Substitution, integration by parts

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Here are some key ideas from sections 5.3 and 5.4

$u = g(x)$
$\frac{du}{dx} = g'(x)$
$du = g'(x)\, dx$

- The substitution method is a way to undo the chain rule. First, some motivation:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \Rightarrow \int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

Now for the rule: suppose $u = g(x)$ is the "inside function." If u is differentiable, then:

$$\int f'(u)\, du = f(u) + C$$

⋆ We can use the substitution method for definite integrals, but we would need to change the bounds of integration!

- Integration by parts is a way to undo the product rule. First, some motivation:

$$\frac{d}{dx} f(x)g(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

Now for the rule: if $u = f(x)$ and $v = g(x)$, then $du = f'(x)\, dx$ and $dv = g'(x)\, dx$, so

$u = f(x) \Rightarrow du = f'(x)\, dx$
$v = g(x) \Rightarrow dv = g'(x)\, dx$

$$\int v\, du + \int u\, dv = uv \quad \text{-OR-} \quad \int u\, dv = uv - \int v\, du$$

. . . .

Trig practice: Solve $2\sin^2 x - \sin x - 1 = 0$ for $x \in [0, 2\pi]$.

Problem 1: (Stewart 5.4) This problem will walk you through evaluating $\int 2x\sqrt{1+x^2}\, dx$ using **substitution**.

1. For the substitution method to work, we need the integrand (stuff inside the integral) to have a composition of functions and the derivative of the inside function. What is the composition $f(g(x))$? What is $g'(x)$?
2. Let u be the inside function $g(x)$. Then $du = g'(x)\, dx$. Find du.
3. Rewrite the integral in terms of u, and evaluate the antiderivative (still in terms of u).
4. Substitute $g(x)$ for u. Celebrate, you did it!

My Attempt:

Solution:

① The composition is $\sqrt{1+x^2}$. The inside function is $g(x) = 1+x^2$. The outside function is $f(x) = \sqrt{x}$. Then $g'(x) = 2x$.

② $u = g(x) = 1+x^2$. Then $du = g'(x)\, dx = 2x\, dx$.

③ $\int 2x\sqrt{1+x^2}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{\frac{3}{2}} + C$

④ $\frac{2}{3}u^{\frac{3}{2}} + C = \boxed{\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C}$ ◡̈

Problem 2: (Stewart 5.5) This problem will walk you through evaluating $\int x \sin x \, dx$ using **integration by parts**.

1. For integration by parts to work, we need a product of functions in the integrand. We will let ~~dv~~ u correspond to the function that is "easier" to differentiate, and ~~u~~ dv will be the other one. Find u and dv.
2. Using your choice of dv from the previous part, find v and du.
3. Use the formula $\int u \, dv = uv - \int v \, du$ to find the antiderivative. Yay, you did it again!

My Attempt:

Solution:

1. $u = x, \; dv = \sin x \, dx$
2. $du = 1 \cdot dx, \; v = -\cos x$
3. $$\int x \sin x \, dx = \int u \, dv = uv - \int v \, du$$
$$= -x\cos x - \int -\cos x \, dx$$
$$= -x\cos x + \sin x + C$$

Problem 3: (Stewart 5.4) Evaluate the following integrals. *Hint: remember to change the bounds of integration accordingly!*

a) $\int_0^1 \cos(\pi t/2) \, dt$; b) $\int_0^1 (3t+1)^5 \, dt$; c) $\int_0^1 \frac{e^z + 1}{e^z + z} \, dz$.

My Attempt:

Solution:

a) $u = \pi t/2 \Rightarrow du = \pi/2 \, dt \Rightarrow dt = 2/\pi \, du$

$$\int_0^1 \cos(\pi t/2) \, dt = \int_{t=0}^{t=1} \cos u \, \frac{2}{\pi} \, du = \frac{2}{\pi} \int_0^{\pi/2} \cos u \, du$$
$$= \frac{2}{\pi} [\sin u]_0^{\pi/2} = \frac{2}{\pi} \cdot (1-0) = \frac{2}{\pi}$$

b) $u = 3t+1 \Rightarrow du = 3 \, dt \Rightarrow dt = du/3$

$$\int_0^1 (3t+1)^5 \, dt = \int_{t=0}^{t=1} u^5 \, du/3 = \frac{1}{3} \int_1^4 u^5 \, du = \frac{1}{3}\left[\frac{u^6}{6}\right]_1^4 = \frac{4^6 - 1}{18}$$

c) $u = e^z + z, \; du = e^z + 1 \, dz$

$$\int_0^1 \frac{e^z+1}{e^z+z} \, dz = \int_{z=0}^{z=1} \frac{du}{u} = \int_1^{e+1} \frac{du}{u} = [\ln u]_1^{e+1} = \ln(e+1) - 0 = \ln(e+1)$$

Problem 4: (Stewart 5.4)

a) If f is continuous and $\int_0^9 f(x) \, dx = 4$, find $\int_0^3 x f(x^2) \, dx$.

b) If f is continuous and $\int_0^4 f(x) \, dx = 10$, find $\int_0^2 f(2x) \, dx$.

My Attempt:

Solution:

a) $u = x^2 \Rightarrow du = 2x \, dx$

$$\int_0^3 x f(x^2) \, dx = \frac{1}{2} \int_0^3 2x f(x^2) \, dx = \frac{1}{2} \int_{x=0}^{x=3} f(u) \, du$$
$$= \frac{1}{2} \int_0^9 f(u) \, du = \frac{1}{2}(4) = 2$$

b) $u = 2x \Rightarrow du = 2 \, dx$

$$\int_0^2 f(2x) \, dx = \frac{1}{2} \int_0^2 2 f(2x) \, dx = \frac{1}{2} \int_{x=0}^{x=2} f(u) \, du$$
$$= \frac{1}{2} \int_0^4 f(u) \, du = \frac{1}{2} \cdot 10 = 5$$

Problem 5: (Stewart 5.5) Evaluate the following integrals.

a) $\displaystyle\int_1^2 \frac{\ln x}{x^2}\,dx$; b) $\displaystyle\int_0^\pi e^{\cos t}\sin 2t\,dt$; c) $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta$; d) $\displaystyle\int \sin(\ln x)\,dx$.

My Attempt:

a) $u = \ln x$ $dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx$ $v = \frac{-1}{x}$

$\int \frac{\ln x}{x^2}\,dx = \int u\,dv = uv - \int v\,du = \frac{-\ln x}{x} - \int \frac{-1}{x}\cdot\frac{1}{x}\,dx$

$= \frac{-\ln x}{x} - \frac{1}{x}$

Then $\int_1^2 \frac{\ln x}{x^2}\,dx = \left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^2 = \frac{-\ln 2}{2} - \frac{1}{2} + \frac{\ln 1}{1} + 1 = \boxed{\frac{-\ln 2}{2} + \frac{1}{2}}$

b) $\sin 2t = 2\sin t\cos t \Rightarrow$ Find $\int e^{\cos t}(2\sin t\cos t)\,dt$

let $w = \cos t \Rightarrow dw = -\sin t\,dt$

Then $\int e^{\cos t}(2\sin t\cos t)\,dt = \int e^w\cdot 2w\cdot(-dw) = 2\int we^w\,dw$

let $u = w \Rightarrow du = dw$

let $dv = e^w\,dw \Rightarrow v = e^w$

$2\int u\,dv = 2\left(uv - \int v\,du\right) = 2\left(we^w - \int e^w\,dw\right)$

$= 2we^w - 2e^w = 2\cos t\,e^{\cos t} - 2e^{\cos t}$

$\int \to \left[2\cos t\,e^{\cos t} - 2e^{\cos t}\right]_0^\pi$

$= 2(-1)e^{-1} - 2e^{(-1)} + 2(1)e^1 + 2e^1$

$= -2e^{-1} - 2e^{-1} + 2e^1 + 2e^1$

$= \boxed{4e - 4e^{-1}}$

Solution:

c) let $w = \theta^2$, $dw = 2\theta\,d\theta \Rightarrow \int \theta^3\cos\theta^2\,d\theta = \frac{1}{2}\int w\cos w\,dw$

let $u = w$, $du = dw$

$dv = \cos w\,dw$, $v = \sin w$

Then $\frac{1}{2}\int w\cos w\,dw = \frac{1}{2}\left(w\sin w - \int \sin w\,dw\right) = \frac{1}{2}(w\sin w + \cos w)$

$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos\theta^2\,d\theta = \left[\frac{1}{2}(w\sin w + \cos w)\right]_{\pi/2}^{\pi} = \frac{1}{2}\left(\pi(0) + (-1) - (\pi/2 + 0)\right) = \boxed{\frac{(-1-\pi/2)}{2}}$

d) $w = \ln x$, $dw = \frac{1}{x}\,dx$

$\int \sin(\ln x)\,dx = \int \sin(\ln x)\cdot x\cdot\frac{1}{x}\,dx = \int \sin w\cdot e^w\,dw$

let $u = \sin w$ $du = \cos w\,dw$

$dv = e^w\,dw$ $v = e^w$

$\int \sin w\,e^w\,dw = \sin w\,e^w - \int e^w\cos w\,dw$ integrate by parts again (steps not shown)

$= \sin w\,e^w - e^w\cos w - \int e^w\sin w\,du$

Thus $\int \sin w\,e^w = \frac{1}{2}\left(e^w\sin w - e^w\cos w\right) = \frac{1}{2}\left(e^{\ln x}\sin\ln x - e^{\ln x}\cos\ln x\right)$

$= \boxed{\frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right)}$

Problem 6: (Stewart 5.4) Evaluate $\displaystyle\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx$.

My Attempt:

Solution:

$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx$

$= \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$

First, find

$\int x\sqrt{4-x^2}\,dx$

let $u = 4-x^2 \Rightarrow du = -2x\,dx$

$-\frac{1}{2}\int u^{\frac{1}{2}}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(4-x^2)^{\frac{3}{2}} + C$

$\int_{-2}^{2} x\sqrt{4-x^2}\,dx = \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_{-2}^{2} = 0$

Now find $\int_{-2}^{2} 3\sqrt{4-x^2}\,dx$

Notice $\sqrt{4-x^2}$ is a half circle

So $3\int_{-2}^{2}\sqrt{4-x^2}\,dx = 3\cdot\frac{\pi\cdot 2^2}{2} = \boxed{6\pi}$

Then

$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = 6\pi + 0 = \boxed{6\pi}$

Problem 7: (Stewart 5.5) Use integration by parts to prove the reduction formula $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

My Attempt:

Solution:

let $u = x^n \Rightarrow du = nx^{n-1}\,dx$

$dv = e^x\,dx \Rightarrow v = e^x$

$\int x^n e^x\,dx = x^n e^x - \int e^x\cdot nx^{n-1}\,dx$

$= x^n e^x - n\int x^{n-1}e^x\,dx$ ✓

Challenge problem: If a and b are positive numbers, show that $\displaystyle\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.

let $u = 1-x \Rightarrow du = -x\,dx$

Then $\int_0^1 x^a(1-x)^b\,dx = \int_{x=0}^{x=1} -(1-u)^a u^b\,du = -\int_1^0 (1-u)^a u^b\,du = \int_0^1 (1-u)^a u^b\,du = \int_0^1 (1-x)^a x^b\,dx$