Comments on section 6.4 of [SN]

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Here a brief overview of section 6.4. In section 6.4, we specialize to the case of spherically symmetric potentials. The Hamiltonian hence commutes with the angular momentum operator $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, and it is advantageous to use simultaneous eigenstates $|E, l, m\rangle$ of the operators H_0, $\mathbf{L}^2$, and L_z that all commute with each other. Note that we should here always work with L being infinite, as we do not want to solve a problem with spherical symmetry in a space formed as a cube. Section 6.4.1 derives how one can transform between the momentum eigenkets $|\mathbf{k}\rangle$ and $|E, l, m\rangle$ as well as between the position eigenkets $|\mathbf{x}\rangle$ and $|E, l, m\rangle$. Equation (6.104) is also important. Note that further information about the various spherical functions can be found in appendix B.

According to equation (6.57), we have

$$\langle \mathbf{x}|\psi^{(+)}\rangle \to \frac{1}{(2\pi)^{3/2}} \left(e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r} f(\mathbf{k}', \mathbf{k}) \right) \quad \text{for large } r. \tag{1}$$

Section 6.4.2 reexpresses this equation into partial waves:

$$\langle \mathbf{x}|\psi^{(+)}\rangle \to \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) \frac{P_l(\cos(\theta))}{2ik} \left((1 + 2ikf_l(k)) \frac{e^{ikr}}{r} - \frac{e^{-i(kr - l\pi)}}{r} \right) \quad \text{for large } r. \tag{2}$$

Remember that $|\psi^{(+)}\rangle$ reduces to the plane wave $|\mathbf{k}\rangle$ in the limit of no potential, i.e. when $f_l(k) = 0$. At this point, we have expressed the formal solution to the scattering problem in terms of $f_l(k)$, but we have still not solved the problem, because we do not yet know what the $f_l(k)$ are. Section 6.4.3 argues that $|1 + 2ikf_l(k)| = 1$, such that we can write

$$1 + 2ikf_l(k) \equiv e^{2i\delta_l}, \tag{3}$$

where δ_l is real.

As we saw earlier, we can find $|\psi^{(+)}\rangle$ for all r by solving the time independent Schrödinger equation. Since the Hamiltonian is spherically symmetric, the eigenstates can be written as a product of an angular and a radial part, but we can also make linear combinations of states with the same energy. Here, we are interested in the solution that reduces to $|\mathbf{k}\rangle$ in the limit of no potential. As we choose $\mathbf{k}$ to point in the $\hat{\mathbf{z}}$ direction, only spherical harmonics with L_z eigenvalue zero will contribute, and they are

$$Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos(\theta)). \tag{4}$$

The eigenstate we are interested in, can hence be written as

$$\langle \mathbf{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}} \sum_l i^l (2l+1) A_l(r) P_l(\cos(\theta)), \tag{5}$$

where

$$A_l(r) = \frac{u_l(r)}{r} \tag{6}$$

and u_l fulfils the radial equation

$$\frac{d^2 u_l(r)}{dr^2} + \left(\frac{2mE}{\hbar^2} - \frac{2mV}{\hbar^2} - \frac{l(l+1)}{r} \right) u_l(r) = 0 \tag{7}$$

with $E = \hbar^2 k^2/(2m)$ and boundary condition

$$u_l(0) = 0. \tag{8}$$

The coefficients in (5) have been chosen such that $A_l(r)$ reduces to $j_l(kr)$ in the limit $V = 0$. Section 6.4.4 explains that one should solve (7) to determine $A_l(r)$ in (5) and then compare (5) for large r to (2) to determine all $f_l(k)$, or equivalently all δ_l.

Section 6.4.4 also considers the special case, where the potential is zero outside some radius R. In that case, equation (7) is always the same for $r > R$, namely (7) with $V = 0$, and the solution is given by (6.135). Comparing (6.135) to (2), one concludes

$$A_l(r) = e^{i\delta_l} (\cos(\delta_l) j_l(kr) - \sin(\delta_l) n_l(kr)), \quad \text{for } r > R. \tag{9}$$

Hence, instead of comparing the solution of the Schrödinger equation to the asymptotic behavior for $r \to \infty$, one can now find δ_l by comparing the solution for $r < R$ to (9) at $r = R$. Specifically, if the potential is finite at $r = R$, both $u_l(r)$ and the first derivative of $u_l(r)$ should be continuous at $r = R$. The book considers the logarithmic derivative rather than the derivative, which has the advantage of eliminating the normalization constants.

Finally, section 6.4.5 considers an example, where the potential is infinite for $r < R$ and zero for $r > R$.

References

[SN] J. J. Sakurai and J. Napolitano, Modern Quantum Mechanics, third edition, Cambridge University Press (2021).