

Describe how the coupling of angular momenta works in quantum mechanics, and discuss the Clebsch–Gordan coefficients.

Coupling of angular momenta in quantum mechanics:

Coupling of angular momentum is the procedure used to multiply two angular momentum operators. This is especially useful in atomic physics where we have to deal with multiple electrons, each having their own angular momentum. Or with the case of spin-orbit coupling (nuclear physics), where we have to deal with the coupling of the orbital angular momentum and the spin angular momentum of a single electron.

Uncoupled basis:

I will start with the general case of two angular momenta \vec{J}_1 and \vec{J}_2 , with quantum numbers j_1, m_1 and j_2, m_2 respectively. m being the magnetic quantum number ($m = -j, -j+1, \dots, j$) and j the total angular momentum quantum number ($j = 0, 1/2, 1, 3/2, 2, \dots$). It is important that these two angular momentum operators satisfy the usual angular momentum commutation relations:

$$[J_{1i}, J_{1j}] = i\hbar\epsilon_{ijk}J_{1k} \quad [J_{2i}, J_{2j}] = i\hbar\epsilon_{ijk}J_{2k} \quad [J_{1i}, J_{2j}] = 0$$

The last commutation relation holds because they come from different sources (e.g. acts on different particles) and therefore do not interfere with each other.

Proof:

$$\begin{aligned} [J_{1i}, J_{2j}] &= [J_{1i} \otimes I, I \otimes J_{2j}] \\ &= (J_{1i} \otimes I)(I \otimes J_{2j}) - (I \otimes J_{2j})(J_{1i} \otimes I) \\ &= J_{1i} \otimes J_{2j} - J_{1i} \otimes J_{2j} \\ &= 0 \end{aligned}$$

This holds because $(A \otimes I)(I \otimes B) = (AI) \otimes (IB) = A \otimes B$. If we add these two angular momenta together, we get a new angular momentum operator $\vec{J} = \vec{J}_1 + \vec{J}_2$, with quantum numbers j, m . Here it is important to note that the total angular momentum operator \vec{J} also satisfies the angular momentum commutation relations:

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$$

However, this is not just a simple matrix addition, since the two momentum operators live in different sub spaces with different dimensions (e.g. $d_1 = 2j_1 + 1$, $d_2 = 2j_2 + 1$). The total ket that describes these two angular momenta independently is called the uncoupled ket, and is given by:

$$|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad (1)$$

The dimension of this ket is $d = d_1 \times d_2 = (2j_1 + 1)(2j_2 + 1)$. In this case each ket is an eigenket of the individual angular momentum operators \vec{J}_1 and \vec{J}_2 :

$$\begin{aligned} \vec{J}_1^2 |j_1, m_1; j_2, m_2\rangle &= \hbar^2 j_1(j_1 + 1) |j_1, m_1; j_2, m_2\rangle \\ J_{1z} |j_1, m_1; j_2, m_2\rangle &= \hbar m_1 |j_1, m_1; j_2, m_2\rangle \\ \vec{J}_2^2 |j_1, m_1; j_2, m_2\rangle &= \hbar^2 j_2(j_2 + 1) |j_1, m_1; j_2, m_2\rangle \\ J_{2z} |j_1, m_1; j_2, m_2\rangle &= \hbar m_2 |j_1, m_1; j_2, m_2\rangle \end{aligned}$$

In this basis we have these commutation relations $[\vec{J}_1^2, \vec{J}_2^2] = 0$, $[J_{1z}, J_{2z}] = 0$, which can be seen from the fact that the individual angular momentum operators act on different subspaces.

Coupled basis: The total angular momentum operator \vec{J} has its own set of quantum numbers j, m . The total ket that describes the coupled angular momenta is called the coupled ket, and is given by:

$$|j_1, j_2; j, m\rangle \quad (2)$$

So it can diagonalize the total angular momentum operator \vec{J} but not necessarily the individual angular momentum operators \vec{J}_1 and \vec{J}_2 . Here we have a ket that is an eigenket of \vec{J}^2 , J_z , \vec{J}_1^2 and \vec{J}_2^2 :

$$\begin{aligned}\vec{J}^2|j_1, j_2; j, m\rangle &= \hbar^2 j(j+1)|j_1, j_2; j, m\rangle \\ J_z|j_1, j_2; j, m\rangle &= \hbar m|j_1, j_2; j, m\rangle \\ \vec{J}_1^2|j_1, j_2; j, m\rangle &= \hbar^2 j_1(j_1+1)|j_1, j_2; j, m\rangle \\ \vec{J}_2^2|j_1, j_2; j, m\rangle &= \hbar^2 j_2(j_2+1)|j_1, j_2; j, m\rangle\end{aligned}$$

The reason behind keeping j_1 and j_2 in the coupled ket is that they are needed to specify the subspace we are working in. In this basis we have this commutation relation $[J^2, J_1^2] = 0$, which can be seen if we write J^2 as $J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$. The relation between these two basis is given by the Clebsch-Gordan coefficients:

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} |j_1, m_1; j_2, m_2\rangle \langle j_1, m_1; j_2, m_2|j_1, j_2; j, m\rangle \quad (3)$$

Where we have inserted the identity operator in the uncoupled basis $\sum_{m_1, m_2} |j_1, m_1; j_2, m_2\rangle \langle j_1, m_1; j_2, m_2| = I$, and get the Clebsch-Gordan coefficients $\langle j_1, m_1; j_2, m_2|j_1, j_2; j, m\rangle$ as the expansion coefficients.

Clebsch-Gordan coefficients (CGC):

We know that both the coupled and uncoupled basis form a complete orthonormal set of basis kets, and are normalized, which means that transformation between these two basis is unitary. Under unitary transformations the probability/norm is conserved and the two basis are both before and after transformation orthonormal and complete (they both live in the same Hilbert space). In addition is the transformation reversible, which means that we can move freely between the two basis without loss of information of the two states.

The Clebsch-Gordan coefficients therefore have the following properties:

- They are real numbers \Rightarrow they are equal to their complex conjugate.
- They satisfy the selection rules:
- They satisfy the orthogonality relations:

By using the orthonormality of the uncoupled states and the completion of the coupled states we can see that each row of the transformation matrix is normalized.

Selection rules for CGC:

So we start by finding the possible values of j and m for given j_1 and j_2 . We do this by looking at the eigenvalues of the operators \vec{J}^2 and J_z . By using the transformation made by the CGC, that $J_z = J_{1z} + J_{2z}$, and that J_z works on the coupled ket while J_{1z} and J_{2z} work on the uncoupled ket, we get:

$$\begin{aligned}J_z|j_1, j_2; j, m\rangle &= \sum_{m_1, m_2} (J_{1z} + J_{2z})|j_1, m_1; j_2, m_2\rangle \langle j_1, m_1; j_2, m_2|j_1, j_2; j, m\rangle \\ m\hbar|j_1, j_2; j, m\rangle &= (m_1 + m_2)\hbar|j_1, j_2; j, m\rangle.\end{aligned}$$

So we get that $m = m_1 + m_2$ with $m_{max} = m_{1,max} + m_{2,max} = j_1 + j_2$, with $j = m_{max}$ which leaves us with $j_{max} = j_1 + j_2$. We now want to find the number of states. We know that for a given j there are $2j + 1$ possible values of m and thereby $2j + 1$ number of states $|j_1, j_2; j, m\rangle$. The total number of states in the coupled basis is therefore:

$$\begin{aligned}N &= \sum_{j=j_{min}}^{j_{max}} (2j + 1) = (2j_1 + 1)(2j_2 + 1) \\ &= (j_{max} - j_{min} + 1)(j_{max} + j_{min} + 1) = (j_1 + j_2 + 1)^2 - j_{min}^2.\end{aligned}$$

From this we can solve for j_{min} and get that $j_{min} = |j_1 - j_2|$. So all possible values of j are given by:

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2.$$

If we go back to look at $J_z \Rightarrow J_z - J_{1z} - J_{2z} = 0$ we can see that this means that the only non-zero Clebsch-Gordan coefficients are those that satisfy $m = m_1 + m_2$:

$$\begin{aligned}\langle j_1, m_1; j_2, m_2 | J_z - J_{1z} - J_{2z} | j_1, j_2; j, m \rangle &= 0 \\ (m - m_1 - m_2) \langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m \rangle &= 0\end{aligned}$$

Therefore, if $m \neq m_1 + m_2$ then $\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m \rangle = 0$. So only the CGC with $m = m_1 + m_2$ are non-zero. So we end up with the following selection rules for the CGC (where the CGC is non-zero):

- $|j_1 - j_2| \leq j \leq j_1 + j_2$
- $m = m_1 + m_2$

Claculating CGC:

I do not understand this part well enough to write it down... How the hell do i calculate this???????

Coupling of spin-1 and spin-1/2 particles:

Maybe also spin orbit coupling