Comments on section 6.3 of [SN]

Anne E. B. Nielsen (Aarhus University)

1 Equation (6.84)

According to the first order Born approximation

$$\langle \mathbf{x}|\psi^{(+)}\rangle \approx \langle \mathbf{x}|\mathbf{k}\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}')\langle \mathbf{x}'|\mathbf{k}\rangle. \tag{1}$$

We consider a spherically symmetric potential $V(\mathbf{x}') \to V(|\mathbf{x}'|)$ with typical strength V_0 and range a, and we aim to estimate when the first order Born approximation is valid. To be valid, the norm of the second term on the right hand side of (1) must be small compared to the norm of the first term. This should apply for all $\mathbf{x}$. For large $|\mathbf{x}|$, the outgoing wave is approximately a spherical wave, so the amplitude of the wave will decrease with distance. A natural point for checking the relative size of the two terms is therefore $\mathbf{x} = 0$, which also simplifies the integration. Utilizing

$$\langle \mathbf{x}|\mathbf{k}\rangle = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{L^{3/2}}, \tag{2}$$

we can then formulate our condition for the validity of the first order Born approximation as

$$1 \gg \left| \frac{2m}{\hbar^2}\int d^3x' \frac{e^{ik|\mathbf{x}'|+i\mathbf{k}\cdot\mathbf{x}'}}{4\pi|\mathbf{x}'|} V(|\mathbf{x}'|)\right|. \tag{3}$$

Performing the angular integration, this reduces to

$$1 \gg \left| \frac{m}{ik\hbar^2}\int_0^\infty (e^{2ikr'}-1)V(r')dr'\right|. \tag{4}$$

At low energies ($ka \ll 1$), we can Taylor expand $e^{2ikr'}$ to obtain

$$1 \gg \left| \frac{m}{ik\hbar^2}\int_0^\infty 2ikr'V(r')dr'\right| \approx \frac{m|V_0|a^2}{\hbar^2}. \tag{5}$$

At high energies ($ka \gg 1$), the exponential oscillates so fast that we can ignore it. Then

$$1 \gg \left| \frac{m}{k\hbar^2}\int_0^\infty V(r')dr'\right| \approx \frac{m|V_0|a}{k\hbar^2}. \tag{6}$$

This expression has a $\ln(ka)$ less than equation (6.85). Note, however, that $\ln(ka)$ is not that different from 1.

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).