

Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Adapted for MODULE(S^2)

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Part I

Introduction to Complex Numbers

Basics of Complex Numbers

Defining Complex Numbers

First, a bit of review. The set of natural numbers, \mathbb{N} , consists of all of the counting numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

The set of integers, \mathbb{Z} , consists of the natural numbers and their additive inverses:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The rational numbers are those numbers that can be written as a fraction:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Some numbers cannot be written as a ratio of two integers, and we call those numbers irrational. For example, $\sqrt{2}$ is not a rational number; it is irrational. But, finally, we have the set of real numbers, \mathbb{R} , which consists of all the rational numbers together with all of the irrational numbers. So, we have the following containments:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

We often model the real numbers on a line, called the real number line.

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Real numbers can added, subtracted, multiplied and divided. As suggested by the number line model, there is also an ordering of the real numbers with the following properties:

- For any real numbers a and b , if $a \leq b$ and $b \leq a$, then $a = b$. (Antisymmetry)
- For any real numbers a , b , and c , if $a \leq b$ and $b \leq c$, then $a \leq c$. (Transitivity)
- For any real numbers a and b , $a \leq b$ or $b \leq a$. (Totality)

These properties fit our intuition for how objects on a number line should behave, and for that reason we call an ordering with these properties a **linear order**. So, we say the real numbers are “linearly ordered.” The real numbers also have a very important property called “completeness,” which allows us to conclude that there are no “gaps” in the real numbers. Although the real numbers are complete in this sense, it turns out that there are some problems which can’t be solved with real numbers.

Inquiry: Sums and Products

Find two numbers whose sum is 10 and whose product is 16.

Find two numbers whose sum is 10 and whose product is 25.

Find two numbers whose sum is 10 and whose product is 40.

Instructor note. Distribute handout with this question. As teachers work on it, circulate and listen to the questions and comments they make. They may say and do things that will lead into a discussion on clarifying the question, precision, and also what it means to have less or more satisfying answers to a question.

In the last task in the inquiry above, the equation

$$x^2 - 10x + 40 = 0$$

is found to have no real solutions. So are there two numbers which satisfy the requirements? Well, there are no **real** numbers that work, but we can find two **complex numbers**: $5 + i\sqrt{15}$ and $5 - i\sqrt{15}$. In the sections that follow we will introduce complex numbers and study their properties.

ORDERED PAIRS

You may know that complex numbers have the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. And, you may also know that we can model complex numbers in the plane. We will begin our study of complex numbers by viewing them as points or vectors in the plane. Thus, we can say that a **complex** number z is an ordered pair

$$z = (a, b)$$

of real numbers a and b . This representation allows us to very naturally graph complex numbers in the plane:

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When we graph complex numbers in the plane, we call that plane the “complex plane.”

Also, in this representation, it is natural to identify a complex number of the form $(a, 0)$ with the real number a . Thus we identify the complex number $(3, 0)$ with the real number 3 and we identify the complex number $(\frac{1}{3}, 0)$ with the real number $\frac{1}{3}$.

Thus, if we denote the set of complex numbers by the symbol \mathbb{C} , then we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

In this way, the complex numbers contain the real numbers as a subset. In the complex plane we say that the set of real numbers are found along the “real axis.”

Complex numbers of the form $(0, b)$ are called **pure imaginary numbers**, or just **imaginary numbers**. In the complex plane, the set of imaginary numbers are found along the “imaginary axis.”

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If we are given a complex number $z = (a, b)$ we call a the **real part** of z and we call b the **imaginary part** of z . The notation for this is

$$\operatorname{Re} z = a \text{ and } \operatorname{Im} z = b.$$

Two complex numbers are said to be equal when they have the same real parts and the same imaginary parts. That is (just like ordered pairs), we say

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d.$$

Now, notice that

$$z = (a, b) = a \cdot (1, 0) + b \cdot (0, 1).$$

We already know that we can identify the complex number $(1, 0)$ with the real number 1. We will also let i denote the pure imaginary number $(0, 1)$. Thus, we can write

$$\begin{aligned} z &= (a, b) \\ &= a \cdot (1, 0) + b \cdot (0, 1) \\ &= a \cdot 1 + b \cdot i \\ &= a + bi \end{aligned}$$

Many students are more familiar with the form $z = a + bi$ for the complex number (a, b) . Both representations are useful. We will usually use the form $z = a + bi$, but we will also often use the form $z = (a, b)$. The notation $a + bi$ is very flexible, but there are some conventions. For example, $5 + 3i$, $5 + i3$, $3i + 5$ and $i3 + 5$ are all equal, but we usually write it as $5 + 3i$. Instead of writing something like $8 + 1i$, we would usually just write $8 + i$. And, rather than $3 + 0i$ or $0 + 0i$ we typically just write 3 or 0, respectively.

- Let $z = a + bi$ where $a, b \in \mathbb{R}$. There are two triangles that contain z as a vertex and whose legs are parallel or perpendicular to the real and imaginary axes. What are the coordinates of the other vertices of these triangles in terms of complex numbers?
- In the above question, what are the lengths of the legs and hypotenuses of the right triangles?

Given a complex number $z = (a, b)$ the **complex conjugate** of z is the complex number $\bar{z} = (a, -b)$.

- What is the complex conjugate of $z = 3 - 4i$? Graph z and \bar{z} on the same set of axes.
- What is the complex conjugate of $z = (0, 1)$? Graph z and \bar{z} on the same set of axes.
- What transformation of the plane maps z to \bar{z} ?

Addition and Subtraction of Complex Numbers

Complex numbers are identified with points in the plane for some very good reasons. One of those reasons is that addition works as expected: we add them **component wise**. That is, given complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$ we add them as follows:

$$\begin{aligned} z_1 + z_2 &= (a, b) + (c, d) \\ &= (a + c, b + d) \end{aligned}$$

If we had represented these complex numbers as $z_1 = a + bi$ and $z_2 = c + di$ then addition would work as follows

$$\begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \end{aligned}$$

In any number system, we say a number \mathbf{n} is an **additive identity** if $x + \mathbf{n} = \mathbf{n} + x = x$ for any number x in that number system.

- What is the additive identity of the natural numbers? What about the integers, rationals, and reals?
- What is the additive identity in the set of complex numbers \mathbb{C} ? Write it in two ways.

Suppose x is a number of any type. It can be an integer, rational, real or complex number. We define the additive inverse of x to be the number y so that $x + y = \mathbf{0}$.

- Given a complex number $z = a + bi$, what is its additive inverse?
- How would you write z and its additive inverse using ordered pairs?
- How would you define subtraction of complex numbers? Write the definition using both the $a + bi$ representation and the ordered pair representation.
- Given a complex number z , what is $z + \bar{z}$? What is $z - \bar{z}$? Write your answers in terms of $\text{Re } z$ and $\text{Im } z$.

Finally we define the **modulus** of a complex number $z = a + bi$ to be $|z| = \sqrt{a^2 + b^2}$. The modulus goes by many names. You will also see the modulus of z referred to as the **magnitude of z** , the **norm of z** , the **length of z** , and sometimes the **absolute value of z** .

- Use the Pythagorean theorem to explain why it makes sense that the modulus of z is also called the magnitude (or norm, or length) of z .
- What is the relationship between the concept of the absolute value of a real number and the concept of the modulus of a complex number?

Multiplication of Complex Numbers

I

Part II

Representations of Complex Numbers

Part III

Roots and polynomials