Extrema

October 29th, 2024

Here are some key ideas from section 4.1.

- If $f(c)$ is greater than all the other values of $f(x)$, then it is an absolute maximum. If it is greater than all the other values in its neighborhood, then it is a local maximum.
- If $f(c)$ is less than all the other values of $f(x)$, then it is an absolute minimum. If it is less than all the other values in its neighborhood, then it is a local minimum.
- The Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value and an absolute minimum on that interval.
- We say that c is a critical value if $f'(c) = 0$ or $f'(c) =$ undefined.
- Local extrema can only exist at critical values.
- Absolute extrema can exist at critical values or endpoints.

Problem 1: Here's a general algorithm for finding absolute extrema on a closed interval $[a, b]$.

1. Find the critical values of $f(x)$.
2. For each critical value c, find the corresponding y-value given by $f(c)$.
3. Find the values of f at the endpoints of the interval.
4. The largest value in the Steps 2 and 3 is the absolute maximum. The smallest value is the absolute minimum.

(Stewart 4.1) Find the absolute maximum and absolute minimum values of $f(x) = 12 + 4x - x^2$ on $[0, 5]$.

My Attempt:

Solution:

① Find where $f'(c)$ is 0 or undefined.

$f'(c) = 4 - 2c$

$4 - 2c = 0 \Rightarrow c = 2$

$f'(c)$ is never undefined!

critical value: $c = 2$

② $f(2) = 12 + 8 - 4 = 16$

③ $f(0) = 12$

$f(5) = 12 + 20 - 25 = 7$

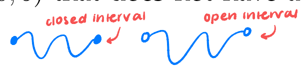
④ Abs. max = 16

Abs. min = 7

Problem 2:

a) How does the algorithm in Problem 1 change if we are finding absolute extrema on an open interval (a, b)?

b) Draw the graph of a function defined on the open interval $(0, 5)$ that does not have any absolute or local extrema.

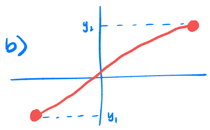


My Attempt:

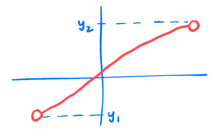
Solution:

a) After executing the algorithm, if an extremum seems to occur endpoint, that extremum does not exist. For example:

b)



On a closed interval: the abs. min is @ y_1; the abs. max. is @ y_2



On an open interval: the function approaches but never reaches y_1; the function approaches but never reaches y_2; so it has no absolute extrema; we can also see it has no local extrema

Problem 3: (Stewart 4.1) Find the critical values of the following functions.

a) $g(t) = |3t - 4|$; b) $g(\theta) = 4\theta - \tan\theta$; c) $f(x) = x^2 e^{-3x}$.

My Attempt:

a) $g(t) = \begin{cases} 3t-4 & \text{if } 3t-4 \geq 0 \to 3t \geq 4 \Rightarrow t \geq \frac{4}{3} \\ -(3t-4) & \text{if } 3t-4 < 0 \to 3t < 4 \Rightarrow t < \frac{4}{3} \end{cases}$

$= \begin{cases} 3t-4 & \text{if } t \geq \frac{4}{3} \\ -(3t-4) & \text{if } t < \frac{4}{3} \end{cases}$

$g'(t) = \begin{cases} 3 & \text{if } t \geq \frac{4}{3} \\ -3 & \text{if } t < \frac{4}{3} \end{cases} \Rightarrow$ $g'(t)$ is never 0, is only undef. for $\boxed{t = \frac{4}{3}}$

b) $g(\theta) = 4\theta - \tan\theta \Rightarrow g'(\theta) = 4 - \sec^2\theta$

① $4 - \sec^2\theta = 0$

$4 = \sec^2\theta = \dfrac{1}{(\cos\theta)^2}$

sqrt both sides

$\dfrac{1}{\cos\theta} = 2$ or $\dfrac{1}{\cos\theta} = -2$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = \frac{-1}{2}$

1st, 4th, 2nd, 4th

$\boxed{\theta = \frac{\pi}{3} + 2\pi k, \ -\frac{\pi}{3} + 2\pi k, \ \frac{2\pi}{3} + 2\pi k, \ \frac{4\pi}{3} + 2\pi k}$

② $4 - \sec^2\theta$ DNE

$4 - \dfrac{1}{\cos^2\theta}$ DNE

$\cos^2\theta = 0 \Rightarrow \boxed{\frac{\pi}{2} + 2\pi k, \ \frac{3\pi}{2} + 2\pi k}$

Solution:

c) Use the product rule:

$f'(x) = x^2(-3e^{-3x}) + 2xe^{-3x}$

$= xe^{-3x}(-3x + 2)$

$f'(x)$ is never undefined

$f'(x) = 0 \Rightarrow \boxed{x = 0}$ or $e^{-3x} = 0$ or $-3x + 2 = 0$

impossible

$\boxed{x = \frac{2}{3}}$

So the critical pts are $x = 0, x = 2\pi/3$

Problem 4: (Stewart 4.1) Find the absolute extrema of $f(t) = t + \cot(t/2)$ on the closed interval $[\pi/4, 7\pi/4]$.

My Attempt:

calculator friendly ☺

Solution:

$f'(t) = 1 - \csc^2(t/2) \cdot 1/2 = 1 - \frac{1}{2}\csc^2\left(\frac{t}{2}\right) = 1 - \dfrac{1}{2\sin^2\left(\frac{t}{2}\right)}$

Two cases

① $f'(t) = 0$

$1 - \dfrac{1}{2\sin^2\left(\frac{t}{2}\right)} = 0$

$\sin^2\left(\frac{t}{2}\right) = \frac{1}{2}$

$\sin(t/2) = \pm\frac{1}{\sqrt{2}}$

$\frac{t}{2} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow t = 2\left(\frac{\pi}{4}\right) + 2\pi k, \ 2\left(\frac{3\pi}{4}\right) + 2\pi k$

value in $[\frac{\pi}{4}, \frac{7\pi}{4}]$: $\frac{\pi}{2}$; value in $[\frac{\pi}{4}, \frac{7\pi}{4}]$: $\frac{3\pi}{2}$

$\frac{t}{2} = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) \Rightarrow t = 2\left(\frac{-\pi}{4}\right) + 2\pi k, \ 2\left(\frac{5\pi}{4}\right) + 2\pi k$

value in $[\frac{\pi}{4}, \frac{7\pi}{4}]$: $\frac{3\pi}{2}$; value in $[\frac{\pi}{4}, \frac{7\pi}{4}]$: $\frac{\pi}{2}$

② $f'(t)$ is undefined

$1 - \dfrac{1}{2\sin^2\left(\frac{t}{2}\right)}$ is undefined

when $\sin^2\left(\frac{t}{2}\right) = 0$

so $\sin\left(\frac{t}{2}\right) = 0$

$t = \cancel{2\pi k}$

not in $[\frac{\pi}{4}, \frac{7\pi}{4}]$

Observe the critical values and endpoints

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cot\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + 1 \approx 2.5708$

$f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} + \cot\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2} - 1 \approx 3.7124$

$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \cot\left(\frac{\pi}{8}\right) \approx 3.1996$

$f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} + \cot\left(\frac{7\pi}{16}\right) \approx 3.0836$

Problem 5: (Stewart 4.1) Find the maximum value of $f(x) = x^a(1-x)^b$ on the interval $[0, 1]$, assuming a and b are both positive numbers.

My Attempt:

Solution:

① Find critical points:

$f'(x) = ax^{a-1}(1-x)^b + x^a b(1-x)^{b-1}$

$= ax^{a-1}(1-x)^{b-1}(1-x) + b\,\underline{x^{a-1}}\,\underline{x(1-x)^{b-1}}$

$x^a = x^{a-1}x$

$(1-x)^b = (1-x)^{b-1}(1-x)$

$= x^{a-1}(1-x)^{b-1}\left(a(1-x)+bx\right)$

$= x^{a-1}(1-x)^{b-1}\left(a - ax + bx\right)$

$= x^{a-1}(1-x)^{b-1}\left(a - x(a+b)\right)$

$f'(x)$ is never undefined.

$f'(x) = 0 \Rightarrow x^{a-1}(1-x)^{b-1}\left(a - x(a+b)\right) = 0$

$x^{a-1} = 0 \Rightarrow x = 0$

$(1-x)^{b-1} = 0 \Rightarrow x = 1$

$a - x(a+b) = 0 \Rightarrow x(a+b) = a \Rightarrow x = \frac{a}{a+b}$

② Absolute max can be at $\underbrace{\text{critical points}}_{0,\,1,\,\frac{a}{a+b}}$ or $\underbrace{\text{endpoints}}_{0,\,1}$

$f(0) = 0^a(1-0)^b = 0$

$f(1) = 1^a(1-1)^b = 0$

$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \underbrace{\left(1-\frac{a}{a+b}\right)^b}_{\text{this is strictly positive}} > 0 \Rightarrow$ the maximum is $\boxed{\left(\frac{a}{a+b}\right)^a\left(1-\frac{a}{a+b}\right)^b}$

Problem 6: Find the absolute extrema of $f(x) = \frac{x^2-4}{x^2+4}$ on the interval $[-4, 4]$.

My Attempt:

Solution:

critical points: LO DI HI HI DI LO

$$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} \leftarrow \text{LO LO}$$

$$= \frac{2x(x^2+4-x^2+4)}{(x^2+4)^2}$$

$$= \frac{16x}{(x^2+4)^2}$$

So $f'(x)$ is never undefined for real x

$f'(x) = 0$ for $x = 0$, the only critical value

possible locations of extrema: $\underset{\text{critical value}}{x=0}$, $\underbrace{x=-4,\ x=4}_{\text{endpoints}}$

$f(0) = \frac{0-4}{0+4} = -1$

$f(-4) = \frac{16-4}{16+4} = \frac{12}{20} = \frac{3}{5}$

$f(4) = \frac{16-4}{16+4} = \frac{3}{5}$

$\boxed{\text{Absolute max: } \frac{3}{5}, \text{ Absolute min: } -1}$

Challenge problem: (Stewart Chapter 4) If x, y, and z are positive numbers, then prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8.$$

The left-hand side is simply $\left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right)$

Show that the absolute min of $\frac{x^2+1}{x}$ is 2 assuming $x > 0$

By directly substituting variables, we see that the mins of $\left(\frac{y^2+1}{y}\right)$ and $\left(\frac{z^2+1}{z}\right)$ are both 2

So the min of the product is $2 \cdot 2 \cdot 2 = 8$.