Partial fractions

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Here are some key ideas from sections 5.6.

a) Here's a video on partial fraction decomposition.

b) Here's a video on integrating partial fractions.

Trig practice: Find all values of $\sin, \cos$, and $\tan$ for for the following values of θ: $0, \pi/6, \pi/4, \pi/3, \pi/2$.

Problem 1: (Stewart 5.6) This problem will walk you through evaluating $\int \frac{x+5}{x^2+x-2}\,dx$ using **partial fractions**.

1. Factor the denominator! You should get $(x-a)(x-b)$, so find a and b.
2. Each linear term will be the denominator of a fraction in your partial fraction decomposition. You should get something that looks like
$$\frac{x+5}{x^2+x-2} = \frac{A}{x-a} + \frac{B}{x-b}.$$
Find A and B.
3. Integrate your partial fraction decomposition term by term! Don't forget to add a constant C.
4. Give yourself a well-earned pat on the back.

My Attempt:

Solution:

① $x^2+x-2 = \boxed{(x-1)(x+2)}$

② $\frac{x+5}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$

$= \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x+2)(x-1)}$

$= \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$

$= \frac{x(A+B)+2A+B}{x^2+x-2}$

$\Rightarrow$ So $2A+B=5$
$A+B=1$
$A = 4$
Then $B=1-A=-3$

Thus $\frac{x+5}{x^2+x-2} = \frac{4}{x+1} - \frac{3}{x+2}$

③ $\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{4}{x-1}\,dx - \int \frac{3}{x+2}\,dx$

$= \boxed{4\ln|x-1| - 3\ln|x+2| + C}$

④ ☺

Problem 2: (Stewart 5.6) Write each fraction as a sum of partial fractions.

a) $\frac{1}{x^2-1}$; b) $\frac{2}{x^2+x}$; c) $\frac{2-x}{x^2-2x\,\cancel{+}\,{-}8}$; d) $\frac{x}{x^2+x-2}$.

My Attempt:

Solution:

ⓐ $x^2-1=(x+1)(x-1)$

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{A(x-1)}{(x+1)(x-1)}+\frac{B(x+1)}{(x-1)(x-1)}=\frac{x(A+B)+(-A+B)}{x^2-1}$$

Thus $A+B=0 \Rightarrow A=-B$

$-A+B=1 \Rightarrow 2B=1 \Rightarrow B=1/2 \Rightarrow A=-1/2$

Then $\frac{1}{x^2-1}=\boxed{\frac{-1}{2(x+1)}+\frac{1}{2(x-1)}}$

ⓑ $x^2+x=x(x+1)$

$$\frac{2}{x^2+x}=\frac{A}{x}+\frac{B}{x+1}=\frac{A(x+1)}{x(x+1)}+\frac{Bx}{x(x+1)}=\frac{x(A+B)+A}{x(x+1)}$$

Thus $A=2$

$A+B=0 \Rightarrow B=-2$

then $\frac{2}{x^2+x}=\boxed{\frac{2}{x}-\frac{2}{x+1}}$

ⓒ $x^2-2x-8=(x-4)(x+2)$

$$\frac{2-x}{x^2-2x-8}=\frac{A}{x-4}+\frac{B}{x+2}=\frac{A(x+2)}{(x-4)(x+2)}+\frac{B(x-4)}{(x-4)(x+2)}=\frac{x(A+B)+2A-4B}{(x-4)(x-2)}$$

Thus $A+B=-1 \Rightarrow B=-1-A$

$2A-4B=2 \Rightarrow A-2B=1 \Rightarrow A-2(-1-A)=1 \Rightarrow A+2+2A=1 \Rightarrow 3A=-1 \Rightarrow A=-1/3$

Then $B=-1+1/3=-2/3$

So $\frac{2-x}{x^2-2x-8}=\boxed{\frac{-1}{3(x-4)}-\frac{2}{3(x+2)}}$

ⓓ $x^2+x-2=(x-1)(x+2)$

Then $\frac{x}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}=\frac{A(x+2)+B(x-1)}{(x-1)(x-2)}=\frac{x(A+B)+2A-B}{(x-1)(x-2)}$

Then $2A-B=0 \Rightarrow B=2A$

$A+B=1 \Rightarrow A+2A=1 \Rightarrow A=1/3 \Rightarrow B=2/3$

$\frac{x}{x^2+x-2}=\boxed{\frac{1}{3(x-1)}+\frac{2}{3(x+2)}}$

Problem 3: Find the antiderivative of each fraction in the previous problem using the decompositions you found.

My Attempt:

Solution:

a) $\int\left(\frac{-1}{2(x+1)}+\frac{1}{2(x-1)}\right)dx=\boxed{\frac{-1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C}$

b) $\int\left(\frac{-1}{3(x-4)}-\frac{2}{3(x+2)}\right)dx=\boxed{\frac{-1}{3}\ln|x-4|-\frac{2}{3}\ln|x+2|+C}$

c) $\int\left(\frac{-1}{3(x-4)}-\frac{2}{3(x+2)}\right)dx=\boxed{\frac{-1}{3}\ln|x-4|-\frac{2}{3}\ln|x+2|+C}$

d) $\int\left(\frac{1}{3(x-1)}+\frac{2}{3(x+2)}\right)dx=\boxed{\frac{1}{3}\ln|x-1|+\frac{2}{3}\ln|x+2|+C}$

Problem 4: (Stewart 5.6) Evaluate the following integrals.

a) $\displaystyle\int \frac{ax}{x^2 - bx}\,dx$; b) $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$; c) $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$.

My Attempt:

Solution:

ⓐ $$\int \frac{a}{x-b}\,dx = \boxed{a \ln|x-b| + C}$$

ⓑ Decompose into partial fractions

Then $$\frac{1}{(x+a)(x+b)} = \frac{1}{(b-a)(x+a)} - \frac{1}{(b-a)(x+b)}$$

$$\int \frac{1}{(x+a)(x+b)}\,dx = \boxed{\frac{1}{b-a}\ln|x+a| - \frac{1}{b-a}\ln|x+b| + C}$$

ⓒ Decompose into partial fractions

$$\int \frac{2}{2x^2+3x+1}\,dx = \int \frac{4}{2x+1}\,dx - \int \frac{2}{x+1}\,dx$$

$$= 2\ln|2x+1| - 2\ln|x+1| + C$$

Then $$\Big[2\ln|2x+1| - 2\ln|x+1|\Big]_0^1 = \boxed{-2\ln\left(\tfrac{2}{3}\right)}$$

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Problem 5: (Stewart 5.6) Use both the substitution rule and partial fractions to evaluate the following integrals.

a) $\displaystyle\int_9^1 6\frac{\sqrt{x}}{x-4}\,dx$; b) $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x}\,dx$.

My Attempt:

Solution:

ⓐ let $u = \sqrt{x}$

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = 2\int_3^4 \left(1 + \frac{4}{u^2-4}\right) du$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)}$$

$$= 2 + 8\int_3^4 \left(\frac{-1}{4(u+2)} + \frac{1}{4(u-2)}\right) du$$

$$= 2 + 2\ln\tfrac{5}{3} + 2\ln\left(\tfrac{5}{3}\right)^2 + C$$

so $$6\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx$$

$$= \boxed{6\left(2 + 2\ln\tfrac{5}{3} + 2\ln\left(\tfrac{5}{3}\right)^2\right) + C}$$

ⓑ $u = \sin x \quad du = \cos x\,dx$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u}$$

$$= \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \boxed{\ln|\sin x| - \ln|\sin x + 1| + C}$$

Challenge problem: Evaluate $\displaystyle\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2}\,dx$. Set fraction $= $ to $\frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$. $A = 3,\ B = -1,\ C = 2$

Integrate to get $\boxed{\frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C}$