Characteristic polynomials, iteration

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Here are some key ideas from sections 8.7 and 8.8.

- One way we can solve for eigenvalues is by using **characteristic polynomials**. Recall that to have an eigenvector (which is nonzero), we need that $\det(A - \lambda I) = 0$

  where λ is an eigenvalue. Now consider the matrix $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Solving, we get:

  $$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \lambda I\right) = \det\left(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right)$$

  $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

  $$= (2-\lambda)(3-\lambda) - (2)(1) = 6 - 2\lambda - 3\lambda + \lambda^2 - 2 = \lambda^2 - 5\lambda + 4$$

  characteristic poly!

  $$(\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, \lambda = 4$$

- Matrices express transformations. If we start with a vector $\vec{n}_0$, then we can keep applying the transformation:

  $$\vec{n}_1 = A\vec{n}_0, \quad \vec{n}_2 = A\vec{n}_1, \quad \vec{n}_3 = A\vec{n}_2, \ldots$$

- The **recursion** for this formula is $\vec{n}_{t+1} = A\vec{n}_t$. Now let $\vec{v}_1$ be an eigenvalue with eigenvector λ_1. Likewise, let $\vec{v}_2$ be an eigenvalue with eigenvector λ_2. For

  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{P} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

  $$P = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}; \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\vec{n}_0 \quad ;$$

  the solution to the recursion is:

  $$\vec{n}_t = c_1 \begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix} \lambda_1^t + c_2 \begin{bmatrix} | \\ \vec{v}_2 \\ | \end{bmatrix} \lambda_2^t$$

Problem 1: (Stewart & Day 8.7) For the example matrix $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, solve for the eigenvalues and the corresponding eigenvectors.

$(A - \lambda I)\vec{v} = 0$

My Attempt:

Solution:

$$\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \lambda I\right) = 0 \Rightarrow \det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

Solve: $6 - 5\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$,

so $(\lambda - 1)(\lambda - 4) = 0 \Rightarrow \lambda = 1, \lambda = 4$

Recall $(A - \lambda I)\vec{v} = \vec{0}$.

$\lambda = 1$: $\begin{bmatrix} 2-1 & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

so $v_1 + 2v_2 = 0$, $v_1 + 2v_2 = 0$ $\Rightarrow v_1 = -2v_2$. Pick $v_1 = 2$ to get $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda = 4$: $\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

so $-2v_1 + 2v_2 = 0$, $v_1 - v_2 = 0$ $\Rightarrow v_1 = v_2$. Pick $v_1 = 1$ to get $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem 2: (Stewart & Day 8.7) Suppose that $A^2 = 0$ for some matrix A. Show that the only possible eigenvalue of A is 0.

My Attempt:

Solution:

$A\vec{v} = \lambda\vec{v} \Rightarrow A^2\vec{v} = A\lambda\vec{v}$

$\Rightarrow A^2\vec{v} = \lambda A\vec{v}$ (why can we switch λ and A?)

$\Rightarrow A^2\vec{v} = \lambda(\lambda\vec{v})$

$\Rightarrow 0 = \lambda^2\vec{v}$

Since $\vec{v}$ is nonzero, $\lambda = 0$.

Problem 3: (Stewart & Day 8.8) Show that $A = PDP^{-1}$, where P ~~is a diagonal matrix~~ whose columns are the eigenvectors of A and D is a diagonal matrix with the corresponding eigenvalues.

a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$.

My Attempt:

Solution:

a) char poly: $(2-\lambda)(1-\lambda) = 0$. so $\lambda_1 = 1, \lambda_2 = 2$.

solve to find corresp. e-values: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

verify $PDP^{-1} = A$.

b) repeat the process from (a):

char poly: $(1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$.

so $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$.

verify $PDP^{-1} = A$

Problem 4: (Stewart & Day 8.8) Suppose $\vec{n}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Express the solution to the recursion $\vec{n}_{t+1} = A\vec{n}_t$ in terms of the eigenvectors and eigenvalues of A.

My Attempt:

Solution:

char poly: $\lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$

Find corresp. e-values: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\vec{n}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. so we have

$$\vec{n}_t = c_1 \begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix} \lambda_1^t + c_2 \begin{bmatrix} | \\ \vec{v}_2 \\ | \end{bmatrix} \lambda_2^t$$

$$= 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^t + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1)^t = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Problem 5: (Stewart & Day 8.7) Suppose that $\vec{v}$ is an eigenvector of matrix A with eigenvalue λ_A, and it is also an eigenvector of matrix B with eigenvalue λ_B. Show that $\vec{v}$ is an eigenvector of $A+B$ and find its associated eigenvalue. Then show that $\vec{v}$ is an eigenvector of AB and find its associated eigenvalue.

My Attempt:

Solution:

For $A+B$:

$$(A+B)\vec{v} \overset{\text{distribute}}{=} A\vec{v} + B\vec{v} \overset{\text{use given e-value}}{=} \lambda_A \vec{v} + \lambda_B \vec{v} \overset{\text{factor}}{=} (\lambda_A + \lambda_B)\vec{v}$$

so $\vec{v}$ is an e-vector with e-value $\lambda_A + \lambda_B$.

For AB:

$$(AB)\vec{v} \overset{\text{associative prop}}{=} A(B\vec{v}) \overset{\text{use given e-value}}{=} A(\lambda_B \vec{v}) \overset{\text{scalars commute}}{=} \lambda_B A\vec{v} \overset{\text{use given e-value}}{=} \lambda_B \lambda_A \vec{v}$$

so $\vec{v}$ is an e-vector with e-value $\lambda_B \lambda_A$.

Problem 6: (Stewart & Day 8.7) (Stewart & Day 8.8) Suppose $\vec{n}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$, for $a \neq 0$ and $b \neq 1$. Express the solution to the recursion $\vec{n}_{t+1} = A\vec{n}_t$ in terms of the eigenvectors and eigenvalues of A.

My Attempt:

Solution:

e-values must satisfy $(1-\lambda)(b-\lambda) = 0$, which has solutions $\lambda_1 = 1$ and $\lambda_2 = b$. Solve for e-vectors to get $\vec{v}_1 = [1, 0]$ and $\vec{v}_2 = [a, b-1]$.

Then $D = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$, $P = \begin{bmatrix} 1 & a \\ 0 & b-1 \end{bmatrix}$

$$P^{-1} = \frac{1}{b-1}\begin{bmatrix} b-1 & -a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a/b-1 \\ 0 & 1/b-1 \end{bmatrix}$$

Also, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\vec{n}_0 = \begin{bmatrix} 1 & -a/b-1 \\ 0 & 1/b-1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-a/b-1 \\ 1/b-1 \end{bmatrix}$

$$\vec{n}_t = c_1 \begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix} \lambda_1^t + c_2 \begin{bmatrix} | \\ \vec{v}_2 \\ | \end{bmatrix} \lambda_2^t$$

$$= \left(1 - \frac{a}{b-1}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} 1^t + \left(\frac{1}{b-1}\right)\begin{bmatrix} a \\ b-1 \end{bmatrix} b^t$$

Challenge Problem: (Stewart & Day 8.8) Suppose that $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, with $a \neq 0$ and $b \neq 0$. Show that the eigenvalues of T are $\lambda = a \pm bi$. Then show that T can be written as $T = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow (a-\lambda)^2 + b^2 = 0 \Rightarrow a - \lambda = \pm bi$, so $\lambda = a \pm bi$.

Represent $a + bi$ as pts in the Cartesian plane to get $a = r\cos\theta$ and $b = r\sin\theta$, where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}(b/a)$.

Then $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$