Practice Problems

A very typical problem of electromechanical position control is electric motor driving a load that has one dominant vibration mode. The problem arises in computer-disk-head control, reel-to-reel tape drives, and many other applications. A schematic diagram is sketched in Figure 8. The motor has an electrical constant K_e , a torque constant K_t, an armature inductance L_a, and a resistance R_a. The rotor has an inertia J_1 and a viscous friction B. The load has an inertia J_2. The two inertias are connected by a shaft with a spring constant k and an equivalent viscous damping b. Use the state vector $x = \begin{bmatrix}\theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 & i_a\end{bmatrix}$. The output is $y = [\theta_1 \quad \theta_2 \quad i_a]$ and the input is a sinusoidal wave with frequency $20\, rad/s$ and amplitude 1. Write the equations of motion and state space representation of the model. Then simulate the model using both state space block and differential equations.

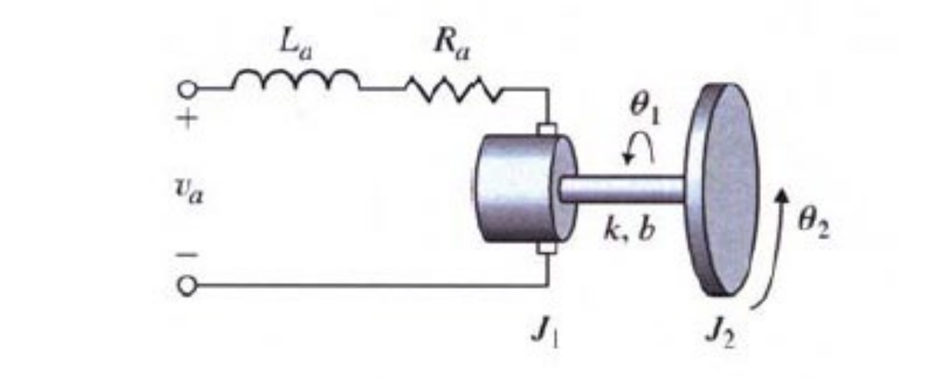

Figure 8. Motor with flexible load

Table 2. Parameters of DC motor with external load

Parameter	Value
k	8
b	0.08
B	0.0030 $N.m.s/rad$
J_1	0.222 $kg.m^2$
J_2	0.022 $kg.m^2$
R_a	11.2 Ω
L_a	1.21 H
K_e	1.28 $N.m/A$
K_t	1.25 $V.s/rad$

Electrical Equation

$$V_a - L_a\frac{di_a}{dt} - i_a R_a - V_m = 0$$

$$V_m = K_e \dot{\theta}_1$$

$$L_a\frac{di_a}{dt} + R_a i_a + K_e\dot{\theta}_1 = V_a \quad \text{— (1)}$$

Mechanical Equations

Motor

$$J_1\ddot{\theta}_1 = -K(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2) - B\dot{\theta}_1 + T_m$$

$$T_m = K_t i_a$$

$$J_1\ddot{\theta}_1 = -K\theta_1 + K\theta_2 - b\dot{\theta}_1 + b\dot{\theta}_2 - B\dot{\theta}_1 + K_t i_a$$

$$J_1\ddot{\theta}_1 = -(b+B)\dot{\theta}_1 + b\dot{\theta}_2 - K\theta_1 + K\theta_2 + K_t i_a \quad \text{— (2)}$$

Load

$$J_2\ddot{\theta}_2 = -K(\theta_2 - \theta_1) - b(\dot{\theta}_2 - \dot{\theta}_1)$$

$$J_2\ddot{\theta}_2 = -K\theta_2 + K\theta_1 - b\dot{\theta}_2 + b\dot{\theta}_1 \quad \text{— (3)}$$

All equations

$$L_a\frac{di_a}{dt} + R_a i_a + K_e\dot{\theta}_1 = V_a$$

$$J_1\ddot{\theta}_1 = -(b+B)\dot{\theta}_1 + b\dot{\theta}_2 - K\theta_1 + K\theta_2 + K_t i_a$$

$$J_2\ddot{\theta}_2 = -k\theta_2 + k\theta_1 - b\dot{\theta}_2 + b\dot{\theta}_1$$

$$x_1 = \theta_1 \quad x_2 = \dot{\theta}_1 \quad x_3 = \theta_2 \quad x_4 = \dot{\theta}_2 \quad x_5 = i_a$$

$$\dot{x}_1 = \dot{\theta}_1 = x_2$$

$$\dot{x}_2 = \ddot{\theta}_1 = ? \text{ but } J_1\dot{x}_2 = -(b+B)x_2 + bx_4 - kx_1 + kx_3 + K_t x_5$$

$$\therefore \dot{x}_2 = \frac{-k}{J_1}x_1 - \frac{b+B}{J_1}x_2 + \frac{k}{J_1}x_3 + \frac{b}{J_1}x_4 + \frac{K_t}{J_1}x_5$$

$$\dot{x}_3 = \dot{\theta}_2 = x_4$$

$$\dot{x}_4 = \ddot{\theta}_2 = ? \text{ but } J_2\dot{x}_4 = -kx_3 + kx_1 - bx_4 + bx_2$$

$$\therefore \dot{x}_4 = \frac{k}{J_2}x_1 + \frac{b}{J_2}x_2 - \frac{k}{J_2}x_3 - \frac{b}{J_2}x_4$$

$$\dot{x}_5 = \dot{i}_a = ? \text{ but } L_a\dot{x}_5 + R_a x_5 + K_e x_2 = u$$

$$L_a\dot{x}_5 = -K_e x_2 - R_a x_5 + u$$

$$\dot{x}_5 = -\frac{K_e}{L_a}x_2 - \frac{R_a}{L_a}x_5 + \frac{1}{L_a}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -k/J_1 & -\frac{b+B}{J_1} & \frac{k}{J_1} & \frac{b}{J_1} & \frac{K_t}{J_1} \\ 0 & 0 & 0 & 1 & 0 \\ k/J_2 & b/J_2 & -k/J_2 & -b/J_2 & 0 \\ 0 & -K_e/L_a & 0 & 0 & -\frac{R_a}{L_a} \end{bmatrix} x + \begin{bmatrix} 0\\0\\0\\0\\1/L_a \end{bmatrix} u$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -K/J_1 & -\frac{b+B}{J_1} & K/J_1 & b/J_1 & K_t/J_1 \\ 0 & 0 & 0 & 1 & 0 \\ K/J_2 & b/J_2 & -K/J_2 & -b/J_2 & 0 \\ 0 & -K_e/L_a & 0 & 0 & \frac{-R_a}{L_a} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/L_a \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For DE modelling in MATLAB

$$L_a \frac{di_a}{dt} + R_a i_a + K_e \dot{\theta}_1 = V_a$$

$$\frac{di_a}{dt} = \frac{1}{L_a}\left[V_a - R_a i_a - K_e \dot{\theta}_1 \right] \quad \text{— i}$$

$$J_1 \ddot{\theta}_1 = -(b+B)\dot{\theta}_1 + b\dot{\theta}_2 - K\theta_1 + K\theta_2 + K_t i_a$$

$$\ddot{\theta}_1 = -\frac{(b+B)}{J_1}\dot{\theta}_1 + \frac{b}{J_1}\dot{\theta}_2 - \frac{K}{J_1}\theta_1 + \frac{K\theta_2}{J_1} + \frac{K_t}{J_1} i_a \quad \text{— ii}$$

$$J_2\ddot{\theta}_2 = -k\theta_2 + k\theta_1 - b\dot{\theta}_2 + b\dot{\theta}_1$$

$$\ddot{\theta}_2 = -\frac{k}{J_2}\theta_2 + \frac{k}{J_2}\theta_1 - \frac{b}{J_2}\dot{\theta}_2 + \frac{b}{J_2}\dot{\theta}_1 \quad \text{— iii}$$