Math 10A Fall 2024 Worksheet 3

CJ Dowd and Reed Jacobs

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1 Vectors in triangles

In each part (a), (b), (c), (d) below, you are given three points P, Q, R in 2D or 3D space. For each part:

(i) Write down the vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{PR}$. Make sure that the sign is correct, e.g. for $\overrightarrow{PQ}$ the vector should go from P to Q and not the other way around.

(ii) Determine whether P, Q, R are collinear, and thus determine whether PQR is a (non-degenerate) triangle.

(iii) Find the lengths of the sides of PQR. If this shape is a triangle, determine whether it is equilateral, isosceles, or scalene.

(iv) Compute the angles $\angle QPR, \angle PQR$, and $\angle QRP$. (Hint: take appropriate dot products and use the angle formula. Pay attention to the direction vectors are facing.) If PQR is a triangle, determine whether it is an acute, right, or obtuse triangle.

(v) In the 2D cases, plot P, Q, R, draw the vectors from part (i), and check that your answers in (ii), (iii), and (iv) visually make sense. (Challenge: do this in the 3D case too.) Using a calculator if necessary, verify that the three angles you found in part (d) add to π rad $= 180°$.

(a) $P(0,0)$, $Q(-1,2)$, $R(2,1)$

(b) $P(4,2)$, $Q(3,3)$, $R(2,4)$

(c) $P(-5,2)$, $Q(-3,-1)$, $R(-4,0)$

(d) $P(1,1,0)$, $Q(3,3,1)$, $R(0,3,7)$

2 Standard unit vector notation

In 3D space, we set $\mathbf{i} = [1,0,0], \mathbf{j} = [0,1,0], \mathbf{k} = [0,0,1]$. These are the *standard unit vectors*.

1. Let a, b, c be scalars. Show that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = [a, b, c]$. Therefore the standard unit vectors provide an alternative notation for writing vectors.

2. Show that $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$. Also show that the standard unit vectors are orthogonal to each other.

3. (Stewart Exercise 8.10) Compute the dot product $\vec{v} \cdot \vec{w}$, where $\vec{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\vec{w} = 4\mathbf{i} + 5\mathbf{k}$.

3 Projection onto a vector

Let $\mathbf{v}, \mathbf{w}$ be vectors, say in $\mathbb{R}^2$ (but the setup works for any $\mathbb{R}^n$). The "projection of $\mathbf{w}$ onto $\mathbf{v}$", denoted $\text{proj}_{\mathbf{v}}(\mathbf{w})$, is given by the formula

$$\text{proj}_{\mathbf{v}}(\mathbf{w}) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

1. Check that $\text{proj}_{\mathbf{v}}(\mathbf{w})$ makes sense as a formula. Does it output a scalar or a vector?

2. Compute $\text{proj}_{\mathbf{v}}(\mathbf{w})$ for each of the following pairs of $\mathbf{v}$ and $\mathbf{w}$, some in 2D and some in 3D or 4D.

   (a) $\mathbf{v} = (2, 4), \mathbf{w} = (-1, 3)$

   (b) $\mathbf{v} = (1, 7), \mathbf{w} = (0, -5)$

   (c) $\mathbf{v} = (3, 5, -2), \mathbf{w} = (4, -1, 0)$

   (d) $\mathbf{v} = (2, 6, 1, 0), \mathbf{w} = (-3, -3, 1, -1)$

3. For any vector $\mathbf{v}$, what is $\text{proj}_{\mathbf{v}}(\mathbf{v})$? What about $\text{proj}_{\mathbf{0}}(\mathbf{v})$ and $\text{proj}_{\mathbf{v}}(\mathbf{0})$?

4. For any vector $\mathbf{v}$, compute $\text{proj}_{\mathbf{v}}(2\mathbf{v})$ and $\text{proj}_{2\mathbf{v}}(\mathbf{v})$. Your answers should each be a formula with a $\mathbf{v}$ in it.

5. To do this problem, draw pictures for 3.2.a and 3.2.b. (For a challenge, draw the picture for 3.2.c.) Why does the algebraic formula for $\text{proj}_{\mathbf{v}}$ deserve to be called projection? In other words, what is $\text{proj}_{\mathbf{v}}$ doing geometrically to its input?

6. Find a vector $\mathbf{w} \in \mathbb{R}^2$ such that $\text{proj}_{(2,1)}(\mathbf{w}) = \mathbf{0}$, then plot the vector you found and the vector $(2, 1)$. What does $\text{proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{0}$ mean for the geometry of $\mathbf{v}$ and $\mathbf{w}$?

7. Find a vector $\mathbf{w} \in \mathbb{R}^3$ such that $\text{proj}_{(3,-1,2)}(\mathbf{w}) = (0, -1, 2)$.

8. (a) Describe the set of all vectors $\mathbf{w} \in \mathbb{R}^2$ such that $\text{proj}_{(3,4)}(\mathbf{w}) = 0$. Your answer should be an equation defining a subset of the plane.

(b) Describe the set of all vectors $\mathbf{w}$ such that $\text{proj}_{(1,2,0)}(\mathbf{w}) = 0w.Youranswershouldbeanequationdefiningasubsetof3Dsp$

Solutions

1 Vectors in triangles

For part (b), the points are collinear if and only if all of the vectors are scalar multiples of each other, i.e. pointing in exactly the same direction or pointing exactly in the opposite direction. For part (d), the formulas for the angles $\theta = \angle QPR, \varphi = \angle PQR, \psi = \angle QRP$ are

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|}$$
$$\cos\varphi = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\|\overrightarrow{QP}\|\|\overrightarrow{QR}\|}$$
$$\cos\psi = \frac{\overrightarrow{RP} \cdot \overrightarrow{RQ}}{\|\overrightarrow{RP}\|\|\overrightarrow{RQ}\|}$$

Notice that the order of the points is switched in some of these, which corresponds to flipping the sign of the vector, e.g. $\overrightarrow{PQ} = -\overrightarrow{QP}$.

(a) (i) $\overrightarrow{PQ} = [-1, 2], \overrightarrow{QR} = [3, -1], \overrightarrow{PR} = [2, 1]$

(ii) Not collinear.

(iii) $\|\overrightarrow{PQ}\| = \sqrt{5}, \|\overrightarrow{QR}\| = \sqrt{10}, \|\overrightarrow{PR}\| = \sqrt{5}$. Isosceles triangle.

(iv) $\angle QPR = \pi/2$ rad $= 90°$ since $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 0$, so this is a right triangle. Since we already know the triangle is isosceles, this tells us the other two angles are $\pi/4$ rad $= 45°$. Alternatively, you can use the angle formula again.

(b) (i) $\overrightarrow{PQ} = [-1, 1], \overrightarrow{QR} = [-1, 1], \overrightarrow{PR} = [-2, 2]$.

(ii) Collinear.

(iii) $\|\overrightarrow{PQ}\| = \sqrt{2}, \|\overrightarrow{QR}\| = \sqrt{2}, \|\overrightarrow{PR}\| = 2\sqrt{2}$. Not a triangle.

(iv) $\angle QPR = 0$, $\angle PQR = \pi$ rad $= 180°, \angle QRP = 0$. Not a triangle.

(c) (i) $\overrightarrow{PQ} = [2, -3], \overrightarrow{QR} = [-1, 1], \overrightarrow{PR} = [1, -2]$

(ii) Not collinear.

(iii) $\|\overrightarrow{PQ}\| = \sqrt{13}, \|\overrightarrow{QR}\| = \sqrt{2}, \|\overrightarrow{PR}\| = \sqrt{5}$. Scalene.

(iv) $\angle QPR = \arccos\left(\frac{8}{\sqrt{65}}\right), \angle PQR = \arccos\left(\frac{5}{\sqrt{26}}\right), \angle QRP = \arccos\left(\frac{-3}{\sqrt{10}}\right)$. The last angle is obtuse since arccos of a number in the range $(-1, 0)$ is an angle strictly between $90°$ and $180°$, so the triangle is obtuse.

(d) (i) $\overrightarrow{PQ} = [2, 2, 1], \overrightarrow{QR} = [-3, 0, 6], \overrightarrow{PR} = [-1, 2, 7]$

(ii) Not collinear.

(iii) $\|\overrightarrow{PQ}\| = 3, \|\overrightarrow{QR}\| = \sqrt{45}, \|\overrightarrow{PR}\| = \sqrt{54}$. Scalene.

(iv) $\angle QPR = \arccos\left(\frac{1}{\sqrt{6}}\right), \angle PQR$ is a right angle, $\angle QRP = \arccos\left(\frac{5}{\sqrt{30}}\right)$. Right triangle.

2 Standard unit vectors

1. $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a[1, 0, 0] + b[0, 1, 0] + c[0, 0, 1] = [a, 0, 0] + [0, b, 0] + [0, 0, c] = [a, b, c]$.

2. $\|\mathbf{i}\| = \|[1,0,0]\| = \sqrt{1^2+0^2+0^2} = 1$, and similarly for the other two cases. The dot product of any two different standard unit vectors is 0 because none of the standard unit vectors share nonzero coordinates, e.g. $\mathbf{i}\cdot\mathbf{j} = (1)(0)+(0)(1)+(0)(0)$ vanishes because every term in the sum is 0. Thus, all of them are orthogonal to each other.

3. $\vec{v}\cdot\vec{w} = (3)(4)+(2)(0)+(-1)(5) = 7$.

3 Projection onto a vector

1. The fraction $\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|^2}$ is a scalar, since dot product and norm both output scalars. As long as $\mathbf{v}\neq\mathbf{0}$, so that we are not dividing by 0, this fraction will be a scalar that scales $\mathbf{v}$, so this makes sense.

2. (a) $\frac{2\cdot -1+4\cdot 3}{\sqrt{2^2+4^2}}(2,4) = (\sqrt{20}, 2\sqrt{20})$
   (b) $\frac{1\cdot 0+7\cdot -5}{\sqrt{1^2+7^2}}(1,7) = (\frac{-35}{\sqrt{50}}, \frac{-245}{\sqrt{50}})$
   (c) $\frac{3\cdot 4+5\cdot -1+-2\cdot 0}{\sqrt{3^2+5^2+(-2)^2}}(3,5,-2) = (\frac{21}{\sqrt{38}}, \frac{35}{\sqrt{38}}, -\frac{14}{\sqrt{38}})$
   (d) $\frac{2\cdot -3+6\cdot -3+1\cdot 1+0\cdot -1}{\sqrt{2^2+6^2+1^2+0^2}}(2,6,1,0) = (-\frac{46}{\sqrt{41}}, -\frac{138}{\sqrt{41}}, -\frac{23}{\sqrt{41}}, 0)$

3. $\mathrm{proj}_{\mathbf{v}}(\mathbf{v}) = \mathbf{v}$ and $\mathrm{proj}_{\mathbf{v}}(\mathbf{0}) = \mathbf{0}$. $\mathrm{proj}_{\mathbf{0}}(\mathbf{v})$ doesn't make sense since in the formula you would divide by 0, but it is defined to be $\mathbf{0}$ by convention.

4. Through using how dot product and norm behave under scaling, you can see $\mathrm{proj}_{\mathbf{v}}(2\mathbf{v}) = 2\mathbf{v}$ and $\mathrm{proj}_{2\mathbf{v}}(\mathbf{v}) = \mathbf{v}$. Another way to see this is the geometry of the projection map: Projection onto $\mathbf{v}$ fixes *every* scalar multiple of $\mathbf{v}$, because there is nothing to push down onto the line containing $\mathbf{v}$, those vectors already lie on that line.

5. Take the unique line ℓ containing $\mathbf{v}$. For any vector $\mathbf{w}$, find the point P it points to, then draw the shortest possible line segment from P to ℓ (it is perpendicular to ℓ), say this line segment htis ℓ at a point Q. $\mathrm{proj}_{\mathbf{v}}(\mathbf{w})$ sends $\mathbf{w}$ to the vector pointing to Q.

   Vectors lying on line ℓ– i.e. all the scalar multiples of $\mathbf{v}$– do not change, and any other vector gets collapsed the least possible amount to make it a scalar multiple of $\mathbf{v}$.

6. Many $\mathbf{w}$ work; one is $(-1,2)$. $\mathrm{proj}_{\mathbf{v}}(\mathbf{w}) = 0$ means that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular (orthogonal).

7. It is impossible to find such a $\mathbf{w}$! This is because all vectors project to scalar multiples of $(3,-1,2)$, but $(0,-1,2)$ is not a scalar multiple of $(3,-1,2)$.

8. (a) Writing $\mathbf{w} = (x,y)$ and using the formula for projection, the condition $\mathrm{proj}_{(3,4)}(\mathbf{w})$ is equivalent to
   $$\frac{3x+4y}{\sqrt{3^2+4^2}}(3,4) = 0.$$
   The only way this can happen is if $3x+4y=0$, which is the equation of a line in the plane of slope $-3/4$. Indeed, this is perpendicular to the line containing $(3,4)$, which is $y = 4/3\cdot x$, since $4/3\cdot -3/4 = -1$.
   (b) Similarly to the 2D example, these $\mathbf{w}$ are all the vectors lying in the plane orthogonal to $(1,2,0)$. This plane has equation $x+2y=0$.