Chapter 12

True Role of Relaxation

The role of relaxation in multigrid processes has often been stated: It is to *smooth the error*; i.e., to reduce that part of the error (the "high-frequency" part) which cannot be well approximated on the next coarser grid. Some elaboration and clarification of this statement is important.

What is the "error" we want to smooth? It is usually thought of as the *algebraic* error, i.e., the difference $u^h - \tilde{u}^h$ between our calculated solution $\tilde{u}^h$ and the discrete solution u^h (the exact solution to the discrete equations). However, in view of the double discretization scheme (Sec §10.2), where u^h is not well-defined, it becomes clear that what relaxation should really do is to smooth the *differential* error, i.e., the difference $u - \tilde{u}^h$, where u is the solution to the given differential equations. In fact, this is the true role of relaxation even when double discretization is not used, if what we want to approximate is u, not u^h: It is the smoothness of $u - \tilde{u}^h$ which permits its efficient reduction via the coarser grids.

Thus; the important measure of relaxation efficiency is not the algebraic smoothing factor $\overline{\mu}$, but the differential smoothing factor, the factor by which the high-frequency part of $u - \tilde{u}^h$ is reduced per sweep. This is not usually recognized because the latter factor is not constant: It approximately equals $\overline{\mu}$ when the high-frequencies in $u - \tilde{u}^h$ are large compared with those in $u - u^h$ (where u^h is the local solution to the discrete equations employed in relaxation), but below this level $\overline{\mu}$ may mislead, and when $\tilde{u}^h$ is closer to u than to u^h in their non-smooth components, the differential factor may even be larger than 1. For example, in solving a singular perturbation problem with strong alignment (see §2.1), we can reduce the algebraic smoothing factors of point Gauss-Seidel relaxation by taking a larger artificial viscosity and, more importantly, by taking it *isotropically* instead of anisotropically. This would not however improve the overall performance of our double-discretization FMG algorithm (see the experiments in [Bra81a, §7]), since it would not reduce the *differential* smoothing factors.

The differential smoothing is the purpose of relaxation not only on the finest grid h_*. On any grid h, its relaxation should reduce the scale-h high-frequency components of the error $u - \tilde{u}^{h_*}$ where we interpret changes in $\tilde{u}^h$ as changes in $\tilde{u}^{h_*}$ via the interpolation relations.

We can here also elaborate on what are those scale-h *"high-frequency components"* (of the differential error) that should be converged by relaxation on grid h. Generally speaking we say that these are the components "invisible" on the next coarser grid H, i.e.., Fourier components $\exp(i\underline{\theta}\cdot\underline{x}/\underline{h})$ which on grid H coincide with lower components, that is to say components with $\pi < \max_j |\theta_j| H_j/h_j$, $max_j|\theta_j| \le \pi$ (cf. (3.3)).

More precisely, we should include in the "high-frequency" range all those components that are not efficiently reduced by relaxation on the other grids, which can for example be any range of the form

$$\left\{(\theta_1, \ldots, \theta_d) : \exists j \text{ s.t. } \frac{\alpha_j h_j}{H_j} \le \theta_j \le \alpha_j\right\}, \tag{12.1}$$

where each $0 < \alpha_j \le \pi$ is fixed (assuming H_j/h_j is the same for all levels). In other words, we can allow some of the highest frequency components on any intermediate level not to converge efficiently by relaxation ($\alpha_j < \pi$), as long as those components efficiently converge by the next-finer-level relaxation. This may leave the highest frequencies on the finest grid uncontrolled, but they are unimportant and can be eliminated by averaging the final results. Examples where this further understanding of mode analysis is relevant are mentioned in §18.6 and in [Bra81a, §5.7].

The range of frequencies to be reduced by relaxation may also change by modified coarse-grid functions of the type mentioned in §4.2.2. In such cases relaxation may not reduce some high-frequency error components; but the unreduced components are very special ones, hence they are described by few parameters. This is a general property of relaxation (see §1.1). Very generally we can thus say that the role of relaxation is to reduce the *information content* of the error, so that it becomes approximable by a lower dimensional approximation space.

Another important point to clarify is that relaxation should be efficient only as long as the high-frequency error components have relatively large amplitudes: When the high-frequency errors are too small compared with the low-frequency ones, relaxation cannot usually be efficient because of certain *feeding from low to high components.* Such feeding is caused by interaction with boundaries, and by non-constant coefficients, and by the high-frequency harmonics generated when the low-frequency error is corrected via the coarse-grid cycle (see observation (D) in §4.3). Sometimes such feeding is even caused by the interior relaxation itself; e.g., red-black relaxation of an order-m differential equation produces $O(h^m)$ high-frequency errors from $O(1)$ low-frequency errors. When the size of high-frequency amplitudes approaches the size fed from low frequencies, relaxation should be stopped; this is the point where the coarse-grid correction should be made. If relaxation is stopped in time, then the range of strong interactions with low-frequencies is not entered. It is also only then that the multigrid convergence rates can accurately be deduced from the smoothing-rate analyses.

Exchange-rate criteria for stopping relaxation in time are described in §9.6. In most cases, though, the practical approach is to stop relaxation as soon as its amount of computations becomes substantially larger than the amount invested in the coarse-grid correction, which simply means to limit the number of relaxation sweeps per cycle to be less than 3 or 4.

Finally, even though smoothing is the main role of relaxation, we should not forget its influence on other components. Some relaxation schemes with extremely good smoothing factors are either unstable or they cause large amplification of some low-frequency errors (see §3.2).

We can thus say in summary that *the role of relaxation is to reduce large amplitudes of certain components of the differential error (those components not efficiently reduced by relaxation at other levels), while avoiding from significantly amplifying its other components.*

Stability of the difference equations used in relaxation is only a tool in performing this role, not an end by itself.