Linear approximations, Taylor polynomials

October 22nd, 2024

Here are some key ideas from section 3.8.

- The linearization (also known as the tangent line approximation) of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.
- Newton's method can be used to find a root, or zero, of a function $f(x)$:
  1. Make a guess for the root, and call it x_1.
  2. Successively calculate $x_{n+1} = x_n - \frac{x_n}{f'(x_n)}$.
  3. After enough iterations, and with an appropriate initial guess, x_n gets closer to a zero.
- The nth degree Taylor polynomial is $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$, where $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and so on.

  $\frac{f(0)}{0!}$ $\frac{f'(0)}{1!}$ $\frac{f''(0)}{2!}$ $\frac{f^{(n)}(0)}{n!}$

Problem 1: Use a linear approximation to estimate $(2.001)^5$.

My Attempt:

Solution:

Let $f(x) = x^5$, $a = 2$

Then $L(x) = f(a) + f'(a)(x-a)$
$= 2^5 + 5(2)^4(x-2)$
$= 32 + 80(x-2)$

$L(2.01) = 32 + 80(2.001 - 2)$
$= 32 + 80(0.001)$
$= 32.08$

Problem 2: (Stewart Chapter 3) At what point on the curve $y = [(\ln(x+4)]^2$ is the tangent line horizontal?

My Attempt:

Solution:

$\frac{dy}{dx} = 2\ln(x+4) \cdot \frac{1}{x+4} = \frac{2}{x+4}\ln(x+4)$

$\frac{dy}{dx} = 0$ means $\frac{2}{x+4}\ln(x+4) = 0$

so $\frac{2}{x+4} = 0$ or $\ln(x+4) = 0$ ✓

impossible

$\ln(x+4) = 0 \Rightarrow e^0 = x+4$
$\Rightarrow 1 = x+4$
$\Rightarrow x = -3$
$y = (\ln(1))^2 = 0$

$(-3, 0)$

Problem 3: (Stewart Chapter 3) Find the derivative of $\sin^2\left(\cos\sqrt{\sin \pi x}\right)$. $= \left(\sin\left(\cos\left(\sin \pi x\right)^{\frac{1}{2}}\right)\right)^2$

My Attempt:

Solution:

$2\sin\left(\cos\sqrt{\sin \pi x}\right)$

$\cdot \cos\left(\cos\sqrt{\sin \pi x}\right)$

$\cdot \left(-\sin\left(\sin \pi x\right)^{\frac{1}{2}}\right)$

$\cdot \frac{1}{2}\left(\sin \pi x\right)^{-\frac{1}{2}}$

$\cdot \cos \pi x$

$\cdot \pi$

Problem 4: (Stewart 3.8) Find the first three terms of the Taylor polynomial of degree n for $f(x) = e^x$, centered at a.

My Attempt:

Solution:

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$

$= \boxed{1 + x + \frac{x^2}{2}}$

Problem 5: (Stewart 3.8) Find an initial value of x_1 such that Newton's method fails on the function $x^3 - 3x + 6$.

My Attempt:

Solution:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

If $f'(x_1) = 0$, then x_2 is undefined.

$f'(x_1) = 0 \Rightarrow 3x_1^2 - 3 = 0 \Rightarrow \boxed{x_1 = \pm 1}$

Problem 6: ★ (Stewart Chapter 3) Find h' in terms of f' and g'.

$$h(x) = \frac{f(x)g(x)}{f(x)+g(x)} \begin{matrix} \text{HI} \\ \text{LO} \end{matrix}$$

My Attempt:

Solution:

$$\frac{\overset{\text{LO}}{(f(x)+g(x))}\overset{\text{DI}}{(f'(x)g(x)+f(x)g'(x))} - \overset{\text{HI}}{f(x)g(x)}\overset{\text{DI LO}}{(f'(x)+g'(x))}}{\underset{\text{LO}^2}{(f(x)+g(x))^2}}$$

Problem 7: ★ (Stewart Chapter 3) Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3}.$$

My Attempt:

Solution:

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} \cdot \frac{(\sqrt{1+\tan x} + \sqrt{1+\sin x})}{(\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x\to 0} \frac{1+\tan x - (1+\sin x)}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})}$$

$$= \lim_{x\to 0} \frac{\tan x - \sin x}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x\to 0} \frac{\sin x(\sec x - 1)}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})}$$

$$= \lim_{x\to 0} \overset{1}{\cancel{\frac{\sin x}{x}}} \cdot \frac{1}{x^2} \cdot \overset{\frac{1}{2}}{\cancel{\frac{1}{(\sqrt{1+\tan x} + \sqrt{1+\sin x})}}} \cdot (\sec x - 1) = \frac{1}{2} \lim_{x\to 0} \frac{\sec x - 1}{x^2}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{1}{2} \lim_{x\to 0} \frac{\overset{\tan^2 x}{\cancel{\sec^2 x - 1}}}{x^2(\sec x + 1)} = \frac{1}{2} \lim_{x\to 0} \frac{\frac{\sin^2 x}{\cos^2 x}}{x^2(\sec x + 1)}$$

$$= \frac{1}{2} \lim_{x\to 0} \left(\overset{1}{\cancel{\frac{\sin x}{x}}} \cdot \overset{1}{\cancel{\frac{\sin x}{x}}} \cdot \frac{\frac{1}{\cos^2 x}}{\sec x + 1} \right) = \frac{1}{2} \cdot \frac{1}{1+1} = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$$

Problem 8: ★ (Stewart Chapter 3) Show that

$$\frac{d}{dx}\left(\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}\right)=-\cos 2x.$$

My Attempt:

Solution:

$$y = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$$

$$= \frac{\sin^2 x}{1+\frac{1}{\tan x}} + \frac{\cos^2 x}{1+\tan x}$$

$$= \frac{\sin^2 x\,(\tan x) + \cos^2 x}{1+\tan x}$$

$$= \frac{\sin^3 x + \cos^3 x}{\cos x\,(1+\tan x)}$$

$$= \frac{\sin^3 x + \cos^3 x}{\cos x + \sin x}$$

$$= \frac{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)}{\cos x + \sin x}$$

$$= 1 - \sin x \cos x$$

$$= 1 - \tfrac{1}{2}\sin 2x \Rightarrow \frac{d}{dx}\left(1-\tfrac{1}{2}\sin 2x\right) = -\cos 2x$$

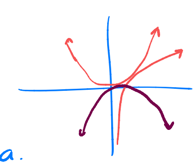
Problem 9: ★ (Stewart Chapter 3) For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

My Attempt:

Solution:

When does $y = \ln x$ intersect $y = cx^2$ at one point?

Suppose a is the x-value. Then $\ln a = ca^2$, so a is the unique soln.



If $c>0$, then:

The tangents have the same slope, so

$\ln x$ and cx^2 have the same slope at $x=a$.

Thus $\frac{1}{a} = 2ca \Rightarrow a = e^{1/2} \Rightarrow c = \frac{\ln a}{a^2} = \frac{1}{2e}$

If $c<0$, then they intersect at one point.

The same holds for $c=0$.

So we may have $c = \frac{1}{2e}$ or $c \le 0$