Chapter 19

Steady-State Incompressible Navier-Stokes Equations

19.1 The differential problem

Using the notation of §18.1, the steady-state incompressible Navier-Stokes equations in d dimensions can be written in the form

$$\underline{\nabla} \cdot \underline{u} = F_0 \tag{19.1a}$$

$$Q\underline{u} + \underline{\nabla} p = \underline{F}, \tag{19.1b}$$

$$Q := -\frac{1}{R}\Delta + \underline{u} \cdot \underline{\nabla} = -\frac{1}{R}\Delta + \sum_{j=1}^{d} u_j \partial_j, \tag{19.2}$$

R being the Reynolds number; i.e., R^{-1} is a scaled viscosity-over-density coefficient. (It is assumed that time and distance are scaled so that $\underline{u}$ is $O(1)$ and the domain dimensions are $O(1)$.) The principal part of this system is the Stokes system (18.1) (rescaling $p \leftarrow Rp$), hence for $R \leq O(1)$ the solution processes of §18 are directly applicable here (using FAS of course to deal with the nonlinearity). But we will be interested in solving also for large R, hence we will look at the corresponding *subprincipal operator* (cf. §2.1)

$$L = \begin{pmatrix} 0 & \partial_1 & \cdots & \cdots & \partial_d \\ \partial_1 & Q & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \partial_d & 0 & \cdots & 0 & Q \end{pmatrix}. \tag{19.3}$$

Observe indeed that (19.3) is not the full Newton linearization for (19.1); only subprincipal terms are kept. Since $\det L = \Delta Q^{d-1}$, (19.1) is again an elliptic system of order $2d$ and therefore requires d boundary conditions.

Usually the velocities are given on the boundary as in (18.4), leading again to the computability condition (18.5).

For $R \to \infty$, $\det L \to -\Delta(\underline{u} \cdot \underline{\nabla})^{d-1}$, hence the streamlines are the subcharacteristics (characteristic lines of the reduced equation), and in the limit only one condition is needed all along the boundary, plus $d-1$ conditions at points where the flow enter the domain (because $\underline{u} \cdot \underline{\nabla}$ is a streamwise derivative which, with the singular perturbation $-\frac{1}{R}\Delta$, permits discontinuity only at the exit end of each streamline). Typically, for $R \to \infty$, the velocities $\underline{u}$ are given at entrance points, and either the normal velocity or the pressure at other boundary points. See for example [Tem77] for theoretical investigations of this system.

19.2 Staggered finite-difference approximations

The discretization is carried out on the same staggered grid as before (Fig. 18.1), using the difference equations

$$\sum_{j=1}^{d} \partial_j^h u_j^h = F_0^h \qquad \text{at cell centers} \tag{19.4a}$$

$$Q^h u_j^h + \partial_j^h p = F_j^h \quad \text{at centers of } j\text{-faces}, \quad (j = 1, \ldots, d) \tag{19.4b}$$

where Q^h is some difference approximation to Q. (Non-staggered grids are discussed in §19.5.) Since $det L^h = -\Delta^h (Q^h)^{d-1}$, it is clear that L^h has good h-ellipticity measure if and only if Q^h does. Hence, all we have to construct is a good approximation to Q. For small to moderate $hR|\underline{u}|$ (i.e., $hR|\underline{u}|$ not much larger than 1) this can be done by central differencing. But for larger $hR|\underline{u}|$, upstream differencing or explicit artificial viscosity terms should be used.

The artificial viscosity terms may be anisotropic, so that the total (physical and artificial) viscosity has the form $-\sum_i \overline{\beta}_i h(\partial_i^h)^2$. For stability of the *simplest* DGS schemes (§19.3), $\overline{\beta}(i) \geq .5 \max |u_i(y)|$ is needed, where the maximum is taken over all y neighboring x. Upstream differencing is the same as $\overline{\beta}_i = .5|u_i|$. For large R, sharp cross-stream changes (large solution changes per cross-stream meshsize) can travel with the stream, and one may like to avoid smearing them by preventing cross-stream artificial viscosity. This is only possible by *strong alignment* (cf. §2.1), i.e., by using a grid (sometimes through the method of §9.3) such that one of its principal directions is (nearly) aligned with the stream throughout a large subdomain. One can then use $\overline{\beta}_i = O(h|\underline{u}|)$ (necessary for stability) in the stream direction, together with zero (or small) cross-stream artificial viscosity. In three-dimensional problems it may sometimes be difficult and unnecessary to have the flow (nearly) aligned with one grid direction, but a grid can be used so that each streamline (nearly) stay in one grid plane.

We call this *plane alignment*. In this case only sharp changes perpendicular to these planes are resolvable. For that purpose, zero artificial viscosity perpendicular to the planes should be used.

Another, more special case of strong alignment may arise near boundaries. Namely, to obtain sharp numerical boundary layers on grids that do not resolve the *physical* boundary layer, the Q^h operators near the boundary should be constructed so that tangential components of $Q^h \underline{u}$ do not straddle the boundary layer, i.e., do not include boundary values. Such a choice of Q^h can be made without aligning the grid. It should similarly be made at any *known* layer of sharp cross-stream transition.

The artificial viscosity may introduce $O(h)$ error, but such errors can be eliminated by omitting the artificial terms from the residuals calculated for the transfer to the next coarser grid (cf. §10.2); except that the rule of not straddling the boundary layer should still be kept by those residuals, too.

For a full discussion of the discretization and multigrid procedures for the convection-diffusion operator Q, see [Bra81a].

In case the boundary-layer interaction is important (e.g., in the driven cavity problem, where this interaction determines the entire flow), the boundary layer should be resolved. One can use then anisotropic local refinements (see §9.2), with local coordinate transformations in case the boundary is not along grid directions (§9.3).

19.3 Distributive relaxation

Generalizing the scheme in §18.3 to *any* operator Q^h, relaxation is guided by the distribution operator

$$M^h = \begin{pmatrix} Q^h & 0 & \cdots & \cdots & 0 \\ -\partial_1^h & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\partial_d^h & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{19.5}$$

and thus proceeds as follows.

The *j-th* momentum equation (19.4b) is relaxed by changing $\tilde{u}_j^h$ only, in any order and manner suitable for the operator Q^h. Thus, for $hR|\underline{\tilde{u}}| \leq O(1)$ the best perhaps is the Red-Black (RB) Gauss-Seidel scheme: $\tilde{u}_j^h$ is changed at each point so as to satisfy (19.4b) at that point, the red points being scanned first, the black next. For $hR|\underline{\tilde{u}}| \gg 1$, this relaxation is still one of the best, except for cases of intended strong alignment described above (§19.2).

In those cases, block relaxation must be used, but only in the specific strong-alignment direction, in its specific subdomain. This means streamwise line relaxation, except for plane relaxation in cases of plane alignment.

Furthermore, to obtain sharp numerical boundary layers, one should relax in blocks (i.e., simultaneously, in one or several blocks) exactly those equations where the special Q^h was constructed for that end (cf. §19.2). In many cases, the block relaxation can be replaced by a suitable downstream or ILU scheme (cf. §3.3).

In the case of plane alignment, the plane relaxation could be replaced by point relaxation if the grid is not coarsened in the direction perpendicular to the planes (cf. §4.2.1 and 3.3). Complicated alternating-direction line relaxation schemes are needed only if a fast solution is desired with errors far below truncation errors. A simple RB scheme, fully parallelizable and vectorizable, can thus most often be used.

Having relaxed in this way one pass per each momentum equation ($j = 1, \ldots, d$), we then make a pass of relaxation for the continuity equation (19.4a), by scanning the *cells* one by one, preferably in redblack ordering. At each cell the distributive relaxation step resulting from (19.5) is a generalization of Fig. 18.2: denoting by ξ^h the characteristic function of the relaxed cell (i.e., $\xi^h = 1$ at that cell center, and $\xi^h = 0$ at all other cell centers), the relaxation step changes all functions $\tilde{u}_j^h$ and $\tilde{p}^h$ by the prescription

$$\tilde{u}_j^h \leftarrow \tilde{u}_j^h - \delta h \partial_j^h \xi^h, \quad (j = 1, \ldots, d) \tag{19.6a}$$

$$\tilde{p}^h \leftarrow \tilde{p}^h + \delta h Q^h \xi^h, \tag{19.6b}$$

where δ is still given by (18.12).

The smoothing factor $\overline{\mu}$ is the slowest among the factors obtained for the triangular operator

$$L^h M^h = \begin{pmatrix} -\Delta^h & \partial_1^h & \cdots & \cdots & \partial_d^h \\ 0 & Q^h & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & Q^h \end{pmatrix}. \tag{19.7}$$

Hence, $\overline{\mu} = \max\{\overline{\mu}^\Delta, \overline{\mu}^Q\}$, where $\overline{\mu}^\Delta$ is the smoothing factor of RB relaxation for Δ^h, and $\overline{\mu}^Q$ is the smoothing factor of the relaxation of $Q^h \tilde{u}_j^h$. Hence $\overline{\mu}^\Delta = .25$ if one or two sweeps are performed per cycle, while $\overline{\mu}^\Delta = .32$ in case of three sweeps per cycle. If down-stream relaxation is used for the momentum equations (hence for Q^h), one can obtain $\overline{\mu}_Q \le \overline{\mu}^\Delta$ and hence $\overline{\mu} = \overline{\mu}^\Delta$. As discussed above, however, especially in cases of widely varying stream directions, it is not important to obtain perfect smoothing by relaxing in all these directions. The relaxation rules specified above, deviating from simple (RB) point relaxation only in some cases of strong alignment, ensure fast reduction of all high-frequency components, except for nearly characteristic components which anyway have large truncation

errors almost everywhere in the flow field (cf. §3.3). For this to hold the total viscosity coefficients $\underline{\beta}_i$ (see §19.2) should be slightly larger than the minimum, e.g. $\underline{\beta}_i = .7|u_i|$. In case the minimum $\underline{\beta}_i = .5|u_i|$ is used, as in upstream differencing, some highest frequencies are not reduced (unless downstream relaxation ordering is everywhere ensured); but this is not important from the point of view of *differential* smoothing (cf. §12), which will in fact be damaged if $\underline{\beta}_i$ is increased too much ($\underline{\beta}_i = .7|u_i|$ is still good). In any case, the distribution matrix (19.5) reduces the problem of relaxing the Navier-Stokes system into consideration concerning the relaxation of the scalar convection-diffusion operator Q^h (which is in detail studied in [Bra81a]).

19.4 Multigrid procedures and numerical results

The grids, their relative positions and the interpolation procedures between them are generally the same as for the Stokes equations (§18.4). Because of the nonlinearity, FAS is of course used (see §8), and the full weighting (18.14) is preferred over (18.13) in the fine-to-coarse transfers of both the velocities and the momentum residual functions.

For $R > O(h^{-1})$, the momentum residuals themselves can be calculated in two ways, using either the same $O(h)$ approximation Q^h used in relaxation (see §19.2), or the $O(h^2)$ central approximation to Q^h_* (i.e., Q^h with zero artificial viscosity). The latter is the "double-discretization" scheme (see §10.2) which exploits the fact that Q^h is a better approximation for the high-frequency components, hence used in relaxation, while Q^h_* is a better (higher) approximation for the low-frequency components which converge through the interaction with the coarse grid. Q^h_* should, however, respect intended discontinuities (boundary layers) in the same way that Q^h does, even if it means $O(h)$ local truncation errors; the global discretization errors (e.g., the L_1-norms of $u - \tilde{u}^h$, $v - \tilde{v}^h$ and $p - \tilde{p}^h$) will still be $O(h^2)$.

For large hR it is also advisable to use W cycles or accommodative algorithms (see §6.2). On very coarse grids, where velocity changes per meshsize are comparable to the velocity itself, BGS relaxation (see §3.4 and the end of §5.6) is safer than the DGS relaxation described above.

Our 1978 numerical results are reported in [Din79], with two examples in [BD79]. They clearly show convergence to below truncation errors in an FMG algorithm with one accommodative cycle per level, making only two relaxation sweeps on the finest grid. At the time we were worried about asymptotic convergence rates, but we should have not been; see the end of §4.1. The results will be tabulated in a separate report.

19.5 Results for non-staggered grids

The non-staggered approach to the Stokes equations (§18.6) can also be used for the Navier-Stokes equations. On a non-staggered grid the short-

central difference operator ∂_j^h in (19.4) should throughout be replaced by the long-central ∂_j^{2h}. Hence, ∂_j^h should be replaced by ∂_j^{2h} in the distribution operator (19.5), too (yielding M^h as in (18.20) with Δ^h replaced by Q^h). As in Stokes equations, error components around (18.18) are not efficiently reduced by relaxation; but in the Navier-Stokes case, non-linear interactions can actually amplify these components. Therefore, the pressure averaging (18.19) must now be applied not only to the final results but also to any pressure correction, just before it is interpolated to a finer grid.

This approach was tested with RB ordering, full residual weighting (4.6), bilinear interpolation of corrections, W(2,0) cycles and an FMG algorithm with bi-cubic interpolation of solutions. For clarity, Table 19.1 shows the results for the case of periodic boundary conditions with the smooth non-aligned solution $u = v = p = 1 + .2\sin(\cos(x + 2y))$ and $R = \infty$. Compatibility conditions were ignored (see §18.6), but the correct averages of u, v and p were enforced, so as to make the periodic problem well posed. Double discretization was used, with $\underline{\beta} = \beta|u_i|$ in relaxation and $\underline{\beta} = \beta_*|u_i|$ in residual transfers.

h	grid	$(\beta, \beta_*) \rightarrow$	(1., 1.)		(.7, .7)		(1., 0.)
		# cycles $\rightarrow$	1	2	1	2	1
1/16	16 × 16			.1602		.1101	.1233
1/32	32 × 32			.0809		.0583	.0309
1/64	64 × 64		.0420	.0416	.0298	.0291	.0067

Table 19.1. ***Differential error in the FMG Algorithm for the two-dimensional Stokes equations on non-staggered grids.***

The error $\|u - \tilde{u}^h\|_1 + \|v - \tilde{v}^h\|_1 + \|p - \tilde{p}^h\|_1$ *obtained on increasingly finer FMG levels* h.