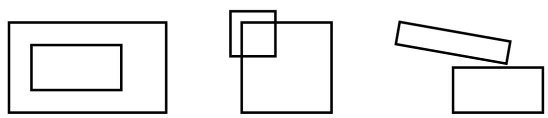
Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Adapted for MODULE(S^2)

Version Spring 2018



This work is licensed under a Creative Commons Attribution-ShareAlike 3.0 Unported License.

The Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools (MODULE(S^2)) project is partially supported by funding from a collaborative grant of the National Science Foundation under Grant Nos. DUE-1726707,1726804, 1726252, 1726723, 1726744, and 1726098. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Todo list

Contents

Part I

Introduction to Complex Numbers

Basics of Complex Numbers

Defining Complex Numbers

First, a bit of review. The set of natural numbers, $\mathbb{N}$, consists of all of the counting numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The set of integers, $\mathbb{Z}$, consists of the natural numbers and their additive inverses:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The rational numbers are those numbers that can be written as a fraction:

$$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\, b \neq 0\}.$$

Some numbers cannot be written as a ratio of two integers, and we call those numbers irrational. For example, $\sqrt{2}$ is not a rational number; it is irrational. But, finally, we have the set of real numbers, $\mathbb{R}$, which consists of all the rational numbers together with all of the irrational numbers. So, we have the following containments:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

We often model the real numbers on a line, called the real number line.

INSERT IMAGE

Real numbers can added, subtracted, multiplied and divided. As suggested by the number line model, there is also an ordering of the real numbers with the following properties:

- For any real numbers a and b, if $a \leq b$ and $b \leq a$, then $a = b$. (Antisymmetry)
- For any real numbers a, b, and c, if $a \leq b$ and $b \leq c$, then $a \leq c$. (Transitivity)
- For any real numbers a and b, $a \leq b$ or $b \leq a$. (Totality)

The real numbers have a very important property called "completeness," which allows us to conclude that there are no "gaps" in the real numbers. Although the real numbers are complete in this sense, it turns out that there are some problems which can't be solved with real numbers.

Find two numbers whose sum is 10 and whose product is 16.

Find two numbers whose sum is 10 and whose product is 25.

Find two numbers whose sum is 10 and whose product is 40.

In the last task in the inquiry above, the equation

$$x^2 - 10x + 40 = 0$$

is found to have no real solutions. So are there two numbers which satisfy the requirements? Well, there are no **real** numbers that work, but we can find two **complex numbers**: $5 + i\sqrt{15}$ and $5 - i\sqrt{15}$. In the sections that follow we will introduce complex numbers and study their properties.

Ordered Pairs

You may know that complex numbers have the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. And, you may also know that we can model complex numbers in the plane. We will begin our study of complex numbers by viewing them as points in the plain. Thus, we say that a **complex** number z is an orderd pair

$$z = (a, b)$$

of real numbers a and b. This representation allows us to very naturally graph complex numbers:

INSERT IMAGE

Also, in this representation, it is natural to identify a complex number of the form $(a, 0)$ with the real number a. Thus we identify the complex number $(3, 0)$ with the real number 3 and we identify the complex number $\left(\frac{1}{3}, 0\right)$ with the real number $\frac{1}{3}$. In this way, the complex numbers contain the real numbers as a subset.

Thus, if we denote the set of complex numbers by the symbol $\mathbb{C}$, then we have,

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

By the way, complex numbers of the form $(0, b)$ are called **pure imaginary numbers**, or just **imaginary numbers**.

Consistent with this, if we are given a complex number $z = (a, b)$ we call a the **real part** of z and we call b the **imaginary part** of z. The notation for this is

$$\text{Re } z = a \text{ and } \text{Im } = b.$$

Two complex numbers are said to be equal when they have the same real parts and the same imaginary parts. That is (just like ordered pairs), we say

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d.$$

Now, notice that

$$z = (a, b) = a \cdot (1, 0) + b \cdot (0, 1).$$

We already know that we can identify the complex number $(1, 0)$ with the real number 1. We will also **let i denote the pure imaginary number $(0, 1)$.** Thus, we can write

$$\begin{aligned} z &= (a, b) \\ &= a \cdot (1, 0) + b \cdot (0, 1) \\ &= a \cdot 1 + b \cdot i \\ &= a + bi \end{aligned}$$

Many students are more familiar with the form $z = a + bi$ for the complex number (a, b). Both representations are useful. We will usually use the form $z = a + bi$, but we will also often use the form $z = (a, b)$. The notation $a + bi$ is very flexible, but there are some conventions. For example, $5 + 3i$, $5 + i3$, $3i + 5$ and $i3 + 5$ are all equal, but we usually write it as $5 + 3i$. Instead of writing something like $8 + 1i$, we would usually just write $8 + i$. And, rather than $3 + 0i$ or $0 + 0i$ we typically just write 3 or 0, respectively.

Addition and Subtraction of Complex Numbers

Complex numbers are identified with points in the plane for some very good reasons. One of those reasons is that addition works as expected: we add them **component wise**. That is, given complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$ we add them as follows:

$$\begin{aligned} z_1 + z_2 &= (a, b) + (c, d) \\ &= (a + c, b + d) \end{aligned}$$

If we had represented these complex numbers as $z_1 = a + bi$ and $z_2 = c + di$ then addition would work as follows

$$\begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \end{aligned}$$

In any number system, we say a number $\mathbf{n}$ is an **additive identity** if $x + \mathbf{n} = \mathbf{n} + x = x$ for any number x in that number system.

- What is the additive identity of the natural numbers? What about the integers, rationals, and reals?
- What is the additive identity in the set of complex numbers $\mathbb{C}$? Write it in two ways.

Suppose x is a number of any type. It can be an integer, rational, real or complex number. We define the additive inverse of x to be the number y so that $x + y = \mathbf{0}$.

- Given a complex number $z = a + bi$, what is its additive inverse?
- How would you write z and its additive inverse using ordered pairs?

How would you define subtraction of complex numbers? Write the definition using both the $a + bi$ representation and the orderd pair representation.

Multiplication of Complex Numbers

I

Complex Conjugate

Modulus

Part II

Representations of Complex Numbers

Part III

Roots and polynomials