Comments on section 6.6 of [SN]

Anne E. B. Nielsen (Aarhus University)

1 Equations (6.193) to (6.197)

Equation (6.194) does, of course, not apply all the way to $r \to \infty$, and (6.195) does not look like (6.194). This can be fixed as follows. We assume a spherically symmetric potential and consider S-wave scattering (i.e. $l = 0$). We also assume that the potential is zero outside some radius R. From (6.138), we have

$$u_0 = e^{i\delta_0} \frac{\cos(\delta_0)\sin(kr) + \sin(\delta_0)\cos(kr)}{k}, \quad \text{for } r > R, \tag{1}$$

which we rearrange to

$$u_0 = e^{i\delta_0} \cos(\delta_0) \left(\frac{\sin(kr)}{k} + \frac{\cos(kr)}{k\cot(\delta_0)} \right), \quad \text{for } r > R. \tag{2}$$

Now assume low energy, such that $kR \ll 1$, and consider r to be only slightly larger than R, such that $kr \ll 1$ also applies. Then

$$u_0 \approx e^{i\delta_0} \cos(\delta_0) \left(r + \frac{1}{k\cot(\delta_0)} \right) \quad (\text{for } k \to 0 \text{ and } r \text{ slightly larger than } R). \tag{3}$$

Defining the scattering length

$$a = -\lim_{k\to 0} \frac{1}{k\cot(\delta_0)} = -\lim_{k\to 0} \frac{\tan(\delta_0)}{k}, \tag{4}$$

we hence have

$$u_0 \approx e^{i\delta_0} \cos(\delta_0)\,(r - a) \quad (\text{for } k \to 0 \text{ and } r \text{ slightly larger than } R). \tag{5}$$

Conclusion: For low energy and r slightly larger than R the outside wavefunction is close to a straight line. To find the scattering length, one should draw the tangent line of the outside wavefunction at $r = R$ and read off where that line crosses the r-axis.

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).