Home Exercise 2

Green's functions

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1 Problem formulation

The following ODE is to be solved

$$\begin{cases} (\frac{d^2}{dr^2} - a^2)\phi = -4\pi r\rho, \quad a \in \mathbb{R} \\ \rho(r) = \frac{1}{8\pi}e^{-r}, \\ \phi(r=0) = \phi(r \to \infty) = 0. \end{cases} \tag{1}$$

The solutions to the corresponding homogenous ODE are

$$\begin{cases} \phi^h_<(r_<) = \frac{1}{\sqrt{2a}}(e^{ar_<} - e^{-ar_<}), \\ \phi^h_>(r_>) = -\frac{1}{\sqrt{2a}} - e^{-ar_>}. \end{cases} \tag{2}$$

The numeric solution terms are derived from the Wronskian and obtained by using numeric integration of the formula

$$\phi(r) = \phi^h_>(r)\int_0^r \phi^h_<(r')S(r')dr' + \phi^h_<(r)\int_r^\infty \phi^h_>(r')S(r')dr' \tag{3}$$

which can then be compared with the following given analytic solution

$$\phi(r) = \Big(\frac{1}{1-a^2}\Big)^2\Big(e^{-ar} - e^{-r}\Big[1 + \frac{1}{2}(1-a^2)\Big]\Big). \tag{4}$$

Using parameters $a = 4$ and $r_{max} = 30$ for the numerical integration. The numerical integration must be accurate and therefore Bode's rule

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}\Big[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\Big] \tag{5}$$

is a good candidate if the domain can be divided into N subdomains, where N is divisible by 4. The numerical implementation in MATLAB uses $N = 1000$.

2 Results

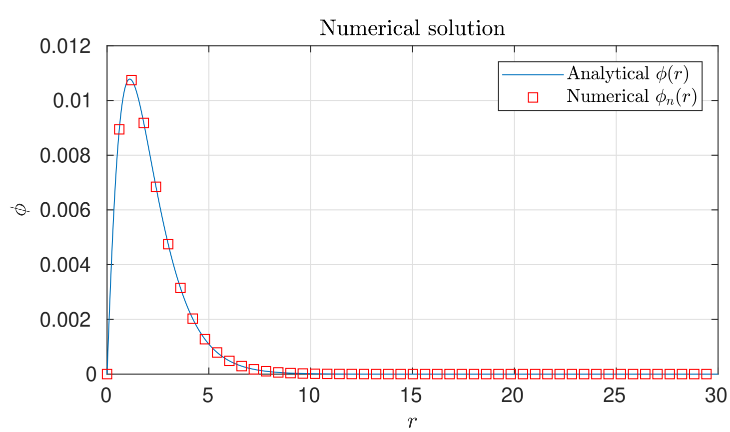

Figure 1: Comparison of numerical and analytical solution for $N = 1000$

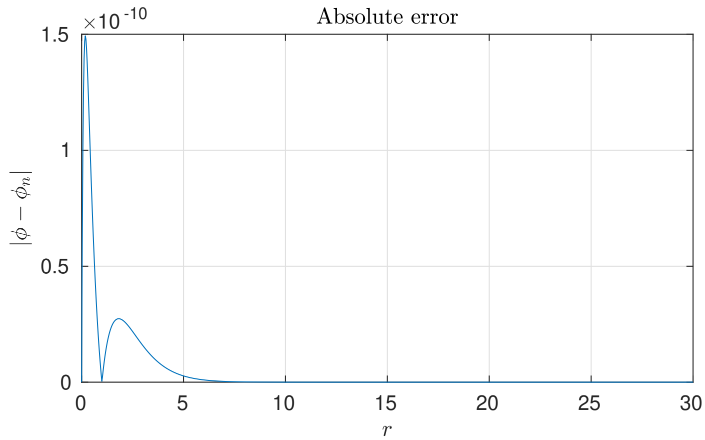

Figure 2: Error plot for a grid size of $N = 1000$

3 Error analysis

In Figure 2, the absolute error of the magnitude is 10^{-10}, which is very small for the grid size $N = 1000$. For reference, the error peaks when the solution in Figure 1 become stiff.

4 Summary and outlook

Overall, the integration performed well, and accuracy was suitable for the given equations, i.e. the numerical and analytical solution did not deviate considerably. However, the implemented solution requires finding the homogeneous solution in order before implementing the numeric method. For more complicated equations, theses values has to be approximated in another way. Furthermore, Boole's method required that the integration steps were divisible by 4, which can be limiting if the spatial dimension is odd and does not allow modification. Neither of these potential issues were a hindrance in this simple case. The problem was analyzed using a grid size of $N = 1000$ and had a reasonably fast execution time on a modern laptop. If a fine grid size would have been a problem, an adaptive step size integration method would have been investigated.