Derivative shortcuts

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Here are some key ideas from sections 3.3.

a) $\frac{d}{dx}c =$ 0

b) $\frac{d}{dx}x^n =$ nx^{n-1}

c) $\frac{d}{dx}(f(x)+g(x)) =$ $\frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

d) $\frac{d}{dx}(f(x)-g(x)) =$ $\frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

e) $\frac{d}{dx}e^x =$ e^x

f) $\frac{d}{dx}\sin x =$ $\cos x$

g) $\frac{d}{dx}\cos x =$ $-\sin x$

Midterm practice (Apostol): Suppose that the height of a projectile is given by $f(t)$ at t seconds after being fired directly upward from the ground. If the initial velocity of the projectile is v_0, then

$$f(t) = v_0 t - 16t^2 \text{ ft}/\cancel{\text{sec}}$$

1. Show that the average velocity of the projectile during a time interval from t to $t+h$ is $v_0 - 32t - 16h$ ft/sec. *Hint: the velocity is the instantaneous rate of change of the height function.*
2. What is the velocity at the moment the projectile returns to the ground?
3. What must the initial velocity of the projectile be for it to return to the ground after s seconds?
4. The acceleration is the rate of change of velocity. Show that the acceleration of this projectile is constant.
5. Find a formula for a height function $g(t)$ which has a constant acceleration of -20 ft/sec.

My Attempt: | Solution:

① $\frac{f(t+h)-f(t)}{t+h-t} = \frac{v_0(t+h) - 16(t+h)^2 - v_0 t + 16t^2}{h} = v_0 - 32t - 16h$ ✓

② $v_0 t - 16t^2 = 0 \Rightarrow t(v_0 - 16t) = 0 \Rightarrow \cancel{t=0}$, $v_0 - 16t = 0$, $v_0 = 16t$, $t = v_0/16$ *

* velocity

$f'(t) = v_0 - 32t$

$f'(v_0/16) = v_0 - 32(v_0/16) = v_0 - 2v_0 = -v_0$

③ $f(s) = 0 \Rightarrow v_0 s - 16s^2 = 0 \Rightarrow v_0 s = 16s^2 \Rightarrow v_0 = 16s$

④ $f''(t) = -32$ ☺

⑤ $g''(t) = -20$

$g'(t) = -20t$

$g(t) = -10t^2$

Problem 1: (Apostol) Let $f(x) = 2 + x - x^2$. Compute

a) $f'(0)$; b) $f'(1/2)$; c) $f'(1)$; d) $f'(-10)$.

My Attempt:

Solution:

$f'(x) = 0 + 1 - 2x = 1 - 2x$

a) $f'(0) = 1 - 0 = 1$

b) $f'(1/2) = 1 - 2(1/2) = 0$

c) $f'(1) = 1 - 2(1) = -1$

d) $f'(-10) = 1 - 2(-10) = 21$

Problem 2: (Apostol) Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$, find all x for which

a) $f'(x) = 0$; b) $f'(x) = -2$; c) $f'(x) = 10$.

My Attempt:

Solution:

$f'(x) = x^2 + x - 2$

a) $f'(x) = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x-1)(x+2) = 0$
$\Rightarrow x = 1, x = 2$

b) $f'(x) = -2 \Rightarrow x^2 + x = 0 \Rightarrow x(x+1) = 0$
$\Rightarrow x = 0, x = 1.$

c) $f'(x) = 10 \Rightarrow x^2 + x - 12 = 0 \Rightarrow (x-3)(x+4) = 0$
$\Rightarrow x = 3, x = -4$

Problem 3: (Apostol) Find the derivative of

$$f(x) = \frac{\sqrt{x}}{x^{7/2}}.$$

My Attempt:

Solution: Notice $\sqrt{x} = x^{1/2}$. Then

$f(x) = x^{1/2} / x^{7/2} = x^{-6/2} = x^{-3}$

$f'(x) = -3x^{-4}$

Problem 4: (Apostol) Suppose $P(x) = ax^3 + bx^2 + cx + d$. Moreover, $P(0) = P(1) = -2$, $P'(0) = -1$, and $P''(0) = 10$. Find a, b, c, and d.

My Attempt:

Solution:

We get $P'(x) = 3ax^2 + 2bx + c$
$P''(x) = 6ax + 2b$

$P(0) = -2 \Rightarrow a(0) + b(0) + c(0) + d = -2 \Rightarrow d = -2$

$P'(0) = -1 \Rightarrow 3a(0) + 2b(0) + c = -1 \Rightarrow c = -1$

$P''(0) = 10 \Rightarrow 6a(0) + 2b = 10 \Rightarrow b = 5.$

At this point, we know $b = 5, c = -1, d = -2$. Then

$P(1) = -2 \Rightarrow a(1)^3 + 5(1)^2 + -1(1) + -2 = -2$

$\Rightarrow a + 5 - 1 - 2 = -2$

$\Rightarrow a = -4$

Problem 5: (Apostol) Evaluate

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1}.$$

My Attempt:

Solution: one formula for the derivative that we learned is $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. match the colors:

$f(x) = x^{1000}$
$f(a) = 1$
$a = 1$

Thus the given limit is $f'(1)$ for $f(x) = x^{1000}$ $\Rightarrow$ we know $f'(x) = 1000x^{999}$, so $f'(1) = 1000$.

Problem 6: (Apostol) For each function below, find an equation of the tangent line to the curve at the given point.

a) $y = 8\cos x$, $(\pi/3, 4)$;
b) $y = x^2 - x^5$, $(1, 0)$.

My Attempt:

Solution: recall that the derivative tells us the slope of the tangent line.

a) $\left.\frac{dy}{dx}\right|_{x=\pi/3} = -8\sin\frac{\pi}{3} = \frac{-8\sqrt{3}}{2} = \frac{-4\sqrt{3}}{2}$.

use point-slope form: $\boxed{y-4 = \frac{-4\sqrt{3}}{2}(x-\pi/3)}$

b) $\left.\frac{dy}{dx}\right|_{x=1} = 2(1) - 5(1)^4 = 2-5 = -3$

use point-slope form: $\boxed{y = -3(x-1)}$

Problem 7: (Apostol) Find the first five derivatives of $\frac{1}{x}$. Then find a formula for the nth derivative of $\frac{1}{x}$.

My Attempt:

Solution: let $f(x) = x^{-1}$

Function: x^{-1}
1st deriv: $(-1)x^{-2}$
2nd deriv: $(-1)(-2)x^{-3}$
3rd deriv: $(-1)(-2)(-3)x^{-4}$
4th deriv: $(-1)(-2)(-3)(-4)x^{-5}$
5th deriv: $(-1)(-2)(-3)(-4)(-5)x^{-6}$

Now find a pattern!
For the nth derivative, our power of x is $x^{-(n+1)}$
And our coefficient is $(-1)^n n!$
so $\boxed{f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}}$

another way to write the nth derivative.

Challenge problem: Prove that the derivative of $\cos x$ is $-\sin x$. To do this, first find a limit expression for the derivative in the form

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Then use the identity $\cos(a+b) = \cos a\cos b - \sin a\sin b$. Finally, simplify your expression using the following known limits:

$$\lim_{a\to 0}\frac{\sin a}{a} = 1; \quad \lim_{a\to 0}\frac{\cos a - 1}{a} = 0.$$

① $\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

② $= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h}$

$= \lim_{h\to 0} \cos x \cancel{\frac{(\cos h - 1)}{h}}^{0} - \lim_{h\to 0} \sin x \cancel{\frac{\sin h}{h}}^{1}$

$= 0 - \lim_{h\to 0} \sin x = \boxed{-\sin x}$

 Turns out we can do the same for showing $(\sin x)' = \cos x$!