Potential Exam Problems for Module 3

1. Let n be a positive integer that is not a perfect square. Let $\mathbb{Q}(\sqrt{n})$ be the set of all real numbers x that can be expressed in the form $x = r + s\sqrt{n}$, where r and s are rational numbers.

   (a) Explain why 0 and 1 are in $\mathbb{Q}(\sqrt{n})$.

   > **Solution:** Since $0 = 0 + 0\sqrt{n}$ and $1 = 1 + 0\sqrt{n}$.

   (b) Prove that $+$ and $\cdot$ are closed in $\mathbb{Q}(\sqrt{n})$.

   > **Solution:** Suppose we have $a + b\sqrt{n}$,$c + d\sqrt{n}$ both in $\mathbb{Q}(\sqrt{n})$. Then
   >
   > $$(a + b\sqrt{n}) + (c + d\sqrt{n}) = (a + b) + (c + d)\sqrt{n}$$
   >
   > and
   >
   > $$\begin{aligned}(a + b\sqrt{n})(c + d\sqrt{n}) &= ac + ad\sqrt{n} + bc\sqrt{n} + bdn \\ &= (ac + bdn) + (ad + bc)\sqrt{n}\end{aligned}$$
   >
   > And since a, b, c, d and n are rationals, so are $a + b$, $c + d$, $ac + bdn$ and $ad + bc$.

   (c) Prove that $\langle \mathbb{Q}(\sqrt{n}), +, \cdot \rangle$ is a subfield of the field $\langle \mathbb{R}, +, \cdot \rangle$ of real numbers and that $\mathbb{Q}(\sqrt{n})$ contains the field $\langle \mathbb{Q}, +, \cdot \rangle$ of rationals as a subfield.

   > **Solution:** First, $\mathbb{Q} \subset \mathbb{Q}(\sqrt{n}) \subset \mathbb{R}$. Second, the operations of addition and multiplication on $\mathbb{R}$ restricted to $\mathbb{Q}(\sqrt{n})$ give the field $\langle \mathbb{Q}(\sqrt{n}), +, \cdot \rangle$. Similarly, those operations further restricted to $\mathbb{Q}$ give the field $\langle \mathbb{Q}, +, \cdot \rangle$.

   (d) Prove that $\sqrt{3}$ is not in $\mathbb{Q}(\sqrt{2})$.

   > **Solution:** If $\sqrt{3} = a + b\sqrt{2}$ where a and b are rationals, then
   >
   > $$3 = a^2 + 2ab\sqrt{2} + 2b^2.$$
   >
   > Then
   >
   > $$\sqrt{2} = \frac{3 - a^2 - 2b^2}{2ab}.$$
   >
   > However, this cannot happen since $\sqrt{2}$ is irrational.

2. Suppose that $\langle \mathbb{F}, +, \cdot, \rangle$ is a field with additive identity 0 and multiplicative identify 1. Prove that for any $x \in \mathbb{F}$, $(-1)x = -x$.

Solution: By the distributive axiom,

$$x + (-1)x = x(1 + (-1))$$

By the definition of -1, $1 + (-1) = 0$. Since $0x = 0$ we have

$$\begin{aligned} x + (-1)x &= x(1 + (-1)) \\ &= x(0) = 0 \end{aligned}$$

Since $x + (-1)x = 0$ it follows that $(-1)x$ is the additive inverse of x. That is, $(-1)x = -x$.

3. Suppose that $\langle \mathbb{F}, +, \cdot, < \rangle$ is an ordered field with additive identity 0 and multiplicative identify 1. Using only the field axioms and the ordering axioms, prove that

   (a) if x and y are in $\mathbb{F}$ and are positive, then $x + y$ and xy are positive.

   Solution: Suppose $0 < x$ and $0 < y$. By the ordering axioms: $0 + y < x + y$, and $0 < y$ together with $y < x + y$ implies $0 < x + y$. Similarly, $y(0) < yx$; that is $0 < xy$.

   (b) if $x \neq 0$ is an element of $\mathbb{F}$ then x^2 is positive.

   Solution: If $0 < x$ then $0x < x(x)$. That is, $0 < x^2$. If $x < 0$ then $0 < -x$. So $0(-x) < (-x)(-x)$. But by above, $-x = (-1)x$. So, $(-x)(-x) = (-1)x(-1)x = (-1)^2x^2$. Since $(-1)^2 + -1 = (-1)(-1 + 1) = (-1)(0) = 0$, it follows that $(-1)^2$ is the additive inverse of -1; that is $(-1)^2 = 1$. So $(-x)(-x) = x^2$. Thus, $0 < x^2$.

4. Prove that in an ordered field if x is positive then $\frac{1}{x}$ is positive.

Solution: (This proof uses that a negative times a positive is negative. So just to be sure I'll prove that first. So suppose z is positive and w is negative. Then $-w$ is positive, so $z(-w)$ is positive. Then

$$zw + z(-w) = z(w + (-w)) = z(0) = 0.$$

So zw is the additive inverse of $z(-w)$. Since $z(-w)$ is positive, zw must be negative.) So, suppose $x > 0$ but $\frac{1}{x} < 0$. Then $x\left(\frac{1}{x}\right) = 1 < 0$.

5. Prove that if $\langle \mathbb{F}, +, \cdot \rangle$ is a finite field, then it is not possible to define an order relation on $\mathbb{F}$ that satisfies all of the ordering axioms.

Solution: If $\langle \mathbb{F}, +, \cdot \rangle$ is a finite field with an ordering $<$ then there must be a largest element f. But then consider $f + 1$. It must either be equal to f in which case $1 = 0$ or it must be smaller than f which contradicts one of the ordering axioms (the one about order being preserved when adding).

6. Prove that in an ordered field the product of two negatives is positive.

Solution: Suppose x and y are negative. Then $(-x)(-y)$ is positive. Also,

$$\begin{aligned}(-x)(-y) &= (-1)x(-1)y \\ &= (-1)^2xy \\ &= xy\end{aligned}$$

(See above for a proof that $(-1)^2 = 1$.)

7. Suppose that $\langle \mathbb{F}, +, \cdot, < \rangle$ is an ordered field. Using only the field axioms and ordering axioms, prove that if a and b are negative elements of $\mathbb{F}$ then $\frac{a}{b}$ is positive.

Solution: First, we claim that if b is negative then $\frac{1}{b}$ is negative. If b is negative and $\frac{1}{b}$ is positive then

$$\begin{gathered} b < 0 \\ \frac{1}{b}b < \left(\frac{1}{b}\right)0 \\ 1 < 0 \end{gathered}$$

Contradiction. So, if b is negative, then $\frac{1}{b}$ is negative. Suppose a is negative. Then $-a$ is positive. So, if a and b are negative, then $-a\frac{1}{b}$ is a positive times a negative so is negative. (This last statement follows directly from the axiom that if $x < y$ and $0 < z$ then $zx < zy$.) Finally, And

$$\frac{a}{b} + (-a)\left(\frac{1}{b}\right) = \frac{1}{b}(a + -a) = 0$$

So $(-a)\frac{1}{b}$ is the additive inverse of $\frac{a}{b}$. Since $(-a)\frac{1}{b}$ is negative, $\frac{a}{b}$ must be positive.

8. If an integer n is not a perfect square, explain why the ordered field $\langle \mathbb{Q}(\sqrt{n}), +, \cdot, < \rangle$ is Archimedean but not complete.

Solution: That the field is Archimedian can be seen as follows: Let $r = a + b\sqrt{n} \in \mathbb{Q}(\sqrt{n})$. The number r is also in $\mathbb{R}$. Since $\mathbb{R}$ is Archimedean we can find an integer $m \in \mathbb{R}$ such that $m \leq r < m+1$. But since $m \in \mathbb{Q}(\sqrt{n})$ also, the statement $m \leq r < m + 1$ holds there as well. We now show that $\mathbb{Q}(\sqrt{n})$ is not complete. First, let m be an integer other than n and consider the set

$$S = \{x + \sqrt{n} | x^2 < m\}$$

The least upper bound of this set is $\sqrt{m} + \sqrt{n}$ which is not in $\mathbb{Q}(\sqrt{n})$.

9. (a) Define what it means for an ordered field to be complete.

Solution: An ordered field $\mathbb{F}$ is complete if every nonempty subset $S \subseteq \mathbb{F}$ which is bounded above has a least upper bound.

(b) Define what it means for an ordered field to be Archimedean.

Solution: An ordered field $\mathbb{F}$ is Archimedean if for every $x \in \mathbb{F}$, there is an integral element $n \in \mathbb{F}$ such that $x < n$.

(c) Given an example of an Archimedean field which is not complete. (Just state the example, you do not need to prove it satisifies the requirements.)

Solution: The rationals, $\mathbb{Q}$.

(d) Prove that every complete field is Archimedean.

Solution: Suppose that $\mathbb{F}$ is a complete field. Suppose that $\mathbb{F}$ is not Archimedean. Then there exists an $x \in \mathbb{F}$ such that for all integral elements $n \in \mathbb{F}$, $n^* \leq x$. This means that the set N of interal elements of $\mathbb{F}$ is bounded above. By completeness of $\mathbb{F}$, N must have a least upper bound, say b. Then $b - 1$ is not an upper bound so there exists $m \in N$ such that $b - 1 < m$. Then $b < m + 1$. However, $m + 1$ is an integral element of $\mathbb{F}$, contradicting the definition of b.

10. Consider the number $r = \sqrt[4]{13} + \frac{4}{3}\sqrt{\sqrt{6} + \sqrt{1 + 2\sqrt{7}}}$. Give an explicit sequence of fields $\mathbb{F}_0 = \mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{F}_N$ such that $r \in \mathbb{F}_N$ and for $1 \leq j \leq N - 1$ each $\mathbb{F}_{j+1}$ is a quadratic extension of its predecessor $\mathbb{F}_j$.

Solution: The number $r \in \mathbb{F}_6$ where

$$
\begin{aligned}
\mathbb{F}_0 &= \mathbb{Q} \\
\mathbb{F}_1 &= \mathbb{F}_0(\sqrt{7}) \\
\mathbb{F}_2 &= \mathbb{F}_1\left(\sqrt{1 + 2\sqrt{7}}\right) \\
\mathbb{F}_3 &= \mathbb{F}_2(\sqrt{6}) \\
\mathbb{F}_4 &= \mathbb{F}_3\left(\sqrt{\sqrt{6} + \sqrt{1 + 2\sqrt{7}}}\right) \\
\mathbb{F}_5 &= \mathbb{F}_4(\sqrt{13}) \\
\mathbb{F}_6 &= \mathbb{F}_5(\sqrt[4]{13})
\end{aligned}
$$

11. Explain the main steps of one of the following proofs:

- That a cube cannot be doubled, or
- That an arbitrary angle cannot be trisected

In your explanation, state the result which connects constructible numbers to quadratic extensions of the rationals, and explain how that theorem is applied to prove the result.

Solution:

- Stating main theorem which connects constructible numbers to quadratic extensions: 5

- 2pts for explaining either that
  - Doubling a cube is equivalent to constructing $\sqrt[3]{2}$, or
  - Trisecting an angle 3θ is equivalent to constructing $\cos(theta)$.
- 3 points for explaining either that
  - If $\sqrt[3]{2}$ is in a quadratic extension $\mathbb{F}(\sqrt{k})$, then it is in the base field $\mathbb{F}$, so this implies $\sqrt[3]{2}$ must be rational if it is constructible, or
  - The number $\cos(\theta)$ with $\theta = 20$ deg is the root of a polynomial. We showed that if this polynomial has a root in some quadratic extension $\mathbb{F}(\sqrt{k})$ then it has a root in $\mathbb{F}$. This implies it has a rational root, and we showed this is impossible.

12. Can the cube be "tripled"?

Solution: Nope, same argument as for doubled, but with $\sqrt[3]{3}$.

13. It is clearly possible to divide an arbitrary angle into four equal parts by repeated bisection. Show how this may also be deduced algebraically from the equation relating $\cos(4\theta)$ and $\cos\theta$.

Solution: Assume that an angle 4θ is constructible. By an argument similar to what we have seen in class (i.e. by *geometry*, this implies that $\cos(4\theta)$ is constructible. We also know that dividing the angle into four equal parts would be equivalent to constructing $\cos(\theta)$. So, we argue that $\cos(\theta)$ is constructible: First, using trig identities you can show that

$$\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$$

If we let $x = \cos^2(\theta)$ we obtain the equation,

$$8x^2 - 8x + (1 - \cos(4\theta) = 0$$

The quadratic equation gives

$$x = \frac{8 \pm \sqrt{64 - 32(1 - \cos(4\theta))}}{16}$$

Notice that since $-1 \leq \cos(4\theta) \leq 1$, the discrimiant is at least zero, so the possible values for x are always real numbers. Even more, the values of x can clearly be seen to be in $\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$. Thus those possible values for x are constructible. That gives us that $\cos^2(\theta)$ is constructible, and that implies $\cos(\theta)$ is constructible.