Math 10A Fall 2024 Worksheet 15

October 17, 2024

1 Quotient rule

1. Compute the derivative of the given function.
   (a) $P(x) = \frac{\sin(x)}{x^3-1}$
   (b) $L(x) = \frac{(1-4x)(2+x)}{3+9x}$
   (c) $f(z) = \frac{3z+z^4}{2z^2+1}$
   (d) $g(x) = \frac{1}{\arctan(x)}$
2. Derive the formulas for the derivatives of $\tan(x), \cot(x), \sec(x)$, and $\csc(x)$ by using the quotient rule and the derivatives of $\cos(x)$ and $\sin(x)$.
3. Compute the second derivatives of all six inverse trig functions.
4. Suppose $f(x) = g(x)/h(x)$ and that $g(a) = 1, h(a) = 2, g'(a) = 3$, and $h'(a) = 4$. Compute $f'(a)$.

2 Logarithmic and implicit differentiation

1. Use logarithmic differentiation to compute the derivatives of the following functions. (You can do some of these without logarithmic differentiation, but it might be a lot harder).
   (a) $t(x) = x^x$
   (b) $f(x) = (2x+1)^5(x^4-3)^6$
   (c) $f(z) = \sqrt{z}e^{z^2}(z^2+1)^{10}$
   (d) $h(y) = y^{1/(1+y^2)}$
2. Explain why the chain rule, power rule, and the formula for the derivative of an exponential are unhelpful for computing $\frac{d}{dx}x^x$ without taking logarithms first.
3. (a) Compute $\frac{dy}{dx}$ if $y^x = x^y$ (for $x, y > 0$).
   (b) Is y a function of x?
   (c) Compute the tangent line to this curve at the point $(1, 1)$.
   (d) Something weird should happen when you try to find $\frac{dy}{dx}$ at the point (e, e). What's going on?
   (e) Graph $y^x = x^y$ on Desmos to check your work and get a better sense of what is going on.

3 Linear approximation

1. Use a first-order linear approximation to estimate the following numbers.
   (a) $e^{0.05}$
   (b) $\sin(3.1)$
   (c) $(1.01)^{-20}$
   (d) $\log_2(257)$
   (e) $\arcsin(0.99)$
   (f) $\tan(\pi/4 + 0.02)$
2. Justify the following approximation: $\sin(x) \approx \tan(x) \approx e^x - 1 \approx x$ when $|x|$ is small.
3. In which of the following cases should you suspect that the linearization of $f(x)$ at a might be a poor estimate of $f(a+h)$?
   (a) When $|h|$ is large.
   (b) When $|a|$ is large.
   (c) When $|f(a)|$ is large.
   (d) When $f(x)$ has a jump discontinuity at b for some $a < b < a+h$.
   (e) When $f(x)$ is a polynomial.
   (f) When you have to use the quotient rule to compute the derivative of f at a.

4 Solutions

4.1 Quotient rule

1. (a)
$$P'(x) = \frac{\cos(x)(x^3-1) - 3x^2\sin(x)}{(x^3-1)^2}$$

   (b)
$$L'(x) = \frac{d}{dx}\left(\frac{-4x^2-7x+2}{3+9x}\right) = \frac{(-8x-7)(9x+3)-(-4x^2-7x+2)(9)}{(9x+3)^2}$$

   (c)
$$f'(z) = \frac{(4z^3+3)(2z^2+1)-(4z)(z^4+3z)}{(2z^2+1)^2}$$

   (d) We have $\tan(\tan^{-1}(x)) = x$, so if $f(x) = \tan^{-1}(x)$, the chain rule shows
$$\frac{1}{\cos(\tan^{-1}(x))^2} \cdot f(x) = 1.$$
   From right triangle trigonometry, $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$, so the derivative of $\tan^{-1}(x)$ is $\frac{1}{\sqrt{x^2+1}}$. Therefore, from chain rule
$$g'(x) = \frac{-1}{\tan^{-1}(x)^2 \cdot \sqrt{x^2+1}}.$$

2. Use quotient rule and check your answers online.

3. You do this similarly to 1d, using implicit differentiation, then using right triangle trigonometry to get rid of inverse trig functions appearing in the answer. You can check your own answers online; ask your GSI if you need help.

4. By quotient rule,
$$f'(a) = \frac{g'(a)h(a)-g(a)h'(a)}{h(a)^2} = \frac{3\cdot 2 - 1\cdot 4}{2^2} = \boxed{\frac{1}{2}}.$$

4.2 Logarithmic and implicit differentiation

1. (a) Since $t(x) = x^x$, $\ln(t(x)) = \ln(x^x) = x\ln(x)$. Differentiating with regard to x, and using the chain rule for the $t(x)$ term, we find
$$\frac{t'(x)}{t(x)} = x \cdot 1/x + 1 \cdot \ln(x) = 1 + \ln(x),$$
   so
$$t'(x) = (1+\ln(x))t(x) = \boxed{(1+\ln(x))x^x}.$$

   (b) Take natural logarithm then implicitly differentiate:
$$\begin{aligned}
\ln f(x) &= \ln\left[(2x+1)^5(x^4-3)^6\right] = 5\ln(2x+1) + 6\ln(x^4-3)\\
\frac{f'(x)}{f(x)} &= \frac{5\cdot\frac{d}{dx}(2x+1)}{2x+1} + \frac{6\cdot\frac{d}{dx}(x^4-3)}{x^4-3} = \frac{10}{2x+1} + \frac{24x^3}{x^4-3}\\
f'(x) &= 10(2x+1)^4(x^4-3)^6 + 24x^3(2x+1)^5(x^4-3)^5 = \boxed{(58x^4+24x^3-30)(2x+1)^4(x^4-3)^5}.
\end{aligned}$$
   This one is also pretty easy to do with chain rule and product rule.

(c)

$$\begin{aligned}
\ln f(z) &= \ln\sqrt{z} + \ln e^{z^2} + 10\ln(z^2+1) = z^2 + \tfrac{1}{2}\ln z + 10\ln(z^2+1)\\
\frac{f'(z)}{f(z)} &= 2z + \tfrac{1}{2z} + \frac{20z}{z^2+1}\\
f'(z) &= \boxed{f(z)\left(2z + \tfrac{1}{2z} + \frac{20z}{z^2+1}\right)}
\end{aligned}$$

(d)

$$\begin{aligned}
\ln h(y) &= \ln\left(y^{\frac{1}{1+y^2}}\right) = \frac{\ln y}{y^2+1}\\
\frac{h'(y)}{h(y)} &= \frac{\frac{y^2+1}{y} - 2y\ln y}{(y^2+1)^2}\\
h'(y) &= \boxed{h(y)\left(\frac{\frac{y^2+1}{y} - 2y\ln y}{(y^2+1)^2}\right)}
\end{aligned}$$

In 1c and 1d, we didn't fully expand the answer, but you could do so, since $f(z)$ and $h(y)$ are known functions.

2. In x^x, both the base and the exponent are varying. In contrast, for polynomials, the base is varying and the exponent is fixed, while for exponentials, the exponent is varying and the base is fixed. This means x^x is a genuinely different type of function, so it is necessary to transform it using a logarithm.

3. (a) We differentiated, with regard to x, the functions y^x and x^y using logarithmic differentiation, remembering that y should be treated as a function of x.

$$\begin{aligned}
f(x) &= y^x\\
\ln f(x) &= \ln(y^x) = x\ln y\\
f'(x) &= f(x)\left(\ln y + \tfrac{xy'}{y}\right) = y^x\ln y + xy'y^{x-1}\\
g(x) &= x^y\\
\ln g(x) &= \ln(x^y) = y\ln x\\
g'(x) &= g(x)\left(y'\ln x + \tfrac{y}{x}\right) = x^y y'\ln x + yx^{y-1}.
\end{aligned}$$

So for the curve $y^x = x^y$, implicit differentiation yields

$$\begin{aligned}
y^x\ln y + xy'y^{x-1} &= x^y y'\ln x + yx^{y-1}\\
y^x\ln y - yx^{y-1} &= (x^y\ln x - xy^{x-1})y'\\
y' &= \boxed{\frac{y^x\ln y - yx^{y-1}}{x^y\ln x - xy^{x-1}}}.
\end{aligned}$$

The meaning of this is that if you plug in a point (x, y) that is on the curve $x^y = y^x$ into the expression for y', you will get the slope of the tangent line to the curve at that point.

(b) No, because the curve fails the horizontal line test. For example, the points $(2, 2)$ and $(4, 2)$ both lie on this curve, since $2^2 = 2^2$ and $2^4 = 4^2$.

(c) The tangent line's slope will come from plugging in $(1, 1)$ into the expression from 3a. We get

$$y'(1,1) = \frac{1^1\ln 1 - 1\cdot 1^{1-1}}{1^1\ln 1 - 1\cdot 1^{1-1}} = 1.$$

The tangent line has slope 1 and it passes through $(1, 1)$ so it has equation $\boxed{y = x}$.

(d) If we compute $y'(e,e)$, we get

$$\frac{e^e \ln e - e \cdot e^{e-1}}{e^e \ln e - e \cdot e^{e-1} = \frac{e^e - e^e}{e^e - e^e}} = \frac{0}{0}.$$

This corresponds to the fact that in the graph, there is no well-defined tangent line at the point (e,e). This is because the graph crosses over itself at this point.

(e) `https://www.desmos.com/calculator/lfvdvutxof`

4.3 Linear approximation

1. (a) We want to plug 0.05 into the linearization for e^x at $a = 0$. This linearization is $y = x + 1$ so $e^{0.05} \approx \boxed{1.05}$.

   (b) We want to plug 3.1 into the linearization for $\sin x$ at $a = \pi$. This linearization is $y = (\cos\pi)x + \sin\pi - \pi\cos\pi = -x + \pi$ so $\sin(3.1) \approx -3 + \pi \approx \boxed{0.14159}$.

   (c) We want to plug 1.01 into the linearization for $y = x^{-20}$ at $a = 1$. This linearization is $y = -20x + 21$ so $(1.01)^{-20} \approx -20 \cdot 1.01 + 21 = -20.2 + 21 = \boxed{0.8}$.

   (d) We want to plug 257 into the linearization for $\log_2 x$ at $a = 256$. This linearization is $y = \frac{1}{256\ln 2}x + \log_2 256 - \frac{1}{\ln 2} = \frac{x}{256\ln 2} + 8 - \frac{1}{\ln 2}$ so $\log_2 257 \approx \frac{257}{256\ln 2} + 8 - \frac{1}{\ln 2} = \boxed{8 + \frac{1}{\ln 2}}$. Since $e \approx 2.7$, let's guess that $\ln 2 \approx 1.2$, so $8 + \frac{1}{\ln 2} \approx 8 + \frac{1}{1.2} = \frac{53}{6}$.

   (e) We want to plug 0.99 into the linearization for $\arcsin x$ at $a = 1$. While $y = \arcsin x$ is not differentiable at $a = 1$ (its derivative is $\frac{1}{\sqrt{1-x^2}}$), it still has a tangent line here, namely the vertical line $x = 1$. (Vertical lines have infinite slope, so it makes since the derivative function is not defined– it would output ∞!) There isn't any way to plug 0.99 into the line $x = 1$, since $0.99 \neq 1$, so the only reasonable thing to do is say $\arcsin(0.99) \approx \arcsin(1) = \boxed{\pi/2}$.

   (f) We want to plug $\pi/4 + 0.02$ into the linearization for $\tan x$ at $a = \pi/4$. This linearization is $y = \frac{1}{\cos(\pi/4)^2}x + \tan(\pi/4) - \frac{1}{\cos(\pi/4)^2} \cdot \pi/4 = 2x + 1 - \frac{\pi}{2}$ so $\tan(\pi/4 + 0.02) \approx 2 \cdot \pi/4 + 2 \cdot 0.02 + 1 - \pi/2 = \boxed{1.04}$.

2. We have the known limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$, which says that as $x \to 0$, i.e. when x is very small, $\frac{\sin x}{x}$ is very close to 1, i.e. $\sin x$ is very close to x. If x is close to 0, $\cos x$ is close to 1, so $\tan x = \frac{\sin x}{\cos x}$ is very close to $\frac{\sin x}{1} = \sin x$. For $e^x - 1$, its linearization at 0 is $y = x$, since $\frac{d}{dx}e^x = e^x$ is 1 when evaluated at 0, and $(0,0)$ is the point of tangency. Then as long as x is near 0, $e^x - 1$ is very close to the tangent line, which is given by x.

3. (a) Bad approximation.

   (b) Size of $|a|$ doesn't effect how good the approximation is, it could be good or bad.

   (c) Same as 3b.

   (d) Very bad approximation.

   (e) Good approximation, as long as $h < 1$ and the quadratic and higher degree coefficients are not too big.

   (f) If $f(x) = g(x)/h(x)$, it's a good approximation when the second derivative, $\frac{g'h - gh'}{h^2}\frac{g''h - gh''}{h^4}$, is very small, i.e. when $h(a)$ is large.