Comments on section 6.2 of [SN]

Anne E. B. Nielsen (Aarhus University)

1 Figure 6.4 and equation (6.65)

In figure 6.4, the singularity is at $z_0 = x_0 + i\epsilon$ with $\epsilon \to 0$. For an integral along the real axis, the contour is hence always below the pole. This is why the semicircle in the figure is chosen to be in the lower half plane. If the pole is instead at $z_0 = x_0 - i\epsilon$, the contour along the real axis is always above the pole, and the semicircle is chosen to be in the upper half plane. In this case, the integration in (6.64) is from π to 0, and therefore (6.65) is modified to

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx - i\pi f(x_0). \tag{1}$$

2 Derivation of the optical theorem (6.60)

Our starting point is (6.58)

$$f(\mathbf{k}', \mathbf{k}) = -\frac{mL^3}{2\pi\hbar^2} \langle \mathbf{k}' | V | \psi^{(+)} \rangle \tag{2}$$

and (6.61)

$$\langle \mathbf{k} | V | \psi^{(+)} \rangle = \langle \psi^{(+)} | V | \psi^{(+)} \rangle - \langle \psi^{(+)} | V \frac{1}{E - H_0 - i\epsilon} V | \psi^{(+)} \rangle. \tag{3}$$

Note that $\mathrm{Im}(\langle\psi^{(+)}|V|\psi^{(+)}\rangle) = 0$, because the expectation value of a Hermitian operator is real. We therefore have

$$
\begin{aligned}
\mathrm{Im}(f(\mathbf{k},\mathbf{k})) &= \frac{mL^3}{2\pi\hbar^2}\,\mathrm{Im}\left(\langle\psi^{(+)}|V\frac{1}{E-H_0-i\epsilon}V|\psi^{(+)}\rangle\right)\\
&= \frac{mL^3}{2\pi\hbar^2}\,\mathrm{Im}\left(\sum_{\mathbf{k}'}\langle\psi^{(+)}|V\frac{1}{E-\frac{\hbar^2k'^2}{2m}-i\epsilon}|\mathbf{k}'\rangle\langle\mathbf{k}'|V|\psi^{(+)}\rangle\right)\\
&= \frac{mL^3}{2\pi\hbar^2}\,\mathrm{Im}\left(\sum_{\mathbf{k}'}\frac{1}{E-\frac{\hbar^2k'^2}{2m}-i\epsilon}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right)\\
&= \frac{mL^6}{(2\pi)^4\hbar^2}\,\mathrm{Im}\left(\int d^3k'\frac{1}{E-\frac{\hbar^2k'^2}{2m}-i\epsilon}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right)\\
&= \frac{mL^6}{(2\pi)^4\hbar^2}\,\mathrm{Im}\left(\int d\Omega_{k'}\int dk'(k')^2\frac{1}{E-\frac{\hbar^2k'^2}{2m}-i\epsilon}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right)\\
&= -\frac{mL^6}{(2\pi)^4\hbar^2}\,\mathrm{Im}\left(\int d\Omega_{k'}\int dE'\frac{dk'}{dE'}\frac{2mE'}{\hbar^2}\frac{1}{E'-E+i\epsilon}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right),
\end{aligned}
\tag{4}
$$

where, of course, $\mathbf{k}'$ should be seen as a function of E' and the polar and azimuthal angles in the last line. The integrant has a pole for $E' = E - i\epsilon$, i.e. below the real axis. Considering physical potentials, we assume the remaining part of the integrant is analytical, and we can then use (1). This gives

$$
\begin{aligned}
&\mathrm{Im}(f(\mathbf{k},\mathbf{k})) =\\
&\quad -\frac{mL^6}{(2\pi)^4\hbar^2}\,\mathrm{Im}\left(\mathcal{P}\left(\int d\Omega_{k'}\int dE'\frac{dk'}{dE'}\frac{2mE'}{\hbar^2}\frac{1}{E'-E}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right)\right)\\
&\qquad + \frac{mL^6}{(2\pi)^4\hbar^2}\,\mathrm{Im}\left(i\pi\int d\Omega_{k'}\frac{m}{\hbar^2k}k^2|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2\right)\\
&\quad = \frac{m^2L^6k}{16\pi^3\hbar^4}\int d\Omega_{k'}|\langle\mathbf{k}'|V|\psi^{(+)}\rangle|^2 = \frac{k}{4\pi}\int d\Omega_{k'}|f(\mathbf{k}',\mathbf{k})|^2\\
&\quad = \frac{k}{4\pi}\int d\Omega_{k'}\frac{d\sigma}{d\Omega_{k'}} = \frac{k\sigma_{\mathrm{tot}}}{4\pi}.
\end{aligned}
\tag{5}
$$

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).