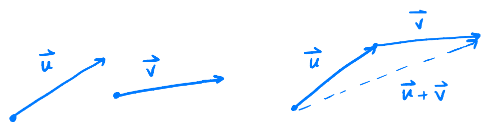
Coordinate systems, vectors solutions

August 29, 2024

Here are some key ideas from sections 8.1 and 8.2.

(0, 0)

- The **distance formula** in n-dimensions tells us how to get from a point $P_1(\cancel{a_1, \cdots}, a_n)$ to $P_2(b_1, \cdots, b_n)$. It says
$$|P_1P_2| = \sqrt{(b_1 - a_1)^2 + \dots + (b_n - a_n)^2}$$
- The set of points at a constant distance from a given point forms a sphere, and its formula is given by
$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad ; \text{ radius} = r, \text{ center} = (a, b, c)$$
- **Vectors** have both magnitude and direction. When we add vectors, we use the tip-to-tail rule.



- When we **scale** a vector by some number c, we multiply the vector's magnitude by c without changing the direction. A unit vector has magnitude 1.
- If $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$, and c is some number, then
$$\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2]$$
$$c\vec{a} = [ca_1, ca_2]$$
This generalizes!

. . . .

Problem 1: (Apostol 12.4) Let $\vec{a} = [1, 3, 6]$, $\vec{b} = [4, -3, 3]$, and $\vec{c} = [2, 1, 5]$ be three vectors in $\mathbb{R}^3$. Determine each of the following:

a) $\vec{a} + \vec{b}$; b) $\vec{a} - \vec{b}$; c) $\vec{a} + \vec{b} - \vec{c}$; d) $7\vec{a} - 2\vec{b} - 3\vec{c}$; e) $2\vec{a} + \vec{b} - 3\vec{c}$.

My Attempt:

Solution:

a) $\vec{a} + \vec{b} = [1, 3, 6] + [4, -3, 3] = [5, 0, 9]$

b) $\vec{a} - \vec{b} = [1, 3, 6] - [4, -3, 3] = [-3, 6, 3]$

c) $\vec{a} + \vec{b} - \vec{c} = [1, 3, 6] + [4, -3, 3] - [2, 1, 5] = [3, -1, 4]$

likewise...

d) $7\vec{a} - 2\vec{b} - 3\vec{c} = [-7, 24, 21]$

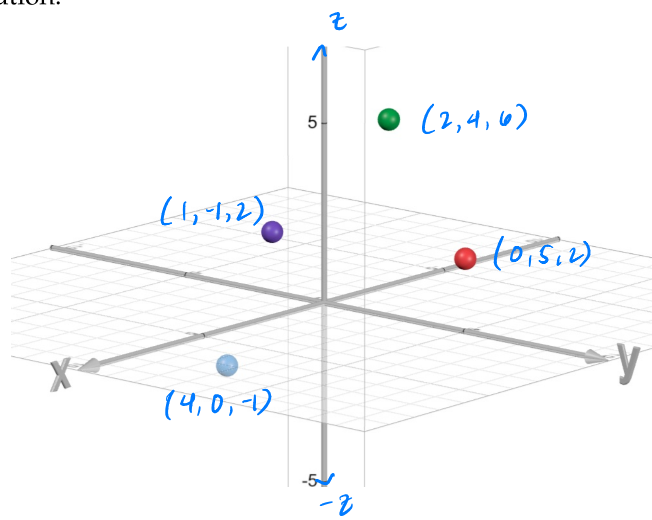
e) $2\vec{a} + \vec{b} - 3\vec{c} = [0, 0, 0]$

Problem 2: (Stewart & Day 8.1) Sketch the points $(0, 5, 2)$, $(4, 0, -1)$, $(2, 4, 6)$ and $(1, -1, 2)$ on a single set of coordinate axes.

desmos.com/3d

My Attempt:

Solution:



Problem 3: (Stewart & Day 8.1) Find an equation of the sphere with center $(2, -6, 4)$ and radius 5.

My Attempt:

Solution:

$$\boxed{(x-2)^2 + (y+6)^2 + (z-4)^2 = 5^2}$$

Problem 4: (Stewart & Day 8.2) Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

My Attempt:

Solution:

$r =$ distance from $(0,0,0)$ to $(1,2,3)$

$$= \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2}$$

$$= \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\boxed{(x-1)^2 + (y-2)^2 + (z-3)^2 = 14}$$

Problem 5: (Stewart & Day 8.2) Find a unit vector that has the same direction as $[-3, 7]$. *Hint: what should we scale the vector by so the magnitude is 1?*

My Attempt:

Solution: Here are the steps:

recall magnitude is distance from origin

① Scale vector by c: $c[-3,7] = [-3c, 7c]$

② Set magnitude equal to 1: $|[-3c, 7c]| = 1$

③ Solve for c:

$$\sqrt{(-3c)^2 + (7c)^2} = 1$$

$$\sqrt{9c^2 + 49c^2} = \sqrt{58c^2} = 1$$

$$c\sqrt{58} = 1 \Rightarrow c = \frac{1}{\sqrt{58}}$$

④ Find the vector: $c[-3,7] = \frac{1}{\sqrt{58}}[-3,7] = \boxed{\left[\frac{-3}{\sqrt{58}}, \frac{7}{\sqrt{58}}\right]}$

Problem 6: (Stewart & Day 8.1) Vaccines tend to protect only against certain pathogens in a defined antigenic space. Suppose the vaccine protects against any strain contained within a sphere of radius 2 centered at $(2, 1, 0)$. For which of the following strains will the vaccine be effective?

a) Strain A at $(0, 0, 0)$; b) Strain B at $(1, 0, 3)$; c) Strain C at $(1, 0, 1)$; d) Strain D at $(1/4, 2, 1)$.

My Attempt:

Solution:

We want the distance from the strain to the point $(2,1,0)$ to be ≤ 2.

a) $D = \sqrt{(2-0)^2 + (1-0)^2 + (0-0)^2} = \sqrt{4+1} = \sqrt{5} > 2$

b) $D = \sqrt{(2-1)^2 + (1-0)^2 + (0-3)^2} = \sqrt{1+1+9} = \sqrt{11} > 2$

c) $D = \sqrt{(2-1)^2 + (1-0)^2 + (0-1)^2} = \sqrt{1+1+1} = \sqrt{3} < 2$

d) $D = \sqrt{(2-\frac{1}{4})^2 + (1-2)^2 + (0-1)^2} = \dots > 2$

Only strain C works.

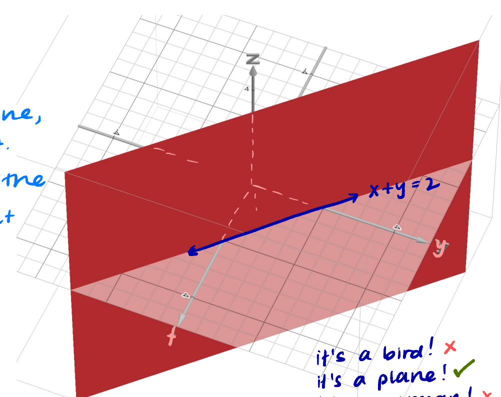
Problem 7: (Stewart & Day 8.1) Describe and sketch the surface in $\mathbb{R}^3$ (three-dimensional space) represented by $x + y = 2$. *Hint: the z-axis is missing from this equation!*

My Attempt:

Solution:

The surface is a plane, kind of like a sheet.

The cross-section is the line $x+y=2$, but it can take on any z-value.



it's a bird! ✗
it's a plane! ✓
it's superman! ✗

Challenge Problem: (Stewart & Day 8.2) Suppose that some vector in $\mathbb{R}^3$ makes angles θ_1, θ_2, and θ_3 with the x, y, and z-axes respectively. Show that $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$.

Lots of approaches for this one. One idea is to set the magnitude to m. Then $m\cos\theta_1$, $m\cos\theta_2$, and $m\cos\theta_3$ are the x, y, and z coordinates. So $\sqrt{m^2\cos^2\theta_1 + m^2\cos^2\theta_2 + m^2\cos^2\theta_3} = m$, and $m\sqrt{\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3} = m$, so $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$.