Solving systems, eigenstuff

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Here are some key ideas from sections 8.6 and 8.7.

- We can write systems of equations in matrix notation as $A\vec{x} = \vec{b}$. A is a matrix, while $\vec{x}$ and $\vec{b}$ are both column vectors. If $\vec{b}$ is not the 0 vector, we say the system is inhomogeneous.
For the system of equations given by

$$3x_1 - 2x_2 = -4$$
$$7x_1 + x_2 = 19,$$

$$A\vec{x} = \vec{b}$$
$$\cancel{A^{-1}A}\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

the corresponding matrix equation is $\begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 19 \end{bmatrix}$ A

- Fill out the following table with the number of solutions to the matrix equation.

	A is invertible	A is not invertible
Homogeneous ($\vec{b} = \vec{0}$)	one! $\vec{x} = \vec{0}$ (trivial)	trivial sol'n, ∞ non-trivial
Inhomogeneous ($\vec{b} \neq \vec{0}$)	one! $\vec{x} = A^{-1}\vec{b}$	Take Math 110 :)

$A\vec{x} = 0$

- An eigenvector is a vector $\vec{v}$ satisfying the equation below. and are nonzero!

$$\cancel{A\vec{x} = \lambda\vec{x}.} \quad A\vec{v} = \lambda\vec{v}$$

The corresponding eigenvalue is λ.

- We can rewrite to get

$$A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \overset{\lambda \cdot I\vec{v}}{\lambda\vec{v}} = \vec{0} \Rightarrow (A - \lambda I)\vec{v} = \vec{0} \rightarrow \text{homogeneous!}$$

which tells us that there are nonzero eigenvectors if and only if $\det(A - \lambda I) = 0$.

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Problem 1: (Stewart & Day 8.6) Solve the discussed system of equations using matrices. *Hint: multiply by A^{-1}.*

$$3x_1 - 2x_2 = -4$$
$$7x_1 + x_2 = 19,$$

$$A\vec{x} = \vec{b} : \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 19 \end{bmatrix}$$

My Attempt:

Solution:

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \left(\begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix}\right)^{-1}\vec{b}$$
$$= \frac{1}{(3)(1)-(-2)(7)}\begin{bmatrix} 1 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ 19 \end{bmatrix}$$
$$= \begin{bmatrix} 1/17 & 2/17 \\ -7/17 & 3/17 \end{bmatrix}\begin{bmatrix} -4 \\ 19 \end{bmatrix}$$
$$= \begin{bmatrix} -4+38/17 \\ 28+57/17 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}$$

Problem 2: (Stewart & Day 8.7) Which of the following scalars k are eigenvalues of their corresponding matrices?

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $k = 3$; b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, $k = 0$; c) $A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$, $k = 2$.

My Attempt:

Solution:

(nonzero) eigenvalue if $\det(A - \lambda I) = 0$.

a) $A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$, $\det(A - 3I) = 0$

so 3 is an eigenvalue.

b) $\det(A - 0I) = \det A = 0$

so 0 is an eigenvalue.

c) $\det(A - 2I) = \det \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} = -3 \neq 0$

so 2 is not an eigenvalue.

Problem 3: (Stewart & Day 8.7) Consider the following system of equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Suppose that the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is nonsingular. Derive expressions for x_1 and x_2.

My Attempt:

Solution:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

so $x_1 = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$

$x_2 = \dfrac{-a_{21}b_1 + a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}}$

Problem 4: (Stewart & Day 8.7) Find the eigenvalues of each matrix.

a) $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$; b) $\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$; c) $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$.

My Attempt:

Solution:

a) If $\det\left(\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} - \lambda I\right) = 0$, then

$\det\begin{bmatrix} 2-\lambda & 0 \\ 3 & -\lambda \end{bmatrix} = 0 \Rightarrow (2-\lambda)(-\lambda) - 0 = 0$

so $\lambda(\lambda-2) = 0 \Longrightarrow$ $\lambda = 0,\ \lambda = -2$

b) If $\det\left(\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix} - \lambda I\right) = 0$, then

$\det\begin{bmatrix} 5-\lambda & -4 \\ 6 & -5-\lambda \end{bmatrix} = 0 \Rightarrow (5-\lambda)(-5-\lambda) = 0$

so $\lambda^2 - 1 = 0 \Rightarrow$ $\lambda = 1,\ \lambda = -1$

c) If $\det\left(\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} - \lambda I\right) = 0$, then

$\det\begin{bmatrix} 3-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) = 0$

so $\lambda = 2,\ \lambda = 3$

Problem 5: (Stewart & Day 8.7) Find an eigenvector associated with the given eigenvalue of A.

a) $A = \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix}$, $\lambda = 9$; b) $A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$, $\lambda = 4 + \sqrt{19}$; c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda = \frac{1+\sqrt{5}}{2}$.

My Attempt:

Solution: must satisfy $A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I)\vec{v} = \vec{0}$.

a) $(A - \lambda I) = 0 \Rightarrow \begin{bmatrix} 9-9 & 0 \\ 2 & 3-9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $2v_1 - 6v_2 = 0$, $v_1 = 3v_2$. Pick $v_1 = 1$ to get $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1-(4+\sqrt{19}) & 5 \\ 2 & 7-(4+\sqrt{19}) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $(-3-\sqrt{19})v_1 + 5v_2 = 0$
$2v_1 + (3-\sqrt{19})v_2 = 0$ $\Rightarrow v_2 = \dfrac{3+\sqrt{19}\ v_1}{5}$ (multiply by conjugate)

Pick $v_1 = 5$ to get $\begin{bmatrix} 5 \\ 3+\sqrt{19} \end{bmatrix}$

c) $\begin{bmatrix} 0-\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & 1-\frac{1+\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(multiply by conjugate)

so $-\left(\frac{1+\sqrt{5}}{2}\right)v_1 + v_2 = 0$
$v_1 + \left(\frac{1-\sqrt{5}}{2}\right)v_2 = 0$ $\Rightarrow v_2 = \left(\frac{1+\sqrt{5}}{2}\right)v_1$

choose $v_2 = 2$ to get $\begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix}$

Challenge Problem: (Stewart & Day 8.7) Derive a general formula for the eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a-d)\lambda + (ad-bc) = 0$

use quadratic formula to get $\lambda = \dfrac{a+d+\sqrt{(a-d)^2+4bc}}{2}$, $\lambda = \dfrac{a+d-\sqrt{(a-d)^2+4bc}}{2}$