Math 10A Fall 2024 Worksheet 20

November 12, 2024

Some of the problems are taken from examples or exercises in Stewart Chapter 4.6.

1 Antiderivatives

1. Why do we include the $+C$ when we take antiderivatives?
2. Suppose f and g are differentiable functions on an interval (a, b).
   (a) If f and g have the same derivatives, is it necessarily true that $f = g$?
   (b) If f and g have the same (family of) antiderivatives, is it necessarily true that $f = g$?
3. It is not always easy to write down an antiderivative of a function. Special techniques covering certain cases are a major part of integral calculus, on which we will spend most of the remainder of the course. Even so, there exist functions without an "elementary" antiderivative, in the sense that we cannot write down the antiderivative using only basic functions like polynomials, exponentials, logs, and trig functions.

   For the functions below, determine which have antiderivatives that are computable using only the techniques and formulas available in Chapter 4 in the textbook. Then write down the general form of the antiderivative in those cases.

   (a) $\sec^2(x)$
   (b) $-\sec^2(x)$
   (c) $\sin(\cos(x))$
   (d) e^{x^2}
   (e) $(e^x)^2$
   (f) $4x + 3e^x$
   (g) $7x^{50} + x^4 + 3x^3 + 10x^2 + 9x + 2 + x^{0.0001}$
   (h) $x + \frac{1}{x^2+1}$
   (i) x^x
   (j) $\sin x$
   (k) $1/x$
   (l) $\sqrt{x^3 - 1}$
   (m) $\sec(x)\tan(x)$

4. Find the general form of an "second antiderivative" of the given function. That is, if the given function is $f''(x)$, find all possibilities for $f(x)$. (Hint: your answers should involve two unknown constants, not just one.)

   (a) $6x + 12x^2$
   (b) e^{2x}
   (c) $\frac{2}{3}x^{-1/3}$
   (d) $\sin x + \cos x$

5. Find the general form of f if

$$f'''(x) = x^3 + 2x + 1.$$

6. (*Challenge.*) Let

$$f(x) = \begin{cases} \ln(-x) + 1 : & x < 0 \\ \ln(x) : & x > 0 \end{cases}$$

   Show that:

   (a) f is differentiable on $(-\infty, 0) \cup (0, \infty)$ with derivative $1/x$.
   (b) There exists no constant C such that $\ln|x| = f(x) + C$.
   (c) Keeping in mind that $\ln|x|$ also has antiderivative $1/x$, explain why (a) and (b) do not contradict Theorem 4.6.1 in the textbook: "If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant."

   (If this example seems confusing, feel free to ignore it. We won't be sticklers about technical cases like these in this course.)

2 Initial value problems

1. In an initial value problem, why is it wrong to include a $+C$ in your final answer?

2. Find f if $f'(x) = e^x + \frac{20}{1+x^2}$ and $f(0) = -2$.

3. A particle moves in a straight line with velocity function $v(t) = \sin t - \cos t$. Its initial displacement is $s(0) = 0$. Find the displacement function $s(t)$.

4. A particle moves along a straight line with acceleration function $a(t) = 5 + 4t - 2t^2$. Its initial velocity is $v(0) = 3$ m/s and its initial displacement is $s(0) = 10$ m. Find its position after t seconds.

5. (*Challenge.*) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of $y = f(x)$.

Solutions

1 Antiderivatives

1. The point is that you are not finding just one antiderivative of a function f, but rather *the family of all functions* F such that $F'(x) = f(x)$. If $F(x)$ is antiderivative of f, then $F(x) + C$ is also an antiderivative of f for any constant C, since the derivative of a constant is 0. Therefore, we always need to allow adding a constant when writing down our family. Conversely, Theorem 4.6.1 in the textbook tells us two different antiderivatives must differ by a constant, so if we can find one particular antiderivative $F(x)$, all the other ones must be form $F(x) + C$ for a constant C (assuming we are working over a single connected interval).

2. (a) No. If f and g differ by a constant, they will have the same derivative. For example, $f(x) = x$ and $g(x) = x + 1$ both have derivative 1.
   (b) Yes. If $F(x)$ is a common antiderivative of f and g, then we must have $f(x) = F'(x) = g(x)$.

3. (a) $\tan(x) + C$
   (b) $-\tan(x) + C$
   (c) No elementary antiderivative
   (d) No elementary antiderivative
   (e) $\frac{1}{2}e^{2x} + C$
   (f) $2x^2 + 3e^x + C$
   (g) $\frac{7}{51}x^{51} + \frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{10}{3}x^3 + \frac{9}{2}x^2 + 2x + \frac{1}{1.0001}x^{1.0001} + C$
   (h) $\frac{1}{2}x^2 + \arctan(x) + C$
   (i) No elementary antiderivative
   (j) $-\cos(x) + C$
   (k) $\ln|x| + C$ (but this is technically not quite the most general antiderivative; see problem 6 for further analysis)
   (l) No elementary antiderivative
   (m) $\sec(x) + C$

4. Below, both C and D are arbitrary constants.
   (a) $x^3 + x^4 + Cx + D$
   (b) $\frac{1}{4}e^{2x} + Cx + D$
   (c) $\frac{3}{5}x^{5/3} + Cx + D$
   (d) $-\sin(x) - \cos(x) + Cx + D$

5. $f(x) = \frac{1}{120}x^6 + \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx^2 + Dx + E$, where C, D, E are arbitrary constants. Note that one can "absorb" the factor of 1/2 into C, since C is already arbitrary.

6. (a) On $(-\infty, 0)$, we have $f(x) = \ln|x| + 1$, which we know to have derivative $1/x$. On $(0, \infty)$, we have $f(x) = \ln|x|$, which also has derivative $1/x$. This means that

$$f'(x) = \begin{cases} 1/x : & x < 0 \\ 1/x : & x > 0 \end{cases}$$

   but this just means $f'(x) = 1/x$, with domain $(-\infty, 0) \cup (0, \infty)$.
   (b) We have $\ln|x| = f(x)$ on the interval $(0, \infty)$, so if such a constant existed it would need to be $C = 0$. However, this constant does not work on the interval $(-\infty, 0)$, since $f'(x) = \ln|x| + 1$ on this domain. Therefore, no single constant C works everywhere.
   (c) The theorem specifies that F needs to be an antiderivative of f *on an interval.* In our case, the function f and its antiderivative $1/x$ are not defined on a single interval, but rather their domain is split into two disconnected pieces. This is enough to make the theorem inapplicable.

2 Initial value problems

1. In an initial value problem, you are not looking for a general family of antiderivatives. Instead, you are looking for one specific antiderivative, using the additional information from the initial conditions to determine exactly what C needs to be set to.

2. The general antiderivative is $e^x + 20\arctan(x) + C$. We need $f(0) = -2$, so we set $x = 0$ and solve for C:

$$\begin{aligned} -2 = f(0) &= e^0 + 20\arctan(0) + C \\ &= 1 + C \end{aligned}$$

   hence $C = -3$. Therefore $f(x) = e^x + 20\arctan(x) - 3$.

3. The general form of $s(t)$ is the antiderivative of $v(t)$, which is $s(t) = -\cos t - \sin t + C$. We need $s(0) = 0$, so we set $x = 0$ and solve for C:

$$\begin{aligned} 0 = s(0) &= -\cos(0) - \sin(0) + C \\ &= -1 + C \end{aligned}$$

   hence $C = 1$ and $s(t) = -\cos(t) - \sin(t) + 1$.

4. Velocity is the antiderivative of acceleration, and position is the antiderivative of velocity. Therefore, we must have

$$\begin{aligned} v(t) &= 5t + 2t^2 - \frac{2}{3}t^3 + C \\ s(t) &= \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + Cx + D \end{aligned}$$

   To determine C, we use the initial condition on velocity:

$$3 = v(0) = 0 + 0 + 0 + C$$

   so $C = 3$. Then to determine D, we use the initial condition for position:

$$10 = s(0) = 0 + 0 + 0 + 0 + D$$

   so $D = 10$. Therefore, the displacement function is $s(t) = \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + 3x + 10$, in meters.

5. The general antiderivative is $f(x) = \frac{1}{4}x^4 + C$. The line $y = -x$ has constant slope -1, and we have $f'(x) = x^3 = -1$ if and only if $x = -1$. Therefore, we need to choose C so that $f(x)$ intersects $y = -x$ at $x = -1$. That is, we need to solve

$$-(-1) = \frac{1}{4}(-1)^4 + C$$

   which gives $C = \frac{3}{4}$. Therefore, $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.