Continuity, the definition of the derivative

October 9th, 2024

Here are some key ideas from sections 2.5, 3.1, and 3.2.

- A function is **continuous** at a if
$$\lim_{x\to a} f(x) = f(a)$$
- If you zoom in on the graph of a **tangent line** to the curve at a, you will see that it touches $f(a)$ at exactly one point.
- The slope of the tangent line to the curve f at a is the slope of the curve at a, written as $f'(a)$. This can be thought of as the instantaneous rate of change.
- We can use secant lines (intersect at two points) to approximate tangent lines.
- Mathematically, this idea is called the **derivative**, and can be written as
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
- Another valid expression is
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad *$$
- We can replace ~~x~~ a with ~~a~~ x in the two equations above to express the derivative as a function.
- A function is differentiable if the derivative exists

Midterm practice (Persson '14 MT1): Find each of the following limits.

(a) $\lim_{x\to-\infty} \frac{x(3x-4)+2}{5x^2-10}$ very good math prof here!

(b) $\lim_{x\to-3} \frac{x^2-9}{x^2+2x-3}$

(c) $\lim_{x\to 1} \frac{\frac{1}{1+x^4}-\frac{1}{2}}{x-1}$

N.B: You do not need to show this much work :)

My Attempt: Solution:

a) $$\lim_{x\to-\infty} \frac{x(3x-4)+2}{5x^2-10} = \lim_{x\to-\infty} \frac{3x^2-4x+2}{5x^2-10}\cdot\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to-\infty}\frac{3-\cancel{\frac{4}{x}}^{\,0}+\cancel{\frac{2}{x^2}}^{\,0}}{5-\cancel{\frac{10}{x^2}}^{\,0}} = \boxed{\frac{3}{5}}$$

b) Direct substitution yields an indeterminate form, so we must modify the function.

$$\lim_{x\to-3}\frac{x^2-9}{x^2+2x-3} = \lim_{x\to-3}\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x-1)} = \lim_{x\to-3}\frac{x-3}{x-1} = \frac{-6}{-4} = \boxed{\frac{3}{2}}$$

c) Direct substitution is indeterminate again, so we must modify.

$$\lim_{x\to1}\frac{\frac{1}{1+x^4}-\frac{1}{2}}{x-1}\cdot\frac{(2(1+x^4))}{(2(1+x^4))}$$
$$= \lim_{x\to1}\frac{2-(1+x^4)}{2(x-1)(1+x^4)}$$
$$= \lim_{x\to1}\frac{1-x^4}{2(x-1)(1+x^4)}$$
$$\lim_{x\to1}\frac{(1-x^2)(1+x^2)}{2(x-1)(1+x^4)}$$
$$= \lim_{x\to1}\frac{(1-x)(1+x)(1+x^2)}{2(x-1)(1+x^4)}$$
$$= \lim_{x\to1}\frac{-\cancel{(x-1)}(1+x)(1+x^2)}{2\cancel{(x-1)}(1+x^4)}$$
$$= -\frac{(2)(2)}{(2)(2)} = \boxed{-1}$$

Problem 1: (Stewart 3.1) A curve has equation $y = f(x)$.

(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.

(b) Write an expression for the slope of the tangent line at P.

My Attempt:

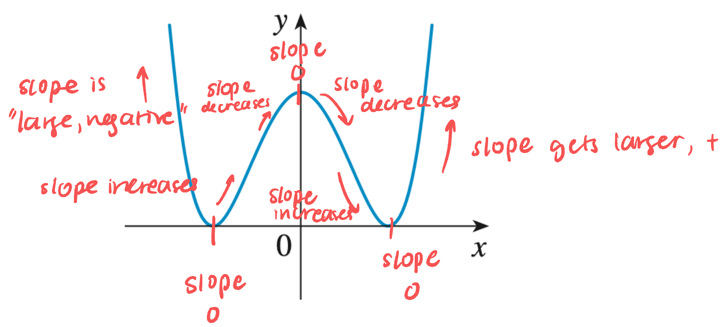
Solution:

a) The secant line connects P and Q.
The slope is $y_2 - y_1 / x_2 - x_1 = \boxed{f(x) - f(3) / x-3}$

b) The tangent line can be found by moving Q towards 3. So the slope is

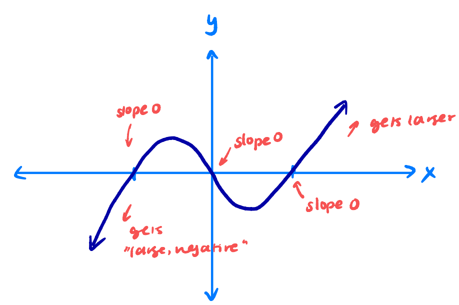
$$\boxed{\lim_{x \to 3} \frac{f(x) - f(3)}{x-3}}$$

Problem 2: (Stewart 3.1) For the given graph of $f(x)$ below, sketch a graph of $f'(x)$.



My Attempt:

Solution:



Problem 3: Use a limit definition of the derivative to find the equation of the tangent line to $y = \frac{2x+1}{x+2}$ at $(1, 1)$.

My Attempt:

Solution:

Let $f(x) = \frac{2x+1}{x+2}$. Then

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{\left(\frac{2x+1}{x+2} - \frac{2(1)+1}{1+2} \right)}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{2x+1}{x+2} - 1}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{2x+1-(x+2)}{x+2}}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{x-1}{x+2}}{x-1}$$

$$= \lim_{x \to 1} \frac{1}{x+2} = \boxed{\frac{1}{3}}$$

Use point-slope:

$$\boxed{y - 1 = \frac{1}{3}(x-1)}$$

Problem 4: (Stewart 3.1) Use a limit definition of the derivative to find the equation of the tangent line to $y = 4x - 3x^2$ at the point $(2, -4)$.

My Attempt:

Solution:

let $f(x) = 4x - 3x^2$.

then $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2}$

$= \lim_{x \to 2} \frac{4x - 3x^2 - 4(2) + 3(2)^2}{x-2}$

$= \lim_{x \to 2} \frac{4x - 3x^2 + 4}{x-2}$

$= \lim_{x \to 2} \frac{-(3x+2)\cancel{(x-2)}}{\cancel{x-2}}$

$= \lim_{x \to 2} -(3x+2) = \boxed{-8}$

Then use point-slope:

$\boxed{y + 4 = -8(x-2)}$

Problem 5: (Stewart 3.1) For the function $f(x) = x^{-2}$, find $f'(a)$ using a limit definition of the derivative.

My Attempt:

Solution:

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$

$= \lim_{x \to a} \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x-a}$

$= \lim_{x \to a} \frac{\frac{a^2 - x^2}{x^2a^2}}{x-a}$

$= \lim_{x \to a} \frac{\frac{(a-x)(a+x)}{x^2a^2}}{x-a}$

$= \lim_{x \to a} \frac{-\cancel{(x-a)}(a+x)}{x^2a^2\cancel{(x-a)}}$

$= \lim_{x \to a} \frac{-(a+x)}{x^2a^2}$

$= \frac{-2a}{a^2a^2} = \boxed{\frac{-2}{a^3}}$

Problem 6: (Stewart 3.2) State the domain of $f(x) = x + \sqrt{x}$ and the domain of its derivative.

$x^{\frac{1}{2}}$ $\frac{1}{2}x^{-\frac{1}{2}}$

My Attempt:

Solution:

1. Find the derivative: you should have gotten $f'(x) = 1 + \frac{1}{2\sqrt{x}}$.
2. Domain of function: $[0, \infty)$, since we cannot take the sqrt of a negative number
3. Domain of derivative: $(0, \infty)$, since we can't divide by zero.

Challenge problem: Assume that

$$f(x) = \begin{cases} \frac{1}{4}x^3 - \frac{1}{2}x^2, & \text{if } x \geq 2 \\ \frac{6x-6}{x^2+2}, & \text{if } x < 2 \end{cases}$$

Not continuous, and therefore not differentiable.

Determine if f is differentiable at $x = 2$, i.e., determine if $f'(2)$ exists.