Quiz 2 study guide

September 15th, 2024

General information

Quiz 2 covers sections 8.4, 8.5, and 8.6. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.
- You are welcome to email me me at dbhatia1089@berkeley.edu if you need help on practice problems.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ The definitions of square matrices, column vectors, row vectors (Section 8.4)
- ☐ Forming the transpose of a matrix (Section 8.4)
- ☐ Matrix addition and scalar multiplication (Section 8.4)
- ☐ Associative, commutative, distributive properties of matrix addition (Section 8.4)
- ☐ How to multiply two matrices and when it is possible to do so (Section 8.4)
- ☐ The form of an identity matrix (Section 8.4)
- ☐ How to draw a Leslie diagram (Section 8.5)
- ☐ The definition of the inverse of a matrix and when matrices can be inverted (Section 8.6)
- ☐ The definition of singular and nonsingular matrices (Section 8.6)
- ☐ How to find the inverse of a 2×2 matrix
- ☐ The determinant of a 1×1 matrix, a 2×2 matrix, and a 3×3 matrix (Section 8.6)
- ☐ Using matrices to represent and solve systems of equations (Section 8.6)

Help! I'm stuck on....

- ...understanding **matrix definitions** (size, rows, columns, etc.): check out this 11 minute video (watch at 2x)
- ...finding the **transpose** of a matrix: check out this 2 minute video and its chill music
- ...matrix **operations—**adding, subtracting, and scalar multiplying: check out this 9 minute video (watch at 2x)
- ...**Leslie matrices**: check out this 10 minute video
- ...finding the **inverse of a** 2×2 **matrix**: check out this 3 minute video
- ...practicing 2×2 and 3×3 **determinants**: check out this 10 minute video
- ...representing **systems of equations** with matrices: check out this 7 minute video

Practice problems

1. *True/False:* Matrix addition is both commutative and associative. (Section 8.4)
2. *True/False:* Scalar multiplication is distributive over matrix addition. (Section 8.4)
3. When is matrix multiplication possible? (Section 8.4)
4. *True/False:* Matrix multiplication is commutative. (Section 8.4)
5. *True/False:* The determinant of a 1×1 matrix is always 0. (Section 8.6)
6. How do you find the inverse of a 2×2 matrix? (Section 8.6)
7. Suppose
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$
Calculate the transpose of A.
8. Given the matrices
$$B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix},$$
compute $2B + 3C$.
9. Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $E = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$. Compute the product DE, and explain why the reverse order is impossible.
10. Let $F = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$. Show that F is invertible, and compute its inverse.
11. For what values of x does the following matrix have no inverse?
$$G = \begin{bmatrix} 1 & 2 \\ 3 & x \end{bmatrix}.$$
12. Consider the system of equations:
$$2x + 3y = 5, \quad 4x + 7y = 9.$$
Represent this system using a matrix equation.
13. Suppose A and B are square matrices of the same size. Which of the following are always correct?
    (a) $(A - B)^2 = A^2 - 2AB + B^2$
    (b) $(AB)^2 = A^2B^2$
    (c) $(A + B)^2 = A^2 + 2AB + B^2$
    (d) $(A + B)^2 = A^2 + AB + BA + B^2$
    (e) $A^2B^2 = A(AB)B$
    (f) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
    (g) $(A + B)(A - B) = A^2 - B^2$
14. Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$ Find all possible 2×2 matrices B satisfying $AB = 0$.

Solutions

1. True. For any matrices A and B of the same dimension, $A + B = B + A$ (commutative) and $A + (B + C) = (A + B) + C$ (associative).

2. True. For a scalar c and matrices A and B, we have $c(A + B) = cA + cB$.

3. False. In general, $AB \neq BA$, so the order of multiplication matters.

4. Matrix multiplication is possible when the number of columns of the first matrix equals the number of rows of the second matrix.

5. False. For a matrix $[a]$, the determinant is simply a.

6. For a matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
the inverse is given by
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$
provided $ad - bc \neq 0$ (i.e., the determinant is nonzero).

7. The transpose of A, denoted A^T, is obtained by swapping the rows and columns to get
$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

8. First, compute $2B$ and $3C$:
$$2B = \begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}, \quad 3C = \begin{bmatrix} 12 & 3 \\ -6 & 15 \end{bmatrix}.$$
Now, add $2B$ and $3C$:
$$2B + 3C = \begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 12 & 3 \\ -6 & 15 \end{bmatrix} = \begin{bmatrix} 16 & 1 \\ -6 & 21 \end{bmatrix}.$$

9. The product DE is computed by taking the dot product of the rows of D with the columns of E:
$$DE = \begin{bmatrix} (1)(7) + (2)(9) & (1)(8) + (2)(10) \\ (3)(7) + (4)(9) & (3)(8) + (4)(10) \\ (5)(7) + (6)(9) & (5)(8) + (6)(10) \end{bmatrix} = \begin{bmatrix} 25 & 28 \\ 57 & 64 \\ 89 & 100 \end{bmatrix}.$$
However, ED is not possible because the number of columns of E (2) does not match the number of rows of D (3), so matrix multiplication is undefined in this order.

10. To check if F is invertible, compute the determinant:
$$\det(F) = (2)(3) - (0)(1) = 6.$$
Since the determinant is non-zero, F is invertible. The inverse of F is given by:
$$F^{-1} = \frac{1}{\det(F)} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$

11. A matrix has no inverse if its determinant is zero. Compute the determinant of G:
$$\det(G) = (1)(x) - (2)(3) = x - 6.$$
Set the determinant to zero and solve for x:
$$x - 6 = 0 \quad \Rightarrow \quad x = 6.$$
Therefore, the matrix has no inverse when $x = 6$.

12. The system of equations can be written as the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$

13. (a) False.
The expansion of $(A - B)^2$ using the distributive property gives:

$$(A - B)^2 = (A - B)(A - B) = A^2 - AB - BA + B^2.$$

The term $-AB - BA$ appears instead of $-2AB$ since matrix multiplication is generally not commutative (i.e., $AB \neq BA$ in general). Thus, $(A - B)^2 \neq A^2 - 2AB + B^2$ unless A and B commute.

(b) False.
For square matrices A and B, the expression $(AB)^2$ expands as:

$$(AB)^2 = ABAB,$$

which is not equal to A^2B^2 unless A and B commute (i.e., $AB = BA$). Therefore, $(AB)^2 \neq A^2B^2$ in general.

(c) False.
This is essentially the same expression as in (d), but with an incorrect claim. The correct expansion should include both AB and BA terms, so the provided form is not correct.

(d) True.
Expanding $(A + B)^2$ using the distributive property gives:

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2.$$

Since A and B are matrices and matrix multiplication is not necessarily commutative, this expansion is correct. Therefore, $(A + B)^2 = A^2 + AB + BA + B^2$.

(e) True.
The expression $A^2B^2 = A(AB)B$ is correct, as A^2B^2 represents the multiplication of A and B in the sequence: $A \cdot A \cdot B \cdot B$, and $A(AB)B$ expresses the same result.

(f) False.
For similar reasons as part (c), the binomial expansion is not correct because order matters when multiplying matrices.

(g) False.
The expression $(A + B)(A - B)$ does not generally follow the difference of squares pattern. It should be $A^2 - AB + BA - B^2$.

14. Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, where b_{ij} are the entries of matrix B. We want to find B such that $AB = 0$, i.e.,

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Multiplying out, we get

$$\begin{bmatrix} (-1)b_{11} + (-1)b_{21} & (-1)b_{12} + (-1)b_{22} \\ (3)b_{11} + (3)b_{21} & (3)b_{12} + (3)b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives the following system of equations:

$$\begin{aligned} -b_{11} - b_{21} &= 0, \\ -b_{12} - b_{22} &= 0, \\ 3b_{11} + 3b_{21} &= 0, \\ 3b_{12} + 3b_{22} &= 0. \end{aligned}$$

From the first equation, $b_{11} = -b_{21}$, and from the second equation, $b_{12} = -b_{22}$. The third and fourth equations are the same as the first and second, so no new information is added.

Thus, any matrix B of the form

$$B = \begin{bmatrix} b_{11} & b_{12} \\ -b_{11} & -b_{12} \end{bmatrix}$$

will satisfy $AB = 0$. Therefore, the general solution for B is

$$B = \begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$$

for any scalars x and y.