Deriving products, quotients, and chains

October 15th, 2024

Here are some key ideas from sections 3.4 and 3.5.

- The derivative of the product of $f(x)$ and $g(x)$ is $\frac{d}{dx}f(x)g(x) =$ $f'(x)g(x) + f(x)g'(x)$.
- The derivative of the quotient of $f(x)$ and $g(x)$ is $\frac{d}{dx}\frac{f(x)}{g(x)} =$ $g(x)f'(x) - f(x)g'(x) / g(x)^2$.
- The derivative of $\tan x$ is $\frac{d}{dx}\tan x =$ $\sec^2 x$.
- The chain rule has to do with compositions of functions.
  If g is differentiable at x and f is differentiable at $g(x)$, then

$$F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x).$$

- ~~The chain rule helps us with implicit differentiation, which is used when we can't isolate the ________ variable. Here's an example:~~

IGNORE ◡̈
we did this in the next worksheet

. . . .

Midterm practice (Paulin MT1 '16): Use a limit definition of the derivative to find $f'(x)$ for $f(x) = x^{3/2}$. Then find the domain of the derivative.

My Attempt:

① Write $\lim_{x\to a} \frac{x^{3/2} - a^{3/2}}{x-a}$ or $\lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h}$

(either should work; I'll use the 2nd one)

② Multiply by the conjugate

$$\lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$$

③ Recall that $(a-b)(a+b) = a^2 - b^2$. Thus

$$\left((x+h)^{3/2} - x^{3/2}\right)\left((x+h)^{3/2} + x^{3/2}\right) = (x+h)^3 - x^3$$

And $(x+h)^3 - x^3 = \cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}$

$$= 3x^2h + 3xh^2 + h^3$$

④ Thus

$$\lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h\left((x+h)^{3/2} + x^{3/2}\right)}$$

Solution:

⑤ Factor out an h in the numerator.

$$\lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h\left((x+h)^{3/2} + x^{3/2}\right)} = \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}\left((x+h)^{3/2} + x^{3/2}\right)}$$

⑥ And then you have — h terms go to 0

$$\lim_{h\to 0} \frac{3x^2 + 3xh + h^2}{\left((x+h)^{3/2} + x^{3/2}\right)} = \frac{3x^2}{x^{3/2} + x^{3/2}} = \frac{3x^{4/2}}{2x^{3/2}} = \boxed{\frac{3}{2}x^{\frac{1}{2}}}$$

Problem 1: Use the rules we talked about today to find the derivatives of

🤡 $\dfrac{\text{lo di hi} - \text{hi di lo}}{\text{lo lo}}$

a) $\sec x$; b) $\csc x$; c) $\cot x$.

My Attempt:

Solution:

a) $\sec x = \dfrac{1}{\cos x}$ ← hi, ← lo

$\dfrac{d}{dx}\sec x = \dfrac{\cos x \cdot (1)' - 1(\cos x)'}{\cos^2 x}$

$= \dfrac{-(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x}$

$= \boxed{\sec x \tan x}$

b) $\csc x = \dfrac{1}{\sin x}$ ← hi, ← lo

$\dfrac{d}{dx}\csc x = \dfrac{\sin x\,(1)' - 1\,(\sin x)'}{\sin^2 x}$

$= \dfrac{-\cos x}{\sin^2 x} = \dfrac{-1}{\sin x} \cdot \dfrac{\cos x}{\sin x}$

$= \boxed{-\csc x \cot x}$

c) $\cot x = \dfrac{\cos x}{\sin x}$

$\dfrac{d}{dx}\cot x = \dfrac{\sin x\,(\cos x)' - \cos x\,(\sin x)'}{(\sin x)^2}$

$= \dfrac{\sin x \cdot (-\sin x) - \cos x\,(\cos x)}{(\sin x)^2}$

$= \dfrac{-\sin^2 x - \cos^2 x}{(\sin x)^2} = -\dfrac{(\sin^2 x + \cos^2 x)}{(\sin x)^2}$

$= \dfrac{1}{(\sin x)^2} = \boxed{-\csc x}$

Problem 2: (Stewart 3.5) Find the derivative of $F(x) = (x^4 + 3x^2 - 2)^5$ (no need to expand).

My Attempt:

Solution: Notice $F(x) = f(g(x))$, where $f(x) = x^5$, $g(x) = x^4 + 3x^2 - 2$.

Then

$f'(x) = 5x^4 \Rightarrow f'(g(x)) = 5(x^4+3x^2-2)^4$

$g'(x) = 4x^3 + 6x$

And $F'(x) = f'(g(x)) \cdot g'(x) = \boxed{5(x^4+3x^2-2)^4 \cdot (4x^3+6x)}$

Problem 3: (Stewart 3.5) Find the derivative of $e^{x\cos x}$.

My Attempt:

Solution: let $f(x) = e^x$, $g(x) = x\cos x$

Then $f'(x) = e^x$

$g'(x) = \cos x - x\sin x$ (product rule)

we get $f'(g(x)) \cdot g'(x) = \boxed{e^{x\cos x} \cdot (\cos x - x\sin x)}$

Problem 4: (Stewart 3.5)

a) Find the 50th derivative of $y = \cos 2x$. b) Find the 1000th derivative of $f(x) = xe^{-x}$.

My Attempt:

Solution:

Look for patterns.

a) $y = \cos 2x$

$y' = 2(-\sin 2x)$

$y'' = 2^2(-\cos 2x)$

$y^{(3)} = 2^3(\sin 2x)$

$y^{(4)} = 2^4(\cos 2x)$

$\vdots \quad \vdots$

$y^{(48)} = 2^{48}(\cos 2x)$

$y^{(49)} = 2^{49}(-\sin 2x)$

$\boxed{y^{50} = 2^{50}(-\cos 2x)}$

The trig functions form a cycle.

If n is a multiple of 4, then $y^{(n)} = 2^n \cos 2x$

b) $y = xe^{-x}$

$y' = (x')e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x}$

$y'' = -e^{-x} - e^{-x} + xe^{-x} = -2e^{-x} + xe^{-x}$

$y^{(3)} = 2e^{-x} + e^{-x} - xe^{-x} = 3e^{-x} - xe^{-x}$

$\vdots \quad \vdots$

$\boxed{y^{(50)} = -50e^{-x} + xe^{-x}}$

Problem 5: (Stewart 3.5) Find an equation of the tangent line to $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $\left(0, \frac{1}{2}\right)$. *Fun fact: this curve is called cardioid because it's shaped like a heart!*

My Attempt:

Solution:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2x^2+2y^2-x)^2$$

$$2x+2y\frac{dy}{dx} = 2(2x^2+2y^2-x)\cdot\left(4x+4y\frac{dy}{dx}-1\right)$$

Plug in $x=0$, $y=1/2$

$$2(0)+2(1/2)\frac{dy}{dx} = 2\left(2(0)+2(1/2)^2-0\right)\cdot\left(4(0)+4(1/2)\frac{dy}{dx}-1\right)$$

$$\frac{dy}{dx} = 2(1/2)\left(2\frac{dy}{dx}-1\right)$$

$$\frac{dy}{dx} = 2\frac{dy}{dx}-1 \Rightarrow \boxed{\frac{dy}{dx}=1}$$

Problem 6: (Stewart 3.5) Find the derivatives of the following functions using the triple chain rule.

a) $y = e^{e^x}$;

b) $y = \sin(\cos(\tan(x)))$;

My Attempt:

Solution:

a) $y' = e^{e^x}\cdot e^x$

b) $y' = \cos(\cos(\tan(x)))\cdot -\sin(\tan(x))\cdot\sec^2 x$

Problem 7: (Stewart 3.5) Find $\frac{dy}{dx}$.

a) $e^y \cos x = 1 + \sin(xy)$;

b) $4\cos x \sin y = 1$;

c) $e^{x/y} = x - y$.

My Attempt:

Solution:

a) $e^y\frac{dy}{dx}\cos x + e^y(-\sin x) = 1+\cos(xy)\cdot\left(y+x\frac{dy}{dx}\right)$

$e^y\frac{dy}{dx}\cos x - e^y\sin x = 1+y\cos(xy)+x\cos(xy)\frac{dy}{dx}$

$e^y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = e^y\sin x + y\cos(xy)+1$

$\frac{dy}{dx}\left(e^y - x\cos(xy)\right) = e^y\sin x + y\cos(xy)+1$

$\frac{dy}{dx} = \frac{e^y\sin x + y\cos(xy)+1}{e^y - x\cos(xy)}$

b) $4\left(\cos x\cos y\frac{dy}{dx} - \sin x\sin y\right) = 1$

$\cos x\cos y\frac{dy}{dx} - \sin x\sin y = \frac{1}{4}$

$\cos x\cos y\frac{dy}{dx} = \frac{1}{4}+\sin x\sin y$

$\boxed{\frac{dy}{dx} = \frac{\frac{1}{4}+\sin x\sin y}{\cos x\cos y}}$

c) $e^{x/y}\cdot\frac{y-x\frac{dy}{dx}}{y^2} = 1-\frac{dy}{dx}$

$e^{x/y}\cdot\left(y-x\frac{dy}{dx}\right) = y^2 - y^2\frac{dy}{dx}$

$e^{x/y}y - xe^{x/y}\frac{dy}{dx} = y^2 - y^2\frac{dy}{dx}$

$e^{x/y}y - y^2 = xe^{x/y}\frac{dy}{dx} - y^2\frac{dy}{dx}$

$= \frac{dy}{dx}\left(xe^{x/y}-y^2\right)$

$\boxed{\frac{dy}{dx} = \frac{e^{x/y}y - y^2}{xe^{x/y}-y^2}}$

Challenge problem: Find a parabola that passes through $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

$y = ax^2+bx+c \Rightarrow y' = 2ax+b. \Rightarrow \begin{matrix} 6 = -2a+b \\ -2 = 10a+b \end{matrix} \Rightarrow \begin{matrix} 30 = -10a+5b \\ -2 = 10a+b \end{matrix} \Rightarrow a = \frac{-2}{3},\ b = \frac{14}{3}$

$4 = \frac{-2}{3}1^2 + \frac{14}{3}1 + 0 \Rightarrow \boxed{y = -\frac{2}{3}x^2+\frac{14}{3}x}$