Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Adapted for University of Nebraska-Lincoln

Version Spring 2018

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Todo list

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Part I

How We Talk and Explore Math

0 Communicating Mathematics in this Course and Beyond

Set and Logical Notation

Set Notation

Definition 1.1. A **set** is a collection of objects, which are called the **elements** of the set.

$x \in D$	"x is an element of the set D" (a proposition about x and its *domain* D)
$P(x)$	A proposition about the variable x; may be true or false depending on x
$\{x \in D : P(x)\}$ $\{x \in D \mid P(x)\}$	The set of all elements of D for which $P(x)$ is true (a subset of D)
$A \subseteq B$	"A is a subset of B" (a proposition about sets A and B)
$A \subsetneq B$	"A is a strict subset of B", i.e., "$A \subseteq B$ and $A \neq B$"
$A \supseteq B$	"A is a superset of B" or "B is a subset of A"
$A \supsetneq B$	"A is a strict superset of B" or "B is a strict subset of A", i.e., "$A \supseteq B$ and $A \neq B$"
$A \cap B$	The intersection of the sets A and B (a set)
$A \cup B$	The union of the sets A and B (a set)
$\varnothing$	The *empty set* (the set with no elements); also known as *null set*
$\lvert A \rvert$	The cardinality ("size") of A. When A is finite, $\lvert A \rvert$ is the number of elements in A.

Note. The notation for subset (without the bottom line) is ambiguous: some people use it to mean $A \subseteq B$ and others use it to mean $A \subsetneq B$. So we don't use it here.

Definition 1.2. Given sets A and B. We say A **is equal to** B if $A \subseteq B$ and $B \subseteq A$.
Notation: $A = B$.

Logical notation

$\neg P(x)$	The negation of $P(x)$
$\forall x, P(x)$	The proposition "For all values of x, $P(x)$ is true."
$\exists x : P(x)$	The proposition "There exists a value of x such that $P(x)$ is true."
$\forall x, P(x) \Rightarrow Q(x)$	The proposition "For all values of x, if $P(x)$ is true then $Q(x)$ is true."
$\forall x, P(x) \Leftrightarrow Q(x)$	The proposition "For all values of x, $P(x)$ is true if and only if $Q(x)$ is true."

Proof structures

To show that ...	Requires showing that ...
$x \in A$	x satisfies set membership rules for A
$x \notin A$	x does not satisfy at least one set membership rule of A
$A \subseteq B$	If $x \in A$, then $x \in B$
$A \subsetneq B$	(1) $A \subseteq B$ (2) there is an element of B that is not in A
$A = B$	(1) $A \subseteq B$ (2) $B \subseteq A$

Sets of numbers

$\mathbb{N}$	The set of *natural numbers* (positive whole numbers)
$\mathbb{Z}$	The set of *integers* (all whole numbers – positive, negative, and zero)
$\mathbb{Q}$	The set of *rational numbers* (all fractions)
$\mathbb{R}$	The set of *real numbers* (all numbers on the real line; equivalently, all decimal numbers)
$\mathbb{C}$	The set of *complex numbers* (all numbers of the form $a + bi$, where a and b are real)

Properties of $\mathbb{R}$ and $\mathbb{Z}$

Operations are well-defined

Well-defined: There is an answer, and there isn't more than one answer.

Operations $+, -, \times$ on $\mathbb{R}$ are well-defined: This means that when we add two numbers, we get exactly one answer (we don't expect there two be two answers to "What is $a + b$?" and we expect that there is an answer); similarly, when we subtract one number from another, or multiply two numbers, we get exactly one answer.

Division by nonzero numbers is well-defined. (There is no good numerical answer to "What is $a/0$?")

Arithmetic Properties of $\mathbb{Z}$ and $\mathbb{R}$

We state them below for $\mathbb{Z}$. They also hold for $\mathbb{R}$.

1	$a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z}$	$\mathbb{Z}$ is closed under addition
2	$a, b, c \in \mathbb{Z} \implies a + (b + c) = (a + b) + c$	Addition in $\mathbb{Z}$ is associative
3	$a, b \in \mathbb{Z} \implies a + b = b + a$	Addition in $\mathbb{Z}$ is commutative
4	$a \in \mathbb{Z} \implies a + 0 = a = 0 + a$	0 is an additive identity in $\mathbb{Z}$
5	$\forall a \in \mathbb{Z}$, the equation $a + x = 0$ has a solution in $\mathbb{Z}$	Additive inverses exist in $\mathbb{Z}$
6	$a, b \in \mathbb{Z} \implies ab \in \mathbb{Z}$	$\mathbb{Z}$ is closed under multiplication
7	$a, b, c \in \mathbb{Z} \implies a(bc) = (ab)c$	Multiplication in $\mathbb{Z}$ is associative
8	$a, b, c \in \mathbb{Z} \implies a(b + c) = ab + ac$ and $(a + b)c = ac + bc$	Distributive property
9	$a, b \in \mathbb{Z} \implies ab = ba$	Multiplication in $\mathbb{Z}$ is commutative
10	$a \in \mathbb{Z} \implies a \cdot 1 = a = 1 \cdot a$	1 is a multiplicative identity in $\mathbb{Z}$
11	$a, b \in \mathbb{Z}, ab = 0 \implies a = 0$ or $b = 0$	$\mathbb{Z}$ has no zero divisors

Divides, Divisor, Factor

- Given $a, b \in \mathbb{Z}$, not both zero. We say *b divides a* if $a = bc$ for some integer c. Notation: $b \mid a$
  These all mean the same thing:
  - b divides a
  - b is a divisor of a
  - b is a factor of a
  - $b \mid a$

  If we want to say that b does not divide a, we write $b \nmid a$.
- A factor of a number is *trivial* if it is ± 1 or the $\pm$ number. A *nontrivial* factor that is not trivial.
- All nonzero natural numbers have a finite number of factors.
- Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Prime, Composite

- An integer p is *prime* if $p \neq 0, \pm 1$ if the only divisors of p are ± 1 and $\pm p$.
  An integer n is *composite* if $n \neq 0, \pm 1$, and it is not prime.
- Let $a \in \mathbb{Z}$. If p, q are primes such that $p \mid a$ and $q \mid a$, and $p \neq q$, then $pq \mid a$.

Even number

An integer n is even if it is divisible by 2.

Fundamental Theorem of Arithmetic

There is only one way to write any whole number as a product of positive primes (reordering doesn't count as a different way).

Sample handwritten proof

Let's use one of the proofs we did in class as an example. We begin with the typed up version and then show one way that this same proof might be handwritten.

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$.

1. *Why $A \subseteq B$*: This was done in the claim we just showed. $\boxed{1}$
2. *Why there is an element of B that is not in A.* If $x \in B$, then x is an even number because if $x = 6k$ for some $k \in \mathbb{Z}$, then x as $x = 2 \cdot (3k)$. Closure of multiplication in $\mathbb{Z}$ implies $3k \in \mathbb{Z}$, so x satisfies the definition of even number.

   However, some members of A are odd numbers: $3, 9, 15, \ldots$.

   Hence there are elements of A that are not in B. $\boxed{2}$

Why this means $B \subsetneq A$: We showed $B \subseteq A$ and found elements of A not in B. By definition of $\subseteq$, we have $A \subsetneq B$.
$\square$

[Handwritten version of this proof goes here]

Ch 0: Create handwritten version of proof, insert in reference for section 1

Good proof communication

Here is the same proof, with key features pointed out. These features are explained at the bottom. In general, you want to incorporate most if not all of these features into any proof you write. Even though it might seem strange at first, you may find eventually that you learn math better when you develop the habits of incorporating these features into your own writing and being aware of these features in proofs you encounter.

[Handwritten version of this proof goes here]

Ch 0: After creating handwritten version of this proof, label the features below by number.

Features of communicating proof well:
(**Essential features in bold**)

1. **Label the claim.**
2. **State the claim precisely.**
3. **Label the proof beginning.**
4. Begin a proof by reminding yourself and readers of the starting point: the conditions of the claim.
5. End the proof with where you need to go: the conclusions of the claim.
6. Summarize your approach to the reader.
7. **Label the proof end.** A traditional way is to use a box.
8. **Write up parts within a proof properly**. **Label when they begin and end.**
   - Give them a name (e.g., Claim A) if it is a proof within a proof
   - **Use labels like** [$\Rightarrow$] **and** [$\Leftarrow$] **if doing an if and only if proof.**
9. Diagrams are good only if you explain what you are showing. Give a caption.

1 Sets, Claims, Negations (Week 1) (Length: 2.5 hours)

Overview

Content

“Parent” relation, implicitly defined as a relation which assigns elements of $\mathbb{N}$ to its factors; used to examine subsets, mathematical statements and their negations, properties of $\mathbb{R}$ and $\mathbb{Z}$, and to engage in mathematical practices.

(Looking ahead:) The parent relation is used in Section 2 to introduce relations and inverse relations.

Subset, **superset**, **strict subset**, and **strict superset**; **equality of sets** A and B, defined as $A \subseteq B$ and $B \subseteq A$.

Mathematical statements, defined as those which can be evaluated as true or false; and

Negation of mathematical statement S, defined as a statement which is false if and only if S is true.

Properties of $\mathbb{R}$ and $\mathbb{Z}$ assumed. (These may have been introduced previously in an abstract algebra course.)

Proof Structures	Mathematical/Teaching Practices
To show that $x \in A$ means showing that x satisfies set membership rules for A; and **to show that** $x \notin A$ means showing that x does not satisfy at least one set membership rule of A. **To show that** $A \subseteq B$ requires showing that if $x \in A$, then $x \in B$. **To show that** $A \subsetneq B$ requires showing that: (1) $A \subseteq B$; (2) there is an element of B that is not in A. **To show that** $A = B$ requires showing that: (1) $A \subseteq B$; (2) $B \subseteq A$.	**Clarifying mathematical questions**, meaning to determine how different interpretations of question statements may have different mathematical consequences. **Conjecturing and being precise**, in the sense of giving "satisfying" answers to mathematical questions **Communicating proofs well**, which includes specifying claims, the body of the proof, and givens and conclusions explicitly, clearly, and correctly.

Summary

We introduce the "parent relation" as a context for engaging in mathematical practices as well as learning how to work with each other on exploratory tasks. The main tasks in this lesson are:

- *Which numbers have more than one pair of parents?*
- *Is one of these sets a subset of the other set? Check the mathematically correct statements. If you put a check in the* $A \neq B$ *column, list an element that is in one but not the other.*

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ multiples of 3, $B =$ multiples of 6							
$A =$ multiples of 6, $B =$ multiples of 9							
$A = \{n^2 : n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n+1) : n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$							

Along the way we introduce notation for sets and subsets, discuss mathematical statements and their negations, and describe properties of $\mathbb{R}$ and $\mathbb{Z}$ assumed for now. There are also tasks in this lesson addressing these ideas.

Acknowledgements. The structure and some tasks of Set notation and Mathematical statements and their negations are from notes from Mira Bernstein and used with permission.

Materials.

- All pages in Section 0: Communicating Mathematics (can be printed double-sided)
- Handouts in In-Class Resources (can be printed double-sided)
- Colored chalk / markers to highlight different parts of good proof communication

Opening inquiry: Number parents

We begin this lesson with the following inquiry:

Two numbers are parents of a child if the child is their product.

A child cannot be its own parent.

Which numbers have more than one pair of parents?

Child	Parents
6	2, 3
4	??
12	4, 3
12	2, 6

Instructor note. Distribute handout with this question. As teachers work on it, circulate and listen to the questions and comments they make. They may say and do things that will lead into a discussion on clarifying the question, precision, and also what it means to have less or more satisfying answers to a question.

As we discussed this question, we learned some issues that arise when asking and answering mathematical questions:

- *Clarifying the question.* Let's assume that we are only working with natural numbers (0, 1, 2, . . .), and that 2, 2 is a set of parents for 4. So we are looking for natural numbers that have more than one pair of parents. We allow pairs of parents to repeat parents.
- *Finding and improving possible answers (conjecturing well).* Here are some possible answers (without explanations) to this question. Which is the most satisfying answer (without explanation)? Why?
  1. 12 has more than one pair of parents.
  2. 12, 18, 20, 28, 30, 42, 44 each have more than one pair of parents.
  3. Any number with at least least three different factors has more than one pair of parents.
  4. Any number with at least three different factors (that aren't itself or 1) has more than one pair of parents.
  5. Any number with at least three different factors (that aren't itself or 1) has more than one pair of parents. There are no other numbers with more than one pair of parents.

Instructor note. The above are answers that prospective teachers in previous courses have given. You might use some of these answers as ringers for your own class discussion, or simply use a variety of answers that teachers in your class have given. The main thing is to have a variety of levels of how satisfying the answers are.

We concluded that an answer is satisfying when it gives the most complete and correct understanding of a situation. We also gave the analogy of answering a question that a child asks, and that the quality of being "satisfying" when giving an answer to a mathematical may well be similar to what makes an answer "satisfying" to a child.

- The first two are dissatisfying because they don't give any sort of pattern or big picture of what's going on. They raise the question: "Are those the only ones?"
- The third one is almost there, but is actually slightly incorrect. The fourth one is getting there, and it is correct. But still, neither answer the question of whether there are more answers.
- The fifth answer is the most satisfying because it provides the big picture of when a number works, and also says, yes, these are the only answers.

We also gave a name to the process of finding and improving answers to mathematical question: the practice of *conjecturing*. Before we get into proving or disproving our conjectures, we first talk about sets. This will give us a structure for addressing this inquiry more completely.

Sets, subsets, supersets, and set equality

Set notation

Definition 1.1. A **set** is a collection of objects, which are called the **elements** of the set.

$x \in D$	"x is an element of the set D" (a proposition about x and its *domain* D)
$P(x)$	A proposition about the variable x; may be true or false depending on x
$\{x \in D : P(x)\}$ $\{x \in D \mid P(x)\}$	The set of all elements of D for which $P(x)$ is true (a subset of D)
$A \subseteq B$	"A is a subset of B" (a proposition about sets A and B)
$A \subsetneq B$	"A is a strict subset of B", i.e., "$A \subseteq B$ and $A \neq B$"
$A \supseteq B$	"A is a superset of B" or "B is a subset of A"
$A \supsetneq B$	"A is a strict superset of B" or "B is a strict subset of A", i.e., "$A \supseteq B$ and $A \neq B$"
$A \cap B$	The intersection of the sets A and B (a set)
$A \cup B$	The union of the sets A and B (a set)
$\emptyset$	The *empty set* (the set with no elements); also known as *null set*
$\lvert A\rvert$	The cardinality ("size") of A. When A is finite, $\lvert A\rvert$ is the number of elements in A.

Note: The notation for subset (without the bottom line) is ambiguous: some people use it to mean $A \subseteq B$ and others use it to mean $A \subsetneq B$. So we don't use it here.

Definition 1.2. Given sets A and B. We say A **is equal to** B if $A \subseteq B$ and $B \subseteq A$. We denote equality with $A = B$.

1. Let $A = \{1, 2, \{3, 4\}, \{5\}\}$. Decide whether each of the following statements is true or false: (**Hint:** There are exactly six true statements.)

$$\begin{array}{cccc} 1 \in A, & \{1,2\} \in A, & \{1,2\} \subseteq A, & \emptyset \in A, \\ 3 \in A, & \{3,4\} \in A, & \{3,4\} \subseteq A, & \emptyset \subseteq A, \\ \{1\} \in A, & \{1\} \subseteq A, & \{5\} \in A, & \{5\} \subseteq A. \end{array}$$

2. True or false? "All students in this class who are under 5 years old are also over 100 years old."

Solution.

1. (a) TRUE (b) false (c) TRUE (d) false
   (e) false (f) TRUE (g) false (h) TRUE
   (i) false (j) TRUE (k) TRUE (l) false

   Reasoning. There are four elements of the set A:
   - 1 (the number 1)
   - 2 (the number 2)
   - $\{3,4\}$ (the set containing the numbers 3, 4)
   - $\{5\}$ (the set containing the number 5)

   The notation $\in$ means "is an element of" is . That's why (a), (f), (k) are TRUE and (b), (d), (e), (i) are false.

   The notation $\subseteq$ means "is a subset of". The set is a subset of A if each of its elements are also elements of A. That's why (c), (j) are TRUE and (g), (l) are false.

   Finally, (h) is TRUE on a technicality. It contains no elements. So all zero of its elements are part of A. The empty set is a subset of any set for this reason.

2. For most sections of mathematics courses at university level, this statement should be TRUE. ■

Note: One helpful metaphor may be thinking of the braces (the { and }) as permanent packaging, like gift wrap that doesn't come off. You can't take out what's inside the packaging. You can only hold the whole package. Even if only one thing is wrapped, you still can't hold the thing by itself, you can only hold it with its gift wrap. But if an object not wrapped, you can hold that object by itself.

Proof Structure: Showing set membership. To show that $x \in S$ means showing that x satisfies set membership rules for S; to show that $x \notin S$ means showing that x does not satisfy at least one set membership rule of A.

> Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
>
> True or false? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $1 \in S$.

Solution. (Partial)

(a) $0.5 \in S$ is TRUE because it can be written as the fraction $\frac{1}{2}$ and $|0.5| < 2$. The number 0.5 satisfies all the rules of membership of S, so it is an element of S.

(b) $3.5 \in S$ is FALSE because even though it can be written as the fraction $\frac{7}{2}$, it does not satisfy the condition $|x| < 2$. The number 3.5 does not satisfy all the rules of membership of S, so it is not an element of S.

(c) $0.25 \in S$ is FALSE. (Why?)

(d) $1 \in S$ is TRUE. (Why? Hint: The fraction does not have to be in lowest terms ...) ■

Subset exploration

Is A a subset of B or vice versa? Complete this table with "yes" or "no" in each cell.

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ multiples of 3, $B =$ multiples of 6							
$A =$ multiples of 6, $B =$ multiples of 9							
$A = \{n^2 \mid n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n+1) \mid n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$							

Teaching the subset exploration task. Take this task one row at a time, emphasizing the mathematical practices of *clarifying the question* and then *finding and improving possible answers (aka conjecturing)*. The goal is first to generate conjectures; then, after generating satisfying conjectures, to *prove (or disprove) the conjectures*.

Rows 1, 2, and 4 can be interpreted in different ways with different mathematical consequences. You may decide with your class to interpret:

- Row 1, 2: Multiples should mean "integer multiples"
- Row 4: a should be considered in two cases, $a \in \mathbb{Z}$ and $a \in \mathbb{Q}$.

This means revising Row 4 and adding a Row 5 to the table:

$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$, where $a \in \mathbb{Z}$					
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$, where $a \in \mathbb{Q}$					

Row 3 may need clarification as far as set notation and what the "..." mean, but is otherwise precisely phrased.

Row 3 may be assigned as homework after discussing what there is to prove.

This task is designed to show why equality of sets requires showing both that $A \subseteq B$ and $B \subseteq A$. Often we have found that students think of showing one direction as sufficient, and that this is reinforced by tasks where containment follows practically tautologically by definition. The examples in rows 3 and 4 do require inference from the definitions, not just the definitions themselves.

Clarifying the question. We found that there were several ways that these questions needed to be clarified: In Row 1 and 2, we asked: what kind of multiples? We decided to consider only integer multiples. In Row 4, we asked: What is a? If $a \in \mathbb{Z}$, there are different consequences than when $a \in \mathbb{Q}$. We added this interpretation as a different row.

Making conjectures/observations and improving them. Possible conjectures about this table include:

- (set of integer multiples of 3) $\supseteq$ (set of integer multiples of 6)
- (set of integer multiples of 3) $\supsetneq$ (set of integer multiples of 6)
- (set of integer multiples of 6) and (set of integer multiples of 9) are not subsets of each other
- (set of perfect squares) $=$ (set of sum of consecutive odd positive numbers)
- When $a \in \mathbb{Z}$, (set of functions of the form $x \mapsto 16^{ax}$) $\subsetneq$ (set of functions of the form $x \mapsto 2^{ax}$)
- When $a \in \mathbb{Q}$, (set of functions of the form $x \mapsto 16^{ax}$) $=$ (set of functions of the form $x \mapsto 2^{ax}$)

Proving conjectures. We will use the properties listed in Section 0.2. We also use the following proof structures.

Proof Structure: Showing one set is a subset or strict subset of another.

- To show that $B \subseteq A$ requires showing: if $x \in B$, then $x \in A$.
- To show that $B \subsetneq A$ requires showing: (1) $B \subseteq A$; (2) there is an element of A that is not in B.

Proof Structure: Showing set equality.

- To show that $A = B$ requires showing: (1) $A \subseteq B$; (2) $B \subseteq A$.

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subseteq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$. Showing that $B \subseteq A$ means showing: if $x \in B$, then $x \in A$. Given $x \in B$. Then:

$$\begin{aligned} x &= 6k, k \in \mathbb{Z}, \text{ by definition of } B \\ &= 3 \cdot 2k \\ &= 3n, n \in \mathbb{Z}, \text{ because } 2 \in \mathbb{Z}, k \in \mathbb{Z}, \text{ and } \mathbb{Z} \text{ is closed under multiplication} \end{aligned}$$

Therefore x satisfies set membership rules of A, implying $x \in A$.

We have shown that if $x \in B$, then $x \in A$. Thus $B \subseteq A$, by definition of subset. □

Claim. If $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$, then $B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$ and $B = \{6n : n \in \mathbb{Z}\}$.

1. *Why* $A \subseteq B$: This was done in the claim we just showed. $\boxed{1}$
2. *Why there is an element of* B *that is not in* A. If $x \in B$, then x is an even number because if $x = 6k$ for some $k \in \mathbb{Z}$, then x as $x = 2 \cdot (3k)$. Closure of multiplication in $\mathbb{Z}$ implies $3k \in \mathbb{Z}$, so x satisfies the definition of even number.
   However, some members of A are odd numbers: $3, 9, 15, \ldots$.
   Hence there are elements of A that are not in B. $\boxed{2}$

Why this means $B \subsetneq A$: We showed $B \subseteq A$ and found elements of A not in B. By definition of $\subseteq$, we have $A \subsetneq B$. □

Claim. If $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$, then $A = B$.

Sketch of proof. Given $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$.
We outline the steps of the proof for you to fill in.

1. *Why* $A \subseteq B$: $\boxed{1}$
2. *Why* $B \subseteq A$: $\boxed{2}$

Why the above means that $A = B$: □

Modeling proof communication.

Ch 1: write instructor note modeling proof communicat use the proof of strict subset, point out relevant handout from Section 0

Mathematical statements and their negations

Logical notation

$P(x)$	A proposition about the variable x; may be true or false depending on x
$\neg P(x)$	The negation of $P(x)$
$\forall x, P(x)$	The proposition "For all values of x, $P(x)$ is true."
$\exists x : P(x)$	The proposition "There exists a value of x such that $P(x)$ is true."
$\forall x, P(x) \Rightarrow Q(x)$	The proposition "For all values of x, if $P(x)$ is true then $Q(x)$ is true."
$\forall x, P(x) \Leftrightarrow Q(x)$	The proposition "For all values of x, $P(x)$ is true if and only if $Q(x)$ is true."

1. For each of the following statements, figure out what it means, and decide whether it is true, false, or neither.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y + x \in \{z \in \mathbb{Z} : z > 0\}$
   (b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : y + x \notin \mathbb{Z}$
   (c) $\forall g : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax}, \exists h : \mathbb{R} \to \mathbb{R}, x \mapsto 4^{bx} : \forall x \in \mathbb{R}, g(x) = h(x)$
   (d) $\forall g : \mathbb{R} \to \mathbb{R}, x \mapsto 4^{ax}, \exists h : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{bx} : \forall x \in \mathbb{R}, g(x) = h(x)$
2. Negate the following statements without using any negative words (“no”, “not”, ”neither … nor”, etc.) Try to make your negation sound as much like normal English as possible.
   (a) Every word on this page starts with a consonant and ends with a vowel.
   (b) The set A is equal to the set B.
   (c) There is a book on this shelf in which every page has a word that starts and ends with a vowel.
   (d) The set A is a strict subset of the set B.

Instructor note. There is typically only time to do one or two of each task, with the rest assigned for homework. For (1), we recommend (a) or (b), and then as time allows, (c) or (d). For (2) We recommend doing least one word negation ((a) or (c)) in class, and then as time allows, one negation having to do with concepts of sets ((b) or (d)).

Solution. (*Partial*)

1. (a) For each real number x, there is a real number y so that $x + y$ is a positive integer. TRUE.
   Reasoning: If $x \in \mathbb{R}$, then take $y = 1 - x$. Then $x + y = 1$, which is a positive integer. Or take any positive integer n and take $y = n - x$.
   (b) What it means: …(fill in the rest). FALSE.
   Reasoning: (Why?)
   (c) For each function $g(x) = 2^{ax}$ there is a function $f(x) = 4^{bx}$ so that $g(x) = h(x)$ on every possible real value of x. NEITHER.
   Reasoning: The truth of this statement depends on the possible values of a and b. If a and b must be integers, then there are some a where 2^{ax} cannot equal 4^{bx}. (All odd integers.) If a and b are rational or real, then for each a, we can take $b = \frac{a}{2}$, and then $2^{ax} = 4^{bx}$
   (d) What it means: …(fill in the rest). TRUE.
   Reasoning: (Why?)
2. (a) THERE IS a word on this page that starts with a vowel OR ends with a consonant.
   (b) The set A has at least one element that is not in B OR the set B has at least one element that is not in A.
   (c) EVERY book on this shelf (…fill in the rest)
   (d) The set A equals B OR (…fill in the rest) ■

Back to the opening inquiry

We have now spent some time discussing set notation and logical notation.

We began this class considering “parents” of numbers. We conjectured that:

> If a number has least three different factors (that are not itself or 1), then it has more than one pair of parents. There are no other numbers with more than one pair of parents.

One way of saying “factors of a number that are not itself or 1” is to say “nontrivial factors”.

Applying set notation. Using set notation, we can interpret the conjecture as saying:

Conjecture 1.3 (Number parent conjecture, take 1)**.** Let

$$\begin{aligned} S &= \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\} \\ T &= \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\} \end{aligned}$$

Then $S = T$.

> How does this way of phrasing the conjecture match up with the original way?
>
> - Look up the definition of set equality. What does $S = T$ mean by definition of set equality?
> - Which part of set equality implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")?
> - Which part of set quality implies the second sentence? ("There are no other numbers with more than one pair of parents")

Solution. By definition, $S = T$ means $S \subseteq T$ and $T \subseteq S$.

$S \subseteq T$ implies that "If a number has least three different nontrivial factors, then it has more than one pair of parents."

$T \subseteq S$ means that "there are no other numbers with more than one pair of parents." ■

Applying logical notation. There is another mathematically equivalent way of saying the conjecture using the logical notation we developed.

Conjecture 1.4 (Number parent conjecture, take 2)**.** $\forall n \in \mathbb{N}$, n has more than one pair of parents $\iff$ n has at least three nontrivial factors.

> How does this way of phrasing the conjecture match up with the original way?
>
> - What does "if and only if" mean?
> - Which part of "iff" implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")? (An abbreviation for "if and only if" is "iff")
> - Which part of "iff" implies the second sentence? ("There are no other numbers with more than one pair of parents")

Solution. By definition, P iff Q means that both $P \implies Q$ and $Q \implies P$ are true statements.

Given $n \in \mathbb{N}$, let the statement P be "n has more than one pair of parents", and the statement Q be statement "n has at least three nontrivial factors".

$Q \implies P$ being true implies that "if a number has least three different nontrivial factors, then it has more than one pair of parents."

$P \implies Q$ being true implies that "there are no other numbers with more than one pair of parents." ■

(The following is stated in two equivalent ways)

Proposition 1.5 (Number parent proposition)**.**

If $S = \{n \in \mathbb{N} : n$ has at least three different non-trivial factors$\}$ and $T = \{n \in \mathbb{N} : n$ has more than one pair of parents$\}$, then $S = T$.	*For all $n \in \mathbb{N}$, n has more than one pair of parents if and only if n has at least three nontrivial factors.*

Proof. Given $S = \{n \in \mathbb{N} : n \textit{ has at least three different non-trivial factors}\}$ and $T = \{n \in \mathbb{N} : n \textit{ has more than one pair of parents}\}$.

1. *Why $S \subseteq T$:*Let $n \in S$. Then there exist distinct $a, b, c \in \mathbb{N}$ such that $a \mid n$, $b \mid n$, and $c \mid n$. Either each of these are paired with another one of a, b, c to be a pair of parents of n or they are not. If they are not paired with any of each other, then n has at least three pairs of parents, which is more than one. If one of them is paired with another, there is still a third factor that cannot be paired with the other two (because they are already paired). So it is part of a second pair of parents. Thus $n \in T$.

   We have shown that if $n \in S$, then $n \in T$. By definition of subset, this shows $S \subseteq T$. $\boxed{1}$

2. *Why $T \subseteq S$:* Let $n \in T$. Then there exist at least two pairs $a, a' \in \mathbb{N}$ and $b, b' \in \mathbb{N}$ such that $aa' = n$ and $b, b' = n$, and $\{a, a'\} \neq \{b, b'\}$.

   If $a \neq a'$ and $b \neq b'$, then n has at least four factors, so $n \in S$.

   It may be true that $a = a'$ or $b = b'$. If $a = a'$, though, then n is a perfect square and $b \neq b'$, since there is only one positive square root possible for every n. Similarly, if $b = b'$, then $a \neq a'$. In either case, n has at least three factors (either a, b, b' or a, a', b), so $n \in S$.

   We have shown that if $n \in T$, then $n \in S$. By definition of subset, this shows $T \subseteq S$. $\boxed{2}$

We have shown that $S \subseteq T$ and $T \subseteq S$. By definition of set equality, we have shown $S = T$. □

Ch 1: Clean up this proof part 1. It is correct but confusing.

Summary of mathematical practices

Clarifying the question

- Make the best sense as you can of the question with what is available.
- Identify what is unambiguous, and then identify what is ambiguous.
- For the ambiguous parts, play around with different possibilities to see what is the most mathematically interesting possibility. Sometimes you may find that there are multiple interesting mathematical possibilities.

Conjecturing and Claim making

- Think of claims as an "I bet" statement.

  If you're the arbitrator for a bet between people, you would want to make absolutely sure that everyone knows exactly what the statement means, and also that everyone would agree on what evidence would count as showing the bet is true or not true!

  The same is true about mathematical statements. A mathematical statement needs to be crystal clear about what it means.
- Mathematical claims should either be true or false; if they "depend" on something, this means that there is often a better claim that can be made.
- The more general a claim, the better it is.

  For instance, "12 has more than one pair of parents" is a true claim, but a better claim is "All numbers with at least three distinct factors have more than one pair of parents" is an even better claim.
- The more "directions" a claim addresses, the better it is.

  For instance, "All numbers with at least three distinct factors have more than one pair of parents" is a true claim, but "A number has more than one pair of parents if and only if it has at least three distinct factors" goes even further to understanding the situation.

Exploring math: Our expectations

- Make claims.
- Try to prove them.
- If you get stuck, consider the negations of the claim.
- Try to prove those.

- Consider the "opposite direction" claim. (The "converse" of the claim.)
- Try to prove those.
- Aim to make the most satisfying claims possible.
- Rewrite, rewrite, rewrite! Use the rewriting process to help things get clear for yourself, your future students, and your future self, and your peers.

Things to keep in mind on the first day. This first lesson is an important place to do what can be called "setting norms and expectations". What this means is communicating to prospective teachers, both implicitly and explicitly, what productive conversation, exploration, questioning, and justification look and feel like. For instance, you may want to teach a class where:

- *Students embrace learning from their own individual and each others' work* – they view their own mistakes courageously and with an open mind; they accept that making errors and learning from them is a natural part of the mathematical process; they recognize what is worthwhile about others' reasoning and what needs further thought, and they do so constructively; they celebrate others' ideas.
- *Students view mathematical reasoning as the ultimate mathematical authority* – they have faith in their ability to learn to reason mathematically; they come back to the mathematics rather than to a perceived authority figure such as an instructor or a "smart" student to figure out what works; they seek precision in language while also understanding that going from informal language to precise language may take some time, may not happen right away, but is a valuable goal.
- *Students persist in seeking mathematical questions and answers* – they accept that setbacks are an important part of learning; they can work for an extended amount of time on one problem in productive ways; they celebrate when they do come to an understanding of a mathematical idea, especially one that is hard-won.

If these are values that you see a productive class expressing, you can do much to foster these values beginning the first day. There are many different things you can do and say, and certainly different things may work better or worse for different instructors and different students. Here are some examples of things to do and say that have helped previous MODULE(S^2)instructors:

- *Praise thoughtful errors.* It's easy to spot "right" answers and there can be a temptation to run with the way that some students have found exactly the "right" way to approach a problem. There is also a temptation to respond to "wrong" answers with saying matter-of-factly, "Not quite; what did others get?" But if you respond in these ways, and exclusively so as your form of interacting with students about their thinking, what message does that send to students about the role of mistakes in the process of working through mathematics? It may well send the message that the best work is the work that is correct the first try, or worse, that the most worthy students are those that only do correct mathematics and make no mistakes. Instead, an alternative approach is to look for thoughtful errors – the kind of thinking that is ultimately mathematically incorrect for some reason, but where thinking through the mistake has the potential to really get at something fundamental about the mathematics at hand or in the future. Moves that you can make to acknowledge thoughtful errors might include:
    - "I am so glad that you brought that up, [student name]. Did everyone understand what [student name] said? Can someone say in their own words what they understand of [student name]'s reasoning?" [If someone raises their hand to counter this idea] "Right now we're not interested in whether we agree or disagree with [student name], we are trying to understand what [student name] is thinking. What might they thinking? Why does it make sense to do this?"
    - "Let's see what happens when we follow this reasoning."
    - "We just learned a really important lesson about doing mathematics because of this reasoning. Thank you, [student name], for sharing your idea. This was incredibly helpful. Let's remember the lesson we learned throughout today and also as we move forward in this class."
- *Do not make a big deal when students get a correct answer right away. Focus on the process of getting to the answer, and on understanding the answer, rather than the answer itself.* The Fields Medalist William Thurston (1994) observed of his colleagues, "I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than the knowledge, people want *personal understanding.*" (p. 51, emphasis by Thurston). The same is true of students, or at least we would like to be a truth about students. Moves that emphasize understanding over the answer might include:
    - (As a matter-of-fact first reaction to the correct answer) "You answered X. What was your reasoning for that answer?" ... "What do others think of this reasoning?"
    - "[Student name] arrived at the solution X, and just shared their reasoning. Did anyone else arrive at this solution? Did you have similar reasoning or different reasoning?"
    - "Let's think back on why this answer makes sense."

- *Relinquish your authority to the students and the mathematics.* A common question instructors hear is, "What do you want?" or "Is this what you are looking for?" Sometimes the answer to these questions really does rest with you, the instructor – especially if it is about specific directions that you are setting for your students that can't be derived from mathematical reasoning. However, answering these questions from your authority as an instructor can be less useful if the questions are actually about mathematical reasoning, for instance, if the question is about whether a proof or solution is correct. In these cases, it can be more productive to return the responsibility of these questions to the students and the mathematics:
    - "Can you tell me more about how you arrived at this?"
    - "Tell me about what's here."
    - "How does this help to give a solution to the question we are working on?"
    - "How complete do you think it is?" . . . "What about your work are you sure about, and what are you less sure about?"
- *Give students ways to work constructively with each other.* Working with each other on mathematics is not necessarily a natural skill; it is a learned skill. Help your students find ways to talk to each other about their thinking. While students are working, stir the pot (meaning, find ways to provoke productive disagreement and/or discussion).
    - "I see that [student A] and [student B] have different answers. It looks like you have something to resolve. [Student A] and [Student B], will you share how you did your work with each other and figure out what's really going on?"
    - "I see that [student A] and [student B] have arrived at the same answer, but it looks like you've done it in different ways. Will you compare what you've done and see how they match each other or do not?"
    - "It looks like [student A] has drawn a graph and [student B] has used mostly equations. Are you thinking about the same thing? Will you talk to each other about how your thinking matches up or not?"
    - "It looks like [Student A] worked on [Case 1] whereas [student B] worked on [Case 2]. Are there more cases to consider? Are both cases necessary? You should talk to each other to figure this out."

In-Class Resources

Opening Inquiry

Two numbers are parents of a child if the child is their product.

A child cannot be its own parent.

Which numbers have more than one pair of parents?

Child	Parents
6	2, 3
4	??
12	4, 3
12	2, 6

Clarifying what it means to be a pair of parents:

Notes on finding and improving answers to mathematical questions:

GETTING TO KNOW SET NOTATION

1. Let $A = \{1, 2, \{3, 4\}, \{5\}\}$. Decide whether each of the following statements is true or false:
   (**Hint:** There are exactly six true statements.)

$$\begin{array}{rrrr} 1 \in A, & \{1,2\} \in A, & \{1,2\} \subseteq A, & \emptyset \in A, \\ 3 \in A, & \{3,4\} \in A, & \{3,4\} \subseteq A, & \emptyset \subseteq A, \\ \{1\} \in A, & \{1\} \subseteq A, & \{5\} \in A, & \{5\} \subseteq A. \end{array}$$

2. True or false? "All students in this class who are under 5 years old are also over 100 years old."

3. Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
   True or false? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $1 \in S$.

GETTING TO KNOW LOGICAL NOTATION

1. For each of the following statements, figure out what it means, and decide whether it is true, false, or neither.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y + x \in \{z \in \mathbb{Z} : z > 0\}$
   (b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : y + x \notin \mathbb{Z}$

2. Negate the following statements without using any negative words ("no", "not", "neither ... nor", etc.) Try to make your negation sound as much like normal English as possible.
   (a) Every word on this page starts with a consonant and ends with a vowel.
   (b) The set A is equal to the set B.

Subset exploration

Is A a subset of B or vice versa? Complete this table with "yes" or "no" in each cell.

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ multiples of 3, $B =$ multiples of 6							
$A =$ multiples of 6, $B =$ multiples of 9							
$A = \{n^2 \mid n \in \mathbb{N}, n > 0\}$, $B = \{1 + 3 + \cdots + (2n+1) \mid n \in \mathbb{N}\}$							
$A =$ functions of the form $x \mapsto 16^{ax}$, $B =$ functions of the form $x \mapsto 2^{ax}$							

Back to the opening inquiry

We began this class considering "parents" of numbers. We conjectured that:

Applying set notation. Using set notation, we can interpret the conjecture as saying:

How does this way of phrasing the conjecture match up with the original way?

- Look up the definition of set equality. What does $S = T$ mean by definition of set equality?
- Which part of set equality implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")?
- Which part of set quality implies the second sentence? ("There are no other numbers with more than one pair of parents")

Applying logical notation. There is another mathematically equivalent way of saying the conjecture using the logical notation we developed:

How does this way of phrasing the conjecture match up with the original way?

- What does "if and only if" mean?
- Which part of "iff" implies the first sentence ("If a number has least three different nontrivial factors, then it has more than one pair of parents.")? (An abbreviation for "if and only if" is "iff")
- Which part of "iff" implies the second sentence? ("There are no other numbers with more than one pair of parents")

Homework

Ch 1: write homework

1. Proving set membership problem
2. Proving subset, subsetneq problem
3. Proving set equality problem (possibly assign even number, consecutive odd numbers); can also assign exponential function problem
4. Proof comprehension question about parent relation proof
5. Something about assigning parents to children, to introduce the idea of a relation as a set of assignments. Possibly the opener to Lesson 2.
6. Something to introduce Cartesian product $D \times R$. (Note that it's also called cross product, but it's not the linear algebra thing.)

Instructor note. For homework, you may want to make sure to assign at least one problem on parent relation and the problem with Cartesian product. These are used in the next lesson, in Section 2, and beyond.

Part II

Relations and Functions

2 Relations (Week 2) (Length: ~2.5 hours)

Overview

Content
Cartesian product of two sets A and B, denoted $A \times B$, defined as the set of ordered pairs $\{(a,b) : a \in A, b \in B\}$.
Relation from a set D to set C, defined from three different perspectives: the "middle school", "high school", and "university"; and their mathematically equivalence. • The "middle school" version is described in terms of a set of arrows between an input and output space. • The "high school" version formalizes arrows to assignments. • The "university" version defines a relation as a subset of the Cartesian product $D \times R$.
We call these definitions the middle school, high school, and university versions to refer to when they most likely arise.
Inverse of a relation, defined from these three perspectives; their mathematically equivalence.
Composition of relations $P : D \to D$ then $Q : D \to D$, defined as the relation that assigns x to z whenever there is a $y \in D$ such that P assigns $x \mapsto y$ and Q assigns $y \mapsto z$. (See p. 30 for why we only consider the case $P : D \to D$ and $Q : D \to D$.)
Graph of a (real) relation, defined as the set of points $(x, y) \in \mathbb{R}^2$ such that the relation assigns x to y.
Graph of an (real) equation in variables x and y, defined as the set of points $(x, y) \in \mathbb{R}^2$ such that evaluating the equation at x and y results in a true statement.
Function from a set D to a set R, defined as a relation from D to R such that each input in D is assigned to no more than one output in R; how this definition can be interpreted from the three perspectives for relation.

Proof Structures	Mathematical/Teaching Practices
To show that a point (x, y) is on a graph of a relation means showing that the relation assigns x to y.	**Connecting mathematically equivalent definitions**, meaning to understand how different but equivalent definitions can serve different pedagogical and mathematical purposes.
To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at x and y results in a true statement.	**Connecting different mathematical representations** of the same concept, meaning to think about different ways of drawing and describing the same mathematical idea.

Summary

One goal of this lesson is to introduce relations and functions from an advanced perspective. However, more importantly, the goal is to connect the advanced perspective to high school and middle school perspectives, so that teachers have a sense of where the math can go.

Using the parent relation as an opening example, we define *relation* in the three (mathematically equivalent) described above. We then define *domain*, *range*, *image* of a point and set, and *preimage* of a point and set.

To highlight the universality of these concepts throughout high school and middle school mathematics, we use examples from algebra (from story problems and also graphs of relations such as $x = y^2$), trigonometry (the relation from $[0, 360)$ to $\mathbb{R}$ defined by equivalence of angle measure in degrees), and geometry (rigid motions).

We then introduce *inverse relations*, *functions*, *graphs of functions*, and *compositions of functions*. For each concept in this lesson, we ask teachers to consider how they might explain the connection between the concept and the middle school, high school, and university conceptions of relation, as well as how they might explain how different representations denote mathematically equivalent ideas.

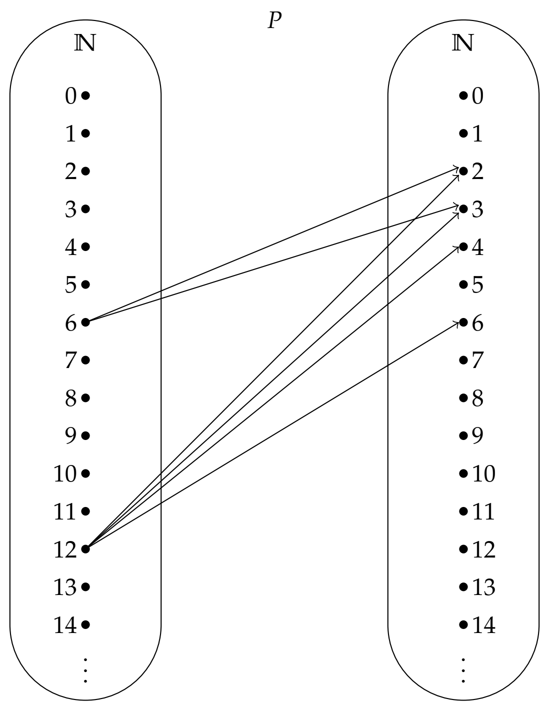
Materials.

- Handouts from In-Class Resources (can be printed double-sided)

Opening example: Parent relation

We learned about natural number parents and children last time.

1. What is the definition of a parent of a natural number child?
2. Let P assign a natural number to each of its parents. We can represent P as a set of arrows from $\mathbb{N}$ to $\mathbb{N}$. Some arrows below have been filled in, for example P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in more arrows.
3. Consider this statement: "Some children have no parents, some children have exactly one parent, and some children have multiple parents."
   Is this statement true or false? Why?
4. How about this statement? "Some numbers have no children, and some numbers have multiple children."



Solution. (*Partial*) Given a number $n \in \mathbb{N}$, a parent of n is a nontrivial factor of n.

The first statement is true:

- n has no parents when n is 0, 1, or prime
- n has exactly one parent when n is a perfect square of a prime number
- n has multiple parents otherwise

These are represented by no arrows starting at a number, exactly one arrow starting at a number, and multiple arrows starting at a number.

The second statement is also true. 0 and 1 have no children. All other numbers have multiple children (infinitely many, in fact). These are represented by arrows ending a number or not. ■

Defining relations

Cartesian products

Let's discuss Cartesian products, which you first saw in your homework from last week.

Definition 2.1. Let D and R be sets. The **Cartesian product** of D and R is defined as the set of ordered pairs $\{(x, y) : x \in D, y \in R\}$. It is denoted $D \times R$.

Let $A = \{5, 6, 10\}$, $B = \{-1, -2, -3\}$, $C = \{-1, 1\}$. Let $\mathbb{N}$ denote natural numbers, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers.

List the elements of the following Cartesian products:

- $A \times B$
- $A \times C$
- $\mathbb{Z} \times C$
- $C \times \mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$.

Which of the above sets contains the element $(6, -1)$? How about $(-1, 10)$?

How would you describe $\mathbb{R} \times \mathbb{R}$?

How about $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$?

Solution. (*Partial*) $(6, -1) \in A \times B, \mathbb{Z} \times C, \mathbb{N} \times \mathbb{N}$. It is not an element of any of the other sets.

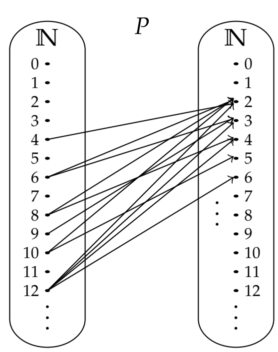
$(-1, 10) \in C \times \mathbb{Z}, \mathbb{N} \times \mathbb{N}$. It is not an element of any of the other sets.

$\mathbb{R} \times \mathbb{R}$ is the coordinate plane.

$\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$ can be thought of as all the coordinates of 3-space. ■

Relations

In middle school, if relations are introduced, they are often done so in the form of a cloud diagram, such as drawn in the opening task. (*Question:* What do the "..." mean in the below diagram?)



In our example, a relation P maps numbers in $\mathbb{N}$ to numbers in $\mathbb{N}$, and the map is represented by arrows connecting input numbers to output numbers.

Definition 2.2 (Relation: Middle school version)**.** A **relation** from a set D to a set R is a set of arrows going from elements of D to elements of R.

If there is an arrow from and element x to an element y, we say the relation **maps** or **assigns** x to y.

We use the notation $r : D \to R$ to mean a relation from D to R called r.

Definition 2.3 (Parent relation)**.** The parent relation $P : \mathbb{N} \to \mathbb{N}$ is the set of arrows from each element of $\mathbb{N}$ to its nontrivial factors.

Note: A relation $P : D \to R$ may map an element of D to no elements of R, exactly one element of R, or multiple elements of R.

An element of R may have no elements of D mapping to it, exactly one element of D mapping to it, or multiple elements of D mapping to it.

Definition 2.4. For a relation $r : D \to R$, we say that

- D is the **candidate domain**;
- R is the **candidate range** (or **codomain**);
- the **image** of an element $x \in D$ is the set of elements of R that x is mapped to. Similarly, the **image** of a subset of $S \subseteq D$ is the subset of R containing the images of all elements of S.
- When a element in D maps to no element in R, we say it has **empty image**. If an element in D does map to at least one element in R, we say it has **nonempty image**.
- the **domain** of r is the subset $D' \subseteq D$ of elements with nonempty image.
- the **preimage** of an element of R is the set of elements of D that map to R. Similarly, the **preimage** of a subset $T \subseteq R$ is the subset of D containing the preimages of all elements of T.
- When a element in R has no element in D mapping to it, we say it has **empty preimage**. If an element in R does have an element in D mapping to it, we say it has **nonempty preimage**.
- the **range** (or **image**) of the relation r is the subset $R \subseteq D$ with nonempty preimage.

Note: In these materials, we will use the terms "codomain" and "candidate range" interchangeably. We will also use the terms "range" and "image" interchangeably. The terms "codomain" and "image" typically do not show up in K-12 materials; they are typically introduced in university or graduate mathematics. The term "range" is standard to middle school and high school materials, though sometimes "range" is used to mean "candidate range" and other times it is used to mean "the set of elements with nonempty preimage". In these materials, "range" only refers to the latter.

> **Instructor note.** In writing these materials, we sought to find a standard term for what we call the "candidate domain." To our knowledge, there is no well-known standard term for this concept. This is perhaps in part because the distinction matters primarily in undergraduate level mathematics and beyond, for instance in defining function composition; and perhaps also because at times at this level, a certain amount of notational interpretation is assumed and we loosen the restriction. For instance, meromorphic functions on C are really holomorphic functions on the complement of a discrete set in $\mathbb{C}$. Functions in $L^2(\mathbb{R})$ are technically only well defined up to equality of all integrals, so they don't have a strict notion of "domain"; however, mathematicians still talk about the "domain" of an L^2 function.
>
> In this context, we want to be careful about differentiating between the "candidate domain" and the "domain", so that we have language for talking about real functions and their domain. The term "candidate domain" was suggested to us by a high school mathematics teacher as a term that would have meaning to high school teachers that could potentially be explained to high school students. Another term we considered was "corange" as a parallel to "codomain", as suggested by some mathematicians. However, we decided to use "candidate domain" instead because the term seemed more down-to-earth, and moreover problematizes the issue of finding the domain. The term suggests the question, "Is this really the domain? If not, how can we fix it to be the domain?"
>
> The term "candidate range" was chosen to mirror "candidate domain", as the phrase "a relation (or function) from a candidate domain to a candidate range" is more straightforward to high school teachers than "a relation (or function) from a candidate domain to a codomain".

We have seen examples of most of these concepts in the parent relation. Other examples of relations might be:

- The relation from the candidate domain of all cars in the world to candidate range of all people in the world, mapping a car to its owner(s).

- The relation from the candidate domain of rooms in the mathematics building to candidate range of courses taking place at 1pm, mapping a room to the course being taught in it at 1pm.

In these examples, we can see how each condition of the note about relations may apply.

What are the domain and range of the car-owner relation and the room-course relation?

Suppose this table contains course assignments to rooms at 1pm. What is the image of Math Bldg Room 100? What is the preimage of Math 996, Math 405, Math 100, and Math 221 under the room-course relation?

Room	Course in room at 1pm
Math Bldg room 100	Math 996
Math Bldg room 104	Math 100
Engineering Bldg room 750	Math 405
not being offered this term	Math 221

Let T be the relation that maps each day of the year 2030 to the its average temperature in $^\circ F$ that day. Describe a possible candidate domain, domain, candidate range, and range of this relation.

Let A be the relation that maps each degree in the interval $[0^\circ, 360^\circ)$ to all degrees in the interval $(-\infty, \infty)$ that give an equivalent angle measure. What is the preimage of 361°? What is the image of 0°?

Let ρ be the relation that maps a point in the plane to its rotation about the origin by 90°. (This means 90° *counter*clockwise.) What is the image of the point $(1, 0)$? What is the preimage of the point $(-2, 0)$?

Let G be the relation that maps x to every y such that $x = y^2$. What is the image of 4? What is the preimage of -6?

Interpret the definitions of candidate domain, domain, image, preimage, candidate range, and range in terms of arrows and their start and end points.

We can think of the definition as a verbal representation of these concepts and the cloud diagrams as another. When connecting different representations, it is often helpful to keep these elements in mind:

- Explain how representations appear different and how they appear the same.
- Identify how the representations can be used highlight different features.
- Go back and forth between the representations: where are the features of one representation located in the other representation?

At the high school level, textbooks generally do not use cloud diagrams any more, nor do they talk about arrows. Instead, discussion of relations (and functions) are in terms of assignments. The definition in high school is mathematically equivalent to the middle school version, but stated in a way that more directly allows for defining concepts like the graph of a relation or later, the behavior of a function. (We note that as we will discuss later, a function is a kind of relation.)

Definition 2.5 (Relation: High school version)**.** A **relation** P from a set D to a set R a set of assignments from elements of D, called inputs, to elements of R, called outputs.

Note: We use the notation $P : D \to R$ to mean a relation from D to R, and the notation $x \mapsto y$ to denote an assignment from $x \in D$ to $y \in R$. Something to keep in mind for "assignment" is that an assignment has to map something to something. So we think of an assignment not just as an "arrow" but as an arrow with specific start and end points.

> What are some example assignments of the relation A mapping each degree in the interval $[0°, 360°)$ to all degrees in the interval $(-\infty, \infty)$ that give an equivalent angle measure? Use the $x \mapsto y$ notation to write down your examples.

Solution. Some examples of assignments are: $0° \mapsto 360°$, $0° \mapsto 0°$, $359° \mapsto -1°$, $90° \mapsto 810°$. ■

> Suppose we were to graph the relation A. What might this graph look like? What are some examples of coordinates that are contained in this graph?

Solution. It would look like the set of all lines of the form $y = x + 360n$, where $n \in \mathbb{Z}$. Some example coordinates are $(0, 360)$, $(0, 0)$, $(359, -1)$, $(90, 810)$. ■

> Suppose we were to graph the parent relation. What might this graph look like? What are some examples of coordinates that are contained in this graph?

In undergraduate courses such as real analysis as well as in graduate courses in analysis, we go one step farther. Rather than defining relations in terms of assignments, we define relations in terms of ordered pairs. The ordered pairs represent the assignments.

Definition 2.6 (Relation: University version)**.** A **relation** $r : D \to R$ is a subset of $D \times R$, i.e., $r \subseteq D \times R$.

One way to think about this definition is that we are defining the relation as its graph in the space $D \times R$.

Note: One question that might come up is: if the candidate domain could be anything, then why bother finding good candidate domains? Why not instead let the candidate domain be the largest set that we could think of? For instance, we might set the candidate domain to be something like this:

$$\mathbb{R} \cup (\text{all cars in the world}) \cup (\text{all rooms in all buildings in the world}) \cup \dots .$$

One reason that we would not want to do this is that eventually, we want to construct and compare graphs of relations and functions. The candidate domain and candidate range of comparable relations and functions are likely to be similar to each other, and the graphs live in the space $D \times R$ where D is the candidate domain and R is the candidate range for these relations or functions.

Building relations from existing relations

There are two main ways to build relations from existing relations: inverses and composition. We will see these again, in more depth, when we discuss functions.

INVERSE OF A RELATION

Definition 2.7 (Inverse relation: Middle school version)**.** If r is a relation from a set D to R, then the **inverse relation** of r is the relation that swaps the direction of the arrows of r. The arrows of the inverse relation go from elements of R to elements of D.

In other words, there is an arrow from x to y in r if and only if there is an arrow from y to x in r^{-1}.

Definition 2.8 (Inverse relation: High school version)**.** Given a relation $r : D \to R$, the **inverse relation** of r is the set of assignments $y \mapsto x$ such that $x \mapsto y$ is an assignment of r. The inverse relation is denoted r^{-1}.

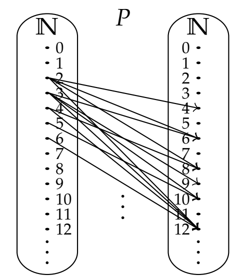
In other words, $x \mapsto y$ is an assignment of a relation r if and only if $y \mapsto x$ is an assignment of r^{-1}.

Definition 2.9 (Inverse relation: University version)**.** Given a relation $r : D \to R$, the **inverse relation** of r is defined

$$r^{-1} = \{(y, x) : (x, y) \in P\}.$$

In other words, $(x, y) \in r$ if and only if $(y, x) \in r^{-1}$.

As an example, let's look at the parent relation. The inverse of the parent relation could be represented like the following. (*Question:* What do the "..." mean in this representation?)



What other arrows does the inverse of the parent relation contain?
What might be a good name for this relation?
What is the inverse of the car-owner relation? How about the relations T, A, ρ, and G?

Instructor note. One reasonable name for the inverse of the parent relation might be the "child relation", as this relation maps natural numbers to their children. The inverse of the car-owner relation is the relation from all people in the world to all cars in the world that map a person to all the cars they own. The relation T^{-1} maps possible temperatures to the days on which that temperature was the day's average temperature. The relation A^{-1} maps an element of $\mathbb{R}$ to the element of $[0, 360)$ which represents its angle measure. The relation ρ^{-1} is rotation about the origin by $-90°$, which is $90°$ clockwise. The relation G^{-1} is the relation that maps y to every x such that $x = y^2$.

Discuss the three versions of the definition of inverse of a relation. What do they each say? How would you represent them? What makes them mathematically equivalent?

Here are some elements to keep in mind when connecting mathematically equivalent definitions; how did they each come up in your discussion?

- Explain how they appear different and how they appear the same.
- Explain why they are mathematically equivalent.
- Analyze the mathematical and pedagogical purposes of each definition and why they may be appropriate for different levels of mathematical study, or why one version makes more sense after having worked with another version.

Let $(P^{-1})^{-1}$ be the inverse of the relation P^{-1}. What is this relation? How do you know? Let $(G^{-1})^{-1}$ be the inverse of the relation G^{-1}. What is this relation? How do you know?

In general, given a relation r, what is $(r^{-1})^{-1}$? Explain how you know.

To show that two relations are the "same", we can go back to two definitions: that of relation, and that of set equality (Definition 1.2). A relation is defined as a set of assignments (or, in the language of middle school, arrows; or, in the language of university, ordered pairs). Two sets are equal if they are both subsets of each other. Hence, to show that two relations are equal to each other, we show that their assignments are both subsets of each other.

Proposition 2.10. *The inverse of the inverse of a relation is the original relation.*

Proof (using high school version). The definition of inverse of a relation states that $x \mapsto y$ is an assignment of a relation r if and only if $y \mapsto x$ is an assignment of r^{-1}. We can also use the definition to say that $y \mapsto x$ is an assignment of r^{-1} if and only if $x \mapsto y$ is an assignment of $(r^{-1})^{-1}$. Hence, combining these if and only if statements, we have that $x \mapsto y$ is an assignment of a relation r if and only if it is also an assignment of $(r^{-1})^{-1}$. The set of assignments of r are equal to the set of assignments of $(r^{-1})^{-1}$, and so $r = (r^{-1})^{-1}$. $\square$

We can interpret this proof in terms of sets. Let A be the set of assignments in r, and B be the set of assignments in $(r^{-1})^{-1}$. We have $x \mapsto y$ is an element of A if and only if it is an element of B. The "if" part of this statement means that $B \subseteq A$; the "only if" part of this statement means $A \subseteq B$. Hence $A = B$, so the set of assignments of r are equal to the set of assignments of $(r^{-1})^{-1}$, and so $r = (r^{-1})^{-1}$.

Composition of relations

In the remainder of this chapter, we work almost exclusively with cases where $D, R, S = \mathbb{N}$, $D, R, S = \mathbb{R}$ or $D, R, S = \mathbb{R}^2$, and We make this choice for two main reasons:

- Most examples of composition in middle school and high school mathematics are those where the candidate domain and candidate range can be both $\mathbb{N}$ (in middle school algebra), both $\mathbb{R}$ (in algebra), or both $\mathbb{R}^2$ (in geometry).
- When the candidate domain and candidate range do not equal each other, the details of some results require more technical bookkeeping.The idea behind these results is more important for high school teaching than learning the technical bookkeeping.

In high school and college, what did you learn that the notation $f \circ g(x)$ means? Circle your answer.
Do f then g Do g then f

Solution. It means to perform g and then f. For example, if $g(x) = x^2$ and $f(x) = 5x$, then $f \circ g(x) = f(x^2) = 5x^2$, wherease $g \circ f(x) = g(5x) = (5x)^2 = 25x^2$. ■

Instructor note. It may seem excessive to do this task recalling the definition of function notation. However, in our experience, it is better to spend a minute making sure that all students recall function notation correctly prior to using it and are primed to use it, rather than losing those students who may not remember. In our experience, almost all students do remember *if prompted*. If not prompted, we have found that there are a few students who do not remember, and then extra time is spend remediating.

Definition 2.11 (composition)**.** Given two relations $P : D \to D$ and $Q : D \to D$, we define the **composition** of P then Q as the relation that assigns x to z whenever there is a $y \in D$ such that P assigns $x \mapsto y$ and Q assigns $y \mapsto z$.

Let P be the parent relation and let $t : \mathbb{N} \to \mathbb{N}, x \mapsto 2x$. How would you represent $t \circ P$ using the middle school version of relation?

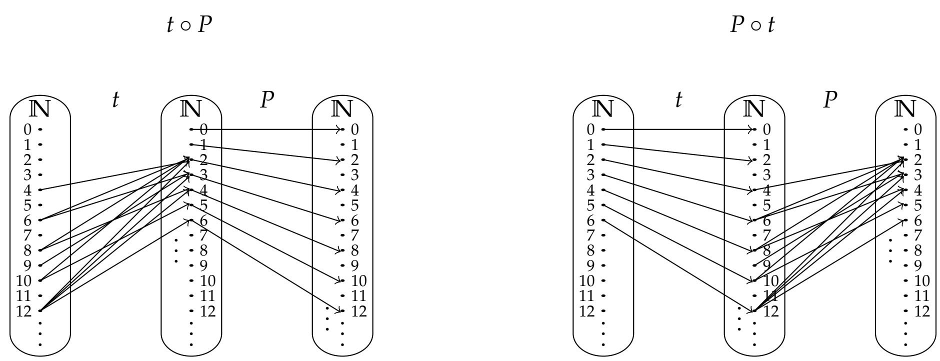
For the relation $t \circ P$, what is the image of 6? Of 12? Of 9?

For the relation $t \circ P$, what is the preimage of 4? Of the set $\{1, 3, 5\}$? Of the set $\{4, 14\}$?

What are the domain and range of the relations $t \circ P$?

How would you represent $P \circ t$? What are its domain and range?

Solution. (*Partial*) We can represent compositions with concatenated cloud diagrams.



composition	domain	range
$t \circ P$	$\{n \in \mathbb{N} : n \neq 0, 1 \text{ and } n \text{ is not prime}\}$	All even natural numbers ≥ 4
$P \circ t$	$\{n \in \mathbb{N} : n \geq 2\}$	$\{n \in \mathbb{N} : n \geq 2\}$

Note that 1 is *not* in the domain of $P \circ t$, because P does not assign $t(1)$ to any element. So $P(t(1))$ is undefined. ■

Instructor note. One way to discuss the domain of $t \circ P$ is to ask, "Raise your hand if 5 was in your domain ... 4? ... 3? ... 2? ... 1? ... 0?" Make a note of any disagreements or wide agreements to the class orally. Then ask for students' reasoning about 5, 1, and 0.

Now let's take on the challenge of combining two ideas that we've been working with: inverse of a relation and composition.

Sketch a representation of $A^{-1} \circ A$. What are its domain and range? What is the image of $45°$? What is the preimage of $45°$?

We often think about inverses as "undoing" something. How well does this analogy work in the case of relations? What goes well with the analogy? What goes wrong with the analogy?

Working with graphs of relations

In this section, we work exclusively with graphs of functions whose candidate domain and candidate range are subsets of $\mathbb{R}$.

Definition 2.12 (graph of a relation)**.** The **graph of a relation** $r : D \to R$ is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that r assigns $a \mapsto b$.

Graph these relations: P, P^{-1}, $t \circ P$, the relation from $\mathbb{R}$ to $\mathbb{R}$ that maps x to y such that $x = y^2$; the relation from $\mathbb{R}$ to $\mathbb{R}$ that maps x to y such that $y - 2^{|x|} = 0$.

Instructor note. It may help to do these examples with sample points. It may be helpful to ask the pre-service teachers to explain why the graph of a relation defined by an equation is the graph of that equation.

As we have just seen, *any equation in x and y defines a relation*! For example, the equation $x = y^2$ defines the relation that maps x to all y such that $x = y^2$. One way to understand this is to think about the "university version" of the definition of relation.

Definition 2.13 (graph of an equation)**.** The **graph of an equation** in x and y is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Proof structure. To show that a point (a, b) is on a graph of a relation means showing that the relation assigns a to b.

As an example of this structure, consider:

What are all the points on the graph of A with x-coordinate $45°$? With y-coordinate $45°$?

Now try:

What are all the points on the graph of A^{-1} with x-coordinate $60°$? With y-coordinate $60°$?

Is the point $(60, 400)$ on the graph of A^{-1}? How about $(430, 70)$? $(70, 430)$? $(10, 200)$? Why or why not?

What is the graph of A^{-1}? How do you know you have graphed all points and not graphed any extra points?

Instructor note. In the above, attend to the reasoning why or why not, and make sure that students are *explicitly* referencing the definition of the graph of a relation. It can be helpful to prompt with questions like, "How are you using the definition of the graph of a relation?" Something else that is helpful if there is hesitation in using the definition is to go around the room and ask each person to state the definition of graph of a relation. This may sound over the top, but when used on occasion, it can be an effective technique for both remembering key definitions and also impressing the importance of particular statements. When students are reluctant to engage in this, it is more about not having done anything like this before than a fundamental aversion; in our experience, students eventually take this in good humor and appreciate being given the time to commit a definition to memory in a public and verbal way.

Show that for any relation $r : D \to D$, if $x \in D$ is in the domain of r, then (x, x) lies on the graph of $r^{-1} \circ r$.

We work with graphs of equations in a similar way to working with graphs of relations.

Proof structure. To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Is the point $(1, 2)$ on the graph of $x^2 + \frac{y^2}{4} = 1$? How about the point $(\frac{1}{2}, \frac{2}{3})$?

Find a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 6x - 4$. Is that the only point? Are there more points? How do you know?

Describe the graph of the equation $x = 0$. Describe the graph of the equation $y = 0$. How do you know that these graphs look this way?

Where does $x^2 + \frac{y^2}{4} = 1$ intersect the x-axis? Where does $x^2 + \frac{y^2}{4} = 1$ intersect the y-axis?

What is a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 5$. Then solve for x when in $y = x^2 + 1$ when y is 5. Explain the numerical coincidence.

Instructor note. Again, in doing this task, attend to pre-teachers' reasoning and make sure that they are *explicitly* referencing the definition of graph of equations, not just plugging in numbers without saying why it makes sense to plug in numbers. It is also important for pre-service teachers to explicitly use the fact that the x-axis (y-axis) consists of all points whose y-coordinate (x-coordinate) is 0.

In this task, we were using these ideas:

Definition 2.14. Given an equation in x and y and its graph, all points $(a, 0)$ on the graph are called **x-intercepts** of the graph. A graph may have 0, 1, or multiple x-intercepts.

Given an equation in x and y and its graph, all points $(0, b)$ on the graph are called **y-intercepts** of the graph. A graph may have 0, 1, or multiple y-intercepts.

Definition 2.15. We say that two graphs **intersect** each other at a point (a, b) when (a, b) is contained in both graphs.

Graphs may intersect at zero points, one point, multiple points, and sometimes even infinitely many points.

What are all the x-intercepts of the graphs of A and A^{-1}?

What are all the y-intercepts of the graphs of A and A^{-1}?

What is the intersection of the graphs A and A^{-1}?

Instructor note. In discussing this task, make sure students are *explicitly* referencing the definition of x- and y-intercepts, as well as writing down the intercepts as coordinates, not numbers. Watch that they are explaining their reasoning, not just plugging in numbers without saying why it makes sense to plug in numbers.

One takeaway from these examples is that when we attending to and explicitly referring to the definition of graph of a relation or graph of an equation, we are in a better position to help students understand concepts such as:

- x-intercepts and y-intercepts
- intersections of graphs
- solving for particular values of an equation

Now let's see how this might show up in an actual classroom.

Instructor note. What follows is an approximately 6 minute clip showing how these definitions can come up in teaching, and why it is important to go back to the definition. If there is time, it is worth showing this clip to students. Otherwise, this also works as a homework assignment, especially followed up by a quick discussion the next day.

Instructor note. When introducing video clips of teaching, we have found it useful to:

- Emphasize that we are *not* viewing videos to judge the teacher or students or their interactions. Instead, we are practicing how to observe *without judgment* to understand what is going on.
- Provide specific viewing questions.
- Emphasize that comments should be based on evidence in the video.

Even in professional development with long-time teachers, it is helpful to provide the reminder about observation rather than judgment; this goes doubly (or more) with novice teachers. We have provided some sample text below for how to explain this way of viewing teaching.

The reason it is helpful to have specific viewing questions is that it directs the conversation and allows you as an instructor to remind the students of the purpose of watching the video and how it fits into the mathematical or other goals of the lesson. Otherwise conversations can have a way of meandering unproductively; setting viewing questions up front provides a structure to hold a productive conversation.

The reason to base discussions on evidence is that otherwise, it is too easy to project one's own experiences and interpretations and leave the video context. However, the only shared context is that of the video, so grounding the conversation in the video provides some insurance for a coherent conversation. Prompts that you may use with the prospective teachers include:

- "Can you say what part of the video you are basing your comment on?"
- "That's interesting. What interaction in the video were you thinking about?"

We will watch a short video of teaching by Ms. Barbara Shreve of San Lorenzo High School. The video shows her teaching an intervention class called Algebra Success. The students in this class have been previously unsuccessful in Algebra 1. They are working on finding intercepts of equations to get ready for working with quadratics.

As you watch the video, it may be tempting to think about what you personally think is good or not as good about the teaching, or what you might have done differently. But before getting to these kinds of judgments, it is more important to simply observe what is going on, what the students' reasoning is, and what the story line is. (This is just like when working with students, as we will see later in this class and you will learn in your methods class: before evaluating students' work, we must first observe and understand students' work without judgment.) Here, we will practice observing the interactions between teachers and

As you watch the video, think about the following questions:

- How does the teacher emphasize to students to explain their reasoning?
- How does the teacher help students feel comfortable sharing their reasoning?
- How was the definition of x-intercept or y-intercept used?

Here is a link to the video: `http://www.insidemathematics.org/classroom-videos/public-lessons/9th-11th-grade-math-quadratic-functions/introduction-part-b`

Again, our discussion will be about the viewing questions. We will have time for general comments later. Let's take the viewing questions one at a time.

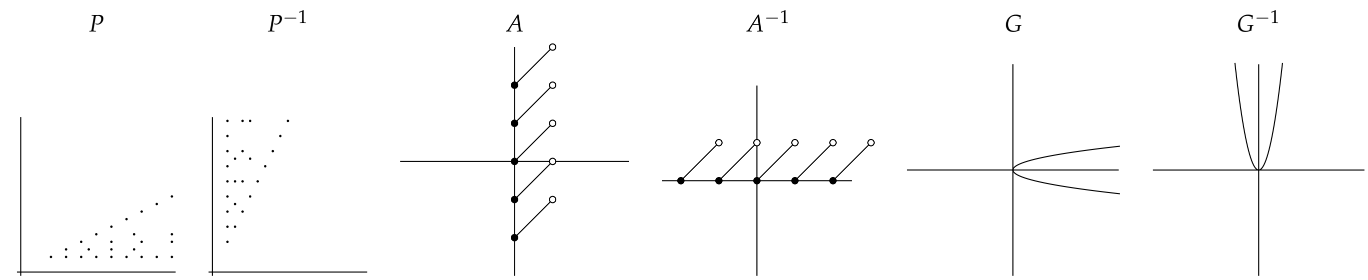
When addressing the viewing questions, be specific about your evidence from the video to support what you are saying.

Now that we have discussed the viewing questions, what other thoughts or questions come to mind?

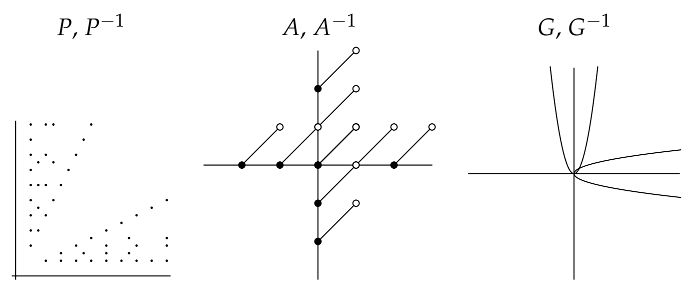
Putting it all together: Investigating graphs of inverses

To finish this chapter, let's investigate the examples we have worked with and try to generalize.

Here are some graphs that we've seen:



And here are those graphs again, this time pairing relations and their inverse relations.



What do you notice about these graphs? Find a way to fold this page so that when you hold up the folded page to the light, the graphs of P and P^{-1} are lined up with each other. Find a way to fold the paper in this way for A and A^{-1}, and then G and G^{-1}.

When you fold the coordinate plane in this way, where does $(1, 2)$ go? Where goes $(-4, 3)$ go? Where goes $(-100, -100001)$ go?

In general, when you fold the coordinate plane in this way, where does the point (a, b) go? Why does this make sense?

Explain why it makes sense that folding the plane this way should always bring a graph of a relation to the graph of its inverse. Do this in two ways:

- First, explain this discovery and why it makes sense using P and P^{-1}, including some specific examples. Make sure to use specific examples and also explain why no other fold will work.
- Then, explain this discovery and why it makes sense in general. Your explanation here should rely only on the definition of relation and its graph, and not depend on any particular examples.

In teaching, it is often useful to be able to provide these two kinds of explanation: specific and general. Specific explanations have to do with a particular example and may help the students be able to keep an image in mind. General explanations help students understand why the reasoning for a specific example applies to a larger class of objects.

Summary

Content

This chapter had a lot going on! We defined relation in three different ways, which we called the middle school, high school, and university ways. We then talked about various properties of relations, such as its domain and range, as well as the image and preimage of points and subsets. We ended by talking about graphs of relations and equations.

Throughout this discussion, we saw algebraic, graphical, and cloud diagram ways of representing relations.

We then discussed inverse relations and compositions of relations, which also can be understood in terms of these different representations.

Another common thread was Cartesian products, which is how we defined ordered pairs. This allowed us to define relations the university way, and it also allowed us to talk meaningfully about graphs of relations. The graph of a relation from $\mathbb{R}$ to $\mathbb{R}$ lives in the space given by the Cartesian product $\mathbb{R} \times \mathbb{R}$, otherwise known as $\mathbb{R}^2$.

The two explorations we did tied together representations and the concepts we discussed:

- Given any relation r, we discovered that the graph of the relation $r^{-1} \circ r$ always contains all points of the form (a, a) where a is in the domain of r.
- Given any relation r, we discovered that the graph of the relation r^{-1} can be obtained by reflecting (folding) the graph of r about the line $y = x$.

This last one may seem familiar: in high school we often teach this statement with the graph of functions. But as you learned, this statement applies to relations in general! You will examine this from a teaching perspective for homework, as well as finish the proofs of these explorations.

In the proofs, you will use the two proof structures we learned:

- To show that a point (x, y) is on a graph of a relation means showing that the relation assigns x to y.
- To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at x and y results in a true statement.

Connecting mathematically equivalent definitions

- Explain how they appear different and how they appear the same.
- Explain why they are mathematically equivalent.
- Analyze the mathematical and pedagogical purposes of each definition and why they may be appropriate for different levels of mathematical study, or why one version makes more sense after having worked with another version.

Connecting different mathematical representations

- Explain how representations appear different and how they appear the same.
- Identify how the representations can be used highlight different features.
- Go back and forth between the representations: where are the features of one representation located in the other representation?

For both of these teaching practices, it is helpful to be able to provide both specific and general explanations.

- Specific explanations have to do with a particular example and may help the students be able to keep an image in mind.
- General explanations help students understand why the reasoning for a specific example applies to a larger class of objects.

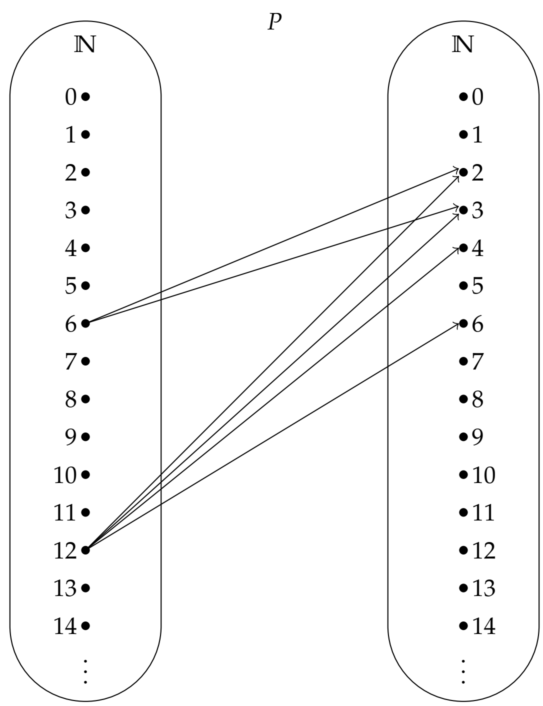
In-Class Resources

Opening Example: Parent Relation

We learned about natural number parents and children last time.

1. What is the definition of a parent of a natural number child?
2. Let P assign a natural number to each of its parents. We can represent P as a set of arrows from $\mathbb{N}$ to $\mathbb{N}$. Some arrows below have been filled in, for example P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in more arrows.
3. Consider this statement: "Some children have no parents, some children have exactly one parent, and some children have multiple parents."

   Is this statement true or false? Why?
4. How about this statement? "Some numbers have no children, and some numbers have multiple children."



Getting familiar with relations and associated concepts

Cartesian products

(We will define A, B C on the board.)

1. (a) List the elements of the following Cartesian products:
   - $A \times B$
   - $A \times C$
   - $\mathbb{Z} \times C$
   - $C \times \mathbb{Z}$
   - $\mathbb{N} \times \mathbb{N}$.

   (b) Which of the above sets contains the element $(6, -1)$? How about $(-1, 10)$?

   (c) How would you describe $\mathbb{R} \times \mathbb{R}$?

   (d) How about $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$?

Domain, range, image, preimage

Do #2. Do not proceed to #3 or #4 yet.

2. (a) What are the domain and range of the car-owner relation and the room-course relation?

   (b) Suppose this table contains course assignments to rooms at 1pm. What is the image of Math Bldg Room 100? What is the preimage of Math 996, Math 405, Math 100, and Math 221 under the room-course relation?

Room	Course in room at 1pm
Math Bldg room 100	Math 996
Math Bldg room 104	Math 100
Engineering Bldg room 750	Math 405
not being offered this term	Math 221

   (c) Let T be the relation that maps each day of the year 2030 to the its average temperature in $^\circ F$ that day. Describe a possible candidate domain, domain, candidate range, and range of this relation.

   (d) Let A be the relation that maps each degree in the interval $[0^\circ, 360^\circ)$ to all degrees in the interval $(-\infty, \infty)$ that give an equivalent angle measure. What is the preimage of 361°? What is the image of 0°?

   (e) Let ρ be the relation that maps a point in the plane to its rotation about the origin by 90°. (This means 90° *counter*clockwise.) What is the image of the point $(1, 0)$? What is the preimage of the point $(-2, 0)$?

   (f) Let G be the relation that maps x to every y such that $x = y^2$. What is the image of 4? What is the preimage of -6?

Connecting different representations

3. Interpret the definitions of candidate domain, domain, image, preimage, candidate range, and range in terms of arrows and their start and end points.
4. Interpret the definitions of these same concepts in terms of the graph of a relation. Use the graphs of A and P to illustrate what you mean.

Mathematical/teaching practice: Connecting different representations

- Explain how representations appear different and how they appear the same.
- Identify how the representations can be used highlight different features.
- Go back and forth between the representations: where are the features of one representation located in the other representation?

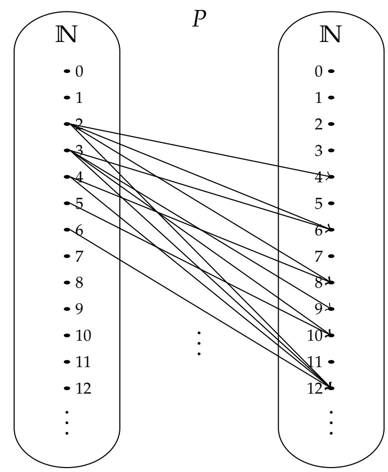
Getting familiar with inverses of relations

Back to the opening example: Inverting the parent relation

The inverse of the parent relation could be represented like this:

What other arrows does the inverse of the parent relation contain?

What might be a good name for this relation?



Connecting different but mathematically equivalent definitions

Discuss the three versions of the definition of inverse of a relation. What do they each say? How would you represent them? What makes them mathematically equivalent?

Mathematical/teaching practice: Connecting different but mathematically equivalent definitions

- Explain how they appear different and how they appear the same.
- Explain why they are mathematically equivalent.
- Analyze the mathematical and pedagogical purposes of each definition and why they may be appropriate for different levels of mathematical study, or why one version makes more sense after having worked with another version.

Working with compositions algebraically

Let P be the parent relation and let $t : \mathbb{N} \to \mathbb{N}, x \mapsto 2x$.

1. (a) For the relation $t \circ P$, what is the image of 6? Of 12? Of 9?
   (b) For the relation $t \circ P$, what is the preimage of 4? Of the set $\{1, 3, 5\}$? Of the set $\{4, 14\}$?
   (c) What are the domain and range of the relations $t \circ P$?

2. How would you represent $P \circ t$? What are its domain and range?

3. (a) Sketch a representation of $A^{-1} \circ A$.
   (b) What are its domain and range?
   (c) What is the image of $45°$?
   (d) What is the preimage of $45°$?

We often think about inverses as "undoing" something. How well does this analogy work in the case of relations? What goes well with the analogy? What goes wrong with the analogy?

Graphs of relations

Proof structures:

To show that a point (a, b) is on a graph of a relation means showing that the relation assigns a to b.

To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

1. What are all the points on the graph of A with x-coordinate $45°$? With y-coordinate $45°$?

2. What are all the points on the graph of A^{-1} with x-coordinate $60°$? With y-coordinate $60°$?
   Is the point $(60, 400)$ on the graph of A^{-1}? How about $(430, 70)$? $(70, 430)$? $(10, 200)$? Why or why not?
   What is the graph of A^{-1}? How do you know you have graphed all points and not graphed any extra points?

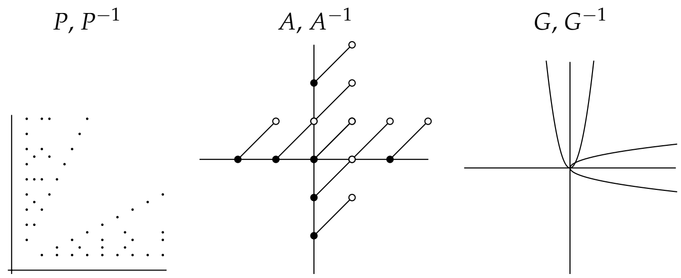
3. Show that for any relation $r : D \to D$, if $x \in D$ is in the domain of r, then (x, x) lies on the graph of $r^{-1} \circ r$.

4. Is the point $(1, 2)$ on the graph of $x^2 + \frac{y^2}{4} = 1$? How about the point $(\frac{1}{2}, \frac{2}{3})$?
   Find a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 6x - 4$. Is that the only point? Are there more points? How do you know?
   Describe the graph of the equation $x = 0$. Describe the graph of the equation $y = 0$. How do you know that these graphs look this way?
   Where does $x^2 + \frac{y^2}{4} = 1$ intersect the x-axis? Where does $x^2 + \frac{y^2}{4} = 1$ intersect the y-axis?
   What is a point that is on the graph of $y = x^2 + 1$ and the graph of $y = 5$. Then solve for x when in $y = x^2 + 1$ when y is 5. Explain the numerical coincidence.

5. What are all the x-intercepts of the graphs of A and A^{-1}?
   What are all the y-intercepts of the graphs of A and A^{-1}?
   What is the intersection of the graphs A and A^{-1}?

CLOSING INQUIRY

To finish this chapter, let's investigate the examples of relations and their inverses. Here are some graphs that we've seen:



What do you notice about these graphs? Find a way to fold this page so that when you hold up the folded page to the light, the graphs of P and P^{-1} are lined up with each other. Find a way to fold the paper in this way for A and A^{-1}, and then G and G^{-1}.

When you fold the coordinate plane in this way, where does $(1, 2)$ go? Where goes $(-4, 3)$ go? Where goes $(-100, -100001)$ go?

In general, when you fold the coordinate plane in this way, where does the point (a, b) go? Why does this make sense?

Explain why it makes sense that folding the plane this way should always bring a graph of a relation to the graph of its inverse. Do this in two ways:

- First, explain this discovery and why it makes sense using P and P^{-1}, including some specific examples. Make sure to use specific examples and also explain why no other fold will work.
- Then, explain this discovery and why it makes sense in general. Your explanation here should rely only on the definition of relation and its graph, and not depend on any particular examples.

Reference: Relations

Definition 2.1. Let D and R be sets. The **Cartesian product** of D and R is defined as the set of ordered pairs $\{(x, y) : x \in D, y \in R\}$. It is denoted $D \times R$.

Definitions 2.2, 2.5, 2.6:

Middle school version	A **relation** from a set D to a set R is a set of arrows going from elements of D to elements of R.
High school version	A **relation** P from a set D to a set R a set of assignments from elements of D, called inputs, to elements of R, called outputs.
University version	A relation $r : D \to R$ is a subset of $D \times R$, i.e., $r \subseteq D \times R$.

Notation: $r : D \to R$ refers to a relation from D to R; and $x \mapsto y$ refers to an assignment from $x \in D$ to $y \in R$

A relation may map an element of D to 0, exactly 1, or multiple elements of R. An element of R may have 0, exactly 1, or multiple elements of D mapping to it.

Definition 2.4. For a relation $r : D \to R$, we say that

- D is the **candidate domain**;
- R is the **candidate range** (or **codomain**);
- the **image** of an element $x \in D$ is the set of elements of R that x is mapped to. Similarly, the **image** of a subset of $S \subseteq D$ is the subset of R containing the images of all elements of S.
- When a element in D maps to no element in R, we say it has **empty image**. Otherwise it has **nonempty image**.
- the **domain** of r is the subset $D' \subseteq D$ of elements with nonempty image.
- the **preimage** of an element of R is the set of elements of D that map to R. Similarly, the **preimage** of a subset $T \subseteq R$ is the subset of D containing the preimages of all elements of T.
- When a element in R has no element in D mapping to it, we say it has **empty preimage**. Otherwise it has **nonempty preimage**.
- the **range** (or **image**) of the relation r is the subset $R \subseteq D$ with nonempty preimage.

Definitions 2.7, 2.8, 2.9:

Middle school version	If r is a relation from a set D to R, then the **inverse relation** of r is the relation that swaps the direction of the arrows of r. The arrows of the inverse go from elements of R to elements of D.
High school version	Given a relation $r : D \to R$, the **inverse relation** of r is the set of assignments $y \mapsto x$ such that $x \mapsto y$ is an assignment of r.
High school version	Given a relation $r : D \to R$, the **inverse relation** of r is defined as the subset $\{(y, x) \subseteq R \times D : (x, y) \in r\}$.

Notation: r^{-1}.

Definition 2.11. Given two relations $P : D \to D$ and $Q : D \to D$, we define the **composition** of P then Q as the relation that assigns x to z whenever there is a $y \in D$ such that P assigns $x \mapsto y$ and Q assigns $y \mapsto z$.

Definition 2.12. The **graph of a relation** $r : D \to R$ is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that r assigns $a \mapsto b$.

Definition 2.13. The **graph of an equation** in x and y is defined as the set of points $(a, b) \in \mathbb{R}^2$ such that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Definition 2.14. Given an equation in x and y and its graph, all points $(a, 0)$ on the graph are called **x-intercepts** of the graph. All points $(0, b)$ on the graph are called **y-intercepts** of the graph. A graph may have 0, 1, or multiple x-intercepts and y-intercepts.

Definition 2.15. Two graphs **intersect** each other at (a, b) when (a, b) is contained in both graphs. Graphs may intersect at zero points, one point, multiple points, and sometimes even infinitely many points.

Proof structures:

- To show that a point (a, b) is on a graph of a relation means showing that the relation assigns a to b.
- To show that a point (x, y) is on the graph of an equation means showing that evaluating the equation at $x = a$ and $y = b$ results in a true statement.

Homework

Ch 2: write homework

0. Read the following definition.

   A **function** f from D to R is a relation from D to R where each input in D is assigned to no more than one output in C.

   Which of the following represents a function?

   [insert cloud diagram 1 - not a function] [insert cloud diagram 2 - is a function] [insert cloud diagram 3 - inverse is a function but it is not itself a function]

1. Graphs of relations and their inverses. Use examples that will be used in Lesson 3: sine, cosine, absolute value, square, cube. In each, label any maxima and minima, then label these corresponding points in their inverse relations. Part (f). Draw images and preimages of intervals in both the graphs of the relations as well as their inverses.

2. Something with intercepts: maybe responding to students' explanations? Using various representations.

3. Proof structure practice: Show that for any relation $r : D \to D$, if $x \in D$ is in the domain of r, then (x, x) lies on the graph of $r^{-1} \circ r$.

4. (Mini SoP?) Proof structure practice: To get from graph of relation to graph of its inverse, reflect over $y = x$. Do this with particular example, explain to class why this is true. Set up a discussion about it. Connect different definitions. Begin by explaining why (a, b) is on the graph of r if and only if (b, a) is on the graph of r^{-1}.

   Given a relation $r : D \to D$. Explain why (a, b) is a point on the graph of r if and only if (b, a) is on the graph of r^{-1}. (Remember that "if and only if" requires two directions of reasoning! What are these two directions?)

5. Composition of graphs (from the canadian textbook; also draw compositions of graphs and their inverses) (probably need to scaffold this more. think about relationship between linear function and linear function, also think about where points go, not sure what else needs to go here. Will probably need to re-assign in Lesson 3) (Maybe instead of doing entire graph, do something like a series of questions – what is $f(g(f(g(1))))$, $g(f(g(1)))$, etc. Do this for a bunch of points, especially for the combination $g(f(x))$ (or whichever composition is easier to do.). Then assign the actual composition in Hwk 3 after discussing correspondence and covariational views and ask students to analyze where they used each viewpoint. In Hwk 3 also assign composition of linear and quadratic, analyze which parts are constant, linear non-constant, quadratic, none of the above.)

   need to introduce definition of function and its notation here, otherwise can't really use this notation.

   Also add a part (b) with g and h where h is relation, and asking to find the image of 1 under $h \circ g$. make h so that $h \circ g$ has multiple values.

6. Something with integral? (Definite integral; evaluating at particular point; relates to definition of graph of an equation)

7. Something with practicing domain and range? Maybe something about teaching them with example $1/x$ or asymptotes or something that is discontinuous?

3 Functions: Introduction to Correspondence and Covariational Views (Weeks 3-4) (Length: ˜5 hours)

Overview

Content

Ch 3: Write overview

Functions, defined as . . .

Partial inverses, defined as . . .

Invertibility of a function, defined as . . .

Additionally, we revisit **composition** and **inverse** so that we can use them to compare and contrast correspondence and covariation views. By these terms, we mean:

- (Correspondence) Conceiving of functions and their behavior primarily in terms of maps from individual elements of the domain to individual elements of the range.
- (Covariation) Conceiving of functions and their behavior primarily in terms of coordinating how changes in the value of one variable impact the value of the other variable.

Constructions	Mathematical/Teaching Practices
Constructing viable partial inverses of functions, meaning . . .	**Introducing a definition**, meaning . . . **Explaining a mathematical "test" of a property**, meaning . . . **Attending to student thinking**, meaning . . . **Recognizing and explaining correspondence and covariation views**, meaning . . .

Summary

This lesson comes in three parts: reviewing key examples from Homework 2, working with functions from a correspondence point of view, and working with functions from a covariation point of view.

Review of Homework 2. We review how graphs can be used to represent relations (and hence functions) using Homework 2 Problem 5. These set up the discussion of how definitions for working with relations specialize to the case when the relation is a function. We also review the functions used in Homework 2 Problem 1 so that the discussion of partial inverses can focus on how to construct partial inverses as opposed to details about the functions themselves, particularly sine and cosine graphs.

Functions and the correspondence view. After defining *functions*, we introduce the teaching practices of *introducing a definition* and *explaining a mathematical "test" of a property*. The latter uses the example of the *vertical line test*. We then define *partial inverses of functions* and discuss *how to construct viable partial inverses of functions*, and then define *invertible function*. We continue practicing how to explain a mathematical test of a property by looking at the *horizontal line test*. We conclude this part of the lesson by revealing that thus far we have been viewing relations and functions from a *correspondence view* of the equation $y = f(x)$, meaning only thinking about functions in terms of maps from elements of the domain to elements of the range.

Covariational view. We use the Morgan Minicase to introduce the difference between correspondence and covariational views. By covariational view, we mean understanding how changing the value of one variable impacts the value of the other variable, and learning to coordinate changes in one variable with changes in the other. This minicase introduces the teaching practice of *recognizing and explaining correspondence and covariation views*. We revisit composition and inverse to compare and contrast these views and provide an opportunity engage in this teaching practice.

Acknowledgements. The Morgan Minicase activities are based on materials in progress for the Content Knowledge for Teaching Minicases project of the Educational Testing Service and is used with permission. The Morgan Minicase is one of a suite of items developed by the Measures of Effective Teaching project and examined through a collaborative grant between the Educational Testing Service and University of Nebraska-Lincoln (NSF #DGE-1445551/1445630).

Materials.

- Handouts from In-Class Resources (can be printed double-sided)

Review of key examples

Using the definition of graph of a relation

In the previous homework, you were given the graphs of two functions f and g and asked to find values such as $f(g(1))$ or $g(g(f(2)))$. Let us now consider how graphs can be used to find these values, and how the reasoning relies on the definition of graph of a relation.

Ch 3: insert graphic from Homework 2 of composition of functions

> Compare your responses to Homework 2 Problem 5. How did you use the definition of graph in your reasoning? Use $g \circ f$ as an example to illustrate your use of these definitions. Then illustrate your reasoning using $h \circ g$.

Solution. (*Partial.*) By definition of a graph of an equation, (a, b) is on the graph of $y = f(x)$ if and only if $b = f(a)$. There is only one coordinate on the graph of f with the x-value 1, so that coordinate's y-value must be $f(1)$.

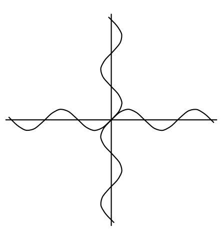
For finding where $h \circ g$ maps 1, we first seek where g maps 1. Just as for f, there is only one coordinate on the graph of g with the x-value 1, so that coordinate's y-value must be $g(1)$. There are multiple possible values for where $h \circ g$ maps 1, because the graph of h contains multiple coordinate points whose x-value is $g(1)$. ■

Some functions and relations we will examine further today

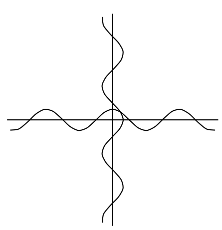
In the homework, you worked with the graphs of $y = \sin(x)$, $y = \cos(x)$, $y = |x|$, $y = x^2$, and $y = x^3$ and their inverse relations. We also learned last time that a point (a, b) is on the graph of a relation r if and only if (b, a) is on the graph of its inverse relation r^{-1}.

You may have obtained graphs that looked like this:

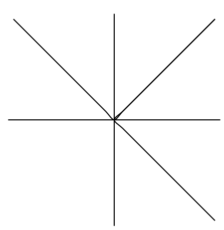
sin and its inverse relation



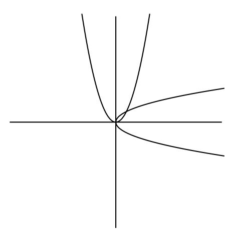
cos and its inverse relation



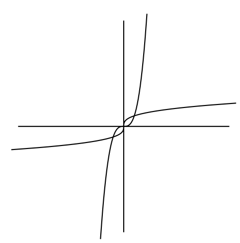
Abs and its inverse relation



Sq and its inverse relation



Cu and its inverse relation



Instructor note. Together with the class, label some maxima and minima of these graphs, as well as a few x-intercepts and y-intercepts. Emphasize again that (x, y) is on the graph of a relation if and only if (y, x) is on the graph of its inverse.

Functions and the correspondence view

A function is a special kind of relation, where every element of the domain is assigned to exactly one element of the range. Here are two ways to think about functions:

Definition 3.1 (Function: Middle and High school version)**.** A **function** f from D to R is a relation from D to R where each input in D is assigned to no more than one output in C.

Definition 3.2 (Function: University version)**.** A **function** is a relation $f : D \to R$, such that if $(x, y), (x, y') \in f$, then $y = y'$.

With relations in general, we can't say exactly where an input is mapped to, because it may be mapped to multiple outputs such as in the parent function. However, with functions, an input determines the output uniquely in the sense that it is either undefined or it is exactly one value. It is because of this property that we have function notation. The notation $f(x)$ means "the at most one value that f maps x to".

> **Instructor note.** We have not typically gone through the following table and definitions in class. This is provided more as a reference than as something to address explicitly in class.

Here is a table summarizing how we can use this notation to interpret concepts associated to relations:

	Relations in general	When the relation is a function
Domain	All elements $a \in D$ with nonempty image	All elements $a \in D$ such that $f(a)$ is defined
Range	All element $b \in R$ with nonempty preimage	All elements $b \in R$ such that $b = f(x)$ has a solution in x
Composition $g \circ f$	The relation contains all assignments $x \to z$ such that there is a y where f maps x to y and g maps y to z	This relation maps x to $f(g(x))$. It is also a function.
Graph of f	The point (a, b) is on the graph of f if and only if f maps a to b	The point (a, b) is on the graph of f if and only if $b = f(a)$. *Or*: The point (a, b) is on the graph of f if and only if a is a solution to $b = f(x)$.

The above table uses the following definitions:

Definition 3.3. Given a function $f : D \to R$, we say that $f(x)$ is **defined** if x has nonempty image. Otherwise, if x has empty image, we say $f(x)$ is **undefined**.

Given $b \in R$, We say that the equation $b = f(x)$ **has a solution in x** if there is an $a \in D$ such that $b = f(a)$ is a true statement. Otherwise, if there is no $a \in D$ such that $b = f(a)$ is true, then we say the equation **has no solution**.

Teaching functions as a case of teaching definitions

Functions and relations are fundamental concepts in middle and high school. When you are teaching the function and relation, you will mostly likely find it useful to discuss their definitions in the context of different representations, such as cloud diagrams, T-charts (table of input and output values) and graphs. As we have been experiencing, teaching definitions often contains these components:

> **Teaching definitions**
>
> - Introductory examples and/or non-examples of the definition
> - Precise statement of the definition
> - Interpreting the precise statement, especially any new terminology or key rules, in terms of the introductory example and/or non-example
> - Interpreting the terminology and rules in terms of the introductory examples, often using different representations that students will continue to encounter.

If you were teaching the definition of function, you might use examples such as the ones we have used today, or other ones that build on or review what your students are familiar with.

Review the definition of the graph of a relation. We can use this definition for functions, too, because all functions are relations.

Graph the relations $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$ and Abs^{-1} on separate graphs. For each, how could you use the graph to understand the definition of function?

Suppose your students are seeing graphs of functions for the very first time. How would you explain the definition of a function using a graph, in general terms? (meaning, not relying on any particular example; an explanation whose wording would apply to all possible cases)

Remember that your students haven't seen anything called a "vertical line test", so they don't know this. However, the above might be a good way of getting to the idea of the "vertical line test."

Instructor note. When asking pre-service teachers to explain their thinking, many often fall back on phrases like "vertical line test". If this happens, then press on what such phrases mean in terms of the definition of function. It may be the case that they are using this phrase to mean a procedure that they do not understand the meaning of; this is useful background information for you as an instructor, because the "vertical line test" and how to teach this procedure with meaning is addressed next. However, whatever the discussion, there must be some sort of complete explanation that uses at least one definition of function explicitly.

Teaching the vertical line test as a case of explaining a mathematical "test" of a property

A common part of high school lessons on graphs is discussing the "vertical line test". This is a "test" in the sense that it will "test" a mathematical thing (a graph of a relation) for a particular property (whether the relation is a function). There are many "tests" like this in high school mathematics; for instance, you may remember "convergence tests" from calculus. In general, here are some elements that may be useful to keep in mind when explaining a mathematical "test" of a property, using the vertical line test as a way to illustrate a way to teach mathematical "tests" of properties.

Explaining a Mathematical "Test" of a Property

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	*Example*: The Vertical Line Test • Today we will learn something called the **Vertical Line Test**. • This test is a way of telling **whether a relation is a function**. • We will test the **graph of a relation** to tell this.
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	Here is how we do the Vertical Line Test: • Graph the relation. • Think about all vertical lines in the plane. Look at whether the vertical lines intersect the graph of the relation. Here is how to pass or fail: • If all of the vertical lines cross the graph zero or one times and no more, then the graph passes. Otherwise the graph fails.
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	If the graph of a relation passes, then the relation is a function. If the graph of a relation fails, then the relation is not a function.
Explain *why* the test "works":	[This explanation is for your homework]

To warm up to understand this table, we first discussed the following:

What does the "vertical line test" do? How do you perform this "test"? How does a graph of a relation to pass or fail the "vertical line test"? What does it mean about the relation if its graph passes or fails? Why is this conclusion true?

As we discussed these issues, we talked about the elements of the table.

Instructor note. The class is unlikely to have time to fill in the next table completely. You might direct them to focus primarily on the first three rows, and then go over them. For the last row, you might instruct them to write down the claim, in the form of an "if-then" statement, that they are to explain.

Based on the discussion we have had, how would you do the following?

Explaining a Mathematical "Test" of a Property: The Vertical Line Test

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	
Explain *why* the test "works":	

What do you think of these elements? What might the purpose of each element be as far as teaching and high school students go?

Invertible functions and the horizontal line test

As we learned in the previous chapter, all functions have inverse *relations*.

This is because all relations have inverse relations, and functions are relations.

Only some functions have inverse *functions*, meaning inverse relations that happen to be functions.

Definition 3.4. A function $f : D \to R$ is **invertible function** if its inverse relation $f^{-1} : R \to D$ is a function.

A function $f : D \to R$ is **non-invertible function** if its inverse relation $f^{-1} : R \to D$ is a not a function.

Examples of functions we have seen include sine, cosine, Abs, Sq, Cu, $x \mapsto \frac{1}{x}$, and $x \mapsto \sqrt{x}$.

How would you use these examples as a way to teach the definitions of *invertible function* and *non-invertible function*? As you think about this, focus on this element of explaining definitions: **interpreting the terminology**

and rules in terms of the introductory examples, using representations that students will continue to encounter.

When you are teaching invertible functions to students who are seeing it for the first time, they may not have heard of anything called a "horizontal line test." It is good practice to try to base explanations of invertibility as much as possible on the definitions of function and invertible/non-invertible functions as a way to help students understand the ideas conceptually. This provides a more solid foundation for ultimately understanding the horizontal line test.

Instructor note. Just as in the discussion about functions in the previous section, it is important to press teachers on how their reasoning explicitly draws on the definitions of function and invertible/non-invertible function.

What is the "horizontal line test"? Explain this test of a mathematical property.

Explaining ______________________________

Introduce *what*: • Name the test. • What is the test supposed to tell us? (Be precise!) • What are you testing? (Be precise!)	
Describe *how*: • How do you do the test? • How do you tell whether the thing passes or fails the test?	
Deliver the *punchline*: What happens when the thing "passes" the test? What happens when the thing "fails" the test?	
Explain *why* the test "works":	

We have discussed previously that inverse often means "undoing". In the case of invertible functions, the "undoing" metaphor works well.

Theorem 3.5 (Non-invertible may not undo, invertible always undos). *Let $f : D \to R$ be a function and f^{-1} be its inverse relation.*

- *When f is non-invertible, $f^{-1} \circ f$ may map an element x to an element other than x.*
- *When f is invertible, we have $f^{-1} \circ f(x) = x$ for all x in the domain of f.*

We can see this in our examples of non-invertible functions, such as Sq, sin, or cos. The proof below summarizes the general phenomenon.

Proof. Given $f : D \to R$ is a function and f^{-1} is its inverse relation. Let a be an element of the domain of f, and suppose $f(a) = b$. When f is non-invertible, f^{-1} may map a to multiple elements, so $f^{-1} \circ f$ may then map a to multiple elements. When f is invertible, $f^{-1}(b) = a$, so $f^{-1} \circ f(a) = a$. □

Instructor note. The above theorem is phrased in a non-standard way, discussing both the cases of non-invertible as well as invertible, as opposed to only invertible functions. The reason for this is that we have found that undergraduate mathematics students in general can ignore the conditions of a theorem to hold (e.g., whether a function is invertible), especially when the conclusion is a very nice and seemingly intuitive one (e.g., $f^{-1} \circ f(x) = x$ for all x in the domain of f). Our solution to this is to phrase theorems to account for the nice and not-so-nice conditions. While not a panacea, we have found it generally helpful in that students are more likely to remember how different conditions may lead to different conclusions.

This brings us to the definition of the inverse of an invertible function seen most often in high school.

Definition 3.6 (Inverse of a function: High school version)**.** If $f : D \to R$ is not an invertible function, then it does not have an inverse function.

If $f : D \to R$ is an invertible function, then the **inverse function** of f is the function f^{-1} such that for all x in the domain of f, we have

$$f^{-1} \circ f(x) = x.$$

CONSTRUCTING PARTIAL INVERSES OF FUNCTIONS (AKA "FAKE INVERSES")

What if a function is non-invertible but we would like to be able to create some sort of inverse anyway?

The function $\mathrm{Sq} : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ does not have an inverse function. However, suppose you wanted to construct an function that is like an inverse, given that we know we can't have a true inverse.

Rank these from best to worst as candidates for an "inverse" for Sq:

(Remember that $\sqrt{x}$ is defined as the zero or positive root of x, when a real root exists.)

- $\mathrm{Rt}_1(x) = -\sqrt{x}$
- $\mathrm{Rt}_2(x) = \sqrt{x}$
- $\mathrm{Rt}_3(x)$ is defined to be $-\sqrt{x}$ when $x \in [0, 1)$ and $\sqrt{x}$ when $x \in [1, \infty)$.
- $\mathrm{Rt}_4(x)$ is defined to be $-\sqrt{-x}$ when $x \leq 0$ and $\sqrt{x}$ when $x \geq 0$.

Each of the functions $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ are called "partial inverses". They are like inverse functions, except that they don't work everywhere; they only work some of the time. So they are "fake" in the sense that they don't work on the entire domain, but they are still an "inverse" in the sense that they will be an inverse in a restricted subset of the domain.

Instructor note. In our experience, pre-service teachers find the nickname "fake inverse" memorable and helpful for understanding partial inverses. It seems prudent to mention that we coined this nickname well before 2017. In 2017, we tried using the nickname "good enough inverse" to avoid reference to current event usage of the term "fake", but the nickname "good enough inverse" did not have the same effect and pre-service teachers forgot both the terms "partial inverse" and "good enough inverse". There is also something to be said here about why we might have nicknames for things that already have perfectly good names. Part of it is as an aid to memory. The concept of "partial inverse" is important to high school mathematics, especially trigonometry. However, if teachers never remember the idea of "partial inverse", then the fact that arcsine or arccosine are not true inverses will not be passed along. Although coming up with nicknames (like "fake inverse" as opposed to "true inverse") can seem little cheesy, we have found it to be more beneficial than detrimental.

Instructor note. We recommend doing this next activity quickly, as a class, with numerical examples.

What is the domain of Sq?

On what subset of the domain of Sq do each of $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ serve as a true inverse? (They satisfy the equation in Definition 3.6.)

On what subset of the domain of Sq are the functions $\mathrm{Rt}_1, \mathrm{Rt}_2, \mathrm{Rt}_3, \mathrm{Rt}_4$ serve as a fake inverse? (They do not satisfy the equation in Definition 3.6.).

Finally, let's apply the ideas of partial inverse to a trigonometric function:

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \sin(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the sine function.

On what subset of the domain of sine do each of your candidates serve as a true inverse? On what subset of the domain of sine do your candidates serve as a fake inverse?

Construct at least three candidates for partial inverses ("fake inverses") for $f(x) = \cos(x)$.

Rank the candidates you have constructed from best to worst "inverse" for the cosine function.

On what subset of the domain of cosine do each of your candidates serve as a true inverse? On what subset of the domain of cosine do your candidates serve as a fake inverse?

Instructor note. The following describes what usually happens with rankings of candidates for inverse of sine.

You may have concluded that the "best" inverse is the candidate whose domain of being a true inverse is $[\frac{\pi}{2}, \frac{\pi}{2}]$. This is in fact how the function that you may have seen before, with the (unfortunate) notation $\sin^{-1}$, is defined. (An alternative notation is arcsin.) In general, when working with functions for which we would like to have an inverse but the functions are not actually invertible, we can define partial inverses ("fake inverses") for them which we take to be standard. Examples of this include $x \mapsto \sqrt{x}$ for Sq, arcsin for sine, and arccosine for cosine. They usually are the candidate where if you looked at the alternatives, you would agree that there's something about that candidate that seems to work better: they are continuous, the subset of the domain on which they are a "true" inverse is close to zero or symmetric around zero, etc.

Covariational view of functions

So far we have been looking at things from a *correspondence view*, meaning that we primarily think of functions in terms of individual inputs and the images to which they are assigned. In other words, we focus on how inputs and outputs correspond. This perspective can be very useful for defining things like domain, range, composition, or inverse. However, this perspective can be at a distinct disadvantage when it comes to looking at the behavior of a function or of understanding how changes in the input and output variables influence each other.

By *covariational view*, we mean understanding how changing the value of one variable impacts the value of the other variable, and learning to coordinate changes in one variable with changes in the other. We use the Morgan Minicase[1] to introduce the difference between correspondence and covariational views. This minicase also introduces the teaching practice of *recognizing and explaining correspondence and covariation views*.

Ms. Morgan's class

During a lesson on writing equations of linear functions represented in tables, Ms. Morgan asked her students to write the equation of the linear function represented in the table below, and to explain how they found their answers.

Students found the correct equation, but they gave different explanations of how they found their answers:

x	y
1	6
2	11
3	16
4	21

Student A:	Each time the value of x goes up by 1, the value of y goes up by 5, so the slope is 5. And if x goes down by 1, then y will have to go down by 5, so the y-intercept is 1. That means the equation is $y = 5x + 1$.
Student B:	I just looked at the value of y and saw that it kept increasing by 5, so $m = 5$. Then I subtracted that number from the first value of y in the table, so $b = 1$. You always put m times x and add the b, so the equation is $y = 5x + 1$.
Student C:	For this function, I saw that you can always multiply the value of x by 5 and then add 1 to get the value of y, so the equation is $y = 5x + 1$.

[1](c) 2013, Educational Testing Service, used with permission

Read through the student responses in Ms. Morgan's class.

- **Observe**: What is the student thinking? How might they have arrived at each step of their solution?
- **Interpret**: What are you sure that each student understands? What are you sure that each student does not understand? What are you unsure that each student understands? Based on what evidence?

  Here are some concepts to consider analyzing for students' understanding:

  - y-intercept
  - Constant rate of change
  - Form of a linear equation
  - How changes in one variable impact changes in the other variable
  - Definition of graph of a function

- **Interpret, continued:** Are the explanations mathematically complete? Why or why not?

Observe: What may Student A/B/C be thinking?

Interpret:

I am sure that Student A/B/C understands ...	I am sure that Student A/B/C does NOT understand ...	I am unsure whether Student A/B/C understands ...

What is complete or incomplete about Student A/B/C's explanation?

There are two useful views on functions: covariation and correspondence. Use this space to take notes on what these mean.

Correspondence:

Covariation:

Then discuss: How do covariation and correspondence views come up in the students' thinking?

Solution. (*Partial*)

Ways that the covariation and correspondence views arise in the Morgan Minicase include the following:

- How changes in one variable impact changes in the other variable: Covariation view; and
- Definition of graph of a function: Correspondence view.

Student A (does provide a mathematically complete explanation)

- understands concept of constant rate of change, y-intercept, form of linear function
- may not understand correspondence view of function

Student B (does *not* provide a mathematically complete explanation)

- does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
- does not understand constant rate of change
- may not understand y-intercept
- does not necessarily understand correspondence view of function

Student C (does provide a mathematically complete explanation)

- does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
- does understand correspondence view of function
- may not understand constant rate of change, y-intercept.

∎

NOTICING STUDENT THINKING AND RECOGNIZING AND EXPLAINING CORRESPONDENCE AND COVARIATION VIEWS

In any teaching, it is important to attend to student work so that you can notice the student thinking. Some things to keep in mind for this are:

> **Noticing student thinking**
>
> - First **observe** what the student's thinking is, **without judgment** as to what they understand or do not understand.
> - Then, **interpret** what they understand, may not understand, or what you are unsure of whether they understand. Always base this interpretation on the evidence of the student thinking you have, and be sure that you know what evidence you are drawing upon.
>
>   Only after this, you might interpret the completeness or correctness of the student's thinking.
> - From here, you might **respond** to the student based on what you have observed and interpreted.

The reason to split up observing without judgement, interpreting, and responding is that it interpretations tend to be more accurate after we have taken a step back to observe what the student may be thinking, without judgment.

In the case of teaching functions, it is helpful to be able to *recognize and explain correspondence and covariation viewpoints*. The Morgan Minicase gave us an opportunity to see how both can show up in response to the same problem as well as how differently they can appear. Throughout this minicase, we kept in mind the following.

> **Recognizing and explaining correspondence and covariation viewpoints**
>
> Correspondence and covariation views can be thought of as the following:
>
> - (Correspondence) Conceiving of functions and their behavior primarily in terms of maps from individual elements of the domain to individual elements of the range.
> - (Covariation) Conceiving of functions and their behavior primarily in terms of coordinating how changes in the value of one variable impact the value of the other variable.
>
> When introducing ideas in class, coming up with examples, or giving explanations, it is helpful to think about whether you are working with a correspondence or covariation view, and then to see what an explanation in the other view might look like.
>
> When noticing student thinking, it can be helpful to interpret whether they are taking a correspondence or covariation view.

We now revisit our ways of building functions to see how the covariation view can come into play.

BUILDING FUNCTIONS: INVERSES AND COMPOSITIONS

Let's begin by looking at compositions of functions that high school students are likely to encounter: linear functions and quadratic functions.

> **Instructor note.** One way to use the following tasks may be the following:
>
> 1. Pose the first question as a quick vote.
> 2. Ask pre-service teachers to talk to a neighbor about the answer.
> 3. Go over the answer and the reasoning.
> 4. Ask pre-service teachers to discuss the reasoning and ask questions about the reasoning.
>
> Then assign the follow-up questions and go over the reasoning as a whole class after giving students individual and/or group time to work on them.
>
> This protocol is based on the Good Questions Project led by Maria Terrell at the Cornell University department of mathematics. She and colleagues observed that instructors who followed this protocol tended to have students

who understood pre-calculus and calculus concepts better than instructors who only did 1 and 2, or who did only 3 or 4, or some other strict subset of these steps. In fact, the students in their observations who fared worst on average were those whose instructors only did 3 and 4.

One way to make sense of these findings is that students who vote and discuss their reasoning have created a personal stake in the mathematics, so they are more primed to listen to reasoning about the question, as well as to compare and contrast their reasoning with someone else's proposed reasoning. They may also be more likely to take the question seriously if they know that they initially got the wrong answer, because they understand what the challenge of the question is. When students only see the answer and the reasoning, even if it makes sense, they may also not appreciate the concepts that the question is getting at. So they are less likely to remember or take the concept seriously.

Suppose that f is a linear function whose constant rate of change is 5 vertical units per horizontal unit; g is a linear function whose constant rate of change is -3 vertical units per horizontal units; and h is a linear function whose constant rate of change is 0 vertical units per horizontal unit.

What is the rate of change of $g \circ f$? Circle your response.

2   -2 15 -15 8 -8 it is not constant   none of the above

What about for $f \circ g$? What is your reasoning?

What about for $h \circ f$? $f \circ h$? What is your reasoning?

Suppose that L is a linear function, and Q is a quadratic function. What is $Q \circ L$? Circle your response.

linear   quadratic   cubic   something else

What is your reasoning?

How would you explain this from a covariation view?

Let sin be the sine function. What is $L \circ \sin$? Circle your response.

linear   sinusoidal   something else

(We can think of sinusoidal as behaving like sine or cosine: it moves "up" and "down" in a period fashion, always to the same maximum and minimum value.)

What is your reasoning?

How would you explain this from a covariation view?

Revisiting a key example

In-Class Resources

Opener

Homework

Note to self about homework followup from Homework 2: Composition of graphs (from the canadian textbook; also draw compositions of graphs and their inverses) (probably need to scaffold this more. think about relationship between linear function and linear function, also think about where points go, not sure what else needs to go here. Will probably need to re-assign in Lesson 3) (Maybe instead of doing entire graph, do something like a series of questions – what is $f(g(f(g(1))))$, $g(f(g(1)))$, etc. Do this for a bunch of points, especially for the combination $g(f(x))$ (or whichever composition is easier to do.). Then assign the actual composition in Hwk 3 after discussing correspondence and covariational views and ask students to analyze where they used each viewpoint. In Hwk 3 also assign composition of linear and quadratic, analyze which parts are constant, linear non-constant, quadratic, none of the above.)

Crib homework problems from combination of Bremigan, Bremigan, and Lorch and Sultan and Artzt. They have some good conceptual problems in there about relations and functions.

Under what conditions can you rotate the graph of a function about the origin, and still have the resulting graph being the graph of a function? If the graph of a function cannot be rotated about the origin without ceasing to be the graph of a function, might there be other points which could act as centre of rotation and preserve the property of being the graph of a function?

Simulation of Practice: Title of Simulation 1

something with vertical line test

Simulation of Practice: Title of Simulation 2

function/relation transformation maybe

Allen item SoP? (wait until Chapter 4 on covariational perspective)

Vertical line test SoP? (wait until next time)

Composition of graphs, again. Correspondence view. Repeat from Homework 2.