The Wigner-Eckart Theorem

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The Wigner-Eckart theorem is discussed in several textbooks, including section 3.11 of [SN]. We here briefly state the theorem.

Consider a tensor $T_q^{(k)}$ with indices k and q, where k is a non-negative integer and $q \in \{-k, -k+1, \ldots, k\}$. The tensor is said to be a spherical tensor if the tensor fulfils the commutation relations

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}, \tag{1}$$

$$[J_\pm, T_q^{(k)}] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} T_{q\pm 1}^{(k)}, \tag{2}$$

where $\vec{J} = (J_x, J_y, J_z)$ is the angular momentum operator.

The Wigner-Eckart theorem: Let $|\alpha, jm\rangle$ be a quantum state fulfilling

$$J^2|\alpha, jm\rangle = \hbar^2 j(j+1)|\alpha, jm\rangle, \tag{3}$$

$$J_z|\alpha, jm\rangle = \hbar m|\alpha, jm\rangle. \tag{4}$$

For spherical tensors $T_q^{(k)}$, we have

$$\langle \alpha', j'm'|T_q^k|\alpha, jm\rangle = \langle jk; mq|jk; j'm'\rangle f(\alpha', j', \alpha, j), \tag{5}$$

where $\langle jk; mq|jk; j'm'\rangle$ are the Clebsch-Gordan coefficients and $f(\alpha', j', \alpha, j)$ is independent of m, m', and q.

In [SN], f is written as

$$f(\alpha', j', \alpha, j) \equiv \frac{\langle \alpha' j'||T_q^{(k)}||\alpha j\rangle}{\sqrt{2j'+1}}. \tag{6}$$

This should *not* be read as the matrix element of the absolute value of $T_q^{(k)}$. It is a double-bar matrix element that means exactly what it needs to mean to make (5) true.

We will not prove the Wigner-Eckart theorem here.

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).