Quiz 3 study guide

September 22nd, 2024

General information

Quiz 2 covers sections 8.6 and 8.7. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.
- You are welcome to email me me at dbhatia1089@berkeley.edu if you need help on practice problems.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Using matrices to represent and solve systems of equations (Section 8.6)
- ☐ If the determinant of A is not zero, then there is a unique solution to $A\vec{x} = \vec{b}$ (Section 8.6)
- ☐ If the determinant of A is zero, then there are either infinitely many solutions or zero solutions to $A\vec{x} = \vec{b}$ (Section 8.6)
- ☐ The definition of an eigenvalue and its corresponding eigenvector (Section 8.7)
- ☐ The definition of a characteristic polynomial (Section 8.7)
- ☐ How to find a characteristic polynomial (Section 8.7)
- ☐ How to find eigenvalues using the characteristic polynomial (Section 8.7)
- ☐ How to find a corresponding eigenvalue given an eigenvector (Section 8.7)
- ☐ How to find a corresponding eigenvector given an eigenvalue (Section 8.7)

Help! I'm stuck on....

- ...finding **eigenvalues from characteristic polynomials**: check out this 4 minute video
- ...**solving** for eigenvectors and eigenvalues: check out parts of this 17 minute video
- ...finding **general forms of eigenvectors**: check out this 6 minute video
- ...solving for **corresponding eigenvectors**: check out this 8 minute video

Practice problems

1. *True/False*: If $\vec{v}$ is an eigenvector, then $\vec{v} \neq 0$.
2. *True/False*: If λ is an eigenvalue, then $\lambda \neq 0$.
3. *True/False*: If the only eigenvalue of a matrix is 0, the matrix is the zero matrix (all entries are zero).
4. *True/False*: If a matrix A is not the identity, then 1 is not an eigenvalue of A.
5. *True/False*: If λ is an eigenvalue of A, then λ is also an eigenvalue of any power of A.
6. *True/False*: If λ is an eigenvalue of A, then λ is also an eigenvalue of the transpose of A.
7. *True/False*: If λ is an eigenvalue of the matrix A, then λ is an eigenvalue of A^2.
8. *True/False*: If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of A^2.
9. Solve the system of equations below using matrices.
$$\begin{aligned} 3x - y &= 2; \\ x + 7y &= 4. \end{aligned}$$
10. Solve the system of equations below using matrices.
$$\begin{aligned} 7x + y &= 0; \\ 14x + 2y &= 0. \end{aligned}$$
11. Solve the system of equations below using matrices.
$$\begin{aligned} 2x + y &= 2; \\ 4x - y &= 4. \end{aligned}$$
12. Find the eigenvectors and eigenvalues for the matrix.
$$A = \begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}.$$
13. Find an expression for the eigenvalues of an arbitrary 2×2 matrix.

Solutions

1. **True**. By definition, an eigenvector must be non-zero.
2. **False**. Eigenvalues can be zero; for example, 0 is an eigenvalue of singular matrices.
3. **False**. A matrix can have only 0 as an eigenvalue without being the zero matrix. For example, any singular matrix may have 0 as its only eigenvalue.
4. **False**. A matrix can have 1 as an eigenvalue without being the identity matrix. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
5. **False**. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
6. **True**. You can see this is true by noticing that the characteristic polynomial of the transpose is the same as the characteristic polynomial of the original matrix.
7. **False**. If λ is an eigenvalue of A, then λ^2 might not be an eigenvalue of A^2. Instead, the eigenvalues of A^2 are the squares of the eigenvalues of A.
8. **True**. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of A^2 with the corresponding eigenvalue being λ^2.
9. The system of equations can be written in matrix form as:
$$\begin{bmatrix} 3 & -1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$
To solve, we first find the inverse of the coefficient matrix:
$$A = \begin{bmatrix} 3 & -1 \\ 1 & 7 \end{bmatrix}, \quad A^{-1} = \frac{1}{(3)(7) - (-1)(1)} \begin{bmatrix} 7 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 7 & 1 \\ -1 & 3 \end{bmatrix}.$$
Now, multiplying both sides by A^{-1}:
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 7 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} (7)(2) + (1)(4) \\ (-1)(2) + (3)(4) \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 18 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{9}{11} \\ \frac{5}{11} \end{bmatrix}.$$
Therefore, the solution is:
$$x = \frac{9}{11}, \quad y = \frac{5}{11}.$$
10. The system of equations can be written in matrix form as:
$$\begin{bmatrix} 7 & 1 \\ 14 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This is a homogeneous system. To solve, we first check the determinant of the coefficient matrix:
$$\det\left(\begin{bmatrix} 7 & 1 \\ 14 & 2 \end{bmatrix}\right) = (7)(2) - (14)(1) = 14 - 14 = 0.$$
Since the determinant is zero, the matrix is singular, meaning the system has either no solutions or infinitely many solutions. In this case, the two equations are dependent (the second is a multiple of the first), so the system has infinitely many solutions.
We solve for y in terms of x from the first equation:
$$7x + y = 0 \implies y = -7x.$$
Therefore, the solution to the system is:
$$x = t, \quad y = -7t \quad \text{for any real number } t.$$
Or, written another way, our solution is all vectors satisfying the form $\begin{bmatrix} x \\ -7x \end{bmatrix}$.

11. The system of equations can be written in matrix form as:
$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$
To solve, we first find the inverse of the coefficient matrix:
$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \quad A^{-1} = \frac{1}{(2)(-1)-(1)(4)} \begin{bmatrix} -1 & -1 \\ -4 & 2 \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -1 & -1 \\ -4 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$
Now, multiplying both sides by A^{-1}:
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} (1)(2)+(1)(4) \\ (4)(2)+(-2)(4) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Therefore, the solution is:
$$x = 1, \quad y = 0.$$

12. To find the eigenvalues, we solve the characteristic equation:
$$\det(A - \lambda I) = 0.$$
Then
$$A - \lambda I = \begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -2 \\ -4 & 4-\lambda \end{bmatrix}.$$
The determinant is:
$$\det(A - \lambda I) = (2-\lambda)(4-\lambda) - (-2)(-4) = (2-\lambda)(4-\lambda) - 8.$$
Expanding:
$$\det(A - \lambda I) = 8 - 6\lambda + \lambda^2 - 8 = \lambda^2 - 6\lambda.$$
Therefore, the characteristic equation is:
$$\lambda^2 - 6\lambda = 0.$$
Factoring:
$$\lambda(\lambda - 6) = 0.$$
So, the eigenvalues are:
$$\lambda_1 = 0, \quad \lambda_2 = 6.$$
Next, we find the eigenvectors for each eigenvalue. For $\lambda_1 = 0$, we solve $(A - 0I)\vec{v} = 0$:
$$A\vec{v} = \begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This gives the system:
$$2v_1 - 2v_2 = 0, \quad -4v_1 + 4v_2 = 0.$$
Both equations reduce to $v_1 = v_2$. So, one eigenvector corresponding to $\lambda_1 = 0$ is:
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Your answer can be any scalar multiple.
For $\lambda_2 = 6$, solve $(A - 6I)\vec{v} = 0$:
$$\begin{bmatrix} 2-6 & -2 \\ -4 & 4-6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This gives the system:
$$-4v_1 - 2v_2 = 0.$$
Solving for v_1 and v_2, we get $v_1 = -\frac{1}{2}v_2$. So, one eigenvector corresponding to $\lambda_2 = 6$ is:
$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$
Your answer can be any scalar multiple.

13. For a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the eigenvalues are the solutions to the characteristic equation:

$$\det(A - \lambda I) = 0,$$

where I is the 2×2 identity matrix. Explicitly, we have:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix},$$

and the determinant is:

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).$$

The eigenvalues are the roots of this quadratic equation:

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$