Home Exercise 1

Differential Equations & Harmonic Motions

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1 Problem formulation

The following system, given by 1 and 2, of two coupled first order differential equations are to be solved numerically using a fourth order Runge-Kutta (RK4) method. The numerical results will be compared to analytic equivalents for different discretization steps h, or number of discretization steps N.

$$\frac{dy}{dt} = p = f(t, p) \tag{1}$$

$$\frac{dp}{dt} = -4\pi^2 y = g(t, y) \tag{2}$$

These two first order differential equations can be combined into a single second order ODE of the form

$$\frac{d^2y}{dt^2} = -4\pi^2 y \tag{3}$$

and the analytic solutions for the displacement and momentum are given as

$$y(t) = c_1 \cos(2\pi t) + c_2 \sin(2\pi t) \tag{4}$$

$$p(t) = 2\pi c_2 \cos(2\pi t) - 2\pi c_1 \sin(2\pi t) \tag{5}$$

The initial conditions are set to $y_0 = 1$ and $p_0 = 1$, both for the initial time $t_0 = 0$. This gives constants $c_1 = 1$ and $c_2 = \frac{1}{2\pi}$.

The algorithm for the RK4 numerical solution is formulated as

$$\begin{aligned}
k_1 &= f(t_n, p_n) \\
k_2 &= f(t_n + \frac{k}{2}, p_n + \frac{h}{2}k_1) \\
k_3 &= f(t_n + \frac{k}{2}, p_n + \frac{h}{2}k_2) \\
k_4 &= f(t_{n+1}, p_n + hk_3) \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned} \tag{6}$$

Equation 6 is the algorithm for calculating a single future displacement value y_{n+1}. The algorithm for momentum is analogous by substituting y_n with p_n. The fact that the algorithm uses both the previous values for momentum p_n and displacement y_n necessitates to calculate both values in tandem. This is implemented in MATLAB.

2 Results

The results are obtained by using different levels of discretization in the time domain. On $t \in [0, 1]$, the oscillations are expected to travel approximately one period. In the family of RK solvers, there exists methods of lower accuracy, i.e RK2 and RK3, that are implemented but not investigated further.

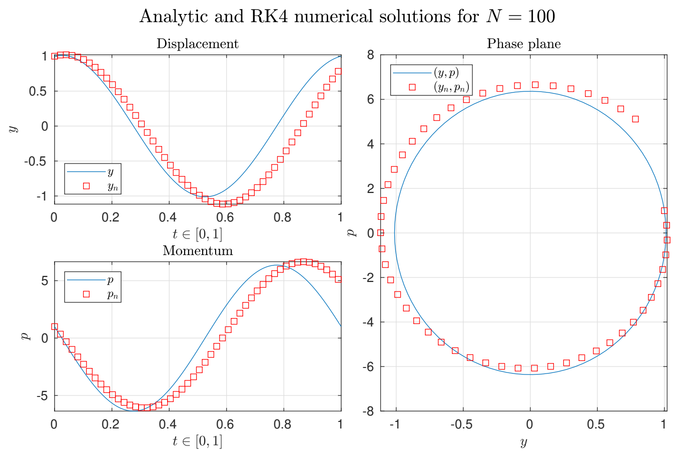

Figure 1: Comparison of analytical and numerical RK4 solutions using a coarse discretization.

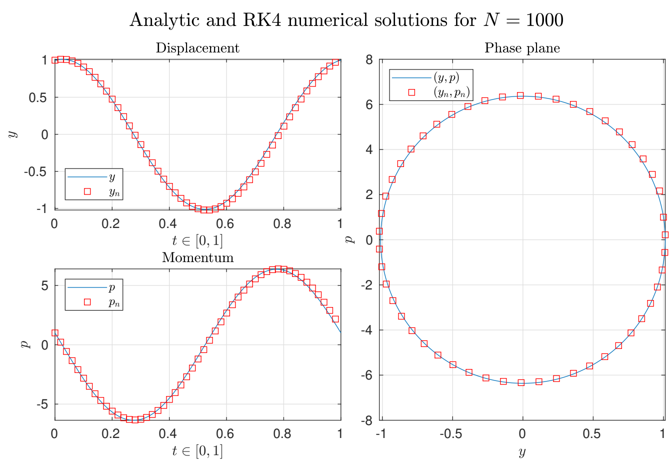

Figure 2: Comparison of analytical and numerical RK4 solutions using a finer discretization.

As seen in Figures 1 and 2, the momentum and displacement are plotted both against time and in the phase plane.

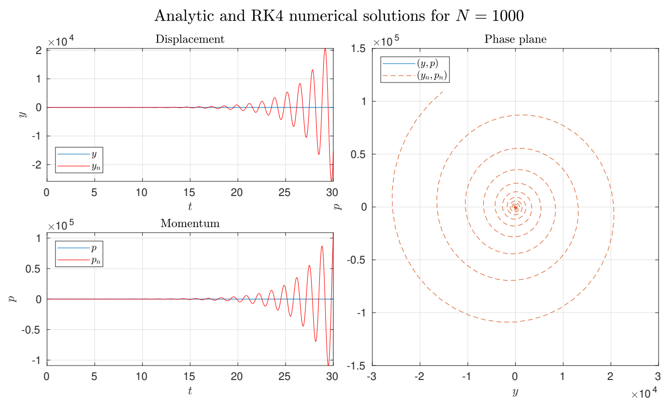

Figure 3: Comparison of analytical and numerical RK4 solutions using a finer discretization in the time domain where $t \in [0, 30]$.

Figure 3 shows a divergent solution. The analytical solution is not visible but forms a small circle in the center of the diverging spiral.

3 Error analysis

3.1 Error measurements

The total distance error $\tilde{e}$ can be computed as the sum of the squared residual

$$\tilde{e}_y = \sum_{n=1}^{N} (y_n - \tilde{y}_n)^2 = \sum_{n=1}^{N} e_{y_n}^2 = e_{y_n}^T e_{y_n} \tag{7}$$

where y_n is the exact value and $\tilde{y}_n$ is the numerical solution at timestep n. The same notation is used for the momentum p_n and $\tilde{p}_n$.

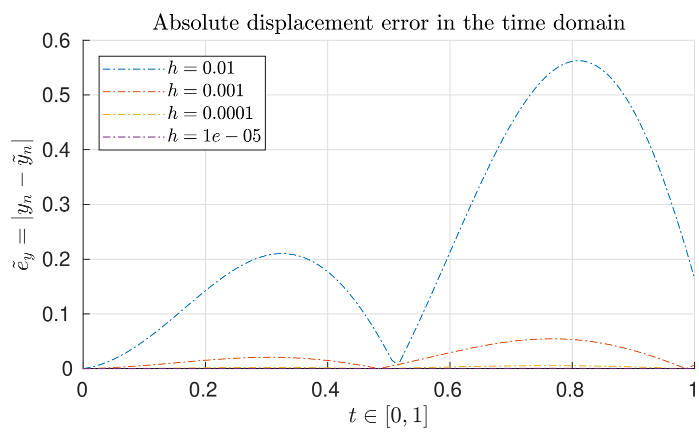

Figure 4: Displacement error for different time steps

Since both the equations describe oscillations in time, the absolute error also shows an oscillatory behaviour. For a coarse discretization, in Figure 4 and 5, both numerical solutions diverge. For a finer step size, the magnitude of the error is reduced significantly but is likely diverging for time domains larger than $t \in [0, 1]$.

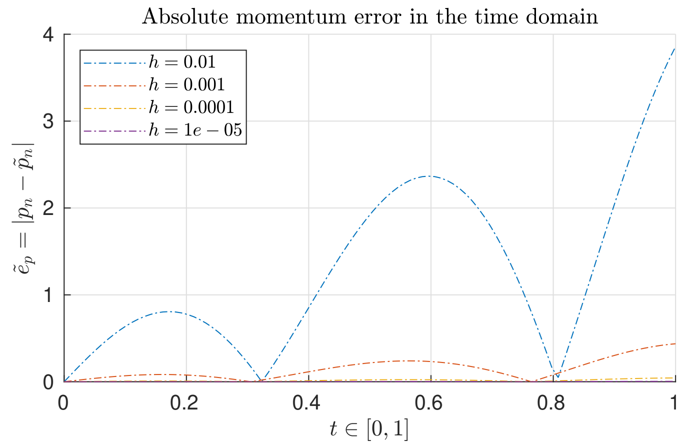

Figure 5: Momentum error for different time steps

3.2 Convergence

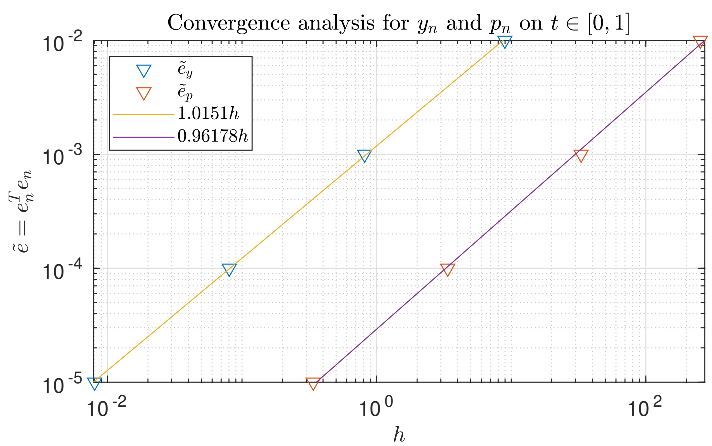

Figure 6: Logarithmic plot of the error of during one oscillation period using log-spaced step sizes.

The displacement error is propagated into the computed momentum and reduces the convergence rate in Figure 6. The fitted curve indicates that RK4 has a first order convergence in time.

4 Summary and outlook

As expected when looking at the results, the finer discretization, the better the solution. The error is further analyzed in Section 3 and shows a first order convergence rate during one oscillation period.

The absolute error of the two equations, which depict oscillations over time, also demonstrates oscillatory behavior. As seen in Figures 3 and 4, when the discretization is rough, the numerical solutions deviate. With a smaller step size, the magnitude of the error decreases significantly, yet it diverges when time becomes large.

In order to get the optimal mix of computational expense och accuracy in combination with RK4 one might consider either using adaptive step sizes or finding the optimum of absolute error and truncation error. As seen, the results attained in the previous section are acceptable for one period, but by using adaptive step sizes and adjusting the error threshold one might be able to attain even more accurate results over larger time domains.