Enduring Relationships in an Economy with Capital and Private Information

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Abstract

We study efficient risk sharing in a model where agents operate linear production technologies with private information about idiosyncratic productivity. Capital is the sole factor of production, and accumulable. We establish a time-invariant, one-to-one mapping between the capital allocated to an agent and his lifetime utility entitlement. The mapping implies properties that are distinct from those in models with private information about endowments. In contrast to the latter, the value of the risk-sharing arrangement in our model always remains above the autarky value. There is no need for long-term commitment. Further, in our model, there are no net expected transfers each period across individuals. This allows us to decentralize the efficient allocation into one-period insurance contracts that do not require long-term commitment on the part of the principal or agent. Furthermore, while the efficient allocation implies an increasing dispersion of lifetime utility entitlements and consumption, this need not lead to declines in individual consumption as in the endowment model. When technology is sufficiently productive, all individuals experience consumption growth.

1 Introduction

We study risk sharing in an environment with production and capital accumulation. Each agent in our economy produces output, using capital, subject to privately observed idiosyncratic productivity shocks. Capital is publicly observable and allocated across agents, each period, before the realization of the shock. After production, aggregate output is allocated as consumption across agents with different histories of shocks and investment. We assume that preferences are represented by constant-relative-risk-aversion (CRRA) utility and that production is linear with shocks that are independently and identically distributed across agents and over time. In this setting, we examine constrained optimal allocation under long-term commitment and the decentralized arrangements that support it.

Over the past three decades, a body of empirical evidence has emerged against full insurance implied by the complete-markets model.[1] This evidence has given rise to a literature, starting with Huggett (1993) and Aiyagari (1994), that studies the effects of incomplete markets on the distribution of consumption, income, and wealth. This literature assumes that households have limited ability to insure against idiosyncratic risk, i.e., some markets are assumed to be missing. A different approach has been to study efficient risk sharing in environments that prevent full insurance against idiosyncratic risk by explicitly modeling barriers that limit risk sharing, such as private information. Beginning with Townsend (1982), Green (1987), Spear and Srivastava (1987), and Thomas and Worrall (1990), this literature has derived efficient risk sharing through incentive-compatible enduring relationships between risk-averse agents and a risk-neutral principal.

While such models of risk sharing offer insights into the sources of uninsurable individual risk, and thus inequality of consumption and wealth, they have been less influential than the incomplete-markets models. One reason is that existing studies of efficient risk sharing through enduring relationships have strong normative predictions that are problematic. Almost all these papers have focused on environments with risky endowments or production

[1] Early examples of such work include Cochrane (1991), Mace (1991), and Hayashi et al. (1996). A more recent test is in Johnson et al. (2006).

without accumulable inputs, where aggregate resources are constant over time. As noted by Townsend (1982), efficient risk sharing can be achieved in such settings through an enduring relationship between a principal and an agent by conditioning future consumption on past history of the agent's reports of private information and using punishments and rewards to incentivize truthtelling. However, diminishing marginal utility implies that punishments are cheaper than rewards, which results in a negative trend in expected lifetime utility and increasing inequality in consumption, i.e., immiseration. Thus, while any agent *initially* benefits from the long-term risk-sharing contract, the continuation value of the contract eventually falls below autarky value. Alternatively, if an agent's ability to commit to a long-term relationship is assumed to be limited, then he reaches the autarky value in finite time and further risk sharing is prevented.

We introduce capital accumulation into the private-information problem and explore efficient risk-sharing arrangements under long-term commitment. Our model builds on the private-information models of Green (1987) and Atkeson and Lucas (1992). In addition to the usual promise-keeping and incentive constraints, the planner in our economy has to ensure that the endogenous aggregate capital stock for next period is sufficient to deliver the continuation of promised utilities. Our capital accumulation model delivers a time-invariant one-to-one mapping between the capital allocated to an individual and his promised utility. The capital has less value for the agent in autarky since he gets no insurance, so the value of the efficient allocation always exceeds autarky value. In other words, we do not require strong assumptions on agents' ability to commit. Moreover, inequality in consumption and capital increase over time in autarky and in our private-information economy, but when agents are sufficiently patient, efficient risk sharing dampens the rise in inequality.[2]

Our first result is that the one-to-one mapping between an agent's promised utility and his capital implies (i) capital is a sufficient statistic for the agent's history of productivity shocks, so wealth can be measured by units of capital, instead of abstract units of promised

[2]Immiseration in private-information endowment economies can be eliminated if the set of promised utilities is bounded below or if the consumption possibilities set is compact. Examples of such assumptions are in Atkeson and Lucas (1995), Phelan (1995), Aiyagari and Alvarez (1995), and Bohacek (2005). As we show later, our environment admits sustained growth and these assumptions do not eliminate immiseration.

utility, and (ii) the capital allocated to each agent does not depend on the entire distribution of promised utility. In contrast, under full information, promised utility is divorced from capital since the expected marginal product of capital is the same for all agents.

Second, the efficient allocation displays a scale invariance: Consumption per unit of capital is constant across agents with the same realized idiosyncratic shock. An agent's mean consumption in each period (before the shock) is proportional to the capital that was allocated to him for production. With CRRA preferences, the efficient provision of incentives imply that the standard deviation of consumption is proportional to mean consumption, i.e., the coefficient of variation in consumption is constant across agents.

Third, on average, there is no transfer of resources between agents, either within a period or across periods. Since agents with different histories are identified by the different levels of capital allocated to them for production, scale invariance implies both expected consumption and expected investment are proportional to expected output. As a result, the expected net transfers across agents, if any, must have the same sign for *all* agents. Since the planner does not have access to external resources, the expected net transfers are zero. Thus, each agent's optimal allocation satisfies consumption plus investment equalling output, in expectation.

Fourth, even though the optimal dispersion in capital increases without bound, our environment allows for sustained growth and there are parameter configurations such that the immiseration is relative, not absolute. Moreover, the planner in our private-information economy can transfer physical resources over time and finds it less costly to provide intertemporal incentives when agents are more patient. For discount factors sufficiently large, the efficient allocation delivers a *slower* increase in capital dispersion relative to capital dispersion in autarky, which also grows without bound.

Fifth, as noted earlier, the assumption of long-term commitment is less restrictive in our model than in endowment models. The efficient allocation will remain unchanged if we introduce the option to go to autarky in any period before the shock is realized. This is because: (i) the expected transfer across agents is zero in every period and (ii) long-term commitment provides partial insurance while autarky provides none. Thus, the long-term arrangement dominates autarky at the start of every date. In contrast, the lifetime utility in autarky is fixed in Green (1987), for instance. Immiseration then implies that autarky will eventually dominate the long-term contract and agents would prefer to be in autarky. While immiseration and lack of voluntary participation in the long-term contract are linked in Green (1987), we obtain voluntary participation despite immiseration.

These results also hold when our economy is subject to a *publicly* observable aggregate shock that follows a finite-state Markov Chain and affects each agent's idiosyncratic productivity. While persistent shocks in environments with private information make incentive problems difficult to solve, we can characterize the solution because the persistent information is public. In each period, the idiosyncratic productivity is drawn from a distribution that is conditional on the aggregate shock, but the conditional distribution of productivity in the next period is independent of current idiosyncratic productivity. For instance, our framework can accommodate the notion that idiosyncratic shocks have a lower mean and higher variance in recessions relative to booms, as in Bloom (2014). Unlike Phelan (1994) who examines a framework with i.i.d. aggregate shocks and without capital, change in the variance of idiosyncratic productivity affects the aggregate growth in our environment.

Finally, for some values of risk aversion the aggregate growth rate is less than that in the full-information economy and less than that in autarky. From Levhari and Srinivasan (1969), we know that when risk aversion equals one (logarithmic preferences), the saving rate and, hence, the growth rate is the same under autarky and full information. The growth rate under private information, however, is less. This is because the incentive problem distorts the intertemporal tradeoff toward current consumption for the low-productivity agent, while the high-productivity agent does not face a distortion. Thus, in the aggregate, the saving rate and the growth rate are lower in the private-information economy. When risk aversion is not equal to one, there is an additional effect. Risk aversion implies the certainty-equivalent rate of return is less than the mean return, and for risk aversion less than one the intertemporal substitution effect dominates the income effect. This adds to the incentive effect, so the saving rate is lower in the private-information economy than that in the full-information economy. The certainty equivalent under autarky is less than that under private information (since autarky offers no insurance), but by continuity, there exist values of risk aversion less than one such that the private-information economy has a lower growth rate compared to both the full-information economy and autarky.

Using a decentralization scheme in which agents enter into binding long-term contracts with risk-neutral intermediaries who can borrow and lend resources between each other, we prove the counterparts to the first and second welfare theorems for our economy. We show that there exists an interest rate that clears the market for intertemporal trade between the component planners and that the equilibrium interest rate is less than the expected marginal product of capital. Furthermore, exploiting the result that there is no cross-subsidization across agents in the efficient allocation, we show that the decentralization can be accomplished through a sequence of *one-period contracts*. The one-period contract has two components: (i) productivity-contingent transfers such that the expected transfer equals zero and (ii) productivity-contingent bounds on next period's capital stock. Unlike Phelan (1995) where agents do not commit and can recontract at the onset of any period, our contract exhibits two-sided lack of commitment, so the recontracting has no effect.

In related work, Smith and Wang (2006) develop a quantitative model with private information on the return to capital and long-term contracts between financial intermediaries and entrepreneurs. In their model, each entrepreneur's project size, or capital requirement, is fixed and long-term commitment in the contract is critical. In contrast, project size is endogenous in our model and the payoff to deviating from truthtelling increases with the project size. Further, long term commitment in the contract is not critical for our results. As noted earlier, (i) agents would voluntarily continue with the contract at the onset of every period and (ii) a sequence of one-period contracts can implement the efficient allocation.

The rest of the paper is organized as follows. We introduce the model in Section 2 and develop our efficiency concept and recursive formulation in Section 3. Properties of the efficient allocation are in Section 4. In Section 5, we present the one-period contract that is a decentralization of the efficient allocation. We conclude in Section 6. The proofs are in Appendix A. Appendix B describes the environment with aggregate shocks, Appendix C contains the results for decentralized long-term contracts with component planners, and Appendix D describes an environment where we allow for randomized allocations.

2 Model

In our economy, there is a continuum of individuals of measure 1. Time is discrete, is indexed by $t = 1, 2, \ldots,$ and runs forever.

Preferences Preferences of every agent are described by

$$E_1 \sum_{s=1}^{\infty} \beta^{s-1}(1-\beta)u\left(c_s\right)$$

where $\beta \in (0,1)$ is the agent's discount factor, c_s is consumption in period s, and u is a utility function of the CRRA class: $u(c) = c^\gamma/\gamma$ for $\gamma \neq 0$ or $u(c) = \log(c)$. (We will treat the $\gamma = 0$ case as synonymous with logarithmic utility.) Let $\mathbf{C} \subset \mathbb{R}$ be the domain of the utility function: $\mathbf{C} = [0, \infty)$ if $\gamma > 0$ and $\mathbf{C} = (0, \infty)$ otherwise. The range of the utility function is denoted $\mathcal{V} \equiv u(\mathbf{C})$; it is also the set of values for expected lifetime utility.

Technology Each individual operates a production technology subject to stochastic productivity. The individual's output in period t is $z_t k_t$, where k_t is the capital stock at the beginning of t and z_t is individual's productivity at time t. Capital is accumulable, must be installed prior to the realization of productivity, and depreciates completely after production. The 100% depreciation is without loss of generality since the technology is linear.

There is no aggregate uncertainty. (We study aggregate shocks in Appendix B.) Productivity is independently and identically distributed over time and across individuals. Specifically, $z_t \in \mathbf{Z} \equiv \{z_1, z_2, \ldots z_n\}$. Let $\mathcal{Z} \equiv 2^{\mathbf{Z}}$ be the complete σ-algebra and define $\mu : \mathcal{Z} \to [0,1]$ as a probability measure on $\mathcal{Z}$, summarized by $\mu(z_i) = \mu_i > 0$. The probability space $(\mathbf{Z}, \mathcal{Z}, \mu)$ generates the finite dimensional product probability spaces $(\mathbf{Z}^t, \mathcal{Z}^t, \mu^t)$, $t = 1, 2, \ldots$, as well as the infinite dimensional probability space of all measurable sequences of productivity $(\mathbf{Z}^\infty, \mathcal{Z}^\infty, \mu^\infty)$ in the standard manner. We denote $z^t = (z_1, z_2, \ldots, z_t) \in \mathbf{Z}^t$.

Information and Endowment The idiosyncratic productivity z is private information and so is consumption. Capital allocated to the agent is publicly observable, so the agent can neither invest privately nor divert the allocated investment to his consumption. (Since capital depreciates completely, investment allocated in the current period is the capital stock for next period's production.)

Each agent is initially entitled to (and identified by) a lifetime utility w_1. The distribution of initial utilities is given by a Borel measure ψ on $\mathcal{V}$.

Commitment We consider direct revelation mechanisms. The planner assigns entire sequences of consumption and capital for $t = 1, 2, \ldots, \infty$ (long-term commitment). In each period, the planner assigns capital for current production and, after receiving the report of productivity, assigns a transfer b for current consumption. Since capital is allocated prior to the realization of productivity, the capital can be conditioned only on the history up to the previous period. Consumption $(zk + b)$ takes place after production, so the transfer can be conditioned on the history including the current period. Agents are treated differently if their reports are different. With this in mind, we define an allocation as follows:

Definition 1 *An allocation* $\mathbf{P}$ *is a sequence of Borel-measurable functions* $\mathbf{P} = \{k_t : \mathcal{V} \times \mathbf{Z}^{t-1} \to \mathbb{R}, b_t : \mathcal{V} \times \mathbf{Z}^t \to \mathbb{R}\}_{t=1}^\infty$ *that satisfies the following constraints:* $k_t(w, z^{t-1}) \geq 0$ *for all* t, $w \in \mathcal{V}$, *and* $z^{t-1} \in \mathbf{Z}^{t-1}$, $b_t(w, z^t) + z_t k_t(w, z^{t-1}) \in \mathbf{C}$ *for all* t, $w \in \mathcal{V}$, $z^t \in \mathbf{Z}^t$. *The set of all allocations is denoted by* Π.

We will use $\mathbf{P}(w_1)$ to refer to the allocation for an agent with initial promised utility w_1. The choice for the agent is his productivity report.

Definition 2 *A reporting strategy is a sequence of functions* $\sigma = \{\sigma_t : \mathbf{Z}^t \to \mathbf{Z}\}_{t=1}^\infty$ *such that* $\sigma_t : \mathbf{Z}^t \to \mathbf{Z}$ *is the time-t report of productivity. We denote* $\sigma^t(z^t) \equiv (\sigma_1(z^1), \sigma_2(z^2), \ldots, \sigma_t(z^t))$

to be the vector of reports up to date t. A reporting strategy σ is feasible for allocation $\mathbf{P}(w_1)$ if for every $z^t \in \mathbf{Z}^t$, $z_t k_t(w_1, \sigma^{t-1}(z^{t-1})) + b_t(w_1, \sigma^t(z^t)) \in \mathbf{C}$. The set of all such feasible reporting strategies is denoted by $\Sigma(\mathbf{P}(w_1))$.

$(w_1, z^{t-1}) \longrightarrow$ Capital is allocated $k_t(w_1, \sigma^{t-1}(z^{t-1}))$ $\longrightarrow$ Productivity shock z is realized $\longrightarrow$ Agent reports $\sigma_t(z^t)$ $\longrightarrow$ Agent consumes $z_i k_t(w_1, \sigma^{t-1}(z^{t-1})) + b_t(w_1, \sigma^t(z^t))$

Figure 1: Sequence of events

The timing of events is illustrated in Figure 1. Denote the truthtelling strategy by σ^*—that is, $\sigma_t^*(z^t) = z_t, \forall z^t \in \mathbf{Z}^t$. Definition 1 implies that $\sigma^* \in \Sigma(\mathbf{P}(w_1))$. Hence, $\Sigma(\mathbf{P}(w_1))$ is nonempty. Each agent's lifetime utility is a function of the allocation $\mathbf{P}(w_1) \in \mathbf{\Pi}$ and his reporting strategy $\sigma \in \Sigma\left(\mathbf{P}(w_1)\right)$:

$$V(\mathbf{P}(w_1), \sigma) = \sum_{s=1}^{\infty} \beta^{s-1} \int_{\mathbf{Z}^s} (1-\beta) u\left(z_s k_s\left[w_1, \sigma^{s-1}(z^{s-1})\right] + b_s\left[w_1, \sigma^s(z^s)\right]\right) \mu^s(dz^s).$$

Incentive compatibility and Promise keeping An allocation $\mathbf{P}$ is *incentive compatible* if the truthtelling strategy satisfies:

$$V(\mathbf{P}(w_1), \sigma^*) \geq V(\mathbf{P}(w_1), \sigma),\ \forall \sigma \in \Sigma\left(\mathbf{P}(w_1)\right),\ \forall w_1 \in \mathcal{V}. \tag{1}$$

An allocation satisfies *promise keeping* if each agent's expected utility satisfies:

$$V(\mathbf{P}(w_1), \sigma^*) \geq w_1,\ \forall w_1 \in \mathcal{V}. \tag{2}$$

The empirical distribution of any variable in the model will coincide, with probability one, with the random process that generates it.[3] Similarly, an analog of the strong law of large numbers will hold—sample averages of all realized variables will be equal to their

[3] See Green (1994) for a construction of a continuum of i.i.d. random vector such that each sample has the same distribution as the population.

expectation almost surely. Hence, an incentive-compatible allocation must satisfy, with probability one, the constraint that the total capital allocated for production is bounded by the total transfers from agents, in every period.

$$-\int_{\mathcal{V}} \left[\int_{\mathbf{Z^t}} b_t(w, z^t)\mu^t(dz^t) \right] \psi(dw) \geq \int_{\mathcal{V}} \left[\int_{\mathbf{Z^t}} k_{t+1}(w, z^t)\mu^t(dz^t) \right] \psi(dw). \tag{3}$$

3 Efficiency

In the spirit of Atkeson and Lucas (1992), our notion of efficiency is the minimum amount of capital necessary to attain a distribution of initial promised utility:

Problem SP

$$\begin{aligned} \varphi^*(\psi) &= \inf_{\mathbf{P} \in \mathbf{\Pi}} \int_{\mathcal{V}} k_1(w_1)\psi(dw) \\ & \quad \text{subject to } (1) - (3). \end{aligned}$$

We call this social planning problem SP.[4] Note that the set of allocations, satisfying (1)-(3) is nonempty, but $\int_{\mathcal{V}} k_1(w_1)\psi(dw)$ is not necessarily finite. If $\Delta(\mathcal{V})$ is the set of all Borel probability measures on $\mathcal{V}$, then $\varphi^* : \Delta(\mathcal{V}) \to \bar{\mathbb{R}}_+$. For the rest of the paper, we restrict ourselves to initial promised-utility distributions that the planner under full information can attain with finite amount of capital which, as we show later, is equivalent to:

Assumption 1 $\int_{\mathcal{V}} u^{-1}(w)\psi(dw) < \infty$.

For every agent, the efficient allocation must satisfy uncountably infinitely many constraints. However, we show that the problem can be reduced to a sequence of two-period problems with essentially just two constraints for every agent. With this formulation the efficient allocation is obtained by solving a functional equation.

[4] If the effects of private information are separable from consumption (e.g., taste shocks in Atkeson and Lucas (1992)), convexity can be restored by an appropriate choice of variables. This is not true in our model: The flow utility of consumption is affected by the true productivity and the reported productivity. In Appendix D, we allow for lotteries and show that the efficient allocation is deterministic conditional on the productivity realizations. That is, even though conditions (1)-(3) seem restrictive, allowing for a complete set of allocations does not change the solution to Problem SP.

3.1 Recursive Formulation

We employ a widely-used method (e.g., Green (1987) and Spear and Srivastava (1987)) to simplify the set of feasible allocations for the planner. We impose the restriction that for every date, agent, and history, *promised utility*–the expected discounted utility from a truthtelling strategy–is a sufficient statistic for future allocations. Then we have an *allocation rule*: At each date, the agent is associated with a point on the real line–his promised utility. Conditional on the promised utility, he is assigned some capital; he then makes a report of productivity; conditional on the report he receives a consumption transfer and a continuation utility. Formally:

Definition 3 *An allocation rule $\mathcal{P}$ is a sequence of Borel-measurable functions $\mathcal{P} = \{k_t(w), b_t(w,z), v_{t+1}(w,z)\}_{t=0}^{\infty}$ that satisfies the following constraints for all t, $w \in \mathcal{V}$, $z \in \mathbf{Z}$: $k_t(w) \geq 0$, $zk_t(w) + b_t(w,z) \in \mathbf{C}$, and $v_{t+1}(w,z) \in \mathcal{V}$.*

The time subscript t in the allocation rule keeps track of the entire distribution of promised utility as of that period. We will show later that the allocation rule will *not* depend the entire distribution of promised utility and, hence, will be time invariant. Allocation rules impose an *equal treatment property*: Two agents with the same promised utility must receive identical future allocations.[5]

Figure 2 illustrates the timing in the recursive formulation. Every allocation rule generates an allocation that can be easily constructed in a recursive fashion.

Promised utility must satisfy the following:

$$w = \int_{\mathbf{Z}} \left[(1-\beta)u(zk_t(w) + b_t(w,z)) + \beta v_{t+1}(w,z))\right] \mu(dz), \quad \forall w \in \mathcal{V}, t. \tag{4}$$

Turning to incentive compatibility, consider an agent who has a promised utility w and shock z_i at an arbitrary date t. A strategy of misreporting only at t should be dominated by

[5] There are allocations that violate this property, but in Appendix D we show that the property is satisfied (up to measure zero) by the efficient allocation.

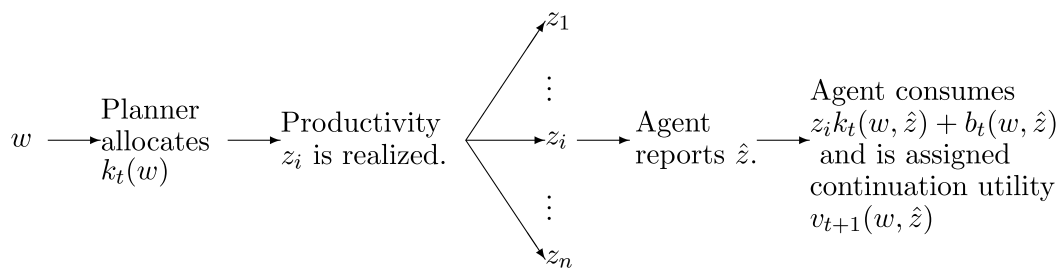

Figure 2: Timeline of events in the recursive formulation

truthtelling, which gives rise to the following *temporary incentive compatibility constraint*:

$$(1-\beta)u(z_i k_t(w) + b_t(w,z_i)) + \beta v_{t+1}(w,z_i)) \geq (1-\beta)u(z_i k_t(w) + b_t(w,\hat{z})) + \beta v_{t+1}(w,\hat{z}). \quad (5)$$

The planner can promise ever higher utility indefinitely, so we need boundedness conditions on promised utility:

$$\lim_{s\to\infty} \beta^s E_t[w_s | w_t, \sigma^*] = 0 \text{ for all } t \text{ and } w_t \in \mathcal{V}, \tag{6}$$

$$\lim_{s\to\infty} \beta^s \inf[w_s | w_1] = 0 \text{ for all } w_1 \in \mathcal{V}. \tag{7}$$

Lemma 1 *Suppose that an allocation* $\mathbf{P}$ *is generated by an allocation rule (definition 3), and conditions* (4)*,* (5)*,* (6)*, and* (7) *hold. Then* $\mathbf{P}$ *is incentive compatible and lifetime utility* $V(\mathbf{P}(w_1), \sigma^*) = w_1$ *for all* $w_1 \in \mathcal{V}$ *.*

Lastly, we turn to the resource constraint. Let ψ_t be the distribution of promised utility induced by an allocation rule. It is easy to see that ψ_t has the following law of motion:

$$\psi_{t+1}(\mathcal{B}) = \int_{\mathcal{V}} \int_{\mathbf{Z}} \chi_{\mathcal{B}}(v_{t+1}(w,z))\mu(dz)\psi_t(dw),$$

where $\mathcal{B}$ is any Borel set and $\chi_{\mathcal{B}}$ is the indicator function. Then an allocation generated by an allocation rule will satisfy the resource constraints if

$$-\int_{\mathcal{V}}\int_{\mathbf{Z}} b_t(w,z)\mu(dz)\psi_t(dw) \geq \int_{\mathcal{V}} k_{t+1}(w)\psi_{t+1}(dw).$$

Functional equation The planner's state variable is the entire distribution of promised utilities. Suppose that starting from tomorrow the capital required to attain a distribution of utilities ψ' is $\varphi(\psi')$. Then the capital necessary to attain the distribution ψ today is:

Problem FE

$$\begin{aligned}
(T\varphi)(\psi) &= \inf_{(k,b,v)} \int_{\mathcal{V}} k(w)\psi(dw) \\
&\quad \text{subject to} \\
k(w) &\geq 0, \quad zk(w)+b(w,z) \in \mathbf{C}, \\
(1-\beta)u\left(zk(w)+b(w,z)\right)+\beta v(w,z) &\geq (1-\beta)u\left(zk(w)+b(w,\hat{z})\right)+\beta v(w,\hat{z}), \quad (8) \\
\int_{\mathbf{Z}} \left[(1-\beta)u\left(zk(w)+b(w,z)\right)+\beta v(w,z)\right]\mu(dz) &= w, \quad (9) \\
-\int_{\mathcal{V}} \left[\int_{\mathbf{Z}} b(w,z)\mu(dz)\right]\psi(dw) &\geq \varphi(\psi'), \quad (10)
\end{aligned}$$

where $k:\mathcal{V}\to\mathbb{R}_+$, $b:\mathcal{V}\times\mathbf{Z}\to\mathbb{R}$, $v:\mathcal{V}\times\mathbf{Z}\to\mathcal{V}$ and the measure of continuation utilities in any Borel set $\mathcal{B}$ is given by:

$$\psi'(\mathcal{B}) = \int_{\mathcal{V}}\int_{\mathbf{Z}} \chi_{\mathcal{B}}[v(w,z)]\mu(dz)\psi(dw).$$

Equation (8) is the temporary incentive compatibility constraint, (9) is the promise keeping constraint, and (10) is the resource constraint.

The usual relationship between the sequential problem and the functional equation holds in our environment. The allocation generated recursively by the functional equation attains the infimum of the SP with the additional boundedness conditions (6) and (7).

Proposition 1 *The minimal φ^* in problem SP solves the functional equation—that is, $T\varphi^* = \varphi^*$. Let $\mathbf{P}$ be an allocation rule generated recursively by iterating on the functional equation above. If (7) holds, then $\mathbf{P}$ attains the infimum in the sequence problem.*

Solving FE There are two obstacles to finding φ^* that solves the functional equation: (i) The operator T is not a contraction and (ii) the state space is infinite dimensional.

To overcome (i), we exploit the fact that the operator is monotone. We start with upper and lower bounds on the infimum function and iterate on the two bounds, as in Atkeson and Lucas (1992). The economics of the problem suggests that the lower bound on φ^* is φ_ℓ, the initial capital necessary to attain ψ with full information, and the upper bound is φ_h, the initial capital under autarky. It is easy to verify that

$$\varphi_i(\psi) = A_i \int_{\mathcal{V}} u^{-1}(w)\psi(dw),\ i = h, \ell$$

for some constants $A_h > A_\ell > 0$. (A_h is well-defined if $\beta^{\frac{1}{1-\gamma}}(Ez)^{\frac{\gamma}{1-\gamma}} < 1$; A_ℓ is well-defined if $\beta^{\frac{1}{1-\gamma}}(E\{z^\gamma\})^{\frac{1}{1-\gamma}} < 1$.)

We gain tractability despite obstacle (ii) since cost functions of the form $A\int_{\mathcal{V}} u^{-1}(w)\psi(dw)$ imply that it is optimal for the planner to consider delivering different promised utilities as separate, unrelated problems, and condition each agent's allocations only on his promised utility. We prove this property in Appendix A; we provide a heuristic derivation here.

For the case when all the agents have the same promised utility, the property holds trivially. Suppose that the promised utility was higher for some mass of agents. The planner will need to provide higher consumption and higher continuation utility to these agents, which requires additional resources, which in turn requires more capital to be allocated. How should this additional capital be allocated? Increasing capital makes the incentive problem more severe and forces a larger spread of consumption and continuation utility. Agents with higher average consumption bear this better, due to decreasing absolute risk aversion. In our environment, CRRA implies that it is optimal to set the variability in consumption exactly proportional to average consumption. As a result, all of the additional capital is allocated to the group with higher promised utility. The allocations for the two groups are thus independent of each other.[6]

[6] In contrast, if preferences were of the Constant-Absolute-Risk-Aversion class, then all agents are allocated the same amount of capital, so the allocations depend on the entire distribution of promised utilities.

The independence of individual cost-minimization problems implies that the only way to ensure the resource constraint (10) is satisfied is for it to be satisfied agent by agent: $-\int_{\mathbf{Z}} b(w,z)\mu(dz) \geq A\int_{\mathbf{Z}} u^{-1}(v(w,z))\mu(dz)$ for (almost) all w. This is the individual accumulation constraint. Therefore, the planner minimizes the amount of capital allocated to each agent subject to the incentive and promise-keeping constraints and the individual accumulation constraint.

The problem can be simplified further by observing that we can perform the minimization at some promised utility level and scale up appropriately for all other w. Denote $(k, b(z), v(z))$ as the optimal allocation for promised utility $u(1)$, i.e., the lifetime utility associated with consuming 1 unit forever. We will focus on the case $\gamma \neq 0$, so that $u(1) = \frac{1}{\gamma}$. Then the allocation $(\hat{k} = (\gamma w)^{\frac{1}{\gamma}} k, \hat{b} = (\gamma w)^{\frac{1}{\gamma}} b(z), \hat{v} = (\gamma w) v(z))$ satisfies the constraints at promised utility w. Since w and $u(1)$ can be exchanged in this construction, the optimal amount of capital for an agent with promised utility w is $k(\gamma w)^{\frac{1}{\gamma}}$. These observations allow the fixed point of Problem FE to be derived from the following auxiliary problem, which operates on a parameter A that identifies the cost of providing $u(1)$ utils:

Problem AP

$$
\begin{aligned}
\phi(A) &= \inf_{(k, b(z), v(z))} k \\
&\quad \text{subject to} \\
(1-\beta)u(zk + b(z)) + \beta v(z) &\geq (1-\beta)u\left(zk + b(\hat{z})\right) + \beta v(\hat{z}), && (11) \\
\int_{\mathbf{Z}} \{(1-\beta)u\left(zk + b(z)\right) + \beta v(z)\}\mu(dz) &= u(1), && (12) \\
-\int_{\mathbf{Z}} b(z)\mu(dz) &\geq A\int_{\mathbf{Z}} u^{-1}(v(z))\mu(dz). && (13)
\end{aligned}
$$

4 Implications

Solving Problem FE, using Problem AP, yields the following decision rules.

Proposition 2 *Let A^* be the unique fixed point of ϕ in $[A_\ell, A_h]$ or A^* is the smallest fixed* *point in $[A_\ell, A_h]$. (i) Suppose $\gamma \neq 0$. (For $\gamma < 0$, A^* is unique.) Then,*

$$\varphi^*(\psi) = A^* \int_{\mathcal{V}} (\gamma w)^{\frac{1}{\gamma}} \psi(dw)$$

and the optimal allocation rule is:

$$k(w) = (\gamma w)^{\frac{1}{\gamma}} k, b(w, z) = b(z)(\gamma w)^{\frac{1}{\gamma}}, v(w, z) = v(z)\gamma w. \tag{14}$$

(ii) Suppose $\gamma = 0$. Then, A^* is unique and

$$\varphi^*(\psi) = A^* \int_{\mathcal{V}} \exp(w)\psi(dw)$$

and the optimal allocation rule is:

$$k(w) = k \exp(w), b(w, z) = b(z) \exp(w), v(w, z) = v(z) + w. \tag{15}$$

In both cases, $(k, b(z), v(z))$ solve problem AP for A^*.

Recall from Problem AP that $(k, b(z), v(z))$ is the optimal allocation to the agent with promised utility $u(1)$. So, Proposition 2 states that the optimal allocation at any level of promised utility is an appropriately scaled version of the one at promised utility $u(1)$.

It is clear from equations (14) and (15) that there is a one-to-one mapping between promised utility and capital: Agents with higher promised utility are allocated more capital.

Two remarks are in order here. (a) Measurement: The one-to-one mapping implies that wealth can be measured by physical units of capital, instead of abstract units of promised utility. (b) Full information: The history of productivity is irrelevant for allocating capital in the full-information economy since the expected marginal product of capital is the same for all agents. So, under full information promised utility is divorced from capital allocation.

In this section, we discuss several implications of our model. First, there are no *expected* transfers across agents: Expected sum of consumption and investment allocated to an agent is equal to his expected output in each period. Second, the efficient allocation displays a scaling property: Agents with higher promised utility receive a "scaled-up" version of the allocations to the agent with lower promised utility. Third, optimal wealth inequality increases without bound, but the immiseration is not necessarily absolute. Fourth, even though the allocations in (14) and (15) are under long-term commitment, they will remain the same if each agent had the option to leave the long-term arrangement and go to autarky. Finally, there exist parameters such that the aggregate growth rate of our economy is less than that in the full-information economy and less than that in autarky.

4.1 No transfers across agents

As noted earlier, the allocations to agents with different promised utilities are solutions to independent problems. Expected net resources for an agent with promised utility w are: $\int_{\mathbf{Z}} (sw, z) + k(v(w, z)) - zk(w))\, \mu(dz)$. Equations (14) and (15) imply that the consumption transfer and investment are both proportional to a nonlinear function of w. So, the expected net transfer for *all* agents has to be of the same sign in *every period.* The sign cannot be positive since the planner does not have access to external resources. Negative net transfers would be suboptimal since the planner can achieve higher promised utility.

Thus, for every agent, *expected* consumption plus *expected* investment equals *expected* output in every period and, hence, there are no expected transfers between agents with different promised utilities. However, there is some transfer of resources across states of productivity, which provides some insurance to the agents.

4.2 Scaling

At every date, the ratio of the consumptions of two agents with the same shock z but different promised utility is given by the ratio of the consumption equivalents of the promised utility. That is, the ratio of consumptions does not depend on the productivity shock, as long as both agents have the same shock.

Recall that an agent with promised utility w and productivity shock z consumes $zk(w)+$

$b(w,z)$. For all $w, \tilde{w} \in \mathcal{V}$ and $z \in \mathbf{Z}$, (14) and (15) imply

$$\frac{k(w)}{k(\tilde{w})} = \frac{c(w,z)}{c(\tilde{w},z)} = \frac{u^{-1}(w)}{u^{-1}(\tilde{w})}.$$

Consumption per unit of capital is constant across agents with the same z and an agent's expected consumption is proportional to his capital.

An implication of the scaling result is that individual consumption follows a geometric random walk. That is, independent of both current distribution of promised utility and initial conditions, there exists some constant x such that for all $w \in \mathcal{V}$, $z, z' \in \mathbf{Z}$,

$$\frac{E[c(v(w,z),z')]}{c(w,z)} = x.$$

With CRRA preferences, another implication is that the standard deviation of consumption is proportional to mean consumption. Hence, the coefficient of variation in consumption is constant across agents.

The scaling property holds for investment as well. Recall that with complete depreciation of capital, current investment is the same as capital $k(v(w,z))$ installed for future production. The ratio $\frac{k(v(w,z))}{k(v(\tilde{w},z))}$ is independent of z.

4.3 Wealth Inequality

Our environment also displays an ever-increasing cross sectional dispersion in promised utility, similar to other models with private information and long-term commitment. However, since promised utility maps monotonically into capital in our environment, this implies an ever-increasing dispersion in capital: An ever-shrinking share of the population is allocated an ever-increasing share of wealth. Formally,

Proposition 3 $\lim_{t\to\infty} \operatorname{Var} \ln(k_t) = \infty$

In our environment $\ln k$ is a random walk. For instance, equation (15) implies that $\ln k' = \ln k + v(z)$.

The immiseration in our case is relative: For some parameter configurations, even the poorest agent's wealth can increase forever. Consider an example with log preferences where $\beta = 0.5$, productivity takes on two values z_H and z_L, each occurring with probability 0.5, mean $Ez = 4$, and $z_H - z_L = 1$. So, the productivity process is such that the lowest possible value of productivity is greater than $\frac{1}{\beta}$. Then, from (13) we can see that $v(z_L) \leq 0$ is dominated; the incentive constraint and promise-keeping constraint are both satisfied when $v(z_L) > 0$. The gross growth rate of k in the low-productivity state is $exp(v(z_L))$ for log utility; see equation (15). Thus, for an agent who had an unlucky string of z_L realizations, the gross growth rate is greater than 1.

Autarky also implies an ever-increasing dispersion in capital. The decision rule in autarky is $k' = szk$, where $s = \beta^{\frac{1}{1-\gamma}} (Ez^{\gamma})^{\frac{1}{1-\gamma}}$. Thus, even under autarky, $\ln k$ is a random walk and $\lim_{t\to\infty} \mathrm{Var} \ln(k_t) = \infty$.

For sufficiently large discount factors, wealth inequality grows slower in our private-information economy than in autarky. To see this, consider an environment with just two productivity levels: $z_H > z_L$. In the efficient allocation, the ratio $\frac{k'(w,z_H)}{k'(w,z_L)}$ is independent of w and is a function only of z_H and z_L; see equations (14) and (15). Under autarky $\frac{k'(w,z_H)}{k'(w,z_L)}$ is just $\frac{z_H}{z_L}$. In both cases, $\frac{k'(w,z_H)}{k'(w,z_L)}$ is constant over time and completely pins down the growth rate of wealth inequality. When the discount factor is close to one, arbitrarily small variation in future capital across states is sufficient to provide incentives. By continuity, when the discount factor is sufficiently large this property still holds. Under autarky, the variation in future capital is the same as variation in z and there is no dampening across states. Hence, the wealth inequality grows slower under private information than in autarky.

If the set of lifetime utility entitlements is assumed to be bounded below, then immiseration in private-information endowment economies can be eliminated. Atkeson and Lucas (1995) motivate the lower bound by the notion that, in a dynasty model, the current generation cannot promise away the utility entitlement of future generations. In our model, lifetime utility is proportional to capital, so the lower on lifetime utility is equivalent to a lower bound on capital. This is stronger than the assumption that agents cannot sell the utility of future generations. Furthermore, with sustained growth, the lower bound on capital will not bind.

Aiyagari and Alvarez (1995) and Bohacek (2005) have shown that delivering an invariant distribution of promised utility in the endowment economies relies on individuals' consumption possibilities sets being compact. In our model with capital accumulation, the possibility of long-run growth makes the assumption of upper bound on consumption unattractive, while the assumption of a lower bound on consumption is eventually non-binding.

4.4 Voluntary participation

Consider the possibility that every agent has the option to leave the long-term arrangement and go to autarky at any point in time *before* productivity is realized (see Figure 2). That is, the agent enters the period with promised utility w and capital $k(w)$ and has to decide whether to continue with the commitment or abandon it and take the capital with him to autarky.

Let $v^{aut}(k)$ denote the expected lifetime utility under autarky of an agent with capital stock k.

$$v^{aut}(k) = \int_{\mathbf{Z}} \left[\max_{k' \in [0, zk]} \left\{ u(z_i k - k') + \beta v^{aut}(k') \right\} \right] \mu(dz).$$

Proposition 4 *The promised utility w associated with capital k under private information is strictly greater than $v^{aut}(k)$.*

This result follows from two properties of the efficient allocation noted above: (i) There are no expected transfers across agents with different promised utilities and (ii) there is insurance across different productivity states. Property (i) holds in autarky in every state; in the private-information economy, it holds only in expectation. Property (ii) holds in the private-information economy since the planner insures the agent across states, whereas autarky offers no insurance. These two properties together imply voluntary participation: The agent strictly prefers the long-term arrangement relative to autarky.

Voluntary participation does not hold in similar private-information economies where the value of autarky is exogenous, such as Green (1987). In our model the value of autarky is endogenous and depends on the capital allocated in the long-term arrangement.

4.5 Aggregate growth

So far, we have considered the properties of the agent's allocation. In this section, we consider the effect of the information friction on the aggregate performance of the economy.[7]

For an agent with promised utility w, or capital $k(w)$, the decision rules in Proposition 2 imply that his capital stock next period will be $k((v(w,z)) \equiv k'(w,z)$. So, the *expected* growth in capital for the agent is:

$$\int \frac{k'(w,z)}{k(w)}\mu(dz) = \begin{cases} \int (\gamma v(z))^{\frac{1}{\gamma}}\mu(dz) & \text{if } \gamma \neq 0, \\ \int \exp(v(z))\mu(dz) & \text{if } \gamma = 0. \end{cases}$$

Thus, the expected growth rate of capital does not depend on w and is the same for all agents. Hence, the expected aggregate growth rate is also given by the above equations.

Proposition 5 *There exists $\underline{\gamma} < 0$ such that the private-information economy grows slower than the full-information economy if $\gamma \in (\underline{\gamma}, 1)$. There exists $\bar{\gamma} \in (0,1]$ such that the private-information economy grows slower than autarky if $\gamma \in (-\infty, \bar{\gamma})$.*

Figure 3 illustrates the relationships between the three growth rates. For log utility, it is well known that the saving rate and, hence, the growth rate is the same under autarky and full information; see Levhari and Srinivasan (1969). As noted in Proposition 5 and Figure 3, the growth rate under private information is less. The reason is the incentive constraint. For simplicity assume that productivity can take only two values: low (z_L) and high (z_H). Increasing $v(z_L)$ by a small amount Δv increases the utility of the z_L-agent and the incentives of the z_H-agent to misreport by a proportional amount $\beta\Delta v$. On the other hand, increasing the transfer $b(z_L)$ (i.e., current consumption) by Δb increases the utility of

[7] Khan and Ravikumar (2001) characterize the effect of incomplete risk-sharing on *equilibrium* growth using long-term contracts between agents and a risk-neutral competitive intermediary for the case of logarithmic utility.

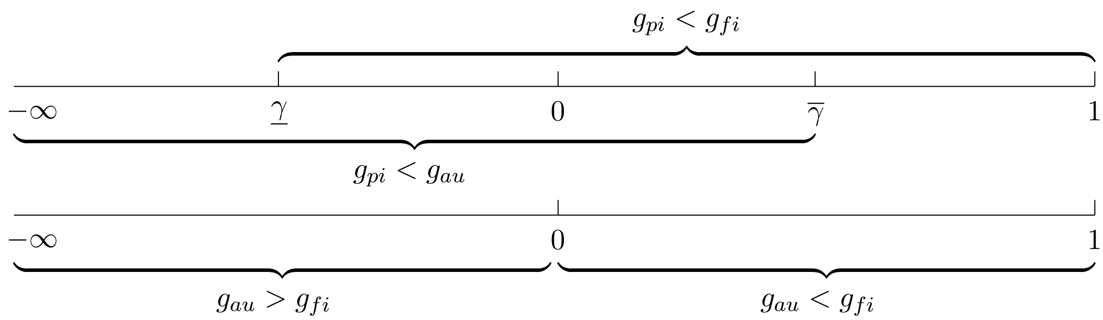

Figure 3: Relationship between growth rates. Parameter γ is on the horizontal axis. The growth rates for autarky, the full-information economy, and the private-information economy are denoted by g_{au}, g_{fi}, and g_{pi}, respectively.

the z_L agent by approximately $(1-\beta)u'(z_Lk + b(z_L))\Delta b$, but the incentive of misreporting by less, $(1-\beta)u'(z_Hk + b(z_L))\Delta b$. Thus, the incentive problem distorts the intertemporal tradeoff toward current consumption for the z_L-agent. The z_H-agent, on the other hand, does not face a distortion toward current consumption. Thus, in the aggregate, the saving rate is lower in the private-information economy.

For $\gamma \neq 0$, in addition to the incentive problems, there is another intertemporal tradeoff. As in Levhari and Srinivasan (1969), risk aversion implies the certainty equivalent rate of return is less than the mean return. For $\gamma \in (0, 1)$ the intertemporal substitution effect dominates the income effect. This works in the same direction as the incentive effect, leading to a lower saving rate in the private-information economy compared to that in the full-information economy. Since autarky offers no insurance, the certainty equivalent under autarky is less than that under private information. So, we have two conflicting forces: On one hand the substitution effect dominates the income effect, so the saving rate in the private-information economy is higher than that under autarky; on the other hand the incentive effect lowers the saving rate. Continuity suggests that there exist values of γ close to but less than zero such that autarky has a lower growth rate compared to the full-information economy, but the private-information economy has a lower growth rate compared to both the full-information economy and autarky. For $\gamma \in (-\infty, 0)$, however, the income effect dominates, so lower certainty equivalent translates to higher saving rate.

Thus, the private-information economy grows slower than autarky.

Inequality and Growth Our model also implies that inequality does not affect expected aggregate growth. To see this, consider two economies—Economy 1 where all agents begin with the same initial promised utility w and Economy 2 where some agents begin with initial promised utility w' and the rest begin with initial promised utility w''. Then, the expected aggregate growth rates in the two economies are the same since, as noted earlier, an agent's expected growth rate of capital does not depend on his promised utility.

5 Decentralization through one-period contracts

We next tackle the question of implementing the private-information economy's efficient allocation. The decentralization can be achieved by enduring relationships between competitive risk-neutral principals, also called component planners in Atkeson and Lucas (1992), and agents. The component planners trade resources with each other intertemporally and we show that there exists an interest rate less than the expected marginal product of capital that clears the market. We demonstrate the first and second welfare theorems for our environment in Appendix C.

As noted earlier, at all dates, every agent's expected output equals expected sum of consumption and investment. So, the decentralization can be accomplished through a sequence of *one-period contracts*. Given a distribution of initial promised utilities, suppose each agent is initially endowed with the capital that the planner would allocate. In our decentralized economy, the agents make decisions on investment that is observable, and they have property rights to the resulting capital stock every period. There is also a mass of risk-neutral competitive insurers.

The insurers and agents can commit to one-period contracts that consist of a net transfer $\tau(k, z)$ to the agent and an investment $\kappa(k, z)$, both of which are conditional on the agent's report z and pre-determined capital k. The observability of investment is crucial: Specifically, an agent with high productivity might want to report low productivity and

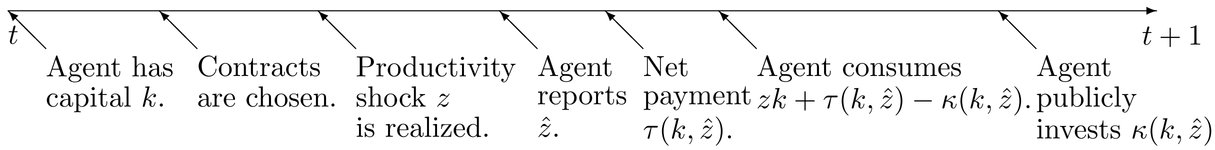

Figure 4: Timeline for the decentralized model with one-period contract

collect a higher net transfer from the insurer, but the insurer would allocate less investment to the agent, which would result in less future capital and, hence, less future consumption.

The timeline in this economy is summarized in Figure 4. Before the productivity shock is realized, competitive insurers offer contracts conditioned on productivity and predetermined capital, and the agents choose contracts (there is no adverse selection); after the report of productivity, the agent is allocated a net transfer and investment.

With risk-neutral competitive insurers, the equilibrium contract maximizes the agent's utility subject to incentive-compatibility and nonegative-profits constraints. Let $v^D(k)$ be the expected utility for an agent with capital k. It is an equilibrium object that is taken as given by the agent and the insurers. The equilibrium contract for an agent with capital k solves the following problem for every z:

Problem C

$$\max_{\tau,\kappa} \int_{\mathbf{Z}} (1-\beta)u(zk+\tau-\kappa) + \beta v^D(\kappa)\mu(dz)$$

subject to

$$\int_{\mathbf{Z}} \tau\mu(dz) \leq 0 \tag{16}$$

$$u(zk+\tau-\kappa) + \beta v^D(\kappa) \geq u(zk+\tau'-\kappa') + \beta v^D(\kappa'). \tag{17}$$

The maximized value of the objective in Problem C is $v^D(k)$ and the maximizers $\tau(k,z)$ and $\kappa(k,z)$ constitute the equilibrium contract.

Let $w(k)$ denote the inverse of the decision rule for k in Proposition 2. Then we have the following:

Proposition 6 *The solution to Problem C is given by functions* $\tau(k,z) = b(w(k),z) + k(v(w(k),z))$ *and* $\kappa(k,z) = k(v(w(k),z))$, *where* b, k, *and* v *are the decision rules from Proposition 2. These functions constitute an equilibrium in the decentralized economy. Moreover,* $v^D(k) = w(k)$.

Proposition 6 implies that in order to satisfy the utility entitlements, the planner can allocate initial capital appropriately and then let the market equilibrium play out. Insurers can replicate the planner's allocation under two conditions: (i) they have an exclusive one-period relationship with the agent, and (ii) the investment is observable.

Phelan (1995) examines a contract with one-sided commitment: The agent does not commit to the long-term contract and is free to recontract his insurance arrangement at the onset of any period. In our model, the one-period insurance contract is such that the expected profit from providing insurance is zero for each firm. Hence, the contract is never rejected by either party at the onset of any period. Our insurance contract thus exhibits two-sided lack of commitment.

6 Concluding remarks

We have examined the efficient allocation in an economy with production and capital accumulation where production is subject to idiosyncratic shocks that are private information. With linear technology, CRRA preferences, and long-term commitment, we established a one-to-one mapping between capital and promised utility: Individuals with higher promised utility are allocated more capital.

Our environment exhibits increasing dispersion in wealth across agents. The one-to-one mapping implies that the value of the autarky option is determined by the capital allocated in the long-term arrangement. Hence, despite the immiseration, agents would prefer the long-term arrangement to autarky since autarky offers no insurance. Furthermore, for some parameter configurations in our model, an agent with a sequence of the lowest possible productivity shock experiences an increase in wealth.

If the set of lifetime utility entitlements is assumed to be bounded below or if individuals' consumption possibilities sets are assumed to be compact, then private-information endowment economies can deliver an invariant distribution of promised utility instead of immiseration. However, since our model exhibits sustained growth, such assumptions do not help eliminate immiseration. For instance, an upper bound on consumption would be incompatible with sustained growth and a lower bound on consumption would not bind.

There is no cross-subsidization across individuals over time in our efficient allocation, so the decentralization can be accomplished through one-period contracts. The expected profit is zero for each firm in the contract. So, the contract exhibits two-sided lack of commitment: The contract is never rejected by the agent or the firm at the onset of any period.

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Appendix A Proofs

A.1 Proof of Lemma 1

Proof. We will denote the allocation generated by an allocation rule by $\{\tilde{k}_t(w, z^{t-1}, \tilde{b}_t(w, z^t)\}_{t=1}^{\infty}$. It can be generated as follows. Let $\tilde{v}_t(w, z^{t-1})$ be the continuation utility of an agent with initial promised utility w and a history of reports z^{t-1}. It is generated recursively by $\tilde{v}_1(w) = w$ and $\tilde{v}_{t+1}(w, z^t) = v_{t+1}(\tilde{v}_t(w, z^{t-1}), z_t)$. Then given $\tilde{v}_t(w, z^{t-1})$, we have $\tilde{k}_t(w, z^{t-1}) = k_t(\tilde{v}_t(w, z^{t-1})); \tilde{b}_t(w, z^t) = b_t(\tilde{v}_t(w, z^{t-1}), z_t)$.

First, we prove that the allocation generated by the allocation rule satisfies promise-keeping constraint (2). We will prove a slightly more general statement that $V_t(w, \sigma^*) = w$ for all $w \in \mathcal{V}$ and t. By repeated application of the promise-keeping constraint (4), we get:

$$
\begin{aligned}
w_t &= (1-\beta)\sum_{s=t}^{T}\beta^{s-t}\int_{\mathbf{Z}^{s-t+1}} u[z_s\tilde{k}_s(w_t, z^{s-t}) + \tilde{b}_s(w_t, z^{s-t+1})]\mu^{s-t+1}(dz^{s-t+1}) \\
&\quad +\beta^{T-t+1}\int_{\mathbf{Z}^{T-t+1}} \tilde{w}_{T+1}(w_t, z^{T-t+1})\mu^{T-t+1}(dz^{T-t+1}) \\
&= (1-\beta)\sum_{s=t}^{\infty}\beta^{s-t}\int_{\mathbf{Z}^{s-t+1}} u[z_s\tilde{k}_s(w_t, z^{s-t}) + \tilde{b}_s(w_t, z^{s-t+1})]\mu^{s-t+1}(dz^{s-t+1}) \\
&\quad +\lim_{T\to\infty}\beta^{T-t+1}\int_{\mathbf{Z}^{T-t+1}} \tilde{w}_{T+1}(w_t, z^{T-t+1})\mu^{T-t+1}(dz^{T-t+1}) \\
&= V_t(w_t, \sigma^*), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.1)}
\end{aligned}
$$

where we used the boundedness condition (6) on expected utility, in the last step.

Next we turn to incentive compatibility. The first step in the proof is to show that truthtelling dominates any strategy of misreporting for finitely many periods. It is immediate from the definition of V_t that

$$
V_t(w_t, \sigma, z^{t-1}) = \int_{\mathbf{Z}^t}\left[(1-\beta)u\left(z_t\tilde{k}_t\left[w_t\right] + \tilde{b}_t\left[w_t, \sigma_t(z^t)\right]\right) + \beta V_{t+1}(\tilde{w}_{t+1}(w_t, \sigma_t(z^t)), \sigma, z^t)\right]\mu^t(dz^t|z^{t-1}), \tag{A.2}
$$

where $\mu^t(\{z^t\}|z^s) = \mu^{t-s}(\{(z_{s+1}, \dots z_t)\})$ if the first $s+1$ elements of z^t coincide with z^s and 0 otherwise. Define $\Sigma_t = \{\sigma \in \Sigma : \sigma_t(z^t) = z_t \text{ if } t \geq T\}$—that is, the set of truthtelling strategies in all periods T and after. Note that $\sigma^* \in \Sigma_0$ and is in fact the only member of Σ_0. For any $\sigma \in \Sigma_T$, using (A.1) and (A.2),

$$
\begin{aligned}
V(\mathbf{P}(w_1), \sigma) &= \sum_{s=1}^{T-1}\beta^{s-1}\int_{\mathbf{Z}^s}(1-\beta)u\left(z_s\tilde{k}_s\left[w_1, \sigma^{s-1}(z^{s-1})\right] + \tilde{b}_s\left[w_1, \sigma^s(z^s)\right]\right)\mu^s(dz^s) + \\
&\quad \beta^{T-1}\int_{\mathbf{Z}^{T-1}} V_T(\tilde{w}_T(w_1, \sigma^{T-1}(z^{T-1})), \sigma, z^{T-1})\mu^{T-1}(dz^{T-1}) \\
&= \sum_{s=1}^{T-1}\beta^{s-1}\int_{\mathbf{Z}^s}(1-\beta)u\left(z_s\tilde{k}_s\left[w_1, \sigma^{s-1}(z^{s-1})\right] + \tilde{b}_s\left[w_1, \sigma^s(z^s)\right]\right)\mu^s(dz^s) + \\
&\quad \beta^{T-1}\int_{\mathbf{Z}^{T-1}} \tilde{w}_T(w_1, \sigma^{T-1}(z^{T-1}))\mu^{T-1}(dz^{T-1}).
\end{aligned}
$$

The rest of the proof is by induction. Strategy σ^* trivially dominates all $\sigma \in \Sigma_0$. Now assume that σ^* dominates all $\sigma \in \Sigma_T$. Consider an arbitrary $\sigma \in \Sigma_{T+1}$. Integrating the t.i.c. constraint (5) over all possible histories and using the promise-keeping constraint (2) we get:

$$
\begin{aligned}
&\int_{\mathbf{Z}^T}\left\{(1-\beta)u\left(z_T\tilde{k}_T(w_1, \sigma^{t-1}(z^{T-1})) + \tilde{b}_T(w_1, \sigma^T(z^T))\right) + \beta\tilde{w}_{T+1}(w_1, \sigma^T(z^T))\right\}\mu^T(dz^T) \leq \\
&\qquad\qquad\qquad\qquad\qquad\qquad \int_{\mathbf{Z}^{T-1}} \tilde{w}_T(w_1, \sigma^{T-1}(z^{T-1}))\mu^{T-1}(dz^{T-1}).
\end{aligned}
$$

Then applying the definition of V, the result above, the inductive assumption, and the fact that (2) is satisfied, we get:

$$\begin{aligned}
V(\mathbf{P}(w_1),\sigma) &= \sum_{s=1}^{T-1}\beta^{s-1}\int_{\mathbf{Z^s}}(1-\beta)u\left(z_s\tilde{k}_s(w_1,\sigma^{s-1}(z^{s-1}))+\tilde{b}_s(w_1,\sigma^s(z^s))\right)\mu^s(dz^s)+\\
&\quad \beta^{T-1}\int_{\mathbf{Z^T}}(1-\beta)u\left(z_T\tilde{k}_T(w_1,\sigma^{T-1}(z^{T-1}))+\tilde{b}_T(w_1,\sigma^T(z^T))\right)\mu^T(dz^T)+\\
&\quad \beta^T\int_{\mathbf{Z}^T}w_{T+1}(w_1,\sigma^T(z^T))\mu^T(dz^T)\\
&\leq \sum_{s=1}^{T-1}\beta^{s-1}\int_{\mathbf{Z^s}}(1-\beta)u\left(z_s\tilde{k}_s(w_1,\sigma^{s-1}(z^{s-1}))+\tilde{b}_s(w_1,\sigma^s(z^s))\right)\mu^s(dz^s)+\\
&\quad +\beta^{T-1}\int_{\mathbf{Z}^{T-1}}w_T(w_1,\sigma^{T-1}(z^{T-1}))\mu^{T-1}(dz^{T-1})\\
&\leq V(\mathbf{P}(w_1),\sigma^*).
\end{aligned}$$

Thus, a strategy of misreporting for T periods is dominated by a strategy of misreporting for $T-1$ periods which in turn is dominated by truthtelling. Then by induction, any strategy in which the agent lies for finite periods is dominated by truthtelling.

Now consider an arbitrary strategy σ. Assume that it dominates σ^*. Then $\exists\ \epsilon > 0$ and T such that

$$\sum_{s=1}^{t}\beta^{s-1}\int_{\mathbf{Z^s}}(1-\beta)u\left(z_s\tilde{k}_s(w_1,\sigma^{s-1}(z^{s-1}))+\tilde{b}_s(w_1,\sigma^s(z^s))\right)\mu^s(dz^s) > w_1+\epsilon$$

for all $t \geq T$. Pick some $T' > T$ such that $\beta^{T'-1}\operatorname{ess\,inf} w_{T'} > -\epsilon/2$. This implies:

$$\beta^{T'-1}\int_{\mathbf{Z}^{T'-1}}w'_T(w_1,\sigma^{T'-1}(z^{T'-1}))\mu^{T'-1}(dz^{T'-1}) > -\epsilon/2.$$

Define σ' to be the strategy identical to σ in all periods up to $T'-1$ and truthtelling afterwards. Then the utility of that strategy is:

$$V(\mathbf{P}(w_1),\sigma') > w_1+\epsilon/2.$$

Thus, misreporting for finite periods strictly dominates truthtelling, which is a contradiction. ■

A.2 Proof of Proposition 1

Proof. The proof is standard. Clearly if the measure on promised utlities ψ gives probability 1 to utility of consuming zero forever, then both $\varphi^*(\psi)=0$ and $T\varphi^*(\psi)=0$. Otherwise, $\varphi^*(\psi)>0$ and $T\varphi^*(\psi)>0$. We need to prove the result only for the latter case.

First, we show that $\varphi^* \geq T\varphi^*$. Suppose not: $\varphi^*(\psi) < T\varphi^*(\psi)$ for some ψ. Then, by definition, there exists an allocation $\mathbf{P}$ that satisfies constraints (1)-(3) and

$$\int_{\mathcal{V}}k_0(w)\psi(dw) < T\varphi^*(\psi).$$

By Proposition D.1, we can assume that it is generated by an allocation rule $\{k_t(w), b_t(w,z), v_{t+1}(w,z)\}_{t=0}^{\infty}$. This allocation rule induces a continuation distribution ψ'. By the aggregate resource constraint (3),

$$-\int_{\mathcal{V}}\int_{\mathbf{Z}}b_1(w,z)\mu(dz)\psi(dw) \geq \int_{\mathcal{V}}k_2(w)\psi'(dw) \geq \varphi^*(\psi').$$

The last inequality follows from the fact that the continuation allocation is attainable and therefore cannot be better than φ^*. Let $\hat{P}=(k_1(w),b_1(w,z),\mathbf{v}_2(w,z))$. Then $\hat{P}$ satisfies the constraints in Problem FE and

$$\int_{\mathcal{V}}k_1(w)\psi(dw) < T\varphi^*(\psi),$$

which is a contradiction. Therefore $\varphi^* \geq T\varphi^*$.

Now we prove that $\varphi^* \leq T\varphi^*$. Suppose not: $T\varphi^*(\psi) < \varphi^*(\psi)$ for some ψ. Let P satisfy the constraints in Problem FE and

$$\int_{\mathcal{V}} k(w)\psi(dw) < T\varphi^*(\psi) + \epsilon < \varphi^*(\psi)$$

for some small enough $\epsilon > 0$. Let ψ' be the continuation distribution induced by P.

The incentive constraint (5) implies that if $k > 0$, then w is above the lower bound of utility. Then there exists a set $M \subseteq \mathcal{V}; \inf V \notin M$ with $\psi(M) > 0$ such that $k(w) > 0$ for $w \in M$. By the Inada conditions, for each $w \in M$, $z \in \mathbf{Z}$, $zk(w) + b(w, z) > 0$. Then for each w and k, it is possible to increase k and $b(z)$ in such a way to satisfy all the constraints. Let $\delta(w)$ be the maximum possible increase in k. This is a Borel-measurable function. Redefine the allocation on M by increasing k at each (w) by $\lambda\delta(w, k)$ and the transfers b to satisfy (2) and (5). For any positive λ, the expected net transfers to the planner increases strictly. We can find a $\lambda > 0$ small enough such that

$$\int_{\mathcal{V}} k(w; \lambda) d\psi < \varphi^*(\psi),$$

where $k(.; \lambda)$ and $b(.; \lambda)$ is the updated allocation of capital, and

$$-\int_{\mathcal{V}} \int_{\mathbf{Z}} b(w, z; \lambda)\mu(dz)\psi(dw) > \varphi^*(\psi').$$

By definition of φ^* there exists some allocation $\mathbf{P}'$ that attains ψ' with capital less than $-\int_{\mathcal{V}} \int_{\mathbf{Z}} b(w, z; \lambda) d\psi d\mu$. Then the allocation $((k_1(.; \lambda), b_1(.; \lambda)), \mathbf{P}')$ dominates $\varphi^*(\psi)$, which is a contradiction. ■

A.3 Proof of Proposition 2

Proof. The proof is long, so we provide a road map. First, we introduce Problem AP2, a generalized form of Problem AP, that allows for net transfers from the agent. In a sequence of claims we show the properties of Problem AP2. Using these properties, we show that if $\varphi(\psi) = A \int_{\mathcal{V}} u^{-1}(w)\psi d(w)$, the decision rule from Problem FE features no net transfers between agents and can be derived from Problem AP. Finally, we show that the function φ^* and the decision rule can be derived from the fixed point of Problem AP.

Problem AP2

$$\hat{k}(w, N; A) = \inf k$$

$$s.t. \;\; [(1-\beta)u(zk + b(z)) + \beta v(z)] \geq \left[(1-\beta)u(zk + b(z')) + \beta v(z')\right], \quad z, z' \in \mathbf{Z}, z \neq z',$$

$$\int_{\mathbf{Z}} \{(1-\beta)u(zk + b(z)) + \beta v(z)\}\mu(dz) = w, \tag{A.3}$$

$$-\int_{\mathbf{Z}} b(z)\mu(dz) \geq N + A \int_{\mathbf{Z}} u^{-1}(v(z))\mu(dz). \tag{A.4}$$

Comparing Problem AP to Problem AP2 we see that $\phi(A) = \hat{k}(u(1), 0; A)$.

Claim 1 *For any feasible allocation in Problem AP2, $b(z_i) \geq b(z_{i+1})$ and $v(z_{i+1}) \geq v(z_i)$ for $i = 1, \ldots, n-1$.*

Claim 2 *In problem AP2 define: $S_{i,j} \equiv (1-\beta)u(z_i k + b(z_i)) + \beta v(z_i) - (1-\beta)u(z_i k + b(z_j)) - \beta v(z_j)$. Suppose that $S_{i,i-1} = 0$, $\forall i = 2, \ldots, n$ and $v(z_{i+1}) \geq v(z_i)$. Then all incentive constraints $S_{i,j} \geq 0$ are satisfied, $b(z_i) \geq b(z_{i+1})$, and $v(z_{i+1}) \geq v(z_i)$ for $i = 1, \ldots, n-1$.*

Claim 3 *In Problem AP2, for every $w \in u(\mathbb{R}_{++})$ and $A > 0$, there exists some $\bar{N} > 0$ such that if $N \leq \bar{N}$, (w, N) is feasible, given A.*

Claim 4 *The minimum in Problem AP2 is attained.*

Claim 5 *Let (k, b, v) be such that $S_{i,i-1} \geq 0$ and $v(z_i) \geq v(z_{i-1})$ for all $i = 2, \ldots n$. Then there exists $(k, \bar{b}, \bar{v})$ such that: (a) All incentive constraints $S_{i,j} \geq 0$ are satisfied and $S_{i,i-1} = 0$, (b) $E\{(1-\beta)u(zk + b(z)) + \beta v(z)\} = E\{(1-\beta)u(zk + \bar{b}(z)) + \beta\bar{v}(z)\}$, (c) $E\{\bar{b}(z) + Au^{-1}(\bar{v}(z))\} \leq E\{b(z) + Au^{-1}(v(z))\}$, and (d) the inequality in (c) is strict if $S_{i,i-1} > 0$ for some i.*

Claim 6 *If $N \geq 0$ at the optimum in Problem AP2, then* (A.4) *binds. Equivalently, $\hat{k}(w, N; A)$ is strictly increasing in N if $N \geq 0$.*

Claim 7 *For $\gamma \neq 0$, $\hat{k}(\lambda w, N\lambda^{\frac{1}{\gamma}}; A) = \lambda^{\frac{1}{\gamma}} k(w, N; A)$ for all $\lambda > 0$. If utility is logarithmic, $\hat{k}(\lambda + w, \exp(\lambda)N; A) = \exp(\lambda)\hat{k}(w, N; A)$ for all $\lambda > 0$.*

Claim 8 *$\hat{k}(w, N; A)$ is convex in N and strictly convex for $N \geq 0$. The vector (k, b, v) that attains the maximum is unique for $N \geq 0$.*

Claim 9 *The solution to Problem AP2 for $(w, 0)$ is $k = \mathbf{k}u^{-1}(w), b(z) = \mathbf{b}(z)u^{-1}(w), v(z) = u(u^{-1}(\mathbf{v}(z))u^{-1}(w))$, where $(\mathbf{k}, \mathbf{b}, \mathbf{v})$ is the solution to Problem AP2 for $(u(1), 0)$.*

Claim 10 *Assume that $\varphi(\psi) = A \int u^{-1}(w)\psi(dw)$ for some $A > 0$. Then the decision rule in Problem FE is given by $k(w) = ku^{-1}(w)$, $b(w, z) = b(z)u^{-1}(w)$, $v(w, z) = u\left(u^{-1}(v(z))u^{-1}(w)\right)$, where $(k, b(z), v(z))$ is the solution to Problem AP. Moreover, $T\varphi(\psi) = \phi(A) \int_{\mathcal{V}} u^{-1}(w)\psi(dw)$.*

Claim 11 *The function $\phi(A)$ is continuous and strictly increasing; $\phi(A_\ell) > A_\ell$ and $\phi(A_h) < A_h$. The set of its fixed points is nonempty and compact. If $\gamma < 0$ or the utility function is logarithmic, there is a unique $A^* \in (A_\ell, A_h)$ such that $\phi(A^*) = A^*$.*

Finally, given these claims, we prove the proposition.

Proof of Claim 1 See Lemma 3 in Thomas and Worrall (1990).

Proof of Claim 2 See Lemma 4 in Thomas and Worrall (1990).

Proof of Claim 3 Clearly if a vector (k, b, v) satisfies the constraints for some (w, N), then it will satisfy the constraints for (w, N') if $N' < N$. Then all we need to show is that (w, N) is feasible for some $N > 0$.

Let $k = 1$, $N = z_1/2$, $b(z) = -(3/4)z_1$, $v(z) = u(z_1/(4A))$. The triple $(k, b(z), v(z))$ delivers utility $\tilde{w} = E(1-\beta)u(z - (3/4)z_1) + \beta u(z_1/(4A))$ and, hence, satisfies the constraints in Problem AP2 for $(\tilde{w}, N)$.

Let $w \neq \inf \mathcal{V}$ be arbitrary. Set $N' = Nu^{-1}(w)/u^{-1}(\tilde{w})$, $k' = ku^{-1}(w)/u^{-1}(\tilde{w})$, $b(z)' = b(z)u^{-1}(w)/u^{-1}(\tilde{w})$, and $v'(z) = u(u^{-1}(v(z))u^{-1}(w)/u^{-1}(\tilde{w}))$. The allocation $(k', b'(z), v'(z))$ satisfies the constraints in Problem AP2 for (w, N'). Moreover, $N' > 0$.

Proof of Claim 4 Take any feasible vector $(\bar{k}, \bar{b}, \bar{v})$. Without loss of generality we can impose the following additional constraint, concentrating on choices that do at least as well as $(\bar{k}, \bar{b}, \bar{v})$: $k \leq \bar{k}$. Feasibility implies that $b_i \geq -z_i k \geq -z_i \bar{k}$.

Since $-Eb_i \geq AEu^{-1}(v_i) + N \geq N$, we see that for all i, $-\mu_i b_i \geq N + \sum_{s \neq i} \mu_s b_s \geq N - \sum_{s \neq i} \mu_s z_s \bar{k}$. Then $b_i \leq \frac{-N + \sum_{s \neq i} \mu_s z_s \bar{k}}{\mu_i}$. $(Ez)\bar{k} \geq -Eb_i \geq AEu^{-1}(v_i) + N \geq \mu_i A u^{-1}(v_i) + N$, therefore $v_i \leq u^{-1}[(Ez\bar{k} - N)/A\mu_i]$.

If $\gamma > 0$, then $v_i \geq 0$. Suppose that $\gamma \leq 0$. Finally, define

$$\psi_i(v_i) = \min \sum_{s=1}^{n} \mu_s c_s + \sum_{s \neq i} \mu_s A u^{-1}(v_s)$$

$$\text{s.t. } (1-\beta) \sum_{s=1}^{n} \mu_s u(c_s) + \beta \sum_{s \neq i} \mu_s v_s \geq w - \mu_i v_i.$$

ψ_i is strictly increasing and $\lim_{v_i \to \inf \mathcal{V}} \psi_i(v_i) = \infty$. This is a problem without an incentive constraint, so for any (k, b, v) satisfying the constraints in AP2, $\sum_{s=1}^{n} \mu_s(z_s k + b_s) + \sum_{s \neq i} \mu_s A u^{-1}(v_s) \geq \psi_i(v_i)$. Moreover,

$$Ez\bar{k} - N \geq Ezk - N \geq \sum_{s=1}^{n} \mu_s(z_s k + b_s) + \sum_{s \neq i} \mu_s A u^{-1}(v_s) \geq \psi_i(v_i).$$

Therefore, $v_i \geq \psi_i^{-1}(Ez\bar{k} - N)$ and hence, v_i must lie in a compact set.

The constraints in Problem AP2 are continuous, so the constraint set is closed. We showed that without loss of generality, we can constrain (k, b, v) to be in a compact subset of $\mathbb{R}^{2N+1}$. The intersection of a closed set and a compact set is compact, so the constraint set is compact. The objective function is continuous, therefore the minimum is attained.

Proof of Claim 5 The proof is by construction. The steps are as below.

1. Set $v'(z_1) = v(z_1)$ and $v'(z_{i+1}) = v'(z_i) + \max\{0, \frac{1-\beta}{\beta}[u(z_{i+1}k + b(z_i)) - u(z_{i+1}k + b(z_{i+1}))]\}$.
2. Set $v''(z_i) = v'(z_i) + E(v(z_i) - v'(z_i))$.
3. Set $b'(z_1) = b(z_1)$ and

$$b'(z_{i+1}) = u^{-1}\left(u(z_{i+1}k + b'(z_i)) + \frac{\beta}{1-\beta}(v''(z_i) - v''(z_{i+1}))\right) - z_{i+1}k.$$

   Note $b'(z_{i+1}) \leq b'(z_i)$, $b'(z) \leq b(z)$, and $S_{i+1,i} = 0$ for all i.

4. Set $b''(z) = b'(z) + \bar{b}$ such that $Eu(zk + b''(z)) = Eu(zk + b(z))$. By concavity of the utility function, $Eb''(z) \leq Eb(z)$; moreover, $S_{i+1,i} \geq 0$ for all i.
5. Set $v'''(z)$ as in steps 1 and 2. Then since $b''(z_{i+1}) \leq b''(z_i)$ and since $S_{i+1,i} \geq 0$ for (k, b'', v''), we have $S_{i+1,i} = 0$ for (k, b'', v''') for all $i = 1, \ldots, n-1$. Also, $AEu^{-1}(v'''(z)) \leq AEu^{-1}(v''(z)) \leq AEu^{-1}(v(z))$.

Set $(k, \bar{b}, \bar{v}) = (k, b'', v''')$. Requirements (b) and (c) in the claim are satisfied by $(k, \bar{b}, \bar{v})$. Incentive compatibility (a) follows from Claim 2. Strict inequality (d) follows from steps 2 and 4, and the strict convexity of the u^{-1} function.

Proof of Claim 6. Suppose that $\gamma > 0$ and $w = 0$. Then the only feasible allocation is $k = 0, b(z) = 0$, and $v(z) = 0$, so (A.4) in Problem AP2 binds trivially. For the rest of the proof we will consider that this is not the case, which implies that $w > \inf \mathcal{V}$.

Let $N > 0$. Constraint (A.4) implies that $-Eb(z_i) \geq N$ and since $b(z_i) \geq -z_i k$ we get $(Ez)k \geq N > 0$, so $k > 0$. If $N = 0$ and $k = 0$, then (A.4) implies that $b(z_i) = 0$, $u^{-1}(v(z_i)) = 0$, which contradicts the promise-keeping constraint (A.3). So it must be that $k > 0$.

Suppose that (A.4) is slack and let (k, b, v) be a minimizer in Problem AP2 (it exists by Claim 4.) We will derive a new allocation that satisfies the constraints in Problem AP2 and lowers the objective function. Since $b(z_{i+1}) \geq b(z_i)$ (Claim 1), $v(z_{i+1}) \geq v(z_i)$.

Let $\epsilon \equiv -\frac{N + \int_{\mathbf{Z}}(Au^{-1}(v(z)) + b(z))\mu(dz)}{Ez} > 0$. Set $k' = k - \epsilon$ and $b'(z) = b(z) + z\epsilon$. Then for (k', b', v), $S_{i+1,i} > 0$ for $i = 1, \ldots, n-1$. Moreover, (k', b', v) satisfies (A.3) and (A.4). Then by Claim 5, $\exists$ $(k', \bar{b}, \bar{v})$ that satisfies the constraints in Problem AP2 and reduces the objective function, a contradiction.

Proof of Claim 7. Let (k, b, v) be the solution to Problem AP2 given $(w, N; A)$. Consider $\gamma \neq 0$ first. Then $(k\lambda^{\frac{1}{\gamma}}, b_i\lambda^{\frac{1}{\gamma}}, v_i\lambda)$ satisfies the constraints in Problem AP2 for $(\lambda w, N\lambda^{\frac{1}{\gamma}})$. Thus, $\hat{k}(\lambda w, N\lambda^{\frac{1}{\gamma}}; A) \leq \lambda^{\frac{1}{\gamma}} k(w, N; A)$. Similarly, $\hat{k}(w, N; A) = \hat{k}((1/\lambda)\lambda w, (1/\lambda)^{\frac{1}{\gamma}} N\lambda^{\frac{1}{\gamma}}; A) \leq (1/\lambda)^{\frac{1}{\gamma}} \hat{k}(\lambda w, N\lambda^{\frac{1}{\gamma}}; A)$. The two inequalities imply the result.

Now, consider $\gamma = 0$. Then, $(\exp(\lambda)k, \exp(\lambda)b_i, v_i + \lambda)$ satisfies the constraints in Problem AP2 for $(\lambda w, \exp(\lambda)N)$. Thus, $k(\lambda + w, \exp(\lambda)N; A) \leq \exp(\lambda)\hat{k}(w, , N; A)$. Similarly, $\hat{k}(w, N; A) = \hat{k}(-\lambda + (w + \lambda), \exp(-\lambda)(N\exp(\lambda)); A) \leq \exp(-\lambda)\hat{k}(w+\lambda, \exp(\lambda)N; A)$. Combining the two inequalities yields the result.

Proof of Claim 8. Let (k', b', v') and (k'', b'', v'') be minimizers for N' and N'', respectively. Let $\lambda \in (0, 1)$. Define $k''' = \lambda k' + (1-\lambda)k''$ and $v'''(z) = \lambda v'(z) + (1-\lambda)v''(z)$. Define $b'''(z)$ implicitly by $u(zk''' + b'''(z)) = \lambda u(zk' + b'(z)) + (1-\lambda)u(zk'' + b''(z))$.

The vector (k''', b''', v''') satisfies the promise-keeping constraint (A.3). Concavity of the utility function implies $z_i k''' + b'''(z_i) \leq \lambda(z_i k' + b'(z_i)) + (1-\lambda)(z_i k'' + b''(z_i))$, so $b'''(z_i) \leq \lambda b'(z_i) + (1-\lambda)b''(z_i)$. (The inequality is strict if $k' \neq k''$.) Therefore:

$$
\begin{aligned}
-\int_{\mathbf{Z}} b'''(z)\mu(dz) &\geq -\lambda \int_{\mathbf{Z}} b'(z)\mu(dz) - (1-\lambda)\int_{\mathbf{Z}} b''(z)\mu(dz) \\
&\geq \lambda \left[N' + A\int_{\mathbf{Z}} u^{-1}(v'(z))\mu(dz)\right] + (1-\lambda)\left[N'' + A\int_{\mathbf{Z}} u^{-1}(v''(z))\mu(dz)\right] \\
&\geq N''' + A\int_{\mathbf{Z}} u^{-1}(v'''(z))\mu(dz).
\end{aligned}
$$

The last inequality follows from the fact that $u^{-1}(x)$ is convex on the relevant range of x. Therefore, (A.4) is satisfied. Next, note

$$
\begin{aligned}
(1-\beta)u(z_{i+1}k''' + b'''(z_{i+1})) + \beta v'''(z_{i+1}) &= \lambda[(1-\beta)u(z_{i+1}k' + b'(z_{i+1})) + \beta v'(z_{i+1})] + \\
&\quad (1-\lambda)[(1-\beta)u(z_{i+1}k' + b'(z_{i+1})) + \beta v'(z_{i+1})] \\
&= (1-\beta)(\lambda u(z_{i+1}k' + b'(z_i)) + \\
&\quad (1-\lambda)u(z_{i+1}k'' + b''(z_i))) + \beta v'''(z_i) \\
&\geq (1-\beta)u(z_{i+1}k''' + b'''(z_i)) + \beta v'''(z_i),
\end{aligned}
$$

where the last inequality follows from the fact that u displays decreasing absolute risk aversion. Therefore for (k''', b''', v'''), $S_{i+1,i} \geq 0$ for all $i = 1\ldots, n-1$. Then by Claim 5 we can modify (k''', b''', v''') further to ensure that all of the incentive constraints in Problem AP2 bind. Therefore $\hat{k}(w, N'''; A) \leq k' = \lambda\hat{k}(w, N'; A) + (1-\lambda)\hat{k}(w, N''; A)$.

Next we show that $\hat{k}(w, N; A)$ is strictly convex in N for $N \geq 0$. If $0 \leq N' < N''$, Claim 6 implies that $k'' > k'$. We showed that $b'''(z) < \lambda E b'(z) + (1-\lambda)E b''(z)$, which implies that (A.4) will be slack. Therefore we can reduce k''' further, which proves that $\hat{k}$ is strictly convex if $N > 0$.

Finally, assume that $N \geq 0$ and that there are two vectors of choice variables attaining the minimum. Set $\lambda = 1/2$ and perform the variation described above. This variation will make (A.4) slack and keep the objective function constant, which is a contradiction.

Proof of Claim 9. By direct inspection, we see that (k, b, v) satisfies the constraints in Problem AP2. Moreover, it is a unique solution to Problem AP2 by Claim 7 and Claim 8.

Proof of Claim 10. If $\varphi(\psi) = A\int_{\mathcal{V}} u^{-1}(w)\psi(dw)$, then the operator in Problem FE can be expressed as:

$$
(T\varphi)(\psi) = \inf_{n(w)} \int_{\mathcal{V}} \hat{k}(w, n(w); A)\psi(dw)
$$

$$
\text{s.t.} \quad \int_{\mathcal{V}} n(w)\psi(dw) \geq 0. \tag{A.5}
$$

The function $\hat{k}(w, N; A)$ is strictly increasing if $N > 0$. Therefore, (A.5) must bind, because otherwise we can reduce $n(w)$ on a set with positive measure where $n(w) > 0$, reducing total required capital.

Define the probability measure ψ' by

$$
\psi'(M) = \frac{\int_M u^{-1}(w)\psi(dw)}{\int_{\mathcal{V}} u^{-1}(w)\psi(dw)},
$$

where $M \subseteq \mathcal{V}$ is an arbitrary Borel set. The function $\frac{u^{-1}(w)}{\int_{\mathcal{V}} u^{-1}(w)\psi(dw)}$ is the Radon-Nikodym derivative of ψ' with respect to ψ. Then by the change-of-variables theorem (Shorack (2000), Theorem 2.2) for any integrable function $f(w)$ we have:

$$
\int f(w)\frac{u^{-1}(w)}{\int_{\mathcal{V}} u^{-1}(w)\psi(dw)}\psi(dw) = \int f(w)\psi'(dw).
$$

By Claim 7, $\hat{k}(w, N; A) = u^{-1}(w)\hat{k}(u(1), N/u^{-1}(w); A)$. So, using the new measure,

$$T\varphi(\psi) = \inf_{n(w)} \left[\int_{\mathcal{V}} u^{-1}(w)\psi(dw)\right] \int_{\mathcal{V}} \hat{k}(u(1), n(w)/u^{-1}(w); A)\psi'(dw)$$

$$\text{s.t.} \quad \left[\int_{\mathcal{V}} u^{-1}(w)\psi(dw)\right] \int_{\mathcal{V}} n(w)/u^{-1}(w)\psi'(dw) = 0.$$

By the fact that $\hat{k}(u(1), x; A)$ is convex in x and strictly convex for $x > 0$, and that ψ' is a probability measure, it follows that it is optimal to set $n(w)/u^{-1}(w) = 0$ a.e. $[\psi']$. But $\psi \ll \psi'$, so it is optimal to set $n(w) = 0$ a.e. $[\psi]$.

So we showed that it is optimal to set $k(w), b(w, z), v(w, z)$ equal to the solution to Problem AP2, for $(w, 0)$ a.e. Then by Claim 9, the decision rule is $k(w) = ku^{-1}(w), b(w, z) = b(z)u^{-1}(w)$, and $v(w, z) = u(u^{-1}(v(z))u^{-1}(w))$, where (k, b, v) is the solution to Problem AP. Plugging the decision rule into the objective function in Problem AP, we immediately get:

$$T\varphi(\psi) = \phi(A)\int_{\mathcal{V}} u^{-1}(w)\psi(dw).$$

Proof of Claim 11.

Continuity of the function ϕ follows directly from the Theorem of the Maximum.

A_ℓ is the fixed point of a relaxed version of Problem AP without the incentive constraint (11). The relaxed problem has a unique solution such that $v_\ell(z) = v_\ell(z')$, $zk_\ell + b_\ell(z) = z'k_\ell + b_\ell(z')$, which violates (11). Hence, all allocations that satisfy the constraints in Problem AP have $k > k_\ell$. Then Claim 4 implies that $\phi(A_\ell) > A_\ell$.

Next, consider $\phi(A_h)$. A_h is the fixed point of a problem in which the resource constraint (13) holds state by state. Let $c(z) = zk+b(z)$. Then $c(z)$ and $v(z)$ maximize the problem $(1-\beta)u(c(z))+\beta v(z)$ subject to $c(z) + A_h u^{-1}(v(z)) \leq zk$ and these maximizers are unique. Consider the incentive constraint $S_{n,n-1}$. If the agent misreports, he consumes $c(z_{n-1}) + (z_n - z_{n-1})k$ and receives $v(z_{n-1})$ continuation utility. Clearly $v(z_{n-1}) < v(z_n)$ and $c(z_{n-1}) + (z_n - z_{n-1})k + A_h u^{-1}(v(z_{n-1})) \leq z_n k$. Therefore,

$$(1-\beta)u((z_n - z_{n-1})k + b(z_{n-1})) + \beta v(z_{n-1}) < (1-\beta)u(z_n k + b(z_n)) + \beta v(z_n).$$

So, $S_{n,n-1} > 0$, and Claims 5 and 6 imply that there is a dominating allocation. This implies $\phi(A_h) < A_h$.

Then, continuity of ϕ, and $\phi(A_\ell) > A_\ell, \phi(A_h) < A_h$ imply that there is at least one fixed point. The compactness of the set of fixed points follows from the fact that the domain of $\phi(A)$ is compact. By Claim 6, the resource constraint (13) is binding, which implies $\phi(A)$ is strictly increasing.

Finally, we show that for $\gamma \leq 0$, there is a unique fixed point $A^* \in (A_\ell, A_h)$. First, consider the case with log utility. Let $A' > 0$ and $A > 0$ be arbitrary and let $(k^*, b^*(z), v^*(z))$ be a solution to Problem AP given A. Define $k' = k^*\left(\frac{A'}{A}\right)^\beta$, $b'(z) = b^*(z)\left(\frac{A'}{A}\right)^\beta$ and $v'(z) = v^*(z) + (\beta - 1)(\log(A') - \log(A))$. Direct inspection shows that $(k', b'(z), v'(z))$ satisfies the constraints Problem AP for A'. Therefore

$$\log(\phi(A')) \leq \log k' = \log k^* + \beta(\log(A') - \log(A)) = \log(\phi(A)) + \beta(\log(A') - \log(A)).$$

Exchanging A' and A, we get that $\log(\phi(A')) = \log(\phi(A)) + \beta(\log(A') - \log(A))$. Setting $A = 1$, we get that $\log(\phi(A)) = \log(\phi(1)) + \beta \log(A)$ or $\phi(A) = \phi(1)A^\beta$, so the unique fixed point is $A^* = [\phi(1)]^{\frac{1}{1-\beta}}$.

Now suppose that $\gamma < 0$. For any (k, b, v), define $\tilde{b}(z_i) = b(z_i)/k; \tilde{v}(z_i) = v(z_i)/k^\gamma$. Clearly, $(k, \tilde{b}, \tilde{v})$ uniquely determine (k, b, v). Abusing notation slightly, we will refer to $(k, \tilde{b}, \tilde{v})$ as a possible vector of choice variables. Then the constraints can be rewritten as:

$$(1-\beta)\frac{(z_i + \tilde{b}(z_i))^\gamma}{\gamma} + \beta\tilde{v}(z_i) \geq (1-\beta)\frac{(z_i + \tilde{b}(z_j))^\gamma}{\gamma} + \beta\tilde{v}(z_j) \tag{A.6}$$

$$-E\tilde{b}(z_i) \geq AE(\gamma\tilde{v}(z_i))^{\frac{1}{\gamma}} \tag{A.7}$$

$$\left((1-\beta)E\frac{(z_i + \tilde{b}(z_i))^\gamma}{\gamma} + \beta E\tilde{v}(z_i)\right)k^\gamma = \frac{1}{\gamma}. \tag{A.8}$$

Equations (A.6), (A.7), and (A.8) are the incentive, resource and promise-keeping constraints, respectively.

Let $A_2 > A_1 > 0$ be arbitrary and let $(k, \tilde{b}, \tilde{v})$ be the solution, given A_1. Define $\tilde{b}(z)' = \tilde{b}(z), \tilde{v}(z)' = (A_1/A_2)^\gamma \tilde{v}(z)$. Given $(\tilde{b}', \tilde{v}')$, (A.8) determines k' uniquely. Note that

$$\begin{aligned}(1-\beta)E(z_i + \tilde{b}_i')^\gamma + \gamma\beta E\tilde{v}_i' &= (1-\beta)E(z_i + \tilde{b}_i)^\gamma + (A_1/A_2)^\gamma \gamma\beta E\tilde{v}_i \\ &< (A_1/A_2)^\gamma \left[(1-\beta)E(z_i + \tilde{b}_i)^\gamma + \gamma\beta E\tilde{v}_i\right].\end{aligned}$$

Then, (A.8) implies

$$\begin{aligned}k' &= \left[(1-\beta)E(z_i + \tilde{b}_i')^\gamma + \gamma\beta E\tilde{v}_i'\right]^{-\frac{1}{\gamma}} \\ &< (A_2/A_1)\left[(1-\beta)E(z_i + \tilde{b}_i)^\gamma + \gamma\beta E\tilde{v}_i\right]^{-\frac{1}{\gamma}} = (A_2/A_1)k.\end{aligned}$$

Constraints (A.7) and (A.8) are satisfied for $(k', \tilde{b}', \tilde{v}')$ at A_2. Next, we consider the incentive constraint (A.6) for an arbitrary $i \geq 2$:

$$\begin{aligned}\beta(\tilde{v}'(z_i) - \tilde{v}'(z_{i-1})) &= \beta(A_1/A_2)^\gamma(\tilde{v}(z_i) - \tilde{v}(z_{i-1})) \\ &> \beta(\tilde{v}(z_i) - \tilde{v}(z_{i-1})) \\ &\geq (1-\beta)\left(\frac{(z_i + \tilde{b}(z_{i-1}))^\gamma}{\gamma} - \frac{(z_i + \tilde{b}(z_i))^\gamma}{\gamma}\right) \\ &= (1-\beta)\left(\frac{(z_i + \tilde{b}'(z_{i-1}))^\gamma}{\gamma} - \frac{(z_i + \tilde{b}'(z_i))^\gamma}{\gamma}\right),\end{aligned}$$

which implies that $S_{i,i-1} > 0$. Then by Claim 5, there exists $(k'', \tilde{b}'', \tilde{v}'')$ that satisfies the constraints in Problem AP and $k'' < k' < (A_2/A_1)k$. This immediately implies $\phi(A_2) \leq k'' < (A_2/A_1)k = (A_2/A_1)\phi(A_1)$.

We proved above the existence of at least one fixed point. Suppose that there are more than one: $A_1^* < A_2^*$. Then we have that

$$A_2^* = \phi(A_2^*) < (A_2^*/A_1^*)\phi(A_1^*) = (A_2^*/A_1^*)A_1^* = A_2^*,$$

which is clearly a contradiction.

Proof of the proposition. Claim 11 states that (a) ϕ is strictly increasing, (b) $\phi(A_\ell) > A_\ell$, and (c) $\phi(A_h) \leq A_h$. Then by repeatedly applying ϕ, we get:

$$A_\ell < \phi(A_\ell) < \phi^{(r)}(A_\ell) < \phi^{(r)}(A_h) \leq A_h.$$

Since ϕ is monotone, $\phi^{r+1}(A_\ell) \geq \phi^r(A_\ell)$ and $\phi^{r+1}(A_h) \leq \phi^r(A_h)$. Therefore the limits $A_\ell^* \equiv \lim_{r\to\infty}\phi^{(r)}(A_\ell)$ and $A_h^* \equiv \lim_{r\to\infty}\phi^{(r)}(A_h)$ exist, with $A_\ell^* \leq A_h^*$. Since ϕ is continuous, $\phi(A_\ell^*) = A_\ell^*$ and $\phi(A_h^*) = A_h^*$. Therefore, Claim 10 implies that $\varphi_i(\psi) = A_i^* \int u^{-1}(w)\psi(dw), i = \ell, h$ is a solution to Problem FE.

For any allowable distribution ψ, $\varphi_\ell(\psi) < \varphi^*(\psi) < \varphi_h(\psi)$. Since the operator T is strictly monotone and φ^* is a fixed point by Proposition 1, $(T^n\varphi_\ell)(\psi) < \varphi^*(\psi) < T^n\varphi_h)(\psi)$. By Claim 10 this implies $\phi^{(n)}(A_\ell)\int u^{-1}(w)\psi(dw) < \varphi^*(\psi) < \phi^{(n)}(A_h)\int u^{-1}(w)\psi(dw)$. Taking limits with respect to n implies that $A_\ell^* \int u^{-1}(w)\psi(dw) \leq \varphi^*(\psi) \leq A_h^* \int u^{-1}(w)\psi(dw)$.

Suppose that the equation $\phi(A) = A$ has only one solution, A^*. This implies $A_\ell^* = A_h^* = A^*$. Then the inequalities above imply that $\varphi^*(\psi) = A^* \int u^{-1}(w)\psi(dw)$.

On the other hand, suppose that $\phi(A) = A$ has more than one solution. Claim 11 shows that this is only possible if $\gamma > 0$. Let $A^* = A_\ell^*$. Let (k, b, v) be the solution to Problem AP for A_ℓ^*. Lemma 1 ensures that the allocation induced by (k, b, v) will satisfy promise-keeping constraint (2). The boundedness condition (6) is satsified if $\beta Ev/u(1) < 1$. The case $\beta Ev/u(1) \geq 1$ violates the assumption that $\beta(Ez)^\gamma < 1$. Since $0 \leq v(z_1) = \inf\{v(z)\} < Ev(z)$ and (7) is also satisfied, the allocation is incentive compatible. The allocation (k, b, v) satisfies (13) and, hence, the aggregate resource constraint (3). Thus, (k, b, v) is incentive compatible, delivers the promised utility, and satisfies the resource constraint. Therefore $A_\ell^* \int u^{-1}(w)\psi(dw) \geq \varphi^*(\psi)$ for every feasible ψ. However, we also proved that $A_\ell^* \int u^{-1}(w)\psi(dw) \leq \varphi^*(\psi)$, which establishes the equality.

Finally, the form of the allocation rule follows from Claim 10. ■

A.4 Proof of Proposition 3

Proof. From Proposition 2, $k_t(w, z^{t-1}) = ku^{-1}(w_t(w, z^{t-1}))$ and $k_{t+1}(w, z^t) = ku^{-1}(w_{t+1}(w, z^t))$. Similarly, $w_{t+1}(w, z^t) = v(w_t(z^{t-1}), z_t) = u[u^{-1}(w_t(w, z^{t-1}))u^{-1}(v(z)]$. Then $k_{t+1}(w, z^t) = ku^{-1}(w_t(w, z^{t-1}))u^{-1}(v(z_t)) = k_t(w, z^{t-1})u^{-1}(v(z_t))$.

Taking logs, $\log(k_{t+1}(w, z^t)) = \log(\tilde{k}_t(w, z^{t-1})) + \log(u^{-1}(v(z_t)))$. Since the shocks are independent across time and agents, $\operatorname{Var}\log(k_{t+1}) = \operatorname{Var}\log(k_t) + \operatorname{Var}(\log u^{-1}(v(z)) = \operatorname{Var}\log(k_1) + (t-1)\operatorname{Var}(\log u^{-1}(v(z))$. We have shown that $v(z_1) < v(z_2) \cdots < v(z_n)$, so $\operatorname{Var}(\log u^{-1}(v(z)) > 0$, which implies the result. ■

A.5 Proof of Proposition 4

Proof. Under private information, $w(k) = u(k/A^*)$; similarly for autarky $v^{aut}(k) = u(k/A_h)$. Claim 11 of the proof of Proposition 2 implies that $A^* < A_h$. Then, since u is strictly increasing, $w(k) > v^{aut}(k)$. ■

A.6 Proof of Proposition 5

Proof. The growth rate of capital allocated to an agent is $E\mathbf{k}(\mathbf{v}(w, z))/\mathbf{k}(w) = Eu^{-1}(v(z))$, where $v(z)$ is the solution to Problem AP for the correct A. Clearly, this is also the aggregate growth rate. We will study this problem to prove the proposition.

Let $\tilde{b}(z_i) = b(z_i)/k$ and $\tilde{v}(z_i) = v_i(z)/k^\gamma$ or $\tilde{v}(z_i) = v(z_i) - \log k$ in the log utility case. Abusing notation slightly, we will denote $\tilde{v}_i = \tilde{v}(z_i); \tilde{b}_i = \tilde{b}(z_i)$. Then constraints (11) and (13) look like:

$$(1-\beta)u(z_i + \tilde{b}_i) + \beta\tilde{v}_i \geq (1-\beta)u(z_i + \tilde{b}_j) + \beta\tilde{v}_j, \tag{A.9}$$

$$-\sum_i \mu_i \tilde{b}_i \geq A \sum_i \mu_i u^{-1}(\tilde{v}_i). \tag{A.10}$$

The promise-keeping constraint (12) can be written as

$$k = \left(\frac{\sum_i \mu_i[(1-\beta)u(z_i + \tilde{b}_i) + \tilde{\beta} v_i]}{u(1)}\right)^{-\frac{1}{\gamma}}, \quad \text{for } \gamma \neq 0,$$

and

$$k = \exp\left(-\sum_i \mu_i[(1-\beta)u(z_i + \tilde{b}_i) + \beta\tilde{v}_i]\right), \quad \text{for } \gamma = 0.$$

Note that $(x/u(1))^{-\frac{1}{\gamma}}$ and $\exp(-x)$ are decreasing in x. So, the optimal $(\tilde{b}_i, \tilde{v}_i)$, $i = 1, \ldots, n$, solve the following problem:

$$\max \sum_i \mu_i[(1-\beta)u(z_i + \tilde{b}_i) + \beta\tilde{v}_i] \tag{A.11}$$

$$\text{subject to } (A.9), (A.10).$$

There are $n(n-1)$ incentive constraints, but by Claim 2 only $n-1$ are binding—the constraints that prevent an agent with shock z_i from reporting z_{i-1}. Earlier, we called this incentive constraint $S_{i,i-1}$ with associated Lagrangian multiplier η_i. For convenience of notation, we will introduce fictitious constraints $S_{1,0}$ and $S_{n+1,n}$ with multipliers $\eta_1 = \eta_{n+1} = 0$. Let λ be the multiplier of the resource constraint (13). Then we have the following first-order conditions:

$$\mu_i u'(z_i + \tilde{b}_i) + \eta_i u'(z_i + \tilde{b}_i) - \eta_{i+1}u'(z_{i+1} + \tilde{b}_i) - \frac{\mu_i\lambda}{1-\beta} = 0, \tag{A.12}$$

$$\beta(\mu_i + \eta_i - \eta_{i+1}) - \mu_i\lambda A \frac{d}{d\tilde{v}_i} u^{-1}(\tilde{v}_i) = 0. \tag{A.13}$$

Since $\eta_2 > 0, \ldots \eta_n > 0$, and $u'(z_{i+1} + \tilde{b}_i) < u'(z_i + \tilde{b}_i)$ we have the following for $i = 1, \ldots, n-1$:

$$\begin{aligned} \mu_i\lambda &= \mu_i(1-\beta)u'(z_i + \tilde{b}_i) + \eta_i(1-\beta)u'(z_i + \tilde{b}_i) - \eta_{i+1}(1-\beta)u'(z_{i+1} + \tilde{b}_i) \\ &> (\mu_i + \eta_i - \eta_{i+1})(1-\beta)u'(z_i + \tilde{b}_i). \end{aligned} \tag{A.14}$$

We can combine (A.13) and (A.14) as follows:

$$\frac{1}{(1-\beta)u'(z_i+\tilde{b}_i)} > \frac{A\frac{d}{d\tilde{v}_i}u^{-1}(\tilde{v}_i)}{\beta},\ i=1,\ldots,n-1, \tag{A.15}$$

with equality for $i=n$.

The solutions $\tilde{b}_i, \tilde{v}_i$, and the fact that A is the minimal capital in Problem AP imply that the growth rate of the private-information economy is $AEu^{-1}(\tilde{v}(z))$.

Let $\Gamma_i \equiv z_i + \tilde{b}_i + Au^{-1}(\tilde{v}_i)$. When $\eta_{i+1}=0$, (A.12) and (A.13) imply that $Au^{-1}(\tilde{v}(z)) = s(A)\Gamma_i$, where

$$s(A) \equiv \frac{[\beta A^{-\gamma}/(1-\beta)]^{\frac{1}{1-\gamma}}}{1+[\beta A^{-\gamma}/(1-\beta)]^{\frac{1}{1-\gamma}}}. \tag{A.16}$$

Since $\eta_{n+1}=0$, $Au^{-1}(\tilde{v}(n)) = s(A)\Gamma_n$. Inequality (A.15) implies that $Au^{-1}(\tilde{v}_i') < s\Gamma_i$ for $i<n$. Constraint (13) implies that $E\Gamma_i = \bar{z}$. Then, the growth rate of the economy with private information is $EAu^{-1}(\tilde{v}(z)) < sE\Gamma(z) = s\bar{z}$.

By similar reasoning, we can establish that under either full information or autarky, $k_z'/k = s(A)\Gamma(z)$ with $A=A_h$ for autarky and $A=A_\ell$ for the full-information case, and $\Gamma(z)=z$ for autarky and $\Gamma(z)=\bar{z}$ for full information. So the growth rates in these respective economies are $s(A_h)\bar{z}$ and $s(A_\ell)\bar{z}$.

Then since $A^* > A_\ell$ for $\gamma \geq 0$, (A.16) implies $s(A^*) \leq s(A_\ell)$ with strict inequality for $\gamma=0$, so $g_{pi} < g_{fi}$ for all $\gamma \geq 0$. Since $s(A^*)/s(A_\ell)$ is arbitrarily close to 1 for γ sufficiently close to 0, the private-information economy grows slower than the full-information economy for all $\gamma \in (\underline{\gamma}, 1)$ with $\underline{\gamma} < 0$.

The comparison with the autarky savings rate s_{au} is similar. For $\gamma \leq 0$, $A^* < A_h$ implies that $s(A^*) \leq s(A_h)$ and, hence, $g_{pi} < g_{au}$. ■

A.7 Proof of Proposition 6

Proof. The functional forms of $\tau(k,z)$ and $\kappa(k,z)$ imply that for every agent's reporting strategy, the consumption process will be the same as in the efficient allocation with a starting utility $w(\kappa(k,z))$. So, from the next period on, it would be optimal for the agent to follow truthtelling and expected continuation utility is $w(\kappa(k,z))$.

The functions k, b, and v satisfy the constraints in Problem AP. Therefor, $\tau(k,z)$ and $\kappa(k,z)$ satisfy (16) and (17).

Now assume that for an agent with some k there is a contract (τ', κ') that is better for the agent and still satisfies the constraints (16) and (17), i.e.,

$$\int_{\mathbf{Z}} \left\{(1-\beta)u(zk+\tau'(k,z)-\kappa'(k,z)) + \beta w(\kappa'(k,z))\right\}\mu(dz) = \tilde{w} > w(k)$$

Define $k' = k/u^{-1}(\tilde{w})$, $b'(z) = b(z)/u^{-1}(\tilde{w})$ and $v'(z) = u[u^{-1}(v(z))/u^{-1}(\tilde{w})]$. Then (k', b', v') satisfies the constraints in Problem AP and $k' < A^*$, which is a contradiction since A^* is a fixed point for Problem AP.
■

Appendix B Aggregate shocks

In this section, we introduce an aggregate shock that affects the probability distribution of idiosyncratic productivity. Assume that there is a *publicly* observed aggregate shock s that takes m possible values $s_1, \ldots s_m$ and evolves according to some Markov process with a transition matrix Q. The aggregate shock s affects the probability distribution of each agent's idiosyncratic productivity shock z, with $z \sim \mu(.|s)$.

For the individual, the state variable is the pair of promised utility w and the aggregate shock s. The allocation rule consists of capital, transfers conditioned on the productivity report, and continuation utility, conditioned on the report and next period's aggregate shock. Then it must satisfy:

$$\int \left\{(1-\beta)u(zk(w,s)+b(w,z,s)) + \beta \sum_{s'} Q(s,s')v(w,z,s,s')\right\} \mu(dz|s) = w \tag{B.1}$$

$$\begin{aligned}(1-\beta)u(zk(w,s)+b(w,z,s)) + \beta \sum_{s'} Q(s,s')v(w,z,s,s') &\geq \\ (1-\beta)u(zk(w,s)+b(w,z',s)) + \beta \sum_{s'} Q(s,s')v(w,z',s,s'). \end{aligned} \tag{B.2}$$

Equation (B.1) is the promise-keeping constraint and (B.2) is the incentive constraint. If we add the correct transversality condition on discounted utility the two conditions above are sufficient.

In this economy the idiosyncratic productivity is not i.i.d. Typically, persistent shocks in an environment with private information make incentive problems infeasible. We can handle this problem, because the persistency is public. The *conditional* distribution of idiosyncratic productivity in the next period is independent of current idiosyncratic productivity.

For the planner, the state variable is the distribution of promised utilities and the aggregate state of the economy. So, as in Section 3, let $\varphi(\psi, s)$ be the infimum of the required capital necessary to satisfy the promised utilities. It will solve the following functional equation:

$$\varphi(\psi,s) = \inf_{(k(w,s),b(w,z,s),v(w,z,s,s'))} \int_{\mathcal{V}} k(w,s)\psi(dw) \tag{B.3}$$

$$\text{s.t. } (B.1), (B.2)\ \forall w \in \mathcal{V}$$

$$-\int_{\mathcal{V}} \left[\int_{\mathbf{Z}} b(w,z,s)\mu(dz|s)\right] \psi(dw) \geq \varphi(\psi'_{s'}, s')\ \forall s'. \tag{B.4}$$

The only substantive difference is in the resource constraint (B.4). The utility promises are conditioned on the next period's aggregate shock, but current transfers cannot be conditioned on the next period's aggregate shock (as they are not observed in advance), so the capital stock accumulated at the end of the period must be sufficient to attain the utility promises for all future aggregate shocks.

The main scaling result still holds.

Proposition B.1 *There exists a vector $A_1^*, \ldots A_m^*$ such that φ^* is given by:*

$$\varphi^*(\psi,s) = A_s^* \int_{\mathcal{V}} u^{-1}(w)\psi(dw)$$

and the decision rule is:

$$k(w,s) = k(s)u^{-1}(w), b(w,z,s) = b(z,s)u^{-1}(w), v(w,z,s,s') = u(u^{-1}(v(z,s,s'))u^{-1}(w))$$

and $(k(s), b(z,s), v(z,s,s'))$ is the decision rule for $(u(1), s)$.

The reason that the convenient characterization is preserved in this environment is that the aggregate shock has a symmetric effect on all agents' idiosyncratic productivity.

B.1 Proof

Proof of Proposition B.1. Denote by $\hat{T}$ the operator defined by (B.1) - (B.4). By the same arguments as in the case without aggregate uncertainty, an analog of Claim 10 of the proof of Proposition 2 is proved, implying that if $\varphi(\psi, s) = A_s \int u^{-1}(w)\psi(dw)$, then $\hat{T}\varphi(\psi, s) = A'_s \int u^{-1}(w)\psi(dw)$.

With this in mind, define the following version of the auxiliary problem:

$$\hat{\phi}(A_1, A_2 \ldots A_n; s) = \inf_{(k,b(z),v(z,s))} k$$

$$\text{subject to } (1-\beta)u(zk+b(z)) + \beta \sum_{s'} Q(s,s')v(z,s') \geq$$

$$(1-\beta)u\left(zk+b(z')\right) + \beta \sum_{s'} Q(s,s')v(z',s'), \quad z, z' \in \mathbf{Z},$$

$$\int_{\mathbf{Z}} \left\{(1-\beta)u\left(zk+b(z)\right) + \beta \sum_{s'} Q(s,s')v(z,s')\right\} \mu(dz) = u(1),$$

$$-\int_{\mathbf{Z}} b(z)\mu(dz) \geq A_i \int_{\mathbf{Z}} u^{-1}(v(z,s_i))\mu(dz), \forall i = 1, \ldots n.$$

We denote a vector $(A_1, \ldots A_n)$ by $\vec{A}$ and let $\hat{\phi}(\vec{A}) = (\hat{\phi}(\vec{A}, s_1), \ldots \hat{\phi}(\vec{A}; s_m)$. In this proof the relations $\leq, <$ and $=$ will be taken to hold elementwise. It is immediate that the value function in autarky and the full-information case is of the form $V_i(k,s) = u(B_{i,s}k)$, where $i = \{au, fi\}$ is the index for either autarky or full-information and that $B_{fi,s} > B_{au,s}$ for all s. Define $A_{l,s} = 1/B_{fi,s}$ and $A_{h,s} = 1/B_{au,s}$.

By repeating the arguments from the case without aggregate uncertainty, we can establish that $\hat{\phi}$ is well-defined for any $\vec{A} > \vec{0}$, continuous, and monotonic. Moreover, $\vec{A}_l < \hat{\phi}(\vec{A}_l) < \hat{\phi}(\vec{A}_h) < \vec{A}_h$. Then let $\vec{A}_0 = \vec{A}_l$ and $\vec{A}_n = \hat{\phi}(\vec{A}_{n-1})$. Monotonicity of the function $\hat{\phi}$ then implies that $\vec{A}_n \leq \vec{A}_{n+1} \leq \hat{\phi}(\vec{A}_h) < \vec{A}_h$. Then, a limit exists and we can set $A^* = \lim_n \vec{A}_n$. Showing that $\varphi^*(\psi, s) = A^*_s \int u^{-1}(w)\psi(dw)$ is done by the same arguments as in the case without aggregate uncertainty. ■

Appendix C Decentralization through long-term contracts

We consider an economy that, in addition to the agents discussed so far, contains a continuum of risk-neutral component planners, as in Atkeson and Lucas (1992). Each component planner is committed to delivering a certain level of promised utility to a particular agent. The component planners can trade excesses or deficits of resources at deterministic prices. The timing of the trading scheme is summarized in Figure 5.

z^{t-1} ⟶ Net trade $b_{t-1}(\sigma^{t-1}(z^{t-1}))$ $+k_t(\sigma^{t-1}(z^{t-1}))$ ⟶ Productivity shock z is realized. ⟶ Agent makes a report and consumes ⟶ Component planner has accumulated $-b_t(\sigma^t(z^t))$

Figure 5: Sequence of events in the decentralization with component planners

The component planner needs $b_{t-1}(\sigma^{t-1}(z^{t-1})) + k_t(\sigma^{t-1}(z^{t-1}))$, on net. Note that in the initial period there is no history and no term b. The component planner chooses an incentive-compatible and promise-keeping allocation for a particular agent that minimizes the expected costs of the net purchases of resources:

Problem DSP

$$C(w_1) = \inf p_1 k_1 + \sum_{t=2}^{\infty} p_t \left[E_1(b_{t-1} + k_t)\right] \quad \text{(C.1)}$$
$$\textit{subject to } (1) \textit{ and } (2).$$

Proposition C.1 *Suppose that there exists a sequence of prices $\{p_t\}_{t=1}^{\infty}$ and an allocation that:*

1. *at the given prices it solves* (C.1)*,*
2. *for all t the aggregate net trades are zero:* $\int_{\mathcal{V}} \int_{\mathbf{Z}^t} \left[b_t(w_1, z^t) + k_{t+1}(w_1, z^t)\right] \mu(dz^t)\psi(dw_1) = 0,$
3. *for every component planner the absolute value of trades is bounded:*

$$p_1 k_1(w) + \sum_{t=2}^{\infty} p_t E_1 \left|b_{t-1}(w, z^{t-1}) + k_t(w, z^{t-1})\right| < \infty, \quad \text{(C.2)}$$

then this allocation attains ψ with minimum initial capital.

The proposition demonstrates the link between the trading environment in this section and the efficient allocation in Section 3, and serves the role of the first welfare theorem for our environment. In the efficient allocation, the tradeoff between current and future consumption is independent of the date and the state of all other agents. This suggests a conjecture that the intertemporal prices in the decentralization are constant, meaning $p_{t+1}/p_t = q$ for all t. If this conjecture holds, the component planner's problem is recursive:

Problem DFE

$$C(w) = \min_{(k,b(z),v(z))} k + qE[b(z) + C(v(z))]$$
$$\textit{subject to}$$
$$E[(1-\beta)u(zk + b(z)) + \beta v(z)] = w \quad \text{(C.3)}$$
$$(1-\beta)u(zk + b(z)) + \beta v(z) \geq (1-\beta)u(zk + b(z')) + \beta v(z'),\ \forall z, z' \in \mathbf{Z} \quad \text{(C.4)}$$
$$k \geq 0, v(z) \in \mathcal{V}, zk + b(z) \in \mathbf{C}. \quad \text{(C.5)}$$

The constraints in Problem DFE are exact counterparts to the recursive incentive and promise-keeping constraints (8) and (9). The problem can be studied with standard techniques. However, the discount factor q is endogenous: It must be such that the minimizer $(k, b(z), v(z))$ of Problem DFE satisfies market clearing. Our next result shows that such a discount factor exists.

Proposition C.2 *Define $q^{FI} = [Ez]^{-1}$ and $q^{AU} = Ez^{\gamma-1}/Ez^{\gamma}$. Assume that $n = 2$ and $q^{AU} < 1$. There exists a price $q^* \in [q^{FB}, q^{AU}]$ such that the solution to Problem DFE is the efficient allocation and there is no trading between component planners.*

The correct discount factor under full information is q^{FI}. In an environment with only a risk-free bond in zero net supply, the correct discount factor is q^{AU}. This result is a counterpart to our results in Section 4—the efficient allocation does not provide full insurance, but it reduces consumption variability relative to the autarky allocation. Similarly, the value of a safe asset is somewhere in between that of full-information environment and autarky. Another way of looking at this result is that it is an analog of the second welfare theorem—it is possible to decentralize the efficient allocation through component planners.

C.1 Proofs

Proof of Proposition C.1. Suppose that the efficient allocation $\mathbf{P}'$ attains the distribution of promised utilities with strictly less initial capital than $\mathbf{P}$. Let $C_p(w)$ denote the value of this allocation at prices $\{p_t\}$:

$$C_p(w) = p_1 k_1'(w) + \sum_{t=2}^{\infty} p_t \left[\int_{\mathbf{Z^{t-1}}} [b_{t-1}'(w, z^{t-1}) + k_t'(w, z^{t-1})] \mu(dz^{t-1}) \right] = p_1 k_1'(w).$$

We have used the fact that for the efficient allocation, $\int_{\mathbf{Z^{t-1}}} [b_{t-1}'(w, z^{t-1}) + k_t'(w, z^{t-1})] \mu(dz^{t-1}) = 0$ for all w and t.

The market-clearing condition and the boundedness condition (C.2) imply that:

$$0 \leq \int C(w)\psi(dw) = \int k_1(w)\psi(dw) < \infty.$$

Then we know that

$$\int C_p(w)\psi(dw) = \int k_1'(w)\psi(dw) < \int k_1(w)\psi(dw) = \int C(w)\psi(dw).$$

This implies some component planner (a positive measure of them) is not optimizing, a contradiction. ■

Proof of Proposition C.2. Define the operator T_q as follows, for some constant $B > 1$:

$$\begin{aligned} T_q C(w) \quad &= \quad \min_{(k,b,v)} k + qE[b(z) + C(v(z))] \qquad &\text{(C.6)} \\ & \qquad \text{subject to (C.3), (C.4),} \\ & \qquad k \leq BC(w), k \geq 0. &\text{(C.7)} \end{aligned}$$

For the proof, we will apply the operator T_q only to the function $C^*(w) = A^* u^{-1}(w)$, where A^* is given by Proposition 2. Then will show that there exists a q^* such that $T_q C^* = C^*$ and that the solution to Problem DFE will be the same as the decision rule in Proposition 2.

Let (k^*, b^*, v^*) denote the solution to Problem DFE, given q. The rest of the proof proceeds in 7 steps. Let $z_1 = z_L$ and $z_2 = z_H$.

1. If $q = q^{FI} = [Ez]^{-1}$, then $E[b^*(z) + A^* u^{-1}(v^*(z))] > 0$.

   Perform the minimization without the incentive constraint (C.4). Within the set of minimizers, $z_i k + b(z_i) = z_j k + b(z_j)$, $v(z_i) = v(z_j)$. Take an arbitrary minimizing vector (k, b, v), and modify it by setting $k' = 0$, $b'(z_i) = z_i k + b(z_i)$. Since $q = (Ez)^{-1}$, this new vector is still a minimizer, but now it is incentive compatible and $E(b(z) + A^* u^{-1}(v(z))) > 0$. Any incentive-compatible vector (k, b, v) with $k > 0$ will have $(1-\beta)u(z_n k + b(z_n)) + \beta v(z_n) > (1-\beta)u(z_1 k + b(z_1)) + \beta v(z_1)$, so it will have higher cost for the component planner. Therefore $k^* = 0$ and $E[b^*(z) + A^* u^{-1}(v^*(z))] > 0$.

2. Define $\bar{b} \equiv E[b(z) + A^* u^{-1}(v(z))]$. If $k^* > 0$, then at the optimum $(z_H k^* + b_H^*)/(z_H k^* + \bar{b}^*) \equiv \alpha_H < \alpha_L \equiv (z_L k^* + b_L^*)/(z_L k^* + \bar{b}^*)$.

   Suppose not: $\alpha_H \geq \alpha_L$. Let $m(z) \equiv [zk^* + b^*(z) + A^* u^{-1}(v^*(z))]/[zk^* + \bar{b}^*]$.

   Forming the Lagrangian of Problem DFE and taking first-order conditions, we see that $C^{*\prime}(v^*(z_L))u'(z_L k^* + b^*(z_L)) < \beta/(1-\beta) = C^{*\prime}(v^*(z_H))u'(z_H k^* + b^*(z_H))$. Then the functional forms of C^* and u imply that $C^*(v^*(z_L))/(z_L k^* + b^*(z_L)) < C^*(v^*(z_H))/(z_H k^* + b^*(z_H))$. This fact and the assumption that $\alpha_H \geq \alpha_L$ imply that $m(z_H) > m(z_L)$. Now assume that $m(z_L) \geq 1$. This implies:

$$(Ez)k^* + \bar{b}^* = E[zk^* + b^*(z) + C^*(v^*(z))] = E[m(z)(zk^* + \bar{b}^*)] > E[zk^* + \bar{b}^*] = (Ez)k^* + \bar{b}^*,$$

which is a contradiction. Therefore, $m(z_L) < 1$. By a similar argument $m(z_H) > 1$. Therefore, if an agent with a high shock misreports, the value of his consumption and continuation utility will be

$$m(z_L)(z_L k^* + \bar{b}^*) + (z_H - z_L)k^* < (z_L k^* + \bar{b}^*) + (z_H - z_L)k^* < m(z_H)(z_H k^* + \bar{b}^*).$$

But by the dual problem,

$$(b^*(z_H), v^*(z_H)) = \arg\max (1-\beta)u(z_H k^* + b^*(z_H)) + \beta v^*(z_H)$$

$$\text{subject to } z_H k^* + b^*(z_H) + C(v(z_H)) \leq m(z_H)(z_H k^* + \bar{b}^*).$$

Therefore, $(1-\beta)u(z_H k^* + b^*(z_H)) + \beta v^*(z_H) > (1-\beta)u(z_L k^* + b^*(z_L)) + \beta v^*(z_L)$. Then $S_{2,1} > 0$, so by Claim 5 the proof of Proposition 2 we can modify the vector of choice variables and reduce costs strictly, which is a contradiction. Therefore, $\alpha_H < \alpha_L$.

3. $z_H k^* + b^*(z_H) > 0$.

   Let $w_H = (1-\beta)u(z_H k^* + b^*(z_H)) + \beta v^*(z_H)$. The promise-keeping and incentive constraints (C.3) and (C.4) imply that $w_H \geq w$. It is immediate that

$$(b^*(z_H), v^*(z_H)) = \arg\min b^*(z_H) + C^*(v^*(z_H))$$

$$\text{subject to } (1-\beta)u(z_H k^* + b^*(z_H)) + \beta v^*(z_H) = w_H.$$

   Substituting $v^*(z_H)$ from the constraint above and taking first-order conditions implies the result.

4. If $k^* \leq C^*(w)$, then $z_L k^* + b^*(z_L) > 0$.

   If $k^* < C^*(w)$ and $\bar{b}^* \leq 0$, the solution to Problem DFE is feasible in Problem FE and strictly preferable to the solution to Problem FE, a contradiction. If $k^* = C^*(w)$ and $\bar{b}^* < 0$, then (k^*, b^*, v^*) is a solution to Problem FE where constraint (10) does not bind, contradicting Claim 6 in the proof of Proposition 2. Therefore, $z_L k^* + \bar{b}^* > 0$.

   From step 3 we know that $\alpha_H = \frac{z_H k^* + b^*(z_H)}{z_H k^* + b^*} > 0$ and from step 2 we know that $\alpha_L > \alpha_H > 0$. So $z_L k + b_L = \alpha_L(z_L k + \bar{b}) > 0$.

5. Define $q^{AU} = E(z^{\gamma-1})/E(z^\gamma)$, with $\gamma = 0$ for the log utility case. If $q = q^{AU}$, then at the optimum, $\bar{b}^* = E(b(z) + C(v(z)) < 0$.

   The solution to Problem FE is always feasible for the component planner and gives a cost of $C^*(w)$. If the component planner's $k > C^*(w)$, it must be that $E(b^*(z) + C(v^*(z))) < 0$.

   Now assume that $k \leq C^*(w)$, which implies $\bar{b} \geq 0$. Consider increasing allocated capital, decreasing transfers $b(z)$ and keeping $v(z)$ constant. Step 4 of the proof shows that this is possible. To satisfy the promise-keeping constraint (C.3), this variation must satisfy:

$$E\left\{u'(zk + b(z))\left[z + \frac{db(z)}{dk}\right]\right\} = 0.$$

   The constraint $S_{2,1} \geq 0$ if:

$$u'(z_H k + b(z_H))\left[z_H + \frac{db(z_H)}{dk}\right] \geq u'(z_H k + b(z_L))\left[z_H + \frac{db(z_L)}{dk}\right].$$

   For ease of exposition we will consider variation in which $db(z_H)/dk = db(z_L)/dk = db/dk$. The second condition above implies that incentive compatibility will be satisfied as long as $db/dk > -z_H$. Constraint (C.3) implies that

$$\frac{db}{dk} = -\frac{Eu'(zk + b(z))z}{Eu'(zk + b(z))}.$$

   so $db/dk > -z_H$ and this variation is feasible and incentive compatible. The marginal change in the component planner's cost from this variation is $1 + q\frac{db}{dk}$, where

$$-\frac{db}{dk} = \frac{E\{(zk + b(z))^{\gamma-1}z\}}{E\{(zk + b(z))^{\gamma-1}\}} = \frac{E\{\alpha_i^{\gamma-1}\left(1 + \frac{\bar{b}}{zk}\right)^{\gamma-1} z^\gamma\}}{E\{\alpha_i^{\gamma-1}\left(1 + \frac{\bar{b}}{zk}\right)^{\gamma-1} z^{\gamma-1}\}}.$$

Since $\bar{b} \geq 0$, $\alpha_L > \alpha_H$ and $\gamma - 1 < 0$, it follows that $p_L \equiv \mu_L \frac{\alpha_L^{\gamma-1}\left(1+\frac{\bar{b}}{z_L k}\right)^{\gamma-1} z_L^{\gamma-1}}{E\{\alpha_i^{\gamma-1}\left(1+\frac{\bar{b}}{zk}\right)^{\gamma-1} z^{\gamma-1}\}} < \mu_L \frac{z_L^{\gamma-1}}{E\{z_i^{\gamma-1}\}} \equiv d_L$ and $p_H \equiv \mu_H \frac{\alpha_H^{\gamma-1}\left(1+\frac{\bar{b}}{z_H k}\right)^{\gamma-1} z_H^{\gamma-1}}{E\{\alpha_i^{\gamma-1}\left(1+\frac{\bar{b}}{z_i k}\right)^{\gamma-1} z_i^{\gamma-1}\}} > \mu_H \frac{z_H^{\gamma-1}}{E\{z^{\gamma-1}\}} \equiv d_H$. $p_L + p_H = d_L + d_H = 1$, so $p_L - d_L = -(p_H - d_H)$. Thus, $-db/dk = p_L z_L + p_H z_H$ and

$$-\frac{db}{dk} - \frac{1}{q} = \frac{db}{dk} - \frac{E\{z^\gamma\}}{E\{z^{\gamma-1}\}} = (p_H - d_H)(z_H - z_L) > 0.$$

However, $-db/dk > 1/q \Rightarrow$ a contradiction: The variation reduces the component planner's cost.

6. For some $q^* \in (q^{AU}, q^{FB})$, $T_q C^*(w) = C^*(w)$ and $k^* = C^*(w)$.

   By the Theorem of the Maximum, $E(b(z) + C^*(v(z)))$ is u.s.c. in q, single-valued, and continuous. Then steps 1 and 5 imply that $E(b(z) + C^*(v(z))) = 0$ for some $q^* \in (q^{AU}, q^{FB})$.

   Next, we show that at this q^*, $T_q C^*(w) = C^*(w)$ and $k^* = C^*(w)$. Clearly $T_{q^*} C^*(w) \leq C^*(w)$. The fact that $E(b(z) + C^*(v(z))) = 0$ implies that the solution to Problem DFE satisfies the constraint (10). Therefore, the solution to Problem Problem DFE satisfies all of the constraints in Problem FE. Therefore $T_q C^*(w) \geq C^*(w)$. Then $T_q C^*(w) = C^*(w)$.

   Finally,
   $$C^*(w) = T_{q^*} C^*(w) = k^* + q^* T_{q^*} C^*(w) = k^*,$$
   which concludes the proof.

■

Appendix D Lotteries

In the main body of this article, we limit our discussion to the allocations that can be generated by allocation rules—that is, allocations that satisfy "the equal treatment property." It is easy to construct allocations that do not satisfy this property. However, in this appendix we show that the efficient allocation has the equal treatment property (up to a set of agents with measure zero).

To do this, we follow Prescott and Townsend (1984*a*,*b*) and generalize the allocation to allow for randomization. Under this specification, the set of allocations satisfying the constraints is convex, which helps us show that for any randomized allocation there exists another randomized allocation that can be generated by an allocation rule and does not require more capital. This proves the equal treatment property in this generalized environment. Finally, we show that the *optimal* allocation rule is deterministic, conditional on productivity. For ease of exposition, we relegate the proofs to Section D.3.

D.1 Randomized allocation

Recall that $\mathbf{C} \subseteq \mathbb{R}$ is the domain of the utility function, so $\mathbf{C} = (0, \infty)$ for $\gamma \leq 0$ and $[0, \infty)$ for $\gamma > 0$. By $\mathbf{C} - x$ we will mean the algebraic sum of the sets $\mathbf{C}$ and $\{-x\}$. Also, we have that $\mathcal{V} = u(\mathbf{C})$.

The randomized allocation is a sequence of lotteries over capital and consumption transfers. Informally, these are $\mathbf{k}_t(.|w_0, k^{t-1}, b^{t-1}, z^{t-1}), \mathbf{b}_t(.|w_0, k^t, b^{t-1}, z^t)$. At any given time t, the planner can condition the capital on an agent's initial promised utility w_0, history of reports z^{t-1}, and vectors of past allocated capital k^{t-1} and consumption transfers b^{t-1}; the consumption transfer is conditioned on this information and also on the allocated capital k_t and the report of productivity z_t. Then a randomized allocation is formally defined as follows.

Definition 4 *Let $\mathcal{B}$ be the Borel sigma algebra on the real line. A randomized allocation is a sequence of Borel probability measures on the real line*

$$\mathbf{P} = \{\mathbf{k}_t : \mathcal{B} \times \mathcal{V} \times \mathbf{R}_+^{t-1} \times \mathbf{R}^{t-1} \times \mathbf{Z}^{t-1} \to [0,1], \mathbf{b}_t : \mathcal{B} \times \mathcal{V} \times \mathbf{R}_+^t \times \mathbf{R}^{t-1} \times \mathbf{Z}^t \to [0,1]\}_{t=0}^{\infty}$$

such that:

1. *For any given $(w, k^{t-1}, b^{t-1}, z^{t-1})$, $\mathbf{k}_t(.|w, k^{t-1}, b^{t-1}, z^{t-1})$ is a Borel probability measure and similarly, for any given (w, k^t, b^{t-1}, z^t), $\mathbf{b}_t(.|w, k^t, b^{t-1}, z^t)$ is a Borel probability measure;*
2. *for any Borel set $B \in \mathcal{B}$, the corresponding measures of B are Borel-measurable functions of $(w, k^{t-1}, b^{t-1}, z^{t-1})$ and (w, k^t, b^{t-1}, z^t), respectively;*
3. *for any $(w, k^{t-1}, b^{t-1}, z^{t-1})$, $\mathbf{k}_t((-\infty, 0)|w, k^{t-1}, b^{t-1}, z^{t-1}) = 0$, and for any (w, k^t, b^{t-1}, z^t), $\mathbf{b}_t(\mathbf{C} - z_t k_t|w, k^t, b^{t-1}, z^t) = 1$.*[8]

If the allocation is deterministic on history of reports, then for any sequence of reports we can determine the values of capital and transfers; therefore, including them as arguments is redundant. This is not the case if the allocation is stochastic. History of capital and transfers are outcomes of lotteries, so we need to keep track of the realizations of the lotteries. Item 3 in the definition above ensures that the probability of allocating negative capital or assigning infeasible consumption to an agent is zero.

In a similar fashion, we need to redefine the agent's reporting strategy. The agent conditions his reports on the history of observed shocks and the realized allocated capital and consumption transfers.

Definition 5 *A reporting strategy is a sequence of Borel-measurable functions $\sigma = \{\sigma_t(k^t, b^{t-1}, z^t) \to \mathbf{Z}\}_{t=0}^{\infty}$. We denote by $\sigma^t(k^t, b^{t-1}, z^t)$ the vector of reports up to date t.*

Since the planner commits to an allocation, there is no benefit to the agent to randomize reports. Since the allocation is not deterministic, the agent needs to keep track of realizations of the allocation in addition to the vector of productivity shocks. The reporting strategy maps $\mathbf{R}_+^t \times \mathbf{R}^{t-1} \times \mathbf{Z}^t$ to $\mathbf{Z}$ for all t.

In the deterministic allocation, the allocation and the reporting strategy are defined for any sequence of productivity or its reports. By the i.i.d. assumption every sequence of productivity realizations z^t has non-zero probability. In order to simplify definitions, we require in the randomized case that the allocation and the reporting strategy be defined for all $(k^{t-1}, b^{t-1}, z^{-t})$ even though some of these vectors can never be reached.

[8] When there is no ambiguity, we suppress the dependence on the agent's initial utility w_0.

Definition 6 *A reporting strategy is feasible for an allocation* $\mathbf{P}$ *if for every* t, $(k^t, b^{t-1}, z^t) \in \mathbf{R}^t_+ \times \mathbf{R}^{t-1} \times \mathbf{Z}^t$, $\mathbf{b}_t(\mathbf{C} - z_t k_k | w, k^t, b^{t-1}, \sigma^t(k^t, b^{t-1}, z^t)) = 1$. *Denote the set of all feasible reporting strategies for an allocation* $\mathbf{P}$ *by* $\Sigma(\mathbf{P})$.

The restrictions on $\mathbf{P}$ ensure that the truthtelling strategy is feasible. Therefore $\Sigma(\mathbf{P}) \neq \emptyset$.

Let P^σ_t be the probability measure over $\mathcal{X}^t \equiv \mathbf{R}^t_+ \times \mathbf{R}^t \times \mathbf{Z}^t$ induced by the allocation, reporting-strategy pair. Define P^σ_t recursively by:

$$\begin{aligned} P^\sigma_{t+1}(M|w_0) \quad = \quad & \int_{\mathcal{X}^t} \int_{\mathbf{R}_+} \int_{\mathbf{Z}} \int_{\mathbf{R}} \chi_M(k^{t+1}, b^{t+1}, z^{t+1}) d\mathbf{b}_{t+1}(b_{t+1}|w, k^{t+1}, b^t, \sigma^{t+1}(k^{t+1}, b^t, z^{t+1})) \\ & \times d\mu(z_{t+1}) d\mathbf{k}_{t+1}(k_{t+1}|w, k^t, b^t, \sigma^t(k^t, b^{t-1}, z^t)) dP^\sigma_t(k^t, b^t, z^t|w_0), \end{aligned}$$

where M is a Borel set and χ_M is its indicator function, and

$$P^\sigma_1(M|w) = \int_{\mathbf{R}_+} \int_{\mathbf{Z}} \int_{\mathbf{R}} \chi_M(k_1, b_1, z_1) d\mathbf{b}_1(b_1|w, k_1, \sigma_1(k_1, z_1)) d\mu(z) d\mathbf{k}_1(k_1|w).$$

By Theorem 8.2. in Stokey et al. (1989), every P^σ_t is a probability measure. If we set the measure $P^{'\sigma}_t$ as $P^{'\sigma}_t(M|w_0) = 1$, for all Borel sets such that $(k^t, b^t, z^t) \in M$, then by iterating on the construction above we can construct the conditional measures $P^\sigma_s(M|k^t, b^t, z^t, w_0)$ for all $s \geq t$.

The agent's choice is an optimal sequence of probability measures. The expected payoff from a strategy σ after a history $(k^t, z^t, b^t) \in \mathbf{R}^t_+ \times \mathbf{R}^t \times \mathbf{Z}^t$ is given by:

$$V_{t+1}(\mathbf{P}, \sigma, w, k^t, b^t, z^t) = (1-\beta) \sum_{s=t+1}^{\infty} \beta^{s-t-1} \int_{\mathcal{X}_s} u(z_s k_s + b_s) dP^\sigma_s(k^s, b^s, z^s | w, k^t, b^t, z^t).$$

Then the counterpart to the promise-keeping constraint (2) is:

$$V_0(\mathbf{P}, \sigma^*, w_0) \geq w_0, \ \forall w_0 \in \mathcal{V}, \tag{D.1}$$

and similarly for the incentive constraint (1):

$$V_0(\mathbf{P}, \sigma^*, w_0) \geq V_0(\mathbf{P}, \sigma, w_0), \ \forall \sigma \in \Sigma(\mathbf{P}), w_0 \in \mathcal{V}. \tag{D.2}$$

The resource constraint (3) is now:

$$-\int_{\mathcal{V}} \int b \, dP^{\sigma^*}_t(b|w_0) d\psi(w_0) \geq \int_{\mathcal{V}} \int k \, dP^{\sigma^*}_{t+1}(k|w_0) d\psi(w_0), \forall t. \tag{D.3}$$

Then the optimization problem would be:

$$\varphi^{**}(\psi) = \inf_{\mathbf{P}} \int_{\mathcal{V}} \int k \, d\mathbf{k}_0(k|w_0) d\psi(w_0), \quad \text{s.t. (D.1), (D.2), (D.3)}.$$

Separable preferences imply that for all $(k^{t-1}, b^{t-1}, z^{t-1}) \in \mathcal{X}^{t-1}$,

$$\begin{aligned} V_t(\mathbf{P}, \sigma, k^{t-1}, b^{t-1}, z^{t-1}, w_0) \quad = \quad & \int_{\mathbf{R}_+} \int_{\mathbf{Z}} \int_{\mathbf{R}} \{(1-\beta) u(z_t k_t + b_t) + \beta V_{t+1}(\mathbf{P}, \sigma, k^t, b^t, z^t, w_0)\} \\ & d\mathbf{b}_t(b_t|w, (k^{t-1}, k_t), b^{t-1}, \sigma^t((k^{t-1}, k_t), b^{t-1}, (z^{t-1}, z_t))) d\mu(z_t) \\ & d\mathbf{k}_t(k_t|w, k^{t-1}, b^{t-1}, \sigma^{t-1}(k^{t-1}, b^{t-1}, z^{t-1})). \end{aligned} \tag{D.4}$$

The stochastic version of the t.i.c. constraint (5) is given by:

$$\begin{aligned} & \int \left[(1-\beta) u(zk + b) + \beta V_{t+1}(\mathbf{P}, \sigma^*, x^{t-1}, k, z, z, b)\right] d\mathbf{b}_t(b|x^{t-1}, k, z) \geq \\ & \quad \int \left[(1-\beta) u(zk + b) + \beta V_{t+1}(\mathbf{P}, \sigma^*, x^{t-1}, k, z, z', b)\right] d\mathbf{b}_t(b|x^{t-1}, k, z'), \end{aligned} \tag{D.5}$$

for all t, $z, z' \in \mathbf{Z}$, $k \geq 0$, $x^{t-1} \in \mathcal{X}^{t-1}$. Here x^{t-1} is a concise expression of the entire vector of conditioning variables—the history of the realized shocks and the realization of the lotteries—up to period t.

Lemma D.1 *If a randomized allocation satisfies incentive compatibility, then it satisfies* (D.5) *for almost all* (x^{t-1}, k) (P^{σ^*}). *For any* $\mathbf{P}$ *that satisfies the constraints* (D.1), (D.2), (D.3), *there exists an allocation* $\mathbf{P}'$ *that also satisfies the constraints* (D.1), (D.2), (D.3), *satisfies* (D.5) *for all* (x^{t-1}, k), *and requires the same amount of initial capital.*

This most general formulation of the problem is cumbersome to work with because the allocation depends on increasingly long vectors of productivity shocks and realizations of the lottery itself. We introduce another version of the allocation in which the planner differentiates between the agents only along one dimension—their expected continuation utility.

Definition 7 *A randomized allocation rule is a sequence of Borel probability measures on the real line,* $\{\mathbf{k}_t(.; w), \mathbf{b}_t(.|w, k, z), \mathbf{v}_{t+1}(.; w, k, z, b)\}$ *such that for any Borel set* $B \subseteq \mathbf{R}$ *the corresponding measures of* B *are Borel-measurable functions and for any* (w, k, z, b), *we have that* $\mathbf{k}_t(\mathbf{R}_+|w, k, z, b) = 1$; $\mathbf{b}_t(\mathbf{C} - zk|w, k, z) = 1$ *and* $\mathbf{v}_{t+1}(\mathcal{V}; w, k, z, b) = 1$.

There is no gain to the agent to condition his strategy on w_t in addition to x^{t-1}, since its value can be determined from the vector $x^{t-1} \in \mathcal{X}^{t-1}$.

Then, as before, we can define probability measures on $\mathcal{X}^t$ induced by a randomized recursive allocation and a reporting strategy. Incentive compatibility and promise-keeping constraints are defined similarly.

Now the recursive analogs to Temporary incentive compatibility (D.5) and promise keeping (D.1) are given by:

$$\int\int [(1-\beta)u(zk+b) + \beta w'] d\mathbf{v}_{t+1}(w'|w, k, z, b) d\mathbf{b}_t(b|w, k, z) \geq \\ \int\int [(1-\beta)u(zk+b) + \beta w'] d\mathbf{v}_{t+1}(w'|w, k, z', b) d\mathbf{b}_t(b|w, k, z') \tag{D.6}$$

and

$$\int\int\int\int [(1-\beta)u(zk+b) + \beta w'] d\mathbf{v}_{t+1}(w'|w, k, z, b) d\mathbf{b}_t(b|w, k, z) d\mu(z) d\mathbf{k}_t(k|w) = w. \tag{D.7}$$

Lemma D.2 *Suppose that a randomized allocation rule satisfies* (D.6), (D.7), *and*

$$\begin{aligned} \lim\nolimits_{t\to\infty} \operatorname{ess\,inf}[\beta^t w_t | w_s] = 0 \\ \lim\nolimits_{t\to\infty} \beta^t E[w_t | w_s] = 0, \end{aligned} \tag{D.8}$$

for almost all w_s, *where the expectations are with respect to the measures* P^{σ^*}. *Then the randomized allocation generated by this allocation rule satisfies* (D.1) *and* (D.2).

Next, we show that without loss of generality we can impose the condition that the allocation must be generated by an allocation rule.

Lemma D.3 *Let* E *be the set of allocations that satisfies* (D.1), (D.2), *and* (D.3). *Then for any* $\mathbf{P} \in E$, *there exists a randomized allocation* $\mathbf{P}' \in E$ *that is generated by a randomized allocation rule, satisfies* (D.8), *and does not require more initial capital.*

If two agents with different initial promised utilities, productivity reports, and outcomes of the lottery have the same expected continuation utility, then they will face the same lottery from that date forward. This is the stochastic version of the equal treatment property. This result uses the fact that the constraints and the planner's objective function are linear in lotteries. We are minimizing a linear functional over a convex subset of a linear space. If we have a set of agents with the same expected discounted utility, we can take a weighted average of their allocations without affecting the constraints or the planner's objective.

The next step is to show that the efficient allocation is deterministic, conditional on the productivity realization.

Proposition D.1 *An optimal randomized allocation exists and it is equivalent to the deterministic efficient allocation.*

The proof is in Section D.3, but we give an intuitive derivation of the result here. First, we show that the lotteries $\mathbf{b}_t(.|w,k,z), \mathbf{v}_{t+1}(.;w,k,z,b)$ are degenerate: After capital and productivity are realized, the consumption transfers and continuation utility are deterministic. Second, given the realized capital, $(b(w,z)$ and $v(w,z)$ are derived from Problem AP2. Finally, since the function $\phi(w,N;A)$ is convex in w and N, it is optimal to make allocated capital deterministic.

Lemma D.3 and Proposition D.1 together imply that the efficient allocation is (i) deterministic conditional on the productivity realization and (ii) generated by an allocation rule, so our approach in Section 3 is justified.

D.2 The Stochastic Functional Equation

The stochastic version of Problem FE is defined below, where the choice variables are **k**, **b**, and **v**):

Problem SFE

$$T^L\varphi(\psi) = \inf_P \int_{\mathcal{V}} \int k d\mathbf{k}(k|w) d\psi(w)$$

$$s.t. \int \int_{\mathcal{V}} \left[(1-\beta)u(zk+b) + \beta w'\right] d\mathbf{v}(w'|w,k,z,b) d\mathbf{b}(b|w,k,z) \geq \int \int_{\mathcal{V}} \left[(1-\beta)u(zk+b) + \beta w'\right] d\mathbf{v}(w'|w,k,z',b) d\mathbf{b}(b|w,k,z'),$$

$$\int \int_{\mathbf{Z}} \int \int_{\mathcal{V}} \left[(1-\beta)u(zk+b) + \beta w'\right] d\mathbf{v}(w'|w,k,z,b) d\mathbf{b}(b|w,k,z) d\mu(z) d\mathbf{k}(k|w) = w,$$

$$-\int_{\mathcal{V}} \int \int_{\mathbf{Z}} \int b d\mathbf{b}(b|w,k,z) d\mu(z) d\mathbf{k}(k|w) d\psi(w) \geq \varphi(\psi'),$$

where ψ' is the distribution over w induced by $(\mathbf{k},\mathbf{b},\mathbf{v})$ and a truthtelling reporting strategy.

D.3 Proofs

First, we need to prove two technical lemmas to deal with the fact that two allocations may differ on sets with probability zero, but they still solve the planner's problem.

Lemma D.4 *Let $M \subseteq \mathcal{X}^t$ be some measurable set such that for some $\sigma \in \Sigma(\mathbf{P})$, $P_t^\sigma(M) > 0$. Then $P_t^{\sigma^*}(M) > 0$.*

Proof. The proof is by induction. From the construction of the measure,

$$P_1^{\sigma^*}(M) \geq P_1^\sigma(M) \frac{\min_z \mu(\{z\})}{\max_z \mu(\{z\})} > 0.$$

Now, suppose that $t \geq 2$. Let $M' \subseteq \mathcal{X}^{t-1}$ be the set of $t-1$-dimensional vectors that have a continuation in M. This set is measurable. By the construction of P_t^σ and the fact that $P_t^\sigma(M) > 0$, it follows that $P_{t-1}^\sigma(M') > 0$. By the inductive assumption then $P_{t-1}^{\sigma^*}(M') > 0$. Then,

$$P_t^{\sigma^*}(M) \geq P_t^\sigma(M) \frac{P_{t-1}^{\sigma^*}(M')}{P_{t-1}^\sigma(M')} \frac{\min_z \mu(\{z\})}{\max_z \mu(\{z\})} > 0,$$

where the last inequality follows from the fact that there are a finitely many productivity levels. ■

Lemma D.5 *Suppose that $\mathbf{P}$ and $\mathbf{P}'$ are randomized allocations such that for any t and any Borel set M, $P_t^{\sigma*}(M) = P_t^{'\sigma*}(M)$. Then for any feasible reporting strategy σ, $P_t^\sigma(M) = P_t^{'\sigma}(M)$.*

Proof. By the construction of P_t, it follows that $\mathbf{k}_t(M|x) = \mathbf{k'}_t(M|x)$ for all Borel sets M a.e. $P_t^{\sigma*}$. (Also a.e. $P_t'^{\sigma*}$.) Similarly for $\mathbf{b}_t$ and $\mathbf{b'}_t$. Then, the conditional measures imply that $P_0^{\sigma}(M) = P_0^{'\sigma}(M)$ for all Borel sets M.

The rest of the proof is by induction. Let $M' \subseteq \mathcal{X}^{t-1}$ be the set of $t-1$-dimensional vectors that have a continuation in M. This set is measurable. By the inductive assumption, $P_{t-1}^{\sigma}(M') = P_{t-1}^{'\sigma}(M')$. Then the fact above and the construction of P_t implies that $P_t^{\sigma}(M) = P_t^{'\sigma}(M)$. ■

Proof of Lemma D.1. The first statement follows from the definition of V. Consider the second statement.

Suppose that the inequality (D.5) is violated for some t for some set with nonzero measure (P^{σ^*}). Then there exists some $\epsilon > 0$ and a set M such that for all $(k, x^{t-1}) \in M$

$$\int \left[(1-\beta)u(z_i k + b) + \beta V_{t+1}(\sigma^*, x^{t-1}, k, z_i, z_j, b)\right] d\mathbf{b}_t(x^{t-1}, k, z_j) \geq$$
$$\int \left[(1-\beta)u(f(k, z_i) + b) + \beta V_{t+1}(\sigma^*, s, z_i, k, b, x^{t-1})\right] d\mathbf{b}_t(s, z_i, k, x^{t-1}) + \epsilon$$

and $P_t^{\sigma^*}(M) = \delta > 0$. Then define a strategy σ' that coincides with σ^* everywhere, except for t, M, and z_i and let $\sigma_t'(z_i, s, k, x^{t-1}) = z_j$ for $(s, k, x^{t-1}) \in M$. Since $\sigma' \in \Sigma(\mathbf{P})$, $V_0(\mathbf{P}(w_0), \sigma') \geq V_0(\mathbf{P}(w_0), \sigma^*) + \beta^t \mu(\{z_i\})\epsilon\delta > V_0(\mathbf{P}(w_0), \sigma^*)$, which contradicts incentive compatibility.

Finally redefine $\mathbf{P}$ to ensure that (D.5) holds everywhere. At date zero, for all k, z such that (D.5) does not hold, redefine $\mathbf{b}$ to give probability 1 to $\{1\}$. Then for all t and vectors x that are continuations for k, z, set $\mathbf{k}_t$ that gives probability 1 to $\{0\}$; similarly redefine $\mathbf{b}_t$ to give probability 1 to $\{1\}$.

Then we repeat the procedure for $t = 1$ and so on. Call the new randomized allocation $\mathbf{P'}$ satisfies (D.5) everywhere and $P_t^{\sigma*}(M) = P_t^{'\sigma*}(M)$ for all Borel sets M. This implies constraints (D.1) and (D.3) are satisfied. Also $\mathbf{P}$ requires the same amount of capital.

By Lemma D.5 then $P_t^{\sigma}(M) = P_t^{'\sigma}(M)$, therefore $V_0(\mathbf{P}, \sigma) = V_0(\mathbf{P'}, \sigma)$, for all σ. Then since the original randomized allocation is incentive compatible, it follows that the new allocation is incentive compatible. ■

Proof of Lemma D.2.

Let $t \in \mathbb{N}$ and $w \in \mathcal{V}$ be arbitrary. Let $\mathbf{P}$ be the randomized allocation generated from the randomized allocation rule starting from date t and utility w. We will show that $V(\mathbf{P}, \sigma^*) = w$.

First, we will show that

$$w = (1-\beta)\sum_{j=t}^{t'} \beta^{j-t} \int_{\mathcal{X}^{j-t+1}} u(z_j k_j + b_j) dP_j^{\sigma^*}(x^j|w) + \beta^{t'+1-t} \int_{\mathcal{X}^{t'-t+2}} w_{t'+1} dP_{t'+1}^{\sigma^*}(x^{t'+1}|w) \qquad \text{(D.9)}$$

holds for all $t' \leq T$.

The proof is by induction. It is true for t by the promise-keeping constraint (D.7). Suppose it holds for some t'. Integrating (D.7) over $w_{t'+1}$ with the measure $P_{t'+1}^{\sigma^*}$ and using the definition of P^{σ^*}, we get

$$\int_{\mathcal{X}^{t'-t+2}} w_{t'} dP_{t'}^{\sigma^*}(x^j|w) = (1-\beta)\int_{\mathcal{X}^{t'-t+2}} u(z_{t'+1}k_{t'+1} + b_{t'+1}) dP_{t'+1}^{\sigma^*}(x^{t'+1}|w) + \beta \int_{\mathcal{X}^{t'-t+3}} w_{t'+2} dP_{t'+2}^{\sigma^*}(x^{t'+2}|w).$$

Substituting this result in (D.9), we establish (D.9) for $t'+1$.

By assumption, $\beta^{t'+1}\int_{\mathcal{Y}^{t'-t+2}} w_{t'+1} dP_{t'+1}^{\sigma^*}(x^{t'+1}|w) \to 0$ as $t' \to \infty$. So taking limits in (D.9) establishes $V_t(\mathbf{P}, \sigma^*, w) = w$.

What is left to show is that truthtelling σ^* dominates any strategy of misreporting for finitely many periods. Define $\Sigma_t = \{\sigma \in \Sigma(\mathbf{P}) : \sigma_j(s, z, k, x^{j-1}) = z \text{ if } j \geq t\}$—that is, the set of strategies of in all periods after (and including) t. Note that $\sigma^* \in \Sigma_0$ (and is in fact the only member of Σ_0). For any $\sigma \in \Sigma_T$:

$$V_0(\mathbf{P}, \sigma) = (1-\beta)\sum_{j=0}^{t-1} \beta^j \int_{\mathcal{X}^{j+1}} u(z_j k_j + b_j) dP_j^{\sigma}(x^j) + \beta^t \int_{\mathcal{X}^{t+1}} V_{t+1}(\mathbf{P}, \sigma^*, x^t) dP_t^{\sigma}(x^t).$$

The proof is by induction. σ^* trivially dominates all $\sigma \in \Sigma_0$. Now assume that σ^* dominates all $\sigma \in \Sigma_t$. We will show that this is true for $t+1$. Let $\sigma \in \Sigma_{t+1}$. The first claim implies that condition (D.6) holds everywhere. Then integrating (D.6) over all possible histories we get:

$$\int_{\mathcal{X}^{t+1}} \left\{(1-\beta)u(zk+b) + \beta V_{t+1}(\mathbf{P}, \sigma^*, x^t)\right\} dP_t^\sigma(x^t) \leq \int_{\mathcal{X}^{t+1}} V_t(\mathbf{P}, \sigma^*, x^{t-1}) dP_{t-1}^\sigma(x^{t-1}).$$

Then applying the definition of V and the result above:

$$\begin{aligned} V_0(\mathbf{P}, \sigma) &\leq (1-\beta)\sum_{j=0}^{t-1} \beta^j \int_{\mathcal{X}^{j+1}} u(z_j k_j + b_j) dP_j^\sigma(x^j) + \beta^t \int_{\mathcal{X}^{t+1}} V_{t+1}(\mathbf{P}, \sigma^*, x^t) dP_t^\sigma(x^t) \\ &\leq V_0(\mathbf{P}(w_0), \sigma^*) \end{aligned}$$

So we showed that a strategy of misreporting for t periods is dominated by a strategy of misreporting for $t-1$ periods which in turn is dominated by truthtelling. Then by induction, any strategy in which the agent lies for finite number of periods is dominated by truthtelling.

Finally suppose that for some $\sigma \in \Sigma(\mathbf{P})$, $V_0(\mathbf{P}, \sigma) - V_0(\mathbf{P}, \sigma^*) = \delta > 0$. Then there exists some T such that for all $t \geq T$,

$$(1-\beta)\sum_{j=0}^{t-1} \beta^j \int_{\mathcal{X}^{j+1}} u(z_j k_j + b_j) dP_j^\sigma(x^j) > V_0(\mathbf{P}, \sigma^*) + (2/3)\delta.$$

By Lemma D.4, equation (D.8), and the first claim, it follows that there exists some $t \geq T$ such that $\beta^t \int_{\mathcal{X}^{t+1}} V_{t+1}(\mathbf{P}, \sigma^*, x^t) dP_t^\sigma(x^t) > -\delta/3$. Let σ' be a reporting strategy identical to σ for all $s \leq t-1$ and truthtelling from date t on. Then:

$$V_0(\mathbf{P}, \sigma') = (1-\beta)\sum_{j=0}^{t-1} \beta^j \int_{\mathcal{X}^{j+1}} u(z_j k_j + b_j) dP_j^\sigma(x^j) + \beta^t \int_{\mathcal{X}^{t+1}} V_{t+1}(\mathbf{P}, \sigma^*, x^t) dP_t^\sigma(x^t) > V_0(\mathbf{P}, \sigma^*) + (1/3)\delta.$$

However, $\sigma' \in \Sigma_t$, so $V_0(\mathbf{P}, \sigma') \leq V_0(\mathbf{P}, \sigma^*)$, which is a contradiction. Therefore $V_0(\mathbf{P}, \sigma) \leq V_0(\mathbf{P}, \sigma^*)$, establishing incentive compatibility. ■

Proof of Lemma D.3. Take an arbitrary allocation $\mathbf{P} \in E$.

By assumption, $\mathbf{v}_1(w, k, z, b) \equiv V_1(\mathbf{P}(w), \sigma^*, k, z, b)$ is a measurable function. The probability measure ψ and the allocation, induce a measure ψ_1 over the real line. Let $\mathcal{B}$ be the Borel sigma-algebra on the real line and let $\mathbf{k}_1'(M; w)$ be a mapping from $\mathcal{B} \times \mathbf{R}$ to $[0, 1]$ that is measurable in w. For any sets $M, B \in \mathcal{B}$, define a randomized allocation rule

$$\int_B \mathbf{k}_1'(M; w) d\psi_1(w) = \int_{\mathcal{V}} \int_{\mathcal{X}} \chi_B(V_1(\mathbf{P}(w_0), \sigma^*, k, b, z)) \mathbf{k}_1(M; w_0, k, z, b) dP_1^{\sigma^*}(k, z, b; w_0) d\psi(w_0).$$

This is the measure on capital, conditional on the agent's promised utility. The Radon-Nikodym theorem implies that this measure exists and is unique almost everywhere.

We construct the measures $\mathbf{b}_1'(M; z, k, w)$ and $\mathbf{v}_2'(M; z, k, b, w)$ similarly. The corresponding measures for all subsequent periods are constructed in the same fashion. Call the new allocation $\mathbf{P}'$. It is a randomized allocation rule and it satisfies (D.1), (D.2), and (D.3). ■

Proof of Proposition D.1. The proposition is proved by 2 claims. Suppose that $\varphi = A \int_{\mathcal{V}} u^{-1}(w)\psi(dw)$, $A > 0$.

Claim 1 *Then in Problem SFE we can impose without loss of generality the additional constraint that b and v are deterministic conditional on allocated capital k and productivity report z.*

Claim 2 *Suppose that $\varphi(\psi) = A \int_{\mathcal{V}} u^{-1}(w)\psi(dw)$. Then the solution to Problem SFE is deterministic conditional on z and given by the solution to Problem AP.*

Proof of Claim 1

Define $v'(w_0,k,z) = \int\int v d\mathbf{v}(v|w_0,k,z,b)d\mathbf{b}(b|w_0,k,z)$. Similarly, define $b'(w_0,k,z)$ implicitly by $u(zk+b'(w_0,k,z)) = \int u(zk+b)d\mathbf{b}(b|w_0,k,z)$. The new allocation (b'v') relaxes the resource constraint and satisfies promise-keeping constraint in Problem SFE. We need to show that it satisfies incentive compatibility in Problem SFE. Let $\tilde{c}_{i+1}$ be the certainty equivalent of the lottery $(z_{i+1}k + b|w_0,k,z_i)$ (the consumption of an agent who misreports). For the truthteller, $z_ik + b(w_0,k,z_i)$ is the certainty equivalent of the lottery $(z_ik + b|w_0,k,z_i)$. Since u displays decreasing absolute risk aversion $\tilde{c}_{i+1} > b(w_0,k,z_i) + z_{i+1}k$, so

$$(1-\beta)u(z_{i+1}k + b(w_0,k,z_i)) + \beta v'(w_0,k,z_i) \leq$$
$$\int [(1-\beta)u(z_{i+1}k + b) + \beta w']d\mathbf{v}(v|w_0,k,z,b)d\mathbf{b}(b|w_0,k,z_i) \leq$$
$$\int [(1-\beta)u(f(k,z_{i+1}) + b) + \beta w']d\mathbf{v}(v|w_0,k,b,z_{i+1})d\mathbf{b}(b|w_0,k,z_{i+1}) =$$
$$(1-\beta)u(z_{i+1}z + b(w_0,k,z_{i+1})) + \beta v'(w_0,k,z_{i+1}).$$

Then the allocation satisfies promise-keeping and $S_{i+1,i} \geq 0$. Also, $v'(w_0,k,z)$ is increasing in z. Then by Claim 5 of the proof of Proposition 2 we can modify this allocation further to satisfy all of the incentive constraints in Problem SFE.

Proof of Claim 2.

First, we define the stochastic analogue to Problem AP2.

Problem APL

$$\hat{k}^L(w,N;A) = \inf_{\mathbf{k},b(k,z),v(k,z)} \int k d\mathbf{k}(k)$$
$$s.\ to\ (1-\beta)u(zk + b(k,z)) + \beta v'(k,z) \geq$$
$$(1-\beta)u(zk + b(k,z')) + \beta v'(k,z'), \quad z,z' \in \mathbf{Z}, z \neq z'$$
$$\int \int_{\mathbf{Z}} \left[(1-\beta)u(zk + b(k,z)) + \beta v'(k,z)\right] \mu(dz)d\mathbf{k}(k) = w,$$
$$-\int \int_{\mathbf{Z}} b(k,z)\mu(dz)d\mathbf{k}(k) \geq N + A\int \int_{\mathbf{Z}} u^{-1}(v'(k,z))\mu(dz)d\mathbf{k}(k),$$

where $\mathbf{k}$ *is a Borel probability measure on* $\mathbf{R}_+$*, and* $b(k,z)$ *and* $v'(k,z)$ *are Borel-measurable functions.*

By Claim 1, we can assume that b and v' are deterministic, conditional on k and z. Then for almost all w, the optimal one-period allocation rule can be derived from Problem APL. Then we have:

$$T^L\varphi(\psi) = \inf_{N(w)} \int_{\mathcal{V}} \hat{k}^L(w,N(w);A)\psi(dw)$$
$$\text{subject to} \int_{\mathcal{V}} N(w)\psi(dw) \geq 0.$$

Since the choice variables in Problem APL are lotteries, it is immediate that $\hat{k}^L(w,N;A)$ is convex in N. By the same logic as in the deterministic case, $\hat{k}^L(w,N;A) = u^{-1}(w)\hat{k}^L(u(1),N(w)/u^{-1}(w);A)$. Then exactly as in Claim 10 of the proof of Proposition 2, it is optimal to set $N(w) = 0$ a.e.

Next, consider the solution to Problem APL for some w (which we will suppress in the notations). Define $N(k) = -\int\int_{\mathbf{Z}} b(k,z)\mu(dz) - A\int\int_{\mathbf{Z}} u^{-1}(v'(k,z))\mu(dz)$ and $w(k) = \int\int_{\mathbf{Z}} [(1-\beta)u(zk + b(k,z)) + \beta v'(k,z)]\,\mu(dz)$. Then the constraint $\int\int_{\mathbf{Z}} [(1-\beta)u(zk + b(k,z)) + \beta v'(k,z)]\,\mu(dz)d\mathbf{k}(k) = w$ is equivalent to $\int w(k)\mathbf{k}(dk) = w$. Similarly, the constraint $-\int\int_{\mathbf{Z}} b(k,z)\mu(dz)d\mathbf{k}(k) \geq N + A\int\int_{\mathbf{Z}} u^{-1}(v'(k,z))\mu(dz)d\mathbf{k}(k)$ is equivalent to $\int N(k)\mathbf{k}(dk) \geq 0$.

It is immediate that the solution to Problem APL must be $k = \hat{k}(w(k),N(k);A)$ almost surely. Then the problem for $N = 0$ can be rewritten as:

$$\hat{k}^L(w,0;A) = \inf_{\lambda(s);N(s)} \int_{\mathcal{V}} \hat{k}(s,N(s);A)\lambda(ds)$$
$$\text{subject to} \int_{\mathcal{V}} s\lambda(ds) = w$$

$$\int_{\mathcal{V}} N(s)\lambda(ds) = 0,$$

where λ is a Borel probability measure over $\mathcal{V}$.

Exactly as in Claim 10 of the proof of Proposition 2, we can show that it is optimal to set $N(s) = 0$ for all s. Then Problem SFE becomes:

$$\inf_{\lambda(s);N(s)} \int_{\mathcal{V}} u^{-1}(w)\hat{k}(u(1), 0; A)\lambda(ds)$$
$$\text{subject to } \int_{\mathcal{V}} s\lambda(ds) = w,$$

where $\hat{k}$ is defined by Problem AP2. Since $u^{-1}(w)$ is strictly convex, it optimal to have λ be the degenerate lottery: $\lambda(\{w\}) = 0$.

Proof of the proposition. The proposition follows from Claim 2 and the replication of the proof of Proposition 2. ■