Week 3

Classical Solutions of Boundary Value Problems

1 Introduction

The *Dirichlet problem for Laplace's equation* consists in the determination of the function u inside a domain Ω satisfying Laplace's equation in Ω and having a given value u_Γ on the boundary of the domain, Γ, i.e Find u such that

$$\nabla^2 u = 0$$

inside Ω and subject to

$$u = u_\Gamma$$

on Γ.

The *Dirichlet problem for Poisson's equation* consists of determining the function u satisfying Poisson's equation in Ω and taking a given value on the boundary Γ, i.e. Find u such that

$$\nabla^2 u + f = 0$$

inside Ω and subject to

$$u = u_\Gamma$$

on Γ.

The *Neumann problem for Laplace's equation* consists in the determination of the function u inside a domain Ω satisfying Laplace's equation in Ω and having a given value of the normal derivative $\partial u/\partial n|_\Gamma = \psi$ on the boundary of the domain, i.e Find u such that

$$\nabla^2 u = 0$$

inside Ω and subject to

$$\frac{\partial u}{\partial n} = \psi$$

on Γ.

As a final example, consider the *mixed problem for Poisson's equation* consisting of determining the function u satisfying Laplace's equation in Ω and subject to various distinct conditions on different portions of the boundary, i.e. Find u such that

$$\nabla^2 u = 0$$

inside Ω and subject to

$$u = u_{\Gamma_1}$$

on Γ_1 and to

$$\frac{\partial u}{\partial n} + \alpha u = \psi$$

on Γ_2 where $\Gamma = \Gamma_1 \bigcup \Gamma_2$.

Note that all the various formal problems above can be represented generically using operator notation as follows, one wants to determine the doubly differentiable function $u \in C^2$ which when operated by the linear differential operator L, transforms into the given data $f \in C^0$, i.e.

$$Lu = f$$

for $x \in \Omega$. The required solution must also satisfy the imposed boundary conditions, i.e.

$$\alpha \frac{\partial u}{\partial n} + \beta u = g$$

for $x \in \Gamma$.

As mentioned before, solutions to Laplace's equation are called *harmonic functions* and they have some very unique properties. Solutions to Dirichlet's problem for Poisson's equation for the case when $f < 0$ are called *sub-harmonic functions.* Also, solutions to Dirichlet's problem for Poisson's equation for the case when $f > 0$ are called *super-harmonic functions.* We expect the solutions to above problems to be unique, single valued and bounded.

A common approach to solving problems involving PDE's consists of first determining a suitable set of particular solutions satisfying some of the specified conditions using, for instance, the method of *separation of variables*, and then combining sets of the found solutions so as to satisfy all the prescribed conditions.

2 The One-Dimensional Limit of Laplace's Equation

In a three-dimensional rectangular Cartesian system of coordinates (x, y, z) Laplace's equation for u is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

For a one-dimensional system $x \in [a, b]$ the above yields

$$\frac{d^2 u}{dx^2} = 0$$

which is readily solved yielding

$$u(x) = Ax + B$$

where the values of the constants A and B must be determined by specifying boundary conditions at $x = a$ and $x = b$. If $u(a) = u_a$ and $u(b) = u_b$, the solution becomes

$$u(x) = \frac{u_b(x-a) + u_a(b-x)}{b-a}$$

3 Laplace's Equation in a Rectangle

Consider Laplace's equation inside a rectangle of width a and height b. The edges at $x = 0$, $x = a$ and $y = 0$ are maintained at $u = 0$ while at the edge $y = b$ $u(x, b) = f(x)$. The required solution $u(x, y)$ must satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

inside the plate, subject to the above stated conditions.

To find u by separation of variables we assume the a *particular solution* can be represented as a product of two functions each depending on a single coordinate, i.e.

$$u_p(x, y) = X(x)Y(y)$$

substituting into Laplace's equation this gives

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

The only way in which the LHS (a function only of x) and the RHS (a function only of y) are going to be equal to each other, is if both terms are equal to a constant. This is called the *separation constant* and will be denoted by k^2.

The constant is selected as k^2 in order to obtain a proper Sturm-Liouville problem for X (with real eigenvalues). A clue about the appropriate choice of the placement of the minus sign above is derived on physical grounds. Note that for a fixed value of y, the solution must be zero at $x = 0$, go through an extremum at $x = a/2$ and become zero again for $x = a$. Therefore, a particular solution in terms of simple trigonometric functions can be expected along the x-direction. On the other hand, along the y-direction, at constant x, a monotonically increasing/ or decreasing function is expected as y changes from zero to b.

With the above the original PDE problem has been transformed into a system of two ODE's, i.e.

$$X'' + k^2 X = 0$$

subject to $X(0) = X(a) = 0$ and

$$Y'' - k^2 Y = 0$$

subject to $Y(0) = 0$.

The solution for $X(x)$ is

$$X = X_n = A_n \sin(\frac{n\pi x}{a})$$

with eigenvalues

$$k_n = \frac{n\pi}{a}$$

for $n = 1, 2, 3, ...$

The solution of $Y(y)$ is

$$Y_n = B_n \sinh(\frac{n\pi y}{a})$$

Therefore, the particular solution obtained is of the form

$$u_p(x, y) = X(x)Y(y) = X_n(x)Y_n(y) = u_n = a_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi y}{a})$$

and since there is one of this for each value of n, the solution actually obtained is a linear combination of the above linearly independent solutions, i.e.

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi y}{a})$$

Finally, the coefficients a_n's are determined by making the above satisfy the remaining nonhomogeneous condition at $y = b$, i.e.

$$f(x) = \sum_{n=1}^{\infty} [a_n \sinh(\frac{n\pi b}{a})] \sin(\frac{n\pi x}{a}) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a})$$

which is a Fourier sine series representation of the given function $f(x)$ with Fourier coefficients

$$c_n = a_n \sinh(\frac{n\pi b}{a}) = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{a})dx$$

Therefore that the final solution to the problem is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{a})\frac{\sinh(\frac{n\pi y}{a})}{\sinh(\frac{n\pi b}{a})}$$

Therefore, as long as $f(x)$ is representable in terms of Fourier series, the obtained solution coverges to the desired solution. Note also that the presence of homogeneous conditions at $x = 0$, $x = a$ made feasible the determination of the required eigenvalues.

4 Laplace's Equation in a Rectangular Parallelepiped

Consider now the problem of solving Laplace's equation inside a rectangular parallelepiped (length a, width b, height c) subject to zero values of u on five faces and the value $u(x,y,c) = f(x,y)$ at $z = c$. Laplace's equation in this case is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

As before, assume a particular solution is of the form

$$u_p = X(x)Y(y)Z(z)$$

With the above one has

$$\frac{\partial^2 u}{\partial x^2} = YZ\frac{d^2X}{dx^2} = YZX''$$

$$\frac{\partial^2 u}{\partial y^2} = XZ\frac{d^2Y}{dy^2} = XZY''$$

and

$$\frac{\partial^2 u}{\partial z^2} = XY\frac{d^2Z}{dz^2} = XYZ''$$

Substituting into the original PDE yields and division by XYZ yields

$$\frac{1}{X}X'' + \frac{1}{Y}Y'' + \frac{1}{Z}Z'' = 0$$

which can be rearranged to

$$\frac{1}{Y}Y'' + \frac{1}{Z}Z'' = -\frac{1}{X}X''$$

But the left hand side is a function of y and z only while the right hand side is a function of x only, therefore, necessarily

$$\frac{1}{Y}Y'' + \frac{1}{Z}Z'' = -\frac{1}{X}X'' = k_1^2$$

Now, rearrangement of the term on the left hand side yields

$$\frac{1}{Z}Z'' - k_1^2 = -\frac{1}{Y}Y''$$

but the term on the left hand side is a function of z only while the one on the right hand side is a function of y only, thus, necessarily

$$\frac{1}{Z}Z'' - k_1^2 = -\frac{1}{Y}Y'' = k_2^2$$

Two separation constants k_1 and k_2 are needed in this case giving

$$X'' + k_1^2 X = 0$$

$$Y'' + k_2^2 Y = 0$$

and

$$Z'' - (k_1^2 + k_2^2)Z = 0$$

Note that homogeneous conditions occur at both boundaries in both the x and y directions leading to

$$X_m = A_m \sin(k_1 x) = A_m \sin(\frac{m\pi x}{a})$$

$$Y_n = B_n \sin(k_2 y) = B_n \sin(\frac{n\pi y}{b})$$

and

$$Z_{mn} = C_{mn} \sinh(k_{mn} z)$$

where $k_{mn}^2 = k_1^2 + k_2^2$. The desired particular solution is then

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \sinh(k_{mn} z)$$

where the coefficients a_{mn} must be determined by incorporating the nonhomogeneous condition. This leads to a double Fourier series representation and the coefficients

$$a_{mn} = \frac{1}{\sinh(k_{mn}c)} \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) dy dx$$

5 Nonhomogeneous Problems

Consider the following mixed problem for Poisson's equation. A rectangular plate has dimensions $a \times b$. Inside the plate, an internal source has a constant value of f. Poisson's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f = 0$$

This must be solved subject to the following boundary conditions: at $x = 0$ for all y

$$\frac{\partial u(0, y)}{\partial x} = 0$$

at $y = 0$ for all x

$$\frac{\partial u(x, 0)}{\partial y} = 0$$

at $x = a$ for all y

$$u(a, y) = 0$$

and at $y = b$ for all x

$$u(x, b) = 0$$

Here we assume a particular solution exists with the following special form

$$u(x, y) = \Psi(x, y) + \Phi(x)$$

such that the original problem can be reformulated as a superposition of two simpler problems; the one-dimensional, nonhomogeneous problem given by

$$\frac{d^2\Phi}{dx^2} + f = 0$$

subject to

$$\frac{d\Phi(0)}{dx} = 0$$

and

$$\Phi(a) = 0$$

with the solution

$$\Phi(x) = \frac{fa^2}{2}[1 - (\frac{x}{a})^2]$$

And the two-dimensional homogeneous problem given by

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

subject to

$$\frac{\partial \Psi(0, y)}{\partial x} = 0$$

at $y = 0$ for all x

$$\frac{\partial \Psi(x, 0)}{\partial y} = 0$$

at $x = a$ for all y

$$\Psi(a, y) = 0$$

and at $y = b$ for all x

$$\Psi(x, b) = -\Phi(x)$$

which can readily by solved by separation of variables assuming a particular solution of the form

$$\Psi(x, y) = X(x)Y(y)$$

to give

$$\Psi(x, y) = -\frac{2f}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^3} \frac{\cos(\lambda_n x)\cosh(\lambda_n y)}{\cosh(\lambda_n b)}$$

where the eigenvalues λ_n are given by

$$\lambda_n = \frac{(2n+1)\pi}{2a}$$

with $n = 0, 1, 2, ...$

Finally the required solution of the original problem is

$$u(x, y) = \frac{fa^2}{2}\{[1 - (\frac{x}{a})^2] - \frac{4}{a^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^3} \frac{\cos(\lambda_n x)\cosh(\lambda_n y)}{\cosh(\lambda_n b)}\}$$

6 Laplace's Equation in a Circular Annulus

Consider now the problem of determining the function u satisfying Laplace's equation inside a circular annulus and subject to specified values at the inner and outer radii r_1 and r_2 given by

$$u(r_1, \theta) = f_1(\theta)$$

$$u(r_2, \theta) = f_2(\theta)$$

Laplace's equation in this case is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Assume now a particular solution is of the form

$$u_p(r, \theta) = R(r)\Theta(\theta)$$

Substituting leads to the following two ODEs

$$r^2 R'' + rR' - k^2 R = 0$$

$$\Theta'' + k^2\Theta = 0$$

where the separation constant as been chosen as k^2 in order to obtain periodic trigonometric functions as the solutions for Θ.

The general solution for R is

$$R(r) = \begin{cases} A_k r^k + B_k r^{-k}; & k \neq 0 \\ A_0 + B_0 \log r; & k = 0 \end{cases}$$

whereas that for Θ is

$$\Theta(\theta) = \begin{cases} C_k \cos(k\theta) + D_k \sin(k\theta); & k \neq 0 \\ C_0 + D_0\theta; & k = 0 \end{cases}$$

The periodicity requirement is satisfied by taking $k = n$, with $n = 1, 2, 3, ...$

The single valued solution obtained is then

$$u = (a_0 + b_0 \log r) + \sum_{n=1}^{\infty} [(a_n r^n + b_n r^{-n}) \cos(n\theta) \\ + (c_n r^n + d_n r^{-n}) \sin(n\theta)]$$

The solution must satisfy the stated boundary conditions. Substituting f_1 and f_2 leads to Fourier series representations and the relationships

$$a_0 + b_0 \ln r_1 = \frac{1}{2\pi} \int_0^{2\pi} f_1(\theta) d\theta$$

$$a_0 + b_0 \ln r_2 = \frac{1}{2\pi} \int_0^{2\pi} f_2(\theta) d\theta$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_1(\theta) \cos(n\theta) d\theta$$

$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_2(\theta) \cos(n\theta) d\theta$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_1(\theta) \sin(n\theta) d\theta$$

$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_2(\theta) \sin(n\theta) d\theta$$

determining the values of the coefficients a_0, b_0, a_n, b_n, c_n, and d_n.

A special case of interest obtained from the above is the solution for the problem of Laplace's equation in a disk ($r_1 = 0$) of radius $r_2 = a$ subject to a value $u(a, \theta) = f(\theta)$ at its boundary. In this case, the solution is

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} (\frac{r}{a})^n [A_n \cos(n\theta) + C_n \sin(n\theta)]$$

with

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta$$

$$C_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta$$

for $n = 1, 2, ...$ Note that the value of u at the center of the disk is simply the average value of u at the boundary.

Poisson's integral formula allows direct determination of the value of the function $u(r, \theta)$ anywhere inside the disk from the knowledge of its values at the boundary. The formula is

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^r}{a^2 - 2ar\cos(\theta - \psi) + r^2} u(a, \psi) d\psi$$

Another important special case is that of computing the function u around a circular hole ($r_2 \to \infty$) of radius $r_1 = a$ satisfying Laplace's equation outside the hole and subject to given values of $u(a, \theta) = f(\theta)$ at the hole boundary. In this case

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} (\frac{a}{r})^n [B_n \cos(n\theta) + D_n \sin(n\theta)]$$

with

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta$$

$$D_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta$$

for $n = 1, 2, ...$ Note that the value of u at infinity is simply the average value at the hole boundary.

7 Laplace's Equation in a Sphere

Consider now the problem of finding the function u satisfying Laplace's equation inside a sphere (radius a) incorporating symmetry in the azimuthal direction (i.e. solution independent of θ). Laplace's equation in this case is

$$\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{\sin\phi} \frac{\partial}{\partial \phi}(\sin\phi \frac{\partial u}{\partial \phi}) = 0$$

At the surface of the sphere, the value of u is prescribed as

$$u(a, \phi) = f(\phi)$$

As before, a particular solution of the form $u_p = R(r)\Phi(\phi)$ is assumed leading to

$$r^2 R'' + 2rR' - k^2 R = 0$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}(\sin\phi\frac{d\Phi}{d\phi}) + k^2\Phi = 0$$

The general bounded solution for R is

$$R(r) = C_n r^n$$

while the general solution for Ψ, finite at $\psi = 0, \pi$ in terms of Legendre polynomials is obtained by making $k^2 = n(n+1)$ giving

$$\Phi(\phi) = A_n P_n(\cos\phi)$$

Therefore the solution obtained is

$$u(r,\phi) = \sum_{n=0}^{\infty} c_n(\frac{r}{a})^n P_n(\cos\phi)$$

The coefficients c_n are determined from the imposed boundary condition as

$$c_n = \frac{2n+1}{2}\int_0^{\pi} f(\phi)P_n(\cos\phi)\sin\phi d\phi$$

8 Finite Difference and Finite Volume Methods

Finite difference methods produce approximations to the desired solutions by directly replacing approximations to the derivatives, obtained by the methods of finite differences or of finite volumes in the governing equations. The original problem is effectively transformed into an algebraic one requiring the solution of systems of intelinked, linear, simultaneous equations.

8.1 One Dimensional Problems

Consider the linear BVP

$$y'' = p(x)y' + q(x)y + r(x)$$

in

$$a \leq x \leq b$$

subject to

$$y(a) = \alpha$$

and

$$y(b) = \beta$$

Introduce a *mesh* in $[a, b]$ by dividing the interval into $N + 1$ equal subintervals of *size* h. This produces two *boundary mesh points* x_0 and x_{N+1}, and N *interior mesh points* x_i, $i = 1, 2, .., N$. The values $y(x_0) = y(a)$ and $y(x_{N+1}) = y(b)$ are known but the values $y(x_i)$, $i = 1, 2, .., N$ must be determined.

From the Taylor expansions for $y(x_{i+1})$ and $y(x_{i-1})$ the following *centered difference formula* is obtained

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] + \frac{h^2}{12}y^{(4)}(\xi_i)$$

for some $\xi_i \in (x_{i-1}, x_{i+1})$.

Similarly, an approximation for $y'(x_i)$ is

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6}y^{(3)}(\eta_i)$$

for some $\eta_i \in (x_{i-1}, x_{i+1})$.

If the higher order terms are discarded from the above formulae and the approximations are used in the original differential equation, the second order accurate finite difference approximation to the BVP becomes a *system of simultaneous linear algebraic equations*

$$-(\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}) + p(x_i)(\frac{w_{i+1} - w_{i-1}}{2h}) + q(x_i)w_i = -r(x_i)$$

or, alternatively

$$-(1 + \frac{h}{2}p(x_i))w_{i-1} + (2 + h^2 q(x_i))w_i - (1 - \frac{h}{2}p(x_i))w_{i+1} = -h^2 r(x_i)$$

for $i = 1, 2, .., N$.

In matrix notation the system becomes

$$A\mathbf{w} = \mathbf{b}$$

where A is a *tridiagonal matrix*.

Theorem of Uniqueness. If $p(x), q(x)$ and $r(x)$ are continuous and $q(x) > 0$ on $[a, b]$ the problem $A\mathbf{w} = \mathbf{b}$ has a unique solution provided $h < 2/L$ where $L = max_{a \leq x \leq b}|p(x)|$. Further, if $y^{(4)}$ is continuous on $[a, b]$ the truncation error is $O(h^2)$.

Linear Finite Difference Method Algorithm.

- Give functions $p(x), q(x), r(x)$, endpoints a, b, boundary conditions α, β, and the number of subintervals $N + 1$.

- Set $h = (b - a)/(N + 1)$; $x = a + h$ $a_1 = 2 + h^2 q$, $b_1 = -1 + (h/2)p$, $d_1 = -h^2 r + [1 + (h/2)p]\alpha$.

- For $i = 2, .., N-1$ set $x = a + ih$ $a_i = 2 + h^2 q$, $b_i = -1 + (h/2)p$, $c_i = -1 - (h/2)p$, $d_i = -h^2 r$.
- Set $x = b - h$, $a_N = 2 + h^2 q$, $c_N = -1 - (h/2)p$, $d_N = -h^2 r + [1 - (h/2)p]\beta$
- Solve tridiagonal system for z_i.
- $w_0 = \alpha$, $w_{N+1} = \beta$, $w_N = z_N$.
- For $i = N-1, ..., 1$; set $w_i = z_i - u_i w_{i+1}$
- Stop.

8.2 Elliptic Partial Differential Equations

The two best known examples of elliptic partial differential equations are *Laplace's equation* and *Poisson's equation*. Consider the case of Poisson's equation for the unknown function $u(x, y)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

on (x, y) in the *interior* $R = \{(x, y) : a < x < b, c < y < d\}$ of a planar region and subject to

$$u(x, y) = g(x, y)$$

along the *boundary* S of the region. As long as f and g are continuous a unique solution exists.

A simple approach to the numerical solution of the above problem consists in partitioning the intervals $[a, b]$ and $[c, d]$ respectively into n and m subintervals with step sizes $\Delta x = h = (b-a)/n$ and $\Delta y = k = (d-c)/m$ so that the *node* or *mesh point* (x_i, y_j) is at $x_i = a + ih$ for $i = 0, 1, .., n$ and $y_j = c + jk$ for $j = 0, 1, .., m$. In practice, a distinction must be made between *interior nodes* (i.e. (x_i, y_j) for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., m-1$) and the *boundary nodes* (represented by nodes with $i = 0$, $i = n$, $j = 1$, or $j = m$).

Using now *centered finite difference* approximations for the partial derivatives the following second order accurate system of simultaneous algebraic equations is obtained for all interior nodes

$$2[(\frac{h}{k})^2 + 1]w_{i,j} - (w_{i+1,j} - w_{i-1,j}) - (w_{i,j+1} - w_{i,j-1})(\frac{h}{k})^2 = -h^2 f(x, y)$$

which is called the *five point formula*.

If the non-homogeneous term is $f(x, y) = 0$ (Laplace's equation) and if the mesh spacings are chosen to be the same (i.e. $h = k$) the following particularly simple system of equations is obtained for the interior nodes:

$$w_{i,j} = \frac{1}{4}[w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}]$$

Exercise 1. Consider the problem involving Laplace's equation thin square plate of edge $= 100$ cm. The edges $x = 0$, $x = l$ and $y = 0$ are maintained at $u = 0$ while at the edge $y = d$ $u(x, d) = 100$. The required solution $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the stated boundary conditions.

This problem is readily solved in closed form by separation of variables yielding

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{100} x) \frac{\sinh(\frac{n\pi}{100} y)}{\sinh(n\pi)}$$

with

$$c_n = \begin{cases} 0 & n \ \ even \\ \frac{400}{n\pi} & n \ \ odd \end{cases}$$

Solve the above problem numerically using the finite difference method and compare your results against the analytical solution.

The five point formulae above are also applicable at the boundary nodes but they must be supplemented by the boundary conditions of the problem in order to obtain the specific finite difference formulae for the boundary nodes.

The boundary conditions imposed along the *boundary nodes* become

$$\begin{aligned} w_{0,j} &= g(x_0, y_j) \quad & j = 0, 1, .., m \\ w_{n,j} &= g(x_n, y_j) \quad & j = 0, 1, .., m \\ w_{i,0} &= g(x_i, y_0) \quad & i = 1, 2, .., n-1 \\ w_{i,m} &= g(x_i, y_m) \quad & i = 1, 2, .., n-1 \end{aligned}$$

To obtain a *banded* matrix the mesh points must be relabeled sequentially from left to right and from top to bottom. The resulting system can be solved by Gaussian elimination if n and m are small and by iterative methods (e.g. SOR iteration) when they are large.

By using Taylor series expansions, the following *error bound* for the numerical solution is obtained for the case when $\Delta x = h = \Delta y = k = h$

$$|w_{i,j} - u(x_i, y_j)| \leq \frac{h^2}{96}(M_{xxxx} + M_{yyyy})$$

where M_{xxxx} and M_{yyyy} are bounds for u_{xxxx} and u_{yyyy}.

8.3 Finite Volume Method

An alternative discretization method is based on the idea of regarding the computation domain as subdivided into a collection of *finite volumes*. In this view, each finite volume is represented by an area in 2D and a volume in 3D. A node, located inside each finite volume is the locus of compuational values. In rectangular cartesian coordinates in 2D the simplest finite volumes are rectangles. For each node, the rectangle faces are formed by drawing perpendiculars through the midpoints between contiguous nodes. Discretization equations are obtained by integrating the original partial differential equation over the span of each finite volume. The method is easily extended to nonlinear problems.

Consider for instance the following problem consisting of determining $u(x,y)$ in $x \in [a,b]$, $y \in [c,d]$ such that

$$\frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(a\frac{\partial u}{\partial y}) = f$$

subject to specified conditions at the boundary. The problem represents steady state heat conduction in a solid with position dependent conductivity a where energy is being internally generated at the rate f, per unit volume.

The typical control volume has dimensions $\Delta x \times \Delta y \times 1$ and the discrete equation is obtained by perfoming an energy balance on the finite volume. The net energy input is obtained by integrating the fluxes $a\partial u/\partial x$ and $a\partial u/\partial y$ into and out from the finite volume in the x and y directions, respectively. Since energy is conserved, the result must equal the rate of energy generation inside the volume which is given by the volume integral of f.

A notation from carthography is used to simplify the discrete formulation. In this notation, the node representing the finite volume is called P and the neighboring nodes along the four coordinate directions are called N, S, E and W, respectively. The following discrete equation is obtained for node P,

$$\frac{a_e(w_E - w_P) + a_w(w_P - w_W)}{\Delta x^2} + \frac{a_n(w_N - w_P) + a_s(w_P - w_S)}{\Delta y^2} = f_P$$

where the subscripts e, w, n, s are used to indicate that the values of a are to be evaluated at the corresponding finite volume faces. Note that if $a_e = a_w = a_n = a_s = a = 1$, $\Delta x = \Delta y = h$ and $f_P = 0$ the above reduces to the simple form

$$w_P = \frac{1}{4}[w_E + w_W + w_N + w_S]$$

which is identical to the expression obtained before using the method of finite differences.

As in the case of finite differences, care is required to properly deal with the boundary conditions at the boundary nodes, specially when derivative conditions are involved.

When the boundaries of the computational domain do not coincide with coordinate lines additional care must be used for the implementation of boundary conditions. Finite difference or finite volume formulae can be derived by ad-hoc methods, by conventional formulae if the irregular boundaries are represented by staicases approximating the actual boundaries or by coordinate transformation methods.