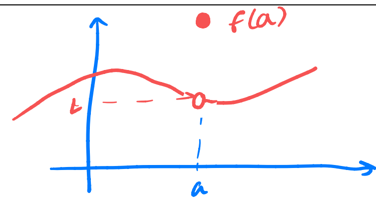
Finite limits, properties of limits

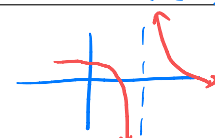
October 3rd, 2024



Here are some key ideas from sections 2.3 and 2.4.

- We write $\lim_{x\to a} f(x) = L$ to mean the function $f(x)$ approaches L as x approaches a.
- If $\lim_{x\to a} f(x) = L$, then $f(x)$ [has to/might not] equal L.
- We write $\lim_{x\to a^+} f(x) = L$ to mean $f(x)$ approaches L from the positive direction (right).
  Likewise, $\lim_{x\to a^-} f(x) = L$ means $f(x)$ approaches L from the negative direction (left).
- The line $x = a$ is a vertical asymptote if

  $$\lim_{x\to a^\pm} f(x) = \pm\infty$$



- If $f(x) \le g(x)$ when x is near a, and if the limits both exist as x approaches a, then $\lim_{x\to a} f(x)$ $\le$ $\lim_{x\to a} g(x)$.
- **Squeeze theorem**: If $f(x) \le g(x) \le h(x)$ when x is near a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.
- Important limit that you should know: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Midterm practice:

(a) For what values of m are the vectors $[-6, m, 2]$ and $[m, m^2, m]$ orthogonal?

(b) Find two unit vectors that make an angle of $60°$ with $\mathbf{v} = [3, 4]$.

My Attempt:

Solution:

(a)
$$-6\cdot m + m\cdot m^2 + 2\cdot m = 0$$
$$-6m + m^3 + 2m = 0$$
$$m^3 - 4m = 0$$
$$m(m^2-4) = 0 \Rightarrow m=0,\ m=2,\ m=-2$$

(b) Want to find $\langle x,y\rangle$ with $\langle x,y\rangle\cdot\langle 3,4\rangle = 0$ and $|\langle x,y\rangle| = 1$.

1. Write the cosine formula:
$$\frac{\langle x,y\rangle\cdot\langle 3,4\rangle}{|\langle x,y\rangle|\cdot|\langle 3,4\rangle|} = \cos\theta$$

2. plug in known values:
$$\frac{\langle x,y\rangle\cdot\langle 3,4\rangle}{|\langle x,y\rangle|\cdot 5} = \cos 60° = \frac{1}{2}$$

3. Simplify angle stuff:
$$\frac{3x+4y}{\sqrt{x^2+y^2}\cdot 5} = \frac{1}{2}$$
unit vector!

4. We know $\sqrt{x^2+y^2} = 1$, so
$$\frac{3x+4y}{1\cdot 5} = \frac{1}{2} \Rightarrow 6x + 8y = 5$$
multiply by 2

Also, $\sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1 \Rightarrow x = \pm\sqrt{1-y^2}$

5. Substitute $\pm\sqrt{1-y^2}$ for x
$$6\left(\pm\sqrt{1-y^2}\right) + 8y = 5 \Rightarrow \pm 6\sqrt{1-y^2} = 5 - 8y$$

6. Solve for y by squaring both sides
$$36(1-y^2) = 25 - 80y + 64y^2$$
$$36 - 36y^2 = 25 - 80y + 64y^2$$
$$100y^2 - 80y - 11 = 0$$
this is a quadratic!

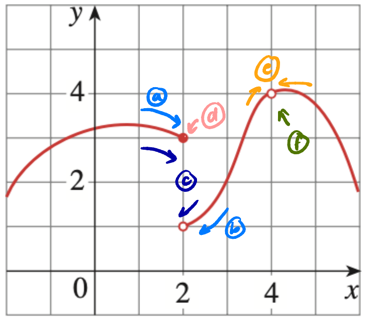
7. Quadratic equation
$$y = \frac{80 \pm \sqrt{6400+4400}}{200} = \frac{80 \pm 10\sqrt{108}}{200} = \frac{4\pm 3\sqrt{3}}{10}$$

8. Solve for x, since $x = \sqrt{1-y^2}$
$$y = \frac{4+3\sqrt{3}}{10} \Rightarrow x = -\sqrt{1-\left(\frac{16+27+24\sqrt{3}}{100}\right)} = -\sqrt{\frac{57-24\sqrt{3}}{100}}$$
$$y = \frac{4-3\sqrt{3}}{10} \Rightarrow x = \sqrt{1-\left(\frac{16+27-24\sqrt{3}}{100}\right)} = \sqrt{\frac{57+24\sqrt{3}}{100}}$$

So we get
$$\left\langle \frac{4+3\sqrt{3}}{10}, -\sqrt{\frac{57-24\sqrt{3}}{100}} \right\rangle$$
$$\left\langle \frac{4-3\sqrt{3}}{10}, \sqrt{\frac{57+24\sqrt{3}}{100}} \right\rangle$$

Problem 1: (Stewart 2.3) For the given graph of the function f, state the value of each quantity if it exists. If it does not, explain why.

a) $\lim_{x\to 2^-} f(x)$ b) $\lim_{x\to 2^+} f(x)$ c) $\lim_{x\to 2} f(x)$ d) $f(2)$ e) $\lim_{x\to 4} f(x)$ f) $f(4)$



My Attempt:

Solution:

a) x approaches 2 from left, f(x) approaches $\boxed{3}$.

b) x approaches 2 from right, f(x) approaches $\boxed{1}$.

c) The one-sided limits differ, so the double-sided limit does not exist

d) $f(2) = \boxed{3}$

e) They approach $\boxed{4}$.

f) f(4) is not defined at x = 4, there's a hole.

Problem 2: (Stewart 2.3) Determine

$$\lim_{x\to -3^+} \frac{x+2}{x+3}.$$

Hint: plug in values to see which infinity it approaches.

My Attempt:

Solution:

since $x > -3$, $x+3 > 0$

$x \to -3^+$ means $x+2$ approaches -1.

$x \to -3^+$ means $x+3$ is a positive number rapidly approaching 0.

Thus $\lim_{x\to -3^+} \frac{x+2}{x+3} = \boxed{-\infty}$ (think $\frac{-1}{0}$)

Problem 3: (Stewart 2.3) Find $\lim_{x\to 3^+} \ln(x^2 - 9)$.

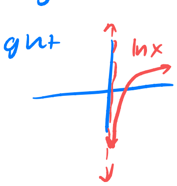
My Attempt:

Solution:

$x \to 3^+$ means $x^2 - 9 \Rightarrow 0$ from the right

Thus we approach ln 0 from the right

so $\lim_{x\to 3^+} \ln(x^2-9) = \boxed{-\infty}$



Problem 4: (Stewart 2.3) Find $\lim_{x\to 2\pi^-} x\csc x$. *Hint:* $\csc(x)$ *is the cosecant function, which is* $1/\sin(x)$.

My Attempt:

Solution:

$\lim_{x\to 2\pi^-} x\csc x = \lim_{x\to 2\pi^-} \frac{x}{\sin x}$

Notice $x \to 2\pi^-$ means $\sin x$ is a negative number rapidly approaching 0.

Thus $\lim_{x\to 2\pi^-} \frac{x}{\sin x} = \boxed{-\infty}$

Problem 5: (Stewart 2.4) Find

$$\lim_{x\to 2^-} \frac{x^2-2x}{x^2-4x+4}.$$

My Attempt:

Solution:

Direct substitution: $\frac{4-4}{4-8+4} = \frac{0}{0}$ (indeterminate)

$\lim_{x\to 2^-} \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x-2)} = \lim_{x\to 2^-} \frac{x}{x-2}$

$x\to 2^-$ means x approaches 2

$x\to 2^-$ means $x-2$ is a negative # rapidly approaching 0. so $-\infty$

Problem 6: (Stewart 2.4) Evaluate

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right).$$

$\frac{1}{0}-\frac{1}{0} = \infty - \infty \Rightarrow$ indeterminate

My Attempt:

Solution:

$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right) = \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}} \cdot \frac{(1+\sqrt{1+t})}{(1+\sqrt{1+t})}$

$= \lim_{t\to 0}\frac{1-(1+t)}{(t\sqrt{1+t})(1+\sqrt{1+t})}$

$= \lim_{t\to 0}\frac{\cancel{t}}{(\cancel{t}\sqrt{1+t})(1+\sqrt{1+t})}$

$= \lim_{t\to 0}\frac{1}{\sqrt{1+t}\,(1+\sqrt{1+t})}$

$= \frac{1}{\sqrt{1}\,(1+\sqrt{1})} = \boxed{\frac{1}{2}}$

Problem 7: (Stewart 2.4) Find

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}.$$

$\frac{0}{0} \Rightarrow$ indeterminate

My Attempt:

Solution:

use $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$

$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} = \lim_{x\to 0}\frac{\cancel{\frac{\sin 4x}{4x}}^{1}\cdot 4x}{\cancel{\frac{\sin 6x}{6x}}^{1}\cdot 6x}$

$= \lim_{x\to 0}\frac{1\cdot 4x}{1\cdot 6x}$

$= \lim_{x\to 0}\frac{4}{6}$

$= \boxed{\frac{2}{3}}$

Challenge problem: (Stewart 2.4) Show that $\lim_{x\to 0} x^4\cos\frac{2}{x} = 0$.

Notice $-1 \le \cos\frac{2}{x} \le 1$. Thus $-x^4 \le x^4\cos\frac{2}{x} \le x^4$. Let $f(x) = -x^4$, $g(x) = x^4\cos\frac{2}{x}$, $h(x) = x^4$.

Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so $\lim_{x\to 0} x^4\cos\frac{2}{x} = 0$ by the Squeeze Theorem.