Comments on equation (5.289) in [SN]

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[SN] uses the result

$$\lim_{\alpha\to\infty} \frac{\sin^2(\alpha x)}{\alpha x^2} = \pi\delta(x) \tag{1}$$

to conclude that

$$''\lim_{t\to\infty} \frac{1}{(E_n - E_i)^2} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] = \frac{\pi t}{2\hbar}\delta(E_n - E_i)''. \tag{2}$$

This statement is incorrect. The problem arises because in going from (1) to (2), the equation is multiplied by α, and α is infinite.

We can, however, go back to the expression in (5.288)

$$4\int \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \frac{|V_{ni}|^2}{|E_n - E_i|^2}\rho(E_n)dE_n. \tag{3}$$

If we assume that $|V_{ni}|^2\rho(E_n)$ is roughly constant within the region, where there are significant contributions to the integral, we can approximate (3) by

$$4\overline{|V_{ni}|^2}\rho(E_i)\int \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \frac{1}{|E_n - E_i|^2}dE_n = \frac{2\pi t}{\hbar}\overline{|V_{ni}|^2}\rho(E_i), \tag{4}$$

which gives the right hand side of (5.291), and from there one obtains Fermi's golden rule.

Let us also take a closer look at the integral

$$\int_{-\infty}^{\infty} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \frac{1}{|E_n - E_i|^2}dE_n = \frac{\pi t}{2\hbar}. \tag{5}$$

Based on figure 5.9 in [SN], we estimate

$$\int_{-2\pi\hbar/t}^{2\pi\hbar/t} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \frac{1}{|E_n - E_i|^2}dE_n \sim \frac{\pi t}{2\hbar}, \tag{6}$$

where the factor of 1/2 on the right hand side is not very accurate. On the other hand, it seems we can approximate

$$2\int_{2\pi\hbar/t}^{\infty} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right] \frac{1}{|E_n - E_i|^2}dE_n \sim \int_{2\pi\hbar/t}^{\infty} \frac{1}{|E_n - E_i|^2}dE_n = \frac{t}{2\pi\hbar}, \tag{7}$$

as the $\sin^2$ is 1/2 on average for $E_n \neq E_i$. This shows that the largest contributions to the integral come from a region of width $4\pi\hbar/t$ around zero, and this width decreases with time. This means that the approximation in (4) becomes better as time passes (provided the transfer is still small enough that first order perturbation theory is accurate). In other words, although the wings in figure 5.9 do not approach zero for large t as the delta function in (5.289) would predict, the area of the peak around $E_n \approx E_i$ increases with time, such that the relative error of the approximation in (4) decreases. Therefore Fermi's golden rule is a reasonable approximation.

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).