Math 10A Fall 2024 Worksheet 15

October 17, 2024

1 Quotient rule

1. Compute the derivative of the given function.

   (a) $P(x) = \frac{\sin(x)}{x^3-1}$

   (b) $L(x) = \frac{(1-4x)(2+x)}{3+9x}$

   (c) $f(z) = \frac{3z+z^4}{2z^2+1}$

   (d) $g(x) = \frac{1}{\arctan(x)}$

2. Derive the formulas for the derivatives of $\tan(x), \cot(x), \sec(x)$, and $\csc(x)$ by using the quotient rule and the derivatives of $\cos(x)$ and $\sin(x)$.

3. Compute the second derivatives of all six inverse trig functions.

4. Suppose $f(x) = g(x)/h(x)$ and that $g(a) = 1, h(a) = 2, g'(a) = 3$, and $h'(a) = 4$. Compute $f'(a)$.

2 Logarithmic and implicit differentiation

1. Use logarithmic differentiation to compute the derivatives of the following functions. (You can do some of these without logarithmic differentiation, but it might be a lot harder).

   (a) x^x

   (b) $f(x) = (2x+1)^5(x^4-3)^6$

   (c) $f(z) = \sqrt{z}e^{z^2}(z^2+1)^{10}$

   (d) $h(y) = y^{1/(1+y^2)}$

2. Explain why the chain rule, power rule, and the formula for the derivative of an exponential are unhelpful for computing $\frac{d}{dx}x^x$ without taking logarithms first.

3. (a) Compute $\frac{dy}{dx}$ if $y^x = x^y$ (for $x, y > 0$).

   (b) Is y a function of x?

   (c) Compute the tangent line to this curve at the point $(1, 1)$.

   (d) Something weird should happen when you try to find $\frac{dy}{dx}$ at the point (e, e). What's going on?

   (e) Graph $y^x = x^y$ on Desmos to check your work and get a better sense of what is going on.

3 Linear approximation

1. Use a first-order linear approximation to estimate the following numbers.
   (a) $e^{0.05}$
   (b) $\sin(3.1)$
   (c) $(1.01)^{-20}$
   (d) $\log_2(257)$
   (e) $\arcsin(0.99)$
   (f) $\tan(\pi/4 + 0.02)$

2. Justify the following approximation: $\sin(x) \approx \tan(x) \approx e^x - 1 \approx x$ when $|x|$ is small.

3. In which of the following cases should you suspect that the linearization of $f(x)$ at a might be a poor estimate of $f(a+h)$?
   (a) When $|h|$ is large.
   (b) When $|a|$ is large.
   (c) When $|f(a)|$ is large.
   (d) When $f(x)$ has a jump discontinuity at b for some $a < b < a + h$.
   (e) When $f(x)$ is a polynomial.
   (f) When you have to use the quotient rule to compute the derivative of f at a.

4 Solutions

4.1 Quotient Rule

1. (a)
$$P'(x) = \frac{\cos(x)(x^3-1) - 3x^2\sin(x)}{(x^3-1)^2}$$

   (b)
$$L'(x) = \frac{d}{dx}\left(\frac{-4x^2-7x+2}{3+9x}\right) = \frac{(-8x-7)(9x+3)-(-4x^2-7x+2)(9)}{(9x+3)^2}$$

   (c)
$$f'(z) = \frac{(4z^3+3)(2z^2+1)-(4z)(z^4+3z)}{(2z^2+1)^2}$$

   (d) We have $\tan(\tan^{-1}(x)) = x$, so if $f(x) = \tan^{-1}(x)$, the chain rule shows
$$\frac{1}{\cos(\tan^{-1}(x))^2} \cdot f(x) = 1.$$
   From right triangle trigonometry, $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$, so the derivative of $\tan^{-1}(x)$ is $\frac{1}{\sqrt{x^2+1}}$. Therefore,
$$g'(x) = \frac{-1}{\tan^{-1}(x)^2 \cdot \sqrt{x^1+1}}.$$

2. (a)
$$\frac{d}{dx}\tan(x) = \frac{\cos(x)^2 - \sin(x)(-\sin(x))}{\cos(x)^2} = \frac{1}{\cos(x)^2}$$

   (b)
$$\frac{d}{dx}\cot(x) = \frac{(-\sin(x))\sin(x) - \cos(x)^2}{\sin(x)^2} = \frac{-1}{\sin(x)^2}$$

   (c)

3.

4.