Quadrature Domains – A Survey

Apaar Shanker

Indian Institute of Science

April 23, 2015

Outline

Outline

Outline

- An exercise in Linear Algebra
- Quadrature Domains
    - A definition and some remarks
    - An example and some questions
    - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

Outline

Outline

- An exercise in Linear Algebra
- Quadrature Domains
  - A definition and some remarks
  - An example and some questions
  - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

Outline

- An exercise in Linear Algebra
- Quadrature Domains
    - A definition and some remarks
    - An example and some questions
    - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

Outline

- An exercise in Linear Algebra
- Quadrature Domains
    - A definition and some remarks
    - An example and some questions
    - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

Outline

- An exercise in Linear Algebra
- Quadrature Domains
    - A definition and some remarks
    - An example and some questions
    - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

Outline

- An exercise in Linear Algebra
- Quadrature Domains
    - A definition and some remarks
    - An example and some questions
    - Relevance of the Cauchy transform
- Properties of quadrature domains
- Constructing quadrature domains
- Examples
- What happens in higher dimensions?

An Exercise in Linear Algebra

Let V_n be the vector space of all real polynomials on $[a, b]$ with degree at most n. Choose distinct points $t_0, t_1, \ldots, t_n$ in $[a, b]$ and consider the following functionals on V_n:

$$L_i(p) = p(t_i),\ 0 \leq i \leq n.$$

- Suppose that

$$\alpha_0 L_0 + \alpha_1 L_1 + \ldots + \alpha_n L_n = 0.$$

- Evaluate this linear combination on the basis elements $\{1, x, x^2, \ldots, x^n\}$ to get

$$\alpha_0 + \alpha_1 t_j + \ldots + \alpha_n t_j^n = 0$$

for all $1 \leq j \leq n$.

An Exercise in Linear Algebra

Let V_n be the vector space of all real polynomials on $[a, b]$ with degree at most n. Choose distinct points $t_0, t_1, \ldots, t_n$ in $[a, b]$ and consider the following functionals on V_n:

$$L_i(p) = p(t_i),\ 0 \leq i \leq n.$$

- Suppose that

$$\alpha_0 L_0 + \alpha_1 L_1 + \ldots + \alpha_n L_n = 0.$$

- Evaluate this linear combination on the basis elements $\{1, x, x^2, \ldots, x^n\}$ to get

$$\alpha_0 + \alpha_1 t_j + \ldots + \alpha_n t_j^n = 0$$

for all $1 \leq j \leq n$.

The Exercise Contd.

- These are $n+1$ equations in the $n+1$ unknowns $\alpha_0, \alpha_1, \ldots, \alpha_n$.
- Since the t_i's were distinct, the only solutions are $\alpha_j = 0$ for all $1 \leq j \leq n$.
- Hence there are constants $c_0, c_1, \ldots, c_n$ such that

$$\int_a^b p(x)\, dx = c_0 p(t_0) + c_1 p(t_1) + \ldots + c_n p(t_n)$$

for all $p \in V_n$.

The Exercise Contd.

- These are $n+1$ equations in the $n+1$ unknowns $\alpha_0, \alpha_1, \ldots, \alpha_n$.
- Since the t_i's were distinct, the only solutions are $\alpha_j = 0$ for all $1 \leq j \leq n$.
- Hence there are constants $c_0, c_1, \ldots, c_n$ such that

$$\int_a^b p(x)\,dx = c_0 p(t_0) + c_1 p(t_1) + \ldots + c_n p(t_n)$$

for all $p \in V_n$.

The Exercise Contd.

- These are $n+1$ equations in the $n+1$ unknowns $\alpha_0, \alpha_1, \dots, \alpha_n$.
- Since the t_i's were distinct, the only solutions are $\alpha_j = 0$ for all $1 \leq j \leq n$.
- Hence there are constants $c_0, c_1, \dots, c_n$ such that

$$\int_a^b p(x)\, dx = c_0 p(t_0) + c_1 p(t_1) + \dots + c_n p(t_n)$$

for all $p \in V_n$.

- Example: For all $p \in V_3$,

$$\int_{-1}^{1} f(t)\,dt = 1/3f(-1) + 4/3f(0) + 1/3f(1)$$

which is a 'Quadrature Identity'.

- This motivates the following ...

- Example: For all $p \in V_3$,

$$\int_{-1}^{1} f(t)\,dt = 1/3f(-1) + 4/3f(0) + 1/3f(1)$$

which is a ‘Quadrature Identity’.

- This motivates the following ...

A Definition and Remarks

- For a domain $D \subset \mathbb{C}$, let $L^1_{\mathcal{O}}(D) = L^1(D) \cap \mathcal{O}(D)$.
- D is a *quadrature domain* for $L^1_{\mathcal{O}}(D)$ if there exist points $z_1, z_2, \ldots, z_m \in D$ and scalars $a_{j,k} \in \mathbb{C}$ such that

$$\int_D f(z)\, d\sigma_z = \sum_{j=1}^{m} \sum_{k=0}^{r_j - 1} a_{j,k} f^{(k)}(z_j)$$

  for every $f \in L^1_{\mathcal{O}}(D)$.
- Quadrature data = The points z_j (called the *nodes*) and the constants $a_{j,k}, r_j$.

A Definition and Remarks

- For a domain $D \subset \mathbb{C}$, let $L^1_{\mathcal{O}}(D) = L^1(D) \cap \mathcal{O}(D)$.
- D is a *quadrature domain* for $L^1_{\mathcal{O}}(D)$ if there exist points $z_1, z_2, \ldots, z_m \in D$ and scalars $a_{j,k} \in \mathbb{C}$ such that

$$\int_D f(z)\, d\sigma_z = \sum_{j=1}^{m} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j)$$

for every $f \in L^1_{\mathcal{O}}(D)$.

- Quadrature data = The points z_j (called the *nodes*) and the constants $a_{j,k}$, r_j.

A Definition and Remarks

- For a domain $D \subset \mathbb{C}$, let $L^1_{\mathcal{O}}(D) = L^1(D) \cap \mathcal{O}(D)$.
- D is a *quadrature domain* for $L^1_{\mathcal{O}}(D)$ if there exist points $z_1, z_2, \ldots, z_m \in D$ and scalars $a_{j,k} \in \mathbb{C}$ such that

$$\int_D f(z)\, d\sigma_z = \sum_{j=1}^{m} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j)$$

  for every $f \in L^1_{\mathcal{O}}(D)$.
- Quadrature data = The points z_j (called the *nodes*) and the constants $a_{j,k}, r_j$.

An Example and Some Questions

- An Example: Take $D = B(a, r) \subset \mathbb{C}$. Then for all such f,

$$\int_D f(z)\, d\sigma_z = \pi r^2 f(a).$$

- Are there other examples of domains that admit such quadrature identities? How many are there?
- What are the analytic properties of such domains?
- If there are many such examples, is there a procedure to construct them given the quadrature data?

An Example and Some Questions

- An Example: Take $D = B(a, r) \subset \mathbb{C}$. Then for all such f,

$$\int_D f(z)\, d\sigma_z = \pi r^2 f(a).$$

- Are there other examples of domains that admit such quadrature identities? How many are there?
- What are the analytic properties of such domains?
- If there are many such examples, is there a procedure to construct them given the quadrature data?

An Example and Some Questions

- An Example: Take $D = B(a, r) \subset \mathbb{C}$. Then for all such f,

$$\int_D f(z)\, d\sigma_z = \pi r^2 f(a).$$

- Are there other examples of domains that admit such quadrature identities? How many are there?
- What are the analytic properties of such domains?
- If there are many such examples, is there a procedure to construct them given the quadrature data?

An Example and Some Questions

- An Example: Take $D = B(a, r) \subset \mathbb{C}$. Then for all such f,

$$\int_D f(z)\, d\sigma_z = \pi r^2 f(a).$$

- Are there other examples of domains that admit such quadrature identities? How many are there?
- What are the analytic properties of such domains?
- If there are many such examples, is there a procedure to construct them given the quadrature data?

Relevance of the Cauchy transform

- Think of the quadrature identity as: The integral of f on D is a finite linear sum of Dirac masses and their derivatives at the nodes z_j.
- Take a quadrature domain $D \subset \mathbb{C}$. Pick $\zeta \in \mathbb{C} \setminus D$ and use

$$f(z) = 1/(z - \zeta)$$

  in the quadrature identity.
- Then

$$\int_D \frac{d\sigma_z}{z - \zeta} = R(\zeta)$$

  where $R(\zeta)$ is rational.
- The left side is exactly the Cauchy transform of χ_D, the characteristic function of D!

Relevance of the Cauchy transform

- Think of the quadrature identity as: The integral of f on D is a finite linear sum of Dirac masses and their derivatives at the nodes z_j.
- Take a quadrature domain $D \subset \mathbb{C}$. Pick $\zeta \in \mathbb{C} \setminus D$ and use

  $$f(z) = 1/(z - \zeta)$$

  in the quadrature identity.
- Then

  $$\int_D \frac{d\sigma_z}{z - \zeta} = R(\zeta)$$

  where $R(\zeta)$ is rational.
- The left side is exactly the Cauchy transform of χ_D, the characteristic function of D!

Relevance of the Cauchy transform

- Think of the quadrature identity as: The integral of f on D is a finite linear sum of Dirac masses and their derivatives at the nodes z_j.
- Take a quadrature domain $D \subset \mathbb{C}$. Pick $\zeta \in \mathbb{C} \setminus D$ and use

$$f(z) = 1/(z - \zeta)$$

in the quadrature identity.
- Then

$$\int_D \frac{d\sigma_z}{z - \zeta} = R(\zeta)$$

where $R(\zeta)$ is rational.
- The left side is exactly the Cauchy transform of χ_D, the characteristic function of D!

Relevance of the Cauchy transform

- Think of the quadrature identity as: The integral of f on D is a finite linear sum of Dirac masses and their derivatives at the nodes z_j.
- Take a quadrature domain $D \subset \mathbb{C}$. Pick $\zeta \in \mathbb{C} \setminus D$ and use

$$f(z) = 1/(z - \zeta)$$

  in the quadrature identity.
- Then

$$\int_D \frac{d\sigma_z}{z - \zeta} = R(\zeta)$$

  where $R(\zeta)$ is rational.
- The left side is exactly the Cauchy transform of χ_D, the characteristic function of D!

Need to understand the Cauchy transform of χ_D ...

Lemma

Let $D \subset \mathbb{C}$ be such that

$$\int_D \frac{d\sigma_z}{|z|} < \infty.$$

Define

$$S(\zeta) = \int_D \frac{d\sigma_z}{z - \zeta}.$$

Then $S(\zeta)$ is continuous on $\mathbb{C}$, is $O(|\zeta|^{1/2})$ as $\zeta \to \infty$ and for $\zeta \in D$ satisfies

$$S(\zeta) = -\pi\overline{\zeta} + g(\zeta)$$

where g is continuous on $\overline{D}$ and holomorphic on D.

Proof

- $S(\zeta) = \chi_D * 1/z$ and hence S is continuous on $\mathbb{C}$.
- Take a test function ϕ and compute

$$\frac{\partial S}{\partial \bar{\zeta}}(\phi) = -\int_{\mathbb{C}} \left(\frac{\partial \phi}{\partial \bar{\zeta}} \int_{\mathbb{C}} \frac{\chi_D(z)}{z-\zeta}\, d\sigma_z \right) d\sigma_\zeta$$
$$= -\int_{\mathbb{C}} \chi_D(z) \left(\int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial \phi}{\partial \bar{\zeta}}\, d\sigma_\zeta \right) d\sigma_z.$$

- The inner integral $= \pi\phi(z)$ and hence $\frac{\partial S}{\partial \bar{\zeta}} = -\pi\chi_D(\zeta)$ as a distribution.
- $S(\zeta)$ is holomorphic for $\zeta \in \mathbb{C} \setminus D$ and in D, the function $g = S + \pi\bar{\zeta}$ satisfies $\partial g/\partial\bar{\zeta} = 0$.

Proof

- $S(\zeta) = \chi_D * 1/z$ and hence S is continuous on $\mathbb{C}$.
- Take a test function ϕ and compute

$$\begin{aligned}\frac{\partial S}{\partial \bar{\zeta}}(\phi) &= -\int_{\mathbb{C}} \left(\frac{\partial \phi}{\partial \bar{\zeta}} \int_{\mathbb{C}} \frac{\chi_D(z)}{z - \zeta}\, d\sigma_z \right) d\sigma_\zeta \\ &= -\int_{\mathbb{C}} \chi_D(z) \left(\int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial \phi}{\partial \bar{\zeta}}\, d\sigma_\zeta \right) d\sigma_z.\end{aligned}$$

- The inner integral $= \pi\phi(z)$ and hence $\frac{\partial S}{\partial \bar{\zeta}} = -\pi\chi_D(\zeta)$ as a distribution.
- $S(\zeta)$ is holomorphic for $\zeta \in \mathbb{C} \setminus D$ and in D, the function $g = S + \pi\bar{\zeta}$ satisfies $\partial g/\partial\bar{\zeta} = 0$.

Proof

- $S(\zeta) = \chi_D * 1/z$ and hence S is continuous on $\mathbb{C}$.
- Take a test function ϕ and compute

$$\begin{aligned}\frac{\partial S}{\partial \bar{\zeta}}(\phi) &= -\int_{\mathbb{C}} \left(\frac{\partial \phi}{\partial \bar{\zeta}} \int_{\mathbb{C}} \frac{\chi_D(z)}{z-\zeta}\, d\sigma_z \right) d\sigma_\zeta \\ &= -\int_{\mathbb{C}} \chi_D(z) \left(\int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial \phi}{\partial \bar{\zeta}}\, d\sigma_\zeta \right) d\sigma_z.\end{aligned}$$

- The inner integral $= \pi\phi(z)$ and hence $\frac{\partial S}{\partial \bar{\zeta}} = -\pi \chi_D(\zeta)$ as a distribution.
- $S(\zeta)$ is holomorphic for $\zeta \in \mathbb{C} \setminus D$ and in D, the function $g = S + \pi\bar{\zeta}$ satisfies $\partial g/\partial \bar{\zeta} = 0$.

Proof

- $S(\zeta) = \chi_D * 1/z$ and hence S is continuous on $\mathbb{C}$.
- Take a test function ϕ and compute

$$\frac{\partial S}{\partial \bar{\zeta}}(\phi) = -\int_{\mathbb{C}} \left(\frac{\partial \phi}{\partial \bar{\zeta}} \int_{\mathbb{C}} \frac{\chi_D(z)}{z-\zeta}\, d\sigma_z \right) d\sigma_\zeta$$
$$= -\int_{\mathbb{C}} \chi_D(z) \left(\int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial \phi}{\partial \bar{\zeta}}\, d\sigma_\zeta \right) d\sigma_z.$$

- The inner integral $= \pi\phi(z)$ and hence $\frac{\partial S}{\partial \bar{\zeta}} = -\pi\chi_D(\zeta)$ as a distribution.
- $S(\zeta)$ is holomorphic for $\zeta \in \mathbb{C} \setminus D$ and in D, the function $g = S + \pi\bar{\zeta}$ satisfies $\partial g / \partial \bar{\zeta} = 0$.

Proof Contd.

- Let $D_1 = \{z : |z-\zeta| \leq |\zeta|^{1/2}\}$ and $D_2 = \mathbb{C} \setminus D_1$.
- Then
$$|S(\zeta)| \leq \int_{D_1} \frac{|\chi_D(z)|}{|z-\zeta|}\, d\sigma_z + \int_{D_2} *.$$
- The first integral is at most
$$\int_{|w|\leq|\zeta|^{1/2}} \frac{d\sigma_w}{|w|} \lesssim |\zeta|^{1/2},$$

Proof Contd.

- Let $D_1 = \{z : |z - \zeta| \leq |\zeta|^{1/2}\}$ and $D_2 = \mathbb{C} \setminus D_1$.
- Then

$$|S(\zeta)| \leq \int_{D_1} \frac{|\chi_D(z)|}{|z-\zeta|}\, d\sigma_z + \int_{D_2} *.$$

- The first integral is at most

$$\int_{|w| \leq |\zeta|^{1/2}} \frac{d\sigma_w}{|w|} \lesssim |\zeta|^{1/2},$$

Proof Contd.

- Let $D_1 = \{z : |z-\zeta| \leq |\zeta|^{1/2}\}$ and $D_2 = \mathbb{C} \setminus D_1$.
- Then
$$|S(\zeta)| \leq \int_{D_1} \frac{|\chi_D(z)|}{|z-\zeta|}\, d\sigma_z + \int_{D_2} *.$$
- The first integral is at most
$$\int_{|w|\leq|\zeta|^{1/2}} \frac{d\sigma_w}{|w|} \lesssim |\zeta|^{1/2},$$

- while the second one can be written as

$$\int_{D_2} * \leq \int_{\mathbb{C}} \frac{|z|}{|z-\zeta|} \cdot \frac{|\chi_D(z)|}{|z|}\, d\sigma_z$$

where the term

$$\frac{|z|}{|z-\zeta|} \leq 1 + \frac{|\zeta|}{|z-\zeta|} \leq 1 + |\zeta|^{1/2}.$$

- Hence $S(\zeta) = O(|\zeta|^{1/2})$ for large ζ.

- while the second one can be written as

$$\int_{D_2} * \leq \int_{\mathbb{C}} \frac{|z|}{|z-\zeta|} \cdot \frac{|\chi_D(z)|}{|z|}\, d\sigma_z$$

  where the term

$$\frac{|z|}{|z-\zeta|} \leq 1 + \frac{|\zeta|}{|z-\zeta|} \leq 1 + |\zeta|^{1/2}.$$

- Hence $S(\zeta) = O(|\zeta|^{1/2})$ for large ζ.

The Aharonov–Shapiro Theorem

The following are equivalent:

(i) D is a quadrature domain.

(ii) There is a distribution α supported at finitely many points in D (the nodes!) and which is a finite linear combination of Dirac masses and their derivatives at these points, such that

$$\int_D f = \langle \alpha, f \rangle.$$

The Aharonov–Shapiro Theorem

The following are equivalent:

(i) *D* is a quadrature domain.

(ii) There is a distribution α supported at finitely many points in *D* (the nodes!) and which is a finite linear combination of Dirac masses and their derivatives at these points, such that

$$\int_D f = \langle \alpha, f \rangle.$$

(iii) There is a rational function $R(\zeta)$ with all poles in D such that $S(\zeta) = R(\zeta)$ for $\zeta \in \mathbb{C} \setminus D$.

(iv) There exists a meromorphic function in D, say h with finitely many poles in D such that h is continuous on $\overline{D}$ and

$$h(z) = \overline{z}$$

for $z \in \partial D$.

Remark: Connection with the Schwarz reflection principle and the ‘analytic’ Dirichlet problem.

(iii) There is a rational function $R(\zeta)$ with all poles in D such that $S(\zeta) = R(\zeta)$ for $\zeta \in \mathbb{C} \setminus D$.

(iv) There exists a meromorphic function in D, say h with finitely many poles in D such that h is continuous on $\overline{D}$ and

$$h(z) = \overline{z}$$

for $z \in \partial D$.

Remark: Connection with the Schwarz reflection principle and the 'analytic' Dirichlet problem.

Pointers to the Proof:

- Suppose (ii) holds. Take $f(z) = 1/(z-\zeta)$ for $\zeta \in \mathbb{C} \setminus D$. Then
$$S(\zeta) = \int_D \frac{d\sigma_z}{z-\zeta} = \langle \alpha, 1/(z-\zeta) \rangle$$
which is a rational function, say $R(\zeta)$ with poles exactly at the nodes. Thus (iii) holds.
- Conversely, if (iii) holds, then
$$S(\zeta) = \int_D \frac{d\sigma_z}{z-\zeta} = R(\zeta) = p(\zeta)/q(\zeta).$$

Pointers to the Proof:

- Suppose (ii) holds. Take $f(z) = 1/(z - \zeta)$ for $\zeta \in \mathbb{C} \setminus D$. Then
$$S(\zeta) = \int_D \frac{d\sigma_z}{z - \zeta} = \langle \alpha, 1/(z - \zeta) \rangle$$
which is a rational function, say $R(\zeta)$ with poles exactly at the nodes. Thus (iii) holds.
- Conversely, if (iii) holds, then
$$S(\zeta) = \int_D \frac{d\sigma_z}{z - \zeta} = R(\zeta) = p(\zeta)/q(\zeta).$$

Since $S(\zeta) = O(|\zeta|^{1/2})$, $\deg p \leq \deg q$, i.e., the partial fraction decomposition of R does not have a holomorphic part. Thus,

$$R(\zeta) = \langle \alpha, 1/(z - \zeta) \rangle$$

for an appropriate α. The support of α is exactly the set of poles of $R(\zeta)$. Thus the quadrature identity holds for functions of the form $1/(z - \zeta)$, $\zeta \in \mathbb{C} \setminus D$. By Ahlfors–Bers, such functions are dense in our test class and hence (ii) holds.

Why are (iii) and (iv) equivalent?

Suppose (iii) holds, i.e., $S(\zeta) = R(\zeta)$ on $\mathbb{C} \setminus D$. Let

$$h(\zeta) = \pi^{-1}(g(\zeta) - R(\zeta)).$$

- Then h is meromorphic on D and has finitely many poles in D.
- h extends continuously to ∂D since g does and on ∂D,

$$h(\zeta) = \pi^{-1}(S(\zeta) + \pi\bar{\zeta} - R(\zeta)) = \bar{\zeta}.$$

Why are (iii) and (iv) equivalent?

Suppose (iii) holds, i.e., $S(\zeta) = R(\zeta)$ on $\mathbb{C} \setminus D$. Let

$$h(\zeta) = \pi^{-1}(g(\zeta) - R(\zeta)).$$

- Then h is meromorphic on D and has finitely many poles in D.
- h extends continuously to ∂D since g does and on ∂D,

$$h(\zeta) = \pi^{-1}(S(\zeta) + \pi\overline{\zeta} - R(\zeta)) = \overline{\zeta}.$$

Boundaries of Quadrature Domains

Theorem

Let D be a quadrature domain. Then there exists a nonconstant polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that

$$\partial D \subset \{z = x + iy : P(x, y) = 0\}.$$

Assuming this –

- Let
$$F(z) = \begin{cases} S(z), & \text{if } z \in \mathbb{C} \setminus D; \\ g(z) - \pi h(z), & \text{if } z \in D. \end{cases}$$
- F is meromorphic away from ∂D, continuous on $\mathbb{C}$ and is $O(|\zeta|^{1/2})$ far away. Hence it is rational!

Boundaries of Quadrature Domains

Theorem

Let D be a quadrature domain. Then there exists a nonconstant polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that

$$\partial D \subset \{z = x + iy : P(x, y) = 0\}.$$

Assuming this –

- Let
$$F(z) = \begin{cases} S(z), & \text{if } z \in \mathbb{C} \setminus D; \\ g(z) - \pi h(z), & \text{if } z \in D. \end{cases}$$
- F is meromorphic away from ∂D, continuous on $\mathbb{C}$ and is $O(|\zeta|^{1/2})$ far away. Hence it is rational!

Boundaries of Quadrature Domains

Theorem

Let D be a quadrature domain. Then there exists a nonconstant polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that

$$\partial D \subset \{z = x + iy : P(x, y) = 0\}.$$

Assuming this –

- Let
$$F(z) = \begin{cases} S(z), & \text{if } z \in \mathbb{C} \setminus D; \\ g(z) - \pi h(z), & \text{if } z \in D. \end{cases}$$
- F is meromorphic away from ∂D, continuous on $\mathbb{C}$ and is $O(|\zeta|^{1/2})$ far away. Hence it is rational!

Proof of the Theorem

Lemma

Let $D \subset \mathbb{C}$ be bounded. Suppose $f, g \in \mathcal{M}(D)$ with only finitely many poles. Suppose further that both extend continuously to ∂D and take real values there. Then there is a polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that $P(f(z), g(z)) = 0$ on D.

- $m =$ the no. of poles of f, g with multiplicity and $\mathcal{P} =$ the set of poles of f, g.
- For $n \geq 1$, consider $\{f^j g^k : j, k \geq 0, 1 \leq j + k \leq n\}$.
- Exactly $n(n+3)/2$ such functions with poles in $\mathcal{P}$ and the total multiplicity is at most mn.

Proof of the Theorem

Lemma

Let $D \subset \mathbb{C}$ be bounded. Suppose $f, g \in \mathcal{M}(D)$ with only finitely many poles. Suppose further that both extend continuously to ∂D and take real values there. Then there is a polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that $P(f(z), g(z)) = 0$ on D.

- $m =$ the no. of poles of f, g with multiplicity and $\mathcal{P} =$ the set of poles of f, g.
- For $n \geq 1$, consider $\{f^j g^k : j, k \geq 0, 1 \leq j + k \leq n\}$.
- Exactly $n(n+3)/2$ such functions with poles in $\mathcal{P}$ and the total multiplicity is at most mn.

Proof of the Theorem

Lemma

Let $D \subset \mathbb{C}$ be bounded. Suppose $f, g \in \mathcal{M}(D)$ with only finitely many poles. Suppose further that both extend continuously to ∂D and take real values there. Then there is a polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that $P(f(z), g(z)) = 0$ on D.

- $m =$ the no. of poles of f, g with multiplicity and $\mathcal{P} =$ the set of poles of f, g.
- For $n \geq 1$, consider $\{f^j g^k : j, k \geq 0, 1 \leq j + k \leq n\}$.
- Exactly $n(n+3)/2$ such functions with poles in $\mathcal{P}$ and the total multiplicity is at most mn.

Proof of the Theorem

Lemma

Let $D \subset \mathbb{C}$ be bounded. Suppose $f, g \in \mathcal{M}(D)$ with only finitely many poles. Suppose further that both extend continuously to ∂D and take real values there. Then there is a polynomial $P \in \mathbb{R}[X, Y]$, irreducible over $\mathbb{C}$ such that $P(f(z), g(z)) = 0$ on D.

- $m =$ the no. of poles of f, g with multiplicity and $\mathcal{P} =$ the set of poles of f, g.
- For $n \geq 1$, consider $\{f^j g^k : j, k \geq 0, 1 \leq j + k \leq n\}$.
- Exactly $n(n+3)/2$ such functions with poles in $\mathcal{P}$ and the total multiplicity is at most mn.

- The vector space generated by the principal parts of $f^j g^k$ has real dimension $\leq 2mn$.
- Choose n so that $n(n+3)/2 > 2mn$!
- Then there is $Q \in \mathbb{R}[X, Y]$ such that $Q(f, g) \in \mathcal{O}(D)$ and is real on ∂D. Then $Q(f, g) \equiv 0$ on D.
- Choose such a P with least degree d. Suppose that $P = P_1 P_2$ where $P_1, P_2 \in \mathbb{C}[X, Y]$ and have degrees $< d$.

- The vector space generated by the principal parts of $f^j g^k$ has real dimension $\leq 2mn$.
- Choose n so that $n(n+3)/2 > 2mn$!
- Then there is $Q \in \mathbb{R}[X, Y]$ such that $Q(f, g) \in \mathcal{O}(D)$ and is real on ∂D. Then $Q(f, g) \equiv 0$ on D.
- Choose such a P with least degree d. Suppose that $P = P_1 P_2$ where $P_1, P_2 \in \mathbb{C}[X, Y]$ and have degrees $< d$.

- The vector space generated by the principal parts of $f^j g^k$ has real dimension $\leq 2mn$.
- Choose n so that $n(n+3)/2 > 2mn$!
- Then there is $Q \in \mathbb{R}[X, Y]$ such that $Q(f, g) \in \mathcal{O}(D)$ and is real on ∂D. Then $Q(f, g) \equiv 0$ on D.
- Choose such a P with least degree d. Suppose that $P = P_1 P_2$ where $P_1, P_2 \in \mathbb{C}[X, Y]$ and have degrees $< d$.

- The vector space generated by the principal parts of $f^j g^k$ has real dimension $\leq 2mn$.
- Choose n so that $n(n+3)/2 > 2mn$!
- Then there is $Q \in \mathbb{R}[X, Y]$ such that $Q(f, g) \in \mathcal{O}(D)$ and is real on ∂D. Then $Q(f, g) \equiv 0$ on D.
- Choose such a P with least degree d. Suppose that $P = P_1 P_2$ where $P_1, P_2 \in \mathbb{C}[X, Y]$ and have degrees $< d$.

- Both $P_1(f,g), P_2(f,g)$ are meromorphic on D and $P(f,g) = 0$ on D. Hence, say $P_1(f,g) \equiv 0$ on D.
- Write $P_1 = P_3 + iP_4$ and note that

$$P_3(f,g) + iP_4(f,g) = 0$$

on ∂D. Since f,g are real on ∂D, it follows that $P_3(f,g), P_4(f,g) \equiv 0$ on D.
- But $\deg P_3, \deg P_4 < d$!

- Both $P_1(f,g), P_2(f,g)$ are meromorphic on D and $P(f,g) = 0$ on D. Hence, say $P_1(f,g) \equiv 0$ on D.

- Write $P_1 = P_3 + iP_4$ and note that

$$P_3(f,g) + iP_4(f,g) = 0$$

  on ∂D. Since f, g are real on ∂D, it follows that $P_3(f,g), P_4(f,g) \equiv 0$ on D.

- But $\deg P_3, \deg P_4 < d$!

- Both $P_1(f,g), P_2(f,g)$ are meromorphic on D and $P(f,g) = 0$ on D. Hence, say $P_1(f,g) \equiv 0$ on D.

- Write $P_1 = P_3 + iP_4$ and note that

$$P_3(f,g) + iP_4(f,g) = 0$$

  on ∂D. Since f, g are real on ∂D, it follows that $P_3(f,g), P_4(f,g) \equiv 0$ on D.

- But $\deg P_3, \deg P_4 < d$!

- Take h meromorphic on D with finitely many poles and $h(z) = \bar{z}$ on ∂D.
- Let $f(z) = (z + h(z))/2$ and $g(z) = (z - h(z))/2$. Note that $f(z) = x, g(z) = y$ on ∂D.
- There is a $P \in \mathbb{R}[X, Y]$ such that $P(f, g) = P(x, y) = 0$ on ∂D.

- Take h meromorphic on D with finitely many poles and $h(z) = \bar{z}$ on ∂D.
- Let $f(z) = (z + h(z))/2$ and $g(z) = (z - h(z))/2$. Note that $f(z) = x, g(z) = y$ on ∂D.
- There is a $P \in \mathbb{R}[X, Y]$ such that $P(f, g) = P(x, y) = 0$ on ∂D.

- Take h meromorphic on D with finitely many poles and $h(z) = \bar{z}$ on ∂D.
- Let $f(z) = (z + h(z))/2$ and $g(z) = (z - h(z))/2$. Note that $f(z) = x, g(z) = y$ on ∂D.
- There is a $P \in \mathbb{R}[X, Y]$ such that $P(f, g) = P(x, y) = 0$ on ∂D.

Remarks

- The region bounded by a triangle is not a quadrature domain.
- An annulus is not a quadrature domain.
- The region bounded by an ellipse is also not a quadrature domain. Use: If

$$P = P_n + P_{n-1} + \ldots + P_0$$

is the decomposition into homogeneous terms, then P_n is divisible by $x^2 + y^2$.

Remarks

- The region bounded by a triangle is not a quadrature domain.
- An annulus is not a quadrature domain.
- The region bounded by an ellipse is also not a quadrature domain. Use: If

$$P = P_n + P_{n-1} + \ldots + P_0$$

is the decomposition into homogeneous terms, then P_n is divisible by $x^2 + y^2$.

Remarks

- The region bounded by a triangle is not a quadrature domain.
- An annulus is not a quadrature domain.
- The region bounded by an ellipse is also not a quadrature domain. Use: If

$$P = P_n + P_{n-1} + \ldots + P_0$$

  is the decomposition into homogeneous terms, then P_n is divisible by $x^2 + y^2$.

A characterization of the disc: Epstein–Schiffer

Suppose that $0 \in D$, that D is bounded and

$$\int_D f d\sigma_z = af(0)$$

for all f in our test class. Then h has a simple pole at $z = 0$ and $h(z) = \bar{z}$ on ∂D. So $zh(z) = z\bar{z} =$ constant on ∂D. This means that D is a disc.

Constructing Quadrature Domains

Theorem

A bounded simply connected domain $D \subset \mathbb{C}$ is a quadrature domain if and only if its Riemann map

$$\phi : \Delta \to D$$

is rational with all poles outside $\overline{\Delta}$. In fact, if z_i's are the nodes in D and $t_i = \phi^{-1}(z_i)$, then the poles of ϕ are exactly at $1/\bar{t}_i$.

Proof

- ϕ extends continuously up to $\partial\Delta$.
- Then $h(\phi(t)) = \overline{\phi(t)}$ for $|t| = 1$.
- Define
$$R(t) = \begin{cases} \overline{\phi(1/\bar{t})}, & \text{if } |t| > 1; \\ h(\phi(t)), & \text{if } |t| \leq 1. \end{cases}$$
- R is rational. So is $\phi(t) = \overline{R(1/\bar{t})}$.

Proof

- ϕ extends continuously up to $\partial\Delta$.
- Then $h(\phi(t)) = \overline{\phi(t)}$ for $|t| = 1$.
- Define

$$R(t) = \begin{cases} \overline{\phi(1/\bar{t})}, & \text{if } |t| > 1; \\ h(\phi(t)), & \text{if } |t| \leq 1. \end{cases}$$

- R is rational. So is $\phi(t) = \overline{R(1/\bar{t})}$.

Proof

- ϕ extends continuously up to $\partial\Delta$.
- Then $h(\phi(t)) = \overline{\phi(t)}$ for $|t| = 1$.
- Define
$$R(t) = \begin{cases} \overline{\phi(1/\bar{t})}, & \text{if } |t| > 1; \\ h(\phi(t)), & \text{if } |t| \leq 1. \end{cases}$$
- R is rational. So is $\phi(t) = \overline{R(1/\bar{t})}$.

Proof

- ϕ extends continuously up to $\partial\Delta$.
- Then $h(\phi(t)) = \overline{\phi(t)}$ for $|t| = 1$.
- Define

$$R(t) = \begin{cases} \overline{\phi(1/\bar{t})}, & \text{if } |t| > 1; \\ h(\phi(t)), & \text{if } |t| \leq 1. \end{cases}$$

- R is rational. So is $\phi(t) = \overline{R(1/\bar{t})}$.

Illustrating this technique

- Recall that $\psi(z) = (z + 1/z)/2$ conformally maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus [-1, 1]$.
- For $R > 1$, the circle C_R is mapped onto an ellipse E_R where

$$x = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta, y = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta.$$

- Thus $\psi(Rz)$ maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus \overline{E}_R$.

Illustrating this technique

- Recall that $\psi(z) = (z + 1/z)/2$ conformally maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus [-1, 1]$.
- For $R > 1$, the circle C_R is mapped onto an ellipse E_R where

$$x = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta, y = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta.$$

- Thus $\psi(Rz)$ maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus \overline{E}_R$.

Illustrating this technique

- Recall that $\psi(z) = (z + 1/z)/2$ conformally maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus [-1, 1]$.
- For $R > 1$, the circle C_R is mapped onto an ellipse E_R where

$$x = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta, y = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta.$$

- Thus $\psi(Rz)$ maps $\mathbb{C} \setminus \overline{\Delta}$ onto $\mathbb{C} \setminus \overline{E}_R$.

- Thus $z \mapsto 1/\overline{\psi(R/\bar{z})}$ is a conformal map from Δ onto the region bounded by inverting E_R under the map $z \mapsto 1/\bar{z}$.
- The boundary of this region, say N_R is given by

$$(x^2+y^2)^2 - a^{-2}x^2 - b^{-2}y^2 = 0$$

  where $a = (R+1/R)/2$ and $b = (R-1/R)/2$.
- The conformal map on the disc is $\phi(z) = 2Rz/(R^2+z^2)$ which has poles at $z = \pm iR$.

- Thus $z \mapsto 1/\overline{\psi(R/\overline{z})}$ is a conformal map from Δ onto the region bounded by inverting E_R under the map $z \mapsto 1/\overline{z}$.
- The boundary of this region, say N_R is given by

$$(x^2 + y^2)^2 - a^{-2}x^2 - b^{-2}y^2 = 0$$

  where $a = (R + 1/R)/2$ and $b = (R - 1/R)/2$.
- The conformal map on the disc is $\phi(z) = 2Rz/(R^2 + z^2)$ which has poles at $z = \pm iR$.

- Thus $z \mapsto 1/\overline{\psi(R/\overline{z})}$ is a conformal map from Δ onto the region bounded by inverting E_R under the map $z \mapsto 1/\overline{z}$.
- The boundary of this region, say N_R is given by

$$(x^2 + y^2)^2 - a^{-2}x^2 - b^{-2}y^2 = 0$$

  where $a = (R + 1/R)/2$ and $b = (R - 1/R)/2$.
- The conformal map on the disc is $\phi(z) = 2Rz/(R^2 + z^2)$ which has poles at $z = \pm iR$.

- Invert the poles to get $\pm i/R$ and note that $\phi(\pm i/R) = \pm 2i/(R^2 - R^{-2})$. These are the nodes in N_R!
- To get the quadrature identity, take f holomorphic in a neighbourhood of $\overline{N}_R$ and note

$$\begin{aligned} \int_{N_R} f \, d\sigma_z &= (1/2i) \int_{\partial N_R} f(z)\overline{z} \, dz \\ &= (1/2i) \int_{\partial N_R} f(z)h(z) \, dz \\ &= 2\pi(a_1 f(z_1) + a_2 f(z_2)) \end{aligned}$$

where a_1, a_2 are the residues of $h(z)$ at the simple poles z_1, z_2 above.

- Invert the poles to get $\pm i/R$ and note that $\phi(\pm i/R) = \pm 2i/(R^2 - R^{-2})$. These are the nodes in N_R!
- To get the quadrature identity, take f holomorphic in a neighbourhood of $\overline{N}_R$ and note

$$\begin{aligned} \int_{N_R} f\, d\sigma_z &= (1/2i) \int_{\partial N_R} f(z)\bar{z}\, dz \\ &= (1/2i) \int_{\partial N_R} f(z)h(z)\, dz \\ &= 2\pi(a_1 f(z_1) + a_2 f(z_2)) \end{aligned}$$

where a_1, a_2 are the residues of $h(z)$ at the simple poles z_1, z_2 above.

Ubiquity of quadrature domains: Gustafsson's Theorem

Theorem

Let D be a smoothly bounded domain with finite connectivity. Then there exists a quadrature domain G as close as we want to D in the C^∞ topology.

Remark: An annulus is not a quadrature domain, but there are plenty of such domains nearby!

The story in higher dimensions

. . . is yet to take off.

References

- D. Aharonov, H. Shapiro: *Domains on which analytic functions satisfy quadrature identities*, J. Anal. Math. **30**, (1976), 39–73.
- P. J. Davis: *The Schwarz function and its applications*, The Carus Math. Monographs, **17**, (1974).

References

- D. Aharonov, H. Shapiro: *Domains on which analytic functions satisfy quadrature identities*, J. Anal. Math. **30**, (1976), 39–73.
- P. J. Davis: *The Schwarz function and its applications*, The Carus Math. Monographs, **17**, (1974).

Thank You