An Introduction to Category Theory and Lawvere's Fixed Point Theorem

A Senior Comprehensive Project
by
Aubrey P. Collins
Allegheny College
Meadville, PA

January 26, 2019

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science.

Project Advisor: Dr. Craig Dodge
Second Reader: Dr. Tamara Lakins

I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

---

name

Acknowledgements

Thank you to the mathematics faculty for contributing to my education over the past four years. In particular, thank you to Professor Dodge and Professor Lakins for their guidance and feedback on this project.

Thank you to my friends and family for listening to me complain about "abstract nonsense".

Abstract

This senior project aims to provide an introduction to many of the basic concepts of category theory, up to the level of understanding Lawvere's Fixed Point Theorem. In particular, we cover functors, natural transformations, universal arrows, products, limits, adjunctions, and Cartesian closed categories. Finally, in Chapter 4, we state and prove the aforementioned theorem, which provides a generalization of diagonalization arguments such as Cantor's Theorem.

Contents

1 Introduction

The goal of this project is to go over all of the background information necessary to understand the statement and proof of Lawvere's Fixed Point Theorem. To that end, we make use of many definitions, theorems, exercises, and examples from Mac Lane's textbook [1] throughout the first chapters, and Lawvere's paper [2] in the last chapter. Work which is drawn from this sources is cited as such, both in the statement and the proof; however, minor changes in the notation and presentation of these arguments will generally be made for the purpose of clarification. Examples, theorems, and proofs which are not explicitly cited have been written without referencing outside sources.

1.1 Categories

Definition 1.1 ([1, Section 1.1])**.** A **category** C consists of a collection of **objects** and a collection of **arrows** (also called **morphisms**) between these objects. In particular, we write "$c \in C$" when c is an object and "f in C" when f is an arrow. Furthermore, a category must satisfy the following properties:

1. Each arrow f in C has a domain $\text{dom } f = a$ and codomain $\text{cod } f = b$, where $a, b \in C$. We denote such an arrow as $f : a \to b$. Note that while this notation is commonly used for functions, functions are a special case of arrows, and should be thought of separately.

2. For each pair of **composable** ($\text{cod } f = \text{dom } g$) arrows $f : a \to b$, $g : b \to c$, there is a unique **composition arrow** $g \circ f : a \to c$ in C.

Composition of arrows is associative. Moreover, we write composition from right to left, so that $g \circ f$ means "g after f".

3. For each object $a \in C$, there is an **identity arrow** $1_a : a \to a$ in C such that any arrows $f : a \to b, g : c \to a$ satisfy $f \circ 1_a = f$ and $1_a \circ g = g$.

We may represent finite categories with a directed graph diagram whose vertex set is the set of objects and whose edge set is the set of arrows between objects:

$$\begin{tikzcd}
a \arrow[loop left, "1_a"] \arrow[r, "f"] \arrow[d, "g\circ f"] & b \arrow[loop right, "1_b"] \arrow[dl, "g"] \\
c \arrow[loop left, "1_c"] &
\end{tikzcd}$$

For sake of simplicity, we will generally not draw identity and composition arrows in category diagrams. In cases where their properties are specifically relevant, however, we will explicitly include them.

We will often make use of **commutative diagrams** to demonstrate certain properties of categories. Such a diagram shows a certain subset of objects and arrows in a category, illustrating that paths with the same origin and destination in the diagram commute. For example, the commutative diagram

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "j"] & b \arrow[r, "g"] \arrow[d, "i"] & c \arrow[d, "h"] \\
x \arrow[r, "k"] & y \arrow[r, "l"] & z
\end{tikzcd}$$

shows that the following equalities hold:

$$i \circ f = k \circ j,$$

$$h \circ g = l \circ i,$$

$$h \circ g \circ f = l \circ i \circ f = l \circ k \circ j.$$

In commutative diagrams, we occasionally stylize arrows in certain ways to convey additional information. In particular, dashed lines will be used to indicate that (given the other solid arrows) a particular arrow exists which makes the diagram commute, such as in the following diagrams describing an invertible arrow f.

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "1_a"] & \\ b \arrow[r, dashed, "f^{-1}"'] & a \end{tikzcd} \qquad \begin{tikzcd} b \arrow[d, dashed, "f^{-1}"'] \arrow[dr, "1_b"] & \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

1.2 Category Theory

Categories can be used to model many kinds of mathematical structures on different levels of abstraction. For example, any set is equivalent to a category with an object for each set element and no non-identity arrows. But the collection of all sets (to be made more precise in Section 1.3) also forms a category, namely, the category **Set** whose objects are all sets and whose arrows are all functions between those sets.

In particular, category theory can be immediately applied as a generalization of many kinds of algebraic structures. Any group $(G, *)$ is equivalent to a category with a single object g. The group's identity element is then the identity arrow 1_g, and any other group element is an arrow $g \to g$, with $a * b = a \circ b$ for all such elements/arrows a, b in G. As before, we may also move to a higher level of abstraction by defining a category **Grp** whose objects are all groups and whose arrows are all homomorphisms between groups.

By translating groups and other structures into categories in this way, we may prove results about them strictly in terms of category theoretic properties. In doing so, we may generalize these results and connect seemingly unrelated areas of mathematics (for example, in Corollary 3.2 we prove that Cayley's Theorem follows as a special case of Yoneda's Lemma).

1.3 Examples of Categories

Many mathematical structures can be interpreted as categories. The following examples provide a sample of some of the categories which will be referred to throughout this paper.

- **0**, the category with no objects and no arrows.
- **1**, the category consisting of a single object and a single arrow (the identity).
- **2**, the category consisting of two objects a, b with their identity arrows and one additional arrow $a \to b$.
- **Set**, the category of all sets, with arrows being functions between sets. Note that by "all sets", we really mean all sets contained in a particular "universe" set U, and we call an object *small* if it is contained in U. For the rest of this paper, we will assume that a sufficiently small universe exists so that the categories of sets, groups, vector spaces, and other structures we discuss can be said to have sets of (small)

objects and sets of arrows.

- **Grp**, the category of all groups with arrows being group homomorphisms.
- $\mathbf{Vet_K}$, the category of all vector spaces over a particular field K, with arrows between them being linear transformations.
- A **preorder** is a category in which, for any objects a and b, there is at most one arrow $a \to b$. We define a reflexive and transitive relation $\leq$ on the set of objects such that $a \leq b$ when there is an arrow $a \to b$. If $f : a \to b$ and $g : b \to c$ are arrows in a preorder, then the composition $g \circ f : a \to c$ is the unique arrow $a \to c$. In terms of the relation $\leq$, this composition indicates that $a \leq b \leq c$ if and only if $a \leq b$ and $b \leq c$.

2 Functors and Natural Transformations

2.1 Functors

As noted in the previous section, we can think of groups on different levels of abstraction: the "internal" structure of a group G, i.e. the relation between its elements, can be understood by representing G as a category; its "external" structure, i.e. how it relates to other groups, can be understood by considering G as an object in the category **Grp** of groups and homomorphisms between groups. To help bridge the gap between these two perspectives, we introduce the concept of a functor.

Definition 2.1 ([1, Section 1.3]). Let C and B be categories. A **covariant functor** (hereafter referred to as just a *functor*) $\mathcal{T} : C \to B$ is a morphism between the two categories, consisting of two parts. The first part, called the object function of $\mathcal{T}$, assigns each object in $c \in C$ to an object $\mathcal{T}c \in B$. Second, the arrow function of $\mathcal{T}$ assigns each arrow $f : c \to c'$ in C to an arrow $\mathcal{T}f : \mathcal{T}c \to \mathcal{T}c'$ in B. Moreover, $\mathcal{T}$ must preserve composition and identities, meaning:

1. For all composable arrows f, g in C, $\mathcal{T}(g \circ f) = \mathcal{T}g \circ \mathcal{T}f$.

2. For any object $c \in C$ with identity arrow 1_c, $\mathcal{T}1_c = 1_{\mathcal{T}c}$.

Essentially, a functor embeds an image of one category in another in a way which preserves its structure. The next two examples show how this allows us to move between different levels of abstraction.

Example 2.2. Let S and T be sets. Note that these sets can be understood as categories in which each set element is an object and there are no arrows except the identities. Then any function $f : S \to T$ is equivalent to a functor $\mathcal{F}$ between their corresponding category representations: use f as the object function of $\mathcal{F}$, and have the arrow function map identity arrows to identity arrows in the natural way. Hence, the functor axioms are met, since $\mathcal{F}$ preserves identities by definition, and the categories have no other composable arrows. Note then that the arrows from the object associated with S to the object associated with T in the category **Set** are the same as the functors from the category S to the category T.

Example 2.3. Let $(G, *)$ and $(H, \cdot)$ be groups; equivalently, define G to be a category with a single object g, where each group element is an arrow $g \to g$, and define a category representation for H in the same way. Then any homomorphism $\varphi : G \to H$ is equivalent to a functor $\mathcal{P} : G \to H$. Specifically, this functor has arrow function φ and object function $g \mapsto h$. To see that this satisfies the functor properties, first observe that since $\varphi(x * y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in G$, $\mathcal{P}(x \circ y) = \mathcal{P}x \circ \mathcal{P}y$. Moreover, since φ maps the identity element e_G of G to the identity element e_H of H, we have $\mathcal{P}1_g = 1_h = 1_{\mathcal{P}g}$. Hence $\mathcal{P}$ preserves composition and identities, and so it is a functor as claimed. Note that the arrows $G \to H$ in **Grp** are then equivalent to the functors from the category representation of G to the category representation of H.

Example 2.4 ([1, Exercise 1.4.1])**.** For a fixed set S, let X^S be the set of all functions $S \to X$. Then there is a functor $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ such that for any set X, $\mathcal{T}X = X^S$.

Proof. To show this, we must provide an arrow function which maps each arrow $f : X \to Y$ to an arrow $\mathcal{T}f : X^S \to Y^S$. For any $(h : S \to X) \in X^S$, define the arrow function by $(\mathcal{T}f)(h) = f \circ h$. Note that $f \circ h$ is a function $S \to Y$, i.e. it is an element of Y^S. Hence, the domain and codomain of $\mathcal{T}f$ are as claimed. $\mathcal{T}f$ also satisfies the identity property $\mathcal{T}1_X = 1_{\mathcal{T}X}$, since for all $h \in X^S$, $(\mathcal{T}1_X)(h) = 1_X \circ h = h$. Finally, the arrow function preserves composition, since for any $f : X \to Y, g : Y \to Z$ we have

$$\begin{aligned}
\mathcal{T}(g \circ f)(h) &= (g \circ f) \circ h \\
&= g \circ (f \circ h) \\
&= g \circ (\mathcal{T}f(h)) \\
&= \mathcal{T}g(\mathcal{T}f(h)) \\
&= (\mathcal{T}g \circ \mathcal{T}f)(h).
\end{aligned}$$

□

Definition 2.5 ([1, Section 1.2])**.** Let C be a category. For any two objects $a, b \in C$, the **hom-set** $\hom(a, b)$ is the set of all arrows in C with domain a and codomain b. By replacing b with a generic object, each object a also defines a function $\hom(a, -)(x) = \hom(a, x)$; likewise, replacing a with a generic object defines a function $\hom(-, b)(x) = \hom(x, b)$.

Definition 2.6 ([1, Section 1.3])**.** Let C and B be categories, and $\mathcal{T}$ be a functor $\mathcal{T} : C \to B$. Note that for each pair of objects c, c' in C, we can think of the functor $\mathcal{T}$ as defining a function $\mathcal{T}_{c,c'} : \hom(c, c') \to \hom(\mathcal{T}c, \mathcal{T}c')$ under which each arrow $f : c \to c'$ has $f \mapsto \mathcal{T}f$. By considering all such $\mathcal{T}_{c,c'}$, we can observe several special cases of functors which are worth noting.

If $\mathcal{T}_{c,c'}$ is one-to-one for all $c, c' \in C$, then we say $\mathcal{T}$ is **faithful**. If every such $\mathcal{T}_{c,c'}$ is onto, then $\mathcal{T}$ is **full**. Finally, when each $\mathcal{T}_{c,c'}$ is both one-to-one and onto, we call $\mathcal{T}$ **fully faithful**. The following examples illustrate these properties.

Example 2.7. Let C be a category, and $\mathcal{I}_C : C \to C$ be the **identity functor**, i.e. the functor such that for all objects $c \in C$ and arrows f in C, $\mathcal{I}_C(c) = c$ and $\mathcal{I}_C(f) = f$. Then $\mathcal{I}_C$ is fully faithful.

Example 2.8. Let C and B be, respectively, the two categories depicted below.

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, "g"] & b \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, bend left, "f'"] \arrow[r, "g'"] \arrow[r, bend right, "h'"'] & b' \end{tikzcd}$$

Let $\mathcal{T} : C \to B$ be the functor whose object function maps $a \mapsto a'$ and $b \mapsto b'$, and whose arrow function maps $f \mapsto f'$ and $g \mapsto g'$. Observe that $\mathcal{T}_{a,b} : \{f, g\} \to \{f', g', h'\}$ is one-to-one (since $\mathcal{T}f \neq \mathcal{T}g$), but not onto (since $\mathcal{T}$ does not map an arrow to h'). Thus $\mathcal{T}$ is faithful, but not full.

On the other hand, suppose $\mathcal{S} : B \to C$ is the functor with object function mapping $a' \mapsto a$ and $b' \mapsto b$, and arrow function mapping $f' \mapsto f, g' \mapsto g$, and $h' \mapsto g$. Then $\mathcal{S}_{a',b'}$ is onto, but not one-to-one (since $\mathcal{S}h' = \mathcal{S}g'$). Hence, $\mathcal{S}$ is full, but not faithful.

We also define a property of functors which describes a much stronger form of structural similarity.

Definition 2.9 ([1, Section 1.3])**.** Let $\mathcal{T} : C \to B$ be a functor. We say $\mathcal{T}$ is an **isomorphism** if its object function and its arrow function are both bijections.

In practice, the requirements for an isomorphism of categories are often too strong for it to be a useful tool. In Section 3.3, we will define the notion of an adjunction, which gives a more lenient way to compare functors and identify structural similarities in categories, and which has a plethora of examples throughout diverse areas of mathematics.

2.2 Contravariant Functors

So far, we have only discussed covariant functors, which preserve the direction of arrows. In contrast, *contravariant* functors *reverse* the direction of arrows.

Definition 2.10 ([1, Section 2.2])**.** A **contravariant functor** $\mathcal{G} : C \to B$ has an object function which assigns each object $c \in C$ to an object $\mathcal{G}c \in B$. Additionally, it has an arrow function which reverses the direction of the arrows so that any $f : c \to c'$ in C maps to $\mathcal{G}f : \mathcal{G}c' \to \mathcal{G}c$ in B. Like covariant functors, contravariant functors preserve identity arrows. However, a contravariant functor must also reverse composition: for all composable arrows f, g in C, the contravariant functor $\mathcal{G}$ has $\mathcal{G}(g \circ f) = \mathcal{G}f \circ \mathcal{G}g$.

A clear example of contravariance and covariance can be found in the notion of *hom-functors*. Recall that the hom-set $\hom(a, b)$ is the set of all arrows in a category C having domain a and codomain b. Then we may get one function $C \to \mathbf{Set}$ by keeping a fixed and varying the codomain object, and another by keeping b fixed and varying the domain object. The following definition uses these functions as the object maps of two related functors.

Definition 2.11 ([1, Section 2.2])**.** Given any category C and an object $a \in C$, the **covariant hom-functor** $C(a, -) : C \to \mathbf{Set}$ maps each object $b \in C$ to the hom-set $\hom(a, b) = C(a, b)$, and maps each arrow $f : x \to y$ in C to the *composition function* $C(a, f) : \hom(a, x) \to \hom(a, y)$ defined by $C(a, f)(g) = f \circ g$ for all $g \in \hom(a, x)$.

The **contravariant hom-functor** $C(-, b) : C \to Set$ likewise maps each object $a \in C$ to the hom-set $\hom(a, b)$. However, it maps arrows using *pre-composition*, so that each $f : x \to y$ maps to the function $C(f, b) : \hom(y, b) \to \hom(x, b)$ defined by $C(f, b)(g) = g \circ f$ for all $g \in \hom(y, b)$. In the following examples we verify that these functors are covariant and contravariant as claimed.

Example 2.12. The hom-functor $C(a, -) : C \to \mathbf{Set}$ is covariant.

Proof. To prove this, we must show that $C(a, -)$ preserves identities and compositions. Note that $C(a, -)(1_a) = C(a, 1_a)$ has $C(a, 1_a)(g) = 1_a \circ g = g$ for any $g \in C(a, a)$, so that $C(a, 1_a) = 1_{C(a,a)}$. Next, let $f : x \to y$ and $g : y \to z$ be arrows in C. Observe that for all $h \in C(a, x)$,

$$\begin{aligned}
[C(a, g)(C(a, f))](h) &= g \circ C(a, f)(h) \\
&= g \circ f \circ h \\
&= C(a, g \circ f)(h),
\end{aligned}$$

thereby completing the proof. $\square$

Example 2.13. The hom-functor $C(-, b) : C \to \mathbf{Set}$ is contravariant.

Proof. We must show that $C(-,b)$ preserves identities and reverses composition. In a similar manner to the previous proof, we have $C(1_b,b)(f) = f \circ 1_b = f$ for all $f \in C(b,b)$, so that $1_{C(b,b)} = C(1_b,b)$. Next, let $f : x \to y$ and $g : y \to z$ be arrows in C. Observe that for all $h \in C(z,b)$,

$$\begin{aligned}[C(f,b)(C(g,b))](h) &= C(f,b)(C(g,b)(h)) \\ &= C(f,b)(h \circ g) \\ &= h \circ g \circ f \\ &= C(g \circ f, b)(h),\end{aligned}$$

showing that $C(-,b)$ reverses composition and hence is contravariant. □

2.3 Natural Transformations

In previous sections, we have dealt with morphisms between objects (arrows) and morphisms between categories (functors); the notion of a natural transformation allows us to define morphisms between *functors*.

Definition 2.14 ([1, Section 1.4])**.** Let C and B be categories, and let $\mathcal{G}, \mathcal{F} : C \to B$ be functors. A **natural transformation** $\tau : \mathcal{F} \dot{\to} \mathcal{G}$ is a function assigning each object $c \in C$ to an arrow (called a **component** of the transformation) $\tau_c : \mathcal{F}c \to \mathcal{G}c$ in such a way that the following diagram commutes for every choice of arrow $f : c \to c'$.

$$\begin{tikzcd} c \arrow[d, "f"] & \mathcal{F}c \arrow[r, "\tau_c"] \arrow[d, "\mathcal{F}f"] & \mathcal{G}c \arrow[d, "\mathcal{G}f"] \\ c' & \mathcal{F}c' \arrow[r, "\tau_{c'}"] & \mathcal{G}c' \end{tikzcd}$$

Note that a natural transformation τ is not itself an arrow in the category C or the category B; however, each component τ_c *is* an arrow in B, and hence provides a way to translate between the two functors "naturally", so that the structure of B is preserved. We say that each component τ_c is **natural in** c. Moreover, if each component τ_c is invertible (having an inverse $\tau_c^{-1} : \mathcal{G}c \to \mathcal{F}c$ so that $\tau_c^{-1} \circ \tau_c = 1_{\mathcal{F}c},\ \tau_c \circ \tau_c^{-1} = 1_{\mathcal{G}c}$), then we say the transformation is a **natural isomorphism** $\tau : \mathcal{F} \cong \mathcal{G}$.

Example 2.15. Let C and B, respectively, be the categories

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, "f'"] & b' \end{tikzcd}$$

and let $\mathcal{F}, \mathcal{G} : B \to C$ be functors defined by

$$\mathcal{G}a' = a \qquad \mathcal{G}b' = b \qquad \mathcal{G}f' = f$$

$$\mathcal{F}a' = a \qquad \mathcal{F}b' = c \qquad \mathcal{F}f' = h.$$

Observe that there exists a natural transformation $\tau : \mathcal{G} \dot{\to} \mathcal{F}$, since the following diagram is commutative using $\tau a' = 1_a$ and $\tau b' = g$:

$$\begin{tikzcd} \mathcal{G}a' \arrow[r, "\tau a'"] \arrow[d, "\mathcal{G}f'"] & \mathcal{F}a' \arrow[d, "\mathcal{F}f'"] \\ \mathcal{G}b' \arrow[r, "\tau b'"] & \mathcal{F}b' \end{tikzcd}$$

However, notice that there is no natural transformation $\tau' : \mathcal{F} \dot{\to} \mathcal{G}$, since there is no arrow $\mathcal{F}b' \to \mathcal{G}b'$ in C.

Example 2.16 ([1, Exercise 1.4.4])**.** Let $\mathcal{F}, \mathcal{G} : C \to P$ be functors from a category C to a preorder (defined in Section 1.3) P. Show that there exists a unique natural transformation $\mathcal{F} \dot{\to} \mathcal{G}$ if and only if $\mathcal{F}c \leq \mathcal{G}c$ for all objects $c \in C$.

Proof. Recall that the arrows in a preorder represent the relation $\leq$, i.e. there exists an arrow $f : m \to n$ in P if and only if $m \leq n$. Moreover, since there is at most one such f arrow between any pair of objects $m, n \in P$, we may write $m \leq n$ to represent f.

First, suppose there is a unique natural transformation $\tau : \mathcal{F} \dot{\to} \mathcal{G}$. Then for any object $c \in C$, there is an arrow $\tau c : \mathcal{F}c \to \mathcal{G}c$ in P, which indicates that $\mathcal{F}c \leq \mathcal{G}c$ since P is a preorder. For the converse, choose an object $c \in C$, and suppose $\mathcal{F}c \leq \mathcal{G}c$. By this assumption, there exists an arrow $\tau c : \mathcal{F}c \to \mathcal{G}c$ in P. Choose an arrow $g : c \to c'$ in C, and consider the following diagram:

$$\begin{tikzcd}
c \arrow[d, "g"] & \mathcal{F}c \arrow[r, "\tau c"] \arrow[d, "\mathcal{F}g"] & \mathcal{G}c \arrow[d, "\mathcal{G}g"] \\
c' & \mathcal{F}c' \arrow[r, "\tau c'"] & \mathcal{F}c'
\end{tikzcd}$$

Since P is a preorder, it has at most one arrow $\mathcal{F}c \to \mathcal{F}c'$, and so $\mathcal{G}g \circ \tau c = \tau c' \circ \mathcal{F}g$. This shows that the diagram commutes, and so τ is a natural transformation. Moreover, since there is at most one arrow $\mathcal{F}c \to \mathcal{G}c$, τc is uniquely determined for each c, so τ is the unique natural transformation. $\square$

Example 2.17 ([1, Exercise 1.4.1])**.** Let S be a fixed set, and X^S be the set of all functions $S \to X$. Show that the evaluation function $e_x : (X^S \times S) \dot{\to} X$

defined by $e_x(h, s) = h(s)$ is a natural transformation.

Proof. Define the functor $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ as in Example 2.4. Let $\mathcal{I}$ be the identity functor on $\mathbf{Set}$, and let $\mathcal{U} : \mathbf{Set} \to \mathbf{Set}$ be the functor with object function defined by $\mathcal{U}X = X^S \times S$ and arrow function $(\mathcal{U}f)(h, s) = ((\mathcal{T}f)(h), s)$. We claim that the following diagram commutes for all $f : X \to Y$.

$$\begin{array}{ccc} \mathcal{U}X & \xrightarrow{e_X} & \mathcal{I}X \\ \downarrow{\scriptstyle \mathcal{U}f} & & \downarrow{\scriptstyle \mathcal{I}f} \\ \mathcal{U}Y & \xrightarrow{e_Y} & \mathcal{I}Y \end{array}$$

To see this, choose $(h, x) \in X^S \times X$. Then

$$\begin{aligned} (\mathcal{I}f \circ e_x)(h, s) &= If(e_x(h, s)) \\ &= If(h(s)) \\ &= (f \circ h)(s) \\ &= e_Y(f \circ h)(s) \\ &= e_Y(\mathcal{T}f(h), s) \\ &= e_Y(\mathcal{U}f(h, s)) \\ &= (e_Y \circ \mathcal{U}f)(h, s) \end{aligned}$$

Hence the diagram commutes, and so e_X is a natural transformation. $\square$

Example 2.18. Let A, B, C, D be categories with functors

$$A \xrightarrow{\mathcal{J}} B \overset{\mathcal{F}}{\underset{\mathcal{G}}{\rightrightarrows}} C \xrightarrow{\mathcal{H}} D$$

and a natural transformation $\tau : \mathcal{F} \dot{\to} \mathcal{G}$. Then $\mathcal{H}$ may be composed with τ

to define a natural transformation $\mathcal{H}\tau : \mathcal{H}\mathcal{F} \dot{\to} \mathcal{H}\mathcal{G}$, for the compositions of functors $\mathcal{H}\mathcal{F} : B \to D$ and $\mathcal{H}\mathcal{F} : B \to D$. Likewise, τ may be composed with $\mathcal{J}$ to define a natural transformation $\tau\mathcal{J} : \mathcal{F}\mathcal{J} \dot{\to} \mathcal{G}\mathcal{J}$.

Proof. Since τ is a natural transformation, the following diagram commutes for any arrow $f : b \to b'$ in B.

$$\begin{tikzcd}
\mathcal{F}b \arrow[r, "\tau_b"] \arrow[d, "\mathcal{F}f"] & \mathcal{G}b \arrow[d, "\mathcal{G}f"] \\
\mathcal{F}b' \arrow[r, "\tau_{b'}"] & \mathcal{G}b'
\end{tikzcd}$$

Hence the image under $\mathcal{H}$ of this diagram

$$\begin{tikzcd}
\mathcal{H}\mathcal{F}b \arrow[r, "\mathcal{H}\tau_b"] \arrow[d, "\mathcal{H}\mathcal{F}f"] & \mathcal{H}\mathcal{G}b \arrow[d, "\mathcal{H}\mathcal{G}f"] \\
\mathcal{H}\mathcal{F}b' \arrow[r, "\mathcal{H}\tau_{b'}"] & \mathcal{H}\mathcal{G}b'
\end{tikzcd}$$

also commutes, and defines a natural transformation $\mathcal{H}\tau : \mathcal{H}\mathcal{F} \dot{\to} \mathcal{H}\mathcal{G}$ with components $(H\tau)_b = H(\tau_b)$.

Since the image under $\mathcal{J}$ of any arrow $g : a \to a'$ in A is an arrow in B, we may again invoke the naturality of τ to get the following commutative diagram.

$$\begin{tikzcd}
\mathcal{F}\mathcal{J}a \arrow[r, "\tau_{\mathcal{J}a}"] \arrow[d, "\mathcal{F}\mathcal{J}g"] & \mathcal{G}\mathcal{J}a \arrow[d, "\mathcal{G}\mathcal{J}g"] \\
\mathcal{F}\mathcal{J}a' \arrow[r, "\tau_{\mathcal{J}a'}"] & \mathcal{G}\mathcal{J}a'
\end{tikzcd}$$

Hence we may define a natural transformation $\tau\mathcal{J} : \mathcal{F}\mathcal{J} \dot{\to} \mathcal{G}\mathcal{J}$ with components $(\tau\mathcal{J})_a = \tau_{\mathcal{J}a}$. $\square$

2.4 Functor Categories

Finally, we may use natural transformations to define the notion of a *functor category*. In the past few sections we have been discussing how we may move between different levels of abstraction. In Example 2.1, we saw that a group homomorphism is the same thing as a functor between category representations of the groups. We noted how **Grp** forms a category whose objects are these category representations and whose arrows are the functors between them. In contrast to this "category of categories", a functor category is a "category of functors," with arrows being natural transformations. For this notion to make sense, we must first prove the following fact.

Theorem 2.19. *Let C, D be categories with functors $\mathcal{F}, \mathcal{G}, \mathcal{H} : C \to D$ and natural transformations $\tau : \mathcal{F} \dot{\to} \mathcal{G}$ and $\theta : \mathcal{G} \dot{\to} \mathcal{H}$. Then the composition $\theta \circ \tau : \mathcal{F} \dot{\to} \mathcal{H}$ is also a natural transformation.*

Proof. For any arrow $f : c \to c'$ in C, the fact that τ and θ are natural transformations ensures that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}c \arrow[r, "\tau_c"] \arrow[d, "\mathcal{F}f"] & \mathcal{G}c \arrow[r, "\theta_c"] \arrow[d, "\mathcal{G}f"] & \mathcal{H}c \arrow[d, "\mathcal{H}f"] \\
\mathcal{F}c' \arrow[r, "\tau_{c'}"] & \mathcal{G}c' \arrow[r, "\theta_{c'}"] & \mathcal{H}c'
\end{tikzcd}$$

This diagram shows that $\theta \circ \tau$ is a natural transformation, with $(\theta \circ \tau)_c = \theta_c \circ \tau_c$. $\square$

Definition 2.20 ([1, Section 2.4])**.** Let C, J be categories. Then the **functor category** C^J has the functors $J \to C$ has its objects, and the natural transformations between these functors as its arrows. Typically we consider cases in which J is a finite category. We then call J an **index category**, and call its functors **diagrams**.

The following examples begin to showcase the usefulness of functor categories. They will be applied further in the next chapter, helping to the notion of a *limit* (Definition 3.3).

Example 2.21 ([1, Section 2.4])**.** Let C be a category. Recall that $\mathbf{1}$ is the category with a single object $*$ and its identity arrow. Any object $x \in C$ can thus be thought of as the functor $\mathcal{X} : \mathbf{1} \to C$ which maps $* \mapsto x$. From this perspective, an arrow $f : x \to y$ in C is equivalent to a natural transformation $F : \mathcal{X} \dot{\to} \mathcal{Y}$ between the functors $* \mapsto x$ and $* \mapsto y$, and all the natural transformations in $C^{\mathbf{1}}$ can be described as an arrow of C in this way. Thus, the functor category $C^{\mathbf{1}}$ is isomorphic to C.

Example 2.22 ([1, Section 2.4])**.** Recall that $\mathbf{2}$ is the category consisting of two objects with their identities and an additional arrow between them. Then the functor category C^{2} has as its objects all the arrows in C, and has as its arrows $f \to f'$ all the pairs of arrows $\langle h, k\rangle$ in C which make the following square commute.

$$\begin{tikzcd}
a \arrow[r, "h"] \arrow[d, "f"] & a' \arrow[d, "f'"] \\
b \arrow[r, "k"] & b'
\end{tikzcd}$$

Example 2.23. Let C be a category, with index category J being the category consisting of two objects and just their identity arrows. Then $C^{\mathbf{J}}$

has as its diagrams all the pairs of objects $\langle a, b\rangle$ $(a, b \in C)$, and has all pairs $\langle f, g\rangle$

$$\begin{array}{ccc} a & b & \langle a,b\rangle \\ \downarrow f & \downarrow g & \downarrow \langle f,g\rangle \\ a' & b' & \langle a',b'\rangle \end{array}$$

as its arrows. Note that C^J is better known as the Cartesian product $C \times C$. Using this setup, we may define the *diagonal functor* $C \to C \times C$, which will be useful in future sections.

Definition 2.24 ([1, Section 3.3])**.** Let C be a category. The **diagonal functor** $\Delta : C \to C \times C$ is the functor assigning each object $c \in C$ to $\Delta c = \langle c, c\rangle$, and each arrow $f : c \to c'$ in C to $\Delta f = \langle f, f\rangle$.

More generally, for an index category J, the diagonal functor $\Delta : C \to C^J$ assigns every object $c \in C$ to an object $\Delta c \in C^J$, i.e. to a functor $\Delta c : J \to C$. Specifically, each Δc is a "constant functor," sending every object in J to c and every arrow in J to 1_c. Moreover, Δ maps each arrow $f : c \to c'$ in C to a natural transformation $\Delta f : \Delta c \dot{\to} \Delta c'$. Note that the natural transformation diagram below commutes when each component of Δf is defined to be f.

$$\begin{array}{cccc} j & (\Delta c)j & \xrightarrow{(\Delta f)_j} & (\Delta c')j \\ \downarrow g & \downarrow 1_c & & \downarrow 1_c \\ j' & (\Delta c)j' & \xrightarrow{(\Delta f)_{j'}} & (\Delta c')j' \end{array}$$

3 Universals and Adjunctions

A highly useful way to analyze the structure of a particular category is by understanding its *universal properties.* Such properties allow us to make general statements about the existence and (most importantly) uniqueness of certain arrows in the category. A straightforward example can be seen through the concept of *initial* and *terminal* objects, whose universal properties ensure they have a unique arrow to or from every other object.

Definition 3.1. Let C be a category. An object $i \in C$ is called **initial** if it has a unique arrow $i \to c$ for each object $c \in C$. Dually, an object $t \in C$ is **terminal** if every $c \in C$ has a unique arrow $c \to t$.

Example 3.2 ([1, Section 1.5])**.** For any set S, there is exactly one function $\emptyset \to S$. Hence $\emptyset$ is an initial object in **Set**. It is easy to show that this is the only initial object, since any nonempty set defines multiple functions to a set containing two or more elements. Additionally, for any singleton set $\{*\}$ there is exactly one function $S \to \{*\}$, namely the function $s \mapsto *$ for all $s \in S$. Thus, each singleton set is a terminal object in **Set**.

3.1 Universal Arrows

Definition 3.3 ([1, Section 3.1])**.** Let D and C be categories, $c \in C$, $r \in D$, and $\mathcal{S}$ be a functor $D \to C$. An arrow $u : c \to \mathcal{S}r$ in C is said to be **universal from c to $\mathcal{S}$** if, given any arrow $f : c \to \mathcal{S}d$ in C, there is a unique arrow $f' : r \to d$ in D such that $\mathcal{S}f' \circ u = f$, as in the commutative diagram:

$$\begin{tikzcd} r \arrow[d, dashed, "f'"] & c \arrow[r, "u"] \arrow[dr, "f"'] & \mathcal{S}r \arrow[d, dashed, "\mathcal{S}f'"] \\ d & & \mathcal{S}d \end{tikzcd}$$

Dually, an arrow $v : \mathcal{S}r \to c$ in C is said to be **universal from $\mathcal{S}$ to c** if for any arrow $g : \mathcal{S}d \to c$ in C, there is a unique arrow $g' : d \to r$ in D such that $v \circ \mathcal{S}g' = g$, as in the commutative diagram:

$$\begin{tikzcd} r & c & \mathcal{S}r \arrow[l, "v"'] \\ d \arrow[u, dashed, "g'"'] & & \mathcal{S}d \arrow[ul, "g"] \arrow[u, dashed, "\mathcal{S}g'"'] \end{tikzcd}$$

Example 3.4. The universal property of terminal and initial objects can be described using universal arrows. Let C be a category with a terminal object $*$. If $T : C \to C$ is the functor sending every object to $*$ and arrow to 1_*, then the unique arrow $u : c \to *$ for each object c is universal from c to T, according to the following commutative diagram.

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[dr, "f"'] & * \arrow[d, dashed, "1_*"] \\ & * \end{tikzcd}$$

For a less trivial example of universal arrows in action, recall the following linear algebra theorem:

Theorem 3.5 ([3, Theorem LTDB])**.** *Let W be a vector space with basis $\{w_1, w_2, \ldots, w_n\}$ and V be a vector space containing vectors $v_1, v_2, \ldots, v_n$. Then there is a unique linear transformation $T : W \to V$ such that each $T(w_i) = v_i$ $(1 \le i \le n)$.*

Example 3.6 ([1, Section 3.1])**.** To understand the above theorem in category theoretic way, consider the category $\mathbf{Vet}_K$ (defined in Section 1.3) of all vector spaces over a field K together with all linear transformations between them. We may define a forgetful functor $\mathcal{U} : \mathbf{Vet}_K \to \mathbf{Set}$ which maps each vector space to the set of its elements ("forgetting" the vector space operations), and each linear transformation to its associated function between such sets. Fix a finite set X, and define V_X to be the vector space with X as a basis, i.e. the vector space of formal sums of elements in X with coefficients in K. Then there is an arrow $j : X \to \mathcal{U}(V_X)$ in **Set** such that each element $x_i \in X$ is mapped to its associated vector $\vec{x_i} = 0x_1 + 0x_2 + \cdots + 1x_i + \cdots + 0x_n$. We claim that j is universal.

For another vector space W, consider any function $f : X \to \mathcal{U}(W)$ in **Set**. By the above theorem, there is unique linear transformation $f' : V_X \to W$ such that $f'(\vec{x_i}) = f(x_i)$ for each $x_i \in X$. Moreover, since j is the function which maps each element of X onto a basis vector in V_X, we have $f'(j(x_i)) = f'(\vec{x_i}) = f(x_i)$, i.e. $f' \circ j = f$. Moreover, $\mathcal{U}f' = f'$, since the functor $\mathcal{U}$ simply "forgets" that f' is a linear transformation while preserving maps. Therefore $\mathcal{U}f' \circ j = f$, meaning j is universal. Note that the equivalence of this theorem also works in the other direction: if we assume that j is a universal arrow, then each function $f : X \to \mathcal{U}(W)$ determines a unique linear transformation $f' : V_X \to W$ such that each $x_i \in X$ has $f(x_i) = f'(\vec{x_i})$, which is exactly the statement of the theorem.

Lastly, we will prove that universal arrows are unique up to a unique isomorphism. This will allow us to demonstrate equality of arrows by showing that they satisfy the same universal property. Moreover, proving this will extend uniqueness to other types of universal constructions which are defined via universal arrows.

Theorem 3.7. *If $u : c \to \mathcal{S}r$ is universal from c to $\mathcal{S}$, then u is unique up to a unique isomorphism.*

Proof. Suppose $u : c \to \mathcal{S}r$ and $v : c \to \mathcal{S}z$ are both universal from c to $\mathcal{S}$. Then there exist unique arrows $\mathcal{S}v'$ and $\mathcal{S}u'$ which make the following diagrams commute.

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[dr, "v"'] & \mathcal{S}r \arrow[d, dashed, "\mathcal{S}v'"] \\ & \mathcal{S}z \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "v"] \arrow[dr, "u"'] & \mathcal{S}z \arrow[d, dashed, "\mathcal{S}u'"] \\ & \mathcal{S}r \end{tikzcd}$$

It then follows that $\mathcal{S}u' \circ \mathcal{S}v' \circ u = u$ and $\mathcal{S}u' \circ \mathcal{S}v' \circ v = v$. Moreover, by the universality of u and v, $\mathcal{S}v' \circ \mathcal{S}u'$ and $\mathcal{S}u' \circ \mathcal{S}v'$ must be the unique arrows for which these equations hold. Hence, $\mathcal{S}v' \circ \mathcal{S}u' = 1_{\mathcal{S}z}$ and $\mathcal{S}u' \circ \mathcal{S}v' = 1_{\mathcal{S}r}$, so that $\mathcal{S}v' = (\mathcal{S}u')^{-1}$. Thus $\mathcal{S}r$ and $\mathcal{S}z$ are isomorphic objects in the category. This isomorphism is unique due to the uniqueness of $\mathcal{S}v'$ and $\mathcal{S}u'$.

□

3.2 The Yoneda Lemma

Definition 3.8 ([1, Section 2.4])**.** If C, D are categories and $\mathcal{F}, \mathcal{G}$ are functors $D \to C$, then $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$ is the set of all natural transformations $\mathcal{F} \dot{\to} \mathcal{G}$. Alternatively, this may be understood in terms of a hom-set in the functor category (Definition 2.4), i.e.

$$\mathrm{Nat}(\mathcal{F},\mathcal{G}) = C^D(\mathcal{F},\mathcal{G})$$

(using the hom-functor notation of Definition 2.2).

Lemma 3.9 (Yoneda's Lemma, [1])**.** *Let D be a category, $\mathcal{K}$ be a functor $D \to$ **Set**, and r be an object in D. Then there exists a bijection $y : Nat(D(r,-),\mathcal{K}) \to \mathcal{K}r$.*

Proof ([1, Section 3.2]). Let $\alpha \in \mathrm{Nat}(D(r,-),\mathcal{K})$. Since α is a natural transformation, any arrow $f : r \to d$ in D yields the following commutative diagram:

$$\begin{tikzcd} D(r,r) \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"] & \mathcal{K}r \arrow[d, "\mathcal{K}f"] \\ D(r,d) \arrow[r, "\alpha_d"] & \mathcal{K}d \end{tikzcd}$$

Using this, we may narrow our focus to the image of the identity arrow $1_r \in D(r,r)$, generating a new commutative diagram:

$$\begin{tikzcd} 1_r \arrow[r, mapsto, "\alpha_r"] \arrow[d, mapsto, "D(r,f)"] & \alpha_r(1_r) \arrow[d, mapsto, "\mathcal{K}f"] \\ f \arrow[r, mapsto, "\alpha_d"] & \alpha_d(f) \end{tikzcd}$$

Hence $\alpha_d(f) = \mathcal{K}f(\alpha_r(1_r))$, so any natural transformation $D(r,-)\dot{\to}\mathcal{K}$ is determined by the choice of where to send the identity arrow 1_r, in addition to the functor $\mathcal{K}$ itself. Any element $x \in \mathcal{K}r$ can then be used to define a unique natural transformation by choosing $\alpha_r(1_r) = x$. This shows that the map y^{-1} is one-to-one. Moreover, the map is also onto since each natural transformation can be described in this way, according to the above diagram.

□

The same argument may be applied to prove a contravariant version of Yoneda's lemma, namely that for two contravariant functors $D(-, r)$ and $\mathcal{K}$ we have $\mathrm{Nat}(D(-, r), \mathcal{K}) \cong \mathcal{K}r$. To see this, simply set up above the commutative diagram for a natural transformation α using an arrow $f : d \to r$ and follow the image of the identity through both sides of the diagram.

Yoneda's Lemma is one of the most famous results in category theory, offering the insight that a functor $D \to \mathbf{Set}$ can be understood in terms of its natural transformations from the hom-functor. In particular, this insight can be viewed as a generalization of Cayley's Theorem, as shown in the following corollary.

Corollary 3.10 (Cayley's Theorem)**.** *Any group is isomorphic to a permutation group.*

Proof. Let $(G, *)$ be a group; equivalently, G is a category consisting of a single object g such that every arrow $g \to g$ is a group element, meaning all arrows $f, j : g \to g$ are invertible and have $f \circ j = f * j$. We first claim that the set $\mathrm{Nat}(G(-, g), G(-, g))$ is a set of permutation functions on the group elements $G(g, g)$. Each transformation α in this set has a single component $\alpha_g : G(g, g) \to G(g, g)$, meaning the transformation is uniquely determined by a function on the set of group elements. Thus it suffices to show α_g is a bijection. As noted above, each such α is uniquely determined by the image of the identity. Hence, for any group element f, $\alpha_g(f) = G(f,g)(\alpha_g(1_g)) = \alpha_g(1_g) \circ f$ (since we are using the *contravariant* hom-functor). We may then define an inverse transformation by $\alpha_g^{-1}(1_g) = (\alpha_g(1_g))^{-1}$, where the inverse operation on the righthand side refers to the *group* inverse. Hence α is equivalent to an invertible function on the group elements, i.e. a permutation. Moreover, since there is an identity transformation and the composition of transformations is also a transformation, $\mathrm{Nat}(G(-,g), G(-,g))$ forms a permutation group.

Applying Yoneda's lemma to the contravariant hom-functor $G(-,g)$, we see that $\mathrm{Nat}(G(-,g), G(-,g)) \cong G(g,g)$, i.e. there is a bijection y mapping each natural transformation α onto the group element $\alpha_g(1_g)$. We must verify that this isomorphism of sets is also an isomorphism of groups. Let $\alpha, \beta \in \mathrm{Nat}(G(-,g), G(-,g))$. Then (recalling from Theorem 2.4 that the composition of natural transformations is a natural transformation), we have

$$\begin{aligned} y(\alpha \circ \beta) &= (\alpha \circ \beta)_g(1_g) \\ &= (\alpha_g \circ \beta_g)(1_g) \\ &= \alpha_g(\beta_g(1_g)) \\ &= \alpha_g(1_g) \circ \beta_g(1_g) \\ &= y(\alpha_g) \circ y(\beta_g) \end{aligned}$$

This shows that y preserves the group structure, since the composition of arrows $y(\alpha_g) \circ y(\beta_g)$ in the category is equivalent to the product $y(\alpha_g) * y(\beta_g)$ in the group. Hence, y is an isomorphism between G and the permutation group $\mathrm{Nat}(G(-,g), G(-,g))$. $\square$

3.3 Products and Limits

In Definition 2.4 and its examples, we saw the notion of a category of functors and the natural transformations between them. We then introduced the diagonal functor $\Delta : C \to C^J$ in Definition 2.23. Now that we have discussed universality, we are ready to apply this notion in order to define a *limit*.

Definition 3.11 ([1, Section 3.2])**.** Let C be a category with an index category J. If $c \in C$ and $\mathcal{F}$ is a diagram in C^J (i.e. a functor $J \to C$), then we call each natural transformation $\tau : \Delta c \dot{\to} \mathcal{F}$ a **cone** with **base** $\mathcal{F}$ and **vertex** c. Since Δc is the constant functor, each component τ_j $(j \in J)$ is an arrow $c \to \mathcal{F}j$ in C.

Thus, we can think of a cone as providing natural map from its vertex to the image of the index J under its base functor. We may then ask if there is a choice of vertex which gives us the most general map, i.e. whether there is a cone which is universal among cones with the same base. From this concept we get the dual notions of *limit* and *colimit*.

Definition 3.12 ([1, Section 3.4])**.** A **limit** is a *universal cone*, i.e. a natural transformation $u : \Delta r \dot{\to} \mathcal{F}$ which is universal among all natural transformations $\tau : \Delta c \dot{\to} \mathcal{F}$. In other words, for every $f : x \to y$ in the index category J (and thus for every commutative triangle in the cone) there exists a unique $g : c \to r$ which makes the following commute.

$$\begin{tikzcd}
r \arrow[d, "u_x"'] \arrow[dr, "u_y"' near start] & c \arrow[l, dashed, "g"'] \arrow[dl, "\tau_x"' near start] \arrow[d, "\tau_y"] \\
\mathcal{F}x \arrow[r, "\mathcal{F}f"'] & \mathcal{F}y
\end{tikzcd}$$

When this holds, we write $r = \lim \mathcal{F}$ in order to indicate that $\lim \mathcal{F}$ is the vertex of the universal cone.

Dually, a **colimit** is a natural transformation $v : \mathcal{F} \dot{\to} \Delta s$ which is universal among natural transformations $\tau : \mathcal{F} \dot{\to} \Delta c$, as shown in the commutative diagram below. Note that the duality reverses the arrows in the cones as well as the universal arrow g.

$$\begin{tikzcd}
s \arrow[rr, dashed, "g"] & & c \\
\mathcal{F}x \arrow[u, "v_x"] \arrow[urr, "\tau_x" near start] \arrow[rr, "\mathcal{F}f"'] & & \mathcal{F}y \arrow[u, "\tau_y"'] \arrow[ull, "v_y" near start]
\end{tikzcd}$$

When s satisfies this property, we write $s = \mathrm{colim}\mathcal{F}$.

It is worth noting briefly that when they exist, the limit and colimit are both unique up to a unique isomorphism. This follows from the uniqueness of universal arrows proved in Theorem 3.1.

Example 3.13 ([1, Exercise 3.4.3])**.** If s is an initial object in an index category J, any functor $\mathcal{F} : J \to C$ has limit $\mathcal{F}s$.

Proof. For any object c in C, consider the cone $\tau : \Delta c \dot{\to} \mathcal{F}$ with apex c and base $\mathcal{F}$. For each arrow $g : s \to x$, this cone contains a commutative triangle

$$\begin{tikzcd}
c \arrow[d, "\tau_s"] \arrow[dr, "\tau_x"] & \\
\mathcal{F}s \arrow[r, "\mathcal{F}g"] & \mathcal{F}x
\end{tikzcd}$$

which shows $\tau_x = \mathcal{F}g \circ \tau_s$. Observe that $\mathcal{F}s$ forms another cone μ with vertex $\mathcal{F}s$ and base $\mathcal{F}$

$$\begin{tikzcd}
c \arrow[d, "\tau_s"] \arrow[ddr, "\tau_x"] & \\
\mathcal{F}s \arrow[d, "\mu_s"] \arrow[dr, "\mathcal{F}g"] & \\
\mathcal{F}s \arrow[r, "\mu_x"] & \mathcal{F}x
\end{tikzcd}$$

using $\mu_s = 1_{\mathcal{F}s}$ and $\mu_x = \mathcal{F}g$. Each cone τ is then uniquely determined by the component τ_s mapping the vertex of the cone τ onto the vertex $\mathcal{F}s$ of this new cone μ. Moreover, the diagram commutes because the two triangles commute on their own and

$$\mathcal{F}g \circ \tau_s = \mathcal{F}g \circ 1_{\mathcal{F}s} \circ \tau_s = \mu_x \circ \mu_s \circ \tau_s.$$

Therefore μ is the universal cone, i.e. $\mathcal{F}s = \lim\ \mathcal{F}$. By the dual construction, a terminal object $t \in J$ has $\mathcal{F}t = \text{colim}\ \mathcal{F}$ for any functor $\mathcal{F} : J \to C$. $\square$

Example 3.14. In Example 2.4, we noted that when the index category J is the category with two objects a, b and only their identity arrows, the functors $\mathcal{F} : J \to C$ (diagrams in C^J) are equivalent to all the ways of choosing pairs of objects and arrows in C. Thus, for an object $c \in C$, a cone $\tau : \Delta c \dot{\to} \mathcal{F}$ consists of two arrows $\tau_a : c \to \mathcal{F}a$ and $\tau_b : c \to \mathcal{F}a$ from the vertex to the respective objects in such a pair. Suppose the limit of $\mathcal{F}$ exists, with a universal cone $\pi : \Delta(\lim \mathcal{F}) \to \mathcal{F}$. Then there is a unique arrow $h : c \to \lim \mathcal{F}$ making the following diagram commute.

$$\begin{tikzcd}
 & c \arrow[dl, "\tau_a"'] \arrow[d, dashed, "h"] \arrow[dr, "\tau_b"] & \\
\mathcal{F}a & \lim \mathcal{F} \arrow[l, "\pi_a"] \arrow[r, "\pi_b"'] & \mathcal{F}b
\end{tikzcd}$$

This construction gives us the notion of a *product of objects* in a category. In the definition below, the projection arrows p, q are the components of the universal cone π, and the arrows f, g are the components of another cone τ.

Definition 3.15 ([1, Section 3.4])**.** A **product** of objects $a, b \in C$ is an object $a \times b \in C$ together with its **projection arrows** $p : a \times b \to a$ and $q : a \times b \to b$. The product satisfies the universal property that for any object $c \in C$ with arrows $f : c \to a$ and $g : c \to b$, there exists a unique arrow $f \times g : c \to a \times b$ making the following diagram commute.

$$\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[d, dashed, "f\times g"] \arrow[dr, "g"] & \\
a & a \times b \arrow[l, "p"] \arrow[r, "q"'] & b
\end{tikzcd}$$

If C has a product $a \times b$ for any two objects a, b, we say the category **has finite products**. To avoid ambiguity in such cases, we write $p_{a,b}$ and $q_{a,b}$ for the projection arrows belonging to the particular product $a \times b$.

Example 3.16 ([1, Section 3.4])**.** The category **Set** has finite products, and the product $A \times B$ of sets A, B is the familiar Cartesian product, containing all pairs $\langle a, b \rangle$ where $a \in A, b \in B$. The projection functions are then defined by $p(a, b) = a$ and $q(a, b) = b$. Clearly, any pair of functions $f : X \to A, g : X \to B$ can be uniquely factored through the Cartesian product so that $p \circ (f \times g) = f$ and $q \circ (f \times g) = q$.

Definition 3.17 ([1, Section 4.1])**.** Let C be a category with finite products, and recall that $C \times C$ is the Cartesian product containing all pairs of objects and pairs of arrows in C. Then we may define a **product functor** $\times : C \times C \to C$ which maps each pair $\langle a, b \rangle$ to the product $a \times b$, and maps each pair of arrows $\langle f, g \rangle : \langle a, b \rangle \to \langle c, d \rangle$ to the the unique arrow $f \times g : a \times b \to c \times d$ which makes the following diagram commute.

$$
\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "p_{a,b}"'] \arrow[r, "q_{a,b}"] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "p_{c,d}"'] \arrow[r, "q_{c,d}"] & d
\end{tikzcd}
$$

Additionally, for a fixed object $b \in C$, we may define another product functor $(- \times b)$. This functor maps objects $a \in C$ to the product $a \times b$, and maps arrows $f : y \to z$ in C to $f \times 1_b : y \times b \to z \times b$.

3.4 Adjunctions

In Definition 2.1, we saw one way to show that two categories A and X are structurally similar: an isomorphism of categories via an invertible functor $\mathcal{S} : A \to X$. However, as we noted earlier, this condition is often too strong to be useful. In this section, we introduce *adjunctions* as a more flexible way to compare two categories.

Definition 3.18 ([1, Section 4.1])**.** An **adjunction** is a triple $\langle \mathcal{F}, \mathcal{G}, \varphi \rangle$, consisting of functors $\mathcal{F} : X \to A$, $\mathcal{G} : A \to X$ along with a function φ assigning each pair of objects $x \in X, a \in A$ to a bijection of hom-sets

$$\varphi_{x,a} : A(\mathcal{F}x, a) \cong X(x, \mathcal{G}a),$$

natural in x and a. In particular, this naturality means that the following two diagrams commute for all $k : a \to a'$ and $h : x \to x'$.

$$\begin{array}{ccc} A(\mathcal{F}x, a) & \xrightarrow{\varphi_{x,a}} & X(x, \mathcal{G}a) \\ \Big\downarrow{\scriptstyle A(\mathcal{F}x,k)} & & \Big\downarrow{\scriptstyle X(x,\mathcal{G}k)} \\ A(\mathcal{F}x, a') & \xrightarrow{\varphi_{x,a'}} & X(x, \mathcal{G}a') \end{array} \qquad \begin{array}{ccc} A(\mathcal{F}x, a) & \xrightarrow{\varphi_{x,a}} & X(x, \mathcal{G}a) \\ \Big\downarrow{\scriptstyle A(\mathcal{F}x,\mathcal{F}h)} & & \Big\downarrow{\scriptstyle X(x,h)} \\ A(\mathcal{F}x', a) & \xrightarrow{\varphi_{x',a}} & X(x', \mathcal{G}a') \end{array}$$

We then say that $\mathcal{F}$ and $\mathcal{G}$ are **adjoint** functors. In particular, $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is right adjoint to $\mathcal{F}$, written as $\mathcal{F} \dashv \mathcal{G}$.

Example 3.19. Suppose categories A and X are isomorphic via an invertible functor $\mathcal{S} : A \to X$. Then $\mathcal{S}^{-1}$ and $\mathcal{S}$ are adjoint functors.

Proof. Since the categories A and X are isomorphic via $\mathcal{S}$, we have a bijection of hom-sets $A(\mathcal{S}^{-1}x, \mathcal{S}^{-1}x') \cong X(x, x')$, natural in x and x'. Given any object $x' \in X$, there exists $a \in A$ with $x' = \mathcal{S}a$, namely $a = \mathcal{S}^{-1}x'$. Substituting into the bijection of hom-sets, we get $A(\mathcal{S}^{-1}x, a) \cong X(x, \mathcal{S}a)$, natural in x and a. This is exactly the form of an adjunction $\mathcal{S}^{-1} \dashv \mathcal{S}$. $\square$

Example 3.20. Recall that $\mathbf{1}$ is the category with a single object $*$ whose only arrow is the identity arrow 1_*. Let $\mathcal{F} : C \to \mathbf{1}$ be the functor defined by $\mathcal{F}c = *$, $\mathcal{F}f = 1_*$ for all objects $c \in C$ and arrows f in C. Then this functor has a right adjoint $\mathbf{1} \to C$ mapping $*$ to a terminal object in C. Moreover, the right adjoints of $\mathcal{F}$ are naturally isomorphic to the terminal objects in C.

Proof. Assume an adjunction $\langle \mathcal{F}, \mathcal{G}, \varphi \rangle$ exists with a suitable functor $\mathcal{G}$:

$\mathbf{1} \to C$. Then there is a bijection of hom-sets

$$\varphi_{*,c} : \mathbf{1}(\mathcal{F}c, *) \cong C(c, \mathcal{G}*)$$

which is natural in c. Hence, since $\mathbf{1}(\mathcal{F}c, *) = \mathbf{1}(*, *) = \{1_*\}$, this bijection ensures that each $C(c, \mathcal{G}*)$ has only one element. In other words, there is a unique arrow $c \to \mathcal{G}*$ for each object $c \in C$, and therefore $\mathcal{G}*$ is a terminal object of C. To complete the definition of $\mathcal{G}$, note that its arrow function is determined by $\mathcal{G}1_* = 1_{\mathcal{G}*}$. Furthermore, observe that each unique adjoint functor $\mathcal{G}$ has its own bijection of hom-sets and hence is mapped to a unique terminal object. Conversely, each terminal object in C corresponds to a unique right adjoint of $\mathcal{F}$. Thus, the right adjoints of $\mathcal{F}$ are isomorphic to the terminal objects in C. $\square$

So far we have dealt with adjunctions from the perspective of a particular bijection of hom-sets. We may also formulate an equivalent definition in terms of two natural transformations: the **unit** η and **counit** ϵ.

Theorem 3.21 ([1, Theorem 4.2.2])**.** *An adjunction $\langle \mathcal{F}, \mathcal{G}, \varphi \rangle$ determines a natural transformation $\eta : I_X \dot{\to} \mathcal{G}\mathcal{F}$ and a natural transformation $\epsilon : \mathcal{F}\mathcal{G} \dot{\to} I_A$. Moreover, each component η_x is a universal arrow from x to $\mathcal{G}$, and each component ϵ_a is universal from $\mathcal{F}$ to a.*

Proof ([1]). Suppose $\mathcal{F} \dashv \mathcal{G}$ for functors $\mathcal{F} : X \to A$ and $\mathcal{G} : A \to X$. Then we have a bijection of hom-sets

$$\varphi_{x,\mathcal{F}x} : A(\mathcal{F}x, \mathcal{F}x) \cong X(x, \mathcal{G}\mathcal{F}x).$$

Moreover, this bijection is natural in x, meaning the following diagrams commute for all objects $a \in A, x, \in X$ and arrows $k : \mathcal{F}x \to a, h : x \to \mathcal{G}a$.

$$\begin{array}{ccc} A(\mathcal{F}x, \mathcal{F}x) & \xrightarrow{\varphi_{x,\mathcal{F}x}} & X(x, \mathcal{G}\mathcal{F}x) \\ \big\downarrow{\scriptstyle A(\mathcal{F}x,k)} & & \big\downarrow{\scriptstyle X(x,\mathcal{G}k)} \\ A(\mathcal{F}x, a) & \xrightarrow{\varphi_{x,a}} & X(x, \mathcal{G}a) \end{array} \qquad \begin{array}{ccc} A(\mathcal{F}x, \mathcal{F}x) & \xrightarrow{\varphi_{x,\mathcal{F}x}} & X(x, \mathcal{G}\mathcal{F}x) \\ \big\downarrow{\scriptstyle A(\mathcal{F}h,\mathcal{F}x)} & & \big\downarrow{\scriptstyle X(h,\mathcal{G}\mathcal{F}x)} \\ A(\mathcal{F}\mathcal{G}a, \mathcal{F}x) & \xrightarrow{\varphi_{x,a}} & X(\mathcal{G}a, \mathcal{G}\mathcal{F}x) \end{array}$$

Define $\eta_x = \varphi_{x,\mathcal{F}x}(1_{\mathcal{F}x})$. By following the image of $1_{\mathcal{F}x}$ through both sides of the commutative squares, we get

$$\begin{aligned} \varphi(k) &= \mathcal{G}k \circ \eta_x \\ \varphi(\mathcal{F}h) &= \eta_x \circ h \end{aligned} \tag{1}$$

(keeping in mind that the contravariant hom-functor *pre-composes*). Hence, for any $f : x \to x'$, the diagram

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & \mathcal{G}\mathcal{F}x \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathcal{G}\mathcal{F}f} \\ x' & \xrightarrow{\eta_{x'}} & \mathcal{G}\mathcal{F}x' \end{array}$$

commutes, because

$$\begin{aligned} \eta_{x'} \circ f &= \varphi(\mathcal{F}f) \\ &= \mathcal{G}\mathcal{F}f \circ \eta_x. \end{aligned}$$

Thus the image of $1_{\mathcal{F}x}$ under $\varphi_{x,\mathcal{F}x}$ defines a component η_x of a natural transformation $\eta : I_X \dot{\to} \mathcal{G}\mathcal{F}$. Furthermore, by (1) and the fact that each φ is a bijection, any arrow $f : x \to \mathcal{G}a$ corresponds to a unique $f' : \mathcal{G}\mathcal{F}x \to \mathcal{G}a$

such that $f = \varphi(f') = \mathcal{G}f' \circ \eta_x$. Therefore, each component η_x is universal from x to $\mathcal{G}$. A similar proof shows that ϵ may be defined with components $\epsilon_a = \varphi^{-1}(Ga)$, and that ϵ satisfies the naturality and universality conditions.

Finally, using the fact that $\varphi(k) = \mathcal{G}k \circ \eta_x$, taking $x = \mathcal{G}a$ and $k = \epsilon_a$ gives

$$\begin{aligned}
1_{\mathcal{G}a} &= \varphi(\varphi^{-1}(1_{\mathcal{G}}a)) \\
&= \varphi(\epsilon_a) \\
&= \mathcal{G}\epsilon_a \circ \eta_{\mathcal{G}a} \\
&= (\mathcal{G}\epsilon \circ \eta\mathcal{G})_a.
\end{aligned}$$

Note that the last line of this equation is a component of a composition of natural transformations (Theorem 2.4), and $G\epsilon$ and ηG are themselves compositions of transformations with the functor G (Example 2.18). Additionally, $1_{\mathcal{G}a}$ is a component of the identity functor $1_{\mathcal{G}} : \mathcal{G} \dot{\to} \mathcal{G}$. Hence, we get that the following *triangle identity*, whose objects are functors and arrows natural transformations, commutes.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\eta\mathcal{G}"] \arrow[d, "1_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[dl, "G\epsilon"] \\
\mathcal{G} &
\end{tikzcd}$$

Likewise, applying similiar reasoning with $1_{\mathcal{F}x}$ gives the triangle identity below.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathcal{F}\eta"] \arrow[d, "1_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[dl, "\epsilon F"] \\
\mathcal{F} &
\end{tikzcd}$$

$\square$

Example 3.22 ([1, Exercise 4.1.3])**.** Let C be a category with finite products. For the adjunction $\langle \Delta, \times, \varphi \rangle$ (where $\times$ is the product functor of Definition 3.3 and Δ the diagonal functor of Definition 2.23), each component δ_c of the unit δ is the unique arrow which makes the following diagram commute.

$$\begin{tikzcd} & c \arrow[dl, "1_c"'] \arrow[d, dashed, "\delta_c"] \arrow[dr, "1_c"] & \\ c & c \times c \arrow[l, "p"] \arrow[r, "q"'] & c \end{tikzcd} \tag{2}$$

Proof. We will first show that the counit of this adjunction is defined by all the pairs of projection arrows p and q. To do this, we must show such pairs satisfy the naturality and universality conditions described in Theorem 3.4. First, let $a, b, c, d \in C$, and suppose C has arrows $f : a \to c$, $g : b \to d$. Then $C \times C$ has a corresponding arrow $\langle f, g \rangle : \langle a, b \rangle \to \langle c, d \rangle$. The image of this arrow under the functor $\times$ is defined as the unique arrow $f \times g : a \times b \to c \times d$ which makes the following diagram in C commute.

$$\begin{tikzcd} a \arrow[d, "f"] & a \times b \arrow[l, "p_{a,b}"'] \arrow[r, "q_{a,b}"] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\ c & c \times d \arrow[l, "p_{c,d}"'] \arrow[r, "q_{c,d}"] & d \end{tikzcd}$$

Hence, the following diagram in $C \times C$ must also commute.

$$\begin{tikzcd} \langle a \times b, a \times b \rangle \arrow[r, "\langle p_{a,b}, q_{a,b} \rangle"] \arrow[d, "\langle f \times g, f \times g \rangle"'] & \langle a, b \rangle \arrow[d, "\langle f, g \rangle"] \\ \langle c \times d, c \times d \rangle \arrow[r, "\langle p_{c,d}, q_{c,d} \rangle"'] & \langle c, d \rangle \end{tikzcd}$$

In this diagram, the objects on the left side are the images of $\langle a, b\rangle$ and $\langle c, d\rangle$ under the composition of functors $\Delta\times$, whereas the objects on the right are the images of those objects under the identity functor. Hence, each arrow $\langle p_{a,b}, q_{a,b}\rangle$ defines a component $\epsilon_{\langle a,b\rangle}$ of a natural transformation $\epsilon : \Delta \times \dot{\to} I_{C\times C}$.

Next we must show that ϵ is universal to $\langle a, b\rangle$ from Δ. For all objects $a, b, c \in C$ and arrows $j : c \to a,\ k : c \to b$, the definition of the product $a \times b$ ensures there is a unique arrow h such that $p_{a,b} \circ h = j$ and $q_{a,b} \circ h = k$, as in the commutative diagram:

$$\begin{tikzcd}
 & c \arrow[dl, "j"'] \arrow[d, dashed, "h"] \arrow[dr, "k"] & \\
a & a \times b \arrow[l, "p_{a,b}"] \arrow[r, "q_{a,b}"'] & b
\end{tikzcd}$$

Hence there is a unique arrow $\langle h, h\rangle : \Delta c \to \Delta(a \times b)$ such that $\langle j, k\rangle = \epsilon_{\langle a,b\rangle} \circ \langle h, h\rangle$, which is exactly what we needed to show in order to prove the universality of ϵ. Therefore ϵ is the counit of the adjunction, as claimed.

Recall from Example 2.18 that certain functors may be composed and pre-composed with natural transformations. Since the unit of the adjunction is a natural transformations $\delta : I_C \dot{\to} \times \Delta$, composing the functor Δ with δ gives a natural transformation $\Delta\delta : \Delta \dot{\to} \Delta \times \Delta$. Likewise, composing the counit with Δ gives a natural transformation $\epsilon\Delta : \Delta \times \Delta \dot{\to} \Delta$. With these compositions, we may construct the following triangle identity, as shown in Theorem 3.4.

$$\begin{tikzcd}
\Delta \arrow[r, "\Delta\delta"] \arrow[d, "1_\Delta"'] & \Delta \times \Delta \arrow[dl, "\epsilon\Delta"] \\
\Delta &
\end{tikzcd}$$

In particular, consider the components of these transformations determined by a pair $\langle c, c \rangle \in C \times C$. The diagram above gives $\epsilon_{\langle c,c \rangle} \circ \langle \delta_c, \delta_c \rangle = \langle 1_c, 1_c \rangle$. Equivalently, $p_{c,c} \circ \delta_c = 1_c$ and $q_{c,c} \circ \delta_c = 1_c$, making diagram (2) commute and completing the proof. $\square$

4 Lawvere's Fixed Point Theorem

4.1 Cartesian Closed Categories

Lawvere's Fixed Point Theorem offers a generalization of diagonalization arguments to all *Cartesian closed categories*. Before we may understand the theorem, we must define this notion.

Definition 4.1 ([1, Section 4.6])**.** A category is called **Cartesian closed** if it has a terminal object, finite products, and finite exponentials.

We have already seen the notion of a product $a \times b$ in a category in Definition 3.3; the first requirement of this definition states that such a product must exist for any two objects a, b, in which case we say the category "has finite products". Similarly, the second requirement states that the category must have an *exponential* object a^b for all objects a, b. In Example 2.4, we dealt with the specific case of an exponential object in **Set** as a set of all functions from one set to another — in other words, a hom-set. In general, the notion of an exponential object serves to capture this sense in which an object can behave like a collection of arrows between other objects.

Definition 4.2 ([1, Section 4.6])**.** An object a^b together with an **evaluation arrow** $e_{a,b} : (a^b \times b) \to a$ is called an **exponential object** if the evaluation arrow is natural in a and universal from $- \times b \to a$ (the product functor with a fixed object, defined in Definition 3.3). This universality condition says that for any arrow $g : a \times b \to a$, there exists a unique arrow $\lambda g : a \to a^b$

such that the following diagram commutes.

$$\begin{tikzcd}
a \times b \arrow[d, dashed, "\lambda g \times 1_b"'] \arrow[dr, "g"] & \\
a^b \times b \arrow[r, "e_{a,b}"'] & a
\end{tikzcd}$$

Note that in the case of **Set** (replacing the lowercase objects with uppercase sets), the above commutative diagram describes the process of *currying functions*, common in computer science and the study of functional programming. To curry a function $g(a, b)$ of two variables ($a \in A, b \in B$), we construct a function λg assigning each a to a function $B \to A$ so that $e_{A,B}(\lambda g(a), b) = \lambda g(a)(b) = g(a, b)$.

4.2 Point-Surjectivity

Definition 4.3 ([2, Section 1])**.** Let C be a Cartesian closed category with a terminal object $*$. An arrow $g : x \to z$ in C is called **point-surjective** if and only if for every $k : * \to z$ there exists $j : * \to x$ such that $j \circ g = k$.

Example 4.4. In **Set**, point-surjectivity is familiar: an arrow $g : X \to Z$ is point-surjective if and only if it is a surjective function.

Proof. Since a terminal object in this category is a set containing a single element (say, $* = \{0\}$), each function $* \to S$ simply chooses an element of S, and each element corresponds to such a function. Thus, we can treat arrows $* \to S$ as equivalent to elements of S, according to a bijection

$$\hom(*, S) \cong S.$$

Suppose that g is point-surjective, so that each arrow $z : * \to Z$ has some $x : * \to X$ such that $g \circ x = z$. Then there are unique set elements $z' \in Z$, $x' \in X$ such that $z(0) = z'$ and $x(0) = x'$, and thus $g(x') = z'$. Note that we may also work backwards here to define an arrow $z : * \to Z$ given any choice of set element $z' \in Z$. Therefore, point-surjectivity of g implies that g is a surjective function. The argument for the converse is similar, making use of our ability to translate between set elements and maps from a terminal object. $\square$

Definition 4.5 ([2, Section 1])**.** Let C be a Cartesian closed category with a terminal object $*$. An arrow $g : x \to y^a$ is called **weakly point-surjective** if and only if for every arrow $f : a \to y$ there exists $j : * \to x$ such that for all $l : * \to a$,

$$e_{y,a} \circ (gj \times l) = f \circ l.$$

This requirement essentially states that the arrow g must allow the exponential y^a to behave similar to a hom-set: for any arrow $f : a \to y$, we may find an "input" j such that evaluating the composition $g \circ j$ at any "input" l achieves the same result as composing f with l.

4.3 Fixed Points

Definition 4.6 ([2, Section 1])**.** An object y is said to have the **fixed point property** if for any arrow $f : y \to y$ there exists $h : * \to y$ such that $f \circ h = h$.

With these definitions, we are now ready to state and prove Lawvere's Fixed Point Theorem.

Theorem 4.7 (Lawvere's Fixed Point Theorem, [2, Theorem 1.1])**.** *Let C be a Cartesian closed category. If C has a weakly point-surjective arrow $g : a \to y^a$, then y has the fixed point property.*

Proof ([2]). Assume $g : a \to y^a$ is a weakly point-surjective arrow in C and $*$ is a terminal object in C. Since C is Cartesian closed, it has an arrow $g' : a \times a \to y$ such that $g = \lambda g'$. Then we have $g' = e_{y,a} \circ (g \times 1_a)$ making the following diagram commute.

$$\begin{tikzcd} a \times a \arrow[d, "g \times 1_a"'] \arrow[dr, "g'"] & \\ y^a \times a \arrow[r, "e_{y,a}"'] & y \end{tikzcd}$$

Let t be some arrow $y \to y$. Define $f : a \to y$ as the composition $f = t \circ g' \circ \delta_a$, where δ is the unit of the adjunction $\langle \Delta, \times, \varphi \rangle$, as described in Example 3.4. Then because of the above commutative diagram and the assumption that g is weakly point-surjective, there exists $j : * \to a$ such that for all $l : * \to a$,

$$\begin{aligned} g' \circ (j \times l) &= e_{y,a} \circ (gj \times l) \\ &= f \circ l \\ &= t \circ g' \circ (\delta_a \circ l) \\ &= t \circ g' \circ (l \times l). \end{aligned}$$

Thus, defining an arrow $h : * \to y$ by $h = g' \circ (j \times j)$, we have $h = t \circ h$. Therefore, y has the fixed point property. □

4.4 Diagonalization Arguments

The following corollary describes the contrapositive of Theorem 4.7 in the category **Set**.

Corollary 4.8. *Suppose Y is a set with a function $f : Y \to Y$ such that $f(y) \neq y$ for all $y \in Y$. Then for any set A, there is no surjective function from A to the set of all functions $A \to Y$.*

The following two examples illustrate how this result can be used to generalize diagonalization arguments.

Theorem 4.9 (Cantor)**.** *For any set A, there is no bijection between A and its power set $\mathcal{P}(A)$.*

Proof ([2, Section 1]). Let $Y = \{0, 1\}$. Note that there is an equivalence between functions $g : A \to Y$ and subsets of A, as defined by $\{a \in A \mid g(a) = 1\} \subseteq A$. Each such function determines a unique subset of A, and vice versa. Hence $Y^A \cong \mathcal{P}(A)$. Finally, define a function $f : Y \to Y$ by $f(0) = 1$ and $f(1) = 0$. By Lawvere's theorem, there is no surjective function $A \to \mathcal{P}(A)$, and so there is no bijection. □

Theorem 4.10 (Cantor)**.** *There is no bijection between $\mathbb{N}$ and $\mathbb{R}$.*

Proof. Let $Y = \{0, 1, \ldots, 9\}$, the set of base-10 digits. Observe that a function $g : \mathbb{N} \to Y$ corresponds to an assignment of digits to a real number

$$0.g(1)g(2)g(3)\cdots \in [0, 1]$$

and conversely we may use the digits of any real $x \in [0,1]$ to construct such a function. Hence $Y^{\mathbb{N}} \cong [0,1]$. To apply Lawvere's theorem, we may define a function $f : Y \to Y$ by

$$f(n) = \begin{cases} n+1 & \text{if } 0 \leq n \leq 8 \\ 0 & \text{if } n = 9 \end{cases}$$

so that f does not have a fixed point. It follows that there is no surjective function $\mathbb{N} \to [0,1] \subseteq \mathbb{R}$, and thus there is no bijection between $\mathbb{R}$ and $\mathbb{N}$. $\square$

References

[1] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[2] F. William Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications, II (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Two)*, pages 134–145. Springer, Berlin, 1969.

[3] Robert Arnold Beezer. *A first course in linear algebra*. 3.5th edition, 2016.