-1} D P)$$

$$= P^{-1} D \underbrace{(P P^{-1})}_{I} D P$$

$$= P^{-1} D D P = P^{-1} D^2 P$$

$$A^3 = A \cdot A^2 = (P^{-1} D P)(P^{-1} D^2 P)$$

$$= P^{-1} D (P P^{-1}) D^2 P = P^{-1} D^3 P$$

So, in general, we have

$$A^k = P^{-1} D^k P \quad \text{for } k \geq 1$$

if A is similar to a diagonal matrix.

Notes

- This is a computationally cheap way to compute A^k in many settings (CS, Physics, etc.)
- Applications: finding connectivity in complex networks

Def.

An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix; i.e., there exist invertible P and diagonal matrix D such that

$$A = P^{-1} D P$$

Thm (Diagonalization Theorem)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

The matrix $P = \begin{vmatrix} \underline{v}_1 & \cdots & \underline{v}_p \end{vmatrix}$ (invertible) is formed by the eigenvectors of A.

and $D = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix}$ has the corresponding eigenvalues.

That is

$$P^{-1}AP = D = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{vmatrix}$$

<u>Pf.</u> (next class...)