

## Chapter 10

# Higher-Order Techniques

A sound way of constructing high-order approximations to a given differential problem  $LU = F$ , is first to construct a multigrid program with a low approximation order, and then convert it into a high-order program. The lower order is easier to develop and is also useful as a component in the higher-order program. Such programs are usually more efficient than programs which use high-order difference operators throughout. We mostly recommend the method of §10.2 below, especially for non-elliptic and singular perturbation problems.

### 10.1 Fine-grid defect corrections. Pseudo spectral methods

Given a program for solving the (linear or nonlinear) low-order (order  $p_0$ ) discrete system  $L_0^h u^h = f^h$ , an obvious multigrid approach for raising the approximation order is by high-order “deferred” (or “defect”) corrections introduced once per cycle on the currently-finest grid [Bra79b, §3.4]. That is, we add to  $f^h$  the correction

$$\sigma_{1,0}^h(x^h) := L_0^h \tilde{u}^h(x^h) - L_1^h \tilde{u}^h(x^h), \quad (10.1)$$

where  $L_1^h$  is the higher-order operator, its approximation order (consistency order) being  $p_1 > p_0$ , and  $\tilde{u}^h$  is the current approximate solution. A similar correction is of course introduced to the discrete boundary conditions, too. To save h-cycles one should employ an FMG algorithm (§7), and use corrections like (10.1) at all the FMG stages (i.e., for every currently-finest grid). The total amount of work is then still basically given by (7.4).

Note that that work is proportional to the approximation order  $p_1$ . However, this count does not take into account the calculation of (10.1) once per cycle. For lower  $p_1$  this extra work may be less than the other

work within the cycle (a couple of sweeps on each level), but for high  $p_1$  it becomes dominant and makes the amount of work per cycle proportional to  $p_1$  (assuming spectral-type methods cannot be used and the complexity of calculating  $L_1^h$  is thus proportional to  $p_1$ ), hence the total work is in principle  $O(p_1^2)$ . Furthermore, for higher  $p_1$  we have in principle to use higher computer precision, making the work of each arithmetic operation (in calculating (10.1)) again proportional to  $p_1$ , bringing the total work to  $O(p_1^3)$ . This can be reduced to just  $O(p_1^2)$  by a method of *compound deferred corrections*, i.e., taking  $p_0 = p_1/2$  and solving for  $L_0^h$  by deferred corrections to a system of order  $p_0/2$ , and so on recursively. In the normally used range of  $p_1$ , however, the work of even the uncompounded deferred correction is often still dominated by relaxation and hence still proportional to  $p_1$ .

This technique can in particular be applied to **pseudo-spectral approximations**  $L_1^h$ , i.e., approximations attaining very high order (proportional to  $1/h$ ) through a discrete spectral (Fourier, Chebyshev, etc.) representation of the solution, using fast numerical transformers (e.g., FFT – the Fast Fourier transform) to obtain that representation and to calculate from it the approximate derivatives at gridpoints (cf. e.g., [GO77]). Using reasonably high order in  $L_0^h$  (e.g.,  $p_0 = 4$ , itself calculated by deferred corrections to a second-order operator) one can attain the spectral approximation order with just few applications of the spectral operator. (Other spectral multigrid methods are described in [BFT83], [SZH83], [ZWH82], [ZWH83].)

The deferred correction technique (suggested by L. Fox) is a special case of the concept of defect corrections (see [Lin76], [Ste78], [AS82]). An important advantage of such a technique is that the higher-order operator  $L_1^h$  (and the corresponding higher order boundary conditions) need not be stable. This gives much freedom in the relatively difficult task of calculating  $L_1^h$ . This freedom is especially welcome in nonelliptic and singular perturbation cases, where convenient central approximations are unstable.

The reason  $L_1^h$  need not be stable is that the convergence of the defect correction iterations, to the solution corresponding to  $L_1^h$ , is fast only in the smooth components (for which  $L_0^h$  is a good approximation to  $L_1^h$ ) and is very slow in the high-frequency components. Since instability is a property of high-frequencies, it can creep in only very slowly. The growth of unstable modes within the few solution cycles is negligible.

The whole purpose of defect corrections is in fact to correct low-frequency components; only for such components higher-order approximation, such as  $L_1^h$ , are much better than lower-order approximations like  $L_0^h$ . Recognizing this and the fact that in multigrid processes low frequencies are converged via the coarse-grid corrections, we see that the main effect of the defect corrections can be obtained by applying them only at the stage of transferring residuals to the coarser grids. This would save about two work-units per cycle, and would give better approximations in case  $L_1^h$  is unstable. This idea, from a different point of view, is described in the next

section.

## 10.2 Double discretization: High-order residual transfers

On any given grid participating in multigrid interactions, discrete approximations to the continuous operator  $L$  are used in two different processes: in relaxation sweeps, and in calculating residuals transferred to coarser grids. The two discretization schemes need not be the same [BD79, §3.11]. The discretization  $L_0^h$  employed in the relaxation sweeps must be stable (see §12), but its accuracy may be lower than the one we wish to generate. The discretization  $L_1^h$  used in calculating the transferred residuals determines the accuracy of our numerical solution, but it need not be stable. This “double discretization” scheme is especially useful in dealing with non-elliptic and singular perturbation problems: One can use the most convenient (but sometimes unstable) central differencing for  $L_1^h$ , and add artificial viscosities (see §2.1) only to  $L_0^h$ . This will ensure stable solutions which still have the accuracy of the central differencing.

Note that such a multigrid process will not converge to zero residuals, since it uses two conflicting difference schemes. The very point is, indeed, that the solution produced may be a better approximation to the *differential* solution than can be produced by any of the two schemes. More importantly, during running a double discretization solver, instead of checking algebraic convergence, *one should directly check convergence to the differential solution* through the sequence of solutions produced at different stages of the FMG algorithm (see §1.6).

The lack of algebraic convergence makes the usual two-level mode analysis irrelevant for double discretization schemes. Instead they can be analyzed by the two-level FMG mode analysis (§7.4).

Double discretization schemes can of course similarly be applied to **boundary conditions**; e.g., to Neumann conditions: Simple first-order schemes can be used in relaxation, while second-order Neumann conditions (which are sometimes complicated and may sometimes be unstable) can be used to transfer boundary-condition residuals to coarser grids.

The double discretization scheme need not be confined to the currently finest level; it can **also be used on coarser levels**. This will give better coarse-grid corrections, and hence faster algebraic convergence. (In non-elliptic and singular perturbation cases the algebraic convergence is usually determined by the quality of the coarse-grid correction [Bra81a, §5.1].) It is also more convenient to program, since the same residual transfer routine, based on  $L_1^h$ , is used on all levels.

Moreover, if only  $L_0^h$  is used on coarser levels, the gain in approximation order per cycle cannot be more than  $p_0$ ; hence the final approximation order cannot exceed  $2p_0 + r_0$ , where  $r_0$  is the convergence order of relax-

ation [Bra81a, §2.2]. Such a restriction does not exist if  $L_1^h$  is used for residual transfers on all levels. The approximation order  $p_1$  can then be attained, perhaps even in one cycle, no matter how high  $p_1$  is. In particular, pseudo-spectral approximations can be used in  $L_1^h$ , yielding very high approximation orders in few cycles.

In order to obtain the high approximation orders several rules should be observed: Suitable interpolation orders and residual transfer orders should be employed. The right orders can be derived by crude mode analysis, as in §4.3, but with particular attention to boundary (see in particular rule (C) in §4.3). FMG algorithms with  $W(\nu, 0)$  cycles should be used (see §6.2), to ensure accurate enough solution of the course-grid equations and to avoid degradation of the approximation by terminal relaxation. Also, when double discretization is used on all levels together with the Full Approximation Scheme (see §8), notice that two different right-hand sides should be used on coarser grids, one for relaxation and a different one for residual transfers [Bra81a, §2.1].

In case  $L_1^h$  is a better approximation than  $L_0^h$  not only for smooth components but also in the high-frequency range, the method of fine-grid defect corrections (§10.1) will eventually give smaller errors than the coarse-grid defect correction described here. But the gain will hardly justify the extra work involved in calculating (10.1) separately from the calculation of residuals. In problems where  $L_1^h$  is unstable, the present method is both faster and more accurate.

Double discretization schemes have already been used successfully in various cases, including fourth and sixth order approximations to Poisson equation [Sch82]; second-order approximations to simple singular perturbation problems [Bra77a, §7], [Bör81, §7]; and second-order approximations to incompressible Navier-Stokes equations with high Reynolds numbers. Also, the  $\lambda$  extrapolation (§8.4) can be viewed as a special case, where  $L_1^h = (2^{p_0} L^h - L^{2h}) / (2^{p_0} - 1)$ .

### 10.3 Relaxation with only subprincipal terms

A particularly useful application of the above techniques is to employ a simple relaxation operator  $L_0^h$  where non-principal terms are neglected; more precisely, to employ the simplest stable  $L_0^h$  which approximate the subprincipal terms of the differential operator (see §2.1). Other terms need to be approximated only in  $L_1^h$ . For some fluid-dynamics systems this procedure can save a substantial amount of work. The techniques of either §10.1 or §10.2 can be used with this relaxation; more work is saved by the latter, but the former is safer. On very coarse grids, as the non-principal terms become more important, this type of relaxation may give worse performance. In such cases use more sweeps or reintroduce the neglected non-principal terms.