Planes and projections

September 3rd, 2024

Here are some key ideas from section 8.3.

- Dot products have some cool properties!
  - $\vec{a} \cdot \vec{a} =$ $|\vec{a}|^2$.
  - $\vec{0} \cdot \vec{a} =$ 0.
  - $\vec{a} \cdot (\vec{b} + \vec{c}) =$ $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
  - $(c\vec{a}) \cdot \vec{b} =$ $c(\vec{a} \cdot \vec{b})$ $=$ $\vec{a} \cdot (c\vec{b})$.
- The dot product helps us write equations of **planes**. A plane is determined by a point and a vector $\vec{n} = \langle a, b, c \rangle$ that is normal (perpendicular) to the plane.
- An equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\langle a, b, c \rangle$ is

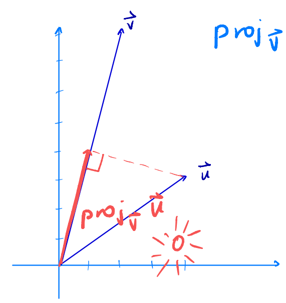
$$\langle a, b, c\rangle \cdot \langle (x-x_0), (y-y_0), (z-z_0)\rangle = 0 \iff a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

- The **vector projection** of $\vec{b}$ onto $\vec{a}$ is given by $\text{proj}_{\vec{a}} \vec{b} = \left(\vec{a} \cdot \vec{b} / |\vec{a}|^2\right) \vec{a}$. You can think of it as the shadow that $\vec{b}$ casts on $\vec{a}$.
- The **scalar projection** (component) is the magnitude of the vector projection, and is given by $\text{comp}_{\vec{a}} \vec{b} = \vec{a} \cdot \vec{b} / |\vec{a}|$.

Problem 1: (LibreTexts) For the vectors $\vec{u} = \langle 4, 3 \rangle$ onto $\vec{v} = \langle 2, 8 \rangle$, first sketch and then calculate the vector projection of $\vec{u}$ onto $\vec{v}$. Then calculate the scalar projection.

My Attempt:

Solution:



$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$$

$$= \frac{\langle 4, 3\rangle \cdot \langle 2, 8\rangle}{\left(\sqrt{2^2 + 8^2}\right)^2} \langle 2, 8 \rangle$$

$$= \boxed{\frac{32}{68} \langle 2, 8 \rangle}$$

$$= \boxed{\frac{16}{17} \langle 1, 4 \rangle}$$

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \boxed{\frac{32}{\sqrt{68}}}$$

Problem 2: (LibreTexts) Find the scalar projection and vector projection of $\vec{b} = \langle 0, 1, \frac{1}{2} \rangle$ onto $\vec{a} = \langle 2, -1, 4 \rangle$.

My Attempt:

Solution:

$\langle \vec{a} \cdot \vec{b} \rangle = 0 + (-1) + 2 = 1$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$

$\text{proj}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} = \dfrac{1}{21} \vec{a} = \boxed{\dfrac{1}{21} \langle 2, -1, 4 \rangle}$

$\text{comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \boxed{\dfrac{1}{\sqrt{21}}}$

Problem 3: (Stewart & Day 8.3) Suppose that $\vec{a}$ and $\vec{b}$ are nonzero vectors. When does $\text{comp}_{\vec{a}} \vec{b} = \cancel{\text{comp}_{\vec{a}} \vec{b}}$ $\text{comp}_{\vec{b}} \vec{a}$? When does $\text{proj}_{\vec{a}} \vec{b} = \cancel{\text{proj}_{\vec{a}} \vec{b}}$ $\text{proj}_{\vec{b}} \vec{a}$?

My Attempt:

This problem is hard! It's okay if you didn't get it right away.

For scalar projections

$\text{comp}_{\vec{a}} \vec{b} = \text{comp}_{\vec{b}} \vec{a}$ means $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

This is true when $\vec{a} \cdot \vec{b} = 0$, which happens when $\vec{a}$ and $\vec{b}$ are perpendicular.

Otherwise, divide out by $\vec{a} \cdot \vec{b}$:

$\dfrac{\cancel{\vec{a} \cdot \vec{b}}}{|\vec{a}|} = \dfrac{\cancel{\vec{a} \cdot \vec{b}}}{|\vec{b}|} \Rightarrow \dfrac{1}{|\vec{a}|} = \dfrac{1}{|\vec{b}|}$

$\Rightarrow |\vec{a}| = |\vec{b}|$

In summary, we get
1) $\vec{a}$ and $\vec{b}$ perpendicular
2) $\vec{a}$ and $\vec{b}$ have the same magnitude

Solution:

In summary, we get
1) $\vec{a}$ and $\vec{b}$ are perpendicular
2) $\vec{a}$ and $\vec{b}$ are equal

For vector projections

$\text{proj}_{\vec{a}} \vec{b} = \text{proj}_{\vec{b}} \vec{a}$ means $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$

There are two possibilities here again, if $\vec{a} \cdot \vec{b} = 0$ (perpendicular), the projections are equal. Otherwise,

$\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} \Rightarrow \dfrac{\vec{a}}{|\vec{a}|^2} = \dfrac{\vec{b}}{|\vec{b}|^2}$ (*)

$\Rightarrow \vec{a} = \dfrac{|\vec{a}|^2}{|\vec{b}|^2} \vec{b}$

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are scalar multiples

Then $\vec{a} = c\vec{b}$

Plug back into (*):

$\dfrac{c\vec{b}}{|c\vec{b}|^2} = \dfrac{\vec{b}}{|\vec{b}|^2} \Rightarrow \dfrac{c\cancel{\vec{b}}}{|c|^2 \cancel{|\vec{b}|^2}} = \dfrac{\cancel{\vec{b}}}{\cancel{|\vec{b}|^2}} \Rightarrow \dfrac{c}{c^2} = 1 \Rightarrow c = 1$

So $\vec{a} = 1 \cdot \vec{b} = \vec{b}$, the vectors are equal!

Problem 4: (Stewart & Day 8.3) Find the equation of the plane that passes through the origin and is perpendicular to the vector $\langle 1, -2, 5 \rangle$.

My Attempt:

Solution: Plug & chug into the formula on p.1

$P(x_0, y_0, z_0) = (0, 0, 0)$

$\vec{n} = \langle 1, -2, 5 \rangle$

Then we get

$\vec{n} \cdot \langle (x-0), (y-0), (z-0) \rangle = 0$

$1 \cdot x + (-2) y + 5z = 0$

$\boxed{x - 2y + 5z = 0}$

Problem 5: (Stewart & Day 8.3) The orthogonal projection of $\vec{b}$ onto $\vec{a}$ is $\text{orth}_{\vec{a}} \vec{b} = \vec{b} - \text{proj}_{\vec{a}} \vec{b}$. Show that the orthogonal projection is, in fact, orthogonal to $\vec{a}$.

My Attempt:

Solution:

$\text{orth}_{\vec{a}} \vec{b}$ is perpendicular to $\vec{a}$ if and only if $\text{orth}_{\vec{a}} \vec{b} \cdot \vec{a} = 0$

Let's find that dot product!

$$\begin{aligned}(\vec{b} - \text{proj}_{\vec{a}} \vec{b}) \cdot \vec{a} &= \vec{a} \cdot \vec{b} - \vec{a} \cdot \text{proj}_{\vec{a}} \vec{b} \\ &= \vec{a} \cdot \vec{b} - \vec{a} \cdot \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} \right) \\ &= \vec{a} \cdot \vec{b} - \frac{\vec{a} \cdot \vec{b}}{\cancel{|\vec{a}|^2}} \cdot \cancel{|\vec{a}|^2} \\ &= 0 \checkmark \end{aligned}$$

So they are orthogonal.

Problem 6: (Stewart & Day 8.3) For $\vec{a} = \langle 1, 2, 0 \rangle$, find a vector $\vec{b}$ such that $\text{comp}_{\vec{a}}\vec{b} = 2$. Then describe the set of all vectors $\vec{w}$ such that $\text{proj}_{\langle 1,2,0\rangle}(\vec{w}) = \vec{0}$.

My Attempt:

If $\text{comp}_{\vec{a}}\vec{b} = 2$, we need

$$2 = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\langle 1,2,0\rangle \cdot \langle b_1, b_2, b_3\rangle}{\sqrt{1^2+2^2+0^2}} = \frac{b_1+2b_2}{\sqrt{5}}$$

So $2\sqrt{5} = b_1 + 2b_2$. There are many possibilities, and one of them is $b_1 = 0$, $b_2 = \sqrt{5}$. So $\langle 0, \sqrt{5}, 0\rangle$ is one answer.

Solution:

If $\text{proj}_{\vec{w}}\langle 1,2,0\rangle = 0$, then $\frac{\langle 1,2,0\rangle\langle w_1,w_2,w_3\rangle}{|\vec{w}|^2}\vec{w} = 0$

so either $\vec{w} = 0$, or $\langle 1,2,0\rangle\langle w_1, w_2, w_3\rangle = 0$

In summary:

1. $\vec{w} = 0$, or
2. $\vec{w}$ is orthogonal to $\langle 1, 2, 0\rangle$.

Problem 7: Muscle fibers contract in various directions. Suppose a muscle fiber contracts along a vector $\vec{v}$ with a magnitude of 5 cm. The direction of contraction makes an angle of $\pi/4$ with a bone. Find the scalar projection of the contraction vector $\vec{v}$ onto the bone's direction, which represents the effective shortening of the muscle along the bone's axis.

My Attempt:

Solution: Let the bone be the vector $\vec{u} = \langle 0, 1\rangle$.

Then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Then $\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{u}|}$

$= |\vec{v}|\cos\theta$

$= 5\cos\frac{\pi}{4}$

$= \frac{5\sqrt{2}}{2}$

Problem 8: Find an equation of the plane containing the points $\overset{P}{(1,0,0)}$, $\overset{Q}{(1,1,1)}$, and $\overset{R}{(1,1,0)}$.

My Attempt:

Solution:

Let $P = (1,0,0)$, $Q = (1,1,1)$, $R = (1,1,0)$.

$\vec{PQ} = \langle 0,1,1\rangle$, $\vec{PR} = \langle 0,1,0\rangle$

We want a vector normal to both of these, and one of them is $\langle 1,0,0\rangle$ since the dot products are 0

Then $\langle 1,0,0\rangle \cdot \langle (x-1), (y-0), (z-0)\rangle = 0$ is an eq. of the plane. Simplify:

$x - 1 + 0 + 0 = 0$

$x = 1$

Challenge Problem: (Stewart & Day Chapter 8) Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}.$$

If $P_2(x_2, y_2)$ is also on the line, then the projection of $\vec{P_1P_2}$ onto $\vec{n}$ is the distance, if $\vec{n}$ is normal to the line.

$\vec{n} = \langle a, b\rangle$. So

$\text{comp}_{\vec{n}}\vec{P_1P_2} = |\langle a,b\rangle \cdot \langle x_1-x_2, y_1-y_2\rangle| / \sqrt{a^2+b^2}$

$= |ax_1 - ax_2 + by_1 - by_2| / \sqrt{a^2+b^2}$

$= |ax_1 + by_1 + c| / \sqrt{a^2+b^2}$