Matrix algebra

September 10th, 2024

Here are some key ideas from sections 8.4 and 8.5.

• A **matrix** is an array of vectors. For a given matrix A having m rows and n columns, the entry a_{ij} is in the th row and the column.

$$A = \begin{bmatrix} 0 & 7 & 1 \\ 2 & 9 & 2 \end{bmatrix}_{r_{1}}^{r_{1}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \end{bmatrix}.$$

• If the number of rows is the same as the number of columns, we say the matrix is square

• The **transpose** of A is written as A and is obtained by interchanging the rows and columns. The transpose of the above matrix is

$$A^{T} = \begin{bmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{3} & \mathbf{9} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \qquad A : \mathbf{m} \times \mathbf{m}$$

$$A^{T} : \mathbf{n} \times \mathbf{m}$$

• When we **multiply** two matrices A and B, we need that the number of <u>columns</u> in A is equal to the number of <u>nows</u> in B. A, B $m \times n$ $n \times p$

The entry in the ith row and jth column of AB is the dot product of the ith row of A and the ith row of B. For example,

$$A = \begin{bmatrix} 2 & 7 \\ 9 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 & 2 \\ 1 & 5 & 9 \end{bmatrix}, AB = \begin{bmatrix} 13 & 21 & 17 \\ 24 & -78 & -9 \end{bmatrix}.$$

Problem 1: (Stewart & Day 8.4) Just as with vectors, adding matrices and multiplying by scalar are done coordinate-wise. Suppose we have

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}, \ B = \begin{bmatrix} 7 & x \\ a & 5 \end{bmatrix}, \ C = \begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}.$$

Find A - 3C and 5B - A.

My Attempt:

Solution:

$$A - 3C = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} - 3 \begin{bmatrix} 9 & 2 \\ 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} - \begin{bmatrix} 27 & 6 \\ 21 & 30 \end{bmatrix} = \begin{bmatrix} -15 & -1 \\ -10 & -23 \end{bmatrix}$$

$$5B - A = 5 \begin{bmatrix} 7 & x \\ a & 5 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$$

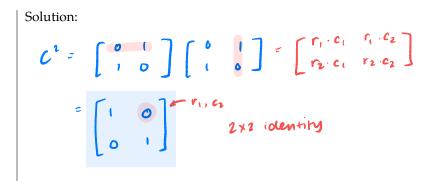
$$= \begin{bmatrix} 35 & 5x \\ 5a & 25 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 33 & 5x - 5 \\ 5a - 1 & 18 \end{bmatrix}$$

Problem 2: (Stewart & Day 8.4) For a matrix C, the quantity C^n represents multiplying C by itself n times.

$$C = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{c_2}$$

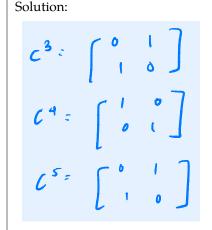
find C^2 .

My Attempt:



Problem 3: (Stewart & Day 8.4) For the same matrix C as in Problem 2, find C^3 , C^4 , and C^5 .

My Attempt:



Problem 4: (Stewart & Day 8.4) For arbitrary 2×2 matrices A, B, and C, verify the following identity:

$$A(B+C) = AB + AC.$$

Hint: set the entries equal to variables.

My Attempt:
$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{21} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ a_{21} & b_{21} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} \begin{bmatrix} b_{11}+c_{21} & b_{12}+c_{22} \\ a_{21} & b_{11}+c_{21} & b_{21} + a_{12} & c_{21} \end{bmatrix}$$

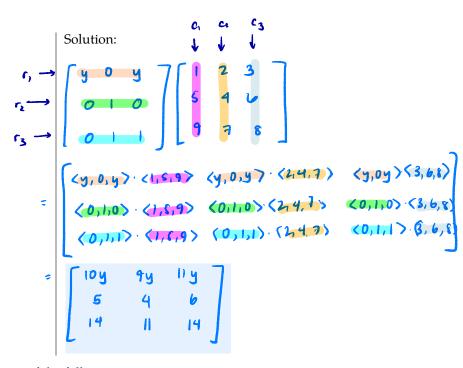
$$= \begin{bmatrix} a_{11} & b_{11}+a_{21} & c_{11} \\ a_{21} & b_{11}+a_{21} & c_{11} \end{bmatrix} + \begin{bmatrix} a_{12} & a_{22} & c_{21} \\ a_{21} & b_{12}+a_{12} & c_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & c_{12}+a_{12} & c_{22} \\ a_{21} & b_{11}+a_{22} & c_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & c_{12}+a_{12} & c_{22} \\ a_{21} & c_{11} & a_{22} & c_{21} \end{bmatrix} + \begin{bmatrix} a_{11} & c_{12}+a_{12} & c_{22} \\ a_{21} & c_{11} & a_{22} & c_{21} \end{bmatrix} = AB + AC$$

Problem 5: (Stewart & Day 8.4) For

$$F = \begin{bmatrix} y & 0 & y \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 9 & 7 & 8, \end{bmatrix}$$

find FG.

My Attempt:



Problem 6: (Stewart & Day 8.4) Find the transposes of the following matrices.

a)
$$\begin{bmatrix} 3X \\ 1 \\ 2 \end{bmatrix}$$

c)
$$\begin{bmatrix} 2 & 1 & 7 \\ 8 & 3 & 6 \end{bmatrix}$$
 d) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

$$d) \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

My Attempt:

Solution:

a)
$$\begin{bmatrix} 3x & 1 & 2 \end{bmatrix}$$

b) $\begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$
c) $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 7 & 6 \end{bmatrix}$
d) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Challenge Problem: (Stewart & Day 8.4) Show that for any $n \times n$ matrices A and B,

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For $C = AB$, $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. So $(AB)^T = B^T A^T$. If $E = B^T A^T$ then $e_{ij} = \sum_{k=1}^{n} b_{ki} a_{jk}$. So $C_{ij}^T = \sum_{k=1}^{n} a_{jk} b_{ki}$.