Antiderivatives

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Here are some key ideas from sections 4.6.

- We say that $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
- If $F(x)$ is one of the antiderivatives of $f(x)$, then all of the antiderivatives of $f(x)$ have the form $F(x) + C$.

  this is called the "family" of antiderivatives
- Let's do an example. One of the antiderivatives of $f(x) = \sin x$ is:

  I claim $F(x) = -\cos x$ works.
  To check this, notice $F'(x) = (-\cos x)' = -(-\sin x) = \sin x = f(x)$ ✓

  The most general antiderivative of $\sin x$ is:

  $F(x) = -\cos x + C$

. . . .

Trig practice: Find $\sec(225°)$ and $\tan(225°)$.

Problem 1: (Stewart 4.6) Find the most general antiderivatives of the following functions. Remember that the derivative of the antiderivative is the original function.

a) $cf(x)$; b) $f(x) + g(x)$; c) x^n, for $n \neq -1$; d) $1/x$;

e) e^x; f) e^{cx}; g) $\cos x$; h) $\sin x$;

i) $\sec^2 x$; j) $\sec x \tan x$ k) $\frac{1}{1+x^2}$.

My Attempt:

Solution:

You can find this table in section 4.6 of the book.

(2) Table of Antidifferentiation Formulas

Function	General antiderivative	Function	General antiderivative
$cf(x)$	$cF(x) + C$	$\cos x$	$\sin x + C$
$f(x) + g(x)$	$F(x) + G(x) + C$	$\sin x$	$-\cos x + C$
$x^n\ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$	$\sec^2 x$	$\tan x + C$
$1/x$	$\ln\lvert x\rvert + C$	$\sec x \tan x$	$\sec x + C$
e^x	$e^x + C$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x + C$
e^{cx}	$\dfrac{1}{c}e^{cx} + C$		

don't forget the abs. value

As an exercise, check that the derivatives of the antiderivatives yield the original functions.

Problem 2: (Stewart 4.6) Find the most general antiderivative of $\frac{3}{t^2}$, assuming $t > 0$.

My Attempt:

Solution:

Write it as $3t^{-2}$

The (particular) antiderivative of t^{-2} is $\frac{t^{-1}}{-1} = -t^{-1}$

Then the (general) antiderivative of $3t^{-2}$ is

$-3t^{-1} + C$

Problem 3: (Stewart 4.6) Find the most general antiderivative of $f(x) = x(2-x)^2$.

My Attempt:

Solution:

Expand to get $f(x) = x(4 - 4x + x^2) = 4x - 4x^2 + x^3$

Find the antiderivative by term:

- Antiderivative of $4x$: $4 \cdot \frac{x^2}{2} = 2x^2$
- Antiderivative of $-4x^2$: $-4 \cdot \frac{x^3}{3} = -\frac{4x^3}{3}$
- Antiderivative of x^3: $\frac{x^4}{4}$

Combine to get $2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} + C$

Problem 4: (Stewart 4.6) Find the most general antiderivative of $f(x) = 2\sqrt{x} + 6\cos x$.

My Attempt:

Solution:

$f(x) = 2x^{\frac{1}{2}} + 6\cos x$

Find the antiderivative by term:

- Antiderivative of $2x^{\frac{1}{2}}$: $2 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{4}{3}x^{\frac{3}{2}}$
- Antiderivative of $6\cos x$: $6\sin x$

Combine to get $\frac{4}{3}x^{\frac{3}{2}} + 6\sin x + C$

Problem 5: (Stewart 4.6) Suppose $\frac{dr}{d\theta} = \cos\theta + \sec\theta\tan\theta$, where $0 < \theta < \pi/2$ and $r(\pi/3) = 4$. Find $r(\theta)$. *Hint: this is an inital value problem—find the general antiderivative first and then use the provided "initial value."*

My Attempt:

Solution:

The antiderivative of $dr/d\theta$ is $r(\theta) + C$

We can find the antiderivative by term:

- Antiderivative of $\cos\theta$: $\sin\theta$
- Antiderivative of $\sec\theta\tan\theta$: $\sec\theta$

Combine to get $r(\theta) = \sin\theta + \sec\theta + C$

To solve for C, use the given point $(\frac{\pi}{3}, 4)$:

$r(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) + \sec(\frac{\pi}{3}) + C = \frac{\sqrt{3}}{2} + 2 + C = 4$

So $C = 4 - 2 - \frac{\sqrt{3}}{2} = 2 - \frac{\sqrt{3}}{2}$

Then $r(\theta) = \sin\theta + \sec\theta + 2 - \frac{\sqrt{3}}{2}$

Problem 6: (Stewart 4.6) For each of the functions $f''(x)$ below, find the most general expression for $f(x)$. *Hint: you will need to "undo" the derivative twice.*

a) $f''(x) = 6x + 12x^2$

b) $f''(x) = 6x + \sin x$.

My Attempt:

Solution: key idea: the antiderivative of f'' is f'; and the antiderivative of f' is f.

a) f' is the antiderivative of f'', so

$$f'(\theta) = 3x^2 + 4x^3 + C$$

f is the antiderivative of f', so

$$\boxed{f(\theta) = x^3 + x^4 + Cx + D.}$$

b) f' is the antiderivative of f'', so

$$f'(\theta) = 3x^2 - \cos x + C$$

f is the antiderivative of f', so

$$\boxed{f(\theta) = x^3 - \sin x + Cx + D}$$

Problem 7: (Stewart 4.6) Suppose a sample of a radiocative substance with initial mass of 75 mg decays t years later at a rate of $1.7325e^{-0.0231t}$ mg/year. Find the mass of the sample after 20 years.

My Attempt:

Solution:

Let $m(t)$ be the mass of the particle at time t.

Then $\frac{dm}{dt} = -1.7325e^{-0.0231t}$

negative because it decays!

$m(t)$ is the antiderivative, which is

$$m(t) = -1.7325 \frac{e^{-0.0231t}}{-0.0231} + C$$

$$= 75e^{-0.0231t} + C$$

Use the initial point to solve for C, so

$$m(0) = 75e^0 + C = 75 \Rightarrow C = 0$$

so $m(t) = 75e^{-0.0231t}$

$$m(20) = \boxed{75e^{-0.0231 \cdot 20}}$$

Challenge problem: (Stewart 4.6) Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$, but $\frac{dV}{dt} = -k \cdot (4\pi r^2)$. Thus $\frac{dr}{dt} = -k$. Then $r(t) = -kt + C$.

Since $V(0) = \frac{1}{2}V(0)$, we know $[r(2)]^3 = \frac{1}{2}[r(0)]^3$. Then $(-2k+C) = \frac{1}{2}C^3 \Rightarrow k = \frac{1}{2}(1-2^{-1/3}C)$. We want t such that $r(t) = 0$, so

$-\frac{1}{2}(1-2^{-1/3})Ct + C = 0 \Rightarrow \frac{1}{2}(1-2^{-1/3})t = 1 \Rightarrow \boxed{t = \frac{4}{2-2^{2/3}}}$