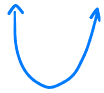
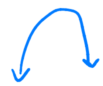
# Shapes of graphs

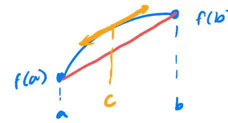
concave up:  concave down: 

***October 31st, 2024***

Here are some spooky key ideas from section 4.2.

The Mean Value Theorem says that if $f$ is differentiable on the interval $[a, b]$, then there exists some $c$ between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ (slope of line between f(a) and f(b))

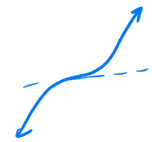



$\frac{f(b)-f(a)}{b-a}$

On a given interval, we say $f$ is $\begin{cases} \text{increasing} & \text{if } f'(x) > 0 \\ \text{decreasing} & \text{if } f'(x) < 0 \end{cases}$ on that interval.

The **first derivative test** is used to find extrema. Recall that $c$ is a critical number if $f'(c)$ is 0 or undefined.

We say $f$ has a $\begin{cases} \text{maximum} & \text{at } c \text{ if } f' \text{ changes from positive to negative} \\ \text{minimum} & \text{at } c \text{ if } f' \text{ changes from negative to positive} \end{cases}$ at $c$.



But if $f'$ does not change sign at $c$, then we can't make a conclusion.

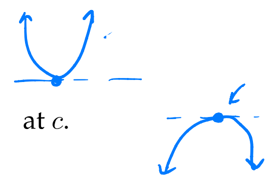
We can also define concavity in terms of second derivatives.

On a given interval, we say $f$ is $\begin{cases} \text{concave up} & \text{if } f''(x) > 0 \\ \text{concave down} & \text{if } f''(x) < 0 \end{cases}$ on that interval.

The second-derivative analogue for critical points is *inflection points*. A point $P$ on a curve $f(x)$ is an inflection point if $f$ is continuous at $P$ and concavity changes sign.

Here is the **second derivative test** for a function $f$ continuous near $c$:

We say $f$ has a $\begin{cases} \text{minimum} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) > 0 \\ \text{maximum} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) < 0 \end{cases}$ at $c$.



---

**Trig practice:** Show that $(\sin^4\theta - \cos^4\theta + 1)\csc^2\theta = 2$.

**Problem 1:** (Stewart 4.2) Suppose that the derivative of a function $f$ is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. On what interval is $f$ increasing?

My Attempt:

Solution:

① Find critical points: — f'(x) is never undefined!

$f'(x) = 0 \Rightarrow (x+1)^2(x-3)^5(x-6)^4 = 0$

so $x = -1, 3, 6$

② consider intervals around critical points

$f'(x)$: number line with marks at $-1$, $3$, $6$

$(-\infty, -1)$ $(-1, 3)$ $(3, 6)$ $(6, \infty)$

Notice the derivative cannot change sign on each of these intervals (otherwise there would be more critical pts)

③ Test values of $f'(x)$ on each interval:

| Interval | Test value ($x = c$) | sign of $f'(c)$ |
|---|---|---|
| $(-\infty, -1)$ | $c = -2$ | $\oplus \times \ominus \times \oplus = \ominus$ |
| $(-1, 3)$ | $c = 0$ | $\oplus \times \ominus \times \oplus = \ominus$ |
| $(3, 6)$ | $c = 4$ | $\oplus \times \oplus \times \oplus = \oplus$ |
| $(6, \infty)$ | $c = 7$ | $\oplus \times \oplus \times \oplus = \oplus$ |

$(x+1)^2(x-3)^5(x-6)^4$, $x = -2$
$(-2-1)^2$ is positive $\oplus$
$(-2-3)^5$ is negative $\ominus$
$(-2-6)^4$ is positive $\oplus$
$\oplus \times \ominus \times \oplus = \ominus$
(repeat for others)

increasing where $f'(x) > 0$
so $(3, 6) \cup (6, \infty)$

**Problem 2:** (Stewart 4.2) Find the intervals of increase and the intervals of decrease of $f(x) = \sin x + \cos x$, for $0 \leq x \leq 2\pi$.

My Attempt:

Solution: ① Find critical points

$f'(x) = \cos x - \sin x$

Notice $f'(x)$ is never undefined, so the only critical pts occur when $f'(x) = 0$

Thus $\cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \underbrace{\frac{\pi}{4}, \frac{5\pi}{4}}_{\text{on given interval}}$

② Plot critical points

$0 \quad \frac{\pi}{4} \quad \frac{5\pi}{4} \quad 2\pi$

$[0, \frac{\pi}{4}) \quad (\frac{\pi}{4}, \frac{5\pi}{4}) \quad (\frac{5\pi}{4}, 2\pi]$

③ Test points

| Interval | Test Point | Sign of derivative | |
|---|---|---|---|
| $[0, \frac{\pi}{4})$ | $c = 0$ | $\cos 0 - \sin 0 = 1$ ⊕ | increasing |
| $(\frac{\pi}{4}, \frac{5\pi}{4})$ | $c = \pi$ | $\cos \pi - \sin \pi = -1$ ⊖ | decreasing |
| $(\frac{5\pi}{4}, 2\pi]$ | $c = 2\pi$ | $\cos 2\pi - \sin 2\pi = 1$ ⊕ | increasing |

So the intervals of increase are $[0, \frac{\pi}{4})$ and $(\frac{5\pi}{4}, 2\pi]$

The interval of decrease is $(\frac{\pi}{4}, \frac{5\pi}{4})$

**Problem 3:** (Stewart 4.2) Find the intervals of concavity and the inflection points of $f(x) = 4x^3 + 3x^2 - 6x + 1$.

My Attempt:

Solution:

① Find candidate inflection points

$f'(x) = 12x^2 + 6x - 6$

$f''(x) = 24x + 6$

$f''(x) = 0 \Rightarrow 24x + 6 = 0 \Rightarrow x = -1/4$

② Find intervals around $-1/4$

$f''$: $\quad (-\infty, -\frac{1}{4}) \quad -\frac{1}{4} \quad (-\frac{1}{4}, \infty)$

③ Test points

| Interval | Test point | Sign of $f''$ | |
|---|---|---|---|
| $(-\infty, -\frac{1}{4})$ | $c = -1$ | $f''(-1) = -24 + 6$ ⊖ | concave down on $(-\infty, -1/4)$ |
| $(-\frac{1}{4}, \infty)$ | $c = 0$ | $f''(0) = 6$ ⊕ | concave up on $(-1/4, \infty)$ |

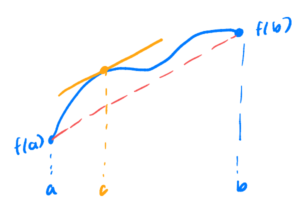
$f(-\frac{1}{4}) = 4(-\frac{1}{4})^3 + 3(-\frac{1}{4})^2 - 6(-\frac{1}{4}) + 1 = \frac{21}{8}$

Inflection pt at $(-\frac{1}{4}, f(-\frac{1}{4}))$
$(-\frac{1}{4}, \frac{21}{8})$

**Problem 4:** (Stewart 4.2) Suppose that $3 \leq f'(x) \leq 5$ for all $x$. Show that $18 \leq f(8) - f(2) \leq 30$. *Hint: use the Mean Value Theorem!*

My Attempt:

Solution:




Mean Value Theorem

There is $c$ between $a$ and $b$ with $f'(c) = \frac{f(b) - f(a)}{b - a}$

Examine $f(8) - f(2)$.

Divide by $8-2$ to get $\frac{f(8) - f(2)}{8-2}$

By MVT, there must be some $c$ in $(2, 8)$ such that $f'(c) = \frac{f(8) - f(2)}{8-2}$. But notice $3 \leq f'(c) \leq 5$.

So

$3 \leq \frac{f(8) - f(2)}{8-2} \leq 5 \Rightarrow 3(8-2) \leq f(8) - f(2) \leq 5(8-2)$

$\Rightarrow 18 \leq f(8) - f(2) \leq 30$ ✓

**Problem 5:** (Stewart 4.2) For what values of $a$ and $b$ does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

My Attempt:

Solution:

We are given that there is an extremum at $x = 2$.

But extrema only occur at critical points.

Thus there is a critical point at $x = 2$.

So $f'(2) = 0$ or $f'(2)$ is undefined. impossible since $f$ is everywhere differentiable

Notice $f'(x) = a\left((x)'e^{bx^2} + x\left(e^{bx^2}\right)'\right)$

$= a\left(e^{bx^2} + xe^{bx^2} \cdot 2bx\right)$

$= ae^{bx^2}(1 + x \cdot 2bx)$

$= ae^{bx^2}(1 + 2bx^2)$

$f'(2) = 0 \Rightarrow ae^{4b}(1 + 8b) = 0$

so $a = 0$, $e^{4b} = 0$, or $1 + 8b = 0$

$a = 0$: impossible, else $f(x) = 0$ thus $f(2) \neq 1$

$e^{4b} = 0$: impossible generally

$1 + 8b = 0$: only possibility! so $8b = -1 \Rightarrow \boxed{b = -1/8}$

Then $f(2) = 1 \Rightarrow a \cdot 2 \cdot e^{-\frac{1}{8} \cdot 2^2} = 1 \Rightarrow 2ae^{-\frac{1}{2}} = 1 \Rightarrow \boxed{a = \frac{1}{2}e^{\frac{1}{2}}}$

**Problem 6:** (Stewart 4.2) Find the intervals of concavity and the inflection points of $f(x) = \frac{x^2}{x^2+3}$.

My Attempt:

Solution:

$$f'(x) = \frac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2}$$

$$= \frac{2x^3 + 6x - 2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$$f''(x) = \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4}$$

$$= \frac{(x^2+3)\left(6(x^2+3) - 6x \cdot 2 \cdot 2x\right)}{(x^2+3)^4}$$

$$= \frac{18(-x^2+1)}{(x^2+3)^3}$$

never undefined, but zero for $x = \pm 1$.

$f(1) = \frac{1}{1+3} = \frac{1}{4}$

$f(-1) = \frac{1}{1+3} = \frac{1}{4}$

$\boxed{\text{Inflection points at } (1, \tfrac{1}{4}) \text{ and } (-1, \tfrac{1}{4})}$

$f''$: $\ominus$ $-1$ $\oplus$ $1$ $\ominus$

$\boxed{\text{concave down on } (-\infty, -1) \cup (1, \infty)}$

$\boxed{\text{concave up on } (-1, 1)}$

**Challenge problem:** (Stewart 4.2) Find the $x$-coordinate of the inflection point of a cubic function with real roots $x_1$, $x_2$, and $x_3$.

$f(x) = k(x - x_1)(x - x_2)(x - x_3)$

Show $f''(x) = k\left(6x - 2(x_1 + x_2 + x_3)\right)$

Then $6x - 2(x_1 + x_2 + x_3) = 0 \Rightarrow x = \boxed{\frac{x_1 + x_2 + x_3}{3}}$