Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Adapted for MODULE(S^2)

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Part I

Introduction to Fields

Overview

Content

Field, a set of objects together with two operations - addition ($+$) and multiplication ($\cdot$) which satisfy the following properties

(A) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutative laws)

(B) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative laws)

(C) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law)

(D) There are distinct elements, called 0 and 1, such that $a + 0 = a$ and $a \cdot 1 = a$ for all a.

(E) For each a there is an element b such that $a + b = 0$ and if $a \neq 0$, there is an element c such that $a \cdot c = 1$.

A field is properly specified by listing its set of objects, together with its two operations. We will often use the symbol $\mathbb{F}$ to denote the set of objects of an arbitrary field, and complete mathematical structure is most properly denoted $\langle \mathbb{F}, +, \cdot \rangle$. However, when no confusion will result we will often just use the symbol $\mathbb{F}$ to refer to the field.

A **Subfield** is essentially defined as a "field within a field;" a subfield $\mathbb{G}$ of a field $\mathbb{F}$ is a subset of $\mathbb{F}$ which is also a field when the operations of $\mathbb{F}$ are restricted to $\mathbb{G}$. For example, the rationals $\mathbb{Q}$ are a subfield of the reals $\mathbb{R}$.

A **Field extension**, is an extension of a field to a larger field. For example, the field of real numbers $\mathbb{R}$ is an extension of the field of rational numbers $\mathbb{Q}$.

The **additive identity** is a special element of a field $\mathbb{F}$, often denoted 0, with the property that for any $a \in \mathbb{F}$, $a + 0 = 0 + a = a$.

Every element of a field has an **additive inverse**; that is, for every element a in a field $\mathbb{F}$ there is some element $b \in \mathbb{F}$ such that $a + b = b + a = 0$. Since every element a in a field $\mathbb{F}$ has a unique additive identity, we usually denote it by "$-a$."

The **multiplicative identity** is another special element of a field $\mathbb{F}$. In this case, we usually denote the multiplicative identity by 1, and it has the property that for every $a \in \mathbb{F}$, $a \cdot 1 = 1 \cdot a = a$.

Every nonzero element in a field $\mathbb{F}$ has a unique **multiplicative inverse**. That is, for every $a \in \mathbb{F}$, $a \neq 0$, there exists $b \in \mathbb{F}$ with the property that $a \cdot b = b \cdot a = 1$. The multiplicative identity of a nonzero element a is unique to a, so we will usually denote it by a^{-1}.

An **ordered field** is a field $\mathbb{F}$ with an ordering relation $<$ which satisfies the following properties for any $a, b, c \in \mathbb{F}$:

- (Trichotomy) Exactly one of the following holds: $a < b$, $a = b$ or $b < a$.
- (Transitivity) $a < b$ and $b < c$ implies that $a < c$.
- (Addition) If $a < b$, then $a + b < a + c$.
- (Multiplication) If $a < b$, $0 < c$, then $ac < bc$.

Summary

In this section we introduce the notion of a field. The emphasis is on justifying well known facts from high school algebra using the formal properties of fields. We then introduce ordered fields and relate these to issues often studied in high school mathematics.

Acknowledgements. The material in this part is inspired by the notes from courses we've taught. The textbooks used in those courses include

Hadlock, C. R. (1978). Field Theory and Its Classical Problems. Washington: Mathematical Association of America.

Madden, D. J. & Aubrey, J. A. (2017). An introduction to proof through real analysis (1st ed.). Hoboken, NJ: Wiley.

Peressini, A. L. (2003). Mathematics for high school teachers: An advanced perspective. Prentice Hall.

Materials. In this part there are three inquiries to print and give to students. They are

- Opening Inquiry: Solving Equations,
- Inquiry: Finite Fields,
- Inquiry: $\mathbb{Q}(\sqrt{2})$.

Opening inquiry: Solving Equations

Consider the equation:

$$3x + 8 = 14$$

It's not hard to see that the solution to this equation is $x = 2$: $3(2) + 8 = 14$. Let's think very carefully through some steps we could take to solve this equation, justifying each step along the way.

First we could subtract 8 from both sides (Step 1):

$$(3x + 8) - 8 = 14 - 8.$$

Second, we could re-write the left side of the equation to get (Step 2):

$$3x + (8 - 8) = 6.$$

We know that $8 - 8 = 0$ so we have (Step 3):

$$3x + 0 = 6.$$

We know that adding 0 to any number returns the value of the number that we added it to. So we have (Step 4):

$$3x = 6.$$

We now could multiply each side by $1/3$ to obtain (Step 5):

$$\frac{1}{3}(3x) = \frac{1}{3} \cdot 6.$$

We can re-write the left side of the equation (and simplify the right side) to obtain (Step 6):

$$\left(\frac{1}{3} \cdot 3\right) x = 2.$$

We know that $\frac{1}{3} \cdot 3$ is equal to 1, so we have (Step 7):

$$1x = 2.$$

We also know that the number 1 multiplied times any other number returns the value of that number, so we conclude that (Step 8):

$$x = 2.$$

Now, analyze this situation even more carefully by answering the following:

1. How many operations did the above solution use? List them.
2. How many "laws" or "special properties" of numbers did the above solution use? List them in each step.
3. Can you solve this equation by using other operations than you have listed above. How many operations did you use? What were they?
4. When you solved this equation using other operations, did you gain any more "special properties?" If so, what were they?
5. Try to solve this equation without using a rational number (NOTE: the $1/3$ above is a rational number).

Instructor note. The goal of this opening inquiry is to get students to articulate the properties of addition and multiplication that they use when solving an equation such as this. It is not necessary to start with the solution outlined above. A good way to use this inquiry in the classroom would by to have the class solve the equation together "live" with the instructor taking suggestions from students about what steps to follow and their justifications. A nice feature of this particular example, which is mentioned later in the text, is that the equation $3x + 8 = 14$ can be solved over the integers (without using multiplicative inverses), and can be solved over the natural numbers (not using additive inverses). This feature is discussed later, and can be leveraged into a classroom discussion of how students in different grades learn to solve equations.

Introduction to Fields

In this section we will begin our study of **fields**. You've already encountered fields in your mathematical studies: the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$ are fields, as is the set of complex numbers $\mathbb{C}$. In the opening inquiry to this section, you saw that $\mathbb{Z}_n$ is a field for some values of n. The sets $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_n$ are different in many ways, but here we will focus on the ways in which they are similar.

The study of fields is motivated by the desire to provide justification for the steps we use in solving equations with the two arithmetic operations of addition and multiplication. What rules do we need to follow in order to solve such equations? When can we guarantee that such an equation will always have a solution?

The goal of this section is to explore the foundations of the number systems we typically use for solving equations. This exploration will allow us to provide well grounded, thorough, and pedagogically appropriate justifications for the steps we use in algebra every day to solve equations. But it will also allow us to explore exciting extensions of our ordinary mathematical practices and allow us to connect equation solving to geometry in an intriguing way.

To begin, consider the equation

$$3x + 8 = 14.$$

It's not hard to see that the solution to this equation is $x = 2$: $3(2) + 8 = 14$. Let's us solve this equation step-by-step, justifying each step along the way. First we will subtract 8 from both sides:

$$(3x + 8) - 8 = 14 - 8.$$

(Note that we could also view this as adding -8 to both sides. The number -8 is known as the **additive inverse** of 8.) Applying the associative law on the left-hand side gives

$$3x + (8 - 8) = 6.$$

We know that $8 - 8 = 0$ so we have

$$3x + 0 = 6.$$

The number 0 is an **additive identity**. That means adding 0 returns the value we added it to. So we have

$$3x = 6.$$

We now multiply each side by $1/3$ to obtain

$$\frac{1}{3}(3x) = \frac{1}{3} \cdot 6.$$

Multiplication is associative, so we can write this as

$$\left(\frac{1}{3} \cdot 3\right) x = 2.$$

The number $1/3$ is the **multiplicative inverse** of 3, meaning that $\frac{1}{3} \cdot 3$ is equal to the **multiplicative identity**; that is, $\frac{1}{3} \cdot 3 = 1$. Thus we have

$$1x = 2.$$

The number 1 is the **multiplicative identity** meaning that $1x = x$. So we conclude that

$$x = 2.$$

Let us analyze this situation more carefully. First note that the equation $3x + 8 = 14$ uses two operations, called addition and multiplication. (Subtraction can always be defined in terms of addition, and division can be defined in terms of multiplication.) We used some familiar properties of addition and multiplication such as associativity of addition and multiplication.

Above we multiplied by $1/3$ at point in the solution. Since $1/3$ is a rational number, we say that we solved this equation "over the rationals." But, notice that in this example we didn't really need to do this. Next we give a solution to the equation $3x + 8 = 14$ "over the integers." We begin the same way:

$$\begin{aligned} 3x + 8 &= 14 \\ (3x + 8) - 8 &= 14 - 8 \\ 3x + (8 - 8) &= 6 \\ 3x + 0 &= 6 \\ 3x &= 6. \end{aligned}$$

Next we observe that $6 = 3(2)$ so we have

$$3x = 3(2).$$

We have the following property of integer multiplication: if a, b, and c are integers and $ab = ac$ then $b = c$. Using just that fact, we can conclude that

$$x = 2.$$

Instructor note. When students first learn multiplication, they are often taught to solve simple equations such as $ax = b$, where a and b are natural numbers. For example, a fourth grade student might be asked to find the number that should go in the box:

$$3 \cdot \square = 6.$$

Of course, 2 is the answer, but it may be interesting to discuss with students how we know that 2 is the only possible answer. One way to articulate the property we're using is this: if a, b, and c are integers and $ab = ac$ then $b = c$. A similar fact is that if $a + b = a + c$ then $b = c$. This can be easily proved in the integers using additive inverses, but it's also true in the natural numbers. At this level, restricting students from using multiplicative inverses or additive inverses may seem artificial, but doing so recalls the experience of learning the operations. Young students learn addition before subtraction but are still asked to reason about equations such as $4 + \square = 6$. Students learn about multiplication before division, and are still asked about equations such as $3 \cdot \square = 6$.

- Can you solve the equation $3x + 8 = 14$ over the natural numbers? (Here you're not allowed to use additive inverses!)

The equation $3x + 8 = 14$ can be solved over the rationals, integers, or natural numbers, but notice that the equation $3x + 8 = 10$ cannot be solved over the integers or natural numbers. The solution $x = 2/3$ is a rational number and is not a natural number or integer. Notice that so long as a, b and c are always rational numbers, $ax + b = c$ will always have a rational solution. The same goes for equations with real or complex coefficients. On the other hand, if a, b, and c are integers, that does not guarantee that $ax + b = c$ will have an integer solution. We want to determine all of the properties necessary on a set of numbers for an equation such as $ax + b = c$ to always have a solution in that set. That is, we want to figure out what makes a set of numbers like $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ in this regard. We will call such a set of numbers a field.

We begin with some important terminology. It is important to recognize that learning proper terminology in mathematics is crucial and goes beyond simple vocabulary building. The Common Core Standards for Mathematical Practice includes mathematical practice MP6 Attend to Precision, which includes the following:

> Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.

and later

> In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

In a small group, discuss why it is important for high school students to learn and use definitions. Why is it important for teachers?

We call 0 an additive identity because for any number n, $n + 0 = 0 + n = n$. The number 0 is also an additive identity in the set of complex numbers, although more formally it is $0 + 0i$. A corresponding notion for multiplication exists - the multiplicative identity.

- Consider the collection of all 2×2 matrices whose entries are real numbers. Write down the additive identity of this set.
- How would you define the general notion of a **multiplicative identity**? What is a multiplicative identity in $\mathbb{Q}$?
- Is there a multiplicative identity for the set of all 2×2 matrices with real entries?
- Recall that a rational function is a function of the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. The set of rational functions is sometimes called $\mathbb{Q}(x)$. What do you think is the multiplicative identity in $\mathbb{Q}(x)$?

Once we have a notion of an additive identity, we can define the notion of an additive inverse. We say that a number b is an additive inverse of a number a if and only if $a + b = b + a = 0$.

How would you define the notion of a **multiplicative inverse**? Give an example of a number a and its multiplicative inverse b.

We now define the notion of a **field**. We already know about some fields: the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$. Some of the properties these sets have in common will be taken as axioms. Axioms are just rules that we accept as defining the crucial properties of a class of mathematical objects. In this case, our axioms will *define* what it means for a mathematical structure to be a field.

A **field** $\mathbb{F}$ is a collection of mathematical objects (possibly numbers, matrices, functions, etc.) with two operations, called addition $(+)$ and multiplication $(\cdot)$, in which we can always solve an equation of the form

$$ax + b = c$$

where $a, b, c \in \mathbb{F}$ and $a \neq 0$. The properties we need to make this happen are given in the following definition.

Definition 0.1. A field $\mathbb{F}$ is a nonempty set together with two operations addition $+$ and multiplication $\cdot$ which satisfy the following properties, called the field axioms:

(A) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutative laws)

(B) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative laws)

(C) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law)

(D) There are distinct elements, called 0 and 1, such that $a + 0 = a$ and $a \cdot 1 = a$ for all a.

(E) For each a there is an element b such that $a + b = 0$ and if $a \neq 0$, there is an element c such that $a \cdot c = 1$.

Of course, you have seen fields before: the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are both fields under their usual operations of addition and multiplication. But there are fields which are quite different from these familiar ones, as seen in the next inquiry.

Inquiry: Finite Fields

We are interested in mathematical structures with operations of addition ($+$) and multiplication ($\cdot$) that satisfy the following properties:

(A) $a+b=b+a$ and $a \cdot b = b \cdot a$ (commutative laws)

(B) $(a+b)+c=a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative laws)

(C) $a \cdot (b+c) = a \cdot b + a \cdot c$ (distributive law)

(D) There are distinct elements, called 0 and 1, such that $a+0=a$ and $a \cdot 1 = a$ for all a.

(E) For each a there is an element b such that $a+b=0$ and if $a \neq 0$, there is an element c such that $a \cdot c = 1$.

Structures which satisfy these properties are called **<u>fields</u>**. We now define a field called $\mathbb{Z}_5$. We will avoid the complexities of a formal definition of $\mathbb{Z}_5$ and simply assert that $\mathbb{Z}_5 = \{0,1,2,3,4\}$. The operations of addition ($+$) and multiplication ($\cdot$) are defined "modulo 5." That means we find the regular sum or product and then divide by 5 and take the remainder as our answer. For example, in $\mathbb{Z}_5$:

$$3+4=7$$
$$7 \div 5 = 1 \text{ R } 2.$$

Since the sum of 3 and 4 is 7 and the remainder when we divide 7 by 5 is 2, we conclude that in $\mathbb{Z}_5$, $3+4=2$. Using that procedure, fill in the following addition table:

$+$	0	1	2	3	4
0					
1					
2					
3					2
4					

1. Find a number $a \in \mathbb{Z}_5$ with the property that $a+2=2+a=0$. The number a that you find is called the **<u>additive inverse</u>** of 2 in $\mathbb{Z}_5$. That is, it is the value of -2 in $\mathbb{Z}_5$.

   (a) $-2 =$ ______ in $\mathbb{Z}_5$ because $2 +$ ______ $=$ ______ $+ 2 =$ ______.

2. Find the value of -1, -3, and -4 in $\mathbb{Z}_5$.

   (a) $-1 =$ ______ in $\mathbb{Z}_5$ because $1 +$ ______ $=$ ______ $+ 1 =$ ______.

   (b) $-3 =$ ______ in $\mathbb{Z}_5$ because $3 +$ ______ $=$ ______ $+ 3 =$ ______.

   (c) $-4 =$ ______ in $\mathbb{Z}_5$ because $4 +$ ______ $=$ ______ $+ 4 =$ ______.

We follow the analogous process for multiplication. For example, $4 \cdot 4 = 16$. When we divide 16 by 5 we get 3 with a remainder of 1. So we conclude that in $\mathbb{Z}_5$, $4 \cdot 4 = 1$. Using that procedure, fill in the following multiplication table:

$\cdot$	0	1	2	3	4
0					
1					
2					
3					
4					1

Since $4 \cdot 4 = 1$ in $\mathbb{Z}_5$, we conclude that 4 is its own **multiplicative inverse** in $\mathbb{Z}_5$. That is, $4^{-1} = 4$ in $\mathbb{Z}_5$!

3. Find the value of 2^{-1} in $\mathbb{Z}_5$ and the value of 3^{-1} in $\mathbb{Z}_5$.
   (a) $2^{-1} =$ ______ in $\mathbb{Z}_5$ because $2 \cdot$ ______ $=$ ______ $\cdot 2 =$ ______
   (b) $3^{-1} =$ ______ in $\mathbb{Z}_5$ because $3 \cdot$ ______ $=$ ______ $\cdot 3 =$ ______

One can see by inspection of the addition and multiplication tables for $\mathbb{Z}_7$ that every element of $\mathbb{Z}_7$ has an additive inverse, and every nonzero element of $\mathbb{Z}_7$ has a multiplicative inverse. That is, $\mathbb{Z}_7$ satisfies (E) above. It is also easy to see by inspecting those tables that for all $a \in \mathbb{Z}_7$ $0 + a = a + 0 = a$ and $a \cdot 1 = 1 \cdot a = a$. That is, property (D) above is true in $\mathbb{Z}_7$.

4. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 \cdot (1 + 4) = 3 \cdot 1 + 3 \cdot 4$
   - $5 \cdot (2 + 6) = 5 \cdot 2 + 5 \cdot 7$
5. Make up two number sentences with the same form using numbers $0 - 6$ and verify that they are true in $\mathbb{Z}_7$.
6. Do you believe that the distributive law (C) holds in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of the distributive law are there in $\mathbb{Z}_7$?
7. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 + (1 + 4) = (3 + 1) + 4$ and $5 + (2 + 6) = (5 + 2) + 6$.
   - $3 \cdot (1 \cdot 4) = (3 \cdot 1) \cdot 4$ and $5 \cdot (2 \cdot 6) = (5 \cdot 2) \cdot 6$.
8. Do you believe that the associative laws (B) hold in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of each of the two distributive laws are there in $\mathbb{Z}_7$?
9. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 + 4 = 4 + 3$ and $5 + 6 = 6 + 5$.
   - $3 \cdot 4 = 4 \cdot 5$ and $5 \cdot 6 = 6 \cdot 5$.
10. Do you believe that the commutative laws (A) hold in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of each of the two commutative laws are there in $\mathbb{Z}_7$?
11. Based on your work above, does $\mathbb{Z}_7$ satisfy properties (A)-(E)? Is it a field?
12. Create addition and multiplication tables for $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and answer appropriate versions of questions 1-11 for these two structures. Once you have determined whether or not $\mathbb{Z}_4$ and $\mathbb{Z}_4$ are fields, form a conjecture for which values of n is $\mathbb{Z}_n$ a field?
13. Test your conjecture with another value of n and be prepared to discuss your conjecture with the class.

Instructor note. Distribute handout with this question. The activity should be done in groups of 3-4. As students work on it, circulate and listen to the questions and comments they make. They may say and do things that will lead into a discussion about additive inverses, multiplicative inverses and other properties of fields. Groups may conjecture that $\mathbb{Z}_n$ is a field whenever n is odd. Be prepared to discuss this conjecture.

More on Identities and Inverses

We all know that in the rational numbers there is only one additive identity: the number 0. But is zero the only additive identity? Consider rational numbers of the form

$$\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots$$

An additive identity in the rationals is a rational number a so that

$$a + b = b + a = b$$

for any other rational number b. Suppose we add $\frac{0}{5}$ and $\frac{1}{2}$:

$$\frac{0}{5} + \frac{1}{2} = \frac{2(0) + 5(1)}{5(2)} = \frac{5}{10} = \frac{1}{2}.$$

In fact, it's not hard to see that if we add $\frac{0}{5}$ to any rational number b then the result is b. That is, $\frac{0}{5}$ fits the definition of an additive identity. In fact, the same goes for all of the numbers

$$\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots$$

They are all additive identities - they all behave exactly like 0. Well, are they zero?

We have the following proposition which says that in any field (such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$, etc.) there is in fact only one additive identity.

Proposition 0.2. *In any field $\mathbb{F}$, then additive identity is unique.*

Proof. Suppose that we have additive identities 0 and z in $\mathbb{F}$. Since 0 is an additive identity, we know that

$$0 + z = z.$$

But since z is also an additive identity, we also know that

$$0 + z = 0.$$

So, we have that

$$z = 0 + z = 0.$$

This proves that the additive identity in any field is unique. □

- Discuss in a small group what is going on here with the field of rational numbers $\mathbb{Q}$. How is it that

$$\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots$$

  all fulfill the definition of an additive identity when the theorem above says there is only one additive identity in any field?
- Consider the field of rational functions over the rationals, $\mathbb{Q}(x)$. (We have not yet proved this is a field, but we will later.) What is the additive identity in $\mathbb{Q}(x)$? How is this situation similar to $\mathbb{Q}$?

There are a couple of observations to make about the proof above. First, a good general strategy for proving that something is unique is to assume that there are two of them and then prove that they are equal. If needed, you can also assume that your two proposed objects are not equal and derive a contradiction, but notice that we did not need to do that in the proof above. Second, observe that besides using the definition of an additive identity, the only other property we used to prove the proposition above is that addition is commutative.

Since the additive identity in any field is unique, we will always use the usual symbol 0 to represent it, unless we have a good reason not to.

Use the proof above as a model to show that in any field the multiplicative identity is unique.

Similarly, since the multiplicative identity in any field is unique, we will almost always use the usual symbol 1 to represent it.

There is a similar fact to observe with respect to additive and multiplicative inverses. For example, there is only one rational number whose sum with $-\frac{1}{2}$ is 0, namely $\frac{1}{2}$. Similarly, there is only one rational number whose product with $-\frac{1}{2}$ is 1, namely -2. Above you may have noticed that we said "an additive inverse" instead of "the additive inverse," and "a multiplicative inverse" instead of "the multiplicative inverse." We didn't want to suggest that they are unique, and were hoping that a reader might notice our strange locution and question it. But now we are at a point where we are happy to admit that additive and multiplicative inverses are, in a sense, unique:

Proposition 0.3. *If $\mathbb{F}$ is a field and $a \in \mathbb{F}$, then its additive inverse is unique to it.*

Proof. Suppose that $\mathbb{F}$ is a field and that $a \in \mathbb{F}$. We want to prove that there is only one element $b \in \mathbb{F}$ so that

$$a + b = b + a = 0.$$

To this end, suppose that there are two such elements $b, c \in \mathbb{F}$. Then we have both:

$$a + b = b + a = 0$$
$$a + c = c + a = 0$$

Consider the sum $b + a + c$. On one hand we have

$$b + a + c = (b + a) + c = 0 + c = c.$$

On the other hand we have

$$b + a + c = b + (a + c) = b + 0 = b.$$

Thus we conclude that $c = b$ and that every element in a field has a unique additive inverse. □

Solve the equation

$$3x + 6 = 8.$$

How do you know that there is only one solution to this equation?

If b is the additive inverse of a we write $b = -a$. Note that $-a$ may be positive or negative. For example, the additive inverse of 4 is -4, but the additive inverse of -5 is 5. This brings up an important point. When people see "$-a$" it is common to read it as "minus a," or "negative a." The least common thing for people to say is "the additive inverse of a." But, that's what we want you to do because it really helps to keep things straight as, for example, in the following proposition.

Proposition 0.4. *Suppose $a, b \in \mathbb{F}$. Then*

(a) $-(-a) = a$

(b) $-a = (-1)a$

(c) $-(a + b) = (-a) + (-b)$

(d) $-(a \cdot b) = (-a) \cdot b = a \cdot (-b)$.

Proof. The proof of item (a) is really an exercise in understanding the definition of the additive inverse. The expression $-a$ means "the additive inverse of a." So the expression "$-(-a)$ means the additive inverse of $-a$. What is the additive inverse of $-a$? It's a of course. That's because

$$a + (-a) = (-a) + a = 0.$$

To prove (b), we want to so that $(-1)a$ is the additive inverse of a. How would we do that? Well, we must show that

$$a + (-1)a = 0.$$

Here it goes:

$$\begin{aligned} a + (-1)a &= 1a + (-1)a \\ &= (1 + (-1))a \\ &= 0 \cdot a \\ &= 0 \end{aligned}$$

Thus $a + (-1)a = 0$. Since addition is commutative we know that $a + (-1)a = (-1)a + a = 0$. So $(-1)a$ fits the definition of an additive inverse of a. Since additive inverses are unique we conclude that $(-1)a = -a$. The proofs of parts (c) and (d) are left as exercises. □

It's a good idea to translate the statements in Proposition 0.4 into statements in ordinary language. In the task below, fill in the blanks.

Mathematical Statement	English Statement
$-(-a) = a$	The additive inverse of the additive inverse of a is a itself.
$-a = (-1)a$	______________________________
______________	The additive inverse of a sum is the sum of the additive inverses.
$-(a \cdot b) = (-a) \cdot b$	______________________________

Notice that in the proof above we had a nice, if slightly tricky, application of the distributive property. That trick is really helpful. Here's another application of it.

Proposition 0.5. *If $a \in \mathbb{F}$, then $a \cdot 0 = 0 \cdot a = 0$.*

Proof. We have

$$\begin{aligned} a &= a \cdot 1 \\ &= a \cdot (1 + 0) \\ &= a \cdot 1 + a \cdot 0 \\ &= a + a \cdot 0. \end{aligned}$$

Thus, $a = a + a \cdot 0$. Now add $-a$ to both sides:

$$\begin{aligned} (-a) + a &= (-a) + (a + a \cdot 0) \\ 0 &= (-a + a) + a \cdot 0 \\ 0 &= 0 + a \cdot 0 \\ 0 &= a \cdot 0 \end{aligned}$$

□

In the proof above we only used field axioms, but did not identify which ones we used as we went along. For each step in the proof above, identify the field axioms that justify the step.

Now let's discuss multiplicative inverses. The fundamental facts about multiplicative inverses largely parallel the fundamental facts about multiplicative inverses. In a field, there is a unique multiplicative identity. We'll always call it 1 unless we have a good reason not to. And, in a field every element except the additive identity has a multiplicative inverse which is unique to it. We will denote the multiplicative inverse of a as a^{-1}. Because our experience with fields is mostly limited to $\mathbb{Q}$ and $\mathbb{R}$, it is common to reflexively think that

$$a^{-1} = \frac{1}{a}.$$

For example, $2^{-1} = 1/2$. And it is true that in $\mathbb{Q}$ and $\mathbb{R}$ (and even in $\mathbb{C}$), $a^{-1} = 1/a$ for nonzero a. However, as we saw in the opening inquiry, it is not true in every field that $a = 1/a$.

Example 0.6. In the finite fields inquiry, we defined a field called $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Recall that the operations of addition ($+$) and multiplication ($\cdot$) are defined "modulo 5." That means we find the regular sum or product and then divide by 5 and take the remainder as our answer. For example, in $\mathbb{Z}_5$:

$$\begin{aligned} &3 + 4 = 7 \\ &7 \div 5 = 1 \text{ R } 2. \end{aligned}$$

Since the sum of 3 and 4 is 7 and the remainder when we divide 7 by 5 is 2, we conclude that in $\mathbb{Z}_5$, $3 + 4 = 2$. Using that procedure, we developed the following addition table:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Using, the addition table above identify additive inverses in $\mathbb{Z}_5$:

- $-1 = \underline{4}$
- $-2 =$ ____
- $-3 =$ ____
- $-4 =$ ____

We followed the analogous process for multiplication. For example, $4 \cdot 4 = 16$. When we divide 16 by 5 we get 3 with a remainder of 1. So we conclude that in $\mathbb{Z}_5$, $4 \cdot 4 = 1$. Using that procedure, we developed the following multiplication table:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Since $4 \cdot 4 = 1$ in $\mathbb{Z}_5$, we conclude that 4 is its own multiplicative inverse in $\mathbb{Z}_5$. That is, $4^{-1} = 4$ in $\mathbb{Z}_5$! Similarly, in $\mathbb{Z}_5$, $1^{-1} = 1$, $2^{-1} = 3$ and $3^{-1} = 2$.

Using, the multiplication table above identify multiplicative inverses in $\mathbb{Z}_5$:

- $1^{-1} =$ ____
- $2^{-1} =$ ____
- $3^{-1} =$ ____
- $4^{-1} = \underline{4}$

The main point of the previous example is that $-a$ does not always mean "the negative of a," and a^{-1} does not always mean $1/a$. In every context it's safe to read $-a$ as "the additive inverse of a" and to read a^{-1} as "the multiplicative inverse of a."

We have the following facts about multiplicative inverses. The proof of this proposition is left as an exercise.

Proposition 0.7. *Suppose that $\mathbb{F}$ is a field and that $a, b \in \mathbb{F}$ and $a, b \neq 0$.*

(a) $(a^{-1})^{-1} = a$

(b) $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$

(c) $(-a)^{-1} = -a^{-1}$

Again, it's helpful here to state the parts of this proposition in ordinary language. For example, part (a) asserts that "the multiplicative inverse of the multiplicative inverse of a is a itself." Do the same for parts (b) and (c).

Field Extensions

Instructor note. We are introducing field extensions here because students will use quadratic extensions extensively in Part 3 of this module.

Sometimes we will be working with more than one field at a time. If we have two fields, say $\mathbb{F}$ and $\mathbb{G}$ and

- $\mathbb{F} \subseteq \mathbb{G}$, and
- The operations on $\mathbb{F}$ are the operations on $\mathbb{G}$ restricted to $\mathbb{F}$

then we say that $\mathbb{G}$ is an **extension** of $\mathbb{F}$. If $\mathbb{G}$ is an **extension** of $\mathbb{F}$, then we say that $\mathbb{F}$ is a **subfield** of $\mathbb{G}$.

As an example, $\mathbb{Q}$ is a subfield of $\mathbb{R}$. Similarly, $\mathbb{C}$ is an extension of $\mathbb{R}$.

Inquiry: $\mathbb{Q}(\sqrt{2})$

In this section we develop the idea of a **quadratic extension** of the field of rational numbers. This construction will be very important to us later in this module. We begin with a task in which students convince themselves that $\mathbb{Q}(\sqrt{2})$ is a field.

Consider the set $\mathbb{Q}(\sqrt{2})$ defined as

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

Given $a + b\sqrt{2}$ and $c + d\sqrt{2}$ in $\mathbb{Q}(\sqrt{2})$ we define

- $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$, and
- $(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (ad + bd)\sqrt{2}$.

The definition of addition here is quite natural, but the definition of multiplication might seem confusing until you realize that it is just the result of the distributive law:

$$\begin{aligned}(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) &= a(c + d\sqrt{2}) + b\sqrt{2}(c + d\sqrt{2}) \\ &= ac + ad\sqrt{2} + bc\sqrt{2} + bd\sqrt{2}\sqrt{2} \\ &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\ &= (ac + 2bd) + (ad + bd)\sqrt{2}\end{aligned}$$

With these definitions, explore the following questions in small groups.

1. Does $\mathbb{Q}(\sqrt{2})$ satisfy the commutative laws? Convince yourself that the commutative law for multiplication is true in $\mathbb{Q}(\sqrt{2})$ by computing $(a + b\sqrt{2}) \cdot (c + d\sqrt{2})$ and $(c + d\sqrt{2}) \cdot (a + b\sqrt{2})$ and showing that they have the same value.
2. Does $\mathbb{Q}(\sqrt{2})$ satisfy the associative laws? Convince yourself that the associative law for multiplication is true in $\mathbb{Q}(\sqrt{2})$ by computing $((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) \cdot (e + f\sqrt{2})$ and $(a + b\sqrt{2}) \cdot ((c + d\sqrt{2}) \cdot (e + f\sqrt{2}))$ and showing that they have the same value.
3. What is the additive identity in $\mathbb{Q}(\sqrt{2})$? What is the multiplicative identity in $\mathbb{Q}(\sqrt{2})$?
4. If $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ what is its additive inverse is clearly $-a - b\sqrt{2}$ and this is in $\mathbb{Q}(\sqrt{2})$. Suppose we know that $a + b\sqrt{2} \neq 0$. What is the multiplicative inverse of $a + b\sqrt{2}$? That is, what is the value of $(a + b\sqrt{2})^{-1}$? Is the multiplicative inverse of $a + b\sqrt{2}$ an element of $\mathbb{Q}(\sqrt{2})$?

5. You can now the equation $x^2 = 2$ in this field. Can you solve the equation $x^2 = 4\sqrt{2}$ in the field $\mathbb{Q}(\sqrt{2})$? Give examples of other equations you can solve in this field. Give examples of equations you cannot solve in this field.

In the inquiry above, we showed that $\mathbb{Q}(\sqrt{2})$ is a field. The field $\mathbb{Q}(\sqrt{2})$ is called a **quadratic extension** of $\mathbb{Q}$. First, $\mathbb{Q}(\sqrt{2})$ is an extension of $\mathbb{Q}$ because $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$ and the operations of addition and multiplication on $\mathbb{Q}$ are the same as those on $\mathbb{Q}(\sqrt{2})$, just restricted to $\mathbb{Q}$. We call $\mathbb{Q}(\sqrt{2})$ a quadratic extension because we extended $\mathbb{Q}$ by **adjoining** the root of a quadratic polynomial over $\mathbb{Q}$. In this case, we adjoined a root of

$$x^2 - 2$$

to $\mathbb{Q}$.

Let n be a positive integer and define $\mathbb{Q}(\sqrt{n})$. Show that $\mathbb{Q}(\sqrt{n})$ is a field. When does $\mathbb{Q}(\sqrt{n}) = \mathbb{Q}$?

Ordered Fields

Instructor note. We are introducing ordered fields here because students will study them in Part 2 of this module.

Above we gave the axioms that define a field. We add to those the following **order axioms** to create an **ordered field**.

Definition 0.8. Suppose that $\mathbb{F}$ is a field. Then $\mathbb{F}$ is an **ordered field** if there is an order relation $<$ on $\mathbb{F}$ that satisfies the following properties for any $a, b, c \in \mathbb{F}$:

- (Trichotomy) Exactly one of the following holds: $a < b$, $a = b$ or $b < a$.
- (Transitivity) $a < b$ and $b < c$ implies that $a < c$.
- (Addition) If $a < b$, then $a + b < a + c$.
- (Multiplication) If $a < b$, $0 < c$, then $ac < bc$.

Definition 0.9. Suppose that $\mathbb{F}$ is an ordered field. An element $a \in \mathbb{F}$ is positive if $0 < a$ and a is negative if $a < 0$.

Proposition 0.10. *Suppose that $\mathbb{F}$ is an ordered field. Then*

(a) *$x \in \mathbb{F}$ is positive if and only if $-x$ is negative.*

(b) *If $x, y \in \mathbb{F}$, then $x + y$ and xy are positive.*

(c) *If $x \neq 0$, then x^2 is positive.*

(d) *1 is positive.*

Proof. For part (a) we proceed as follows. Suppose that $x \in \mathbb{F}$ is positive. By definition that means $0 < x$. Since $x \in \mathbb{F}$ it's additive inverse $-x$ is also in $\mathbb{F}$. By the addition axiom for ordering, since $0 < x$ we have

$$-x + 0 < -x + x.$$

We know that $-x + 0 = -x$ and that $-x + x = 0$. So we have $-x < 0$. Thus, if x is positive, then $-x$ is negative. The proofs of the remaining items are left as exercises. $\square$

Use the field axioms and the ordering axioms to prove parts (b)-(d) of the previous proposition.

Proposition 0.11. *Suppose that $\mathbb{F}$ is an ordered field. Then for all $a, b, c \in \mathbb{F}$ if $a < b$ and $c < 0$, then $ac > bc$.*

Proof. To see this, suppose that $a < b$ and that $c < 0$. Then $0 < -c$. So multiplication by $-c$ preserves order:

$$(-c)a < (-c)b.$$

But $(-c)a = -ca$ and $(-c)b = -cb$. Thus, we have $-ca < -cb$. But then

$$(ca + cb) - ca < (ca + cb) - cb.$$

Then by commutativity and associativity of addition

$$(ca - ca) + cb < ca + (cb - cb).$$

But $ca - ca = 0$ and $cb - cb = 0$ so we have $cb < ca$. By commutativity of multiplication we can restate this as

$$ac > bc.$$

$\square$

Corollary 0.12. $\mathbb{C}$ *is not an ordered field.*

Proof. This statement means that it is impossible to define an ordering on $\mathbb{C}$ which satisfies the four ordering axioms above. To see this, suppose by way of contradiction that we have found an ordering $<$ on $\mathbb{C}$ that satisfies all four of the ordering axioms. Then either $0 < i$ or $i < 0$. If $0 < i$, then $i \cdot 0 < i \cdot i$. So $0 < -1$. But since 1 is positive, we know that -1 must be negative, and so we have a contradiction. Now suppose that $i < 0$. Then since we are multiplying by a negative element, the inequality is reversed. So $i \cdot i > i \cdot 0$. That is, $-1 > 0$. Again, this is a contradiction. $\square$

Proposition 0.13. *In an ordered field*

(a) the product of a positive element and a negative element is negative.

(b) the product of a negative element and a negative element is positive.

Proof. To prove (a), let us suppose that $x > 0$ and $y < 0$. We claim that $xy < 0$. Suppose not. Then $xy \geq 0$. Since $x > 0$ we know that $x^{-1} > 0$. Thus $x^{-1}(xy) \geq x^{-1}(0) = 0$. Then by associativity, $(x^{-1}x)y \geq 0$. That is $y \geq 0$. This contradicts our assumption that $y < 0$.

We leave part (b) as an exercise. $\square$

Homework

1. Recall that we have defined **fields** to be mathematical structures with operations of addition $(+)$ and multiplication $(\cdot)$ that satisfy the following properties:

   (A) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutative laws)

   (B) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative laws)

   (C) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law)

   (D) There are distinct elements, called 0 and 1, such that $a + 0 = a$ and $a \cdot 1 = a$ for all a.

   (E) For each a there is an element b such that $a + b = 0$ and if $a \neq 0$, there is an element c such that $a \cdot c = 1$.

   In light of these properties, consider the following equation

   $$8x + 3 = 2x + 21.$$

   i. Carefully solve this equation one step at a time, indicating which of the above properties (A - E) that you utilized in each step.

   ii. How many of the above properties are **necessary** in order to solve the equation (i.e. did you need to use them all)? If not, which properties were not necessary?

   iii. Are these properties **sufficient** for solving this equation (i.e., did you need to apply other properties that are not listed in A - E in order to solve the equation)? If so, what other properties were required?

2. Suppose you are teaching multiplication in fourth grade and a student comes to class with the following solution to $3x = 6$:

   $$\begin{aligned} 3x &= 6 \\ \frac{3x}{3} &= \frac{6}{3} \\ x &= 2 \end{aligned}$$

   The students have not yet learned division, but the student says his older sister showed him how to solve it this way. How would you handle this situation?

3. Suppose that $\mathbb{F}$ is a field, and that $k \in \mathbb{F}$. Let

   $$\mathbb{F}(\sqrt{k}) = \{a + b\sqrt{k} \mid a, b \in \mathbb{F}\}.$$

   We define addition and multiplication on $\mathbb{F}(\sqrt{k})$ as follows:

   $$\begin{aligned} (a + b\sqrt{k}) + (c + d\sqrt{k}) &= (a + c) + (b + d)\sqrt{k} \\ (a + b\sqrt{k}) \cdot (c + d\sqrt{k}) &= (ac + bdk) + (ad + bc)\sqrt{k} \end{aligned}$$

   Here we show that $\mathbb{F}(\sqrt{k})$ is a field.

   (a) Show that addition and multiplication are commutative on $\mathbb{F}(\sqrt{k})$.

   (b) Show that addition and multiplication are associative on $\mathbb{F}(\sqrt{k})$.

   (c) Show that the distributive property holds in $\mathbb{F}(\sqrt{k})$.

   (d) What is the additive identity in $\mathbb{F}(\sqrt{k})$? What is the multiplicative identity in $\mathbb{F}(\sqrt{k})$?

   (e) If $a + b\sqrt{k} \in \mathbb{F}(\sqrt{k})$, what is its additive inverse?

   (f) If $a + b\sqrt{k} \in \mathbb{F}(\sqrt{k})$, and a, b are not both zero, then what is the multiplicative inverse of $a + b\sqrt{k}$? (Hint: For this one, begin by solving the equation $(a + b\sqrt{k})x = 1$ for x. Of course, the solution to that equation is $x = \frac{1}{a+b\sqrt{k}}$. The problem is that written this way, we don't know x is in $\mathbb{F}(\sqrt{k})$. Try to rewrite x so that it is clearly in $\mathbb{F}(\sqrt{k})$.)

4. Let $\mathbb{F}$ be an ordered field and $a, b \in \mathbb{F}$. Prove the following.

   (a) $(a^{-1})^{-1} = a$.

(b) $(ab)^{-1} = b^{-1}a^{-1}$

(c) $(-a)^{-1} = -(a^{-1})$

5. Prove that in an ordered field $\mathbb{F}$ if a is positive, then so is a^{-1}, and that if a is negative then a^{-1} is negative.
6. Prove that in an ordered field $\mathbb{F}$, if a and b are positive, then ab^{-1} and ba^{-1} are positive.
7. Prove that in an ordered field the product of two negative elements of a field is a positive element.
8. Carefully use the field axioms to prove the "Average Theorem."

   Suppose that $\mathbb{F}$ is an ordered field and that $a, b \in \mathbb{F}$ with $a < b$. Then there is some $r \in \mathbb{F}$ with $a < r < b$. In fact $r = \frac{a+b}{2}$ is one such element of $\mathbb{F}$.

9. Suppose that $\mathbb{F}$ is an ordered field. Then for $x \in \mathbb{F}$ we can define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

   Prove the following

   (a) If $a \in \mathbb{F}$, then $|a| \geq 0$.

   (b) If $a \in \mathbb{F}$, then $-|a| \leq a \leq |a|$.

   (c) Let $a, r \in \mathbb{F}$ with $r \geq 0$. Then for all $x \in \mathbb{F}$, $|x - a| \leq r$ if and only if $a - r \leq x \leq a + r$.

   (d) For all $a, b \in \mathbb{F}$, $|ab| = |a||b|$.

   (e) For all $a, b \in \mathbb{F}$, $|a + b| \leq |a| + |b|$. This is called the "Triangle Inequality."

In-Class Resources

Opening Inquiry: Solving Equations

Consider the equation:

$$3x + 8 = 14$$

It's not hard to see that the solution to this equation is $x = 2$: $3(2) + 8 = 14$. Let's think very carefully through some steps we could take to solve this equation, justifying each step along the way.

First we could subtract 8 from both sides (Step 1):

$$(3x + 8) - 8 = 14 - 8.$$

Second, we could re-write the left side of the equation to get (Step 2):

$$3x + (8 - 8) = 6.$$

We know that $8 - 8 = 0$ so we have (Step 3):

$$3x + 0 = 6.$$

We know that adding 0 to any number returns the value of the number that we added it to. So we have (Step 4):

$$3x = 6.$$

We now could multiply each side by $1/3$ to obtain (Step 5):

$$\frac{1}{3}(3x) = \frac{1}{3} \cdot 6.$$

We can re-write the left side of the equation (and simplify the right side) to obtain (Step 6):

$$\left(\frac{1}{3} \cdot 3\right) x = 2.$$

We know that $\frac{1}{3} \cdot 3$ is equal to 1, so we have (Step 7):

$$1x = 2.$$

We also know that the number 1 multiplied times any other number returns the value of that number, so we conclude that (Step 8):

$$x = 2.$$

Now, analyze this situation even more carefully by answering the following:

1. How many operations did the above solution use? List them.
2. How many "laws" or "special properties" of numbers did the above solution use? List them in each step.
3. Can you solve this equation by using other operations than you have listed above. How many operations did you use? What were they?
4. When you solved this equation using other operations, did you gain any more "special properties?" If so, what were they?
5. Try to solve this equation without using a rational number (NOTE: the $1/3$ above is a rational number).

We are interested in mathematical structures with operations of addition ($+$) and multiplication ($\cdot$) that satisfy the following properties:

(A) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutative laws)

(B) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative laws)

(C) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributive law)

(D) There are distinct elements, called 0 and 1, such that $a + 0 = a$ and $a \cdot 1 = a$ for all a.

(E) For each a there is an element b such that $a + b = 0$ and if $a \neq 0$, there is an element c such that $a \cdot c = 1$.

Structures which satisfy these properties are called **fields**. We now define a field called $\mathbb{Z}_5$. We will avoid the complexities of a formal definition of $\mathbb{Z}_5$ and simply assert that $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. The operations of addition ($+$) and multiplication ($\cdot$) are defined "modulo 5." That means we find the regular sum or product and then divide by 5 and take the remainder as our answer. For example, in $\mathbb{Z}_5$:

$$3 + 4 = 7$$
$$7 \div 5 = 1 \text{ R } 2.$$

Since the sum of 3 and 4 is 7 and the remainder when we divide 7 by 5 is 2, we conclude that in $\mathbb{Z}_5$, $3 + 4 = 2$. Using that procedure, fill in the following addition table:

+	0	1	2	3	4
0					
1					
2					
3					2
4					

1. Find a number $a \in \mathbb{Z}_5$ with the property that $a + 2 = 2 + a = 0$. The number a that you find is called the **additive inverse** of 2 in $\mathbb{Z}_5$. That is, it is the value of -2 in $\mathbb{Z}_5$.

   (a) $-2 =$ ______ in $\mathbb{Z}_5$ because $2 +$ ______ $=$ ______ $+ 2 =$ ______.

2. Find the value of -1, -3, and -4 in $\mathbb{Z}_5$.

   (a) $-1 =$ ______ in $\mathbb{Z}_5$ because $1 +$ ______ $=$ ______ $+ 1 =$ ______.

   (b) $-3 =$ ______ in $\mathbb{Z}_5$ because $3 +$ ______ $=$ ______ $+ 3 =$ ______.

   (c) $-4 =$ ______ in $\mathbb{Z}_5$ because $4 +$ ______ $=$ ______ $+ 4 =$ ______.

We follow the analogous process for multiplication. For example, $4 \cdot 4 = 16$. When we divide 16 by 5 we get 3 with a remainder of 1. So we conclude that in $\mathbb{Z}_5$, $4 \cdot 4 = 1$. Using that procedure, fill in the following multiplication table:

$\cdot$	0	1	2	3	4
0					
1					
2					
3					
4					1

Since $4 \cdot 4 = 1$ in $\mathbb{Z}_5$, we conclude that 4 is its own **multiplicative inverse** in $\mathbb{Z}_5$. That is, $4^{-1} = 4$ in $\mathbb{Z}_5$!

3. Find the value of 2^{-1} in $\mathbb{Z}_5$ and the value of 3^{-1} in $\mathbb{Z}_5$.
   (a) $2^{-1} =$ ______ in $\mathbb{Z}_5$ because $2 \cdot$ ______ $=$ ______ $\cdot 2 =$ ______
   (b) $3^{-1} =$ ______ in $\mathbb{Z}_5$ because $3 \cdot$ ______ $=$ ______ $\cdot 3 =$ ______

One can see by inspection of the addition and multiplication tables for $\mathbb{Z}_7$ that every element of $\mathbb{Z}_7$ has an additive inverse, and every nonzero element of $\mathbb{Z}_7$ has a multiplicative inverse. That is, $\mathbb{Z}_7$ satisfies (E) above. It is also easy to see by inspecting those tables that for all $a \in \mathbb{Z}_7$ $0 + a = a + 0 = a$ and $a \cdot 1 = 1 \cdot a = a$. That is, property (D) above is true in $\mathbb{Z}_7$.

4. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 \cdot (1 + 4) = 3 \cdot 1 + 3 \cdot 4$
   - $5 \cdot (2 + 6) = 5 \cdot 2 + 5 \cdot 7$
5. Make up two number sentences with the same form using numbers $0 - 6$ and verify that they are true in $\mathbb{Z}_7$.

6. Do you believe that the distributive law (C) holds in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of the distributive law are there in $\mathbb{Z}_7$?

7. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 + (1 + 4) = (3 + 1) + 4$ and $5 + (2 + 6) = (5 + 2) + 6$.
   - $3 \cdot (1 \cdot 4) = (3 \cdot 1) \cdot 4$ and $5 \cdot (2 \cdot 6) = (5 \cdot 2) \cdot 6$.

8. Do you believe that the associative laws (B) hold in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of each of the two distributive laws are there in $\mathbb{Z}_7$?

9. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share you work with the class.
   - $3 + 4 = 4 + 3$ and $5 + 6 = 6 + 5$.
   - $3 \cdot 4 = 4 \cdot 5$ and $5 \cdot 6 = 6 \cdot 5$.

10. Do you believe that the commutative laws (A) hold in $\mathbb{Z}_7$? Have you proved it? Exactly how many different instances of each of the two commutative laws are there in $\mathbb{Z}_7$?

11. Based on your work above, does $\mathbb{Z}_7$ satisfy properties (A)-(E)? Is it a field?

12. Create addition and multiplication tables for $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and answer appropriate versions of questions 1-11 for this structure.
13. Create addition and multiplication tables for $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and answer appropriate versions of questions 1-11 for this structure.
14. Once you have determined whether or not $\mathbb{Z}_4$ and $\mathbb{Z}_5$ are fields, form a conjecture about which values of n make $\mathbb{Z}_n$ a field. Test your conjecture with another value of n and be prepared to discuss your conjecture with the class.

Inquiry: $\mathbb{Q}(\sqrt{2})$

Consider the set $\mathbb{Q}(\sqrt{2})$ defined as

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

Given $a + b\sqrt{2}$ and $c + d\sqrt{2}$ in $\mathbb{Q}(\sqrt{2})$ we define

- $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$, and
- $(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (ad + bd)\sqrt{2}$.

The definition of addition here is quite natural, but the definition of multiplication might seem confusing until you realize that it is just the result of the distributive law:

$$\begin{aligned}(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) &= a(c + d\sqrt{2}) + b\sqrt{2}(c + d\sqrt{2}) \\ &= ac + ad\sqrt{2} + bc\sqrt{2} + bd\sqrt{2}\sqrt{2} \\ &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\ &= (ac + 2bd) + (ad + bd)\sqrt{2}\end{aligned}$$

With these definitions, explore the following questions in small groups.

1. Does $\mathbb{Q}(\sqrt{2})$ satisfy the commutative laws? Convince yourself that the commutative law for multiplication is true in $\mathbb{Q}(\sqrt{2})$ by computing $(a + b\sqrt{2}) \cdot (c + d\sqrt{2})$ and $(c + d\sqrt{2}) \cdot (a + b\sqrt{2})$ and showing that they have the same value.

   - $(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) =$
   - $(c + d\sqrt{2}) \cdot (a + b\sqrt{2}) =$

   Now start with the result of $(a + b\sqrt{2}) \cdot (c + d\sqrt{2})$ and manipulate it algebraically to derive the result of $(c + d\sqrt{2}) \cdot (a + b\sqrt{2})$.

2. Does $\mathbb{Q}(\sqrt{2})$ satisfy the associative laws? Convince yourself that the associative law for multiplication is true in $\mathbb{Q}(\sqrt{2})$ by computing $((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) \cdot (e + f\sqrt{2})$ and $(a + b\sqrt{2}) \cdot ((c + d\sqrt{2}) \cdot (e + f\sqrt{2}))$ and showing that they have the same value.

   - $(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) =$
   - $((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) \cdot (e + f\sqrt{2}) =$

   - $(c + d\sqrt{2}) \cdot (e + f\sqrt{2}) =$
   - $(a + b\sqrt{2}) \cdot ((c + d\sqrt{2}) \cdot (e + f\sqrt{2})) =$

   Now start with the result of $((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) \cdot (e + f\sqrt{2})$ and manipulate it algebraically to derive the result of $(a + b\sqrt{2}) \cdot ((c + d\sqrt{2}) \cdot (e + f\sqrt{2}))$.

3. What is the additive identity in $\mathbb{Q}(\sqrt{2})$? What is the multiplicative identity in $\mathbb{Q}(\sqrt{2})$?

4. If $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ what is its additive inverse is clearly $-a - b\sqrt{2}$ and this is in $\mathbb{Q}(\sqrt{2})$. Suppose we know that $a + b\sqrt{2} \neq 0$. What is the multiplicative inverse of $a + b\sqrt{2}$? That is, what is the value of $(a + b\sqrt{2})^{-1}$? Is the multiplicative inverse of $a + b\sqrt{2}$ an element of $\mathbb{Q}(\sqrt{2})$?
Hint: Start with the equation $(a + b\sqrt{2})x = 1$ and solve for x. Yes, of course, after one step you get $x = \frac{1}{a+b\sqrt{2}}$. The crucial part though is showing that you can put this into the form $c + d\sqrt{2}$ for some $c, d \in \mathbb{Q}$.

Part II

Semi-Advanced Topics in Ordered Fields

Overview

Content

This part entirely studies **ordered fields** which are fields $\mathbb{F}$ with an ordering relation $<$ which satisfies the following properties for any $a, b, c \in \mathbb{F}$:

- (Trichotomy) Exactly one of the following holds: $a < b$, $a = b$ or $b < a$.
- (Transitivity) $a < b$ and $b < c$ implies that $a < c$.
- (Addition) If $a < b$, then $a + b < a + c$.
- (Multiplication) If $a < b$, $0 < c$, then $ac < bc$.

We look at the **field of rational functions over the rationals,** $\mathbb{Q}(x)$

The **minimum** of a set is the least element in a set. The **maximum** of a set is the greatest element in a set.

A **lower bound** of a set is an element which is less than every element of the set.

An **upper bound** of a set is an element which is greater than every element of the set.

The **greatest lower bound** of a set, if it exists, is the unique lower bound which is greater than or equal to every lower bound.

The **least upper bound** of a set, if it exists, is the unique upper bound of a set which is less than or equal to every upper bound.

An **Archimedean** field $\mathbb{F}$ has the property that for every $x \in \mathbb{F}$ there is an integral element $n \in \mathbb{F}$ so that $x < n$. $\mathbb{Q}$ is Archimedean while $\mathbb{Q}(x)$ is not.

A field is **complete** if every nonempty subset of the field which is bounded above has a least upper bound.

Summary

In this part we explore the crucial concepts of the minimum and maximum of a set, upper and lower bounds, and least upper bounds and greatest lower bounds. These concepts are important in themselves in high-school mathematics. However, we take these notions a bit further to explore the question of what makes the rational numbers and the real numbers special. We discuss Archimedean fields, complete fields, and explore examples of these. These properties are of crucial importance for nearly all of the algebra we do in high-school mathematics, but they are rarely discussed except at an advanced level. We explore these properties and conclude by indicating the way in which the real numbers is truly special - it is the unique complete ordered field.

Acknowledgements. The material in this part is inspired by the notes from courses we've taught. The textbooks used in those courses include

Hadlock, C. R. (1978). Field Theory and Its Classical Problems. Washington: Mathematical Association of America.

Madden, D. J. & Aubrey, J. A. (2017). An introduction to proof through real analysis (1st ed.). Hoboken, NJ: Wiley.

Peressini, A. L. (2003). Mathematics for high school teachers: An advanced perspective. Prentice Hall.

Materials. In this part there are six inquiries to print and give to students. They are

- Opening Inquiry: $\mathbb{Q}(x)$
- Inquiry: Minimums and Maximums
- Inquiry Upper and Lower Bounds
- Inquiry: Least Upper Bounds and Greatest Lower Bounds
- Inquiry: Completeness
- Inquiry: Reviewing our Fields

In this section we will study some very deep and interesting properties of ordered fields that we use all the time in algebra, but whose important roles are often overlooked. One such property we will study is the Archimedean principle. The Archimedean principle is very easy to state and it seems obviously true. However, it is a fact that needs proof. And, although it does hold in the field of rational numbers and the field of real numbers, it does not hold in every ordered field.

The Archimedean principle for the real numbers states:

> For every real number x there exists a natural number n so that $x < n$.

To get a sense for the meaning of the Archimedean principle, discuss the following questions:

- Is ∞ a real number?
- Is there an infinitely large real number?
- Is there an infinitely small real number?

In the opening inquiry, we explore the Archimedean principle from another point of view.

Opening Inquiry: $\mathbb{Q}(x)$

In $\mathbb{Q}$ and $\mathbb{R}$ we know that we have the integers as a subset. And, each positive integer can be built up by repeatedly adding 1 to itself. That is,

$$\begin{aligned} 1+1 &= 2 \\ 1+1+1 &= 3 \end{aligned}$$

and so on. Then, once we have the positive integers, we know we have the negative integers because in a field every element has an additive inverse.

We can do exactly the same construction in any field. That's because any arbitrary field $\mathbb{F}$ has a multiplicative identity, which we call 1. (As we have seen, it may actually be quite different from the number 1.) Then we can form special elements by repeatedly adding the multiplicative identity to itself over and over. So for example, in any field we can define

$$\begin{aligned} 2 &= 1+1 \\ 3 &= 1+1+1 \end{aligned}$$

and so on. We call these elements **integral elements**. In this inquiry we want to study the integral elements of a somewhat strange field - the field of rational functions with rational coefficients. We begin by defining this field, which we denote $\mathbb{Q}(x)$.

DEFINING $\mathbb{Q}(x)$

First we define the set of objects $\mathbb{Q}(x)$:

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x) \text{ and } q(x) \text{ are polynomials with rational coefficients} \right\}.$$

1. Which of the following are elements of $\mathbb{Q}(x)$?
   - $r(x) = x^2 + 1$
   - $s(x) = \frac{3x^2+\pi x+1}{4x^3+2x^2+x+2}$
   - $t(x) = \frac{x^{1/2}+2x+1}{\frac{3}{2}x+5}$
   - $u(x) = \frac{\frac{2}{3}x^2+2x+1}{\frac{3}{2}x+5}$

We define the equality of two rational functions, as well as their sums and products, like we do with fractions. That is, given $r_1(x), r_2(x) \in \mathbb{Q}(x)$ with $r_1(x) = \frac{p_1(x)}{q_1(x)}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)}$ we define

- $r_1(x) = r_2(x)$ if and only if $p_1(x)q_2(x) - q_1(x)p_2(x) = 0$
- $(r_1 + r_2)(x) = \frac{p_1(x)q_2(x)+q_1(x)p_2(x)}{q_1(x)q_2(x)}$
- $(r_1 \cdot r_2)(x) = \frac{p_1(x)p_2(x)}{q_1(x)q_2(x)}$

2. Suppose that
$$r_1(x) = \frac{x^2 + \frac{1}{2}x + 2}{x+1}, \quad r_2(x) = \frac{x+2}{x^2+4}, \text{ and } r_3(x) = \frac{2x^2+x+4}{2x+2}.$$
Is $r_1(x) = r_2(x)$? Is $r_1(x) = r_3(x)$? Is $r_2(x) = r_3(x)$?
3. Suppose that
$$r_1(x) = \frac{3x^2+2x+1}{x+5} \text{ and } r_2(x) = \frac{8x+2}{4x^2+\frac{1}{2}}.$$
Compute $(r_1 + r_2)(x)$ and $(r_1 \cdot r_2)(x)$.

Next we define an ordering on $\mathbb{Q}(x)$ as follows: Given $r_1(x), r_2(x) \in \mathbb{Q}(x)$ with $r_1(x) = \frac{p_1(x)}{q_1(x)}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)}$, suppose that the leading coefficients of $q_1(x)$ and $q_2(x)$ are both positive. Then

$$r_1(x) < r_2(x)$$

if and only if the leading coefficient of

$$p_2(x)q_1(x) - p_1(x)q_2(x)$$

is positive. This ordering can be difficult to get a handle on. In practice you want to (a) make sure to leading coefficients in the denominator are positive, and if not multiply by $-1/-1$ to make it so and then (b) cross multiply and subtract. Here are two examples, followed by some exercises for you to try.

Example 0.14. Let us compare $r_1(x) = \frac{2x^2+11x+1}{x+5}$ and $r_2(x) = 2x + 1$. First, we observe that as a rational function $r_2(x) = \frac{2x+1}{1}$ and that both $r_1(x)$ and $r_2(x)$ have positive leading coefficients in the denominator. Now we cross multiply and obtain

$$(x+5)(2x+1) = 2x^2 + 11x + 5$$
$$(1)(2x^2 + 11x + 1) = 2x^2 + 11x + 1$$

We see that $(2x^2 + 11x + 5) - (2x^2 + 11x + 1) = 4$ while subtracting in the other direction gives a negative. So we conclude that

$$\frac{2x^2 + 11x + 1}{x + 5} < 2x + 1.$$

Example 0.15. Let us compare $r_1(x) = \frac{-x^2+3}{-2x+1}$ and $r_2(x) = \frac{2x+1}{2x-1}$. First, we multiply $r_1(x)$ by $-1/-1$ to rewrite it as $r_1(x) = \frac{x^2-3}{2x-1}$. Now we cross multiply and obtain

$$(2x-1)(x^2-3) = 2x^3 - x^2 - 6x + 4$$
$$(2x-1)(2x+1) = 4x^2 - 1$$

We see that $(2x^3 - x^2 - 6x + 4) - (4x^2 - 1) = 2x^3 - 5x^2 - 6x + 3$ has positive leading coefficient. So we conclude that

$$r_2(x) = \frac{2x+1}{2x-1} < r_1(x) = \frac{-x^2+3}{-2x+1}.$$

4. It is important to play with this ordering a bit to get used to it. Try to work through these examples in small groups. For each pair $r_1(x)$ and $r_2(x)$ determine whether $r_1(x) < r_2(x)$, $r_1(x) = r_2(x)$ or $r_2(x) < r_1(x)$.

   (a) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x+5}{3x^2+x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x+6}{3x^2+x-2}$.

   (b) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{3x^2+x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x+5}{3x^2+x-2}$.

   (c) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{3x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x^2+5}{3x^2-2}$.

   (d) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{-3x+2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x^2+5}{3x-2}$.

The purpose of this opening task is to give you a sense of how the field $\mathbb{Q}(x)$ works. The fact that these operations make it an ordered field is a theorem which is not difficult to prove, but is somewhat tedious. So here we simply state it without proof.

Theorem. The set of rational functions over the rationals,

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x) \text{ and } q(x) \text{ are polynomials with rational coefficients} \right\}.$$

together with the operations of addition $(+)$ and multiplication $(\cdot)$ defined above form a field. If we add in the ordering $<$ defined above, $\mathbb{Q}(x)$ is an ordered field.

We opened this inquiry by talking about integral elements. Here, in $\mathbb{Q}(x)$, the integral elements are of the form

$$q_n(x) = n$$

for $n \in \mathbb{N}$. (Notice that these are degree 0 polynomials, and are also rational functions since $q_n(x) = \frac{n}{1}$.) As your final task in this inquiry, prove the following facts:

5. Suppose that $r(x) = x^2 + 2x + 3$. Prove that $q_{500}(x) < r(x)$
6. Suppose that $r(x) = 2x + 3$. Prove that for any $n \in \mathbb{N}$, $q_n(x) < r(x)$.
7. Suppose that $r(x) = a_m x^m + a_{m-1} x_{m-1} + \cdots a_1 x + a_0$ is an element of $\mathbb{Q}(x)$ with $m \geq 1$. Prove that if $a_m > 0$, then $q_{500}(x) < r(x)$
8. Suppose that $r(x) = a_m x^m + a_{m-1} x_{m-1} + \cdots a_1 x + a_0$ is an element of $\mathbb{Q}(x)$ with $m \geq 1$. Prove that for any $n \in \mathbb{N}$, if $a_m > 0$, then $q_n(x) < r(x)$

You've just shown that in $\mathbb{Q}(x)$ every polynomial with degree at least 1 and with positive leading coefficient is greater than every integral element.

Let's talk a bit more about integral elements of a field because they will play an important role below. The idea is that every field contains a multiplicative identity, 1, and an addition operation $+$. So, we can add 1 to itself as often as we like. It can get tedious to write expressions like $1+1$, $1+1+1$, $1+1+1+1$, and so on, so we give them special names:

$$\begin{aligned}2&=1+1\\3&=1+1+1\\4&=1+1+1+1\\5&=1+1+1+1+1\end{aligned}$$

and so on. In infinite fields, we can generate infinitely many integral elements in this manner. In finite fields, the expressions on the right are still valid - we can add 1 to itself as often as we like - but the values of those sums will cycle through the elements of the field. Notice that in $\mathbb{Q}$ if somebody gives you any $x \in \mathbb{Q}$, no matter how large, you can always find an integral element n so that $x < n$. Below we will call a field with this property an "Archimedean field." Notice that you showed that this property does not hold in $\mathbb{Q}(x)$. In the inquiry above, you showed that there are some rational functions that are larger than every integral element (namely polynomials of degree at least 1 with positive leading coefficients). Thus, we will call $\mathbb{Q}(x)$ a *non-Archimedean* field.

1 Minimums and Maximums, Upper and Lower Bounds

Once we have a notion of order in a field, we can talk about minimums and maximums and upper and lower bounds.

Inquiry: Minimums and Maximums

We haven't formally defined the notions of minimum and maximum elements of a set, but with some thought about the intended meanings of these words we can come up with the definitions ourselves.

- Consider the set $S = \{3, 4, 7, 8\}$.
  - What is the minimum of this set? Try to list two facts that make the number you chose the minimum of the set S.
  - Why isn't 2 the minimum of S? Why isn't 4 the minimum of this set?

  Try to write down the definition of the minimum of a set S. In a small group, discuss your definition and be prepared to share it with the class. Must every set have a minimum? Notice that we have talked about "the" minimum of a set, implying that the minimum of a set S is unique. Can sets have more than one minimum? Can you justify your answer?

  - What is the maximum of the set $S = \{3, 4, 7, 8\}$? Try to list two facts that make the number you chose the maximum of the set S.
  - Why isn't 9 the maximum of S? Why isn't 4 the maximum of this set?

  Try to write down the definition of the maximum of a set S. In a small group, discuss your definition and be prepared to share it with the class. Must every set have a maximum? Notice that we have talked about "the" maximum of a set, implying that the maximum of a set S is unique. Can sets have more than one maximum? Can you justify your answer?

  - Suppose that $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$. Based on your definition, what is the minimum and the maximum of $[2, 3)$?
  - Suppose that $(2, 3) = \{x \in \mathbb{R} \mid 2 < x < 3\}$. Based on your definition, what is the minimum and the maximum of $(2, 3)$?
  - Suppose that $(-\infty, 3) = \{x \in \mathbb{R} \mid x < 3\}$. Based on your definition, what is the minimum and the maximum of $(-\infty, 3)$?

- Suppose that $[2, \infty) = \{x \in \mathbb{R} \mid 2 \leq x\}$. Based on your definition, what is the minimum and the maximum of $[2, \infty)$?

Definition 1.1. Given a set S

- m is a minimum of S if
  - $m \in S$, and
  - for all $x \in S$, $m \leq x$.
- M is a maximum of S if
  - $M \in S$, and
  - for all $x \in S$, $x \leq M$.

Lemma 1.2. *If a set S has a minimum, then it is unique.*

Proof. A good strategy for trying to prove that something is unique is to start by supposing there are two of them. So suppose that m_1 and m_2 are minimums of S. Since m_1 and m_2 are both minimums of S they must both be members of S.

Since m_1 is a minimum of S we must also have that $m_1 \leq x$ for every $x \in S$. But we know that $m_2 \in S$, so we must have $m_1 \leq m_2$.

Since m_2 is a minimum of S we must also have that $m_2 \leq x$ for every $x \in S$. But we know that $m_1 \in S$, so we must have $m_2 \leq m_1$.

Thus we have $m_1 \leq m_2$ and $m_2 \leq m_1$. By trichotomy, this implies that $m_1 = m_2$. Thus the minimum of a set is unique.
□

The proof that the maximum of a set is unique is similar, and we leave it as an exercise.

Inquiry: Upper and Lower Bounds

To illustrate the idea of upper and lower bounds, consider the following questions. Don't look up any answers on the internet, just give your best answers based on what you know of the world.

- Give an upper bound on the height of all humans that have ever lived.
- Give a lower bound on the mass of the earth.
- Give upper and lower bounds on the cost of raising a child.

Let's consider the first question for a moment. It is impossible to know how tall the tallest human ever was. Of course, we could look up the height of the tallest known human, but recorded history goes back only so far. And, even if the tallest person ever did in fact live during the period of recorded history, maybe nobody ever recorded his or her height or maybe the records were lost or forgotten. Even so, we can still give a reasonably good answer to the first question. For example, we can probably all agree that there was never a human who grew to be over 1 million feet tall. So, 1 million feet is an upper bound on the height of all humans that ever lived. Similarly, we can be confident that all humans that ever lived were less than 100 feet tall. So 100 feet is also an upper bound on the height of all humans that ever lived. What about 20 feet? 10 feet? The point is that an upper bound on a set just tells us that everything in the set is less than that value. With this discussion in mind, consider the other two questions and be prepared to discuss and share your answers.

- Consider the set $S = \{3, 5, 8, 10\}$.
  - Is 15 an upper bound of this set?
  - Is 11 an upper bound of this set?
  - Is 10 an upper bound of this set?
  - Is 9 an upper bound of this set?

- Consider the set $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - Is 5 an upper bound of this set?
  - Is 3 an upper bound of this set?
  - Is 3 the maximum of this set?
  - Is there an upper bound of this set less than 3?
- Try to complete the definition of an upper bound of a set: An element u is an upper bound of a set S if...
- Consider the set $S = \{3, 5, 8, 10\}$.
  - Is 0 a lower bound of this set?
  - Is 1 a lower bound of this set?
  - Is 3 a lower bound of this set?
  - Is 4 a lower bound of this set?
- Consider the set $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - Is 0 a lower bound of this set?
  - Is 1.99 a lower bound of this set?
  - Is 2 a lower bound of this set?
  - Is there a lower bound of this set greater than 2?
- Try to complete the definition of a lower bound of a set: An element ℓ is a lower bound of a set S if...

We now give the formal definitions of an upper and lower bound of a set.

Definition 1.3. Let $\mathbb{F}$ be an ordered field and suppose that $S \subseteq \mathbb{F}$. Then

- u is an upper bound of S if $x \leq u$ for all $x \in S$.
- l is a lower bound of S if $l \leq x$ for all $x \in S$.

Example 1.4. Consider the set $S = \{x \in \mathbb{Q} \mid 2 \leq x < 3\}$. Notice that 3 is an upper bound of S, but so is 3.5, 4, and indeed any number larger than 3. Similarly, 2 is a lower bound of $\mathbb{Q}$ as is $1\frac{3}{4}$ and any number less than 2. Upper bounds and lower bounds may or not be members of the sets they bound.

INQUIRY: LEAST UPPER BOUNDS AND GREATEST LOWER BOUNDS

- Some lower bounds are special in that they are the greatest lower bounds of a set. That is, not only are they lower bounds, but nothing larger is a lower bound.
- Similarly, some upper bounds are special in that they are the least upper bounds. That is, not only are they upper bounds, but nothing lower is an upper bound.

Let us explore these ideas in this inquiry.

- Consider the set $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - What is the least upper bound of this set?
  - Why isn't 2.95 the least upper bound of this set? What about 2.999995?
  - The greatest lower bound of this set is 2. There are two ways to say this.
    - First, we could express this by writing a mathematical statement which says that no number larger than 2 is a lower bound of this set. That is, if $x > 2$, then x is not a lower bound of $[2, 3)$. What does it mean to not be a lower bound of a set? Can you complete a mathematical sentence which says that no number larger than 2 is a lower bound of this set? It begins "If $x > 2$, then..."

- ∗ Second, we could express this by saying that every lower bound of this set is less than or equal to 2. Can you write a mathematical statement which says this? Do you see why these two ways of expressing the idea of a greatest lower bound are the equivalent?

- Consider the set $[2, 5] = \{x \in \mathbb{R} \mid 2 \leq x \leq 5\}$.
  - What is the greatest lower bound of this set?
  - Why isn't 2.05 the greatest lower bound? What about 2.000001?
  - The least upper bound of this set is 5. There are two ways to say this
    - ∗ First, we want to write a mathematical statement which says that no number less than 5 is an upper bound of this set. That is, if $x < 5$, then x is not a lower bound of $[2, 5]$. What does it mean to not be a lower bound of a set? Can you complete a mathematical sentence which says that no number less than 5 is an upper bound of this set? It begins "If $x < 5$, then…"
    - ∗ Second, we could express this by saying that every upper bound of this set is greater than or equal to 5. Can you write a mathematical statement which says this? Do you see why these two ways of expressing the idea of a least upper bound are equivalent?

Before we move on to the formal definition of the least upper bound and greatest lower bound, let us explore these ideas as they relate to sequences of numbers.

Consider the infinite sequence of numbers

$$0.1, 0.12, 0.121, 0.1212, 0.12121, \ldots$$

- Notice that every number in the sequence is a finite decimal, but the sequence appears to be tending toward a number which has an infinite decimal expansion. What is the number ℓ that this sequence is tending toward?
- Can you write the number ℓ as a fraction?
- Is the number ℓ a member of the sequence we started with? Is ℓ is an upper bound of the sequence, lower bound of the sequence or neither?

Definition 1.5. Let $\mathbb{F}$ be an ordered field and suppose that $S \subseteq \mathbb{F}$. Then

u is a least upper bound of S if

1. $x \leq u$ for all $x \in S$, and
2. If b is an upper bound of S, then $u \leq b$.

l is a greatest lower bound of S

1. if $l \leq x$ for all $x \in S$, and
2. if b is a lower bound of S, then $b \leq l$.

Example 1.6. Consider the set $S = \{x \in \mathbb{Q} \mid 2 \leq x < 3\}$. Notice that 3 is an upper bound of S, and any other upper bound of S is greater than or equal to 3. Thus, 3 is the least upper bound of S. Similarly, 2 is a lower bound of $\mathbb{Q}$ and any other lower bound of S must be less than or equal to 2. Thus, 2 is the greatest lower bound of S. Note that a greatest lower bound or least upper bound of a set may be in the set or not in the set.

Theorem 1.7. *If S has a greatest lower bound, then it is unique. If S has a least upper bound, then it is unique.*

Proof. We leave the proof of this theorem as an exercise. □

The Archimedean Property and Completeness

We have seen this definition before.

Definition 1.8. An ordered field $\mathbb{F}$ is Archimedean if and only if for each positive $x \in \mathbb{F}$ there is an integral element $k \in \mathbb{F}$ such that $x < k$.

The following theorem says that in an Archimedean field, the ceiling and floor functions are well defined.

Theorem 1.9. *For each positive element x in an Archimedean field $\mathbb{F}$ there is a unique integral element n such that*

$$n \leq x < n + 1.$$

Proof. We leave the proof as an exercise. $\square$

INQUIRY: COMPLETENESS

Let us consider the set

$$S = \{r \in \mathbb{Q} \mid r > 0 \text{ and } r^2 < 2\}.$$

Note that

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766\ldots$$

where the decimal expansion goes on forever with no pattern. That's because $\sqrt{2}$ is irrational - it's not rational. Here we could write $\sqrt{2} \notin \mathbb{Q}$ to say that $\sqrt{2}$ is not a rational number. We could be more specific and say $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ to say that $\sqrt{2}$ is a real number but not a rational number.

Our goal here is to figure out the least upper bound of S. First, for each of the numbers r listed below, determine whether or not $r^2 < 2$.

- $r = 1.414$
- $r = 1.41421$
- $r = 1.4142135$
- $r = 1.414213562$
- $r = 1.41421356237$

Notice that each of the numbers r above satisfies $r < \sqrt{2}$. Now let's consider some numbers that are barely above $\sqrt{2}$ and determine whether or not they satisfy $r^2 < 2$.

- $r = 1.415$
- $r = 1.41422$
- $r = 1.4142136$
- $r = 1.414213563$
- $r = 1.41421356238$

Complete the following sentences:

- If r is less than $\sqrt{2}$, even barely, then $r^2 \ldots$.
- If r is greater than $\sqrt{2}$, even barely, then r^2 is. . . .

We now claim that $\sqrt{2}$ is an upper bound of S. Suppose $s \in S$ and $s > \sqrt{2}$

- There is an ordering axiom which implies that $s \cdot s > s \cdot \sqrt{2}$. Which is it?
- There is an ordering axiom which implies that $s \cdot \sqrt{2} > \sqrt{2} \cdot \sqrt{2}$. Which is it?
- There is an ordering axiom given the previous two facts implies that $s^2 > 2$. Which is it?

Thus, if $s \in S$, then we must have $s \leq \sqrt{2}$. So $\sqrt{2}$ is an upper bound of S.

We now claim that if $x < \sqrt{2}$, then x is not an upper bound of S. To see this, suppose that $x < \sqrt{2}$. Then there is some $\epsilon > 0$ so that $x + \epsilon = \sqrt{2}$. Notice that $x < x + \epsilon/2 < \sqrt{2}$. Fill in the blanks:

$$(x + \epsilon/2)^2 = \underline{\hspace{8cm}} < x^2 + 2x\epsilon + \epsilon^2 = (x + \epsilon)^2.$$

But $(x + \epsilon)^2 =$ ________________. Thus,

$$(x + \epsilon/2)^2 < 2.$$

Thus $x + \epsilon/2$ is in S but greater than x.

So, if $x < \sqrt{2}$, then x is not an upper bound of S. So, $\sqrt{2}$ is the least upper bound of S. The moral of the story is that S is a subset of $\mathbb{Q}$, but its least upper bound is not in $\mathbb{Q}$. That means that $\mathbb{Q}$ is not **complete** in the sense defined below.

Definition 1.10. An ordered field $\mathbb{F}$ is complete if and only if every subset of $\mathbb{F}$ that is bounded above has a least upper bound in $\mathbb{F}$.

There is nothing special about least upper bounds in this respect. We have the following theorem.

Theorem 1.11. *An ordered field $\mathbb{F}$ is complete if and only if every subset of $\mathbb{F}$ that is bounded below has a greatest lower bound in $\mathbb{F}$.*

Notice that $\mathbb{Q}$ shows that not every Archimedean field is complete. However, we have the following.

Theorem 1.12. *Any complete ordered field is Archimedean.*

Proof. Suppose that $\mathbb{F}$ is a complete ordered field. Suppose to the contrary that $\mathbb{F}$ is not Archimedean. Then, there is some $x \in \mathbb{F}$ so that no integral element n is above x. Thus,

$$N = \{n \in \mathbb{F} \mid n \text{ is an integral element}\}$$

is bounded above by x. Since N is a nonempty set which is bounded above and $\mathbb{F}$ is a complete field, it follows that N has a least upper bound, say b. Consider $b - 1$. Since $b - 1 < b$ and b is the least upper bound of N, it follows that there is some $n \in N$ so that $b - 1 < n$. But then $b < n + 1$. This is a contradiction because $n + 1$ is an integral element and it is greater than b which was suppose to be an upper bound of N. So, we have a contradiction and our assumption that $\mathbb{F}$ is not Archimedean must be false. So, we conclude that if $\mathbb{F}$ is complete, then it is Archimedean. $\square$

INQUIRY: REVIEWING OUR FIELDS

We have just seen that every complete ordered field is Archimedean. Try to answer as many of the following questions as you can without looking at your notes.

- Give two examples of fields that are not ordered fields.
- Define what it means for a field to be Archimedean.
- Give an example of an ordered field that is not Archimedean.
- Define what it means for a field to be complete.
- Give an example of an Archimedean field that is not complete.

The real numbers $\mathbb{R}$ is a complete ordered field. In fact, it is the unique complete ordered field.

As stated in the inquiry above, we have the following theorem that makes the real numbers special. It says that there is essentially only one complete ordered field, and it is the field of real numbers.

Theorem 1.13. *Any complete ordered field is isomorphic to the ordered field of real numbers.*

Again, the purpose of stating this theorem is to highlight the uniqueness of the real numbers. Essentially, the real numbers contain every number we might need for measurement. The proof of this theorem is beyond the scope of these notes.

Homework

1. For each of the following sets, find their minimums and maximums and greatest lower bounds and least upper bounds, if they exist.
   (a) $\mathbb{N}$
   (b) $[0, 1)$
   (c) The set of irrational numbers in $[0, 1]$.
2. Show that 0 is the greatest lower bound of $S = \{x \in \mathbb{R} \mid x^{-1} \in \mathbb{N}\}$.
3. Let A be a set of real numbers.
   (a) Prove that if A has an upper bound, then A has an upper bound that is a natural number.
   (b) Prove that if A has a lower bound, then A has a lower bound that is an integer.
   (c) Prove that if A has a lower bound and an upper bound, then there is a natural number n so that n is an upper bound and $-n$ is a lower bound of A.
   (d) Prove that A is bounded (above and below) if and only if there is $n \in \mathbb{N}$ so that for all $x \in A$, $-n \leq x \leq n$.
   (e) Prove that A is bounded (above and below) if and only if there is $n \in \mathbb{N}$ so that for all $x \in A$, $-n < x < n$.
4. Prove that if $a \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ so that $a < 10^n$.
5. Let $a \in \mathbb{R}$.
   (a) Prove that for all $n \in \mathbb{N}$, there is $m \in \mathbb{Z}$ so that $10^n m \leq a \leq 10^n(m+1)$.
   (b) What does part a say about approximating real numbers with decimals?
6. Let $S = \{r \in \mathbb{Q} \mid \text{there is an } n \in \mathbb{N} \text{ such that } r = \frac{10^n - 1}{10^n}\}$.
   (a) Prove that S has at least one element.
   (b) Prove that S has at least one upper bound.
   (c) Prove that S has a real number that is its least upper bound.
   (d) What does this tell you about $u = 0.99999\ldots$?
   (e) Can you guess what the real number u is?
7. Is $\mathbb{Q}(\sqrt{2})$ Archimedean? Is it complete?

2 In-Class Resources

Opening Inquiry: $\mathbb{Q}(x)$

In $\mathbb{Q}$ and $\mathbb{R}$ we know that we have the integers as a subset. And, each positive integer can be built up by repeatedly adding 1 to itself. That is,

$$1 + 1 = 2$$
$$1 + 1 + 1 = 3$$

and so on. Then, once we have the positive integers, we know we have the negative integers because in a field every element has an additive inverse.

We can do exactly the same construction in any field. That's because any arbitrary field $\mathbb{F}$ has a multiplicative identity, which we call 1. (As we have seen, it may actually be quite different from the number 1.) Then we can form special elements by repeatedly adding the multiplicative identity to itself over and over. So for example, in any field we can define

$$2 = 1 + 1$$
$$3 = 1 + 1 + 1$$

and so on. We call these elements **integral elements**. In this inquiry we want to study the integral elements of a somewhat strange field - the field of rational functions with rational coefficients. We begin by defining this field, which we denote $\mathbb{Q}(x)$.

Defining $\mathbb{Q}(x)$

First we define the set of objects $\mathbb{Q}(x)$:

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x) \text{ and } q(x) \text{ are polynomials with rational coefficients} \right\}.$$

1. Which of the following are elements of $\mathbb{Q}(x)$?
   - $r(x) = x^2 + 1$
   - $s(x) = \frac{3x^2+\pi x+1}{4x^3+2x^2+x+2}$
   - $t(x) = \frac{x^{1/2}+2x+1}{\frac{3}{2}x+5}$
   - $u(x) = \frac{\frac{2}{3}x^2+2x+1}{\frac{3}{2}x+5}$

We define the equality of two rational functions, as well as their sums and products, like we do with fractions. That is, given $r_1(x), r_2(x) \in \mathbb{Q}(x)$ with $r_1(x) = \frac{p_1(x)}{q_1(x)}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)}$ we define

- $r_1(x) = r_2(x)$ if and only if $p_1(x)q_2(x) - q_1(x)p_2(x) = 0$

- $(r_1 + r_2)(x) = \frac{p_1(x)q_2(x)+q_1(x)p_2(x)}{q_1(x)q_2(x)}$
- $(r_1 \cdot r_2)(x) = \frac{p_1(x)p_2(x)}{q_1(x)q_2(x)}$

2. Suppose that
$$r_1(x) = \frac{x^2 + \frac{1}{2}x + 2}{x+1}, \quad r_2(x) = \frac{x+2}{x^2+4}, \text{ and } r_3(x) = \frac{2x^2+x+4}{2x+2}.$$
Is $r_1(x) = r_2(x)$? Is $r_1(x) = r_3(x)$? Is $r_2(x) = r_3(x)$?

3. Suppose that
$$r_1(x) = \frac{3x^2+2x+1}{x+5} \text{ and } r_2(x) = \frac{8x+2}{4x^2+\frac{1}{2}}.$$
Compute $(r_1 + r_2)(x)$ and $(r_1 \cdot r_2)(x)$.

Next we define an ordering on $\mathbb{Q}(x)$ as follows: Given $r_1(x), r_2(x) \in \mathbb{Q}(x)$ with $r_1(x) = \frac{p_1(x)}{q_1(x)}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)}$, suppose that the leading coefficients of $q_1(x)$ and $q_2(x)$ are both positive. Then
$$r_1(x) < r_2(x)$$
if and only if the leading coefficient of
$$p_2(x)q_1(x) - p_1(x)q_2(x)$$
is positive. This ordering can be difficult to get a handle on. In practice you want to (a) make sure to leading coefficients in the denominator are positive, and if not multiply by $-1/-1$ to make it so and then (b) cross multiply and subtract. Here are two examples, followed by some exercises for you to try.

Example 2.1. Let us compare $r_1(x) = \frac{2x^2+11x+1}{x+5}$ and $r_2(x) = 2x+1$. First, we observe that as a rational function $r_2(x) = \frac{2x+1}{1}$ and that both $r_1(x)$ and $r_2(x)$ have positive leading coefficients in the denominator. Now we cross multiply and obtain
$$(x+5)(2x+1) = 2x^2 + 11x + 5$$
$$(1)(2x^2 + 11x + 1) = 2x^2 + 11x + 1$$
We see that $(2x^2 + 11x + 5) - (2x^2 + 11x + 1) = 4$ while subtracting in the other direction gives a negative. So we conclude that
$$\frac{2x^2+11x+1}{x+5} < 2x+1.$$

Example 2.2. Let us compare $r_1(x) = \frac{-x^2+3}{-2x+1}$ and $r_2(x) = \frac{2x+1}{2x-1}$. First, we multiply $r_1(x)$ by $-1/-1$ to rewrite it as $r_1(x) = \frac{x^2-3}{2x-1}$. Now we cross multiply and obtain

$$(2x-1)(x^2-3) = 2x^3 - x^2 - 6x + 4$$
$$(2x-1)(2x+1) = 4x^2 - 1$$

We see that $(2x^3 - x^2 - 6x + 4) - (4x^2 - 1) = 2x^3 - 5x^2 - 6x + 3$ has positive leading coefficient. So we conclude that

$$r_2(x) = \frac{2x+1}{2x-1} < r_1(x) = \frac{-x^2+3}{-2x+1}.$$

4. It is important to play with this ordering a bit to get used to it. Try to work through these examples in small groups. For each pair $r_1(x)$ and $r_2(x)$ determine whether $r_1(x) < r_2(x)$, $r_1(x) = r_2(x)$ or $r_2(x) < r_1(x)$.

   (a) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x+5}{3x^2+x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x+6}{3x^2+x-2}$.

   (b) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{3x^2+x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x+5}{3x^2+x-2}$.

   (c) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{3x-2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x^2+5}{3x^2-2}$.

   (d) $r_1(x) = \frac{p_1(x)}{q_1(x)} = \frac{x^2+5}{-3x+2}$ and $r_2(x) = \frac{p_2(x)}{q_2(x)} = \frac{x^2+5}{3x-2}$.

The purpose of this opening task is to give you a sense of how the field $\mathbb{Q}(x)$ works. The fact that these operations make it an ordered field is a theorem which is not difficult to prove, but is somewhat tedious. So here we simply state it without proof.

Theorem. The set of rational functions over the rationals,

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x) \text{ and } q(x) \text{ are polynomials with rational coefficients} \right\}.$$

together with the operations of addition $(+)$ and multiplication $(\cdot)$ defined above form a field. If we add in the ordering $<$ defined above, $\mathbb{Q}(x)$ is an ordered field.

We opened this inquiry by talking about integral elements. Here, in $\mathbb{Q}(x)$, the integral elements are of the form

$$q_n(x) = n$$

for $n \in \mathbb{N}$. (Notice that these are degree 0 polynomials, and are also rational functions since $q_n(x) = \frac{n}{1}$.) As your final task in this inquiry, prove the following facts:

5. Suppose that $r(x) = x^2 + 2x + 3$. Prove that $q_{500}(x) < r(x)$

6. Suppose that $r(x) = 2x + 3$. Prove that for any $n \in \mathbb{N}$, $q_n(x) < r(x)$.

7. Suppose that $r(x) = a_m x^m + a_{m-1} x_{m-1} + \cdots a_1 x + a_0$ is an element of $\mathbb{Q}(x)$ with $m \geq 1$. Prove that if $a_m > 0$, then $q_{500}(x) < r(x)$

8. Suppose that $r(x) = a_m x^m + a_{m-1} x_{m-1} + \cdots a_1 x + a_0$ is an element of $\mathbb{Q}(x)$ with $m \geq 1$. Prove that for any $n \in \mathbb{N}$, if $a_m > 0$, then $q_n(x) < r(x)$

You've just shown that in $\mathbb{Q}(x)$ every polynomial with degree at least 1 and with positive leading coefficient is greater than every integral element.

Inquiry: Minimums and Maximums

We haven't formally defined the notions of minimum and maximum elements of a set, but with some thought about the intended meanings of these words we can come up with the definitions ourselves.

- Consider the set $S = \{3, 4, 7, 8\}$.
  - What is the minimum of this set? Try to list two facts that make the number you chose the minimum of the set S.
  - Why isn't 2 the minimum of S? Why isn't 4 the minimum of this set?

  Try to write down the definition of the minimum of a set S. In a small group, discuss your definition and be prepared to share it with the class. Must every set have a minimum? Notice that we have talked about "the" minimum of a set, implying that the minimum of a set S is unique. Can sets have more than one minimum? Can you justify your answer?

  - What is the maximum of the set $S = \{3, 4, 7, 8\}$? Try to list two facts that make the number you chose the maximum of the set S.
  - Why isn't 9 the maximum of S? Why isn't 4 the maximum of this set?

  Try to write down the definition of the maximum of a set S. In a small group, discuss your definition and be prepared to share it with the class. Must every set have a maximum? Notice that we have talked about "the" maximum of a set, implying that the maximum of a set S is unique. Can sets have more than one maximum? Can you justify your answer?

  - Suppose that $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$. Based on your definition, what is the minimum and the maximum of $[2, 3)$?
  - Suppose that $(2, 3) = \{x \in \mathbb{R} \mid 2 < x < 3\}$. Based on your definition, what is the minimum and the maximum of $(2, 3)$?
  - Suppose that $(-\infty, 3) = \{x \in \mathbb{R} \mid x < 3\}$. Based on your definition, what is the minimum and the maximum of $(-\infty, 3)$?
  - Suppose that $[2, \infty) = \{x \in \mathbb{R} \mid 2 \leq x\}$. Based on your definition, what is the minimum and the maximum of $[2, \infty)$?

Inquiry: Upper and Lower Bounds

To illustrate the idea of upper and lower bounds, consider the following questions. Don't look up any answers on the internet, just give your best answers based on what you know of the world.

- Give an upper bound on the height of all humans that have ever lived.
- Give a lower bound on the mass of the earth.
- Give upper and lower bounds on the cost of raising a child.

Let's consider the first question for a moment. It is impossible to know how tall the tallest human ever was. Of course, we could look up the height of the tallest known human, but recorded history goes back only so far. And, even if the tallest person ever did in fact live during the period of recorded history, maybe nobody ever recorded his or her height or maybe the records were lost or forgotten. Even so, we can still give a reasonably good answer to the first question. For example, we can probably all agree that there was never a human who grew to be over 1 million feet tall. So, 1 million feet is an upper bound on the height of all humans that ever lived. Similarly, we can be confident that all humans that ever lived were less than 100 feet tall. So 100 feet is also an upper bound on the height of all humans that ever lived. What about 20 feet? 10 feet? The point is that an upper bound on a set just tells us that everything in the set is less than that value. With this discussion in mind, consider the other two questions and be prepared to discuss and share your answers.

- Consider the set $S = \{3, 5, 8, 10\}$.
  - Is 15 an upper bound of this set?
  - Is 11 an upper bound of this set?
  - Is 10 an upper bound of this set?
  - Is 9 an upper bound of this set?
- Consider the set $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - Is 5 an upper bound of this set?
  - Is 3 an upper bound of this set?
  - Is 3 the maximum of this set?
  - Is there an upper bound of this set less than 3?
- Try to complete the definition of an upper bound of a set: An element u is an upper bound of a set S if...
- Consider the set $S = \{3, 5, 8, 10\}$.
  - Is 0 a lower bound of this set?
  - Is 1 a lower bound of this set?
  - Is 3 a lower bound of this set?
  - Is 4 a lower bound of this set?
- Consider the set $[2, 3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - Is 0 a lower bound of this set?
  - Is 1.99 a lower bound of this set?
  - Is 2 a lower bound of this set?
  - Is there a lower bound of this set greater than 2?
- Try to complete the definition of a lower bound of a set: An element ℓ is a lower bound of a set S if...

Inquiry: Least Upper Bounds and Greatest Lower Bounds

- Some lower bounds are special in that they are the greatest lower bounds of a set. That is, not only are they lower bounds, but nothing larger is a lower bound.
- Similarly, some upper bounds are special in that they are the least upper bounds. That is, not only are they upper bounds, but nothing lower is an upper bound.

Let us explore these ideas in this inquiry.

- Consider the set $[2,3) = \{x \in \mathbb{R} \mid 2 \leq x < 3\}$.
  - What is the least upper bound of this set?
  - Why isn't 2.95 the least upper bound of this set? What about 2.999995?
  - The greatest lower bound of this set is 2. There are two ways to say this.
    - First, we could express this by writing a mathematical statement which says that no number larger than 2 is a lower bound of this set. That is, if $x > 2$, then x is not a lower bound of $[2,3)$. What does it mean to not be a lower bound of a set? Can you complete a mathematical sentence which says that no number larger than 2 is a lower bound of this set? It begins "If $x > 2$, then…"
    - Second, we could express this by saying that every lower bound of this set is less than or equal to 2. Can you write a mathematical statement which says this? Do you see why these two ways of expressing the idea of a greatest lower bound are the equivalent?
- Consider the set $[2,5] = \{x \in \mathbb{R} \mid 2 \leq x \leq 5\}$.
  - What is the greatest lower bound of this set?
  - Why isn't 2.05 the greatest lower bound? What about 2.000001?
  - The least upper bound of this set is 5. There are two ways to say this
    - First, we want to write a mathematical statement which says that no number less than 5 is an upper bound of this set. That is, if $x < 5$, then x is not a lower bound of $[2,5]$. What does it mean to not be a lower bound of a set? Can you complete a mathematical sentence which says that no number less than 5 is an upper bound of this set? It begins "If $x < 5$, then…"
    - Second, we could express this by saying that every upper bound of this set is greater than or equal to 5. Can you write a mathematical statement which says this? Do you see why these two ways of expressing the idea of a least upper bound are equivalent?

Inquiry: Completeness

Let us consider the set

$$S = \{r \in \mathbb{Q} \mid r > 0 \text{ and } r^2 < 2\}.$$

Note that

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766\ldots$$

where the decimal expansion goes on forever with no pattern. That's because $\sqrt{2}$ is irrational - it's not rational. Here we could write $\sqrt{2} \notin \mathbb{Q}$ to say that $\sqrt{2}$ is not a rational number. We could be more specific and say $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ to say that $\sqrt{2}$ is a real number but not a rational number.

Our goal here is to figure out the least upper bound of S. First, for each of the numbers r listed below, determine whether or not $r^2 < 2$.

- $r = 1.414$
- $r = 1.41421$
- $r = 1.4142135$
- $r = 1.414213562$
- $r = 1.41421356237$

Notice that each of the numbers r above satisfies $r < \sqrt{2}$. Now let's consider some numbers that are barely above $\sqrt{2}$ and determine whether or not they satisfy $r^2 < 2$.

- $r = 1.415$
- $r = 1.41422$
- $r = 1.4142136$
- $r = 1.414213563$
- $r = 1.41421356238$

Complete the following sentences:

- If r is less than $\sqrt{2}$, even barely, then r^2 ....
- If r is greater than $\sqrt{2}$, even barely, then r^2 is....

We now claim that $\sqrt{2}$ is an upper bound of S. Suppose $s \in S$ and $s > \sqrt{2}$

- There is an ordering axiom which implies that $s \cdot s > s \cdot \sqrt{2}$. Which is it?
- There is an ordering axiom which implies that $s \cdot \sqrt{2} > \sqrt{2} \cdot \sqrt{2}$. Which is it?
- There is an ordering axiom given the previous two facts implies that $s^2 > 2$. Which is it?

Thus, if $s \in S$, then we must have $s \leq \sqrt{2}$. So $\sqrt{2}$ is an upper bound of S.

We now claim that if $x < \sqrt{2}$, then x is not an upper bound of S. To see this, suppose that $x < \sqrt{2}$. Then there is some $\epsilon > 0$ so that $x + \epsilon = \sqrt{2}$. Notice that $x < x + \epsilon/2 < \sqrt{2}$. Fill in the blanks:

$$(x + \epsilon/2)^2 = \underline{\hspace{6cm}} < x^2 + 2x\epsilon + \epsilon^2 = (x + \epsilon)^2.$$

But $(x + \epsilon)^2 = \underline{\hspace{3cm}}$. Thus,

$$(x + \epsilon/2)^2 < 2.$$

Thus $x + \epsilon/2$ is in S but greater than x.

So, if $x < \sqrt{2}$, then x is not an upper bound of S. So, $\sqrt{2}$ is the least upper bound of S. The moral of the story is that S is a subset of $\mathbb{Q}$, but its least upper bound is not in $\mathbb{Q}$. That means that $\mathbb{Q}$ is not **complete** in the sense defined below.

Inquiry: Reviewing our Fields

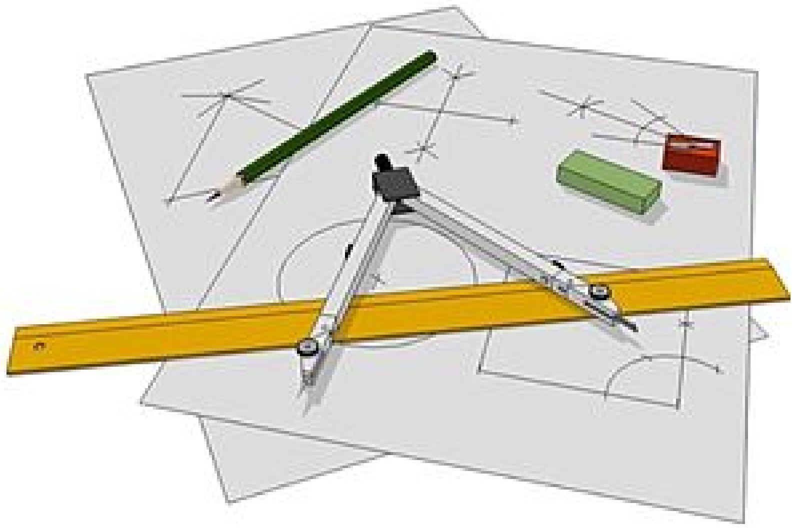
We have just seen that every complete ordered field is Archimedean. Try to answer as many of the following questions as you can without looking at your notes.

- Give two examples of fields that are not ordered fields.
- Define what it means for a field to be Archimedean.
- Give an example of an ordered field that is not Archimedean.
- Define what it means for a field to be complete.
- Give an example of an Archimedean field that is not complete.

The real numbers $\mathbb{R}$ is a complete ordered field. In fact, it is the unique complete ordered field.

Part III

Three Famous Problems about Constructible Numbers



Overview

Content

Compass and straightedge constructions are finite constructions of geometric figures using only a compass and an unmarked straightedge.

There are three **fundamental constructions** which are the only operations we assume we can do with our straightedge and compass.

A number a is said to be **constructible** if and only if a line segment of length $|a|$ can be constructed with compass and straightedge constructions.

We see that all rational numbers can be constructed, and the square root of any constructible number can be constructed.

A **Quadratic extension** of a field $\mathbb{F}$ is a field of the form $\mathbb{F}(\sqrt{k}) = \{a + b\sqrt{k} \mid a, b \in \mathbb{F}\}$ where $k \in \mathbb{F}$ and with appropriately defined operations.

Summary

In this part we study an interesting connection between the theory of fields and an ancient geometric technique for constructing geometric figures - compass and straightedge constructions. We review how to accomplish many common compass and straightedge constructions and then connect this topic to the theory of fields. Using this connection, we give negative answers to two of three famous problems considered by ancient Greek mathematicians.

Acknowledgements. The material in this part is inspired by the notes from courses we've taught. The textbooks used in those courses include

Hadlock, C. R. (1978). Field Theory and Its Classical Problems. Washington: Mathematical Association of America.

Madden, D. J. & Aubrey, J. A. (2017). An introduction to proof through real analysis (1st ed.). Hoboken, NJ: Wiley.

Peressini, A. L. (2003). Mathematics for high school teachers: An advanced perspective. Prentice Hall.

Materials. In this part there are 10 inquiries to print and give to students. They are

- Inquiry: Bisect a line segment
- Inquiry: Construct Angles of 30° and 60°
- Inquiry: Construct a line perpendicular to a given line
- Inquiry: Using Similar Triangles
- Inquiry: Remember the Equilateral Triangle
- Inquiry: Constructing Square Roots
- Inquiry: Am I Constructible?
- Inquiry: Side Length of a Cube
- Inquiry: $\sqrt[3]{2}$
- Inquiry: Remember Your Trig! (Or look it up.)

Now we turn to a collection of problems considered by ancient Greek mathematicians, namely compass-and-straightedge constructions. Here, by the word "compass" we do not mean the navigation tool! The way we're using the word, a compass is a drawing instrument that can be used for inscribing circles or arcs. By a "straightedge" we basically mean a ruler with no markings on it.

The ancient Greeks were curious about the following question which gave rise to these problems: Given a line segment of length 1 in the plane, for what values of a can we construct a line segment of length a using compass and straightedge constructions?

The Greeks discovered how to construct sums, differences, products, ratios, and square roots of given lengths. They could also construct half of a given angle; that is, they could "bisect" any angle. We will see these constructions below, and there are many other interesting constructions they figured out.

However, there were some constructions they could not figure out. For example, they could not figure out how to trisect an arbitrary angle. That is, they could not construct one third of a given angle except in some particular cases. They could not "square a circle." That is, they could not figure out how to construct a square with the same area as a given circle. Nor could they "double a cube." That is, they could not figure out how to construct the side of a cube whose volume would be twice the volume of a cube with a given side.

Constructions using compass and straightedge have a long history in Euclidean geometry, and they reflect the axioms of this system. However, the restriction to using only a straightedge and compass is very artificial – if one is actually interested in constructing a geometrical object for some practical purpose, then there is no reason to limit the tools to use. So why study this subject?

Our interest comes from two directions. First, the value of the constructions themselves lie in the rich supply of problems that can be posed in this way. And, with regard to the constructions as solutions to these problems the important thing is to be able to analyze a construction to see why it works. That is, like much of high-school geometry, the process of compass and straightedge construction is an application of logic.

But, our primary interest in these problems comes from the fact that it took the development of the theory of fields to solve these problems. It was not until the early 19th century that it was proved to be impossible to trisect an arbitrary angle or of double the volume of a cube using straightedge and compass constructions. Then in the late 19th century it was shown that π is a transcendental number, and so it is impossible to use straightedge and compass to construct a square with the same area as a given circle. These proofs are beautiful uses of the theory of fields, and we will explore them below.

The Rules of the Game

In the past, compasses were used in mathematics, drafting, navigation and other purposes. Physical compasses are usually made of metal or plastic, and consist of two parts connected by a hinge which can be adjusted to allow the changing of the radius of the circle drawn. Typically one part has a spike at its end, and the other part a pencil or pen. Unlike physical compasses, we will assume our compasses can be opened arbitrarily wide. We will also assume that our compasses have no markings on them to measure angles or anything else.

We will assume our straightedges are infinitely long, and has no markings on them. Our straightedges can only be used to draw a line segment between two points or to extend an existing segment.

Our constructions must be exact. Sometimes you'll be tempted to "eyeball" it by looking at the construction and guessing at its accuracy, or using some form of measurement, such as the units of measure on a ruler. This is forbidden, and getting close does not count as a solution.

Finally, compass and straightedge construction must have a finite number of steps.

The Fundamental Constructions

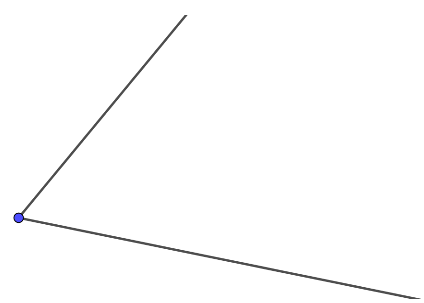
We begin with the three fundamental constructions with which we will build all of our more sophisticated constructions. We will assume that these are the only operations we can perform with our straightedge and compass.

Definition 2.3 (Fundamental Constructions)**.** The following compass and straightedge constructions are known as our three fundamental constructions.

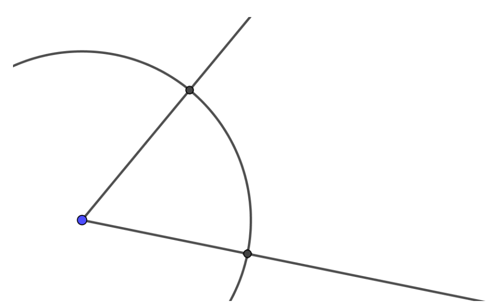
1. Given two points, we may draw a line through them, extending it indefinitely in each direction.
2. Given two points, we may draw the line segment connecting them.
3. Given a point and line segment, we may draw a circle with center at the point and radius equal to the length of the line segment.

We now build up some important basic constructions using the fundamental constructions.

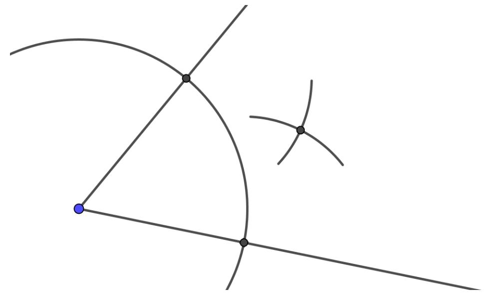
Example 2.4. Using the fundamental constructions, we can bisect any angle.



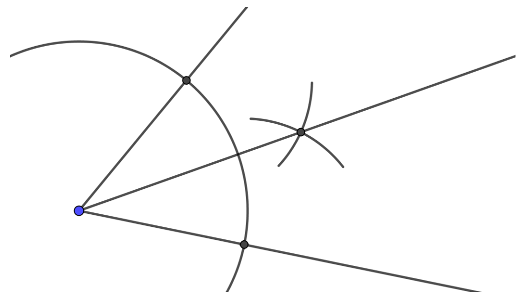
1. We begin by opening the compass to some fixed length, putting the point of the compass at the vertex of the angle, and drawing an arc through the two rays formed by the angle. This determines a point on each ray, equidistant from the vertex. (This uses the third fundamental construction.)



2. Keeping the compass at a fixed length (but perhaps different from the length in step 1), we put the compass point at each point formed in step 1 and create two intersecting arcs between the two rays of our angle. (This uses the third fundamental construction.)



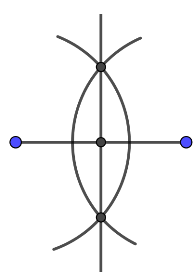
3. Using the straightedge, we draw a ray from the vertex of the angle through the point of intersection of the two arcs. This ray bisects the given angle. (This uses the first fundamental construction.)



Inquiry: Bisect a line segment

To bisect a line segment, follow these steps:

1. Open the compass to a width greater than half the length of the segment.
2. Place the point of the compass at one endpoint of the segment and draw an arc which passes over the midpoint of the segment.
3. Do the same with the point at the other endpoint of the segment without changing the width of the compass.
4. The arcs should intersect at two points. The line between those points bisects the segment.



Inquiry: Construct Angles of 30° and 60°

Follow these steps to construct angles of 30° and 60° using only the fundamental constructions.

1. Using the straightedge, draw a line segment of some length.
2. Open the compass to the length of this segment.
3. Put the point of the compass at one end of the segment and draw an arc over the center of the line

segment.

4. Put the point of the compass at the other end of the segment and draw another arc over the center of the line segment.
5. Draw a line segment from one end of the original segment to the intersection point of the two arcs.

The angle between the two line segments is 60° because the triangle formed by the two endpoints of the original line segment and the intersection point of the arcs is an equilateral triangle. You can now construct a 30° angle by bisecting the 60° angle.

Inquiry: Construct a line perpendicular to a given line

Using the fundamental constructions, we can draw a line perpendicular to a given line at a point on the line. Suppose we begin with a line (or line segment) and a point P on the line.

1. Place your compass point on P and swing an arc of any size below the line that crosses the line twice. You should draw at least a semicircle.
2. Open the compass wider.
3. Place the compass point where the arc crossed the line on one side and make a small arc above the line (the arc could be below the line if you prefer).
4. Without changing the width of the compass, place the compass point where the first arc crossed the line on the other side and make another arc. Your two small arcs should be intersecting.
5. Using a straightedge, connect the intersection of the two small arcs to point P.

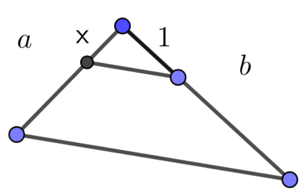
Constructible Lengths

Now we consider the question of what segment lengths we can construct, assuming we have in hand some given segment lengths. The next lemma essentially says that all rational numbers are constructible.

Before stating and proving an important lemma, we have an inquiry which asks you to do a crucial step in one part of the proof.

Inquiry: Using Similar Triangles

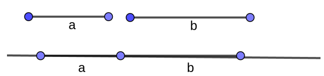
Consider the following image:



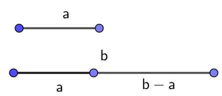
Here one long side of the triangle has length a, and a portion of that side is marked as having length x. Another side of the triangle has length b, and a portion of that side is marked as having length 1. Use similar triangles to prove that in this situation $x = \frac{a}{b}$.

Theorem 2.5. *Given segments of length 1, a and b, it is possible to construct segments of lengths $a+b$, $b-a$ (when $b>a$), ab, and a/b.*

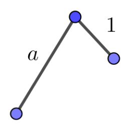
Proof. It is easy to see how to construct segments of length $a+b$ and $a-b$, given segments of length a and b. For the sum, suppose we have segments of length a and b. Extend the segment of length a to a line using the first fundamental construction. Then set the compass with to the length b, put the point at one endpoint of the segment of length a, and use the compass mark a point on the line a distance of b from that endpoint in the direction opposite the other endpoint. The resulting long segment has length $a+b$.



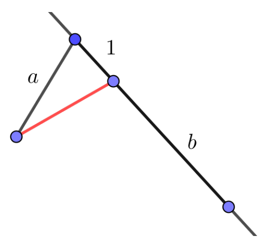
For $b-a$ when $b>a$, draw a segment of length a starting from one endpoint of the segment of length b, toward the other endpoint of that segment. What's left is a segment of length $b-a$.

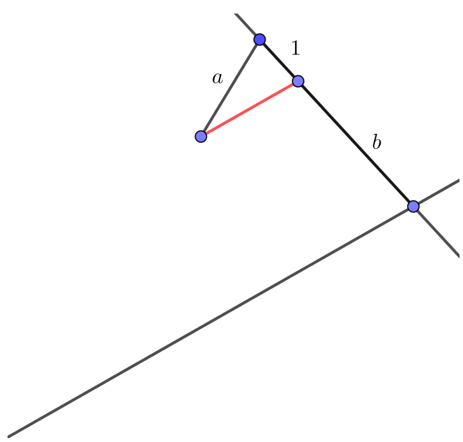


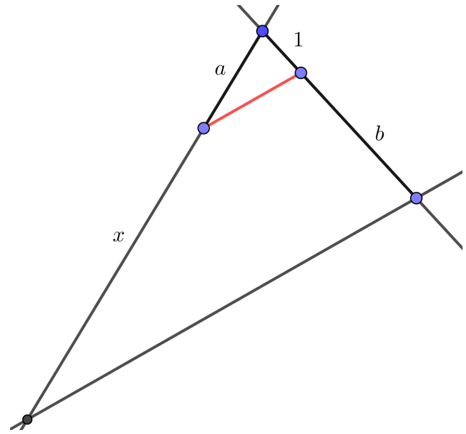
Now suppose that we have given segments of lengths a and b. How can we construct a segment of length ab? This is a little more involved and we will do it in multiple steps. First, we pick a point and draw segments of length a and length 1 emanating from that point with some angle θ between them with $0^\circ < \theta < 180^\circ$



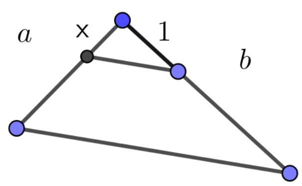
Now we construct two segments. One segment joints the two non-overlapping endpoints of our segment of length a and our segment of length 1. We also construct a segment of length b adjacent to the segment of length 1 and along the line determined by that segment.







This shows that a segment of length of ab can be constructed if we are given segments of length a, b, and 1. As for a/b, the diagram in the inquiry above shows that a segment of length a/b can be constructed.



To complete the proof for this case, you must tie up two loose ends. First, explain how to construct this diagram using the fundamental constructions. Second, the diagram assumes that $b > 1$. We need a diagram which handles the case $b \leq 1$. The tying of these loose ends is left to the reader. $\square$

Definition 2.6. A real number a is constructible if given initially a segment of length 1, it is possible to construct a segment of length $|a|$.

This definition together with Theorem 2.5 allow us to conclude that every rational number is constructible. How is this? It should be clear that if 1 is constructible, then so is 2. After all, it's just $1 + 1$, and if we can construct a segment of length one, then we can construct two of them side-by-side. And similarly, we can construct segments with the length of any positive natural number. This allows us to conclude that every integer is constructible. But ratios of constructible numbers are constructible, so we get that every rational number is constructible. So, we have that if 1 is constructible, then so is every natural number. Well, is 1 a constructible length? Well, recall that our straightedge is not a ruler - there are no distance marks. In particular, the length of "1" is not given to us. We are allowed to decide it. That is, we are free to just create a segment of any length and call it our unit length. Then every other length is measured relative to that. We can create a segment twice its length, three times its length,etc. Then we're off to the races and we can construct any rational length.

This is all great, but suggests the question of whether or not some non-rational numbers are constructible. The answer is yes, but remember that to show that a particular non-rational number is constructible, we have to construct a line segment of that length.

Inquiry: Remember the Equilateral Triangle

Above we showed that we could construct angles of $60°$ and $30°$. We did that by

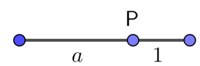
1. constructing an equilateral triangle, and then
2. bisecting one of the angles in the triangle.

Can we generate any constructible numbers from this construction? Use your knowledge of compass and straightedge constructions and trigonometry to find some non-rational constructible numbers from this example.

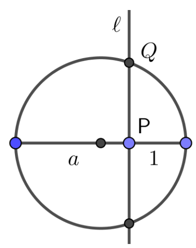
INQUIRY: CONSTRUCTING SQUARE ROOTS

Here we suppose that we are given segments of length 1 and length a, and we construct a segment of length $\sqrt{a}$. To do so, follow these steps.

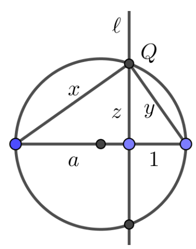
1. Construct a segment of length $a+1$. Let P be the point at which the segment of length a meets the segment of length 1.



2. At point P construct a line ℓ perpendicular to the segment.
3. Bisect the segment, and mark its midpoint.
4. Place the point of the compass at the midpoint of the segment, open the compass to half the width of the segment, and draw a circle centered at the midpoint whose radius is half the length of the segment.
5. The circle intersects the perpendicular bisector ℓ at two points. Pick one of them to work with, and let's call it Q.



6. Let x be the distance from one endpoint of the segment to Q and let y be the distance from Q to the other endpoint. Let z be the distance from the segment to Q.



7. There are three right triangles in your picture. Write down the corresponding three instances of the Pythagorean theorem.
8. Use these equations to show that $z = \sqrt{a}$.

Lemma 2.7. *Given segments of length* 1 *and* a*, a segment of length* $\sqrt{a}$ *may be constructed.*

Proof. You've just proved this in the inquiry above. □

Notice that, for example, since 1 is constructible, so is 2. This theorem allows us to conclude that $\sqrt{2}$ is constructible. So is $\sqrt{3}$, $\sqrt{5}$, etc.

Inquiry: Am I Constructible?

In this task, you will be able to conclude based on our previous work that some of the numbers listed are constructible. For others, you will not be able to draw that conclusion. Below, we'll present a theorem that exactly characterizes which numbers are constructible and which are not. Right now you can only identify if something is constructible. Showing something is not constructible is harder. For each number listed below indicate "Constructible" or "No Conclusion." If the number is constructible, explain how you would construct it given our previous constructions.

- 100
- $\frac{3}{2}$
- $\sqrt{7}$
- $\sqrt[3]{7}$
- $3+\sqrt{7}$
- $\sqrt{3+\sqrt{7}}$
- $\sqrt{\sqrt{\sqrt{3+\sqrt{7}}}}$
- $\sqrt[4]{3+\sqrt{7}}$
- $\sqrt[5]{3+\sqrt{7}}$
- $\sqrt[6]{3+\sqrt{7}}$
- $\sqrt[7]{3+\sqrt{7}}$
- $\sqrt[n]{3+\sqrt{7}}$, where n is even.
- $\sqrt[n]{3+\sqrt{7}}$, where n is odd.
- π

Quadratic Extensions

Now we come to the important relationship between constructible numbers and field extensions, particularly so called "quadratic extensions." First, recall the following result.

Theorem 2.8. *If $\mathbb{F}$ is a field, then so is $\mathbb{F}(\sqrt{k})$.*

Proof. This was proved as homework problem 3 above. □

First we give a sufficient condition for a number to be constructible.

Theorem 2.9. *A number a is constructible if there is a finite sequence of fields $\mathbb{Q}=\mathbb{F}_0\subseteq\mathbb{F}_1\subseteq\ldots\subseteq\mathbb{F}_N$ with $a\in\mathbb{F}_N$ and such that for each j, $0\leq j\leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$.*

Proof. The proof of this is by induction on N. First consider the case where $N=0$. Since $\mathbb{F}_0=\mathbb{Q}$ and we know that every rational number is constructible, it follows that the theorem is true for $N=0$. Suppose now that we have a sequence of field extensions

$$\mathbb{Q}=\mathbb{F}_0\subseteq\mathbb{F}_1\subseteq\ldots\subseteq\mathbb{F}_N$$

and that each j, $0\leq j\leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$. Suppose further that every number in $\mathbb{F}_N$ is constructible, and that $\mathbb{F}_{N+1}$ is a quadratic extension of $\mathbb{F}_N$. In particular, let us suppose that $F_{N+1}=\mathbb{F}_N(\sqrt{k})$ where $k\in\mathbb{F}_N$. Suppose that $a\in\mathbb{F}_{N+1}$. Then

$$a=b+c\sqrt{k}$$

where $b,c\in\mathbb{F}_N$. By hypothesis, every number in $\mathbb{F}_N$ is constructible. So b and c are constructible. Since $k\in\mathbb{F}_N$ it is also constructible. We know that we can construct the square root of any constructible number, so it follows that $\sqrt{k}$ is constructible. Products of constructible numbers are constructible, so $c\sqrt{k}$ is constructible. Sums of constructible numbers are constructible, so

$$a = b + c\sqrt{k}$$

is constructible. It follows then that every number in $\mathbb{F}_{N+1}$ is constructible.

Therefore it follows by induction that for any N if there is a finite sequence of fields $\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \ldots \subseteq \mathbb{F}_N$ such that for each j, $0 \leq j \leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$ and $a \in \mathbb{F}_N$, then a is constructible. □

Definition 2.10. If $\mathbb{F}$ is a field, the plane of $\mathbb{F}$ will denote the set of all points (x, y) in the Cartesian plane so that x and y are in $\mathbb{F}$. By a line in $\mathbb{F}$ we mean a line passing through two points in the plane of $\mathbb{F}$. By a circle in $\mathbb{F}$ we mean a circle with both its center and some point on its circumference in the plane of $\mathbb{F}$.

Note that any fundamental construction using only points in the plane of a field $\mathbb{F}$ involves the construction of a line, line segment, or a circle in $\mathbb{F}$. To see this, recall the three fundamental constructions:

1. Given two points, we may draw a line through them, extending it indefinitely in each direction.
2. Given two points, we may draw the line segment connecting them.
3. Given a point and line segment, we may draw a circle with center at the point and radius equal to the length of the line segment.

If we are using only points in the plane of $\mathbb{F}$, then, in particular, the two points we start with in the first two fundamental constructions must be points in the plane of $\mathbb{F}$. Thus the line or line segment constructed is in $\mathbb{F}$, as defined above. What about the third fundamental construction? In this case, the center of the circle (x, y) must be in the plane of $\mathbb{F}$ and the length of the line segment r that determines the radius must be in $\mathbb{F}$. But then the point $(x + r, y)$ is on the circle and is also in the plane of $\mathbb{F}$.

Lemma 2.11. *Every line in $\mathbb{F}$ can be represented by an equation of the form $ax + by + c = 0$ with $a, b, c \in \mathbb{F}$*

Proof. Suppose that (x_1, y_1) and (x_2, y_2) are points on a line in $\mathbb{F}$ and that these two points are in the plane of $\mathbb{F}$ so that x_1, x_2, y_1, y_2 are all in $\mathbb{F}$. Then, if the line is not vertical, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

That number m is in $\mathbb{F}$ if all of x_1, x_2, y_1, and y_2 are. The point-slope equation of the line is then

$$y - y_1 = m(x - x_1).$$

We can rewrite this as $ax + by + c = 0$ where $a = m$, $b = -1$, and $c = y_1 - x_1$, and these numbers a, b, c are all in $\mathbb{F}$ if x_1, x_2, y_1, and y_2 are. If the line is vertical, then both of the points are of the form (c, y_1), (c, y_2) for some fixed c in the x-coordinate and the equation of the line is $x = c$, or, if you like, $x - c = 0$. □

Lemma 2.12. *Every circle in $\mathbb{F}$ can be represented by an equation of the form $x^2 + y^2 + ax + by + c = 0$ with $a, b, c \in \mathbb{F}$.*

Proof. The proof of this is the same idea as the proof of Lemma 2.11. We start with the center of the circle (x_0, y_0) which is assumed to be in the plane of $\mathbb{F}$ and a point on the circumference of the circle (x_1, y_1) which is also assumed to be in the plane of $\mathbb{F}$. Then the radius of the circle is

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

which is in not necessarily in $\mathbb{F}$, but r^2 definitely is. The equation of the circle is then

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Expanding the terms on the left hand side and rearranging can clearly get us an equation of the form $x^2 + y^2 + ax + by + c = 0$ with $a, b, c \in \mathbb{F}$. □

Theorem 2.13.

1. *The point of intersection of two distinct, nonparallel lines in $\mathbb{F}$ is in the plane of $\mathbb{F}$.*
2. *The points of intersection of a line in $\mathbb{F}$ and a circle in $\mathbb{F}$ are either in the plane of $\mathbb{F}$ or in the plane of some quadratic extension of $\mathbb{F}$.*

3. *The points of intersection of two circles in $\mathbb{F}$ are either in the plane of $\mathbb{F}$ or in the plane of some quadratic extension of $\mathbb{F}$.*

Proof. You can probably imagine how this is proved. For item (1) we start with two distinct, nonparallel lines. As above, these have equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

$a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{F}$. Now we solve for the intersection point. The lines must have exactly one intersection point because they are distinct, nonparallel lines, and that point will have coordinates which use only sums, differences, products and quotients of the coefficients. Since the coefficients are in $\mathbb{F}$ and $\mathbb{F}$ is a field, it follows that each of the coordinates of the intersection point will also be in $\mathbb{F}$.

The situation is similar for item (2), except now we have a line and a circle so our equations look like

$$a_1x + b_1y + c_1 = 0$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

with $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{F}$. If we simultaneously solve these two equations we'll use all $(+, -, \cdot, /)$ of our arithmetic operations but we will also likely have to do a square root. (You can either imagine this or really try it!) If no square roots are needed, then our intersection point(s) are in the plane of $\mathbb{F}$. If we do need to take a square root, then our intersection points will be in the plane of a quadratic extension of $\mathbb{F}$.

In case (3) we now have two circles and our equations look like

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

with $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{F}$. It's not so bad to simultaneously solve these two equations. We start by subtracting the second equation from the first and we get that

$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) = 0.$$

So,

$$y = \frac{c_2 - c_1}{a_1 - a_2} + \frac{b_2 - b_1}{c_1 - c_2}x.$$

Now we plug this into one of our original equations to get a quadratic in x. We can use the quadratic formula to find either zero, one or two values of x. That means the circles intersect in zero points, one point, or two points. After we find the x values we determine the y values. Again, to find the x-values we just take one square root. As a result, our intersection points will either be in the plane of $\mathbb{F}$ or the plane of a quadratic extension of $\mathbb{F}$. □

Theorem 2.14. *The following statements are equivalent:*

1. *The number a is constructible.*
2. *There is a finite sequence of fields $\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \ldots \subseteq \mathbb{F}_N$ with $a \in \mathbb{F}_N$ and such that for each j, $0 \leq j \leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$.*

Proof. We've already proved that (2) implies (1) above. And, we've done most of the hard work for (1) implies (2). To see that (1) implies (2), suppose that a is constructible. That means there was some finite sequence of fundamental constructions that resulted in the construction of a segment of length a. We start with a segment of length 1 whose endpoints we assume to have coordinates $(0, 0)$ and $(1, 0)$. After the first fundamental construction we produce a line segment, line, or circle in the plane of $\mathbb{Q}$ or in the plane of some quadratic extension of $\mathbb{Q}$. At each stage we produce new points by intersecting our existing constructions with our new constructed objects. These intersection points are either in the plane of the field we are working over, or in the plane of some quadratic extension of that field. Since our construction of a segment of length a must terminate after a finite number of steps, it follows that if a is constructible, then there is a finite sequence of fields $\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \ldots \subseteq \mathbb{F}_N$ with $a \in \mathbb{F}_N$ and such that for each j, $0 \leq j \leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$. □

3 Three Famous Problems

Doubling a Cube

The question we want to answer in this section is the following: Given a line segment representing the edge of a cube, is it possible to construct another line segment representing the edge of a cube with exactly twice the volume of the first cube? Let us explore this question a bit in the following inquiry.

INQUIRY: SIDE LENGTH OF A CUBE

- Suppose that a cube, which we'll call Cube 1, has side length s, what is its volume?
- Suppose that a second cube, which we'll call Cube 2, twice the volume of Cube 1. Write an expression for the volume of Cube 2.
- Write an expression for the side length of Cube 2, simplified as much as possible.
- Supposing s is a constructible number, under other number must be constructible for the volume of Cube 2 to be constructible.

So, to "double a cube" means the following: We begin with a cube whose side length s we take to be a constructible number s. Of course, then the volume of the cube is $V = s^3$. When we are asked to "double" that cube, we are being asked to use compass and straightedge constructions to construct a line segment of some length ℓ, so that $\ell^3 = 2V$, which in this case would mean $\ell^3 = 2s^3$. Of course, we can solve for ℓ and we see that we want to construct a line segment of length $\sqrt[3]{2}s$. We know that s is constructible, so we may conclude that we can double our cube if and only if $\{sqrt[3]2$ is constructible.

INQUIRY: $\sqrt[3]{2}$

Let $\mathbb{F}$ be a field, and suppose that $k \in \mathbb{F}$. Also suppose that

$$\sqrt[3]{2} \in \mathbb{F}(\sqrt{k})$$

so that $\sqrt[3]{2} = a + b\sqrt{k}$ with $a, b \in \mathbb{F}$.

- Now cube both sides of the equation above.
- One side of the expression should just be "2." The other side will have four terms, including a term with $\sqrt{k}$ (or $k^{1/2}$) and a term with $k^{3/2}$. Group those two terms together and factor out a $k^{1/2}$ to obtain an equation of the form
$$2 = c + d\sqrt{k}.$$
- Why must the coefficient of $\sqrt{k}$ be zero?
- Explain why this implies that $b = 0$, so that $\sqrt[3]{2} = a$ for $a \in \mathbb{F}$.

Notice that this work shows that if

$$\sqrt[3]{2} \in \mathbb{F}(\sqrt{k})$$

then in fact

$$\sqrt[3]{2} \in \mathbb{F}.$$

In the previous inquiry you've proved:

Theorem 3.1. *Let $\mathbb{F}(\sqrt{k})$ be a quadratic extension of a field $\mathbb{F}$. If $\sqrt[3]{2}$ is in $\mathbb{F}(\sqrt{k})$, then $\sqrt[3]{2}$ must be in $\mathbb{F}$ itself.*

Theorem 3.2. *It is impossible to double the cube.*

Proof. Suppose that it were possible to double the cube. Then, as we saw above, $\sqrt[3]{2}$ would be a constructible number. By Theorem 2.14, there is a finite sequence of fields $\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \ldots \subseteq \mathbb{F}_N$ with $\sqrt[3]{2} \in \mathbb{F}_N$ and such that for each j, $0 \leq j \leq N-1$, $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$. But now we repeatedly apply Theorem 3.1: Since $\sqrt[3]{2} \in \mathbb{F}_N$ and $\mathbb{F}_N = \mathbb{F}_{N-1}(\sqrt{k})$ for some $k \in \mathbb{F}_{N-1}$, it follows by Theorem 3.1 that $\sqrt[3]{2} \in \mathbb{F}_{N-1}$. But $\mathbb{F}_{N-1} = \mathbb{F}_{N-1}(\sqrt{k})$ for some $k \in \mathbb{F}_{N-2}$. So again Theorem 3.1 implies that $\sqrt[3]{2} \in \mathbb{F}_{N-2}$. And so on until we arrive at the conclusion that $\sqrt[3]{2} \in \mathbb{Q}$. But $\sqrt[3]{2} \notin \mathbb{Q}$ - it is an irrational number. □

Trisecting an Angle

We saw above that it is possible to bisect any angle using compass and straightedge constructions. And, it must be admitted that it is possible to trisect *some* angles. For example, we know that we can construct angles of 30° and 60°. It follows that it is indeed possible to trisect angles with measure $3(30°) = 90°$ and $3(60°) = 180°$. But the question here is whether or not there is a general method for trisecting an arbitrary angle? Above we demonstrated a general method for bisecting an arbitrary angle, and that's what we want here, but for trisecting.

It turns out that this is impossible - there is no general method for trisecting an arbitrary angle. To demonstrate this fact, we must find an angle for which we can show that it is impossible trisect that angle with compass and straightedge constructions.

Here we will show that it is impossible to trisect an angle of 60°. Note that if this were possible, then it would be possible to construct a 20° angle. Of course, if we could construct an angle of 20°, then we would be able to construct a segment of length $\cos(20°)$. That is, if it is possible to trisect an angle of 60° with compass and straightedge constructions, then it would be possible to construct a segment of length $\cos(20°)$ with compass and straightedge constructions, meaning that $\cos(20°)$ would be a constructible number.

So, to show that it is impossible to trisect an angle of 60° it suffices to show that $\cos(20°)$ is not constructible. For this we will need some trigonometry.

Inquiry: Remember Your Trig! (Or look it up.)

Our goal in this inquiry is to find a nice polynomial which has as one of its roots the value $\cos(20°)$. We begin by considering

$$\cos(3\theta).$$

Even though we really want to find out something about $\cos(20°)$, we begin by considering $\cos(3\theta)$ because $3(20°) = 60°$ and we know the cosine of 60°. So, we'll use trig identities to write $\cos(3\theta)$ in terms of $\cos(\theta)$.

First let us assemble our arsenal:

A. Write down the most well know trig identity - the one that hopefully know one in class needs to look up. You know the *one*.

B. There are a few well-known version of the identify for $\cos(2\theta)$. We want the one that looks a bit like the previous identity, but with a minus sign. (And be sure you have the order of the subtraction correct!)

C. Now write down the most common identity for $\sin(2\theta)$.

D. Write down the identify for $\cos(\alpha + \beta)$.

Now fill in the following steps:

1. By the sum formula for cosine (identity D above):

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) \\ &= \underline{\hspace{10cm}}\end{aligned}$$

2. In the formula above we see an instance of $\cos(2\theta)$ and an instance of $\sin(2\theta)$. Replacing these with their

equivalent expressions (identities B and C above) we can write:

$\cos(3\theta) =$ ____

3. Simplifying the previous expression we obtain

$$\cos(3\theta) = \cos^{\square}\theta - \square\sin^2\theta\cos\theta$$

4. Now use identity A above to replace $\sin^2\theta$ with an expression involving $\cos^2\theta$. Doing so, we obtain

$\cos(3\theta) =$ ____

5. Simplify the previous expression and fill in the boxes:

$$\cos(3\theta) = \square\cos^3\theta - \square\cos\theta$$

6. Now if we set $\theta = 20^\circ$, then $\cos(3\theta) = \cos(60^\circ) =$ ________________. Rewrite the previous equation but substitute 20° for θ. Since you know the value of $\cos(60^\circ)$, use that value. Just leave $\cos(20^\circ)$ as it is - don't try to replace that expression with a decimal value. Write the new equation here:

7. Now let $u = \cos(20^\circ)$ and rewrite the previous equation here:

8. Now rewrite the previous equation as a polynomial in u set equal to zero:

$$\square u^{\square} - 6u - \square = 0. \tag{1}$$

Notice that this work implies that $\cos(20^\circ)$ is a root of the polynomial on the left-hand side of Equation 1.

In the inquiry above we showed that $u = \cos(20^\circ)$ must be a solution to the equation

$$8u^3 - 6u - 1 = 0$$

Notice that this equation can be written

$$(2u)^3 - 3(2u) - 1 = 0.$$

Therefore, if we take $x = 2u$ we have the polynomial

$$x^3 - 3x - 1 = 0.$$

We have the following theorem about this polynomial.

Theorem 3.3. *Suppose that $k \in \mathbb{F}$ and that the field $\mathbb{F}(\sqrt{k})$ contains a root of $x^3 - 3x - 1 = 0$, then so does $\mathbb{F}$.*

Proof. Suppose that $a + b\sqrt{k} \in \mathbb{F}(\sqrt{k})$ and that $a + b\sqrt{k}$ is a root of the polynomial $x^3 - 3x - 1$. We want to prove that there is some root of the polynomial in $\mathbb{F}$. So if we happened to get lucky and find a root with $b = 0$ then we would be done. So let us suppose that $a + b\sqrt{k} \in \mathbb{F}(\sqrt{k})$ and that $a + b\sqrt{k}$ is a root of the polynomial $x^3 - 3x - 1$ *and* that $b \neq 0$. Now we have some work to do. We are assuming the root we are given is *not* in $\mathbb{F}$ and we have to produce another root which is in $\mathbb{F}$. Well, we claim that in this case $-2a$ is also a root of the polynomial. Notice that since $a \in \mathbb{F}$, it follows that $-2a$ is also in $\mathbb{F}$. So, if we can prove that $-2a$ is also a root (in addition to $a + b\sqrt{k}$) then we will have proved our theorem. To prove this, let's plug in our known root to $x^3 - 3x - 1$. So

$$(a + b\sqrt{k})^3 - 3(a + b\sqrt{k}) - 1 = 0.$$

Let's simplify! We have

$$\begin{aligned}(a + b\sqrt{k})^3 &= a^3 + 3a^2b\sqrt{k} + 3ab^2k + b^3k^{3/2}\\ -3(a + b\sqrt{k}) &= -3a - 3b\sqrt{k}\end{aligned}$$

Thus $(a+b\sqrt{k})^3 - 3(a+b\sqrt{k}) - 1 = 0$ becomes

$$(a^3 + 3a^2b\sqrt{k} + 3ab^2k + b^3k^{3/2}) + (-3a - 3b\sqrt{k}) - 1 = 0.$$

On the left let's group together the terms with $\sqrt{k}$ and $k^{3/2}$:

$$(a^3 + 3ab^2k - 3a - 1) + (3a^2b\sqrt{k} + b^3k^{3/2} - 3b\sqrt{k}) = 0.$$

We can write this as

$$(a^3 + 3ab^2k - 3a - 1) + (3a^2b + b^3k - 3b)\sqrt{k} = 0.$$

For the expression on the left to equal zero we must have

$$3a^2b + b^3k - 3b = 0$$
$$a^3 + 3ab^2k - 3a - 1 = 0$$

Notice we can factor b out of $3a^2b + b^2k - 3b$ so that $b(3a^2 + b^2k - 3) = 3$. Since we are assuming that $b \neq 0$ we must have that $3a^2 - kb^2 - 3 = 0$. Now, notice that the only place b shows up in the second equation is in the term $3ab^2k$. We can use the equation $3a^2 - kb^2 - 3 = 0$ to write $kb^2 = 3 - 3a^2$ so that

$$a^3 + 3ab^2k - 3a - 1 = 0.$$

Becomes

$$a^3 + 3a(3 - 3a^2) - 3a - 1 = 0.$$

This simplifies to

$$-8a^3 + 6a - 1 = 0.$$

We can rewrite this as

$$(-2a)^3 - 3(-2a) - 1 = 0.$$

That is, $-2a$ is a solution to

$$x^3 - 3x - 1 = 0.$$

That's what we wanted to prove! Thus, we have established that whether or not $b = 0$, if there is a root of $x^2 - 3x - 1$ in the field $\mathbb{F}(\sqrt{k})$, then there is one in the field $\mathbb{F}$. □

Theorem 3.4. *It is not possible to trisect and arbitrary angle.*

Proof. If it were possible to trisect a 60° angle by compass and straightedge constructions, then that would imply that $\cos(20^\circ)$ is constructible, and therefore that a root of

$$x^3 - 3x - 1 = 0$$

is constructible. Being a constructible number, by Theorem 2.14 such a root would be in some field $\mathbb{F}_N$ where

$$\mathbb{Q} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \cdots \subseteq \mathbb{F}_N$$

where for $0 \le j \le N-1$ each $\mathbb{F}_{j+1}$ is a quadratic extension of $\mathbb{F}_j$. But, by repeated application of Theorem 3.3, we must have a root in $\mathbb{Q}$. So, suppose that $\frac{m}{n}$ is a rational number in lowest terms which is a solution to $x^3 - 3x - 1 = 0$. That is

$$\left(\frac{m}{n}\right)^3 - 3\left(\frac{m}{n}\right) - 1 = 0.$$

So of course

$$\frac{m^3}{n^3} - \frac{3m}{n} - 1 = 0.$$

Let us multiply both sides of this equation by n^3 to obtain

$$m^3 - 3mn^2 - n^3 = 0.$$

Now we can use this equation to solve for both m^3 and n^3:

$$m^3 = 3mn^2 - n^3 = n(3mn - n^2)$$
$$n^3 = m^3 - 3mn^2 = m(m^2 - 3n^2)$$

Now both m and n have prime factors. But the second equation above tells us that if a prime p divides m, then it divides n^3. And, if a prime divides n^3, then it must divide n. So, any prime that divides m must divide n. Similarly, the first equation tells us that if a prime q divides n, then it divides m^3 and therefore m. So any prime that divides n must divide m. That is, m and n have exactly the same prime factors! Since we are assuming that m/n is in lowest terms, that means that m and n be ± 1. But, neither 1 nor -1 is a solution to $x^3 - 3x - 1 = 0$. So, we have a contradiction, and we conclude there is no rational root to this equation. Therefore there is no constructible root to this equation. Thus, $2\cos(20°)$, which is a root of this equation, is not constructible. So $\cos(20°)$ is not constructible. □

Squaring a Circle

The "third" famous Greek problem is to construct a square that has the same area as the unit circle. Of course, the unit circle has radius $r = 1$ and therefore area

$$A = \pi r^2 = \pi(1)^2 = \pi.$$

When we say "construct a square," what we really mean is that we want to construct a line segment whose length squared is the desired area. So, in this case what we really want to do is to construct a line segment of length $\sqrt{\pi}$. If we could construct $\sqrt{\pi}$, then we could construct π. But, it turns out that π is a so-called "transcendental number."

An "algebraic" number is a real or complex number that is the root of a nonzero polynomial equation with integer (or, equivalently, rational) coefficients. A number is transcendental if it is not algebraic. The numbers π and e are both transcendental. It is beyond the scope of these notes to show that π is transcendental, as is the fact that transcendental numbers are not constructible. These proofs lead us back to some beautiful mathematics and the reader is encouraged to explore further.

Homework

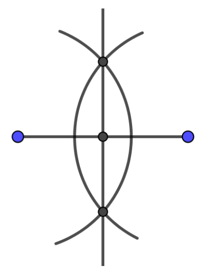
1. Construct a line parallel to a given line using compass and straightedge constructions.
2. Consider the number $r = \sqrt[4]{13} + \frac{4}{3}\sqrt{\sqrt{6} + \sqrt{1 + 2\sqrt{7}}}$. Give an explicit sequence of fields $\mathbb{F}_0 = \mathbb{Q} \subseteq \mathbb{F}_1 \subseteq \mathbb{F}_2 \subseteq \ldots \subseteq \mathbb{F}_N$ such that $r \in \mathbb{F}_N$ and for $1 \leq j \leq N - 1$ each $\mathbb{F}_{j+1}$ is a quadratic extension of its predecessor $\mathbb{F}_j$.
3. Can the cube be "tripled"?
4. It is clearly possible to divide an arbitrary angle into four equal parts by repeated bisection. Show how this may also be deduced algebraically from the equation relating $\cos(4\theta)$ and $\cos\theta$.

In-Class Resources

Inquiry: Bisect a line segment

To bisect a line segment, follow these steps:

1. Open the compass to a width greater than half the length of the segment.
2. Place the point of the compass at one endpoint of the segment and draw an arc which passes over the midpoint of the segment.
3. Do the same with the point at the other endpoint of the segment without changing the width of the compass.
4. The arcs should intersect at two points. The line between those points bisects the segment.



INQUIRY: CONSTRUCT ANGLES OF 30° AND 60°

Follow these steps to construct angles of 30° and 60° using only the fundamental constructions.

1. Using the straightedge, draw a line segment of some length.
2. Open the compass to the length of this segment.
3. Put the point of the compass at one end of the segment and draw an arc over the center of the line segment.
4. Put the point of the compass at the other end of the segment and draw another arc over the center of the line segment.
5. Draw a line segment from one end of the original segment to the intersection point of the two arcs.

The angle between the two line segments is 60° because the triangle formed by the two endpoints of the original line segment and the intersection point of the arcs is an equilateral triangle. You can now construct a 30° angle by bisecting the 60° angle.

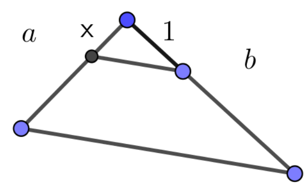
INQUIRY: CONSTRUCT A LINE PERPENDICULAR TO A GIVEN LINE

Using the fundamental constructions, we can draw a line perpendicular to a given line at a point on the line. Suppose we begin with a line (or line segment) and a point P on the line.

1. Place your compass point on P and swing an arc of any size below the line that crosses the line twice. You should draw at least a semicircle.
2. Open the compass wider.
3. Place the compass point where the arc crossed the line on one side and make a small arc above the line (the arc could be below the line if you prefer).
4. Without changing the width of the compass, place the compass point where the first arc crossed the line on the other side and make another arc. Your two small arcs should be intersecting.
5. Using a straightedge, connect the intersection of the two small arcs to point P.

INQUIRY: USING SIMILAR TRIANGLES

Consider the following image:



Here one long side of the triangle has length a, and a portion of that side is marked as having length x. Another side of the triangle has length b, and a portion of that side is marked as having length 1. Use similar triangles to prove that in this situation $x = \frac{a}{b}$.

Inquiry: Remember the Equilateral Triangle

Above we showed that we could construct angles of 60° and 30°. We did that by

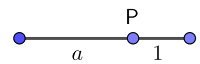
1. constructing an equilateral triangle, and then
2. bisecting one of the angles in the triangle.

Can we generate any constructible numbers from this construction? Use your knowledge of compass and straightedge constructions and trigonometry to find some non-rational constructible numbers from this example.

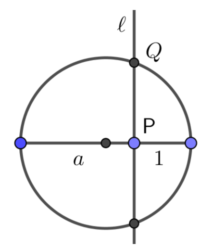
INQUIRY: CONSTRUCTING SQUARE ROOTS

Here we suppose that we are given segments of length 1 and length a, and we construct a segment of length $\sqrt{a}$. To do so, follow these steps.

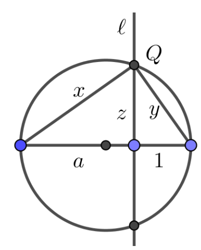
1. Construct a segment of length $a+1$. Let P be the point at which the segment of length a meets the segment of length 1.



2. At point P construct a line ℓ perpendicular to the segment.
3. Bisect the segment, and mark its midpoint.
4. Place the point of the compass at the midpoint of the segment, open the compass to half the width of the segment, and draw a circle centered at the midpoint whose radius is half the length of the segment.
5. The circle intersects the perpendicular bisector ℓ at two points. Pick one of them to work with, and let's call it Q.



6. Let x be the distance from one endpoint of the segment to Q and let y be the distance from Q to the other endpoint. Let z be the distance from the segment to Q.



7. There are three right triangles in your picture. Write down the corresponding three instances of the Pythagorean theorem.
8. Use these equations to show that $z = \sqrt{a}$.

Inquiry: Am I Constructible?

In this task, you will be able to conclude based on our previous work that some of the numbers listed are constructible. For others, you will not be able to draw that conclusion. Below, we'll present a theorem that exactly characterizes which numbers are constructible and which are not. Right now you can only identify if something is constructible. Showing something is not constructible is harder. For each number listed below indicate "Constructible" or "No Conclusion." If the number is constructible, explain how you would construct it given our previous constructions.

- 100
- $\frac{3}{2}$
- $\sqrt{7}$
- $\sqrt[3]{7}$
- $3+\sqrt{7}$
- $\sqrt{3+\sqrt{7}}$
- $\sqrt{\sqrt{\sqrt{3+\sqrt{7}}}}$
- $\sqrt[4]{3+\sqrt{7}}$
- $\sqrt[5]{3+\sqrt{7}}$
- $\sqrt[6]{3+\sqrt{7}}$
- $\sqrt[7]{3+\sqrt{7}}$
- $\sqrt[n]{3+\sqrt{7}}$, where n is even.
- $\sqrt[n]{3+\sqrt{7}}$, where n is odd.
- π

INQUIRY: SIDE LENGTH OF A CUBE

- Suppose that a cube, which we'll call Cube 1, has side length s, what is its volume?
- Suppose that a second cube, which we'll call Cube 2, twice the volume of Cube 1. Write an expression for the volume of Cube 2.
- Write an expression for the side length of Cube 2, simplified as much as possible.
- Supposing s is a constructible number, under other number must be constructible for the volume of Cube 2 to be constructible.

Inquiry: $\sqrt[3]{2}$

Let $\mathbb{F}$ be a field, and suppose that $k \in \mathbb{F}$. Also suppose that

$$\sqrt[3]{2} \in \mathbb{F}(\sqrt{k})$$

so that $\sqrt[3]{2} = a + b\sqrt{k}$ with $a, b \in \mathbb{F}$.

- Now cube both sides of the equation above.
- One side of the expression should just be "2." The other side will have four terms, including a term with $\sqrt{k}$ (or $k^{1/2}$) and a term with $k^{3/2}$. Group those two terms together and factor out a $k^{1/2}$ to obtain an equation of the form
$$2 = c + d\sqrt{k}.$$
- Why must the coefficient of $\sqrt{k}$ be zero?
- Explain why this implies that $b = 0$, so that $\sqrt[3]{2} = a$ for $a \in \mathbb{F}$.

Notice that this work shows that if

$$\sqrt[3]{2} \in \mathbb{F}(\sqrt{k})$$

then in fact

$$\sqrt[3]{2} \in \mathbb{F}.$$

Inquiry: Remember Your Trig! (Or look it up.)

Our goal in this inquiry is to find a nice polynomial which has as one of its roots the value $\cos(20^\circ)$. We begin by considering

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Even though we really want to find out something about $\cos(20^\circ)$, we begin by considering $\cos(3\theta)$ because $3(20^\circ) = 60^\circ$ and we know the cosine of 60°. So, we'll use trig identities to write $\cos(3\theta)$ in terms of $\cos(\theta)$.

First let us assemble our arsenal:

A. Write down the most well know trig identity - the one that hopefully know one in class needs to look up. You know the *one*.

B. There are a few well-known version of the identify for $\cos(2\theta)$. We want the one that looks a bit like the previous identity, but with a minus sign. (And be sure you have the order of the subtraction correct!)

C. Now write down the most common identity for $\sin(2\theta)$.

D. Write down the identify for $\cos(\alpha+\beta)$.

Now fill in the following steps:

1. By the sum formula for cosine (identity D above):

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \underline{\hspace{8cm}}\end{aligned}$$

2. In the formula above we see an instance of $\cos(2\theta)$ and an instance of $\sin(2\theta)$. Replacing these with their equivalent expressions (identities B and C above) we can write:

$$\cos(3\theta) = \underline{\hspace{8cm}}$$

3. Simplify the previous expression and fill in the boxes:

$$\cos(3\theta) = \cos^{\square}\theta - \square\sin^2\theta\cos\theta$$

4. Now use identity A above to replace $\sin^2\theta$ with an expression involving $\cos^2\theta$. Doing so, we obtain

$$\cos(3\theta) = \underline{\hspace{8cm}}$$

5. Simplifying the previous expression we obtain

$$\cos(3\theta) = \square\cos^3\theta - \square\cos\theta$$

6. Now if we set $\theta = 20^\circ$, then $\cos(3\theta) = \cos(60^\circ) = \underline{\hspace{2cm}}$. Rewrite the previous equation but substitute 20° for θ. Since you know the value of $\cos(60^\circ)$, use that value. Just leave $\cos(20^\circ)$ as it is - don't try to replace that expression with a decimal value. Write the new equation here:

7. Now let $u = \cos(20^\circ)$ and rewrite the previous equation here:

8. Now rewrite the previous equation as a polynomial in u set equal to zero:

$$\square u^{\square} - 6u - \square = 0. \tag{2}$$

Notice that this work implies that $\cos(20^\circ)$ is a root of the polynomial on the left-hand side of Equation 2.