Sect 6.2 (Cont...)

Recall, we defined an orthogonal basis for a subspace as a basis where the vectors from an orthogonal set.

We also looked orthogonal projection, the idea that for a vector $\underline{y}$, we could split into the sum of two vectors

$$\underline{y} = \hat{\underline{y}} + \underline{z}$$

such that $\hat{\underline{y}} \cdot \underline{z} = \underline{0}$ and $\hat{\underline{y}} = \alpha \underline{u}$ is the orthogonal projection of $\underline{y}$ onto $\hat{\underline{y}}$. Here, $\alpha = \left(\frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \right)$

Recall that for $\underline{y}$ in a subspace W with orthogonal basis $\{\underline{u}_1 \cdots \underline{u}_p\}$, we could write

$$(*) \quad \underline{y} = \left(\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \right) \underline{u}_1 + \left(\frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \right) \underline{u}_2 + \cdots + \left(\frac{\underline{y} \cdot \underline{u}_p}{\underline{u}_p \cdot \underline{u}_p} \right) \underline{u}_p$$

Note, the expressions $\frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}}$ for orth. proj and $\frac{\underline{y} \cdot \underline{u}_j}{\underline{u}_j \cdot \underline{u}_j}$ for expansion in terms of an orth. basis are very similar. In fact, the formula $(*)$ for writing $\underline{y}$ in terms of an orth. basis is the decomposition of $\underline{y}$ into a sum of orthogonal projections.

Consider $W = \mathbb{R}^2 = \text{span}\{\underline{u}_1, \underline{u}_2\}$ for an orthogonal set $\{\underline{u}_1, \underline{u}_2\}$, then we can write any $\underline{y}$ in $\mathbb{R}^2$ as

$$y = \left(\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right) \underline{u}_1 + \left(\frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\right) \underline{u}_2$$

<u>Ex</u> (standard basis)

Let $\mathcal{B} = \{\underline{e}_1\ \underline{e}_2\}$ where $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
(Note, $\mathcal{B}$ is an orthogonal basis)

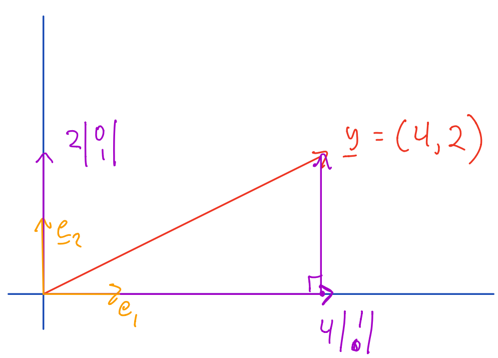
Express $\underline{y} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ in terms of $\mathcal{B}$

$$\underline{y} \cdot \underline{e}_1 = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4 \qquad \underline{e}_1 \cdot \underline{e}_1 = 1$$

$$\underline{y} \cdot \underline{e}_2 = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \qquad \underline{e}_2 \cdot \underline{e}_2 = 1$$

Thus, we have

$$\underline{y} = \frac{4}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{2}{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Ex

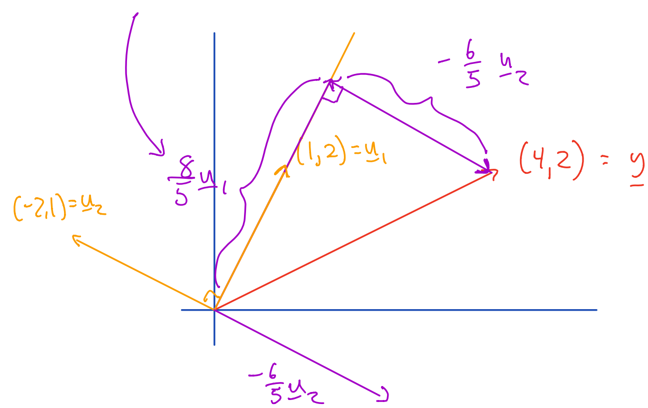
Let $\mathcal{B} = \left\{ \begin{array}{cc} \underline{u}_1 & \underline{u}_2 \\ \begin{vmatrix} 1 \\ 2 \end{vmatrix}, & \begin{vmatrix} -2 \\ 1 \end{vmatrix} \end{array} \right\}$, express $\underline{y} = \begin{vmatrix} 4 \\ 2 \end{vmatrix}$ in terms of the orthogonal basis.

$$\underline{y} \cdot \underline{u}_1 = 8 \qquad \underline{u}_1 \cdot \underline{u}_1 = 5$$

$$\underline{y} \cdot \underline{u}_2 = -6 \qquad \underline{u}_2 \cdot \underline{u}_2 = 5$$

Plugging in, this gives

$$\underline{y} = \frac{8}{5}\underline{u}_1 + \frac{-6}{5}\underline{u}_2 = \begin{vmatrix} 8/5 \\ 16/5 \end{vmatrix} + \begin{vmatrix} 12/5 \\ -6/5 \end{vmatrix} = \begin{vmatrix} 20/5 \\ 10/5 \end{vmatrix} = \begin{vmatrix} 4 \\ 2 \end{vmatrix}$$



Orthonormal Sets

In Sect. 6.1, we defined normalization as taking a vector $\underline{v}$ in $\mathbb{R}^n$ and making a unit vector

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|}$$

such $\|\underline{u}\| = 1$. The direction of $\underline{u}$ is identical to that of $\underline{v}$., i.e, we scaled the length but didn't change direction.

We now combine the idea of normalization with orthogonal sets.

<u>Def</u>

An <u>orthonormal set</u> is an orthogonal set $\{\underline{u}_1 \cdots \underline{u}_p\}$ with each $\underline{u}_j$ normalized, i.e, $\|\underline{u}_j\|=1$

This gives

$$\underline{u}_i \cdot \underline{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

<u>Def</u>

If the orthonormal set $\{\underline{u}_1 \cdots \underline{u}_p\}$ is a basis for a subspace W, we call it an <u>orthonormal basis</u>.

<u>Ex</u>

The standard basis for $\mathbb{R}^n$ given by $\{\underline{e}_1 \cdots \underline{e}_n\}$ where

$$\underline{e}_j = \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j\text{th index} \\ \\ \\ \\ \end{array}$$

is an orthonormal basis. To see this, we note

$$\underline{e}_j \cdot \underline{e}_j = \begin{vmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} = 0 \cdot 0 \cdots 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + \cdots 0 \cdot 0$$

$$\underline{e}_j \cdot \underline{e}_j = 1$$

and

$$\underline{e}_j \cdot \underline{e}_i = \begin{vmatrix} 0 & \cdots & 0 & \underset{\text{jth entry}}{1} & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} = 0 \cdot 0 \cdots \underbrace{0 \cdot 1}_{\text{jth entry}} + \cdots + \underbrace{1 \cdot 0}_{\text{ith entry}} + \cdots 0 \cdot 0$$

ith entry →

$$\underline{e}_j \cdot \underline{e}_i = 0$$

So, the standard basis is an orthonormal basis.

<u>Ex</u> Show that $\{\underline{v}_1\ \underline{v}_2\ \underline{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

$$\underline{v}_1 = \begin{vmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{vmatrix} \qquad \underline{v}_2 = \begin{vmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{vmatrix} \qquad \underline{v}_3 = \begin{vmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{vmatrix}$$

First, show the set is orthogonal

$$\underline{v}_1 \cdot \underline{v}_2 = \frac{-3}{\sqrt{11}\sqrt{6}} + \frac{2}{\sqrt{11}\sqrt{6}} + \frac{1}{\sqrt{11}\sqrt{6}} = 0$$

$$\underline{v}_1 \cdot \underline{v}_3 = \frac{-3}{\sqrt{11}\sqrt{66}} - \frac{4}{\sqrt{11}\sqrt{66}} + \frac{7}{\sqrt{11}\sqrt{66}} = 0$$

$$\underline{v}_2 \cdot \underline{v}_3 = \frac{+1}{\sqrt{6}\sqrt{66}} - \frac{8}{\sqrt{6}\sqrt{66}} + \frac{7}{\sqrt{6}\sqrt{66}} = 0$$

The set is orthogonal. Now, show that each $\underline{v}_j$ is normalized

$$\underline{v}_1 \cdot \underline{v}_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = 1$$

$$\underline{v}_2 \cdot \underline{v}_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

$$\underline{v}_3 \cdot \underline{v}_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$$

This shows $\underline{v}_j$ are unit vectors (i.e., normalized) so $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$

Def

A matrix U is an <u>orthogonal matrix</u> if its columns form an orthonormal set

↳ These matrice have nice theoretical and practical properties.

Thm

An $m \times n$ matrix U is orthogonal if and only if $U^T U = I$ $(n \times n)$

Pf

$$U = \begin{vmatrix} \underline{u}_1 & \cdots & \underline{u}_n \end{vmatrix} \quad \text{where } \underline{u}_j \in \mathbb{R}^m$$

Then

$$U^T U = \begin{vmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_n^T \end{vmatrix} \begin{vmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_n \end{vmatrix} = \begin{vmatrix} \underline{u}_1^T \underline{u}_1 & \underline{u}_1^T \underline{u}_2 & \cdots & \underline{u}_1^T \underline{u}_n \\ \underline{u}_2^T \underline{u}_1 & \underline{u}_2^T \underline{u}_2 & \cdots & \underline{u}_2^T \underline{u}_n \\ \vdots & \vdots & & \\ \underline{u}_n^T \underline{u}_1 & \underline{u}_n^T \underline{u}_2 & \cdots & \underline{u}_n^T \underline{u}_n \end{vmatrix}$$

where the ijth entry of $U^T U$ is given by the dot product $\underline{u}_i^T \underline{u}_j$. From this we see that

$$\begin{vmatrix} \underline{u}_1^T\underline{u}_1 & \underline{u}_1^T\underline{u}_2 & \cdots & \underline{u}_1^T\underline{u}_n \\ \underline{u}_2^T\underline{u}_1 & \underline{u}_2^T\underline{u}_2 & \cdots & \underline{u}_2^T\underline{u}_n \\ \vdots & \vdots & & \\ \underline{u}_n^T\underline{u}_1 & \underline{u}_n^T\underline{u}_2 & \cdots & \underline{u}_n^T\underline{u}_n \end{vmatrix} \doteq \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = I$$

if and only if $\underline{u}_i^T \underline{u}_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

which is the definition of an orthonormal set #