Math 10A Fall 2024 Worksheet 16

October 29, 2024

1 Taylor Series

The point of this exercise is to get you to understand why L'Hopital's rule is true, **if the functions involved have power series**. Suppose the function $h(x) = \frac{f(x)}{g(x)}$, with f and g continuous, is such that the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is a $0/0$ indeterminate form. In other words, $f(a) = g(a) = 0$, so we cannot compute the limit by plugging in $x = a$. Suppose f and g have Taylor series near a, or in other words, if x is very close to a, then

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3 + \frac{f''''(a)}{4\cdot 3\cdot 2}(x-a)^4 + \cdots \\ g(x) &= g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \frac{g'''(a)}{3\cdot 1}(x-a)^3 + \frac{g''''(a)}{4\cdot 3\cdot 2}(x-a)^4 + \cdots. \end{aligned}$$

Remember that since f/g is a $0/0$ indeterminate form,

$$f(a) = g(a) = 0,$$

so both Taylor series really begin with an $(x-a)$ term, not a constant term.

1. Using these Taylor series, show that, assuming $g'(a) \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

2. Suppose that, with everything as above, $f'(a) = g'(a) = 0$, so that $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is also a $0/0$ indeterminate form, but that $g''(a) \neq 0$. Then show using the Taylor series that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}.$$

In the previous exercise, you used Taylor series. This next exercise is to explain where Taylor series come from. The idea of the linearization to $f(x)$ at a is that we find a linear function $g(x) = a_1x + a_0$, such that

$$f(a) = g(a), f'(a) = g'(a).$$

In other words, f and g, and their first derivatives, agree at a, and this gives us the formula

$$g(x) = f(a) + f'(a)(x-a).$$

Similarly, we can ask the following questions.

1. Assume f is twice-differentiable at a, i.e. $f'(a)$ and $f''(a)$ both make sense. Find a quadratic function $g(x) = a_2x^2 + a_1x + a_0$ such that

$$f(a) = g(a), f'(a) = g'(a), f''(a) = g''(a).$$

2. Do the same thing for a cubic function and having all derivatives up to the third derivative match.
3. Compare your answers to (1) and (2) with the formula for the linearization of f at a. What patterns do you notice? (It may help to express your answers as polynomials in $(x-a)$, instead of in x.)
4. Compute the first 8 terms of the Taylor series for $f(x) = \cos(x)$ and $f(x) = \sin(x)$, both at $a = 0$.

2 Min/Max, Curves Analysis

1. Let $f(x) = x^3 - 2x^2$.
   (a) Find the domain of $f(x)$ and $f'(x)$.
   (b) Find all critical points of $f(x)$.
   (c) Find the intervals of increase or decrease.
   (d) Find the intervals of concavity and the inflection points.
   (e) Find the absolute maximum and minimum values on the interval $[-1/2, 5/2]$.
2. Draw four curves $y = f(x)$ illustrating each of the following shapes, and for each case, write a formula for a function illustrating that case.
   (a) $f'(x) > 0, f''(x) > 0$
   (b) $f'(x) > 0, f''(x) < 0$
   (c) $f'(x) < 0, f''(x) > 0$
   (d) $f'(x) < 0, f''(x) < 0$
3. Let $f(x) = x^3$. Show that $x = 0$ is a critical point of f, but not a local minimum nor local maximum. Why does this happen?

3 Solutions

3.1 Taylor Series

1. We compute

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots}{g'(a)(x-a)+\frac{g''(a)}{2}(x-a)^2+\cdots} = \lim_{x\to a}\frac{x-a}{x-a}\cdot\frac{f'(a)+\frac{f''(a)}{2}(x-a)+\cdots}{g'(a)+\frac{g''(a)}{2}(x-a)+\cdots} = \frac{f'(a)}{g'(a)}$$

2. Similar, except now you have $f(a) = g(a) = f'(a) = g'(a) = 0$, so the Taylor series begin at the quadratic term, so you can factor out an $(x-a)^2$ from the numerator and denominator.

1. Suppose the quadratic function we are interested in is $g(x) = a_2(x-a)^2 + a_1(x-a) + a_0$. Then $g''(a) = 2a_2$, so since we want $g''(a) = f''(a)$, $a_2 = \frac{f''(a)}{2}$. The linear and constant term are the same as for the linearization, so the desired quadratic function is

$$g(x) = \boxed{\frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)}.$$

   When you plug in $x = a$, all terms become 0 except $f(a)$. If you differentiate once then plug in $x = a$, all terms become 0 except $f'(a)$. If you differentiate twice, you are left with the constant $f''(a)$.

2. Similarly, we begin by writing $g(x) = a_3(x-a)^3 + a_2(x-a)^2 + a_1(x-a) + a_0$, then differentiate this three times. The lower-degree terms match the quadratic and linear cases.

$$g(x) = \boxed{\frac{f'''(a)}{6}(x-a)^3 + \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)}.$$

3. The general pattern is that, if we are approximating f and all its derivatives near a with an n-degree polynomial in $(x-a)$, the kth degree coefficient is

$$\boxed{\frac{f^{(k)}(a)}{k!}},$$

   where $k! = 1\cdot 2\cdot 3\cdots k$ is the product of the first k positive integers. The reason this occurs is that differentiating x^k k times multiplies by the exponent each time, cumulating in $k!$.

4. The Taylor series for sin and cos are

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

3.2 Min/Max, Curves Analysis

1. (a) $f(x)$ and $f'(x) = 3x^2 - 4x$ are both polynomials, so they each have domain $\boxed{\mathbb{R}}$.

   (b) We have $f'(x) = 3x^2 - 4x$, and the critical points are wherever this is 0, or undefined, or the boundary points if we are working on an interval. This is defined everywhere, and our interval is $\mathbb{R}$ for now so there are no boundary points, so the only thing we need to consider is when $f'(x) = 0$. Note

$$f'(x) = 3x^2 - 4x = x(3x-4),$$

   which is 0 precisely when $x = \boxed{0, 4/3}$.

(c) The intervals of increasing and decreasing are determined by where $f'(x)$ is positive or negative, with boundary points added. There are three intervals to consider, split up by the critical points 0 and 4/3. We get

$$\begin{aligned} \text{On } (-\infty, 0],\ & f(x) \text{ is } \textbf{increasing} \\ \text{On } [0, 4/3],\ & f(x) \text{ is } \textbf{decreasing} \\ \text{On } [4/3, \infty),\ & f(x) \text{ is } \textbf{increasing} \end{aligned}$$

(d) Similarly to increasing/decreasing and $f'(x)$, the concavity intervals are determined by the sign of $f''(x)$. We have

$$f''(x) = 6x - 4,$$

which is 0 precisely when $x = \boxed{2/3}$, that is the only inflection point. The second derivative is negative on $(-\infty, 2/3)$ and positive on $(2/3, \infty)$ so f is **concave up on** $[2/3, \infty)$ and **concave down on** $(-\infty, 2/3]$.

(e) This is a twice-differentiable (even smooth) function, so the maximum and minimum occur at a critical point of f, or at a boundary point $-1/2, 5/2$. So there are four points to test, $-1/2, 0, 4/3, 5/2$. For the critical points,

$$f''(0) = -4 < 0, f''(4/3) = 4 > 0,$$

so there is a local maximum at 0 and a local minimum at 4/3. We have

$$f(-1/2) = -5/8, f(0) = 0, f(4/3) = -32/27, f(5/2) = 25/8,$$

so on $[-1/2, 5/2]$, $\boxed{5/2}$ is the absolute maximum and $\boxed{4/3}$ is the absolute minimum.

2. There's lots of examples you can come up with; all four behaviors occur in the above example $f(x) = x^3 - 2x^2$, and you can use the answers to (c) and (d) to see where each behavior occurs in the graph.

3. This happens because $x = 0$ is both a critical point *and* an inflection point.