Math 10A Fall 2024 Worksheet 19

November 7, 2024

1. Find the most general antiderivative of the function.

   (a) $f(x) = x^3 + x^2 + x + 1$

   (b) $f(x) = e^x - \frac{1}{x^2}$

   (c) $f(x) = \sqrt{3x + 4}$

   (d) $f(x) = \sec^2(2x) - 2\sin x + 3\cos x$

   (e) $f(x) = 2x\cos(x^2) - \frac{2x}{x^2+1}$

   (f) $f(x) = \ln x$ (Hint: what is $(x \ln x)'$?)

2. Let $f(x)$ be a function satisfying

$$f''(x) = \sqrt{x + 1}, \quad f'(0) = f(0) = 1.$$

   Find $f(x)$.

3. What is the maximum area of a rectangle inscribed in a circle of radius 1?

1 Solutions

1. All C's below are constants.

   (a) $F(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$

   (b) $F(x) = e^x + \frac{1}{x} + C$

   (c) $F(x) = \frac{1}{3}\frac{2}{3}(3x+4)^{3/2} + C = \frac{2}{9}(3x+4)^{3/2} + C$

   (d) Recall $\tan(x)' = \sec^2(x)$. $F(x) = \frac{1}{2}\tan(2x) + 2\cos x + 3\sin x + C$.

   (e) Observe that $\sin(g(x))' = g'(x)\cos(g(x))$ and $\ln(h(x))' = h'(x)/h(x)$. You can try to match up with $f(x)$ and find $\sin(x^2)' = 2x\cos(x^2)$ and $\ln(x^2+1)' = 2x/(x^2+1)$, so the antiderivative becomes $F(x) = \sin(x^2) - \ln(x^2+1) + C$. Note that you don't need absolute value on ln because x^2+1 is always positive.

   (f) We have $(x\ln x)' = \ln x + 1$, so $(x\ln x - x)' = \ln x$ and antiderivative is $F(x) = x\ln x - x + C$.

2. From $f''(x) = (f'(x))' = \sqrt{x+1}$, we have $f'(x) = \frac{2}{3}(x+1)^{3/2} + C_1$, and $1 = f'(0) = \frac{2}{3} + C_1$ gives $C_1 = \frac{1}{3}$, $f'(x) = \frac{2}{3}(x+1)^{3/2} + \frac{1}{3}$. Then we have $f(x) = \frac{2}{5}\frac{2}{3}(x+1)^{5/2} + \frac{1}{3}x + C_2 = \frac{4}{15}(x+1)^{5/2} + \frac{1}{3}x + C_2$ and $1 = f(0) = \frac{4}{15} + C_2$ gives $C_2 = \frac{11}{15}$. So the function is

$$f(x) = \frac{4}{15}(x+1)^{5/2} + \frac{1}{3}x + \frac{11}{15}.$$

3. Let a, b be the lengths of the sides of the rectangle. Then the length of the diagonal is $\sqrt{a^2+b^2}$, and this equals 2 by the assumption. Since the area is $S = ab$, our goal is to maximize ab under $\sqrt{a^2+b^2} = 2$. We can express b in terms of a as $b = \sqrt{4-a^2}$, and can view $S = ab = a\sqrt{4-a^2}$ as a function in a, where the domain of $S(a)$ is $0 < a < 2$. We have $S'(a) = \sqrt{4-a^2} + a\frac{1}{2}\frac{-2a}{\sqrt{4-a^2}} = \frac{4-2a^2}{\sqrt{4-a^2}}$, and the critical number of $S(a)$ is $S'(a) = \frac{4-2a^2}{\sqrt{4-a^2}} = 0 \Leftrightarrow a = \sqrt{2}$ (note that a cannot be $-\sqrt{2}$ since length should be positive). One can check that $S(a)$ increases (resp. decreases) for $0 < a < \sqrt{2}$ (resp. $\sqrt{2} < 2$, hence S attains its absolute maximum (not only local maximum) at $a = \sqrt{2}$, which is $S(\sqrt{2}) = 2$. So the maximum area is 2 that is attained by the square of length $\sqrt{2}$.