Math 10A Fall 2024 Worksheet 7

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1 Linear systems

Solve the following linear systems in two different ways:

(a) By substitution, or other methods you learned previously from high school algebra.

(b) By writing the system in matrix form as $A\vec{x} = \vec{b}$ and then computing $\vec{x} = A^{-1}\vec{b}$.

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 5x_2 &= 1 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 &= 0 \\ 25x_1 + 5x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= 1 \\ 2x_1 + 3x_2 &= 6 \end{aligned}$$

Notice whether or not there is a unique solution. You should find that one of these cases is different from the others, and that one of these methods doesn't always work. What's going on?

2 More determinant fun

Determinants have a number of interesting properties that can be useful for computation. Prove the following facts in the 2×2 case writing out the formulas for the determinant explicitly:

1. Switching two rows changes the sign of the determinant. That is, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.
2. Switching two columns changes the sign of the determinant. That is, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} b & a \\ d & c \end{bmatrix}$.
3. The determinant of a matrix is the same as the determinant of its transpose.
4. Scaling a row of a matrix by a scalar r scales the determinant by r.
5. Scaling a column of a matrix by a scalar r scales the determinant by r.
6. Scaling the entire 2×2 matrix by r scales the determinant by c^2 (more generally, by r^n for an $n \times n$ matrix).
7. (Harder) The determinant is multiplicative: we have $\det(MN) = \det(M)\det(N)$.
8. $\det(MN) = \det(NM)$, even though it might not be true that $MN = NM$.

If you feel *very* brave, try proving some of these facts for 3×3 matrices, or even in general. (Don't do this unless you're done with everything else.)

3 Population problems

(Based on Stewart Exercise 8.7.28.) Suppose a population of a certain species has some number of juveniles and adults. Each generation, the population changes according to the rule

$$\begin{bmatrix} 0 & 2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} j_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} j_1 \\ a_1 \end{bmatrix}$$

where j_0, a_0 are the initial populations and j_1, a_1 are the new populations. Let M denote the given matrix in this equation.

1. Describe qualitatively what is going on. Does this model seem reasonable?
2. Write down a rule describing the change in the population after k generations.
3. Write down the characteristic polynomial of M.
4. Find the eigenvalues and eigenvectors of M. Use a calculator to approximate these to 3 decimal places.
5. Write the vector $\begin{bmatrix} 0 \\ 100 \end{bmatrix}$ in the form $xv_1 + yv_2$, where x, y are scalars and v_1, v_2 are the eigenvectors you found. (Use a calculator and the approximations you came up with in part (4)).
6. Suppose the population starts with 0 juveniles and 100 adults. What can you say about the long term behavior of this population? For example, will it eventually go extinct, grow exponentially, or remain stable? What will the balance between adults and juveniles look like in the long run? (Hint: use part (5) to get rewrite the expressions for the population in the k-th generation, and then analyze this expression. Keep using a calculator.)

Bonus: If you still have time after this, repeat this problem, but replace the matrix with the matrix $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/3 \end{bmatrix}$. What does this change mean in the scenario? Can you guess what will happen in the long term before doing any computations?

Solutions

1 Linear systems

1. $x_1 = -2$, $x_2 = 1$ (unique solution).
2. $x_2 = -5x_1$ (non-unique solution: x_1 can be set to any real number). This is the odd case out: the solution is not unique, although it does exist. You'll find method (b) does not work since the matrix related to this system is not invertible, which is related to the non-uniqueness of the solution.
3. $x_1 = 1, x_2 = 4/3$ (unique solution).

2 More determinant fun

1. $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - da = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
2. $\det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = bc - ad = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
3. $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
4. $\det \begin{pmatrix} ra & rb \\ c & d \end{pmatrix} = rad - rbc = r \cdot \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Similarly for other row.
5. $\det \begin{pmatrix} ra & b \\ rc & d \end{pmatrix} = rad - brc = r \cdot \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Similarly for other column.
6. Apply part (4) twice (or part (5) twice).
7. Letting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, N = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, we have $MN = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$. Therefore

   $$\det(MN) = (ax+bz)(cy+dw) - (ay+bw)(cx+dz)$$

   Meanwhile, $\det(M)\det(N) = (ad-bc)(xw-yz)$. Expanding both of these expressions reveals they are the same.
8. Follows immediately from the previous part, since $\det(MN) = \det(M)\det(N) = \det(N)\det(M) = \det(NM)$.

3 Population problems

(a) Every generation, each adult produces, on average, 2 juvenile offspring. Meanwhile, a juvenile has, on average, a 1/2 chance of surviving to adulthood, while each adult has, on average, a 1/3 chance of surviving through until the next generation. No juveniles remain juveniles; they either die or move onto adulthood.

This does seem like a fairly reasonable model, if a bit crude.

(b) If j_0, a_0 is the initial population and j_k, a_k is the population after k generations, then

$$\begin{bmatrix} 0 & 2 \\ 1/2 & 1/3 \end{bmatrix}^k \begin{bmatrix} j_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} j_k \\ a_k \end{bmatrix}.$$

(c) $-\lambda^2 + \frac{1}{3}\lambda - 1$

(d) Eigenvalues: $\lambda = \frac{1 \pm \sqrt{37}}{6}$, or about 1.180 and -0.847. Eigenvectors: $\begin{bmatrix} -1+\sqrt{37} \\ 3 \end{bmatrix}, \begin{bmatrix} -1-\sqrt{37} \\ 3 \end{bmatrix}$, respectively, or about $\begin{bmatrix} 5.083 \\ 3 \end{bmatrix}, \begin{bmatrix} -7.083 \\ 3 \end{bmatrix}$,

(e) To do this, you need to solve the system

$$\begin{aligned} 5.083x - 7.083y &= 0 \\ 3x + 3y &= 100 \end{aligned}$$

The solution ends up being $x \approx 19.41, y \approx 13.93$.

(f) The population after k generations is given by

$$\begin{aligned} \begin{bmatrix} 0 & 2 \\ 1/2 & 1/3 \end{bmatrix}^k \begin{bmatrix} 0 \\ 100 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1/2 & 1/3 \end{bmatrix}^k \left(19.41 \begin{bmatrix} 5.083 \\ 3 \end{bmatrix} + 13.93 \begin{bmatrix} -7.083 \\ 1 \end{bmatrix} \right) \\ &= 19.41(1.180)^k \begin{bmatrix} 5.083 \\ 3 \end{bmatrix} + 13.93(-0.847)^k \begin{bmatrix} -7.083 \\ 1 \end{bmatrix}. \end{aligned}$$

Since $|-0.847| < 1$, the second term eventually becomes extremely small, so the first term dominates in the long run. Since $1.180 > 1$, we conclude that the population eventually grows exponentially. The eventual ratio of juveniles per adult approaches 5.083 juveniles per 3 adults.

In nature, of course, sustained exponential growth is impossible, so realistically our model would break after awhile.

We omit a detailed solution for the bonus problem. The change in the matrix means that each adult now only produces 1 offspring per generation instead of 2, which is probably not good for the long term prospects of this species. It seems reasonable to predict that extinction is inevitable. Indeed, we get eigenvalues of about 0.893 and -0.560. Since neither of these have absolute value at least one, by performing a similar analysis as in part (6) we can conclude that the population decays asymptotically to 0.