Matrix algebra

September 10th, 2024

Here are some key ideas from sections 8.4 and 8.5.

$m \times n$

- A **matrix** is an array of vectors. For a given matrix A having m rows and n columns, the entry a_{ij} is in the i th row and j th column.

$$A = \begin{array}{c c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} & \\ \begin{bmatrix} 0 & 7 & 1 \\ 2 & 9 & 2 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \end{array} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

2×3

- If the number of rows is the same as the number of columns, we say the matrix is square.
- The **transpose** of A is written as A^T and is obtained by interchanging the rows and columns. The transpose of the above matrix is

$$A^T = \begin{bmatrix} 0 & 2 \\ 7 & 9 \\ 1 & 2 \end{bmatrix}$$

3×2

$A: m \times n$
$A^T: n \times m$

- When we **multiply** two matrices A and B, we need that the number of columns in A is equal to the number of rows in B. A, B $\quad \overset{A}{m \times n} \quad \overset{B}{n \times p}$
- The entry in the ith row and jth column of AB is the dot product of the ith row of A and the jth column of B. For example,

$$A = \begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix}, \quad AB = \begin{array}{c c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} & \\ \begin{bmatrix} 13 & 21 & 67 \\ 24 & -78 & -9 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \end{array}.$$

Problem 1: (Stewart & Day 8.4) Just as with vectors, adding matrices and multiplying by scalar are done coordinate-wise. Suppose we have

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & x \\ a & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}.$$

Find $A - 3C$ and $5B - A$.

My Attempt:

Solution:

$$A - 3C = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} - 3\begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} - \begin{bmatrix} 27 & 6 \\ 21 & 30 \end{bmatrix} = \boxed{\begin{bmatrix} -25 & -1 \\ -20 & -23 \end{bmatrix}}$$

$$5B - A = 5\begin{bmatrix} 7 & x \\ a & 5 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 5x \\ 5a & 25 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} 33 & 5x-5 \\ 5a-1 & 18 \end{bmatrix}}$$

Problem 2: (Stewart & Day 8.4) For a matrix C, the quantity C^n represents multiplying C by itself n times.

$$C = \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} , \begin{array}{c} r_1 \\ r_2 \end{array}$$

find C^2.

My Attempt:

Solution:

$$C^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 \\ r_2 \cdot c_1 & r_2 \cdot c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow r_1, c_2$$

2x2 identity

Problem 3: (Stewart & Day 8.4) For the same matrix C as in Problem 2, find C^3, C^4, and C^5.

My Attempt:

Solution:

$$C^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Problem 4: (Stewart & Day 8.4) For arbitrary 2×2 matrices A, B, and C, verify the following identity:

$$A(B + C) = AB + AC.$$

Hint: set the entries equal to variables.

My Attempt:

Solution:

$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} b_{11} + a_{11} c_{11} + a_{12} b_{21} + a_{12} c_{21} & a_{11} b_{12} + a_{11} c_{12} + a_{12} b_{22} + a_{12} c_{22} \\ a_{21} b_{11} + a_{21} c_{11} + a_{22} b_{21} + a_{22} c_{21} & a_{21} b_{12} + a_{21} c_{12} + a_{22} b_{22} + a_{22} c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} c_{12} + a_{12} c_{21} & a_{11} c_{12} + a_{12} c_{22} \\ a_{21} c_{11} + a_{22} c_{21} & a_{21} c_{12} + a_{22} c_{22} \end{bmatrix} = AB + AC$$

Problem 5: (Stewart & Day 8.4) For

$$F = \begin{bmatrix} y & 0 & y \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 9 & 7 & 8, \end{bmatrix}$$

find FG.

My Attempt:

Solution:

$$\begin{array}{cc} & \begin{array}{ccc} c_1 & c_2 & c_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} r_1 \rightarrow \\ r_2 \rightarrow \\ r_3 \rightarrow \end{array} \begin{bmatrix} y & 0 & y \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 9 & 7 & 8 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \langle y,0,y\rangle \cdot \langle 1,5,9\rangle & \langle y,0,y\rangle \cdot \langle 2,4,7\rangle & \langle y,0y\rangle \cdot \langle 3,6,8\rangle \\ \langle 0,1,0\rangle \cdot \langle 1,5,9\rangle & \langle 0,1,0\rangle \cdot \langle 2,4,7\rangle & \langle 0,1,0\rangle \cdot \langle 3,6,8\rangle \\ \langle 0,1,1\rangle \cdot \langle 1,5,9\rangle & \langle 0,1,1\rangle \cdot \langle 2,4,7\rangle & \langle 0,1,1\rangle \cdot \langle 3,6,8\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 10y & 9y & 11y \\ 5 & 4 & 6 \\ 14 & 11 & 14 \end{bmatrix}$$

Problem 6: (Stewart & Day 8.4) Find the transposes of the following matrices.

a) $\begin{bmatrix} 3X \\ 1 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 3 & 9 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 & 7 \\ 8 & 3 & 6 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

My Attempt:

Solution:

a) $\begin{bmatrix} 3x & 1 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 8 \\ 1 & 3 \\ 7 & 6 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Challenge Problem: (Stewart & Day 8.4) Show that for any $n \times n$ matrices A and B,

$$(AB)^T = B^T A^T.$$

For $C = AB$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. So $c_{ij}^T = \sum_{k=1}^{n} a_{jk} b_{ki}$.

If $E = B^T A^T$, then $e_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk}$. So $c_{ij}^T = e_{ij}$ ✓

(use the formula in sec. 8.4)