Quiz 4 study guide

September 29th, 2024

General information

Quiz 4 covers sections 1.1-1.5 and 2.1. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.
- You are welcome to email me me at dbhatia1089@berkeley.edu if you need help on practice problems.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Limit laws for adding, subtracting, multiplying, dividing, and exponentiating sequences (2.1)
- ☐ Limit laws for r^n (2.1)
- ☐ The definitions of convergent and divergent sequences (2.1)
- ☐ Finding $\lim_{n\to\infty} r^n$ for various values of
- ☐ The definitions of functions, domains, ranges, increasing, decreasing, even/odd functions (1.1)
- ☐ The general form of linear functions, polynomials, power functions, rational functions, algebraic functions, trigonometric functions, exponential functions, and logarithmic functions (1.2)
- ☐ How to find the domains and ranges of the above functions (1.2)
- ☐ The exponent rules (e.g. $x^a x^b = x^{a+b}$), logarithm rules (e.g. $\log(a^b) = b \log a$), and trigonometry rules (1.4, 1.5)
- ☐ The general form of horizontal shifts to the left and to the right, vertical shifts up and down, horizontal stretches/squeezes, vertical stretches/squeezes (1.3)
- ☐ Obtaining the equation of the transformation of a function from the graph of its transformation (1.3)
- ☐ How to compose several functions (1.3)
- ☐ The definition of a one-to-one function, and that one-to-one functions have inverses (1.3)
- ☐ How to find the inverse of a function (1.5)

Help! I'm stuck on....

- ...finding limits of sequences: check out this 30 minute video (lots of examples!)
- ...the definition of a **function**: check out this 14 minute video
- ...the **domains and ranges** of different types of functions: check out this 18 minute video, or if you'd like a ton of problems you can skip around this very long video
- ...solving problems with **increasing and decreasing functions**: check out this 11 minute video
- ...determining if a function is **even or odd**: check out this 12 minute video
- ...using **exponent laws**: check out this 13 minute video
- ...using **logarithm laws**: check out this 5 minute video
- ...**squeeze/stretch** transformations: check out this 8 minute video
- ...**compositions** of functions: check out this 5 minute video

Practice problems

1. For the following sequences, determine whether a_n is convergent or divergent. If convergent, find the limit.

   a) $a_n = \frac{1}{3n^4}$ b) $a_n = \frac{n^3-1}{n}$ c) $a_n = \frac{3+5n}{2+7n}$ d) $a_n = \frac{3^{n+2}}{5^n}$ e) $\frac{e^n+e^{-n}}{e^{2n}-1}$.

2. Let $f(x) = \frac{3}{2/x-1}$. Find the domain of $f(x)$ and write your answer in interval notation.

3. Consider the function $f(x) = \sqrt{4-x^2}$. Find the domain and range of this function.

4. Find the domain and range of $A(x) = \frac{4x+|x|}{x}$.

5. For the functions $f(x) = \frac{2}{x}$ and $g(x) = \sin x$, find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, as well as their domains (here, $f \circ g$ represents $f(g(x))$).

6. Starting with the graph of $y = e^x$, write the equation of the graph that results from

   (a) shifting 2 units downward.
   (b) shifting 2 units to the right.
   (c) reflecting about the x-axis.
   (d) reflecting about the y-axis.
   (e) reflecting about the x-axis and then about the y-axis.

7. Find the exact value of $\ln(\ln e^{e^{50}})$.

8. Find the exact value(s) of $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.

9. Prove that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

10. Find the inverse of $\frac{2x+3}{1-5x}$.

11. Solve $\log_2(x^2 - x - 1) = 2$.

Solutions

1. (a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n^4} = \frac{1}{3}\lim_{n\to\infty}\frac{1}{n^4} = 0. \quad \text{Converges.}$$

(b)

$$a_n = \frac{n^3-1}{n} = n^2 - \frac{1}{n} \quad \text{so} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 - \lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty} n^2$$

When n is large, n^2 is large, so $\lim_{n\to\infty} a_n = \infty$ and the sequence diverges.

(c)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+5n}{2+7n} = \lim_{n\to\infty}\frac{\frac{3}{n}+5}{\frac{2}{n}+7} = \lim_{n\to\infty}\frac{\frac{3}{n}+5}{\frac{2}{n}+7}$$

$$= \frac{\lim_{n\to\infty}\frac{3}{n} + \lim_{n\to\infty} 5}{\lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty} 7} = \frac{0+5}{0+7} = \frac{5}{7} \quad \text{Converges.}$$

(d)

$$a_n = \frac{3^{n+2}}{5^n} = \frac{3^2 3^n}{5^n} = 9\left(\frac{3}{5}\right)^n, \quad \text{so} \quad \lim_{n\to\infty} a_n = 9\lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 9\cdot 0 = 0$$

by (3) with $r = \frac{3}{5}$. Converges.

(e)

$$a_n = \frac{e^n+e^{-n}}{e^{2n}-1}\cdot\frac{e^{-n}}{e^{-n}} = \frac{1+e^{-2n}}{e^n-e^{-n}} \to 0 \text{ as } n\to\infty \text{ because } 1+e^{-2n}\to 1 \text{ and } e^n - e^{-n}\to\infty. \text{Converges.}$$

2. To find the domain of the function $f(x) = \frac{3}{\frac{2}{x}-1}$, we need to determine where the denominator $\frac{2}{x}-1$ is defined:

$$\frac{2}{x} - 1 \neq 0.$$

Solving for x:

$$\frac{2}{x} \neq 1.$$

Taking the reciprocal:

$$\frac{x}{2} \neq 1.$$

Simplifying:

$$x \neq 2.$$

In addition, we know that we cannot divide by 0 in the fraction $\frac{2}{x}$, such that we also have:

$$x \neq 0.$$

So, the domain of $f(x)$ is:

$$(-\infty, 0)\cup(0,2)\cup(2,\infty)$$

3. To find the domain of the function, we need to determine the values of x for which the function is defined. In this case, we have a square root function, and the square root of a negative number is undefined in the real number system. Therefore, we need to ensure that the expression under the square root, $4-x^2$, is non-negative:

$$4 - x^2 \geq 0$$

Solving this inequality:

$$\begin{aligned} 4 - x^2 &\geq 0 \\ x^2 &\leq 4 \end{aligned}$$

So, the domain of the function is $-2 \leq x \leq 2$.

Next, let's find the range of the function. The square root of a non-negative number is always non-negative. Therefore, for any valid value of x in the domain, $f(x) = \sqrt{4 - x^2}$ will also be non-negative. In other words, the range of the function is $[0, \infty)$.

4. The domain of a function consists of all the values of x for which the function is defined. In this case, we need to consider two cases. First, when x is positive or zero, the absolute value $|x|$ is equal to x. Second, when x is negative, the absolute value $|x|$ is equal to $-x$.

So, let's consider both cases:

For $x > 0$, we have $|x| = x$, and the function is defined as:

$$A(x) = \frac{4x + |x|}{x} = \frac{4x + x}{x} = \frac{5x}{x} = 5$$

For $x > 0$, the function is always 5.

For $x < 0$, we have $|x| = -x$, and the function is defined as:

$$A(x) = \frac{4x - x}{x} = \frac{3x}{x} = 3$$

For $x < 0$, the function is always 3.

Then the only value for which the function is not defined is $x = 0$, because we cannot divide by 0 in the denominator. So, the domain of $A(x)$ is $(-\infty, 0) \cup (0, \infty)$.

The range of a function consists of all the values that the function can take. In this case, we have shown that the function is constant within its domain. For $x > 0$, $A(x) = 5$. For $x < 0$, $A(x) = 3$. Therefore, the range of $A(x)$ is $\{3, 5\}$.

5. 1. $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f(\sin x) = \frac{2}{\sin x}$$

The domain of $f \circ g$ is all real numbers except where $\sin x = 0$, which occurs at $x = k\pi$, where k is an integer. So the domain is:

$$\text{Domain of } f \circ g : x \in \mathbb{R}, x \neq k\pi, \text{ where } k \in \mathbb{Z}$$

2. $g \circ f$:

$$(g \circ f)(x) = g(f(x)) = \sin\left(\frac{2}{x}\right)$$

The domain of $g \circ f$ is all real numbers x except where $x = 0$ since $\frac{2}{x}$ is undefined at $x = 0$. So the domain is:

$$\text{Domain of } g \circ f : x \in \mathbb{R}, x \neq 0$$

3. $f \circ f$:

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{2}{x}\right) = \frac{2}{\frac{2}{x}} = x$$

The domain of $f \circ f$ is all real numbers except 0, because we cannot calculate $f(0)$, and therefore cannot calculate $f(f(0))$.

Domain of $f \circ f : x \in \mathbb{R}, x \neq 0$

4. $g \circ g$:

$$(g \circ g)(x) = g(g(x)) = \sin(\sin x)$$

The domain of $g \circ g$ is also all real numbers since there are no restrictions on x:

Domain of $g \circ g : x \in \mathbb{R}$

6. These can be found using the transformation rules.

   (a) Shifting 2 units downward:
   $$y = e^x - 2$$

   (b) Shifting 2 units to the right:
   $$y = e^{(x-2)}$$

   (c) Reflecting about the x-axis:
   $$y = -e^x$$

   (d) Reflecting about the y-axis:
   $$y = e^{-x}$$

   (e) Reflecting about the x-axis and then about the y-axis:
   $$y = -e^{-x}$$

7. To find the exact value, we can simplify step by step. Working with the inner value first, notice that $\ln(e^{e^{50}})$ simplifies to e^{50} because $\ln(e^x) = x$. Now we have $\ln(e^{50})$, which again simplifies to 50. So, the exact value of $\ln(\ln(e^{e^{50}}))$ is 50.

8. To find the exact value, we can use the properties of the sine function.

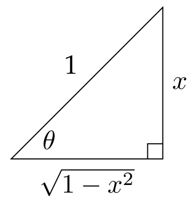
   The sine function $\sin(\theta)$ represents the ratio of the length of the side opposite to angle θ in a right triangle to the length of the hypotenuse. So, we want to find an angle θ such that:
   $$\sin(\theta) = \frac{-1}{\sqrt{2}}$$

   Now, consider the angle $-\frac{\pi}{4}$ radians (or $-45°$). In a unit circle, the coordinates of the point corresponding to this angle are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, which matches the ratio we are looking for.

   Therefore, we have:
   $$\sin\left(-\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

   So, the exact value of $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is $-\frac{\pi}{4}$.

9. Let's consider a right triangle with one angle, θ, such that $\sin(\theta) = x$, where $0 \leq \theta \leq \frac{\pi}{2}$. In this triangle, the side opposite θ has a length of x, and the hypotenuse has a length of 1.



Now, we can use the Pythagorean theorem to find the length of the adjacent side:

$$\sqrt{x^2 + (\text{adjacent side})^2} = 1$$

Solving for the adjacent side:

$$\text{adjacent side} = \sqrt{1-x^2}$$

Now, we can find $\cos(\theta)$ using the definition of cosine in a right triangle:

$$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Therefore, we have shown that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

10. First, replace y with x and x with y:

$$x = \frac{2y+3}{1-5y}$$

Then solve for y. First, cross-multiply:

$$x(1-5y) = 2y+3$$

Distribute x on the left side:

$$x - 5xy = 2y + 3$$

Move all terms involving y to the right side by adding $5xy$ to both sides:

$$x = 2y + 5xy + 3$$

Now, subtract 3 from both sides:

$$x - 3 = 2y + 5xy$$

Move all terms involving y to the left side:

$$2y + 5xy = x - 3$$

Factor out y on the left side:

$$y(2+5x) = x - 3$$

Finally, divide both sides by $(2+5x)$ to solve for y:

$$y = \frac{x-3}{2+5x}$$

Then replace y with $f^{-1}(x)$. So, the inverse function is:

$$f^{-1}(x) = \frac{x-3}{2+5x}.$$

11. Rewrite the equation using the definition of logarithms:

$$2^2 = x^2 - x - 1.$$

Simplify to get

$$x^2 - x - 5 = 0.$$

Solve the quadratic equation. You can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, $a = 1$, $b = -1$, and $c = -5$. Plug these values into the formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)}$$

Simplify further:

$$x = \frac{1 \pm \sqrt{1 + 20}}{2}$$

$$x = \frac{1 \pm \sqrt{21}}{2}$$

So, the solutions are:

$$x_1 = \frac{1 + \sqrt{21}}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{21}}{2}.$$