Final exam study guide

December 12th, 2024

This topics list covers **new content** after the midterm. You still have to know content covered on the midterm, so you may want to review the midterm study guide as well.

Here are some things you should know for the exam (feel free to use this as a checklist):

- ☐ The definition of an absolute maximum/minumum (4.1)
- ☐ The definition of a local maximum/minimum (4.1)
- ☐ The Extreme Value Theorem (4.1)
- ☐ Absolute extrema can be at critical points or endpoints, while local extrema can only be at critical points (otherwise put, endpoints are *not* local extrema) (4.1)
- ☐ Fermat's theorem (4.1)
- ☐ How to find the absolute extrema over a closed interval using the closed interval method (4.1)
- ☐ The Mean Value Theorem (4.2)
- ☐ How to determine where a function is increasing/decreasing (4.2)
- ☐ The First Derivative Test (4.2)
- ☐ How to determine intervals of concavity (4.2)
- ☐ The Second Derivative Test (4.2)
- ☐ L'hôpital's rule (4.3)
- ☐ How to get different types of indeterminate forms to meet the conditions of l'hôpital's rule (4.3)
- ☐ Working with optimization word problems (4.4); here's a process you can use:
    1. List the unknowns and conditions given in the problem.
    2. If possible, draw a diagram and label variables and constants.
    3. Express your target variable in terms of other variables.
    4. Use relationships between variables to write your target variable in terms of exactly one other variable.
    5. Find the required absolute extremum over the correct domain using the first derivative test (or another relevant one).
- ☐ The definition of an antiderivative (4.6)
- ☐ How to write the most general antiderivative $(+C)$ (4.6)
- ☐ The antiderivatives in Table 2 (page 308, 4.6)
- ☐ How to solve for the equation of a function given its derivative (4.6)
- ☐ How to solve for the equation of a function given its second derivative (4.6)
- ☐ Using definite integrals to find areas under curves (5.1)
- ☐ Using rectangles (left, right, and midpoint) to approximate definite integrals (5.1)
- ☐ The limit definition of the area of a region under the graph of a function between $x = a$ and $x = b$ (5.1)
- ☐ Using definite integrals to find distance and displacement (5.1, 5.2)
- ☐ The limit definition of a definite integral (5.2)
- ☐ The definition of integrability (5.2)

- ☐ Properties of the definite integral (5.2)
- ☐ The definition of the indefinite integral (5.3)
- ☐ Summation/difference/constant rules for indefinite integrals (5.3)
- ☐ The second part of the Fundamental Theorem of Calculus, also known as the Evaluation Theorem (section 5.3)
- ☐ Finding definite integrals using the evaluation theorem (section 5.3)
- ☐ The definition of an indefinite integral (section 5.3)
- ☐ Indefinite integrals of common functions (see page 345 of section 5.3)
- ☐ The substitution rule (5.4)
- ☐ Solving indefinite and definite integration problems with substitution (5.4)
- ☐ Integrals of even and odd functions (section 5.4)
- ☐ Integration by parts (5.5)
- ☐ Solving indefinite and definite integration problems with integration by parts (5.5)
- ☐ Solving problems with both substitution and integration by parts (5.5)
- ☐ Decomposing partial fractions by solving for coefficients (5.6)
- ☐ Decomposing fractions using long division (5.6)
- ☐ Integrating partial fractions to get $\ln$ terms (5.6)
- ☐ The definition of an improper integral (5.8)
- ☐ What it means for an improper integral to converge/diverge (5.8)
- ☐ The limit formula for the area between two curves (6.1)
- ☐ How to write the area between two curves as a definite integral (6.1)
- ☐ The formula for the average value of f on $[a, b]$ (6.2)
- ☐ The Mean Value Theorem for integrals (6.2)
- ☐ How to find the volume of a solid of revolution (rotating around either the x or y axis) using definite integration (6.4)

Problem 0: Draw as much of the unit circle as you can in 2 minutes.

Problem 1: (Similar to Practice Exam, 2a and 2d) Evaluate

a) $\displaystyle\int e^x \sin 2x \, dx$; b) $\displaystyle\int x^2\sqrt{x-2}\, dx$.

Problem 2: (Similar to Practice Exam, 3b) Find y'.

a) $e^{x+y} + x^2 - y = \cos(xy^2)$;

b) $y = \displaystyle\int_2^{x^3+e^x} t \ln t \, dt.$

Problem 3: (Similar to Practice Exam, 6b) The Triangle Inequality for vectors says that

$$|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|.$$

Prove that this holds. *Hint: recall that* $|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$*, and then use the distributive property. You may also need the Cauchy-Schwartz inequality, which says* $\vec{a}\cdot\vec{b} \leq |\vec{a}||\vec{b}|$.

Problem 4: (Similar to Practice Exam, 7) Suppose we have a_n and b_n modeling the populations of ants and bears respectively. Moreover, let

$$a_{n+1} = a_n + \frac{1}{2}b_n;$$

$$b_{n+1} = a_n + \frac{3}{2}b_n.$$

Assume $a_0 = 2$ and $b_0 = 1$.

a) Diagonalize the matrix A that models the population of ants and bears. *Hint: the eigenvalues are* $\lambda_1 = 2$ *and* $\lambda_2 = \frac{1}{2}$. *The corresponding eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. *You should know how to find these values for the exam.*

b) Compute A^n.

c) What happens to a_n and b_n as $n \to \infty$?

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Problem 5: (Identical to Practice Exam, 11) A window is comprised of two portions: the lower portion is a rectangle and the upper portion is a semicircle with diameter equal to the length of the top edge of the rectangle. The top edge of the rectangle is connected together with the diameter of the semicircle. The glass from the rectangle is clear, whereas the glass from the semicircle is tinted and only transmits half as much light per unit area as clear glass does. If the outer perimeter of the window is fixed at 50 meters, find the proportions of the window that will admit the most light.

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Problem 6: (Identical to Practice Exam, 8) Oski is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down a straight shoreline from the point nearest the boat. He can row 2 miles per hour and can walk 5 miles per hour. Where should Oski land the boat to reach the village in the least amount of time?

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Problem 7: (Similar to Practice Exam, 4) The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

Problem 8: (Similar to Quiz 7, 1c) Find $\lim_{x \to 0^+} \dfrac{(1 - \cos 5x)(\sin 2x)}{x^2}$.

Problem 9: (Similar to Quiz 9, 3d)

(a) Approximate $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$ in three different ways by using a right, left, and midpoint Riemann sum, each with 2 subintervals. Fully simplify your answers to express each of these approximations as a single fraction. Guess which of these approximations is the best.

(b) Which type of Riemann sum in part (a) is guaranteed to give an overestimate? Which type is guaranteed to give an underestimate?

(c) Find $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$ exactly using the Fundamental Theorem of Calculus.

(d) Compare your answer to part (c) to your three estimates from part (a) and determine which method was most accurate. Try to do this without a calculator first by making rough estimates of your answers from part (a) and approximating $\pi \approx 22/7$. Then check with a calculator. (You won't have a calculator on the exam, so we won't ask a question exactly like this one. However, good number sense is always helpful.)

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Problem 10: (Similar to Quiz 6, 3c and 3e)

(a) Show that if $f(x)$ is a polynomial, then $\dfrac{d^n}{dx^n} f(x) = 0$ for some n. What is the smallest such n?

(b) Find the tangent line to $y = \cos(x)$ at $x = 0$. Show that this line intersects the graph of $y = \cos(x)$ infinitely many times, and describe the points of intersection.

(c) Write down an example of a function $f(x)$ for which $f'(x)$ has a strictly smaller domain than $f(x)$.

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Solutions

Problem 1:

a) To solve this integral, we use integration by parts twice. Set

$$I = \int e^x \sin(2x)\, dx$$

First, let $u_1 = \sin(2x)$ and $dv_1 = e^x dx$. Then:

$$du_1 = 2\cos(2x)dx, \quad v_1 = e^x$$

Using the formula for integration by parts:

$$I = e^x \sin(2x) - \int e^x \cdot 2\cos(2x)\, dx$$

Now apply integration by parts to $\int e^x \cos(2x)\, dx$. Let $u_2 = 2\cos(2x)$ and $dv_2 = e^x dx$. Then

$$du_2 = -4\sin(2x)dx, \quad v_2 = e^x$$

and

$$\int e^x \cos(2x)\, dx = 2e^x \cos(2x) + \int e^x \cdot 4\sin(2x)\, dx = 2e^x \cos(2x) + 4I$$

We substitute this back to get

$$I = e^x \sin(2x) - (2e^x \cos(2x) + 4I)$$

Simplifying, we get

$$5I = e^x \sin(2x) - 2e^x \cos(2x) \implies I = \boxed{\frac{e^x \sin(2x) - 2e^x \cos(2x)}{5} + C}$$

b) Let $u = x - 2$. Then

$$x = u + 2, \quad dx = du.$$

Substituting, we get

$$\int x^2 \sqrt{x-2}\, dx = \int (u+2)^2 \sqrt{u}\, du$$

Expand $(u+2)^2$ to get

$$(u+2)^2 = u^2 + 4u + 4.$$

Thus

$$\int (u+2)^2 \sqrt{u}\, du = \int \left(u^2 + 4u + 4\right) u^{1/2}\, du = \int \left(u^{5/2} + 4u^{3/2} + 4u^{1/2}\right)\, du$$

Integrating term by term, we have

$$\frac{2}{7}u^{7/2} + \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2}, so$$

Substitute $x - 2$ for u to get

$$\int x^2 \sqrt{x-2}\, dx = \boxed{\frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{8}{3}(x-2)^{3/2} + C}$$

Problem 2:

a) This problem is about implicit differentiation. First, we write the derivatives of each term. We get

$$\frac{d}{dx}\left(e^{x+y}\right) = e^{x+y}(1+y'), \qquad \frac{d}{dx}\left(x^2\right) = 2x, \qquad \frac{d}{dx}\left(-y\right) = -y'.$$

For $\frac{d}{dx}\left(\cos(xy^2)\right)$, use the chain rule:

$$\frac{d}{dx}\left(\cos(xy^2)\right) = -\sin(xy^2)\cdot\frac{d}{dx}(xy^2).$$

To find $\frac{d}{dx}(xy^2)$, use the product rule. We have

$$\frac{d}{dx}(xy^2) = \frac{dx}{dx}(y^2) + x\frac{d}{dx}(y^2) = y^2 + 2xyy'.$$

So, putting things together,

$$\frac{d}{dx}\left(\cos(xy^2)\right) = -\sin(xy^2)(y^2+2xyy').$$

So far, the derivatives of both sides are

$$e^{x+y}(1+y') + 2x - y' = -\sin(xy^2)(y^2+2xyy').$$

Now group all terms involving y' to get

$$e^{x+y}y' - y' + \sin(xy^2)\cdot 2xyy' = -e^{x+y} - 2x - \sin(xy^2)y^2.$$

Factor y' on the left-hand side to get

$$y'(e^{x+y} - 1 + \sin(xy^2)\cdot 2xy) = -e^{x+y} - 2x - \sin(xy^2)y^2.$$

Solving for y', we have

$$y' = \boxed{\frac{-e^{x+y} - 2x - \sin(xy^2)y^2}{e^{x+y} - 1 + \sin(xy^2)\cdot 2xy}}.$$

b) We apply the Fundamental Theorem of Calculus and the chain rule. First, differentiate the integral to get

$$y' = (x^3+e^x)\ln(x^3+e^x)\cdot\frac{d}{dx}(x^3+e^x).$$

Differentiating the upper bound of integration, we have

$$\frac{d}{dx}(x^3+e^x) = 3x^2 + e^x.$$

Substitute back to get

$$y' = \boxed{(x^3+e^x)\ln(x^3+e^x)\cdot(3x^2+e^x)}.$$

Problem 3:

Following the hint, we write

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}).$$

Using the distributive property of the dot product, we expand:

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b}.$$

We know that $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ and $\vec{b}\cdot\vec{b} = |\vec{b}|^2$. Substituting, we get:

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2.$$

The dot product satisfies the inequality

$$\vec{a}\cdot\vec{b} \le |\vec{a}||\vec{b}|.$$

Substituting this into the equation, we have

$$|\vec{a}+\vec{b}|^2 \le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2.$$

The right-hand side is a perfect square, so

$$|\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 = (|\vec{a}| + |\vec{b}|)^2.$$

Thus

$$|\vec{a}+\vec{b}|^2 \le (|\vec{a}| + |\vec{b}|)^2.$$

Taking the square root of both sides (and noting that magnitudes are non-negative), we get

$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|.$$

And we are done!

Problem 4:

a) We first construct our matrices P and D, which are

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}.$$

The diagonal matrix is:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

b) Using the diagonalization $A = PDP^{-1}$, we compute:

$$A^n = PD^nP^{-1}.$$

Since D^n is diagonal:

$$D^n = \begin{bmatrix} 2^n & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix}.$$

Thus

$$A^n = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$$

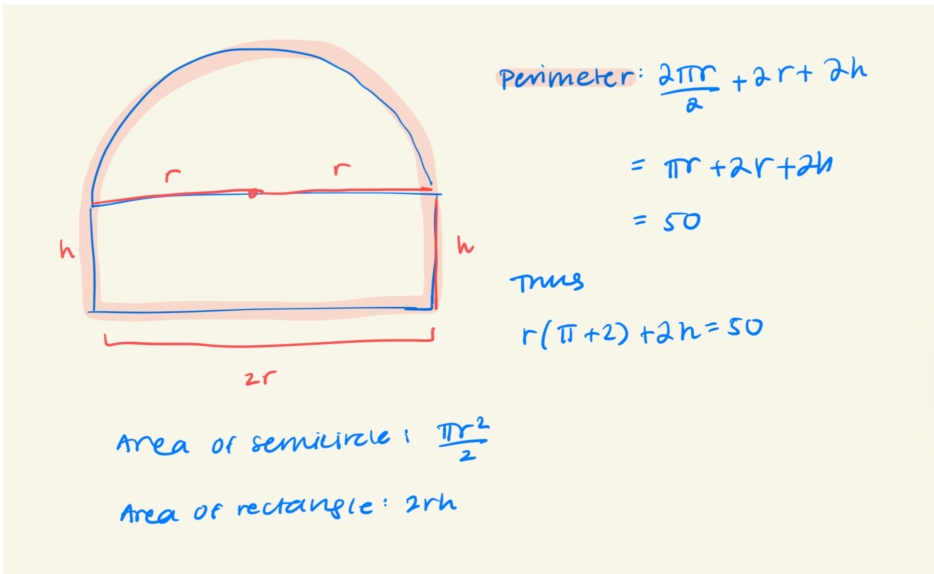
c)

As $n \to \infty$, the term involving $\lambda_2 = \frac{1}{2}$ decays to 0 because $\left(\frac{1}{2}\right)^n \to 0$. Thus it tends to

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^n \\ 2 \cdot 2^n, \end{bmatrix}$$

which tends in the direction of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. In other words, the bear population grows twice as fast as the ant population.

Problem 5: This is what we have.



We are given that twice as much light passes through the rectangle's area than the semicircle's area. Thus the value we want to optimize is (proportional to)

$$L = 2(2rh) + \frac{\pi r^2}{2} = 4rh + \frac{\pi r^2}{2}.$$

Substitute $h = \dfrac{50 - r(\pi + 2)}{2}$ (from the perimeter constraint) into the expression for L to get

$$L = 4r\left(\frac{50 - r(\pi + 2)}{2}\right) + \frac{\pi r^2}{2}.$$

Simplify:

$$L = 2r(50 - r(\pi + 2)) + \frac{\pi r^2}{2} = 100r - 2r^2(\pi + 2) + \frac{\pi r^2}{2}.$$

To maximize L, take the derivative with respect to r:

$$\frac{dL}{dr} = \pi r + 100 - 4r(\pi + 2) = 100 - r(3\pi + 8).$$

Set $\dfrac{dL}{dr} = 0$ to find the critical point, which is when $r = \dfrac{100}{3\pi + 8}$. Then $h = 25 - \dfrac{50}{3\pi + 8}(\pi + 2)$.

Problem 6:

Let Oski land the boat x miles down the shoreline from the point nearest to the boat. The total time T required is the sum of the time spent rowing and the time spent walking, which is

$$T = T_{\text{row}} + T_{\text{walk}}.$$

The rowing distance forms the hypotenuse of a right triangle with vertical side = 2 miles (distance offshore), and horizontal side = x miles. The rowing distance is therefore

$$\sqrt{2^2 + x^2} = \sqrt{4 + x^2}.$$

The rowing speed is 2 miles per hour, so:

$$T_{\text{row}} = \frac{\sqrt{4 + x^2}}{2}.$$

The walking distance is: $6 - x$. The walking speed is 5 miles per hour, so:

$$T_{\text{walk}} = \frac{6 - x}{5}.$$

Thus, the total time is:

$$T(x) = \frac{\sqrt{4 + x^2}}{2} + \frac{6 - x}{5}.$$

To minimize $T(x)$, take the derivative of $T(x)$ with respect to x, set it equal to zero, and solve for x. We get

$$T'(x) = \frac{x}{\sqrt{4 + x^2}} - \frac{1}{5}.$$

Set $T'(x) = 0$:

$$\frac{x}{\sqrt{4 + x^2}} = \frac{1}{5}.$$

Square both sides:

$$\frac{x^2}{4 + x^2} = \frac{1}{25}.$$

Multiply through by $4 + x^2$ and simplify to get

$$x = \frac{\sqrt{6}}{6}.$$

To verify that this value of x minimizes $T(x)$, check the second derivative or confirm that $T'(x)$ changes sign around $x = \dfrac{\sqrt{6}}{6}$.

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Problem 7:
Each horizontal cross-section is an equilateral triangle with side length $s(x)$. The area of an equilateral triangle with side length s is:

$$A = \frac{\sqrt{3}}{4} s^2.$$

You can derive this using the Pythagorean theorem if you don't have it memorized. Substituting $s(x) = \dfrac{1}{4}x$, we get

$$A(x) = \frac{\sqrt{3}}{4}\left(\frac{1}{4}x\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{x^2}{16} = \frac{\sqrt{3}}{64}x^2.$$

To find the volume, we integrate the cross-sectional area $A(x)$ along the height of the monument. The limits of integration are $x = 0$ (top of the monument) to $x = 20$ (base of the monument), so we have

$$V = \int_0^{20} A(x)\,dx = \int_0^{20} \frac{\sqrt{3}}{64}x^2\,dx.$$

We get

$$V = \frac{\sqrt{3}}{64} \cdot \frac{8000}{3} = \frac{\sqrt{3} \cdot 8000}{192}.$$

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Problem 8:
There are a lot of ways to do this problem, and we'll present one of them. We can rewrite the expression as

$$\lim_{x\to 0^+} \frac{(1-\cos 5x)(\sin 2x)}{x^2} = \lim_{x\to 0^+} \frac{1-\cos 5x}{x} \cdot \frac{\sin 2x}{x}.$$

The term $\dfrac{1-\cos 5x}{x}$ results in the indeterminate form $0/0$ as $x \to 0^+$. Applying l'hôpital's rule, we have

$$\lim_{x\to 0^+} \frac{1-\cos 5x}{x} = 0.$$

The term $\dfrac{\sin 2x}{x}$ also results in the indeterminate form $0/0$ as $x \to 0^+$. We apply L'Hôpital's Rule again:

$$\lim_{x\to 0^+} \frac{\sin 2x}{x} = 2.$$

Combining the two limits, we have:

$$\lim_{x\to 0^+} \frac{(1-\cos 5x)(\sin 2x)}{x^2} = \left(\lim_{x\to 0^+} \frac{1-\cos 5x}{x}\right) \cdot \left(\lim_{x\to 0^+} \frac{\sin 2x}{x}\right) = 0 \cdot 2 = 0.$$

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Problem 9:

(a) The two subintervals are $[0, 1/2]$ and $[1/2, 1]$, which have length $1/2$ and midpoints $1/4$ and $3/4$, respectively. Let $f(x) = \dfrac{1}{x^2+1}$.

- Left Riemann sum: We plug the left endpoints of the two intervals into $f(x)$, getting

$$\begin{aligned} f(0) &= \frac{1}{1+0^2} = 1 \\ f(1/2) &= \frac{1}{1+\frac{1^2}{2^2}} = \frac{1}{5/4} = \frac{4}{5}. \end{aligned}$$

  Then the left Riemann sum is

$$L_2 = \frac{1}{2}\left(f(0) + f(1/2)\right) = \frac{1}{2}\left(1 + \frac{4}{5}\right) = \boxed{\frac{9}{10}}$$

- Left Riemann sum: We plug the right endpoints of the two intervals into $f(x)$, getting

$$\begin{aligned} f(1/2) &= \frac{4}{5} \\ f(1) &= \frac{1}{1+1^2} = \frac{1}{2}. \end{aligned}$$

  Then the left Riemann sum is

$$R_2 = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{2}\right) = \boxed{\frac{13}{20}}$$

- Midpoint Riemann sum: We plug the midpoints of the two intervals into $f(x)$, getting

$$f(1/4) = \frac{1}{1+\frac{1^2}{4^2}} = \frac{16}{17}$$
$$f(3/4) = \frac{1}{1+\frac{3^2}{4^2}} = \frac{16}{25}$$

Then the midpoint Riemann sum is

$$M_2 = \frac{1}{2}\left(\frac{16}{17}+\frac{16}{25}\right) = \frac{1}{2}\left(\frac{16\cdot 25+16\cdot 17}{17\cdot 25}\right) = \frac{672}{2\cdot 425} = \boxed{\frac{336}{425}}$$

(b) The function $\frac{1}{x^2+1}$ is decreasing on the interval $(0,1]$ (since the denominator is getting *bigger*). This means that left Riemann sums will *always* overestimate and right Riemann sums will *always* underestimate.

We can't really say much about whether the midpoint Riemann sum is an overestimate or an underestimate without making a more sophisticated argument or checking against the true value.

(c)

$$\int_0^1 \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_0^1 = \arctan(1) - \arctan(0) = \boxed{\pi/4}$$

(d) Approximating $\pi \simeq 22/7$, we approximate the integral as $11/14$, which is more than $3/4$ but less than $4/5$. We'll approximate M_2 as $\frac{330}{420} = \frac{33}{42}$, which is also more than $\frac{3}{4}$ and less than $\frac{4}{5}$. The interval $[3/4, 4/5]$ has length $1/20$, so this means that M_2 is within $1/20$ of being correct, assuming our approximations are good enough. Meanwhile, $L_2 = 9/10$ is $1/10$ more than $4/5$ and $R_2 = \frac{13}{20}$ is $1/10$ less than $3/4$, so both of these Riemann sums are less accurate than M_2. So M_2 should be the best.

Checking with a calculator, we find that

$$L_2 - \pi/4 \approx 0.1146$$
$$R_2 - \pi/4 \approx -0.1354$$
$$M_2 - \pi/4 \approx 0.0051,$$

which confirms our answer (as well as our answer to part (b)).

In this case, it turns out that the midpoint Riemann sum was quite a good approximation, despite only using 2 subintervals.

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Problem 10:

(a) Recall that we define the *degree* of a polynomial to be the largest power appearing in the polynomial. By convention, we consider constants to be polynomials of degree 0. The power rule tells us that $\frac{d}{dx}x^n = nx^{n-1}$, so the derivative reduces the degree by exactly 1 in this case. More generally, taking a derivative reduces the degree of a nonconstant polynomial by 1, and sends a constant function to 0. That means that if $f(x)$ is a polynomial of degree n, then $\frac{d^n}{dx^n}f(x)$ is a nonzero constant function, and one more application of the derivative makes this 0. So the smallest order derivative that kills a polynomial is the degree plus 1.

(b) The derivative is $\frac{d}{dx}\cos(x) = -\sin(x)$, which is 0 at $x = 0$. Therefore, the tangent line at $x = 0$ is the horizontal line passing through $(0, \cos(0)) = (0, 1)$, which is the line $y = 1$. This line intersects the graph $y = \cos(x)$ exactly when $\cos(x) = 1$, i.e. precisely when $x = 2\pi k$ for some integer k. In particular, this happens infinitely many times.

(c) Some possible answers:

- $f(x) = |x|$, which has domain $(-\infty, \infty)$ but is not differentiable at $x = 0$.
- $f(x) = \sqrt[3]{x}$, which has domain $(-\infty, \infty)$ but is not differentiable at $x = 0$.
- Any function that is defined everywhere but has a discontinuity, e.g. a piecewise defined function with a jump discontinuity.