Quiz 4 study guide

September 29th, 2024

General information

Quiz 4 covers sections 8.8 and 1.1-1.5. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.
- You are welcome to email me me at dbhatia1089@berkeley.edu if you need help on practice problems.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ The big formulas (no proofs) from the iterated matrix models section (8.8)
- ☐ The definitions of functions, domains, ranges, increasing, decreasing, even/odd functions (1.1)
- ☐ The general form of linear functions, polynomials, power functions, rational functions, algebraic functions, trigonometric functions, exponential functions, and logarithmic functions (1.2)
- ☐ How to find the domains and ranges of the above functions (1.2)
- ☐ The exponent rules (e.g. $x^a x^b = x^{a+b}$), logarithm rules (e.g. $\log(a^b) = b \log a$), and trigonometry rules (1.4, 1.5)
- ☐ The general form of horizontal shifts to the left and to the right, vertical shifts up and down, horizontal stretches/squeezes, vertical stretches/squeezes (1.3)
- ☐ How to reflect a graph across the x-axis and the y-axis (1.3)
- ☐ Obtaining the equation of the transformation of a function from the graph of its transformation (1.3)
- ☐ How to compose several functions (1.3)
- ☐ The definition of a one-to-one function, and that one-to-one functions have inverses (1.3)
- ☐ How to find the inverse of a function (1.5)

Help! I'm stuck on....

- ...working through iterated matrix/**recursion** examples: check out this 4 minute video
- ...the definition of a **function**: check out this 14 minute video
- ...the **domains and ranges** of different types of functions: check out this 18 minute video, or if you'd like a ton of problems you can skip around this very long video
- ...solving problems with **increasing and decreasing functions**: check out this 11 minute video
- ...determining if a function is **even or odd**: check out this 12 minute video
- ...using **exponent laws**: check out this 13 minute video
- ...using **logarithm laws**: check out this 5 minute video
- ...shift (**translation**) transformations: check out this 9 minute video
- ...**squeeze/stretch** transformations: check out this 8 minute video
- ...**compositions** of functions: check out this 5 minute video
- ...determining if a function is **one-to-one**: check out this 14 minute video

Practice problems

1. In the following problems, write A as PDP^{-1}.

   a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$; b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$; c) $A = \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}$.

2. Suppose $\vec{n}_{t+1} = A^t \vec{n}_t$. Express $\vec{n}_t$ in terms of $\vec{n}_0$.
3. Continuing from the previous problem, suppose $A = PDP^{-1}$. Express $\vec{n}_t$ in terms of P and D.
4. Suppose D is a diagonal matrix. Find an expression for D^t.
5. Let $f(x) = \frac{3}{2/x-1}$. Find the domain of $f(x)$ and write your answer in interval notation.
6. Consider the function $f(x) = \sqrt{4-x^2}$. Find the domain and range of this function.
7. Find the domain and range of $A(x) = \frac{4x+|x|}{x}$.
8. For the functions $f(x) = \frac{2}{x}$ and $g(x) = \sin x$, find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, as well as their domains (here, $f \circ g$ represents $f(g(x))$).
9. Starting with the graph of $y = e^x$, write the equation of the graph that results from
   (a) shifting 2 units downward.
   (b) shifting 2 units to the right.
   (c) reflecting about the x-axis.
   (d) reflecting about the y-axis.
   (e) reflecting about the x-axis and then about the y-axis.
10. Find the exact value of $\ln(\ln e^{e^{50}})$.
11. Find the exact value(s) of $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.
12. Prove that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.
13. Find the inverse of $\frac{2x+3}{1-5x}$.
14. Solve $\log_2(x^2 - x - 1) = 2$.

Solutions

1. (a) The eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ satisfy the characteristic polynomial

$$(1-\lambda)(3-\lambda) = 0,$$

which has solutions $\lambda_1 = 1$ and $\lambda_2 = 3$.
Solving the system $Av = \lambda v$ for each eigenvalue gives the associated eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
So

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Therefore,

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A$$

(b) The eigenvalues of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ satisfy the characteristic polynomial

$$(1-\lambda)^2 + 1 = 0,$$

which has solutions $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$.
Solving the system $Av = \lambda v$ for each eigenvalue gives the associated eigenvectors $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.
So

$$D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

Therefore,

$$PDP^{-1} = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}\begin{bmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}i \end{bmatrix} = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}i + \frac{1}{2}i\right) + \left(\frac{1}{2} + \frac{1}{2}\right) & \left(\frac{1}{2} - \frac{1}{2}i\right) + \left(\frac{1}{2} - \frac{1}{2}i\right) \\ \left(\frac{1}{2} - \frac{1}{2}i\right) + \left(\frac{1}{2} - \frac{1}{2}i\right) & \left(\frac{1}{2} + \frac{1}{2}i\right) + \left(\frac{1}{2} + \frac{1}{2}i\right) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = A$$

(c) The eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}$ satisfy the characteristic polynomial

$$(1-\lambda)(3-\lambda) + 6 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 9 = 0$$

which has solutions $\lambda_1 = 2 + \sqrt{5}i$ and $\lambda_2 = 2 - \sqrt{5}i$.
Solving the system $Av = \lambda v$ for each eigenvalue gives the associated eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1+\sqrt{5}i \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1-\sqrt{5}i \end{bmatrix}$. So

$$D = \begin{bmatrix} 2+\sqrt{5}i & 0 \\ 0 & 2-\sqrt{5}i \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5}i & 1-\sqrt{5}i \end{bmatrix}$$

$$P^{-1} = \frac{1}{-4\sqrt{5}i}\begin{bmatrix} 1-\sqrt{5}i & -2 \\ -1-\sqrt{5}i & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4\sqrt{5}i}+\frac{1}{4} & -\frac{1}{2\sqrt{5}i} \\ \frac{1}{4\sqrt{5}i}+\frac{1}{4} & \frac{1}{2\sqrt{5}i} \end{bmatrix}$$

Therefore,

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5}i & 1-\sqrt{5}i \end{bmatrix}\begin{bmatrix} 2+\sqrt{5}i & 0 \\ 0 & 2-\sqrt{5}i \end{bmatrix}\begin{bmatrix} \frac{1}{4\sqrt{5}i}+\frac{1}{4} & -\frac{1}{2\sqrt{5}i} \\ \frac{1}{4\sqrt{5}i}+\frac{1}{4} & \frac{1}{2\sqrt{5}i} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5}i & 1-\sqrt{5}i \end{bmatrix}\begin{bmatrix} \frac{1}{4\sqrt{5}i}+\frac{1}{4} & -\frac{1}{2\sqrt{5}i} \\ \frac{1}{4\sqrt{5}i}+\frac{1}{4} & \frac{1}{2\sqrt{5}i} \end{bmatrix} \\ &= \begin{bmatrix} 2\left(\frac{1}{4\sqrt{5}i}+\frac{1}{4}\right)+2\left(\frac{1}{4\sqrt{5}i}+\frac{1}{4}\right) & 2\left(-\frac{1}{2\sqrt{5}i}\right)+2\left(\frac{1}{2\sqrt{5}i}\right) \\ (1+\sqrt{5}i)\left(\frac{1}{4\sqrt{5}i}+\frac{1}{4}\right)+(1-\sqrt{5}i)\left(\frac{1}{4\sqrt{5}i}+\frac{1}{4}\right) & (1+\sqrt{5}i)\left(-\frac{1}{2\sqrt{5}i}\right)+(1-\sqrt{5}i)\left(\frac{1}{2\sqrt{5}i}\right) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix} = A \end{aligned}$$

2. Suppose $\vec{n}_{t+1} = A^t\vec{n}_t$. To express $\vec{n}_t$ in terms of $\vec{n}_0$, observe that

$$\vec{n}_t = A^t\vec{n}_0.$$

This follows from the recursive nature of the equation. At $t = 0$, $\vec{n}_0$ is the initial state, and for any subsequent t, we multiply the initial state by A^t.

3. If $A = PDP^{-1}$, then substituting this into the expression for $\vec{n}_t$, we have

$$\vec{n}_t = A^t\vec{n}_0 = (PDP^{-1})^t\vec{n}_0.$$

Let's expand this for a few powers to see a pattern:

$$(PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1}.$$

Since $P^{-1}P = I$ (the identity matrix), this simplifies to

$$PDDP^{-1} = PD^2P^{-1}.$$

Similarly, we have

$$(PDP^{-1})(PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}.$$

In general, for any positive integer t, we find that:

$$(PDP^{-1})^t = PD^tP^{-1}.$$

Therefore, we get

$$\vec{n}_t = PD^tP^{-1}\vec{n}_0.$$

4. Suppose D is a diagonal matrix. Since powers of diagonal matrices are simply the powers of their diagonal entries, we have

$$D^t = \begin{bmatrix} a_1^t & 0 & \cdots & 0 \\ 0 & a_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^t \end{bmatrix}$$

where $a_1, a_2, \cdots, a_n$ are the diagonal entries of D.

5. To find the domain of the function $f(x) = \frac{3}{\frac{2}{x}-1}$, we need to determine where the denominator $\frac{2}{x} - 1$ is defined:

$$\frac{2}{x} - 1 \neq 0.$$

Solving for x:

$$\frac{2}{x} \neq 1.$$

Taking the reciprocal:

$$\frac{x}{2} \neq 1.$$

Simplifying:

$$x \neq 2.$$

In addition, we know that we cannot divide by 0 in the fraction $\frac{2}{x}$, such that we also have:

$$x \neq 0.$$

So, the domain of $f(x)$ is:

$$(-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

6. To find the domain of the function, we need to determine the values of x for which the function is defined. In this case, we have a square root function, and the square root of a negative number is undefined in the real number system. Therefore, we need to ensure that the expression under the square root, $4-x^2$, is non-negative:

$$4 - x^2 \geq 0$$

Solving this inequality:

$$\begin{aligned} 4 - x^2 &\geq 0 \\ x^2 &\leq 4 \end{aligned}$$

So, the domain of the function is $-2 \leq x \leq 2$.

Next, let's find the range of the function. The square root of a non-negative number is always non-negative. Therefore, for any valid value of x in the domain, $f(x) = \sqrt{4 - x^2}$ will also be non-negative. In other words, the range of the function is $[0, \infty)$.

7. The domain of a function consists of all the values of x for which the function is defined. In this case, we need to consider two cases. First, when x is positive or zero, the absolute value $|x|$ is equal to x. Second, when x is negative, the absolute value $|x|$ is equal to $-x$.

So, let's consider both cases:

For $x > 0$, we have $|x| = x$, and the function is defined as:

$$A(x) = \frac{4x + |x|}{x} = \frac{4x + x}{x} = \frac{5x}{x} = 5$$

For $x > 0$, the function is always 5.

For $x < 0$, we have $|x| = -x$, and the function is defined as:

$$A(x) = \frac{4x - x}{x} = \frac{3x}{x} = 3$$

For $x < 0$, the function is always 3.

Then the only value for which the function is not defined is $x = 0$, because we cannot divide by 0 in the denominator. So, the domain of $A(x)$ is $(-\infty, 0) \cup (0, \infty)$.

The range of a function consists of all the values that the function can take. In this case, we have shown that the function is constant within its domain. For $x > 0$, $A(x) = 5$. For $x < 0$, $A(x) = 3$. Therefore, the range of $A(x)$ is $\{3, 5\}$.

8. 1. $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f(\sin x) = \frac{2}{\sin x}$$

The domain of $f \circ g$ is all real numbers except where $\sin x = 0$, which occurs at $x = k\pi$, where k is an integer. So the domain is:

$$\text{Domain of } f \circ g : x \in \mathbb{R}, x \neq k\pi, \text{ where } k \in \mathbb{Z}$$

2. $g \circ f$:

$$(g \circ f)(x) = g(f(x)) = \sin\left(\frac{2}{x}\right)$$

The domain of $g \circ f$ is all real numbers x except where $x = 0$ since $\frac{2}{x}$ is undefined at $x = 0$. So the domain is:

$$\text{Domain of } g \circ f : x \in \mathbb{R}, x \neq 0$$

3. $f \circ f$:

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{2}{x}\right) = \frac{2}{\frac{2}{x}} = x$$

The domain of $f \circ f$ is all real numbers except 0, because we cannot calculate $f(0)$, and therefore cannot calculate $f(f(0))$.

$$\text{Domain of } f \circ f : x \in \mathbb{R}, x \neq 0$$

4. $g \circ g$:

$$(g \circ g)(x) = g(g(x)) = \sin(\sin x)$$

The domain of $g \circ g$ is also all real numbers since there are no restrictions on x:

$$\text{Domain of } g \circ g : x \in \mathbb{R}$$

9. These can be found using the transformation rules.

(a) Shifting 2 units downward:

$$y = e^x - 2$$

(b) Shifting 2 units to the right:

$$y = e^{(x-2)}$$

(c) Reflecting about the x-axis:

$$y = -e^x$$

(d) Reflecting about the y-axis:

$$y = e^{-x}$$

(e) Reflecting about the x-axis and then about the y-axis:

$$y = -e^{-x}$$

10. To find the exact value, we can simplify step by step. Working with the inner value first, notice that $\ln(e^{e^{50}})$ simplifies to e^{50} because $\ln(e^x) = x$. Now we have $\ln(e^{50})$, which again simplifies to 50. So, the exact value of $\ln(\ln(e^{e^{50}}))$ is 50.

11. To find the exact value, we can use the properties of the sine function.

    The sine function $\sin(\theta)$ represents the ratio of the length of the side opposite to angle θ in a right triangle to the length of the hypotenuse. So, we want to find an angle θ such that:

    $$\sin(\theta) = \frac{-1}{\sqrt{2}}$$

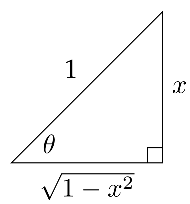
    Now, consider the angle $-\frac{\pi}{4}$ radians (or $-45°$). In a unit circle, the coordinates of the point corresponding to this angle are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, which matches the ratio we are looking for.

    Therefore, we have:

    $$\sin\left(-\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

    So, the exact value of $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is $-\frac{\pi}{4}$.

12. Let's consider a right triangle with one angle, θ, such that $\sin(\theta) = x$, where $0 \leq \theta \leq \frac{\pi}{2}$. In this triangle, the side opposite θ has a length of x, and the hypotenuse has a length of 1.

    Now, we can use the Pythagorean theorem to find the length of the adjacent side:

    $$\sqrt{x^2 + (\text{adjacent side})^2} = 1$$

    Solving for the adjacent side:

    $$\text{adjacent side} = \sqrt{1 - x^2}$$

    Now, we can find $\cos(\theta)$ using the definition of cosine in a right triangle:

    $$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

    Therefore, we have shown that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

13. First, replace y with x and x with y:

    $$x = \frac{2y+3}{1-5y}$$

    Then solve for y. First, cross-multiply:

    $$x(1-5y) = 2y + 3$$

    Distribute x on the left side:

    $$x - 5xy = 2y + 3$$

    Move all terms involving y to the right side by adding $5xy$ to both sides:

$$x = 2y + 5xy + 3$$

Now, subtract 3 from both sides:

$$x - 3 = 2y + 5xy$$

Move all terms involving y to the left side:

$$2y + 5xy = x - 3$$

Factor out y on the left side:

$$y(2 + 5x) = x - 3$$

Finally, divide both sides by $(2 + 5x)$ to solve for y:

$$y = \frac{x-3}{2+5x}$$

Then replace y with $f^{-1}(x)$. So, the inverse function is:

$$f^{-1}(x) = \frac{x-3}{2+5x}.$$

14. Rewrite the equation using the definition of logarithms:

$$2^2 = x^2 - x - 1.$$

Simplify to get

$$x^2 - x - 5 = 0.$$

Solve the quadratic equation. You can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, $a = 1$, $b = -1$, and $c = -5$. Plug these values into the formula:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)}$$

Simplify further:

$$x = \frac{1 \pm \sqrt{1+20}}{2}$$

$$x = \frac{1 \pm \sqrt{21}}{2}$$

So, the solutions are:

$$x_1 = \frac{1+\sqrt{21}}{2} \quad \text{and} \quad x_2 = \frac{1-\sqrt{21}}{2}.$$