(d) $y_1' = y_1 + 2y_3$
$y_2' = y_2 - y_3$
$y_3' = y_1 + y_2 + y_3$
$y_1(0) = y_2(0) = 1, y_3(0) = 4$

3. Given

$$\mathbf{Y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{x}_n$$

is the solution to the initial value problem:

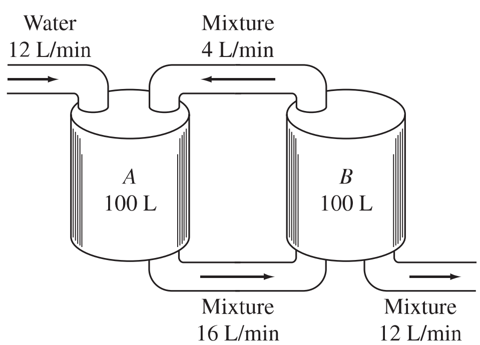
$$\mathbf{Y}' = A\mathbf{Y}, \qquad \mathbf{Y}(0) = \mathbf{Y}_0$$

(a) show that

$$\mathbf{Y}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

(b) let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)^T$. Assuming that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent, show that $\mathbf{c} = X^{-1}\mathbf{Y}_0$.

4. Two tanks each contain 100 liters of a mixture. Initially, the mixture in tank A contains 40 grams of salt while tank B contains 20 grams of salt. Liquid is pumped in and out of the tanks as shown in the accompanying figure. Determine the amount of salt in each tank at time t.



5. Find the general solution of each of the following systems:

(a) $y_1'' = -2y_2$
$y_2'' = y_1 + 3y_2$

(b) $y_1'' = 2y_1 + y_2'$
$y_2'' = 2y_2 + y_1'$

6. Solve the initial value problem

$$y_1'' = -2y_2 + y_1' + 2y_2'$$
$$y_2'' = 2y_1 + 2y_1' - y_2'$$

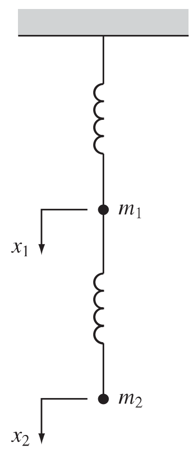
$$y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = -3,\ y_2'(0) = 2$$

7. In Application 2, assume that the solutions are of the form $x_1 = a_1 \sin\sigma t$, $x_2 = a_2 \sin\sigma t$. Substitute these expressions into the system and solve for the frequency σ and the amplitudes a_1 and a_2.

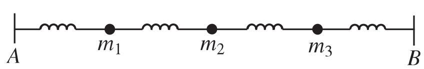
8. Solve the the problem in Application 2, using the initial conditions

$$x_1(0) = x_2(0) = 1,\ x_1'(0) = 4, \text{ and } x_2'(0) = 2$$

9. Two masses are connected by springs as shown in the accompanying diagram. Both springs have the same spring constant, and the end of the first spring is fixed. If x_1 and x_2 represent the displacements from the equilibrium position, derive a system of second-order differential equations that describes the motion of the system.



10. Three masses are connected by a series of springs between two fixed points as shown in the accompanying figure. Assume that the springs all have the same spring constant, and let $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the displacements of the respective masses at time t.



(a) Derive a system of second-order differential equations that describes the motion of this system.

(b) Solve the system if $m_1 = m_3 = \frac{1}{3}$, $m_2 = \frac{1}{4}$, $k = 1$, and

$$x_1(0) = x_2(0) = x_3(0) = 1$$
$$x_1'(0) = x_2'(0) = x_3'(0) = 0$$

11. Transform the nth-order equation

$$y^{(n)} = a_0 y + a_1 y' + \cdots + a_{n-1}y^{(n-1)}$$

into a system of first-order equations by setting $y_1 = y$ and $y_j = y_{j-1}'$ for $j = 2, \ldots, n$. Determine the characteristic polynomial of the coefficient matrix of this system.

SECTION 6.3 EXERCISES

1. In each of the following, factor the matrix A into a product XDX^{-1}, where D is diagonal:

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$ **(d)** $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

(f) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$

2. For each of the matrices in Exercise 1, use the XDX^{-1} factorization to compute A^6.

3. For each of the nonsingular matrices in Exercise 1, use the XDX^{-1} factorization to compute A^{-1}.

4. For each of the following, find a matrix B such that $B^2 = A$.

(a) $A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

5. Let A be a nondefective $n \times n$ matrix with diagonalizing matrix X. Show that the matrix $Y = (X^{-1})^T$ diagonalizes A^T.

6. Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

7. Show that any 3×3 matrix of the form

$$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{pmatrix}$$

is defective.

8. For each of the following, find all possible values of the scalar α that make the matrix defective or show that no such values exist.

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & \alpha \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{pmatrix}$ **(d)** $\begin{pmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & \alpha \end{pmatrix}$

(e) $\begin{pmatrix} 3\alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ **(f)** $\begin{pmatrix} 3\alpha & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$

(g) $\begin{pmatrix} \alpha+2 & 1 & 0 \\ 0 & \alpha+2 & 0 \\ 0 & 0 & 2\alpha \end{pmatrix}$

(h) $\begin{pmatrix} \alpha+2 & 0 & 0 \\ 0 & \alpha+2 & 1 \\ 0 & 0 & 2\alpha \end{pmatrix}$

9. Let A be a 4×4 matrix and let λ be an eigenvalue of multiplicity 3. If $A - \lambda I$ has rank 1, is A defective? Explain.

10. Let A be an $n \times n$ matrix with positive real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\mathbf{x}_i$ be an eigenvector belonging to λ_i for each i, and let $\mathbf{x} = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$.

(a) Show that $A^m\mathbf{x} = \sum_{i=1}^{n} \alpha_i\lambda_i^m\mathbf{x}_i$.

(b) Show that if $\lambda_1 = 1$, then $\lim_{m\to\infty} A^m\mathbf{x} = \alpha_1\mathbf{x}_1$.

11. Let A be a $n \times n$ matrix with real entries and let $\lambda_1 = a + bi$ (where a and b are real and $b \neq 0$) be an eigenvalue of A. Let $\mathbf{z}_1 = \mathbf{x} + i\,\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ both have real entries) be an eigenvector belonging to λ_1 and let $\mathbf{z}_2 = \mathbf{x} - i\,\mathbf{y}$.

(a) Explain why $\mathbf{z}_1$ and $\mathbf{z}_2$ must be linearly independent.

(b) Show that $\mathbf{y} \neq \mathbf{0}$ and that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

12. Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda I$.

13. Show that a nonzero nilpotent matrix is defective.

14. Let A be a diagonalizable matrix and let X be the diagonalizing matrix. Show that the column vectors of X that correspond to nonzero eigenvalues of A form a basis for $R(A)$.

15. It follows from Exercise 14 that for a diagonalizable matrix the number of nonzero eigenvalues (counted according to multiplicity) equals the rank of the matrix. Give an example of a defective matrix whose rank is not equal to the number of nonzero eigenvalues.

16. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A whose eigenspace has dimension k, where $1 < k < n$. Any basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for the eigenspace can be extended to a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$. Let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $B = X^{-1}AX$.

(a) Show that B is of the form

$$\begin{pmatrix} \lambda I & B_{12} \\ O & B_{22} \end{pmatrix}$$

where I is the $k \times k$ identity matrix.