Chapter 15

Chains of Problems. Frozen τ

We often need to solve not just one isolated problem but a sequence of similar problems depending on some parameter. For example, we may be studying the effect of changing some physical parameters on the "performance" of a system, where the performance is measured as some functional of the solution u to a differential problem. We may want to find for what physical parameters the performance is optimal. Or, in "inverse problems", we may desire to find the physical parameters for which the solution best fit some experimentally observed behavior. Or we may need to solve a sequence of problems in a continuation process (see §8.3.2). Or, the most familiar case, the parameter may be the time t, and each problem in the sequence may represent the implicit equations of one time step.

The key to a highly efficient multi-level treatment of such a sequence of problems is to understand the behavior of high-frequency components. Most often, the change in one step (i.e., the change from one problem in the sequence to the next) is a global change, governed by global parameters. In some problems, the relative changes in high-frequency components are therefore comparable to the relative changes in low ones. Hence, for such problems, the absolute high-frequency *changes* in each step are negligible – they are small compared to the high frequencies themselves, and therefore small compared with the discretization errors. In such cases one need not use the finer grids at each step; the finer the level the more rarely it should be activated. Often this is the situation in most parts of the domain, while in some particular parts, such as near boundaries, significant high-frequency changes do take place in every step, hence more refinement levels should more often be activated in those parts only.

The Full Approximation Scheme (FAS) gives us a convenient structure in which to see smooth changes in the solution without (locally) activating finer grids. The way to neglect *changes* in wavelengths smaller than $O(h)$,

without neglecting those components themselves, is to freeze τ_h^{2h} (see §8.2), i.e., to use on grid $2h$ the values of τ_h^{2h} calculated in a previous step, thus avoiding any visit to grid h during the present step. Once in several steps of visiting grid $2h$, a visit can be made to grid h, to update τ_h^{2h}. When visiting grid $2h$, changes in τ_{2h}^{4h} are made, and their cumulative values since the last visit to grid h can serve to decide when a new visit to grid h is needed, using exchange-rate criteria (see §9.5–9.6 and [Bra79a, §3.9]). Since these criteria can be applied locally, one can decide when and *where* to activate increasingly finer levels.

An obvious but important remark: Whether the above procedures are used or not, and whether FAS or CS is employed, in each step (i.e., for each problem in the chain) it is normally more economic to *work on the correction problem*, i.e., taking the previous-step solution as a first approximation. When FAS-FMG is used, this is easily done, even for nonlinear problems, as follows. First, the old values of τ_h^{2h} should be used in the $2h$-stage of the FMG algorithm (i.e., before grid h is ever visited in the present step). Secondly, the FMG interpolation (first interpolation to grid h in this step) should be a FAS-like interpolation, using the old values of $\tilde{u}^h$; i.e., like (8.6), but with possibly higher order $\mathbb{I}_H^h$ replacing I_H^h (cf. §7.1).

Solving the chain of problems we usually need to monitor certain **solution functionals**. In order to calculate such a functional Φ with finest-grid accuracy even at steps not visiting the finest grid, transfers as in (8.15) can be used.