L'Hôpital's rule, optimization

November 5th, 2024

Here are some key ideas from sections 4.3 and 4.4.

- Suppose f and g are differentiable and $g'(x) \neq 0$ near a. L'Hôpital's rule says

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

only when $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate in the form $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$.

- Optimization is about finding absolute extrema. Here are some pointers for optimization problems.

  1. If possible, draw a diagram and identify given quantities on the diagram.
  2. Assign a variable to the value that is to be maximized or minimized, and express it in terms of the independent variable and provided constraints.
  3. Find the domain of the independent variable and proceed to find the absolute extrema.

  ⋆ If the domain is an **open** interval or **half-open** interval, then you must compute the value of f at open endpoints as well. If the absolute extremum appears at the value of an open endpoint, then there may not be an extremum.

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Trig practice: Find $\cos\frac{23\pi}{6}$, $\sin\frac{9\pi}{2}$, and $\cos\frac{-11\pi}{4}$.

Problem 1: (Stewart 4.4) Find two numbers whose difference is 100 and whose product is a minimum.

My Attempt:

Solution:

product

$a - b = 100$, minimize $P = ab$

$\Downarrow$

$a = 100 + b$

$P = (100+b)b = 100b + b^2$

Find critical pts: $\frac{dP}{db} = 100 + 2b = 0 \Rightarrow b = -50$

Test for extrema: $\ominus$ $\oplus$ -50 test $\frac{dP}{db}$ for increase/decrease

so maximum at $b=-50$ and $a = 100 + (-50) \Rightarrow a = 50$

Problem 2: (Stewart 4.4) Find two positive numbers whose product is 100 and whose sum is a minimum.

My Attempt:

Solution:

sum

$ab = 100$, minimize $S = a + b$

$\Downarrow$

$a = \frac{100}{b} = 100b^{-1}$

domain: $b > 0$

undefined @ $b=0$

$S = 100b^{-1} + b \Rightarrow \frac{dS}{db} = -100b^{-2} + 1 = -\frac{100}{b^2} + 1$

critical pts: $1 = \frac{100}{b^2} \Rightarrow b^2 = 100 \Rightarrow b = 10, -10$

only critical value in domain is $b = 10$

$\frac{dS}{db}$ $\ominus$ $\oplus$ 0 10

Minimum @ $b=10$

$a = \frac{100}{10} \Rightarrow a = 10$

Problem 3: (Stewart 4.3) Find $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x}$.

My Attempt:

Solution: indeterminate $\frac{0}{0}$, so use l'hôpital

$$\lim_{x\to 1} \frac{(\ln x)'}{(\sin \pi x)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x}$$

$$= \frac{\frac{1}{1}}{\pi \cos \pi}$$

$$= \boxed{\frac{-1}{\pi}}$$

Problem 4: (Stewart 4.3) Rank the following functions in order of how quickly they grow as $x \to \infty$:

$$y = 2^x, \quad y = 3^x, \quad y = e^{x/2}, \quad y = e^{x/3}.$$

My Attempt:

Solution:

$e^{x/2} = (e^{1/2})^x$

$e^{x/3} = (e^{1/3})^x$

our functions are 2^x, 3^x, $(e^{1/2})^x$, $(e^{1/3})^x$

put bases in increasing order:

$(e^{1/3})^x$, $(e^{1/2})^x$, 2^x, 3^x

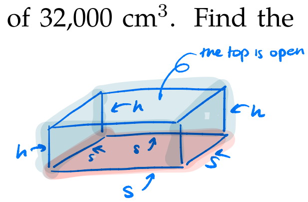
so

$\boxed{e^{x/3},\ e^{x/2},\ 2^x,\ 3^x}$

Problem 5: (Stewart 4.4) A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimizes the amount of material used.

My Attempt:

Solution:



$V = (\text{Area of base}) \cdot h$

$= s^2 h = 32{,}000$

$\Rightarrow h = \frac{32{,}000}{s^2}$

surface Area $= s^2 + 4sh$

Minimize $s^2 + 4sh = s^2 + 4\cancel{s} \cdot \frac{32{,}000}{s^{\cancel{2}}}$

$= s^2 + \frac{128{,}000}{s}$

Find critical pts etc. on domain $(0, \infty)$ to get

surface Area is min @ $\boxed{s = 40}$, $h = \frac{32{,}000}{40^2} \Rightarrow \boxed{h = 20}$

Problem 6: (Stewart 4.3) Find $\lim_{x\to\infty}(xe^{1/x} - x)$.

My Attempt:

Solution:

Indeterminate $\infty - \infty$

indet. $\frac{0}{0}$ use l'hôpital

Rewrite as

$$\lim_{x\to\infty} x(e^{1/x}-1) = \lim_{x\to\infty} \frac{e^{1/x}-1}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{e^{1/x}\left(\cancel{\frac{-1}{x^2}}\right)}{\cancel{\frac{-1}{x^2}}}$$

$$= \lim_{x\to\infty} e^{1/x}$$

$$= e^0 = \boxed{1}$$

Problem 7: (Stewart 4.4) Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

My Attempt:

Solution:

since $y=\sqrt{x}$

Minimize $\sqrt{(y-0)^2+(x-3)^2} = \sqrt{(\sqrt{x})^2+(x-3)^2}$

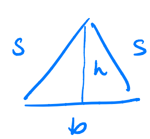
Equivalently, minimize $D = x + (x-3)^2$

$D' = 1+2(x-3) \Rightarrow$ Critical pt: $x = 2.5$

Use first derivative test to show the min is at 2.5

So $\boxed{(2.5, \sqrt{2.5})}$ is closest.

Challenge problem: Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.



1. $P = 2s + b$.
2. $A = \frac{bh}{2} = \frac{b\sqrt{s^2-\left(\frac{b}{2}\right)^2}}{2}$
3. Minimize A using 1st derivative test to get $s = b$.