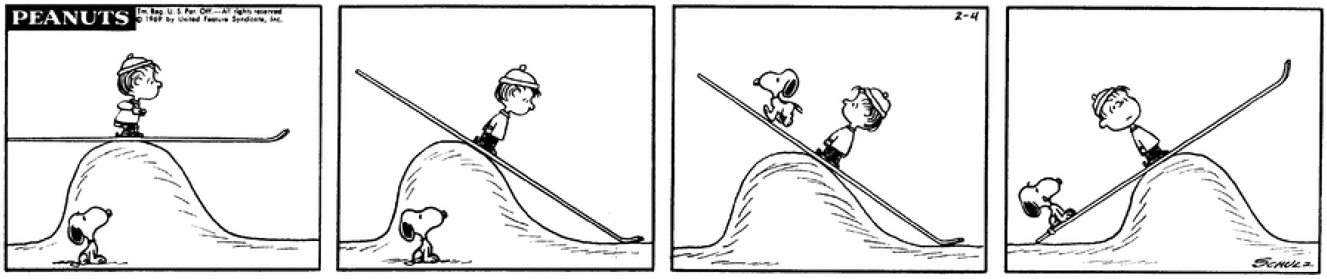

Math 1A: Calculus

Discussion Workbook Solutions

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Contents

Introduction

Welcome to Math 1A!

What are Discussion Sections?

According to Berkeley's GSI Teaching and Resource Center:

> *Discussion sections provide opportunities for collaboration and active learning that do not always take place in a traditional lecture context. The role of section goes beyond clearing up any confusion remaining after the course material has been presented in lecture. Section also provides students with the opportunity to discuss, ask questions, and apply course content, resulting in deeper learning.*

During remote summer sessions, section looks a little bit different than during regular, in-person sessions—rather than working on problems in classrooms, you have been provided with the worksheets in this book to be completed remotely and in tandem with lecture. But all discussion sections have one thing in common: solving problems to further understanding. We that you can take advantage of collaborative platforms, such as Ed discussion, to work through these problems with your peers, increase your confidence with course content, and prepare yourself for exams.

How to Use This Workbook

This workbook is made up of worksheets, one for each section. The worksheets have three major parts:

1. Lecture recap: The material above the dotted line on the first page of each worksheet will ask you to recall major theorems and results from lecture.

2. Mastery problems: Each worksheet has between three and six problems designed to test your understanding of the section.

3. Challenge problems: These problems are usually more difficult than what you can expect on an exam, but can be solved using similar techniques as the mastery problems.

Each worksheet is accompanied with video explanations and written solutions for the mastery problems. It is most effective to attempt the problems on your own or with peers first, and then check your work after.

Strategies for Challenge Problems

Challenge problems are supplements to mastery problems—they may be more difficult than homework and exams. Many challenge problems will ask you to prove statements. Here, we provide some information on a few types of mathematical proofs.

A *direct* proof is perhaps the most straightforward kind of proof: we begin with the information provided, and use facts and inference to arrive at the statement to be proven.

For example, suppose we want to show that the sum of any two odd integers is even. We start with what we know: the two odd integers can be written as $2k+1$ and $2l+1$ where k and l are integers. Then the sum of the integers is $2k+1+2l+1 = 2(k+l)+2 = 2(k+l+1)$, which must be even. Thus the sum of any two odd numbers is even.

A proof by *contradiction* is a way to prove a statement is true by showing that the statement cannot be false. For example, suppose someone claims that they cannot eat an entire pizza in one sitting without getting full. To prove this by contradiction, we suppose on the contrary that they can eat an entire pizza in one sitting without getting full. Then at some point, while eating a pizza, they will get full, which contradicts our hypothesis.

Here's a mathematical example—let's say we want to prove that if x^2 is odd, then x is also odd. To do this, we suppose on the contrary that if x^2 is odd then x can be even. Then we may write $x = 2k$ for some integer k, such that $x^2 = (2k)^2 = 2(2k^2)$, which is necessarily even. But this contradicts our premise: we assumed that x^2 is odd. Thus we can conclude that x must be odd (otherwise, we arrive at a contradiction).

A proof by *mathematical induction* is a way to show that something is true for every natural number. They often have the same general format:

1. Base case: Show that the statement is true for the first or first few cases.
2. Induction hypothesis: Suppose that the statement is true for some arbitrary-numbered case—for example, the kth case.
3. Induction step: Show that the statement is true for the $(k+1)$st case.

To see how this is used in practice, let's do a worked example. We'll prove a familiar identity $1+2+3+...+n = \frac{n(n+1)}{2}$.

1. Base case: We'll show that the statement is true for the first case: for $n = 1$. Notice that $1 = \frac{1(1+1)}{2}$, so we are done.
2. Induction hypothesis: We suppose that the statement is true for the kth case; in other words, we suppose that $1+2+3+...+k = \frac{k(k+1)}{2}$.
3. Induction step: We show that $1+2+3+...+k+(k+1) = \frac{(k+1)(k+2)}{2}$, assuming the induction hypothesis. Notice that
$$1+2+3+...+k+(k+1) = \frac{k(k+1)}{2} + (k+1).$$
Then we can rewrite the right hand side:
$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$
as desired.

Then we can conclude, by the principle of mathematical induction, that $1+2+3+...+n = \frac{n(n+1)}{2}$.

Additional Resources

Here is a short list of resources available to Berkeley students.

- If you are in need of laptops, Wi-Fi hotspots, or other required technologies, you can visit the Student Technology Equity Program (STEP) at https://studenttech.berkeley.edu/step.
- If your name differs from your legal name, you may indicate a preferred name by following the directions provided by the registrar, which can be found at https://registrar.berkeley.edu/academic-records/your-name-on-records-rosters/.
- For free and confidential mental health services, Counseling and Psychological Services (CAPS) is here for you. University Health Services also lists some resources.
- For books, magazines, movies, printing services, museum passes, state park passes, and more, you can apply for a Berkeley Public Library card at https://catalog.berkeleypubliclibrary.org/obr/.

Feedback and Errata

To report any errors in discussion resources, visit the form at https://tinyurl.com/math1a-errata. This form is *not* anonymous in case we need to have some dialogue before making changes.

To provide anonymous feedback, visit the form at https://tinyurl.com/math1a-feedback. This form will verify you are a Berkeley student by checking your email address, but will not collect any information about you.

Four Ways to Represent a Function

Chapter 1, Section 1

Here are some important ideas from lecture:

- **Circle one**: A function is a rule that assigns to each element x from its domain [more than one / exactly one / less than one] element in its range.
- We can represent functions verbally (words), numerically (table of values), visually (graph), or algebraically (explicit formula).
- The Vertical Line Test is a way to tell whether or not a graph in the xy-plane is a function.

Vertical Line Test

An xy-curve is the graph of a function *if and only if* no vertical line intersects the curve more than once.

- **Fill in the blanks**: A function f is *even* if $f(-x) = f(x)$. A function f is *odd* if $f(-x) = -f(x)$. These rules must hold for all x.

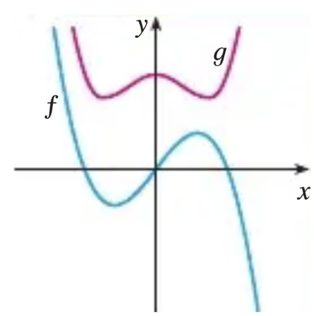
Mnemonic for even and odd functions

Symmetry has to do with the behavior of the function for input values of $-x$ as opposed x. One way to remember what even and odd functions do is **E**ven **E**ats the negative while **O**dd spits it **O**ut.

- A function f is called *increasing* on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
- A function f is called *decreasing* on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

. . . .

Problem 1: (Stewart Section 1.1) Consider the following graph, which depicts the functions f and g. If f even, odd, or neither? Why? Is g even, odd, or neither? Why?



Recall that the symmetry of a function asks what happens when we calculate the function at $-x$ rather than just x.

Testing some values of x, we can see that $f(-x) = -f(x)$ for all x. Thus f is odd. However, $f(-x) \neq f(x)$ for all x, and we can see this by picking $x > 0$. Thus f is not even.

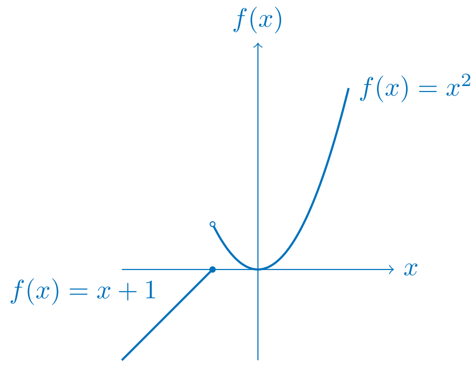
On the other hand, we can see that $g(-x) = g(x)$ for all x, so g is even. However, $g(-x) \neq -g(x)$ for all x, and we can see this by picking $x > 0$. Thus g is not odd.

Problem 2: (Borcherds '05 Midterm 1) Find the domain of the function $g(u) = \sqrt{u} + \sqrt{2-u}$.

Notice that $\sqrt{u}$ is defined for $u \geq 0$, since we cannot take the square root of a negative number. The other term is a little trickier: $\sqrt{2-u}$ is defined for $2 - u \geq 0$, which means $u \leq 2$. In order for the function g to be defined, both of its terms must be defined. Thus we need $u \geq 0$ and $u \leq 2$, which means the domain is in fact $0 \leq u \leq 2$.

Problem 3: (Stewart Section 1.1) Recall that a *piecewise function* splits its domain into pieces and is defined by different formulas for each piece. Sketch the graph of the following piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$



Problem 4: (Stewart Section 1.1) Determine whether $f(x) = x|x|$ is even, odd, or neither.

Notice that

$$f(-x) = (-x)|-x| = (-x)|x| = -x|x| = -f(x)$$

Since $f(-x) = -f(x)$, the function is odd.

Problem 5: (Stewart Section 1.1) Does $x^2 + (y-3)^2 = 5$ define a function? Explain why or why not.

The given equation represents a circle centered at $(0, 3)$ with radius $\sqrt{5}$. The equation of a circle centered at (h, k) with radius r is given by $(x-h)^2 + (y-k)^2 = r^2$. In this case, the center of the circle is $(0, 3)$ and the radius is $\sqrt{5}$. Since for each value of x there are two possible values of y (except at the top and bottom points of the circle), the equation does not define a function by the Vertical Line Test.

Problem 6: (Stewart Section 1.1)
An open rectangular box with volume 2m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

Let x be the length of a side of the square base of the open rectangular box, and let the height of the box be h. Thus, we have the equation $x^2 \cdot h = 2$, and solving for h, we get:

$$h = \frac{2}{x^2}$$

The surface area A is the sum of the areas of the five faces of the box (since it is open):

$$A = x^2 + 4(x \times h) = x^2 + 4(x \times \frac{2}{x^2}) = x^2 + \frac{8}{x}$$

So, the surface area of the box is expressed as a function of the length of a side of the base x as $A(x) = x^2 + \frac{8}{x}$.
Challenge problem: Consider the function $f(x) = 4 + 3x - x^2$. Evaluate the difference quotient given by

$$\frac{f(3+h) - f(3)}{h}.$$

Mathematical Models: a Catalog of Essential Functions

Chapter 1, Section 2

We use *mathematical models* to describe and predict real world phenomena. Below are some of the mathematical models we covered. **Refer to the table at the end of the section for graphs of these functions.**

Type of function	Function definition
Linear	Expression: $y = f(x) = mx + b$. We say m is the slope and b is the y-intercept.
Polynomial	General expression: $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$. We say n is the degree. Quadratic: $P(x) = a_2x^2 + a_1x + a_0$ (degree 2) Cubic: $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ (degree 3)
Power	General expression: $f(x) = x^a$, where a is a constant $n > 0$: If n is a positive interger, then a power function is a polynomial with one term. Root function: $a = 1/n$, where n is a positive integer Reciprocal function: $a = -1$ Inverse square law: $a = -2$. For example, $I = C/x^2$.
Rational	Expression: $f(x) = P(x)/Q(x)$, where P and Q are two polynomials.
Algebraic	Definition: Can be constructed using algebraic operations (i.e. $+, -, \times, \div, \sqrt{\ }$)
Transcendental	Definition: Not algebraic
Trigonometric	Three main examples: $\sin, \cos, \tan$
Exponential	Expression: $f(x) = b^x$, where b is positive
Logarithmic	Expression: $f(x) = \log_b x$, where b is positive

Problem 1: (Stewart Section 1.2) A landlord knows that charging x dollars for a rental space at the market will cause y spaces to be rented. The landlord creates a mathematical model for y which is the equation $y = 200 - 4x$.

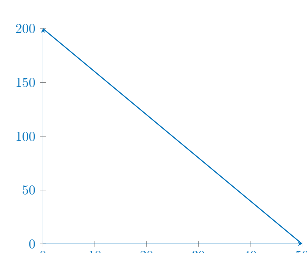
a. What type of function is y?

b. Sketch a graph of the function. *Hint:* what values of x make sense?

c. Practically, what do the slope, y-intercept, and x-intercept represent?

a. y is a linear function.

b.



c. The slope represents the rate of change of the number of rental spaces rented with respect to the rental price. The y-intercept represents the number of rental spaces rented if zero dollars are charged, which may be the number of rental spaces available. The x-intercept is the amount of money to be charged for zero rental spaces to be rented, which may be an upper bound on the cost of a rental space.

Problem 2: (Stewart Section 1.2) Find the domain of the function $f(x) = \frac{\cos x}{1-\sin x}$.

The function is defined for all real numbers except where the denominator is zero, because division by zero is undefined. So, we need to find where $1 - \sin x = 0$. Solving $1 - \sin x = 0$ gives $\sin x = 1$. The solutions to $\sin x = 1$ occur when $x = \frac{\pi}{2} + 2\pi n$, where n is an integer. So, the domain of the function $f(x) = \frac{\cos x}{1-\sin x}$ is all real numbers except $x = \frac{\pi}{2} + 2\pi n$, where n is an integer.

Problem 3: (Stewart Section 1.2) Review the inverse square law from the table on the previous page. Suppose we create a light source and model its illumination from the source using an inverse square law (i.e., our independent variable is distance and our dependent variable is illumination). You are at a distance x from the light source. Then, you move to a distance $x/2$ from the light source. How much brighter is the light?

The inverse square law states that the intensity of illumination from a light source is inversely proportional to the square of the distance from the source. Let I_1 be the initial intensity of illumination when you are at a distance x from the light source. Let I_2 be the intensity of illumination when you are at a distance $\frac{x}{2}$ from the light source. According to the inverse square law, we have:

$$I_1 = \frac{C}{x^2}$$

$$I_2 = \frac{C}{\left(\frac{x}{2}\right)^2} = \frac{4C}{x^2}$$

So, the light is 4 times brighter.

Problem 4: (Stewart Section 1.2) Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

Notice that since $f(-1) = f(0) = f(2) = 0$, we know -1, 0, and 2 are roots of our cubic. Then our function f can be written as $f(x) = k(x+1)(x)(x-2)$. To solve for k, we know $6 = f(1) = k(1+1)(1)(1-2) = k(-2)$, so $k = -3$.

Then our cubic is $f(x) = -3(x+1)(x)(x-2)$.

Problem 5: (Stewart Section 1.3) At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent. Express the water pressure as a function of the depth below the ocean surface.

Let P be the water pressure (in lb/in^2) and d be the depth below the ocean surface (in feet). The relationship between water pressure and depth can be expressed as:

$$P(d) = 15 + \frac{4.34}{10 \times 12} \times d,$$

since we must adjust the 10 ft into 120 inches. Therefore, the water pressure as a function of the depth below the ocean surface is given by:

$$P(d) = 15 + \frac{4.34}{120} d$$

Challenge problem: (Bamler Spring '18 Final Exam) Find the range of $f(x) = \ln(x^2 + 3)$

New Functions From Old Functions

Chapter 1, Section 3

We can get new functions by modifying familiar forms. Here are some examples:

Transformation	Equation of transformation
c units upward	$y = f(x) + c$
c units downward	$y = f(x) - c$
c units to the right	$y = f(x - c)$
c units to the left	$y = f(x + c)$
Vertical stretch by a factor of c	$y = cf(x)$
Vertical shrink by a factor of c	$y = (1/c)f(x)$
Horizontal shrink by a factor of c	$y = f(cx)$
Horizontal stretch by a factor of c	$y = f(x/c)$
Reflection across the x-axis	$y = -f(x)$
Reflection across the y-axis	$y = f(-x)$

Finally, we can combine functions to get other functions. When we modify functions, whether by transformation or by combination, it is important to **consider how the domain and range changes accordingly**.

Function combination	Equation of function combination
Composition of functions	$y = (f \circ g)(x) = f(g(x))$
Sum of functions	$y = f(x) + g(x)$
Difference of functions	$y = f(x) - g(x)$
Quotient of functions	$y = f(x)/g(x)$

. . . .

Problem 1: (Stewart Section 1.3) Relative to $f(x) = \sin(x)$, how have the following graphs been changed?

- $a(x) = \sin(\frac{x}{6})$
- $b(x) = \frac{\sin(x)}{6}$
- $c(x) = \sin(x + 6)$.

- The function $a(x)$ horizontally stretches $f(x)$ by a factor of 6.
- The function $b(x)$ vertically shrinks $f(x)$ by a factor of 6.
- The function $c(x)$ shifts $f(x)$ 6 units to the left.

Problem 2: (Stewart Section 1.3) Under ideal conditions, a certain bacteria population is known to double every 2 hours. Suppose there are initially 700 bacteria. What is the size of the population after t hours?

The population size after t hours is given by the exponential growth model: $P(t) = P_0 \times 2^{\frac{t}{2}}$, where P_0 is the initial population size, which is 700 in this case. Substituting the values, we get:

$$P(t) = 700 \times 2^{\frac{t}{2}}.$$

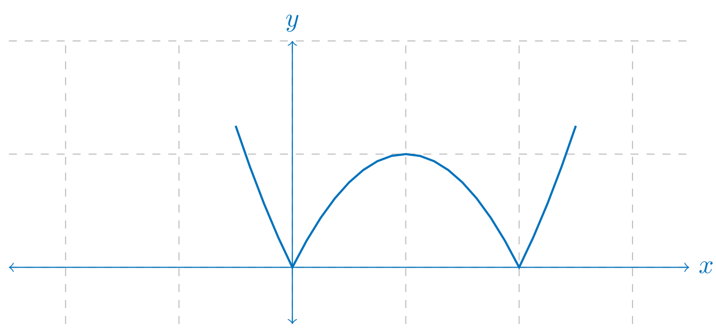
Problem 3: (Stewart Section 1.4) Find the domain of $f(x) = \frac{1+x}{e^{\cos x}}$.

Since both $1 + x$ and $e^{\cos x}$ are defined for all x, notice that this function is only undefined if the denominator is zero. But since $e^{\cos x}$ is never equal to zero, the domain is all real numbers.

Problem 4: (Stewart Section 1.3) Express $R(x) = \sqrt{\sqrt{x} - 1}$ in the form $f \circ g \circ h$.

Observe that $f(x) = \sqrt{x}$, $g(x) = x - 1$, and $h(x) = \sqrt{x}$ works. We can see this is true becuause $f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 1) = \sqrt{\sqrt{x} - 1}$.

Problem 5: (Borcherds '05 Midterm 1) Sketch the graph of $y = |x^2 - 2x|$.



Problem 6: (Stewart Section 1.4) Describe the symmetry of $f(x) = \frac{1-e^{1/x}}{1+e^{1/x}}$. Is it even, odd, or neither?

To determine if $f(x) = \frac{1-e^{1/x}}{1+e^{1/x}}$ is even or odd, evaluate $f(-x)$:

$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{-1/x}}{1 + e^{-1/x}}.$$

This clearly does not equal $f(x)$, so the function is not even. Now observe that

$$-f(x) = -\frac{1 - e^{1/x}}{1 + e^{1/x}} = \frac{-1 + e^{1/x}}{1 + e^{1/x}}.$$

Once again, we see that $f(-x) \neq -f(x)$, so the function is not odd either.

Challenge problem: (Bamler Fall '18 Final Exam) Find the formula for the function $g(x)$ whose graph arises by reflecting $f(x) = \ln x$ about the line $y = 3$.

Exponential Functions

Chapter 1, Section 4

Exponential functions take the form $f(x) = b^x$. These are the important laws of exponents:

1. $b^{x+y} = b^x \cdot b^y$.
2. $b^{x-y} = b^x / b^y$.
3. $(b^x)^y = b^{xy}$.
4. $(ab)^x = a^x b^x$.

The constant e satisfies that the condition that slope of the tangent line to $y = e^x$ at $x = 0$ is 1. We call the function e^x the **natural exponential function**.

Problem 1: (Stewart Section 1.4) Simplify $27^{2/3}$.

We have $27^{2/3} = \left(27^{1/3}\right)^2 = 3^2 = 9$.

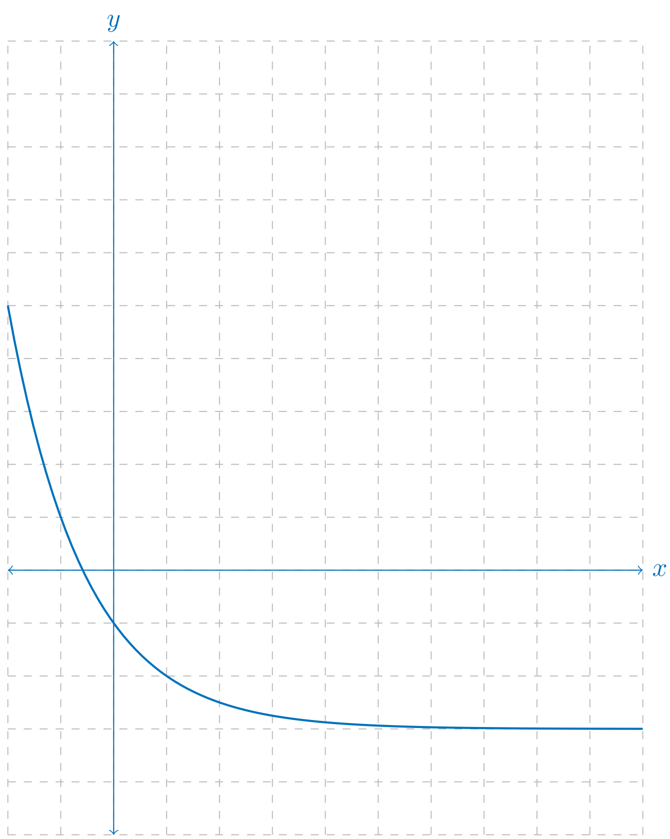
Problem 2: (Stewart Section 1.4) Rewrite and simplify the expression below.

$$\frac{x^3 \cdot x^n}{x^{n+1}}$$

We get

$$\frac{x^3 \cdot x^n}{x^{n+1}} = x^3 \cdot \frac{x^n}{x^{n+1}} = x^3 \cdot x^{n-(n-1)} = x^3 x^{-1} = x^2$$

Problem 3: (Stewart Section 1.4) Use transformation laws to sketch the graph of $h(x) = 2\left(\frac{1}{2}\right)^x - 3$.



Problem 4: (Stewart Section 1.4) Find the domain of the function below.

$$f(x) = \frac{1 - e^{x^2}}{1 - e^{1-x^2}}$$

Notice that $1 - e^{x^2}$ is defined for all x, and so is $1 - e^{1-x^2}$. Thus this function is only undefined for $1 - e^{1-x^2} = 0$, so $e^{1-x^2} = 1$. Then $1 - x^2 = 0$, so $x^2 = 1$, and $x = \pm 1$.

Then the domain is the set of all x such that x is not 1 or -1.

Problem 5: (Stewart Section 1.4) Find the domain of $g(t) = \sin(e^t - 1)$.

Since $e^t - 1$ is defined for all real numbers, and since $\sin(t)$ is defined for all real numbers, their composition is also defined for all real numbers.

Problem 6: (Stewart Section 1.4) If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right).$$

We get

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{5^{x+h} - 5^x}{h} \\ &= \frac{5^x \cdot 5^h - 5^x}{h} \\ &= 5^x \cdot \frac{5^h - 1}{h}. \end{aligned}$$

So, we have shown that

$$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right),$$

as desired.

Challenge problem: (Stewart Chapter 1) Prove that if n is a positive integer, $7^n - 1$ is divisible by 6.

Inverse Functions and Logarithms

Chapter 1, Section 5

- A function f is called *one-to-one* if it never takes the same value twice: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

- Suppose f is a one-to-one function with domain A and range B. Then the *inverse* of f, denoted f^{-1}, has domain B and range A. If $f(x) = y$, then $f^{-1}(y) = x$.
- For an inverse function, the cancellation rules are the following: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.
- The following are the steps to find the inverse of a one-to-one function f.
  1. Write $y = f(x)$.
  2. Solve this equation for x in terms of y.
  3. Switch x and y. Then $y = f^{-1}(x)$.
- In order to make trig functions one-to-one, we have to apply domain restrictions. The restricted domain of $\sin(x)$ is $[\frac{-\pi}{2}, \frac{\pi}{2}]$. The restricted domain of $\cos(x)$ is $[0, \pi]$. The restricted domain of $\tan(x)$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Use the space below to think about what the domain and range of the inverse trig functions are.

  Recall that the domain of the inverse is the range of the original function. Thus the domain of $\sin^{-1}(x)$ is $[-1, 1]$, as is the domain of $\cos^{-1}(x)$. Finally, the domain of $\tan^{-1}$ is $(-\infty, \infty)$.

- The inverse of the exponential function b^x is the logarithmic function $\log_b x$. The following are the laws of logarithms: $\log_b(xy) = \log_b x + \log_b y$, $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, $\log_b(x^r) = r \log_b x$.
- If the base of the logarithm is e (i.e., if $\log_e x$), we can write $\ln x$ instead.
- The change of of base formula says $\log_b x = \ln x / \ln b$.

Problem 1: (Stewart Section 1.5) Find the inverse of $y = 3\ln(x - 2)$.

First, solve for x. Notice that $y = 3\ln(x-2)$ means $e^{\frac{y}{3}} = x - 2$, so $e^{\frac{y}{3}} = x - 2$, and then $x = e^{\frac{y}{3}} + 2$. Switch x and y. We get $y = f^{-1}(x) = e^{\frac{x}{3}} + 2$.

Problem 2: (Bamler Fall '18 Final Exam) If a function f has an inverse function f^{-1}, how many roots can it have?

Suppose x is a root of f. Then $f(x) = 0$, and therefore $x = f^{-1}(0)$. Since there can only be one value that f^{-1} maps 0 to, we know that there is only one possible value of x.

Problem 3: (Bamler Spring '18 Final Exam) Compute and simplify

$$\log_3\left(\frac{\sqrt[5]{27}}{16}\right) + \frac{4}{\log_2 3}.$$

Let's simplify each term separately:

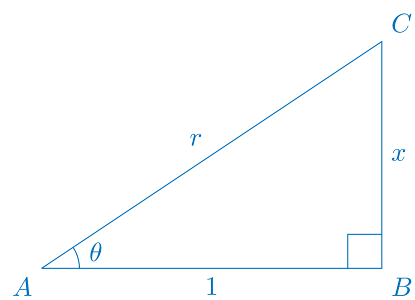
$$\log_3\left(\frac{\sqrt[5]{27}}{16}\right) = \log_3\left(\frac{3^{3/5}}{16}\right) = \log_3\left(3^{3/5}\right) - \log_3 16 = \frac{3}{5} - \frac{\log_2 2^4}{\log_2 3} = \frac{3}{5} - \frac{4}{\log_2 3}.$$

Now, we have

$$\log_3\left(\frac{\sqrt[5]{27}}{16}\right) + \frac{4}{\log_2 3} = \frac{3}{5} - \frac{4}{\log_2 3} + \frac{4}{\log_2 3} = \frac{3}{5}.$$

Problem 4: (Bamler Fall '18 Final Exam) Simplify $\sin(\tan^{-1}(x))$.

To simplify $\sin(\tan^{-1}(x))$, we can visualize it using a right triangle. Let $\theta = \tan^{-1}(x)$. Then, $\tan(\theta) = x$, so we can take x as the length of the side opposite θ and 1 as the length of the side adjacent to θ. According to the Pythagorean theorem, we have $x^2 + 1^2 = r^2$, where r is the length of the hypotenuse. From this triangle, we have $\sin(\theta) = \frac{x}{r}$. Using $x^2 + 1^2 = r^2$, we can solve for r to find $r = \sqrt{1+x^2}$. Thus, $\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$.



Problem 5: (Bamler Fall '18 Final Exam) Find the inverse function f^{-1} of $f(x) = \frac{1}{3}\sqrt{7+e^{5x}}$.

Let $y = f(x)$. Then $3y = \sqrt{7+e^{5x}}$, so $(3y)^2 = 7+e^{5x}$. Simplifying, we get $e^{5x} = 9y^2-7$, and therefore $5x = \ln(9y^2-7)$, so $x = \frac{1}{5}\ln(9y^2-7)$. Therefore, the inverse function f^{-1} is given by:

$$f^{-1}(y) = \frac{1}{5}\ln(9x^2-7)$$

Problem 6: (Bamler Spring '18 Final Exam) Simplify the following expression as much as possible:

$$\frac{1}{2}\ln\left(\frac{12}{e^5}\right) - \ln(\sqrt{3}) - \frac{1}{\log_2 e}$$

We have

$$\begin{aligned}\frac{1}{2}\ln\left(\frac{12}{e^5}\right) - \ln(\sqrt{3}) - \frac{1}{\log_2 e} &= \frac{1}{2}\ln 12 - \frac{5}{2} - \frac{1}{2}\ln 3 - \frac{1}{\log_2 e}\\ &= \frac{1}{2}\ln 4 - \frac{5}{2} - \frac{1}{\log_2 e}\\ &= \frac{\log_2 2}{\log_2 e} - \frac{1}{\log_2 e} - \frac{5}{2}\\ &= -\frac{5}{2}.\end{aligned}$$

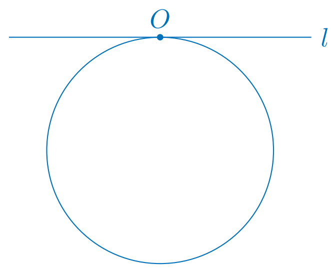
Challenge problem: (Stewart Section 1.5) Solve the inequality $\ln(x^2-2x-2) \le 0$.

The Tangent and Velocity Problems

Chapter 2, Section 1

Here are some important ideas from lecture:

- **Circle one:** A line is *tangent* to a curve if it touches the curve (once / more than once) and follows the direction of the curve at the point of contact. Use the space below to draw a line l that is tangent to a circle:



- **Circle one:** A line is *secant* to a curve if it touches the curve (once / more than once).
- Draw a series of secant lines that get closer and closer to the tangent line in your drawing above. We show that we can use secant lines to approximate tangent lines.

- The instantaneous velocity is defined to be a measure of how fast an object is moving at a point along its path.
- The velocity problem is an application of tangent line approximations.

The Velocity Problem

The velocity problem asks to find the instantaneous velocity of an object at *any* time, provided the position at a *particular* time is given.

- We may sometimes use the average velocity to approximate the instantaneous velocity, which is given by change in positition divided by time elapsed.

. . . .

Problem 1: (Stewart Section 2.1) Suppose the position of an object is given by $y = 10t - 1.86t^2$. Find its average velocity in the interval $[1, 2]$.

We first calculate $s(2)$ and $s(1)$:

$$s(2) = 10(2) - 1.86(2)^2 = 20 - 7.44 = 12.56$$
$$s(1) = 10(1) - 1.86(1)^2 = 10 - 1.86 = 8.14$$

Substituting these values, we get:

$$\text{Average velocity} = \frac{12.56 - 8.14}{2 - 1} = \frac{4.42}{1} = 4.42$$

Therefore, the average velocity in the interval $[1, 2]$ is 4.42 units per time.

Problem 2: (Stewart Section 2.1) We are given that the point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$. Find the slope of the secant line PQ for $x = 0$.

Let Q be the point at $x = 0$. We get:

$$\text{Slope} = \frac{\cos(\pi \cdot 0) - 0}{0 - 0.5} = \frac{\cos(0) - 0}{-0.5} = \frac{1 - 0}{-0.5} = -2$$

Therefore, the slope of the secant line PQ for $x = 0$ is -2.

Problem 3: Do the same as in problem 2, but for following values of x:

- $x = 0.4$
- $x = 1$
- $x = 0.6$

$$\text{Slope for } x = 0.4 = \frac{\cos(0.4\pi) - 0}{-0.1} \approx \frac{-0.309 - 0}{-0.1} \approx 3.09$$
$$\text{Slope for } x = 0.1 = \frac{\cos(\pi) - 0}{0.5} = \frac{-1 - 0}{0.5} = -2$$
$$\text{Slope for } x = 0.6 = \frac{\cos(0.6\pi) - 0}{0.1} \approx \frac{-0.309 - 0}{0.1} \approx -3.09$$

Problem 4: (Stewart Section 2.1) The position of a particle moving back and forth along a straight line is $s = 2 \sin \pi t + 3 \cos \pi t$. Find the average velocity during the interval $[1, 1.1]$.

At $t = 1$:

$$s(1) = -3$$

At $t = 1.1$:

$$s(1.1) = 2 \sin(1.1\pi) + 3 \cos(1.1\pi)$$

Now, we can find the average velocity:

$$\text{Average Displacement} = \frac{s(1.1) - s(1)}{1.1 - 1}$$

$$\text{Average velocity} = \frac{(2 \sin(1.1\pi) + 3 \cos(1.1\pi)) - (-3)}{1.1 - 1} \approx -4.71$$

Challenge problem: Consider a function $f(x)$ defined for $x \geq 0$ such that $f(x)$ represents the height of a staircase at position x. The staircase is formed by stacking unit cubes. The height of the staircase at $x = 0$ is $f(0) = 1$ unit. At each integer x, the height of the staircase increases by one unit, i.e., $f(x) = f(x - 1) + 1$.

Determine the equation of the secant line connecting two points $P(a, f(a))$ and $Q(b, f(b))$ on the staircase.
Explain how the secant line can be used to approximate the slope of the tangent line at the point $P(a, f(a))$. Does this approximation become more accurate as a and b get closer? Why or why not?

The Limit of a Function

Chapter 2, Section 2

In this section, we introduce the idea of limits.

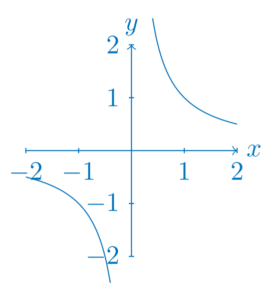
The Limit of a Function

We say the limit of of $f(x)$, as x approaches a, equals L if we can make the values of $f(x)$ as close to L as we like by making x sufficiently close to a but not equal to a.

- One way to estimate the limit of a function is to guess, and making a table might help.
- A one-sided limit only approaches an x-value from one side, either the left or the right.
- We may write
$$\lim_{x\to} f(x) = L \quad \text{if and only if} \quad \lim_{x\to} f(x) = \lim_{x\to} f(x) = L$$
- An **infinite limit** is written as $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinity is *not* a number; instead, it represents growing without bound.
- We say the line $x = a$ is a **vertical asymptote** if:
$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a} f(x) = \pm\infty$$

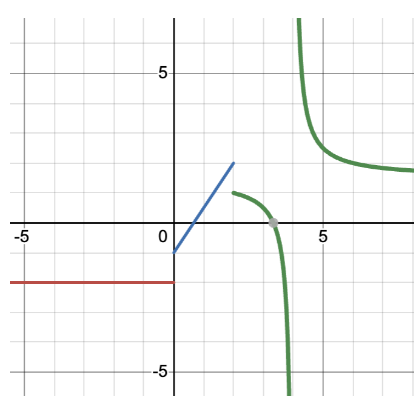
. . . .

Problem 1: Sketch a graph of $f(x) = \frac{1}{x}$. Use your graph to find $\lim_{x\to 0} f(x)$, if it exists.



Here, the limit at 0 does not exist because the one-sided limits differ.

Problem 2: Draw the graph of a function $f(x)$ satisfying the following limits:, $\lim_{x\to 0^-} f(x) = -2$, $\lim_{x\to 0^+} f(x) = -1$, $\lim_{x\to 2^-} f(x) = 2$, $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$



Problem 3: (Stewart Section 2.2) Consider the function given by

$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2-x & \text{if } x \geq 1. \end{cases}$$

Sketch the graph of f, and use your graph to find the x values where the limit does not exist.

We omit the graph here; see the video solution or use a graphing calculator. We find the limit does not exist only at $x = -1$.

Problem 4: (Stewart Section 2.2) Evaluate the following limit.

$$\lim_{x \to 3^-} \frac{\sqrt{x}}{(x-3)^2}$$

Observe that as x approaches 3 from the left side, the numerator approaches $\sqrt{3}$, and the denominator approaches 0. Since dividing by a very small number results in a very large number, we know this limit approaches some infinity. In fact, we can see that the limit approaches ∞, because when we subtract 3 from a value smaller than 3 (i.e. on the left of 3), we get a negative number, and when we square it, it becomes positive.

Problem 5: (Stewart Section 2.2) Let m represent the mass of a particle with velocity v, let m_0 represent its mass at rest, and let c be the speed of light. The theory of relativity is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

What happens as $v \to c^-$?

We see that $v^2 \to c^2$, so $\sqrt{1 - v^2/c^2}$ approaches 0. Thus m will approach some infinity. To see which, observe that since $v < c$ (since we approach from the left side), we get $v^2/c^2 < 1$, and therefore $\sqrt{1 - v^2/c^2} > 0$. Thus our mass will indeed approach $+\infty$.

Challenge problem: (Stewart Section 2.2) Consider the function $f(x) = \tan \frac{1}{x}$.

1. Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$
2. Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \ldots$
3. Make a conclusion about $f(x) = \tan \frac{1}{x}$.

Calculating Limits Using the Limit Laws

Chapter 2, Section 3

The limit laws help us solve limits. They **only** apply when the limits exist and are not $\pm\infty$. Here are some of them, and be sure to refer to the book for a complete list.

- $\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ and $\lim_{x\to a}\big(f(x)-g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}\big(f(x)g(x)\big) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$ and $\lim_{x\to a}\big(f(x)/g(x)\big) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$
- $\lim_{x\to a}\big(cf(x)\big) = c\lim_{x\to a} f(x)$ and $\lim_{x\to a}\big(f(x)\big)^n = \big(\lim_{x\to a} f(x)\big)^n$

Some other properties of limits might help us better understand how to evaluate them.

- Simplify $f(x)$ assuming $x \neq a$, then take the limit with the limit laws.
- Combine fractions to get a rational function. Apply polynomial division if necessary.
- If there is a radical expression $(\sqrt{a}+b)$in the denominator, rationalize by multiplying by the conjugate.

The Squeeze Theorem says that if $f(x) \leq g(x) \leq h(x)$ when x is near a (can exclude a) **and** if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

. . . .

Problem 1: (Stewart Section 2.3) Find $\lim_{x\to\infty}\frac{\sin x}{x}$ using the Squeeze Theorem.

Since $-1 \leq \sin x \leq 1$, we know that $\frac{\sin x}{x}$ always lies between $\frac{-1}{x}$ and $\frac{1}{x}$. And since both $\frac{-1}{x}$ and $\frac{1}{x}$ approach 0 as x approaches ∞, we know by the Squeeze Theorem that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$.

Problem 2: (Stewart Section 2.3) Find the following limit in terms of a.

$$\lim_{h\to 0}\frac{\sqrt{7(a+h)}-\sqrt{7a}}{h}.$$

We rationalize the numerator by multiplying both the numerator and denominator by the conjugate of the numerator expression:

$$\lim_{h\to 0}\frac{\sqrt{7(a+h)}-\sqrt{7a}}{h}\cdot\frac{\sqrt{7(a+h)}+\sqrt{7a}}{\sqrt{7(a+h)}+\sqrt{7a}}.$$

This simplifies to:

$$\lim_{h\to 0}\frac{7(a+h)-7a}{h(\sqrt{7(a+h)}+\sqrt{7a})}.$$

Further simplification yields:

$$\lim_{h\to 0}\frac{7}{\sqrt{7(a+h)}+\sqrt{7a}}.$$

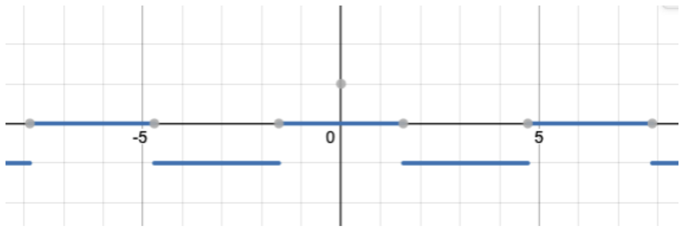
Finally, as h approaches 0, the expression becomes:

$$\frac{7}{\sqrt{7a}+\sqrt{7a}} = \frac{7}{2\sqrt{7a}} = \frac{7}{2\sqrt{7a}}.$$

Problem 3: (Stewart Section 2.3) Is there a value a such that $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exists? Prove it.

Notice that $x^2+x-2 = (x+2)(x-1)$, so we seek a value of a that allows us to cancel $x+2$; in particular, we seek some root c such that $(3x+c)(x+2) = 3x^2+ax+a+3$. Solving for c, we get 9, which leads us to $a = 15$.

Problem 4: (Stewart Section 2.3) Let $[x]$ denote the greatest integer function, where $[x]$ is the largest integer less than or equal to x. Graph $f(x) = [\cos(x)]$ and find the values of a such that $\lim_{x\to a} f(x)$ exists.



We see the limit only exists for $a \neq \{-\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...\}$, so $a \neq \frac{-(2k+1)\pi}{2}$ for $k \in \mathbb{Z}$.

Problem 5: (Stewart Section 2.3) If p is a polynomial, show that the following equation holds (*Hint: recall that any polynomial can be written as* $a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$.)

$$\lim_{x\to a} p(x) = p(a).$$

From the limit laws, we get

$$\begin{aligned}\lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0) &= a_n \lim_{x\to a} x^n + a_{n-1}\lim_{x\to a} x^{n-1} + ... + a_2 \lim_{x\to a} x^2 + a_1 \lim_{x\to a} x + a_0 \\ &= a_n a^n + a_{n-1}a^{n-1} + ... + a_2a^2 + a_1a + a_0 \\ &= p(a).\end{aligned}$$

Problem 6: (Stewart Section 2.3) If the limit below exists, find it. If not, explain why it doesn't exist.

$$\lim_{x\to 0} \frac{2-|x|}{2+x}$$

We get

$$\lim_{x\to 0^-} \frac{2+x}{2+x} = 1,$$

and

$$\lim_{x\to 0^+} \frac{2-x}{2+x} = 1,$$

so the limit is 1.

Challenge problem: (Stewart Section 2.3) Show that the following equation holds.

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0.$$

The Precise Definition of a Limit

Chapter 2, Section 4

In this section, we formalize some concepts about limits that we learned in previous sections.

The Precise Definition of a Limit

If L is the limit at a, the distance between $f(x)$ and L can be made arbitrarily small by letting the distance from x to a be sufficiently small (but nonzero).

Let's write this mathematically. We use ε to represent the distance between the y-values, and δ to represent the distance between the x-values. If $\lim_{x\to a} f(x) = L$, then:

for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

For solving problems, it might be helpful to know that $|x-a| < \delta$ means $a - \delta < x < a + \delta$. Also, $|f(x) - L| < \varepsilon$ means $L - \varepsilon < f(x) < L + \varepsilon$.

....

Let's do an epsilon-delta proof on a linear function.

Problem 1: (Stewart Section 2.4) Prove that $\lim_{x\to 3}(4x-5) = 7$.

The idea of an epsilon-delta proof is the following: if you give me $\varepsilon > 0$ such that $|f(x) - L| < \varepsilon$, then I will give you some $\delta > 0$ such that $0 < |x-a| < \delta$. (*Think: if I tell you the y-values must be a distance of ε from L, then you give me a distance δ from a for the corresponding x-values.*)

Here are the steps for an epsilon-delta proof:

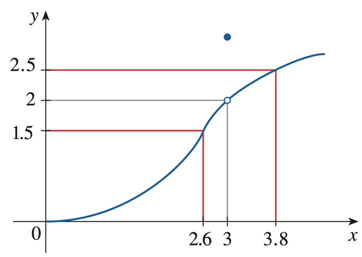
1. First find a possible expression for δ. Suppose you give me $|f(x) - L| < \varepsilon$. Since we know what $f(x)$ is, we can write this as: $|4x - 5 - 7| < \epsilon$, so $|4x - 12| < \epsilon$. Now we will modify this to get our δ:

   We know $|4x - 12| < \epsilon$, so
   $$|4x - 12| = |4(x-3)| = 4|x-3| < \epsilon.$$
   Then dividing both sides by 4, we get $|x - 3| < \frac{\epsilon}{4}$.

2. Check that this value of δ works! Given some arbitrary $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{4}$. If $0 < |x-a| < \delta$, then:

   We have
   $$|x-3| < \frac{\epsilon}{4} \implies 4|x-3| < \epsilon \implies |4x-12| < \epsilon,$$
   so $|4x-12| = |4x-5-7| = |f(x) - L| < \epsilon$, as desired.

Problem 2: (Stewart Section 2.4) Consider the following graph of a function f. Use the graph to find a number δ such that $0 < |x-3| < \delta$ implies $|f(x) - 2| < 0.5$.



Notice that $|f(x) - 2| < 0.5$ means $1.5 < f(x) < 2.5$. And from the graph, we see that the x values such that $f(x)$ is in this range are exactly $2.6 < x < 3.8$, and thus $-0.4 < x - 3 < 0.8$. We need $-\delta < x - 3 < \delta$, and thus the largest value of delta we can have is $\delta = 0.4$. Notice that any bigger value of delta will cause x to be less than 2.6, and thus $f(x)$ would be outside of the range provided.

Recall the definition of an *infinite limit* from previous sections. Now, to define infinite limits rigorously, consider these definitions:

- $\lim_{x \to a} f(x) = \infty$ when for every positive number M there is a positive number δ such that if $0 < |x - a| < \delta$ then $f(x) > M$.
- $\lim_{x \to a} f(x) = -\infty$ when for every negative number M there is a positive number δ such that if $0 < |x-a| < \delta$ then $f(x) < M$.

Problem 3: Prove $\lim_{x \to 0} \frac{1}{x^2} = \infty$. Given some $M > 0$, we want to find delta such that $|x - 0| = |x| < \delta$ implies $\frac{1}{x^2} > M$. But notice that

$$\frac{1}{x^2} > M$$

means $x^2 M < 1$, so

$$x^2 < \frac{1}{M},$$

which certainly holds for

$$-\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}}.$$

Then we may pick $\delta = \frac{1}{\sqrt{M}}$.

Problem 4: (Stewart Section 2.4) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \epsilon$, where $\epsilon = 0.1$.

Given $\epsilon = 0.1$, we want to find δ such that if $|x - 2| < \delta$, then $|4x - 8| < 0.1$. Consider $|4x - 8|$:

$$|4x - 8| = 4|x - 2|$$

So, we need $4|x - 2| < 0.1$.Divide both sides by 4:

$$|x - 2| < \frac{0.1}{4} = 0.025$$

Thus, we can choose $\delta = 0.025$. Therefore, if $|x - 2| < 0.025$, then $|4x - 8| < 0.1$, as required.

Continuity

Chapter 2, Section 5

One way to think about continuity is the following: can we draw the graph without picking up our pencil?

- We say f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. (*Think: if the function approaches $f(a)$ as x gets closer to a, then we can draw the graph without picking up our pencil!*)
- We say f is continuous from the right at a if $\lim_{x\to a^+} f(x) = f(a)$.
- We say f is continuous from the left at a if $\lim_{x\to a^-} f(x) = f(a)$.
- A function is continuous on an interval if it is continuous at every point in the interval.
- If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. Another way to write this is:
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right),$$
so the composition of continuous functions is also continuous.
- The Intermediate Value Theorem says that if f is continuous on $[a, b]$, then if $f(a) \neq f(b)$ and if N is between $f(a)$ and $f(b)$, there exists some c in (a, b) such that $f(c) = N$.

....

Problem 1: (Stewart Section 2.5) Use interval notation to indicate where the function below is continuous.

$$f(x) = \frac{x^2-1}{x^2-7x+6}$$

We need to find where the denominator is not equal to zero. First, let's find where the denominator $x^2 - 7x + 6$ is equal to zero:

$$x^2 - 7x + 6 = 0.$$

Factoring, we get:

$$(x-1)(x-6) = 0.$$

So, the function will be undefined at $x = 1$ and $x = 6$. We'll have two intervals where the function is continuous: $(-\infty, 1) \cup (1, 6) \cup (6, \infty)$.

Problem 2: (Stewart Section 2.5) Find $\lim_{x\to 1} e^{x^2-5x+4}$, and explain how your solution works.

To find the limit $\lim_{x\to 1} e^{x^2-5x+4}$, we can directly substitute $x = 1$ into the expression e^{x^2-5x+4} since it is a continuous function. So,

$$\lim_{x\to 1} e^{x^2-5x+4} = e^{1^2-5\cdot 1+4} = e^{1-5+4} = e^0 = 1$$

The limit evaluates to 1.

Problem 3: (Stewart Section 2.5) Suppose $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Find $(f \circ g)(x)$. Is $f \circ g$ continuous everywhere?

To find $(f \circ g)(x)$, we first need to compute $g(x)$ and then plug it into $f(x)$.

Given $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, we have:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2}\right) = \frac{1}{\left(\frac{1}{x^2}\right)} = x^2$$

So, $(f \circ g)(x) = x^2$.

We should be cautious at $x = 0$ since $f(x)$ and $g(x)$ are undefined at $x = 0$. The composition $f(g(x)) = x^2$ is continuous everywhere except at $x = 0$, where it's not defined.

Problem 4: (Stewart Section 2.5) Prove that $\cos x = x^3$ has at least one real root.

To prove that the equation $\cos(x) = x^3$ has at least one real root, we can use the Intermediate Value Theorem (IVT).

Let $f(x) = \cos(x) - x^3$. We'll show that $f(x)$ changes sign on the interval $[0, \frac{\pi}{2}]$. For $x = 0$, we have $f(0) = \cos(0) - 0^3 = 1$. For $x = \frac{\pi}{2}$, we have $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^3 = -\frac{\pi^3}{8}$. Since $f(0) > 0$ and $f\left(\frac{\pi}{2}\right) < 0$, by the IVT, there exists at least one c in the interval $(0, \frac{\pi}{2})$ such that $f(c) = 0$, which means c is a real root of the equation $\cos(x) = x^3$.

Problem 5: (Stewart Section 2.5) For the function f below, construct a function g that is continuous and "removes the discontinuities" of f.

$$f(x) = \frac{x^2 - 7x + 12}{x - 3}$$

We need to remove the removable singularity at $x = 3$. We can do this by factoring the numerator and canceling out the common factor with the denominator. We get

$$f(x) = \frac{(x-3)(x-4)}{x-3}.$$

Now, we can cancel out the common factor $x - 3$ from the numerator and denominator:

$$f(x) = x - 4$$

So, the function $g(x) = x - 4$ is continuous and "removes the discontinuities" of $f(x)$.

Challenge Problem: Is there a number that is exactly 1 more than its cube?

Limits at Infinity, Horizontal Asymptotes

Chapter 2, Section 6

In a previous section, we discussed infinite limits. Now, we discuss *limits at infinity*.

- "The limit of $f(x)$, as x approaches infinity, is L" is written as: $\lim_{x\to\infty} f(x) = L$.
- "The limit of $f(x)$, as x approaches *negative* infinity, is L" is written as: $\lim_{x\to-\infty} f(x) = L$.
- If we can find L for any infinite limit (i.e. if we can find a value L such that $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote.

When finding limits at infinity, it often helps to manipulate the function.

. . . .

Problem 1: (Stewart Section 2.6) Find the limit below using at least two limit laws. *Hint: divide by the highest power of x in the denominator.*

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

We can use the hint provided and divide both the numerator and denominator by the highest power of x, which is x^2.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

Now, as x approaches infinity, the terms $\frac{1}{x}$ and $\frac{1}{x^2}$ approach 0. So, we have:

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}.$$

Problem 2: (Stewart Section 2.6) Find the limit below. *Hint: how can we make this look like an infinite limit?*

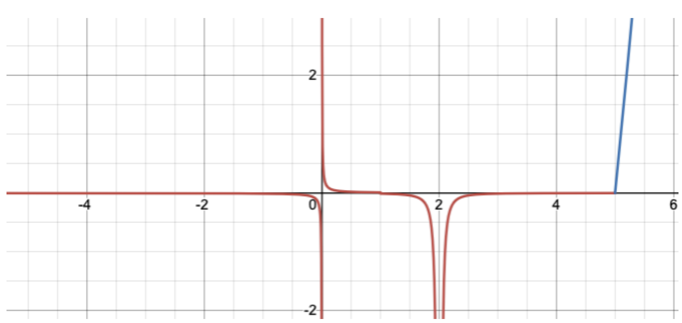
$$\lim_{x\to 2^+} e^{\frac{3}{2-x}}$$

To find the limit $\lim_{x\to 2^+} e^{\frac{3}{2-x}}$, we can observe that as x approaches 2 from the right $(x \to 2^+)$, the expression $\frac{3}{2-x}$ approaches negative infinity. This suggests that we have an infinite limit of the form $e^{-\infty}$. We can rewrite the limit to emphasize this by letting $t = \frac{3}{2-x}$, so as $x \to 2^+$, t approaches 0 from the right. Substituting this into our expression, we get:

$$\lim_{x\to 2^+} e^{\frac{3}{2-x}} = \lim_{t\to-\infty} e^t.$$

From the graph of e^t, we know this limit is 0.

Problem 3: (Stewart Section 2.6) Sketch the graph of a function satisfying the following conditions: $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$



Problem 4: (Stewart Section 2.6) Find $\lim_{x\to\infty}(\sqrt{x^2+5}-\sqrt{x^2-5})$. *Hint: do something with the conjugate!* Given:

$$\lim_{x\to\infty}(\sqrt{x^2+5}-\sqrt{x^2-5})$$

We multiply and divide by the conjugate of the expression, $(\sqrt{x^2+5}+\sqrt{x^2-5})$:

$$\lim_{x\to\infty}\frac{(\sqrt{x^2+5}-\sqrt{x^2-5})(\sqrt{x^2+5}+\sqrt{x^2-5})}{(\sqrt{x^2+5}+\sqrt{x^2-5})}$$

Using the difference of squares formula, we have:

$$\lim_{x\to\infty}\frac{(x^2+5)-(x^2-5)}{\sqrt{x^2+5}+\sqrt{x^2-5}}=\lim_{x\to\infty}\frac{x^2+5-x^2+5}{\sqrt{x^2+5}+\sqrt{x^2-5}}=\lim_{x\to\infty}\frac{10}{\sqrt{x^2+5}+\sqrt{x^2-5}}$$

As x approaches infinity, both $\sqrt{x^2+5}$ and $\sqrt{x^2-5}$ approach infinity, so their sum also approaches infinity. So, $\lim_{x\to\infty}(\sqrt{x^2+5}-\sqrt{x^2-5})=0$.

Problem 5: (Stewart Section 2.6) Suppose we have two polynomials P and Q. Find the limit below first when the degree of P is less than the degree of Q, and second when the degree of P is greater than the degree of Q.

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}$$

Let m be the degree of P. When the degree of P is less than the degree of Q, we get

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{\frac{P(x)}{x^m}}{\frac{Q(x)}{x^n}}=\lim_{x\to\infty}\frac{\text{constant}}{\infty}=0.$$

Since the numerator is a constant and the denominator goes to infinity, the limit approaches zero. When the degree of P is greater than the degree of Q, and for n being the degree of Q, we get

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{\frac{P(x)}{x^n}}{\frac{Q(x)}{x^n}}=\lim_{x\to\infty}\frac{\infty}{\text{constant}}=\pm\infty.$$

Since the numerator goes to infinity and the denominator is a constant, the limit approaches either positive or negative infinity depending on the leading coefficients of P and Q.

Problem 6: (Stewart Section 2.6) Find a *formula* for a function f that satisfies the following conditions:

- $\lim_{x\to\pm\infty} f(x)=0$
- $\lim_{x\to 0} f(x)=-\infty$
- $\lim_{x\to 3^-} f(x)=\infty$
- $\lim_{x\to 3^+} f(x)=-\infty$
- $f(2)=0$.

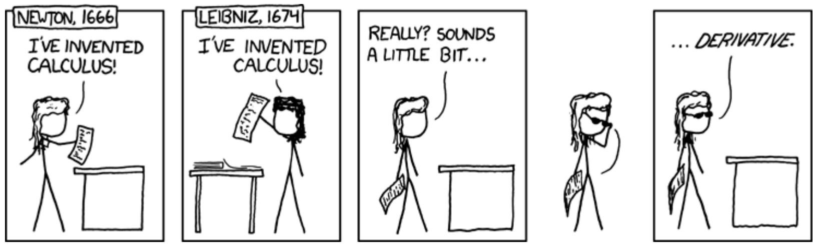
The function below works:

$$f(x)=\frac{-(x-2)}{x^2(x-3)}$$

Challenge problem: (Stewart Section 2.6) Find $\lim_{x\to 2^+}\arctan\left(\frac{1}{x-2}\right)$.

Derivatives and Rates of Change

Chapter 2, Section 7



It's helpful to think of the derivative of a function at a to be *the slope of the tangent line* at a.

- The limit definition of a derivative at a point is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- We can write the derivative of $f(x)$ as:

$$\frac{df}{dx} \qquad \frac{d}{dx}(f) \qquad f'(x) \qquad \dot{y}$$

- A function $f(x)$ is called differentiable at a if:

$f'(a)$ exists.

- If f is differentiable at a, it is also continuous at a. **Circle one:** The reverse (does / does not) hold.

....

Problem 1: Use the limit definition to find the derivative of $f(x) = \sqrt{x}$.

From the definition, we get:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

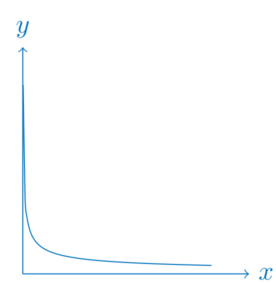
To simplify the expression in the numerator, we multiply by the conjugate:

$$\sqrt{x+h} - \sqrt{x} = \left(\sqrt{x+h} - \sqrt{x}\right) \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} = \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

Substituting this back into the derivative expression, we get:

$$f'(x) = \lim_{h \to 0} \frac{\frac{h}{\sqrt{x+h} + \sqrt{x}}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Problem 2: For $f(x) = \sqrt{x}$, sketch the graph of $f'(x)$.



Problem 3: (Stewart Section 2.7) Find an equation of the tangent line to the curve $y = g(x)$ at $x = 5$ if $g(5) = -3$ an $g'(5) = 4$.

Substituting the given values into point-slope form, we get:

$$\begin{aligned} y - (-3) &= 4(x - 5) \\ y + 3 &= 4(x - 5) \end{aligned}$$

which simplifies to

$$\begin{aligned} y + 3 &= 4x - 20 \\ y &= 4x - 23. \end{aligned}$$

So, the equation of the tangent line to the curve $y = g(x)$ at $x = 5$ is $y = 4x - 23$.

Problem 4: (Stewart Section 2.7) For each of the following limits, find a function f and a value a such that the given expression represents $f'(a)$.

- $\lim_{x \to 2} \frac{x^6 - 64}{x - 2}$
- $\lim_{\theta \to \pi/6} \frac{\sin\theta - 1/2}{\theta - \pi/6}$
- $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

- $f(x) = x^3, a = 2$
- $f(\theta) = \sin\theta, a = \pi/6$
- $f(x) = \cos x, a = \pi$

Problem 5: (Stewart Section 2.7) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

Substituting $g(x) = x^{2/3}$:

$$= \lim_{h \to 0} \frac{(0 + h)^{2/3} - 0^{2/3}}{h}$$

Simplify:

$$= \lim_{h \to 0} \frac{h^{2/3}}{h} = \lim_{h \to 0} \frac{1}{h^{1/3}}.$$

But this limit does not exist, since the one-sided limits are unequal (in fact, if they were equal but infinite, the derivative still would not exist).

Challenge Problem: (Stewart Chapter 2) Suppose f is function satisfying $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$, and then show that $f'(0) = 0$.

The Derivative as a Function

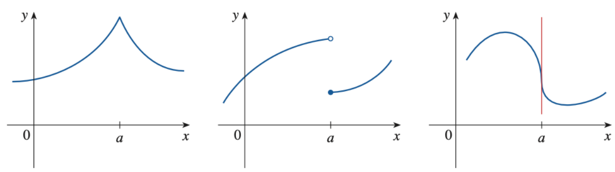
Chapter 2, Section 8

In the previous section, we considered derivatives at a particular point. In this section, we consider the derivative as a *function*, defined on a domain of points.

- The limit definition of the derivative *as a function* is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- We say a function f is differentiable at a if $f'(a)$ exists. We can also find intervals of differentiability.
- A function can fail to be continuous if it has a corner, discontinuity, or vertical tangent. Draw these three possibilities in the space below.



(a) A corner (b) A discontinuity (c) A vertical tangent

- We may take derivatives of derivatives, and so on. The derivatives of a position function f are:

$$f'(x) = \text{velocity}, f''(x) = \text{acceleration}, f'''(x) = \text{jerk}$$

. . . .

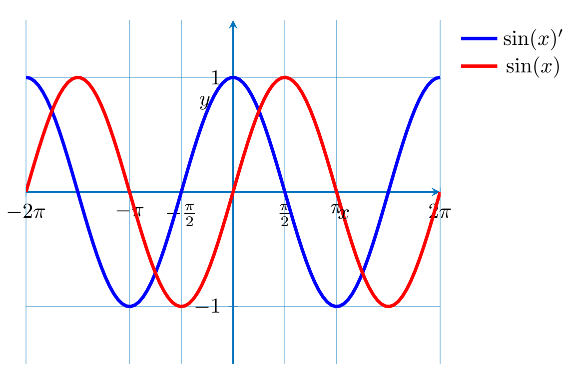
Problem 1: (Stewart Section 2.8) Find the derivative of $f(x) = mx + b$ using the limit definition of a derivative. State the domain of the function and the domain of its derivative.

Substituting, we have:

$$f'(x) = \lim_{h \to 0} \frac{m(x+h) + b - (mx+b)}{h} = \lim_{h \to 0} \frac{mx + mh + b - (mx+b)}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m.$$

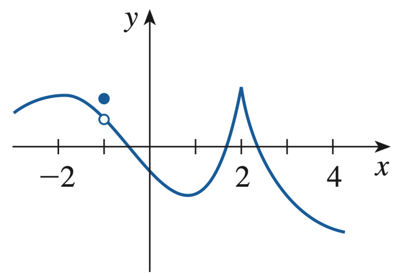
Both the function and its derivative are defined everywhere.

Problem 2: (Stewart Section 2.8) Make a graph of $f(x) = \sin x$, and then sketch its derivative. Use your graph to guess the equation of $f'(x)$.



We can guess that $f'(x) = \cos x$.

Problem 3: (Stewart Section 2.8) Below is the graph of a function $f(x)$. State where (in numbers) the function is not differentiable, and explain why.



The function is not differentiable at $x = -1$ because of a discontinuity, and at $x = 2$ because of a corner.

Problem 4: (Stewart Section 2.8) Prove that the derivative of an even function is an odd function.

By the definition of the derivative, we have:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Now, let's consider $f'(-x)$:

$$\begin{aligned}
f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(-(x-h)) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(x-h) - f(-x)}{-h} \qquad \text{(using the property of even functions)} \\
&= \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \qquad \text{(since } f(-x) = f(x)\text{)} \\
&= -\lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \\
&= -f'(x)
\end{aligned}$$

Therefore, $f'(-x) = -f'(x)$, which proves that the derivative of an even function is an odd function.

Challenge Problem: (Stewart Section 2.8) Let l be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The angle of inclination of l is the angle that ϕ that l makes with the positive direction of the x-axis. Calculate ϕ to the nearest degree.

Differentiation Rules

Chapter 3, Section 1

Now that we know what a derivative is, we can use certain rules to find derivatives efficiently. All these rules can be derived from the limit definition of a derivative.

- For a constant a, the derivative of a with respect to x is $\frac{d}{dx}(a) = 0$.
- The power rule: x is $\frac{d}{dx}(x^n) = nx^{n-1}$.
- The sum/difference rule: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$.
- The derivative of a constant times a function: $\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$.
- The derivative of the natural exponential function: $\frac{d}{dx}e^x = e^x$.

. . . .

Problem 1: (Stewart Section 3.1) Differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$.

To differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$ by first expanding the expression and then using the power rule, we first expand the expression:

$$y = 10x^2 \cdot 2 - 10x^2 \cdot x^2 + 7x \cdot 2 - 7x \cdot x^2 - 2 \cdot 2 + 2 \cdot x^2$$

We find the derivative of each term by using the power rule, to get

$$y' = 40x - 40x^3 + 14 - 21x^2 + 4x$$

which simplifies to

$$y' = -40x^3 - 21x^2 + 44x + 14.$$

Problem 2: (Stewart Section 3.1) Find the derivative of $k(r) = e^r + r^e$.

Let's find the derivative of each term separately: For e^r:

$$\frac{d}{dr}(e^r) = e^r$$

For r^e:

$$\frac{d}{dr}(r^e) = e \cdot r^{e-1}$$

Now, applying the sum rule, the derivative of $k(r) = e^r + r^e$ is:

$$\frac{d}{dr}(e^r + r^e) = e^r + e \cdot r^{e-1}.$$

Problem 3: (Stewart Section 3.1) Find the derivative of $f(x) = \frac{x^2-5x^3-\sqrt{x}}{x^2}$ without using the product rule or the quotient rule.

We can first rewrite the function:

$$f(x) = \frac{x^2}{x^2} - \frac{5x^3}{x^2} - \frac{\sqrt{x}}{x^2} = 1 - 5x - x^{-\frac{3}{2}}$$

Then we use the powe rule to take the derivative each term separately, and the sum rule to add them up:

$$f'(x) = (1)' - (5x)' - (x^{-3/2})' = 0 - 5 - \frac{-3}{2}x^{-\frac{5}{2}} = -5 + \frac{3}{2x^{2.5}}.$$

Problem 4: (Stewart Section 3.1) Let $y = x + e^x$. Find the equation of the line tangent to this function at $x = 0$.

To find the equation of the line tangent to $y = x + e^x$ at $x = 0$, let's first find the derivative of the function:

$$\frac{dy}{dx} = \frac{d}{dx}(x + e^x) = 1 + e^x$$

Evaluating this at $x = 0$, we get $1 + e^0 = 1 + 1 = 2$. So, the slope of the tangent line at $x = 0$ is $m = 2$. Next, we need to find a point on the line. Since the tangent line touches the function at $x = 0$, the corresponding y-coordinate can be found by evaluating the function at $x = 0$ to get $y = 0 + e^0 = 0 + 1 = 1$. Now, using the point-slope form of the equation of a line $(y - y_1) = m(x - x_1)$, where (x_1, y_1) is a point on the line and m is the slope, we can write the equation of the tangent line:

$$(y - 1) = 2(x - 0)$$

Simplifying, we get: $y - 1 = 2x$ so $y = 2x + 1$. So, the equation of the line tangent to the function $y = x + e^x$ at $x = 0$ is $y = 2x + 1$.

Problem 5: (Stewart Section 3.1) Find equations of the tangent line and normal line to $y = x^4 + 2e^x$ at $(0, 2)$. Recall that the normal line is perpendicular to the tangent line.

We first find the derivative of the function:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 2e^x) = 4x^3 + 2e^x$$

We evaluate the derivative at $x = 0$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{x=0} = 4(0)^3 + 2e^0 = 2.$$

Thus our tangent line is

$$y - 2 = 2(x - 0) \implies y = 2x + 2.$$

We also know the slope of the normal line is $-1/2$. Next, we need to find a point on the line. We're given that the point $(0, 2)$ lies on the line. Using the point-slope form of the equation of a line $(y - y_1) = m(x - x_1)$, where (x_1, y_1) is a point on the line and m is the slope, we can write the equation of the tangent line:

$$(y - 2) = (-1/2)(x - 0)$$

Which simplifies to $y = (-1/2)x + 2$.

Challenge problem: Find the second derivative of $\frac{2}{(6+2v-v^2)^4}$.

The Product and Quotient Rules

Chapter 3, Section 2

There are two more derivatives rules that will show up often in this course.

- The product rule: $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$.
- The quotient rule: $\frac{d}{dx}(f(x)/g(x)) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

. . . .

Problem 1: (Stewart Section 3.2) Differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$ using the product rule.

To differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$ using the product rule, we have:

$$\begin{aligned} y' &= (10x^2 + 7x - 2)'(2 - x^2) + (10x^2 + 7x - 2)(2 - x^2)' \\ &= (20x + 7)(2 - x^2) + (10x^2 + 7x - 2)(-2x) \\ &= -40x^3 - 21x^2 + 44x + 14. \end{aligned}$$

Problem 2: (Stewart Section 3.2) If $g(x)$ is a differentiable funciton, find an expression fo the derivative of

$$y = \frac{g(x)}{x}.$$

To find the derivative of $y = \frac{g(x)}{x}$, we'll use the quotient rule:

$$y' = \frac{d}{dx}\left(\frac{g(x)}{x}\right) = \frac{x \cdot g'(x) - g(x) \cdot 1}{x^2} = \frac{xg'(x) - g(x)}{x^2}$$

So, the derivative of y with respect to x is $\frac{xg'(x) - g(x)}{x^2}$.

Problem 3: (Stewart Section 3.2) Suppose $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x. Find $f''(2)$.

To find $f''(2)$, we first differentiate $f'(x) = x^2 f(x)$ with respect to x using the product rule:

$$f''(x) = \frac{d}{dx}(x^2 f(x)) = \frac{d}{dx}(x^2)f(x) + x^2 \frac{d}{dx}(f(x)) = 2xf(x) + x^2 f'(x)$$

Now, we're given that $f'(x) = x^2 f(x)$. Substituting this into the expression for $f''(x)$, we get:

$$f''(x) = 2xf(x) + x^2(x^2 f(x)) = 2xf(x) + x^4 f(x)$$

Now, we can find $f''(2)$ by substituting $x = 2$ into the expression for $f''(x)$:

$$f''(2) = 2 \cdot 2f(2) + 2^4 f(2) = 4f(2) + 16f(2) = 4 \cdot 10 + 16 \cdot 10 = 40 + 160 = 200.$$

Problem 4: (Stewart Section 3.2) For f defined as $f(x) = \sqrt{x}e^x$, find $f'(x)$ and $f''(x)$.

To find $f'(x)$ for $f(x) = \sqrt{x}e^x$, we apply the product rule:

$$f'(x) = \frac{d}{dx}(\sqrt{x}e^x) = \frac{d}{dx}(\sqrt{x}) \cdot e^x + \sqrt{x} \cdot \frac{d}{dx}(e^x) = \frac{1}{2\sqrt{x}} \cdot e^x + \sqrt{x} \cdot e^x = \frac{1}{2}x^{-\frac{1}{2}}e^x + x^{\frac{1}{2}}e^x.$$

To find $f''(x)$, we differentiate $f'(x)$ again using the product rule:

$$f''(x) = \left(\frac{1}{2}x^{-\frac{1}{2}}e^x\right)' + \left(x^{\frac{1}{2}}e^x\right)' = \left(\frac{1}{2} \cdot -\frac{1}{2}x^{-\frac{3}{2}}e^x + \frac{1}{2}x^{-\frac{1}{2}}e^x\right) + \left(\frac{1}{2}x^{-\frac{1}{2}}e^x + x^{\frac{1}{2}}e^x\right)$$

which simplifies to

$$-\frac{e^x}{4x\sqrt{x}} + \frac{e^x}{2\sqrt{x}} + \frac{e^x}{2\sqrt{x}} + e^x\sqrt{x}.$$

Problem 5: (Stewart Section 3.2) Find equations of the tangent lines to the curve f, defined below, that are parallel to the line $x - 2y = 2$.

$$f(x) = \frac{x-1}{x+1}.$$

To find the equations of the tangent lines to the curve f, defined as $f(x) = \frac{x-1}{x+1}$, that are parallel to the line $x - 2y = 2$, we first need to find the derivative of $f(x)$ and then determine the values of x where the derivative matches the slope of the given line.

$$f'(x) = \frac{(x+1) \cdot \frac{d}{dx}(x-1) - (x-1) \cdot \frac{d}{dx}(x+1)}{(x+1)^2} = \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Now, we need to find the values of x where $f'(x)$ is equal to the slope of the given line, which is $m = 1/2$.

$$\frac{2}{(x+1)^2} = \frac{1}{2} \implies 2 = \frac{(x+1)^2}{2} \implies 4 = (x+1)^2 \implies x+1 = \pm 2 \implies x = -1 \pm 2$$

So, we have two possible values of x, $x = 1$ and $x = -3$. Now, we can find the corresponding y-values for these x-values by substituting them into the equation of $f(x)$. When $x = 1$, we get $y = f(-1) = \frac{1-1}{1+1} = 0$.. When $x = -3$, we get $y = f(-3) = \frac{-3-1}{-3+1} = 2$. So, the points where the tangent lines are parallel to the given line are $(1,0)$ and $(3,2)$. Finally, we use point-slope form with these points to find the equations of the tangent lines.

For $(1,0)$:

$$y - 0 = \frac{1}{2}(x-1) \implies y = \frac{1}{2}x - \frac{1}{2}.$$

For $(-3,2)$:

$$y - 2 = \frac{1}{2}(x+3) \implies y = \frac{1}{2}x + \frac{7}{2}.$$

Problem 6: (Stewart Section 3.2) Find equations of the tangent line and normal line to the curve $y = 2xe^x$ at $(0,0)$.

The derivative of the function $y = 2xe^x$ with respect to x is

$$\frac{dy}{dx} = 2e^x + 2xe^x.$$

and evaluating at $x = 0$ we get $2e^0 + 2(0)e^0 = 2$. So, the slope of the tangent line at $(0,0)$ is $m = 2$. The slope of the normal line will be the negative reciprocal of the slope of the tangent line, which is $-1/2$. We can use point-slope form with the point $(0,0)$ and the slopes we found above to get the equations of the tangent and normal lines.

For the tangent line:

$$y - 0 = 2(x-0) \implies y = 2x.$$

For the normal line:

$$y - 0 = -\frac{1}{2}(x-0) \implies y = -\frac{1}{2}x.$$

Challenge problem: (Stewart Section 3.2) Consider $f(x) = x^2e^x$. Find the first 5 derivatives of f, and guess an expression for the nth derivative.

Derivatives of Trigonometric functions

Chapter 3, Section 3

In this section, we learned the derivatives of the trigonometric functions.

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \csc x = -\cot x \csc x$
- $\frac{d}{dx} \sec x = \tan x \sec x$
- $\frac{d}{dx} \cot x = -\cot x \csc x$

We also learned about two important limits.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \text{and} \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

....

Problem 1: (Stewart Section 3.3) Use derivative rules to find $\frac{d}{dx} \csc x$ and $\frac{d}{dx} \sec x$, and check that your answer matches the above.

To find $\frac{d}{dx} \csc x$: We use the quotient rule. First, for $\csc x = \frac{1}{\sin x}$:

$$\frac{d}{dx} \csc(x) = \frac{\sin x \cdot 0 - \cos x \cdot 1}{\sin^2 x} = -\frac{\cos x}{\sin x} \frac{1}{\sin x} = -\cot x \csc x.$$

Then for $\sec x = \frac{1}{\cos x}$:

$$\frac{d}{dx} \sec(x) = \frac{\cos x \cdot 0 - 1 \cdot \sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.$$

Problem 2: (Stewart Section 3.3) Find the values of x such that $f(x) = x + 2 \sin x$ has a horizontal tangent.

To find the values of x such that $f(x) = x + 2\sin(x)$ has a horizontal tangent, we need to find where the derivative $f'(x)$ equals zero. The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \frac{d}{dx}(x) + \frac{d}{dx}(2\sin(x)) = 1 + 2\cos(x).$$

To find the points where $f'(x) = 0$, we solve the equation

$$1 + 2\cos(x) = 0.$$

Simplifying, we have

$$\cos(x) = -\frac{1}{2}$$

The values of x that satisfy this are $x = \frac{2\pi}{3} + 2\pi k$ and $x = \frac{4\pi}{3} + 2\pi k$, where k is an integer.

Problem 3: (Stewart Section 3.3) Find the equation of the tangent line to $y = e^x \cos x$ at $(0, 1)$.

To find the equation of the tangent line to $y = e^x \cos x$ at $(0, 1)$, we first need to find the slope of the tangent line, which is the derivative of y with respect to x, evaluated at $x = 0$. The derivative of $y = e^x \cos x$ can be found with the product rule:

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \cos x) = e^x \cos x - e^x \sin x.$$

Now, we evaluate the derivative at $x = 0$:

$$\left.\frac{dy}{dx}\right|_{x=0} = e^0 \cdot \cos(0) - e^0 \cdot \sin(0) = 1 \cdot 1 - 1 \cdot 0 = 1$$

So, the slope of the tangent line at $(0, 1)$ is $m = 1$. Since we have the slope of the tangent line and a point $(0, 1)$ on the line, we can use point-slope form to find the equation of the tangent line:

$$y - 1 = 1(x - 0) \implies y = x + 1$$

Problem 4: (Stewart Section 3.3) Find $\lim_{x\to 0} \csc x \sin(\sin x)$.

Direct substitution doesn't work here, because $\csc 0$ is undefined. But observe that this can be written as

$$\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}.$$

Now if we let $u = \sin x$, we get $x \to 0 \implies u \to 0$. Then this limit can be written as

$$\lim_{u\to 0} \frac{\sin u}{u},$$

which we know to be 1.

Problem 5: (Stewart Section 3.3) If $H(\theta) = \theta \sin \theta$, find $H'(\theta)$ and $H''(\theta)$.

We use the product rule:

$$\begin{aligned} & H(\theta) = \theta \sin \theta \\ \Rightarrow \quad & H'(\theta) = \frac{d}{d\theta}(\theta \sin \theta) \\ & \qquad\quad = \sin \theta + \theta \cos \theta. \end{aligned}$$

To find $H''(\theta)$, we use the product rule again:

$$\begin{aligned} & H'(\theta) = \sin \theta + \theta \cos \theta \\ \Rightarrow \quad & H''(\theta) = \frac{d}{d\theta}(\sin \theta + \theta \cos \theta) \\ & \qquad\quad = \cos \theta - \theta \sin \theta + \cos \theta \\ & \qquad\quad = 2 \cos \theta - \theta \sin \theta. \end{aligned}$$

Challenge problem: (Stewart Section 3.3) Find constants A and B such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

The Chain Rule

Chapter 3, Section 4

The Chain Rule tells us how to differentiate compositions of functions.

The Chain Rule

Suppose g is differentiable at x and f is differentiable at $g(x)$. If $F = f(g(x))$, then $F'(x) = f'(g(x)) \cdot g'(x)$.

- We can write this in Leibniz notation. If $y = f(u)$ and $u = g(x)$, then:
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

- We can also combine the power rule with the chain rule. For any real number n and differentiable function $u = g(x)$, then
$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$
We can also write this as
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

- One important differentiation rule that comes from the chain rule is
$$\frac{d}{dx}b^x = b^x \ln b$$

. . . .

Problem 1: (Stewart Section 3.4) Find the derivative of $y = e^{\sin(2x)} + \sin(e^{2x})$.

We can differentiate each term separately:
$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin(2x)}\right) + \frac{d}{dx}\left(\sin(e^{2x})\right).$$
Now, applying the chain rule and the derivative of sine function:
$$\frac{dy}{dx} = e^{\sin(2x)} \cdot \frac{d}{dx}(\sin(2x)) + \cos(e^{2x}) \cdot \frac{d}{dx}\left(e^{2x}\right).$$
Using the chain rule and the derivative of exponential function:
$$\frac{dy}{dx} = e^{\sin(2x)} \cdot (2\cos(2x)) + \cos(e^{2x}) \cdot (2e^{2x}) = 2e^{\sin(2x)}\cos(2x) + 2e^{2x}\cos(e^{2x}).$$

Problem 2: (Stewart Section 3.4) Find the derivative of $J(\theta) = \tan^2(n\theta)$.

Using the chain rule, let $u = n\theta$, so $J(\theta) = \tan^2(u)$. Then:
$$\frac{dJ}{d\theta} = \frac{d}{d\theta}(\tan^2(u)) = \frac{d}{du}(\tan^2(u)) \cdot \frac{du}{d\theta}.$$
Now, differentiate $\tan^2(u)$ with respect to u:
$$\frac{d}{du}(\tan^2(u)) = 2\tan(u) \cdot \sec^2(u).$$
Since $u = n\theta$, $\frac{du}{d\theta} = n$:
$$\frac{dJ}{d\theta} = 2\tan(n\theta) \cdot \sec^2(n\theta) \cdot n.$$
Hence, the derivative of $J(\theta)$ with respect to θ is:
$$\frac{dJ}{d\theta} = 2n\tan(n\theta) \cdot \sec^2(n\theta).$$

Problem 3: (Stewart Section 3.4) Find the 1000th derivative of $f(x) = xe^{-x}$.

We find the first few derivatives to identify a pattern:

$$\begin{aligned}
f(x) &= xe^{-x} \\
f'(x) &= e^{-x} - xe^{-x} \\
f''(x) &= -2e^{-x} + xe^{-x} \\
f'''(x) &= 3e^{-x} - xe^{-x} \\
f^{(4)}(x) &= -4e^{-x} + xe^{-x} \\
&\vdots \\
f^{(1000)}(x) &= (-1)^{1001}1000e^{-x} + 1000xe^{-x} = -1000e^{-x} + xe^{-x}
\end{aligned}$$

Problem 4: (Stewart Section 3.4) Find an equation of the tangent line to the curve below at the point $(1, 1)$.

$$y = \frac{|x|}{\sqrt{2 - x^2}}$$

To find the slope of the tangent line, we first find the derivative. Observe that at $x = 1$, we have $|x|$ is positive, so the function behaves like

$$y = \frac{x}{\sqrt{2 - x^2}}$$

Using the chain rule and derivative of absolute value:

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\sqrt{2 - x^2}}\right) &= \frac{1 \cdot \sqrt{2 - x^2} - x \cdot \frac{1}{2\sqrt{2-x^2}} \cdot (-2x)}{(2 - x^2)} = \frac{\sqrt{2 - x^2} + \frac{x^2}{\sqrt{2-x^2}}}{(2 - x^2)} \\
&= \frac{\sqrt{2 - x^2} + \frac{x^2}{\sqrt{2-x^2}}}{(2 - x^2)} \cdot \frac{\sqrt{2 - x^2}}{\sqrt{2 - x^2}} = \frac{(2 - x^2) + x^2}{(2 - x^2)^{3/2}} = \frac{2}{(2 - x^2)^{3/2}}.
\end{aligned}$$

Evaluating at $x = 1$ we have $2/(2 - 1)^{3/2} = 2$. Therefore, the slope of the tangent line is $m = 2$. Since the point $(1, 1)$ lies on the curve, the equation of the tangent line can be written in point-slope form:

$$y - 1 = 2(x - 1) \implies y = 2x - 1.$$

Problem 5: (Stewart Section 3.4) Suppose f is differentiable everywhere. Let $F(x) = f(e^x)$ and let $G(x) = e^{f(x)}$. Find expressions for $F'(x)$ and $G'(x)$.

For $F(x)$:

$$F'(x) = \frac{d}{dx}f(e^x) = f'(e^x) \cdot \frac{d}{dx}e^x = f'(e^x) \cdot e^x = e^x \cdot f'(e^x)$$

For $G(x)$:

$$G'(x) = \frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot \frac{d}{dx}f(x) = e^{f(x)} \cdot f'(x)$$

Challenge problem: (Stewart Section 3.4) One of the reasons for using radian measure in calculus is because it makes differentiation much easier. To see this, use the chain rule to show that if θ is measured in degrees, we may have

$$\frac{d}{d\theta}(\sin\theta) \neq \cos\theta.$$

Implicit Differentiation

Chapter 3, Section 5

Implicit differentiation concerns functions in which it is hard to separate variables.

For example, if $y^2 = x$, then $y = \sqrt{x}$ and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}.$$

Note that the dependent variable is a function of the independent variable. Then we can use the chain rule:

$$2y\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{2y}.$$

Implicit differentiation can be used to find these big results:

- $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2+1}}$
- $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$

....

Problem 1: (Stewart Section 3.5) Use implicit differentiation to find an equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$. *Fun fact: this curve is called a hyperbola!*

Given the equation $x^2 - xy - y^2 = 1$, we differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1) \implies 2x - (x\frac{dy}{dx} + y) - 2y\frac{dy}{dx} = 0 \implies 2x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} = 0.$$

Thus

$$(2x - y) - (x + 2y)\frac{dy}{dx} = 0 \implies (x + 2y)\frac{dy}{dx} = 2x - y \implies \frac{dy}{dx} = \frac{2x - y}{x + 2y}.$$

To find the slope of the tangent line at the point $(2, 1)$, we substitute $x = 2$ and $y = 1$ into $\frac{dy}{dx}$ to get $\frac{dy}{dx} = \frac{2(2)-1}{2+2(1)} = \frac{3}{4}$. So, the slope of the tangent line at the point $(2, 1)$ is $\frac{3}{4}$. Now, we can use point-slope form to find the equation of the tangent line. Plugging in $(x_1, y_1) = (2, 1)$ and $m = \frac{3}{4}$, we get:

$$y - 1 = \frac{3}{4}(x - 2) \implies y = \frac{3}{4}x - \frac{1}{2}.$$

Problem 2: (Stewart Section 3.5) Find the second derivative of the following function by implicit differentiation: $x^2 + 4y^2 = 4$.

Given $x^2 + 4y^2 = 4$, we differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(4) \implies 2x + 8y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}$$

Now, to find the second derivative, we differentiate $\frac{dy}{dx}$ with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{4y}\right) = -\left(\frac{4y\frac{d}{dx}(x) - x\frac{d}{dx}(4y)}{(4y)^2}\right) = -\left(\frac{4y - 4x\frac{-x}{4y}}{16y^2}\right) = \frac{x^2 - 4y^2}{16y^3}.$$

Problem 3: (Stewart Section 3.5) Suppose $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$. Find $f'(1)$.

We differentiate implicitly.

$$\frac{d}{dx}\left(f(x) + x^2[f(x)]^3\right) = \frac{d}{dx}(10) \implies f'(x) + 2x[f(x)]^3 + 3x^2[f(x)]^2 f'(x) = 0$$

Now, we'll plug in $x = 1$ and $f(1) = 2$ into the equation:

$$f'(1) + 2(1)(2)^3 + 3(1)^2(2)^2 f'(1) = 0 \implies f'(1) + 16 + 12f'(1) = 0$$

Solving for $f'(1)$:

$$13f'(1) = -16 \implies f'(1) = -\frac{16}{13}$$

Problem 4: (Stewart Section 3.5) Consider the equation below. Use implicit differentiation to find $\frac{dx}{dy}$ (i.e. treat y as the independent variable).

$$x^4y^2 - x^3y + 2xy^3 = 0.$$

To find $\frac{dx}{dy}$, we'll differentiate both sides of the equation with respect to y:

$$\frac{d}{dy}(x^4y^2) - \frac{d}{dy}(x^3y) + \frac{d}{dy}(2xy^3) = \frac{d}{dy}(0) \implies 4x^3y^2\frac{dx}{dy} + 2x^4y\frac{dy}{dy} - 3x^2y - x^3\frac{dy}{dy} + 2y^3 + 6xy^2\frac{dy}{dy} = 0.$$

Now, let's solve for $\frac{dx}{dy}$:

$$4x^3y^2\frac{dx}{dy} + 2x^4y - 3x^2y - x^3\frac{dy}{dy} + 2y^3 + 6xy^2\frac{dy}{dy} = 0.$$

$$4x^3y^2\frac{dx}{dy} - x^3\frac{dy}{dy} + 6xy^2\frac{dy}{dy} = -2x^4y + 3x^2y - 2y^3.$$

$$\frac{dx}{dy}(4x^3y^2 - x^3 + 6xy^2) = -2x^4y + 3x^2y - 2y^3 = \frac{-2x^4y + 3x^2y - 2y^3}{4x^3y^2 - x^3 + 6xy^2}.$$

Problem 5: (Stewart Section 3.5) Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is c.

Let $P(x_1, y_1)$ be any point on the curve. The equation of the tangent line to the curve at this point can be found using implicit differentiation:

$$\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(\sqrt{c}) \implies \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}}\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}.$$

So, the point-slope form of the equation of the tangent line is: $y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$. To find the x-intercept, set $y = 0$:

$$0 - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) \implies x = \sqrt{x_1}\sqrt{y_1} + x_1.$$

Similarly, to find the y-intercept, set $x = 0$:

$$y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(0 - x_1) \implies y = \sqrt{x_1}\sqrt{y_1} + y_1.$$

Summing up the intercepts:

$$x + y = \sqrt{x_1}\sqrt{y_1} + \sqrt{x_1}\sqrt{y_1} + x_1 + y_1 = x_1 + y_1 + 2\sqrt{x_1}\sqrt{y_1} = (\sqrt{x} + \sqrt{y})^2 = \sqrt{c}^2 = c,$$

as desired.

Challenge problem: (Stewart Section 3.5) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to find an expression for $(f^{-1})'(x)$.

Derivatives of Logarithmic Functions

Chapter 3, Section 6

In this section, we first discuss some differentiation rules for logarithms.

- $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$, and specifically $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$, and more generally $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$
- $\frac{d}{dx}(b^x) = b^x \ln b$, and more generally $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}(\ln b)g'(x)$
- $\frac{d}{dx}\ln|x| = \frac{1}{x}$.

We also learn two different ways to write the number e as a limit:

$$e = \lim_{x\to 0}(1+x)^{1/x} \text{ and } \quad e = \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n$$

Finally, we are introduced to *logarithmic differentiation* as a way to take the derivative of complex expressions. It involves taking the natural logarithm of both sides of an equation $y = f(x)$, differentiating implicitly, and finally solving for y'.

....

Problem 1: (Stewart Section 3.6) Use logarithmic differentiation to find the derivative of $y = x^x$.

To find the derivative of $y = x^x$ using logarithmic differentiation, we take the natural logarithm of both sides:

$$\ln(y) = \ln(x^x)x\ln(x).$$

Now, we differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(x)) \implies \frac{1}{y}\frac{dy}{dx} = \ln(x)+1$$

Now, we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln(x)+1) = \frac{dy}{dx} = x^x(\ln(x)+1)$$

Problem 2: (Stewart Section 3.6) Differentiate $f(x) = \ln\frac{1}{x}$.

Let $u = \frac{1}{x}$, then we have $f(x) = \ln(u)$. Now, let's find $\frac{du}{dx}$:

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Applying the chain rule:

$$\frac{d}{dx}\ln\frac{1}{x} = \frac{1}{u}\frac{du}{dx} = \frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{x}{1}\cdot\frac{1}{x^2} = -\frac{1}{x}.$$

Problem 3: (Stewart Section 3.6) Differentiate $f(x) = \ln\ln x$.

Let $u = \ln x$, then we have $f(x) = \ln(u)$. Now, let's find $\frac{du}{dx}$:

$$\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Applying the chain rule:

$$\frac{d}{dx}\ln(\ln x) = \frac{1}{u}\frac{du}{dx} = \frac{1}{\ln x}\left(\frac{1}{x}\right) = \frac{1}{x\ln x}.$$

Problem 4: (Stewart Section 3.6) Use logarithmic differentiation to find the derivative of $y = (x^2+2)^2(x^4+4)^4$.

We take the natural logarithm of both sides:

$$\ln(y) = \ln\left((x^2+2)^2(x^4+4)^4\right) = 2\ln(x^2+2) + 4\ln(x^4+4)$$

Now, we differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(2\ln(x^2+2) + 4\ln(x^4+4)) \implies \frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x^2+2}\cdot 2x + 4\frac{1}{x^4+4}\cdot(4x^3).$$

Now, we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(2\frac{1}{x^2+2}\cdot 2x + 4\frac{1}{x^4+4}\cdot(4x^3)\right) = (x^2+2)^2(x^4+4)^4\left(\frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}\right) = 4x(x^2+2) + 16x^3(x^4+4)^3.$$

Problem 5: (Stewart Section 3.6) Find y' if $x^y = y^x$.

To find y' if $x^y = y^x$ using logarithmic differentiation, we take the natural logarithm of both sides:

$$\ln(x^y) = \ln(y^x).$$

Using logarithmic properties, we rewrite the equation as:

$$y\ln(x) = x\ln(y) \implies \frac{d}{dx}(y\ln(x)) = \frac{d}{dx}(x\ln(y))$$

Using the product rule and the chain rule, we get:

$$y\frac{1}{x} + \ln(x)\frac{dy}{dx} = \ln(y) + x\frac{1}{y}\frac{dy}{dx}$$

Now, we solve for $\frac{dy}{dx}$:

$$\ln(x)\frac{dy}{dx} - x\frac{1}{y}\frac{dy}{dx} = \ln(y) - y\frac{1}{x} \implies \frac{dy}{dx}(\ln(x) - x\frac{1}{y}) = \ln(y) - y\frac{1}{x} \implies \frac{dy}{dx} = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}.$$

Problem 6: (Stewart Section 3.6) Find

$$\frac{d^9}{dx^9}(x^8\ln x).$$

We can find the first few derivatives:

$$\frac{d}{dx}(x^8\ln x) = x^7 + 8x^7\ln x$$

$$\frac{d^2}{dx^2}(x^8\ln x) = 15x^6 + 8\times 7x^6\ln x$$

$$\frac{d^3}{dx^3}(x^8\ln x) = 146x^5 + 8\times 7\times 6x^5\ln x$$

We see that by the seventh derivative, the polynomial term has power x^1; by the eighth derivative, the polynomial term will be 0. Thus the eighth derivative is $8\times 7\times 6\times \ldots \times 1\ln x = 8!\ln x$, so the ninth derivative is $8!/x$.

Challenge problem: (Stewart Section 3.6) Use the definition of the derivative to prove that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$

Exponential Growth and Decay

Chapter 3, Section 8

Sometimes the rate of change of something depends on the amount of it we have. Then we write

$$\frac{dy}{dt} = ky.$$

One example of exponential growth/decay is population growth, radioactive decay, Newton's law of cooling, compound interest, etc..

- The *differential equation* above expresses the rate of change of the DV is a constant times the DV.
- The solutions to this differential equation are exactly given by $y(t) = y(0)e^{kt}$.

One example of exponential growth is *continuous compounding interest*. For an interest rate r and an initial amount A_0 invested in an account, the amount of money after t years is given by

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

Differentiating, we get

$$\frac{dA}{dt} = rA_0e^{rt} = rA(t).$$

One example of exponential decay is Newton's Law of Cooling. For a surrounding temperature of T_s, a dependent temperature variable T, and a constant k, we get

$$\frac{dT}{dt} = k(T - T_s).$$

To find an equation for the temperature function $T(t)$, we may then use

$$T(t) = T_s + (T_0 - T_s)e^{-kt}.$$

. . . .

Problem 1: (Stewart Section 3.8) Suppose we can write $g'(t) = 13g(t)$ and $g(10) = 13$. Find an equation for $g(t)$.

To solve the differential equation $g'(t) = 13g(t)$ with the initial condition $g(10) = 13$, we can use the fact that this equation represents exponential growth. Given that $g'(t) = 13g(t)$, we can see that $k = 13$. So, the general solution to the differential equation is $g(t) = g_0e^{13t}$. Using the initial condition $g(10) = 13$, we can substitute $t = 10$ and $g(10) = 13$ into the general solution:

$$13 = g_0e^{13\cdot 10} = g_0e^{130} \implies g_0 = \frac{13}{e^{130}}$$

Therefore, the equation for $g(t)$ is:

$$g(t) = \frac{13}{e^{130}}e^{13t}.$$

Problem 2: (Stewart Section 3.8) A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

We know $\frac{dy}{dx} = 2y$, so the curve is given by $y = y_0e^{2x}$. We know that $5 = y_0e^{2\cdot 0} = y_0$, so the equation becomes $y = 5e^{2x}$.

Problem 3: (Stewart Section 3.8) The half-life of cesium-137 is 30 years. Suppose we have a 100mg sample.

(a) Find an expression for the mass that remains after t years.

(b) Find an expression for the amount that will remain after 100 years.

(c) Find an expression for the amount of time it will take for 1mg to remain.

(a) To find an expression for the mass that remains after t years, we can express the remaining mass $M(t)$ as an exponential decay from the initial mass M_0. Since cesium-137 has a half-life of 30 years, we can express $M(t)$ as:

$$M(t) = M_0 \left(\frac{1}{2}\right)^{\frac{t}{30}}.$$

Given that the initial mass M_0 is 100 mg, the expression becomes:

$$M(t) = 100 \left(\frac{1}{2}\right)^{\frac{t}{30}}.$$

(b) To find the amount that will remain after 100 years, we substitute $t = 100$ into the formula:

$$M(100) = 100 \left(\frac{1}{2}\right)^{\frac{100}{30}}.$$

(c) To find the amount of time it will take for 1 mg to remain, we set $M(t) = 1$ and solve for t:

$$1 = 100 \left(\frac{1}{2}\right)^{\frac{t}{30}}.$$

Using properties of logarithms:

$$\left(\frac{1}{2}\right)^{\frac{t}{30}} = \frac{1}{100} \implies \frac{t}{30} = \log_{\frac{1}{2}}\left(\frac{1}{100}\right) \implies t = 30 \cdot \log_{\frac{1}{2}}\left(\frac{1}{100}\right).$$

Problem 4: (Stewart Section 3.8) When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room, its temperature has increased to 10°C. What is the temperature of the drink after 50 minutes? When will its temperature be 15°C?

Substituting $T(25) = 10$, $T_s = 20$, and $T_0 = 5$ into Newton's law of cooling, we have:

$$10 = 20 + (5 - 20)e^{-25k}$$

Solving for k, we get:

$$e^{-25k} = \frac{10-20}{5-20} = \frac{-10}{-15} = \frac{2}{3} \implies -25k = \ln\left(\frac{2}{3}\right) \implies k = -\frac{1}{25}\ln\left(\frac{2}{3}\right).$$

Then our formula is

$$T(t) = 20 + (5-20)e^{-\frac{-1}{25}\ln\left(\frac{2}{3}\right)t} = 20 + (-15)(2/3)^{t/25}$$

Now, to find the temperature of the drink after 50 minutes, we substitute $t = 50$:

$$T(50) = 20 + (-15)(2/3)^{50/25} = \frac{40}{3}.$$

To find when the temperature of the drink will be 15°C, we set $T(t) = 15$ and solve for t:

$$15 = 20 - 15\left(\frac{2}{3}\right)^{\frac{t}{25}} \implies -5 = -15\left(\frac{2}{3}\right)^{\frac{t}{25}} \implies \frac{1}{3} = \left(\frac{2}{3}\right)^{\frac{t}{25}} \implies \frac{t}{25} = \log_{2/3}(1/3) \implies t = 25\log_{2/3} 1/3.$$

Challenge problem: (Stewart Section 3.8) Show with proof what the solutions to the differential equation $\frac{dy}{dx} = ky$ are. *Hint: you already know and used the end result, now prove that it's valid!*

Related Rates

Chapter 3, Section 9

Related rates problems deal with finding a relation between two variables in order to find how the change in one variable affects the change in another variable. Here are the general steps:

1. Find a relation between the variables (if possible, draw a diagram).
2. Differentiate using implicit differentiation.
3. Plug in known values and solve for your target variable.

The best way to master related rates problems is practice!

Problem 1: (Stewart Section 3.9) The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80mm?

The volume V of a sphere with radius r is given by the formula

$$V = \frac{4}{3}\pi r^3.$$

Differentiating both sides of this equation with respect to time t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substituting the given values, where $r = 40$ mm and $\frac{dr}{dt} = 4$ mm/s, we find

$$\frac{dV}{dt} = 4\pi(40)^2 \times 4 = 12800\pi.$$

So, when the diameter is 80 mm, the volume of the sphere is increasing at a rate of 25600π mm^3/s.

Problem 2: (Stewart Section 3.9) A particle is moving along a hyperbolic trajectory given by $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

Let x and y be the coordinates of the particle moving along the hyperbolic trajectory $xy = 8$. We are given that $\frac{dy}{dt} = -3$ cm/s when the particle is at the point $(4, 2)$. Differentiating the equation $xy = 8$ implicitly with respect to time t, we get

$$\frac{d}{dt}(xy) = \frac{d}{dt}8 \implies x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

Substituting the given values, where $x = 4$, $y = 2$, and $\frac{dy}{dt} = -3$, we can solve for $\frac{dx}{dt}$:

$$4(-3) + 2\frac{dx}{dt} = 0 \implies -12 + 2\frac{dx}{dt} = 0 \implies 2\frac{dx}{dt} = 12 \implies \frac{dx}{dt} = 6.$$

Problem 3: (Stewart Section 3.9) Suppose $y = \sqrt{2x+1}$, where x and y are functions of t. If $dx/dt = 3$, find dy/dt when $x = 4$.

Differentiating both sides of $y = \sqrt{2x+1}$ with respect to t, we get

$$\frac{dy}{dt} = \frac{1}{2\sqrt{2x+1}} \cdot \frac{d}{dt}(2x+1).$$

Given $\frac{dx}{dt} = 3$, we can find $\frac{d}{dt}(2x+1)$:

$$\frac{d}{dt}(2x+1) = 2\frac{dx}{dt} = 2 \times 3 = 6$$

Substituting $x = 4$ and $\frac{d(2x+1)}{dt} = 6$, we can find $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{1}{2\sqrt{2(4)+1}} \cdot 6 = \frac{1}{2\sqrt{9}} \cdot 6 = \frac{1}{6} \cdot 6 = 1.$$

Problem 4: (Stewart Section 3.9) If the minute hand of a clock has length r (in cm), find the rate at which it sweeps out area as a function of r.

The area swept out by the minute hand in time t is given by the sector of the circle it traces out. The area of a sector of a circle with radius r and central angle θ (in radians) is given by:

$$A = \frac{1}{2}r^2\theta.$$

Then we can implicitly differentiate to get

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

The minute hand traverses 2π radians every 60 minutes, so $\frac{d\theta}{dt}$ is $\frac{2\pi}{60}$ rad/min. Then

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{2\pi}{60} = \frac{\pi r^2}{60}\frac{\text{cm}^2}{\text{min}}.$$

Problem 5: (Stewart Section 3.9) A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

Let r be the radius of the cylindrical tank (in meters) and h be the height of the water in the tank (in meters). The volume of the cylindrical tank is given by the formula for the volume of a cylinder, $V = \pi r^2 h$. Given that the tank is being filled with water at a rate of $3\,\text{m}^3/\text{min}$, we can express the rate of change of volume ($\frac{dV}{dt}$) with respect to time (t) as $\frac{dV}{dt} = 3\,\text{m}^3/\text{min}$. We want to find how fast the height of the water ($\frac{dh}{dt}$) is increasing. To do this, we differentiate the volume formula with respect to time:

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt},$$

and therefore

$$3 = 25\pi\frac{dh}{dt}.$$

Then

$$\frac{dh}{dt} = \frac{3}{25\pi}.$$

Challenge problem: (Stewart Section 3.9) The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Linear Approximations and Differentials

Chapter 3, Section 10

Here are some big results from the previous lectures:

- The linear approximation, or tangent line approximation, for $f(x)$ is given by:

$$f(x) \approx f(a) + f'(a)(x - a).$$

  When finding this value, choose a value of a that is easy to calculate.

- The differential is given by $dy = f'(x)\, dx$.

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Problem 1: (Stewart Section 3.10) Estimate $(1.999)^4$ using a linear approximation or differentials.

We start by considering the function $f(x) = x^4$. We choose a value close to $x = 1.999$ to make the calculation easier. Let's choose $a = 2$. Let's find $f'(a)$:

$$f'(a) = 4a^3 = 4 \times 2^3 = 32.$$

We use the linear approximation formula:

$$f(x) \approx f(2) + 32(x - 2) = 2^4 + 32(x - 2) = 16 + 32(x - 2).$$

Now, we want to estimate $(1.999)^4$. Let $x = 1.999$:

$$f(1.999) \approx 16 + 32(1.999 - 2) = 16 + 32(-0.001) = 16 - 0.032 = 15.968.$$

Problem 2: (Stewart Section 3.10) Find the linearization $L(x)$ of $f(x) = \sin x$ at $a = \pi/6$.

First, let's find $f(a)$ and $f'(a)$:

$$f(a) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, f'(a) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Now, plug these values into the linearization formula:

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2} \cdot \frac{\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{12}\pi.$$

Problem 3: (Stewart Section 3.10) Find the differential of $y = xe^{-4x}$.

Using the chain rule, we get

$$\frac{dy}{dx} = 1 \cdot e^{-4x} + x \cdot (-4e^{-4x}) = e^{-4x} - 4xe^{-4x}.$$

So, the differential of $y = xe^{-4x}$ is:

$$dy = (e^{-4x} - 4xe^{-4x})dx.$$

Problem 4: (Stewart Section 3.10) Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2+5}$ for all x. Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

We use the linearization formula. For $g(1.95)$:

$$L(1.95) = g(2) + g'(2)(1.95-2) = -4 + \sqrt{2^2+5}(1.95-2).$$

For $g(2.05)$:

$$L(2.05) = g(2) + g'(2)(2.05-2) = -4 + \sqrt{2^2+5}(2.05-2).$$

Problem 5: (Stewart Section 3.10) Are the estimates in the previous problem too large or too small? Explain.

We examine the behavior of the function $g(x)$ near the point $x = 2$. Given that $g'(x) = \sqrt{x^2+5}$, we can observe that $g'(x)$ is an increasing function. Since $g'(2) = \sqrt{2^2+5} = \sqrt{9} = 3$, it indicates that $g(x)$ is increasing at a rate of 3 units per unit change in x.

Now observe that around $x = 2$, the slopes are positive but getting steeper as x increases. Thus the tangent lines lie below the curve, the estimates are too small.

Problem 6: (Stewart Section 3.10) The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk.

The formula for the area of a circular disk is $A = \pi r^2$, where r is the radius. We'll use differentials to find the maximum error in the calculated area. Let $r = 24$ cm be the actual radius of the disk, and let $dr = 0.2$ cm be the maximum error in measurement. Then

$$dA = 2\pi r\, dr$$

Substitute the given values:

$$dA = 2\pi(24) \times 0.2.$$

Thus

$$dA = 2\pi \times 24 \times 0.2 = 9.6\pi$$

So, the maximum error in the calculated area of the disk is approximately 9.6π square centimeters.

Challenge problem: (Stewart Section 3.10) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Maximum and Minimum Values

Chapter 4, Section 1

Today we'll be discussing extrema, which are maximum or minimum values. We learn some theorems that help us find locations and values of extrema.

- The Extreme Value Theorem says that if f is continuous on $[a, b]$, then f must have a maximum and a minimum.
- Fermat's Theorem says that if f has a local minimum or maximum at c, then c is a critical point.
- A local maximum is bigger than the values near it. An absolute maximum is bigger than all the other values (likewise for minima).
- Circle one: endpoints (can/cannot) be local minima or maxima.

Absolute extrema occur at critical points, which are where the derivative is 0 or undefined. On a closed interval, we must also consider the endpoints.

Problem 1: (Stewart Section 4.1) Find the absolute maximum and absolute minimum values of $f(t) = (t^2 - 4)^3$ on the interval $[-2, 3]$.

First, let's find the critical points by taking the derivative of $f(t)$ and setting it equal to zero:

$$f'(t) = \frac{d}{dt}[(t^2 - 4)^3] = 3(t^2 - 4)^2 \cdot 2t = 6t(t^2 - 4)^2.$$

Now, to find the critical points, we set $f'(t)$ equal to zero and solve for t:

$$6t(t^2 - 4)^2 = 0.$$

This equation is satisfied when $t = -2, 0,$ and 2. Next, we need to evaluate $f(t)$ at the critical points and the endpoints of the interval $[-2, 3]$:

$$f(-2) = (-2^2 - 4)^3 = 0 \qquad f(0) = (0^2 - 4)^3 = -64 \qquad f(2) = (2^2 - 4)^3 = 0 \qquad f(3) = (3^2 - 4)^3 = 125.$$

Therefore, the absolute maximum value of $f(t)$ on the interval $[-2, 3]$ is $f(3) = 125$ and it occurs at $t = 3$. The absolute minimum value of $f(t)$ on the interval $[-2, 3]$ is $f(0) = -64$ and it occurs at $t = 0$.

Problem 2: (Stewart Section 4.1) Find the absolute maximum and absolute minimum values of $f(x) = x - \sqrt[3]{x}$ on the interval $[-1, 4]$.

We get $f'(x) = 1 - \frac{1}{3}x^{-\frac{2}{3}}$. To find the critical points, we set $f'(x)$ equal to zero and solve for x:

$$1 - \frac{1}{3}x^{-\frac{2}{3}} = 0 \implies x^{-\frac{2}{3}} = 3 \implies \left(\frac{1}{x}\right)^{\frac{2}{3}} = 3 \implies \left(\frac{1}{x}\right)^{2} = 3^3 \implies \frac{1}{x^2} = 27 \implies x^2 = \frac{1}{27}.$$

Thus the critical points are $x = -\frac{1}{3\sqrt{3}}$ and $x = \frac{1}{3\sqrt{3}}$. Next, we evaluate $f(x)$ at the critical points and the endpoints of the interval $[-1, 4]$:

$$f(-1) = -1 - \sqrt[3]{-1} = -1 - (-1) = 0, f\left(\frac{1}{3\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} - \sqrt[3]{\frac{1}{3\sqrt{3}}} \approx 0.385$$

$$f\left(-\frac{1}{3\sqrt{3}}\right) = -\frac{1}{3\sqrt{3}} - \sqrt[3]{-\frac{1}{3\sqrt{3}}} \approx -0.385, f(4) = 4 - \sqrt[3]{4} \approx 2.413$$

So, the absolute maximum value of $f(x)$ on the interval $[-1, 4]$ is approximately 2.413, and it occurs at $x = 4$. The absolute minimum value is approximately -0.385, and it occurs at $x = -\frac{1}{3\sqrt{3}}$.

Problem 3: (Stewart Section 4.1) Prove that the function $f(x) = x^{101} + x^{51} + x + 1$ has neither a local maximum nor a local minimum.

We find the derivative and analyze its sign. Taking the derivative of $f(x)$ with respect to x, we get:

$$f'(x) = 101x^{100} + 51x^{50} + 1.$$

To find critical points, we set $f'(x) = 0$:

$$101x^{100} + 51x^{50} + 1 = 0.$$

However, since all the terms are positive, $f'(x)$ is always positive, indicating that $f(x)$ is always increasing. Therefore, there are no critical points, and therefore no local maxima or minima.

Problem 4: (Stewart Section 4.1) Find the critical numbers of $f(x) = x^3 + 6x^2 - 15x$.

To find the critical numbers, we first compute the derivative:

$$f'(x) = 3x^2 + 12x - 15.$$

Next, we set $f'(x)$ equal to zero and solve for x:

$$3x^2 + 12x - 15 = 0 = 3(x^2 + 4x - 5) = 3(x + 5)(x - 1) = 0.$$

Setting each factor equal to zero, we find the critical numbers:

$$x = -5\text{and} \quad x = 1.$$

Problem 5: (Stewart Section 4.1) Find the absolute maximum and absolute minimum of $f(x) = \ln x$ on $[1, 2]$.

To find the absolute maximum and absolute minimum of $f(x) = \ln x$ on the interval $[1, 2]$, we need to evaluate the function at the critical points and endpoints. First, let's find the critical points by setting the derivative equal to zero:

$$f'(x) = \frac{1}{x} = 0$$

The derivative is never zero and is not undefined on our interval, so there are no critical points in the interval $[1, 2]$. Next, we evaluate the function at the endpoints:

$$f(1) = \ln 1 = 0$$

$$f(2) = \ln 2$$

Since $\ln 2 > \ln 1$, we get that $\ln 2$ is the absolute maximum and $\ln 1$ is the absolute minimum.

Challenge problem: (Stewart Section 4.1) If a and b are positive numbers, find the maximum vale of $f(x) = x^a(1 - x)^b$, where $0 \leq x \leq 1$.

The Mean Value Theorem

Chapter 4, Section 2

In this section, we learn two important theorems about values of derivatives.

- **Rolle's theorem**: if f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is some $c \in (a,b)$ such that $f'(c) = 0$.
- **The Mean Value theorem:** if f is continuous on $[a,b]$ and differentiable on (a,b), then there is some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We can also characterize graphs based on their derivatives.

- If the derivative is always 0 for a continuous function, then that function is a constant function.
- If $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x) = g(x) + c$.

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Problem 1: (Stewart Section 4.2) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1,1)$ such that $f'(c) = 0$. Explain why this does *not* contradict Rolle's theorem.

We first evaluate $f(-1)$ and $f(1)$:

$$f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0, f(1) = 1 - 1^{2/3} = 1 - 1 = 0$$

So, $f(-1) = f(1)$. Now, let's find $f'(x)$:

$$f'(x) = \frac{d}{dx}(1 - x^{2/3}) = -\frac{2}{3}x^{-1/3}.$$

From this formula, we can see is no value of x in the interval $(-1,1)$ such that $f'(x) = 0$. This observation does not contradict Rolle's theorem because Rolle's theorem requires that the function be continuous on $[a,b]$ and differentiable on (a,b). However, in this case, $f(x) = 1 - x^{2/3}$ is not differentiable at $x = 0$, so the conditions of Rolle's theorem are not met.

Problem 2: (Stewart Section 4.2) Show that $2x + \cos x = 0$ has exactly one real solution.

For any value of x, $2x$ is a continuous function. Additionally, $\cos x$ is also continuous for all real numbers x. Therefore, their sum $2x + \cos x$ is continuous on the entire real number line. Now, let's consider the function $f(x) = 2x + \cos x$. We can see that $f(0) = 2 \cdot 0 + \cos 0 = 1$, and $f(-\pi) = -2\pi + \cos -\pi = 2\pi - (-1)$.

Since $f(x)$ is continuous on the interval $[0,\pi]$, and $f(0) = 1$ and $f(-\pi) = -2\pi + 1$ have opposite signs, by the Intermediate Value Theorem, there exists at least one real number in the interval $(0,\pi)$ whose functional value is 0, meaning there is at least one real solution. But if there were more than one real solution, Rolle's theorem would tell us that there is some number c in $(-\pi, 0)$ such that $f'(c) = 0$. Taking the derivative of $f(x)$, we get $f'(x) = 2 - \sin x$. But since $|\sin x| \leq 1$, we have $f'(x) \geq 1 > 0$ for all $x \in (0,\pi)$. Therefore, $f(x)$ is strictly increasing in the interval $(0,\pi)$, implying that the derivative is never zero. Hence, the equation $2x + \cos x = 0$ has exactly one real solution.

Problem 3: (Stewart Section 4.2) Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

First, we verify the conditions required by Rolle's theorem. The function $f(x) = x^3 - 15x + c$ is continuous on the closed interval $[-2, 2]$, and the function $f(x)$ is differentiable on the open interval $(-2, 2)$. If $f(x)$ has two distinct roots in the interval $[-2, 2]$, then by Rolle's theorem, there must exist at least one point c in $(-2, 2)$ such that $f'(c) = 0$. Let's find the derivative of $f(x)$:

$$f'(x) = 3x^2 - 15.$$

Now, let's find the critical points by setting $f'(x) = 0$:

$$3x^2 - 15 = 0 \implies x^2 = 5 \implies x = \pm\sqrt{5}.$$

Both $\pm\sqrt{5}$ are outside the interval $[-2, 2]$, so there are no critical points in the interval $[-2, 2]$. Since there are no critical points in $(-2, 2)$, Rolle's theorem does not apply. Therefore, if $f(x)$ has two distinct roots in the interval $[-2, 2]$, there is no guarantee that there exists a point c in $(-2, 2)$ where $f'(c) = 0$. Hence, the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

Problem 4: (Stewart Section 4.2) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

To find the smallest possible value of $f(4)$, we can use the Mean Value Theorem (MVT). According to MVT, if $f(x)$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$, then there exists at least one point c in $(1, 4)$ such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} \implies f(4) - f(1) = f'(c) \times (4 - 1).$$

Then

$$f(4) - f(1) \geq 2 \times (4 - 1) \implies f(4) - 10 \geq 6.$$

Solving for $f(4)$, we get $f(4) \geq 16$.

Problem 5: (Stewart Section 4.2) Show that $\sin x < x$ if $0 < x < 2\pi$.

We know that the function $f(x) = \sin x$ is continuous and differentiable on the interval. Now note that for any value of x, we get that there is some number c in $(0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$, so $\sin x = x \cos c$. But $\cos c < 1$ for $0 < c < 2\pi$, meaning $\sin x < 1 \times x = x$.

Challenge problem: (Stewart Section 4.2) Show that a polynomial of degree n has at most n real roots using the results from this section.

How Derivatives Affect the Shape of a Graph

Chapter 4, Section 3

The first derivative helps us decide if a function is increasing or decreasing.

$$\text{On a given interval, we say } f \text{ is } \begin{cases} \text{increasing} & \text{if } f'(x) > 0 \\ \text{decreasing} & \text{if } f'(x) < 0 \end{cases} \quad \text{on that interval.}$$

The *first derivative test* is used to find extrema. Recall that c is a critical number if $f'(c)$ is 0 or undefined.

$$\text{We say } f \text{ has a } \begin{cases} \text{local maximum} & \text{at } c \text{ if } f' \text{ changes from positive to negative} \\ \text{local minimum} & \text{at } c \text{ if } f' \text{ changes from negative to positive} \end{cases} \quad \text{at } c.$$

But if f' does not change sign at c, then we cannot make a conclusion about extrema at c.

We can also define concavity in terms of second derivatives.

$$\text{On a given interval, we say } f \text{ is } \begin{cases} \text{concave up} & \text{if } f''(x) > 0 \\ \text{concave down} & \text{if } f''(x) < 0 \end{cases} \quad \text{on that interval.}$$

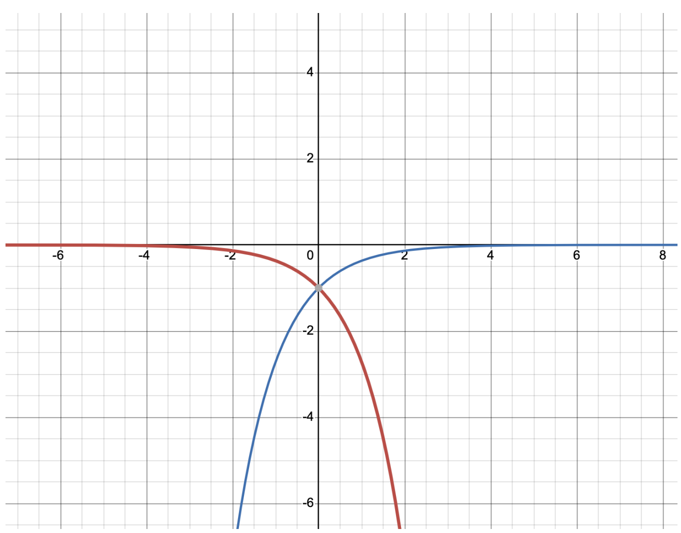
The second-derivative analogue for critical points is *inflection points*. A point P on a curve $f(x)$ is an inflection point if f is continuous at P and concavity changes sign.

Here is the *second derivative test* for a function f continuous near c:

$$\text{We say } f \text{ has a } \begin{cases} \text{local minimum} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) > 0 \\ \text{local maximum} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) < 0 \end{cases} \quad \text{at } c.$$

Problem 1: (Stewart Section 4.3) Sketch the graph of a function that satisfies $f'(x) < 0$ and $f''(x) < 0$ for all x. Then sketch the graph of another function that satisfies $g'(x) > 0$ and $g''(x) < 0$.

Below, they are displayed on the same graph. We have $f(x) = -e^x$ in red, and $g(x) = -e^{-x}$ in blue.



Problem 2: (Stewart Section 4.2) Find the critical numbers of $f(x) = x^4(x-1)^3$. What does the first derivative test tell you?

We need to find where the derivative $f'(x)$ equals zero or is undefined. First, let's find the derivative $f'(x)$:

$$f'(x) = \frac{d}{dx}[x^4(x-1)^3] = 4x^3(x-1)^3 + 3x^4(x-1)^2.$$

Now, let's set $f'(x)$ equal to zero and solve for x: $x^3(x-1)^2(4(x-1)+3x) = 0$. We see that the critical points are 0, 1, and $4/7$. Thus our test intervals for the first derivative test are $(-\infty, 0)$, $(0, \frac{4}{7})$, $(\frac{4}{7}, 1)$, and $(1, \infty)$.

1. For $x \in (-\infty, 0)$, choose a test point $x = -1$. Evaluating $f'(-1)$, we get $f'(-1) = 44$. Since $f'(-1)$ is positive, $f(x)$ is increasing in $(-\infty, 0)$.
2. For $x \in (0, 4/7)$, choose a test point $x = \frac{1}{2}$. Evaluating $f'(\frac{1}{2})$, we get $f'(\frac{1}{2}) = \frac{-1}{16}$. Since $f'(\frac{1}{2})$ is negative, $f(x)$ is decreasing in $(0, 4/7)$.
3. For $x \in (4/7, 1)$, choose a test point $x = \frac{3}{4}$. Evaluating $f'(\frac{3}{4})$, we get $f'(\frac{3}{4}) = \frac{135}{4096}$. Since $f'(\frac{3}{4})$ is positive, $f(x)$ is increasing in $(4/7, 1)$.
4. For $x \in (1, \infty)$, choose a test point $x = 2$. Evaluating $f'(1)$, we get $f'(1) = 80$. Since $f'(2)$ is positive, $f(x)$ is increasing in $(1, \infty)$.

Therefore, using the first derivative test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 4//7$.

Problem 3: (Stewart Section 4.3) Find the intervals of concavity of $f(x) = x^3 - 3x^2 - 9x + 4$.

First, we find the first derivative $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 - 9x + 4) = 3x^2 - 6x - 9.$$

Then we can find the second derivative $f''(x)$ by taking the derivative of $f'(x)$:

$$f''(x) = \frac{d}{dx}(3x^2 - 6x - 9) = 6x - 6.$$

To find the intervals of concavity, we need to find where $f''(x)$ is positive and where it is negative. Setting $f''(x) > 0$, we get:

$$6x - 6 > 0 \implies 6x > 6 \implies x > 1$$

Setting $f''(x) < 0$, we get:

$$6x - 6 < 0 \implies \implies x < 1$$

Therefore, the function $f(x) = x^3 - 3x^2 - 9x + 4$ is concave up for $x > 1$ and concave down for $x < 1$.

Problem 4: (Stewart Section 4.3) Find the intervals of concavity of $f(x) = \sin x + \cos x$ defined on $0 \leq x \leq 2\pi$. First, let's find the first and second derivatives of $f(x)$:

$$f'(x) = \frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x, f''(x) = \frac{d^2}{dx^2}(\cos x - \sin x) = -\sin x - \cos x$$

Now, let's find where $f''(x)$ is positive.

$$-\sin x - \cos x > 0 \implies -\sin x > \cos x \implies \tan x < 0 \implies x \in (\pi/2, \pi) \cup (3\pi/2, 2\pi).$$

since $\sin x$ and $\cos x$ are negative in the third and fourth quadrants. Then we find where $f''(x)$ is negative:

$$-\sin x - \cos x < 0 \implies \tan x > 0 \implies x \in (0, \pi/2) \cup (\pi, 3\pi/2).$$

Therefore, the function $f(x) = \sin x + \cos x$ is concave up on the intervals $(\pi/2, \pi) \cup (3\pi/2, 2\pi)$ and concave down on the intervals $(0, \pi/2) \cup (\pi, 3\pi/2)$.

Challenge problem: (Stewart Section 4.2) Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number x. Prove that f is increasing on the entire interval I.

Indeterminate Forms and L'Hôpital's Rule

Chapter 4, Section 4

L'Hôpital's rule helps us calculate limits having certain types of *indeterminate forms*. We need the following conditions in order to use l'Hôpital's rule:

1. f and g are differentiable near a
2. $g'(x) \neq 0$ if x is close to a
3. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

Then l'Hôpital's rule's says

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

For other types of indeterminate forms, we may come up with strategies to determine whether or not a limit exists.

- Indeterminate products ($0 \cdot \infty$): write the product as a quotient, and use l'Hôpital's rule.
- Indeterminate differences ($\infty - \infty$): convert the difference into a quotient by rationalizing or factoring
- Indeterminate powders (0^0, ∞^0, or 1^∞): take the natural logarithm and write the function as an exponential

Problem 1: (Stewart Section 4.4) Evaluate $\lim_{x\to 0^+} x\ln(x)$ using l'Hôpital's rule, and then use your result to find $\lim_{x\to 0^+} x^x$.

To evaluate $\lim_{x\to 0^+} x\ln(x)$ using L'Hôpital's rule, we rewrite it in the form $\frac{\infty}{\infty}$:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}.$$

Now applying L'Hôpital's rule, we differentiate the numerator and the denominator with respect to x:

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

Thus, $\lim_{x\to 0^+} x\ln(x) = 0$. Now, to find $\lim_{x\to 0^+} x^x$, we can rewrite it as $\lim_{x\to 0^+} e^{\ln(x^x)}$:

$$\lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1.$$

Problem 2: (Stewart Section 4.4) Evaluate

$$\lim_{x\to 0} \frac{x - \sin(x)}{x - \tan(x)}.$$

Notice that both the numerator and the denominator approach 0 as x approaches 0, so we can use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{x - \sin(x)}{x - \tan(x)} = \lim_{x\to 0} \frac{1 - \cos(x)}{1 - \sec^2(x)}.$$

Once again, both the numerator and denominator approach 0. Applying L'Hôpital's rule again (and again):

$$\lim_{x\to 0} \frac{1-\cos(x)}{1-\sec^2(x)} = \lim_{x\to 0} \frac{\sin(x)}{-2\tan(x)\sec^2(x)} = \lim_{x\to 0} \frac{\sin(x)}{-2\frac{\sin(x)}{\cos(x)}\frac{1}{\cos^2(x)}} = \lim_{x\to 0} \frac{1}{-2\frac{1}{\cos(x)}\frac{1}{\cos^2(x)}} = -\frac{1}{2}.$$

Problem 3: (Stewart Section 4.4) Evaluate

$$\lim_{x\to 0^+} x^{\sqrt{x}}.$$

Notice that

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln(x)} = e^{\lim_{x\to 0^+}\sqrt{x}\ln(x)}.$$

Now, we rewrite the limit as:

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}},$$

which allows us to use L'Hopital's rule, since we have the indeterminate $\frac{-\infty}{\infty}$. We differentiate the numerator and the denominator with respect to x:

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}} = \lim_{x\to 0^+} \frac{1/x}{-1/(2x^{3/2})} = \lim_{x\to 0^+} -2\sqrt{x} = 0.$$

Then $\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1$.

Problem 4: (Stewart Section 4.4) Evaluate

$$\lim_{x\to 0}(\csc x - \cot x).$$

We can rewrite the expression using common trigonometric identities:

$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x}.$$

Now, as $x \to 0$, $1-\cos x \to 0$ and $\sin x \to 0$. Then we have the indeterminate $\frac{0}{0}$, and therefore we can rewrite the expression as:

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \lim_{x\to 0} \tan x = 0.$$

Therefore,

$$\lim_{x\to 0}(\csc x - \cot x) = 0.$$

Problem 5: (Stewart Section 4.4) Evaluate

$$\lim_{x\to\infty} x^3 e^{-x^2}.$$

We notice that as x approaches ∞, x^3 approaches ∞ and e^{-x^2} approaches 0. Therefore, we can rewrite the limit as:

$$\lim_{x\to\infty} x^3 e^{-x^2} = \infty \cdot 0.$$

This form is indeterminate, so we can apply L'Hôpital's rule. We rewrite the expression as $\frac{x^3}{e^{x^2}}$, and differentiate the numerator and the denominator with respect to x:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}.$$

This is again indeterminate (∞/∞) we use L'Hôpital's rule again to get:

$$\lim_{x\to\infty} \frac{3x}{2xe^{x^2}} = \lim_{x\to\infty} \frac{3}{2e^{x^2} + 4x^2e^{x^2}} = 0.$$

Therefore, the limit evaluates to 0.

Challenge problem: (Stewart Section 4.4) Show that 0^∞ is not an indeterminate form using L'Hôpital's rule.

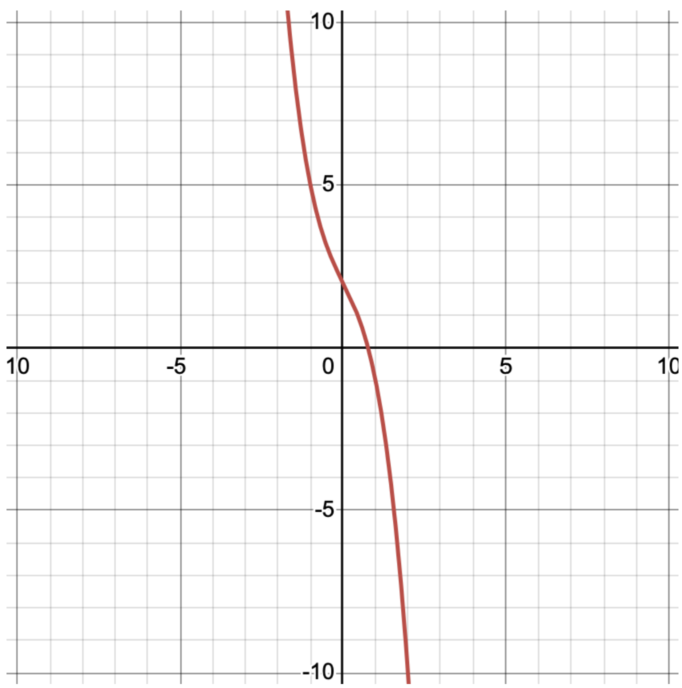
Summary of Curve Sketching

Chapter 4, Section 5

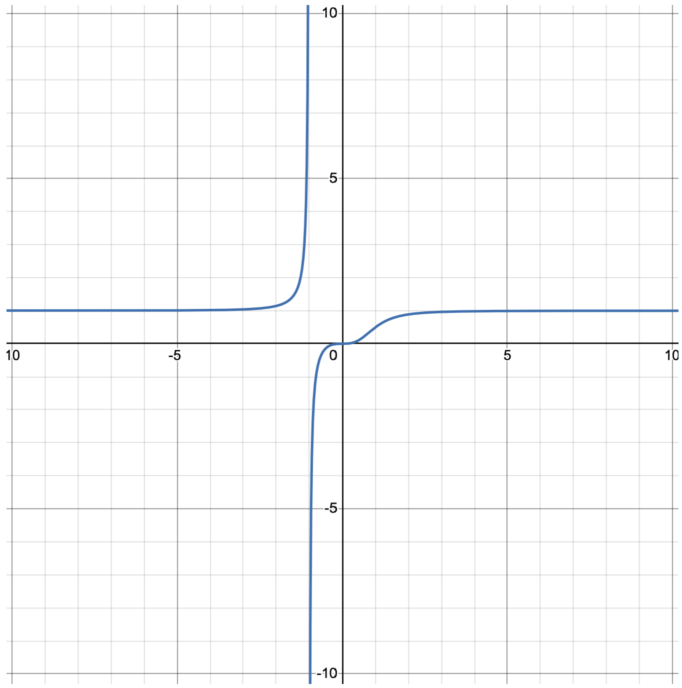
Curve sketching pieces together everything we learned in this chapter so far. Here are the tools that will help you graph a function $f(x)$:

- **Domain** This is where the function is defined.
- **Intercepts** The y-intercept can be found by setting $x = 0$ and the x-intercept can be found by setting $y = 0$.
- **Symmetry** If a function is even, then $f(-x) = f(x)$, and if a function is odd, then $f(-x) = -f(x)$.
- **Asymptotes**: If $\lim_{x\to\infty} f(x) = L$ or if $\lim_{x\to-\infty} f(x) = L$, then we call $y = L$ a horizontal asymptote. If $\lim_{x\to a^{\pm}} f(x) = \pm\infty$ or if $\lim_{x\to a} f(x) = \pm\infty$, we have a vertical asymptote at $x = a$.
- **Intervals of increase or decrease**: Find the first derivative and evaluate the sign of the derivative at the surrounding intervals.
- **Local extrema**: This follows from intervals of increase and decrease, or from using the second derivative test.
- **Concavity**: Find the second derivative and evaluate its sign at different intervals.

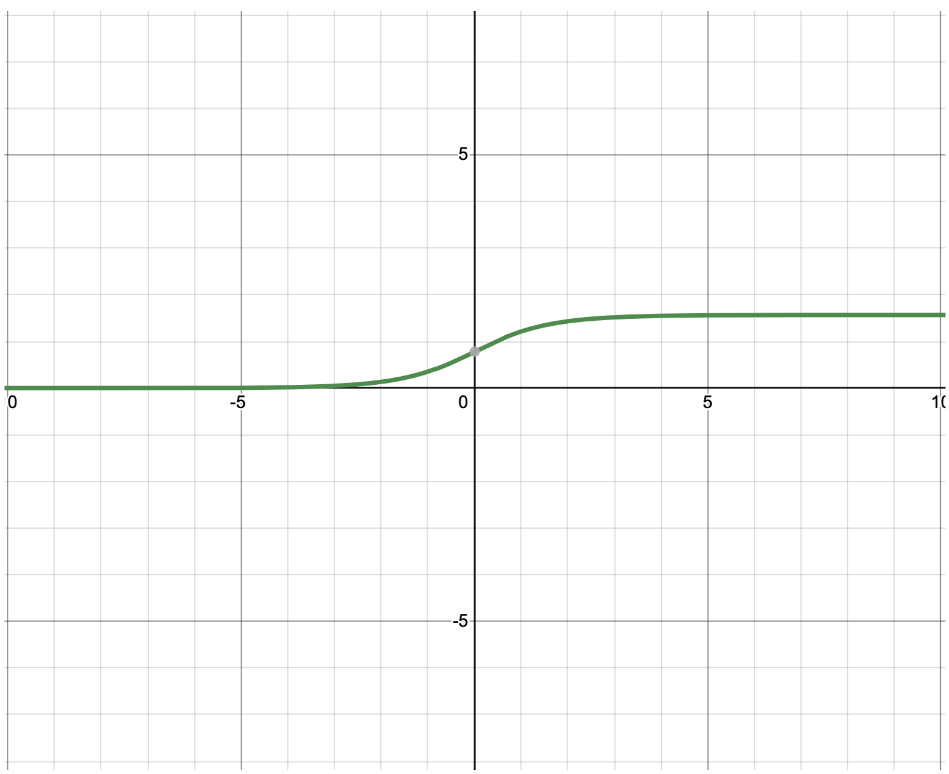
Problem 1: (Stewart Section 4.5) Sketch the curve $y = 2 - 2x - x^3$.



Problem 2: (Stewart Section 4.5) Sketch the curve $y = \frac{x^3}{x^3+1}$.



Problem 3: (Stewart Section 4.5) Sketch the curve $y = \arctan(e^x)$.



Optimization Problems

Chapter 4, Section 7

Optimization is about finding absolute extrema. Here are some pointers for optimization problems.

- Draw a diagram and identify given quantities on the diagram.
- Assign a variable to the value that is to be maximized or minimized, and express it in terms of the independent variable and the provided constants.
- Find the domain of the independent variable. If the domain is an **open** interval or **half-open** interval, then you must compute the value of f at open endpoints as well. If the absolute extremum appears at the value of an open endpoint, then an extremum may not exist.

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Problem 1: (Stewart Section 4.7) Find an equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant.

We consider the equation of the line in slope-intercept form, $y = mx + b$. Observe the line that minimizes the area of the triangle will also minimize the quantity that is twice the area. Let A be exactly this quantity—twice the area of the triangle. Then

$$A = \text{x-intercept} \times \text{y-intercept} = \left(\frac{-b}{m}\right) \times b = \frac{-b^2}{m}.$$

We are given that the line passes through the point $(3, 5)$, so substituting these coordinates into the equation $y = mx + b$, we get:

$$5 = 3m + b \quad \text{so} \quad b = 5 - 3m.$$

Substituting $b = 5 - 3m$ into the expression for the area A, we get:

$$A = -\frac{(5 - 3m)^2}{m} = -\frac{25}{m} + 30 - 9m.$$

To minimize this area, we take the derivative of A with respect to m, set it equal to zero, and solve for m:

$$\frac{dA}{dm} = -\frac{9m^2 - 25}{m^2}$$

Setting the derivative equal to 0, we solve for m to get $m = \frac{5}{3}$ and $m = -\frac{5}{3}$. We perform the first derivative test to see that $m = -\frac{5}{3}$ is where the absolute minimum occurs; we solve for b to get $b = 5 - 3(\frac{-5}{3}) = 10$. Thus the equation of our line is $y = \frac{-5}{3}x + 10$.

Problem 2: (Stewart Section 4.7) Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

Let x be the first positive integer and y be the second positive integer. To maximize the product xy, we can express x in terms of y: $x = 1000 - 4y$. Now, substitute this expression for x into the product xy:

$$xy = (1000 - 4y)y = 1000y - 4y^2.$$

To find the maximum value of xy, we take the derivative of xy with respect to y and set it equal to zero:

$$\frac{d}{dy}xy = 1000 - 8y = 0.$$

Solving this equation for y, we get:

$$y = \frac{1000}{8} = 125.$$

Substitute $y = 125$ back into the expression for x:

$$x = 1000 - 4(125) = 500.$$

So the two positive integers are $x = 500$ and $y = 125$.

Problem 3: (Stewart Section 4.7) What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

To find the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$, we need to find the points where the vertical distance is minimized. Let $D(x)$ be the vertical distance between the two curves at the point x. Then $D(x)$ is given by the absolute difference of the y-coordinates of the two curves:

$$D(x) = |(x^2 + 1) - (x - x^2)| = |2x^2 - x + 1|$$

To minimize $d(x)$, we find the critical points by setting its derivative equal to zero:

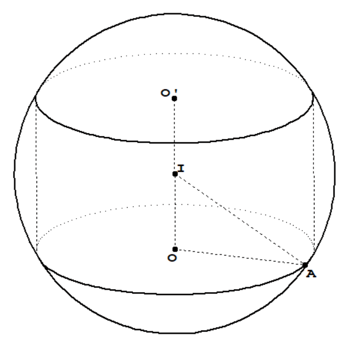
$$\frac{d}{dx}|2x^2 - x + 1| = 0$$

Note that the derivative can only equal zero if the quantity inside the absolute value has derivative 0. Thus we seek to solve

$$\frac{d}{dx}(2x^2 - x + 1) = 0 \implies (4x - 1) = 0 \implies x = \frac{1}{4}.$$

We can use the first derivative test (or test points) to see that the absolute minimum occurs at $x = \frac{1}{4}$, and $D(\frac{1}{4}) = \frac{7}{8}$.

Problem 4: (Stewart Section 4.7) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder. Let V be the volume of the cylinder, r be the radius of the sphere, and h be the height of the cylinder.



Since the cylinder is inscribed, there is a point A such that XA is the radius r of the sphere. Let the radius of the cylinder OA be given by R. Finally, let OX be h, such that the height of the cylinder is in fact $2h$. Then we use the Pythagorean theorem to see that

$$r^2 = h^2 + R^2,$$

which means $R^2 = r^2 - h^2$. The volume of a cylinder is

$$V = \pi R^2(2h) = 2\pi h(r^2 - h^2).$$

We take the first derivative so we can optimize:

$$V = \frac{d}{dh}V = 2\pi(r^2 - 3h^2) = 0.$$

Solving for h, we get $h^2 = r^2/3$, so $h = r/\sqrt{3}$ (since negative height does not make sense). We can use the first derivative test or test points to see that this value of h maximizes the volume. Thus the maximized volume is

$$V = 2\pi h(r^2 - h^2) = 2\pi\left(\frac{r}{\sqrt{3}}\right)\left(r^2 - \left(\frac{r}{\sqrt{3}}\right)^2\right) = \frac{2\pi r}{\sqrt{3}}\left(r^2 - \frac{r^2}{3}\right).$$

Challenge problem: (Stewart Section 4.7) Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

Newton's Method

Chapter 4, Section 8

In this section we learned about Newton's method, which is an algorithm for approximating roots of equations.

- First make a guess, call it x_1.
- Then find a formula for the x-intercept of the tangent line at x_1:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- Then keep going! Given the nth approximation, the $(n+1)$st approximation is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If a suitable x_1 is chosen, then the approximation should get closer as n increases. Otherwise, the approximations will diverge, and a new value of x_1 should be chosen.

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Problem 1: (Stewart Section 4.8) Starting with $x_1 = 2$, find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$.

We have $f'(x) = 3x^2 - 2$. Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(2)^3 - 2(2) - 5}{3(2)^2 - 2} = 2 - \frac{8 - 4 - 5}{12 - 2} = 2 - \frac{-1}{10}, = 2.1$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.1 - \frac{9.261 - 4.2 - 5}{13.86 - 2} = 2.1 - 0.00515 = 2.09485.$$

Problem 2: (Stewart Section 4.8) Explain why Newton's method doesn't work for finding the solution of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_1 = 1$.

The derivative is $f'(x) = 3x^2 - 3$. Now, let's evaluate $f(1)$ and $f'(1)$:

$$f(1) = (1)^3 - 3(1) + 6 = 1 - 3 + 6 = 4$$

$$f'(1) = 3(1)^2 - 3 = 3 - 3 = 0$$

Since $f'(1) = 0$, Newton's method fails because it involves dividing by $f'(x)$. Division by zero is undefined, so the iteration cannot proceed. In this case, the tangent line at $x = 1$ is horizontal, indicating that Newton's method won't converge near this point.

Problem 3: (Stewart Section 4.8) Find, correct to six decimal places, the solution of the equation $\cos(x) = x$. You may use a calculator.

This question is really asking for us to find a root of $f(x) = \cos(x) - x$. The derivative of $f(x)$ is $f'(x) = -\sin(x) - 1$. We approximate using a guess $x_1 = 0.5$. Applying Newton's method:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0.5 - \frac{\cos(0.5) - 0.5}{-\sin(0.5) - 1} \\ &\approx 0.755222417106. \end{aligned}$$

Repeating the process:

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &\approx 0.755222417106 - \frac{\cos(0.755222417106) - 0.755222417106}{-\sin(0.755222417106) - 1} \\ &\approx 0.73914166615 \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &\approx 0.73914166615 - \frac{\cos(0.73914166615) - 0.73914166615}{-\sin(0.73914166615) - 1} \\ &\approx 0.739085133921. \end{aligned}$$

$$\begin{aligned} x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} \\ &\approx 0.739085133921 - \frac{\cos(0.739085133921) - 0.739085133921}{-\sin(0.739085133921) - 1} \\ &\approx 0.739085133215. \end{aligned}$$

At this point, we see that the sixth digit after the decimal point has stopped changing. Thus 0.739085 is an approximation correct to six decimal places.

Challenge problem: (Stewart Section 4.8) Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$ defined on $0 \leq x \leq \pi$, correct to six decimal places. You may use a calculator.

Antiderivatives

Chapter 4, Section 9

Antiderivatives can be thought of as the opposite of derivatives, reversing differentiation. We call a function an antiderivative if $F'(x) = f(x)$. Also, if $F(x)$ is an antiderivative, then so is $F(x) + C$.

You should know the antiderivatives for common functions. Let $F'(x) = f(x)$. Here are some of them.

- For the function $cf(x)$, the general antiderivative is $cF(x) + C$.
- For the function x^n, the general antiderivative is $\frac{x^{n+1}}{n+1} + C$.
- For the function $\cos(x)$, the general antiderivative is $\sin x + C$.
- For the function $\sin(x)$, the general antiderivative is $-\cos x + C$.

⋆ Be sure to check out page 358 of the textbook for a complete list.

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Problem 1: (Stewart Section 4.9) Find the most general antiderivative of $f(x) = 7x^{2/5} + 8x^{-4/5}$, and check your answer by differentiating.

We can integrate each term separately. For the first term:

$$\int 7x^{2/5}\,dx = \frac{7}{\frac{2}{5}+1}x^{\frac{2}{5}+1} + C_1 = \frac{7}{\frac{7}{5}}x^{\frac{7}{5}} + C_1 = 5x^{\frac{7}{5}} + C_1.$$

For the second term $8x^{-4/5}$, we apply the power rule again:

$$\int 8x^{-4/5}\,dx = \frac{8}{-\frac{4}{5}+1}x^{-\frac{4}{5}+1} + C_2 = \frac{8}{\frac{1}{5}}x^{\frac{1}{5}} + C_2 = 40x^{\frac{1}{5}} + C_2.$$

Therefore, the most general antiderivative of $f(x)$ is:

$$\int f(x)\,dx = 5x^{\frac{7}{5}} + 40x^{\frac{1}{5}} + C.$$

To verify our answer, we can differentiate the antiderivative:

$$\frac{d}{dx}\left(5x^{\frac{7}{5}} + 40x^{\frac{1}{5}} + C\right) = \frac{35}{5}x^{\frac{7}{5}-1} + \frac{40}{5}x^{\frac{1}{5}-1} + 0 = 7x^{2/5} + 8x^{-4/5} = f(x).$$

Problem 2: (Stewart Section 4.9) Let $f''(x) = 2x + 3e^x$. Find the most general formula for $f(x)$.

We get

$$f'(x) = \int f''(x)\,dx = \int 2x + 3e^x\,dx = \int 2x\,dx + \int 3e^x\,dx = x^2 + 3e^x + C_1.$$

Integrating again,

$$f(x) = \int f'(x)\,dx = \int x^2 + 3e^x + C_1 = \int x^2\,dx + \int 3e^x\,dx + \int C_1\,dx = \frac{x^3}{3} + 3e^x + C_1x + C_2.$$

Problem 3: (Stewart Section 4.9) Find the most general antiderivative of the function below.

$$f(x) = 1 + 2\sin x + \frac{3}{\sqrt{x}}$$

We integrate each term:

$$\begin{aligned}\int f(x)\,dx &= \int (1 + 2\sin x + 3x^{-1/2})\,dx \\ &= \int 1\,dx + \int 2\sin x\,dx + \int 3x^{-1/2}\,dx \\ &= x - 2\cos x + 3 \cdot 2 \cdot x^{1/2} + C \\ &= x - 2\cos x + 6\sqrt{x} + C.\end{aligned}$$

Problem 4: (Stewart Section 4.9) For $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$, find f.

Integrating three times:

$$\int f'''(x)\,dx = \int \cos x\,dx \implies f''(x) = \sin x + C_1$$

$$\int f''(x)\,dx = \int (\sin x + C_1)\,dx \implies f'(x) = -\cos x + C_1 x + C_2$$

$$\int f'(x)\,dx = \int (-\cos x + C_1 x + C_2)\,dx \implies f(x) = -\sin x + \frac{1}{2}C_1 x^2 + C_2 x + C_3.$$

Applying the initial conditions:

$$\begin{aligned}f(0) = 1 &\Rightarrow C_3 = 1 \\ f'(0) = 2 &\Rightarrow C_2 = 3 \\ f''(0) = 3 &\Rightarrow C_1 = 3.\end{aligned}$$

Therefore, the function $f(x)$ is:

$$f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1.$$

Problem 5: (Stewart Section 4.9) A particle is moving at velocity $v(t) = \sin t - \cos t$. Suppose the position function is given as $s(t)$, where $s(0) = 0$. Find a formula for $s(t)$.

We integrate the velocity function with respect to time.

$$s(t) = \int v(t)\,dt = \int (\sin t - \cos t)\,dt = \int \sin t\,dt - \int \cos t\,dt = -\cos t - \sin t + C.$$

Applying the initial condition $s(0) = 0$:

$$s(0) = -\cos 0 - \sin 0 + C = 0 \Rightarrow C = 1.$$

Therefore, the position function $s(t)$ is:

$$s(t) = -\cos t - \sin t + 1.$$

Challenge problem: (Stewart Section 4.9) What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 seconds?

Areas and Distances

Chapter 5, Section 1

At the end of chapter 4, we defined antiderivatives. In this chapter, we will see that antiderivatives are closely connected to areas under curves. The *area problem* is about finding the area of the region that lies under the curve of any function $y = f(x)$ from $x = a$ to $x = b$.

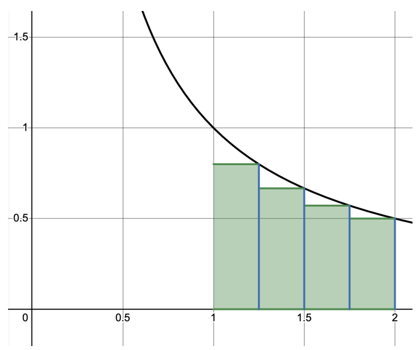
We can approximate areas under curves by drawing rectangles and adding up their areas. We may use left endpoints, right endpoints, or midpoints to find the height of each rectangle.

We say that the *area* A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles. Mathematically, we can write this as:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x]$$

The *distance problem* asks for the distance traveled by an object in a given time interval if the velocity is known at all times. We can calculate displacement by finding the area under the velocity curve, and we can calculate distance by finding the area under the absolute value of velocity curve.

Problem 1: (Stewart Section 5.1) Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is the calculated area an overestimate or an underestimate?



From the drawing, we see that the apprpximation will be an underestimate. The four rectangles have heights equal to the function values at the right endpoints of each subinterval.

$$f(x_1^*) = f\left(1+\frac{1}{4}\right) = f\left(\frac{5}{4}\right) = \frac{1}{\frac{5}{4}} = \frac{4}{5}, \quad f(x_2^*) = f\left(1+\frac{2}{4}\right) = f\left(\frac{6}{4}\right) = \frac{1}{\frac{6}{4}} = \frac{2}{3}$$
$$f(x_3^*) = f\left(1+\frac{3}{4}\right) = f\left(\frac{7}{4}\right) = \frac{1}{\frac{7}{4}} = \frac{4}{7}, \quad f(x_4^*) = f\left(1+\frac{4}{4}\right) = f\left(\frac{8}{4}\right) = \frac{1}{\frac{8}{4}} = \frac{1}{2}$$

The area of each rectangle is $\Delta x \times$ height.

$$A_1 = \frac{1}{4} \times \frac{4}{5} = \frac{1}{5}, \quad A_2 = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$
$$A_3 = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}, \quad A_4 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

The total estimated area is the sum of the areas of the rectangles.

$$A \approx A_1 + A_2 + A_3 + A_4 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \approx 0.634523809524$$

Problem 2: (Stewart Section 5.1) Determine a region whose area is equal to the limit below.

$$\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}$$

We interpret this is a Riemann sum, where the width of each rectangle is $\frac{3}{n}$ and the height of the ith rectangle is $\sqrt{1+\frac{3i}{n}}$. For a right-endpoint approximation on $[1,4]$, notice that the right-endpoint of the ith rectangle is $1+\frac{3i}{n}$. Then using the function $f(x)=\sqrt{x}$ on $[1,4]$, the result follows.

Problem 3: (Stewart Section 5.1) Use the limit definition to find an expression for the area under the graph of $f(x)=\sqrt{\sin x}$ from 0 to π.

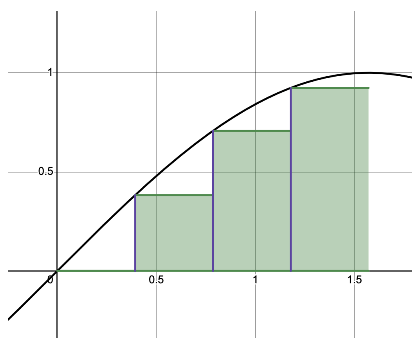
We choose $x_i^* = \frac{i\pi}{n}$, so

$$\Delta x = \frac{\pi}{n}, \quad x_i^* = \frac{i\pi}{n}.$$

Substituting these into the limit definition, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin\left(\frac{i\pi}{n}\right)} \cdot \frac{\pi}{n}.$$

Problem 4: (Stewart Section 5.1) Estimate the area under the graph of $f(x)=\sin x$ from $x=0$ to $x=\pi/2$ using four approximating rectangles and left endpoints. Sketch the graph and the rectangles. Is the calculated area an overestimate or an underestimate?



From the drawing, we see that the apprpximation will be an underestimate. The four rectangles have heights equal to the function values at the left endpoints of each subinterval.

$$f(x_1^*) = f(0+0) = f(0) = \sin 0 = 0, \quad f(x_2^*) = f\left(0+\frac{\pi}{8}\right) = f\left(\frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$$
$$f(x_3^*) = f\left(0+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right), \quad f(x_4^*) = f\left(0+\frac{3\pi}{8}\right) = f\left(\frac{3\pi}{8}\right) = \sin\left(\frac{3\pi}{8}\right)$$

The area of each rectangle is $\Delta x \times$ height.

$$A_1 = \frac{\pi}{8}\times 0 = 0, \quad A_2 = \frac{\pi}{8}\times \sin\left(\frac{\pi}{8}\right)$$
$$A_3 = \frac{\pi}{8}\times \sin\left(\frac{\pi}{4}\right), \quad A_4 = \frac{\pi}{8}\times \sin\left(\frac{3\pi}{8}\right)$$

The total estimated area is the sum of the areas of the rectangles.

$$A \approx A_1 + A_2 + A_3 + A_4 = 0 + \frac{\pi}{8}\times\sin\left(\frac{\pi}{8}\right) + \frac{\pi}{8}\times\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{8}\times\sin\left(\frac{3\pi}{8}\right) \approx 0.790766260123$$

Challenge problem: (Stewart Section 5.1) Find an expression for the area under the curve $y=x^2$ as a limit. Then use a formula for the sum of squares of natural numbers to evaluate the limit.

The Definite Integral

Chapter 5, Section 2

In the previous section, we learned about how rectangles can be used to find sums. In this section, we learn that these approximations are called *Riemann sums*.

The *definite integral* is defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We can only say a function can be integrated if it meets certain conditions.

Conditions for integrability

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$. In other words, the definite integral $\int_a^b f(x)\,dx$ exists.

In this case, for $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$, we may write:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

Certain summation rules can help us evaluate definite integrals from limits. we have

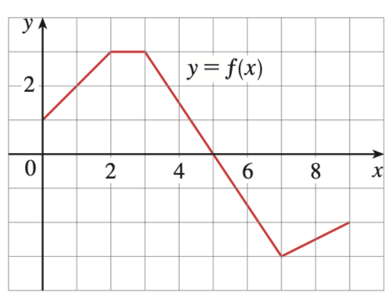
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Definite integrals respect many of the same properties as summations. For example, we may split up the bounds of integration:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Be sure to check out section 5.2 in the book for more of these properties!

Problem 1: (Stewart Section 5.2) Use the graph below to find $\int_0^2 f(x)\,dx$, $\int_0^5 f(x)\,dx$, and $\int_5^7 f(x)\,dx$.



- $\int_0^2 f(x)\,dx$ is made up of a triangle and a rectangle. The sum of these areas is is $2 + 2 = 4$.
- Notice $\int_2^5 f(x)\,dx$ is made up of a rectangle and a triangle. The sum of these areas is $3 + 3 = 6$. We add this to $\int_0^2 f(x)\,dx$ to get $6 + 4 = 10$.
- We have $\int_5^7 f(x)\,dx$ is a triangle below the x-axis. Its area is 3, so the integral is -3.

Problem 2: (Stewart Section 5.2) Prove that

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}.$$

We wish to find the area under the curve $y = x$ from $x = a$ to $x = b$. We may assume that $a \leq b$, and this will not change our result (do you see why?).

This area will in fact be a trapezoid, with one base of length a, the other of length b, and height $a - b$. The area of the trapezoid is

$$\frac{1}{2}(a+b)(b-a) = \frac{1}{2}(b^2 - a^2).$$

Thus

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2).$$

Problem 3: (Stewart Section 5.2) Evaluate

$$\int_{-3}^{0} (1 + \sqrt{9 - x^2})\,dx.$$

We split it into two parts:

$$\int_{-3}^{0} 1\,dx + \int_{-3}^{0} \sqrt{9 - x^2}\,dx.$$

The first part is simply the integral of 1 from -3 to 0, which evaluates to $0 - (-3) = 3$. For the second part, notice that the integrand $\sqrt{9 - x^2}$ represents one quarter of a circle with radius 3 centered at the origin. Thus, this integral represents one quarter of a circle with radius 3. The area of a full circle with radius r is πr^2, so the area of a quarter circle with radius 3 is $\frac{1}{2}\pi(3^2) = \frac{9\pi}{4}$. Putting it all together, we have:

$$\int_{-3}^{0} (1 + \sqrt{9 - x^2})\,dx = 3 + \frac{9\pi}{4} = \frac{9\pi}{4} + 3.$$

Problem 4: (Stewart Section 5.2) Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3. \end{cases}$$

We split the integral into two parts:

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx.$$

For $0 \leq x < 3$, we are given that $f(x)$ is the constant function 3. Thus $\int_0^3 f(x)\,dx$ is simply $3 \times 3 = 9$.

For $3 \leq x \leq 5$, we are given that $f(x)$ is x. As we learned in Problem 2,

$$\int_3^5 f(x)\,dx = \int_3^5 x\,dx = \frac{5^2 - 3^2}{5 - 3} = \frac{16}{2} = 8.$$

Putting it all together:

$$\int_0^5 f(x)\,dx = 9 + 8 = 17.$$

Challenge problem: (Stewart Section 5.2) Express the limit below as a definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$$

The Fundamental Theorem of Calculus

Chapter 5, Section 3

The first part of this section's big theorem tells us about derivatives of definite integrals.

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function $g(x)$ below is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

$$g(x) = \int_a^x f(t)\,dt \qquad \text{defined on } a \le x \le b$$

We can rewrite this first part in Leibniz notation as follows:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

The second part relates definite integrals to antiderivatives.

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any antiderivative of f.

Recall that if $F(x)$ is an antiderivative of $f(x)$, so is $F(x) + C$. We can rewrite the second part as follows:

$$\int_a^x F'(t)\,dt = F(b) - F(a)$$

Problem 1: (Stewart Section 5.3) Find the derivative of the function below.

$$F(x) = \int_x^0 \sqrt{1+\sec t}\,dt$$

We can apply the Fundamental Theorem of Calculus after rewriting the function:

$$\frac{d}{dx}\left(\int_x^0 \sqrt{1+\sec t}\,dt\right) = \frac{d}{dx}\left(-\int_0^x \sqrt{1+\sec t}\,dt\right) = -\frac{d}{dx}\left(\int_0^x \sqrt{1+\sec t}\,dt\right) = -\sqrt{1+\sec x}.$$

Problem 2: (Stewart Section 5.3) Find the derivative of the function below.

$$F(x) = \int_1^{3x+2} \frac{t}{1+t^3}\,dt$$

We view the right-hand side as a composition of functions:

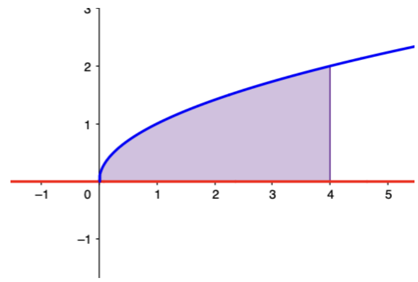
$$F'(x) = \frac{d}{dx}\left(\int_1^{3x+2} \frac{t}{1+t^3}\,dt\right) = \frac{d}{dx}\left(\int_1^{u(x)} \frac{t}{1+t^3}\,dt\right).$$

where $u(x) = 3x+2$. Using the Fundamental Theorem of Calculus and the chain rule, we get:

$$F'(x) = \frac{d}{dx}\left(\int_1^{u(x)} \frac{t}{1+t^3}\,dt\right) = \frac{u(x)}{1+u(x)^3}\cdot\frac{d}{dx}u(x) = \frac{3x+2}{1+(3x+2)^3}\cdot 3 = \frac{3(3x+2)}{1+(3x+2)^3}.$$

Problem 3: (Stewart Section 5.3) Sketch the area enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Find its area.

From the diagram, we see that we need to compute the definite integral of $y = \sqrt{x}$ from $x = 0$ to $x = 4$.



$$\text{Area} = \int_0^4 \sqrt{x}\, dx$$

To evaluate the integral, we can find its antiderivative:

$$\int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C.$$

Then, we evaluate it at the limits of integration:

$$\frac{2}{3}x^{3/2}\bigg|_0^4 = \frac{2}{3} \cdot 4^{3/2} - \frac{2}{3} \cdot 0^{3/2} = \frac{2}{3} \cdot 8 - 0 = \frac{16}{3}.$$

Therefore, the area enclosed by the curves $y = \sqrt{x}$, $y = 0$, and $x = 4$ is $\frac{16}{3}$.

Problem 4: (Stewart Section 5.3) Find the error in the calculation below, and fix it.

$$\int_{-1}^{2} \frac{4}{x^3}\, dx = -\frac{2}{x^2}\bigg]_{-1}^{2} = \frac{3}{2}$$

There is a discontinuity (asymptote) at $x = 0$, so we cannot evaluate the integral that way. The corrected calculation is as follows:

$$\int_{-1}^{2} \frac{4}{x^3}\, dx = \int_{-1}^{0} \frac{4}{x^3}\, dx + \int_{0}^{2} \frac{4}{x^3}\, dx$$

Attempts to evaluate this will result in undefined terms. Ultimately, because of the discontinuity, the integral will not converge and is thus undefined.

Problem 5: (Stewart Section 5.3) On what interval is the function below increasing?

$$f(x) = \int_0^x (t - t^2)e^{t^2}\, dt$$

We need to determine where its derivative $f'(x)$ is positive. Let's first find the derivative of $f(x)$ using the Fundamental Theorem of Calculus:

$$f'(x) = (x - x^2)e^{x^2}.$$

The function e^{x^2} is always positive, so we just need to analyze the sign of $x - x^2$. Notice $x - x^2 = 0$ when $x = 0$ or $x = 1$. Testing points, we see that $x - x^2$ is only positive in $(0, 1)$, so the function $f(x)$ is increasing on $(0, 1)$.

Challenge problem: (Stewart Section 5.3) Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x}.$$

Indefinite Integrals and the Net Change Theorem

Chapter 5, Section 4

As we learned, antiderivatives and derivatives are closely related:

$$\int f(x)\,dx = F(x) \quad \text{means } F'(x) = f(x)$$

We call $\int f(x)\,dx$ an *indefinite integral*, which represents a family of functions. We may find the most general antiderivative by adding a constant to a particular antiderivative.

In this section, we find out that second part of the Fundamental Theorem of Calculus can tell us something about net change.

Net Change Theorem

The integral of a rate of change is the net change.

Mathematically, we can write this as:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Problem 1: (Stewart Section 5.4) Verify by differentiation that the identity below holds.

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

We differentiate the right-hand side:

$$\frac{d}{dx}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right) = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2}(1 + \cos 2x)$$

Using the double-angle identity for cosine, $1 - \cos 2x = 2\sin^2 x$, we know $1 + \cos 2x = 2 - 2\sin^2 x = 2(1 - \sin^2 x)$, so

$$\frac{1}{2}(1 - \cos 2x) = \frac{1}{2} \cdot (2 - 2\sin^2 x) = 1 - \sin^2 x = \cos^2 x,$$

as desired.

Problem 2: (Stewart Section 5.4) Evaluate the integral below.

$$\int_0^\pi (5e^x + 3\sin x)\,dx$$

We get

$$\int_0^\pi (5e^x + 3\sin x)\,dx = \int_0^\pi 5e^x\,dx + \int_0^\pi 3\sin x\,dx = [5e^x]_0^\pi + [-3\cos x]_0^\pi = (5e^\pi - 5e^0) + (-3\cos\pi + 3\cos 0).$$

This evaluates to

$$5e^\pi - 5 + 3(-1) + 3 = 5e^\pi + 1.$$

Problem 3: (Stewart Section 5.4) The linear density of a rod of length 4m is given by $\rho(x) = 9 + 2\sqrt{x}$, measured in kilograms per meter, where x is measured in meters from one end of the rod. What is the total mass of the rod?

We integrate the linear density function over the length of the rod, which is from $x = 0$ to $x = 4$ meters. The total mass m of the rod is given by:

$$m = \int_0^4 \rho(x)\,dx = \int_0^4 (9 + 2\sqrt{x})\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4.$$

We evaluate the definite integral:

$$m = \left(9(4) + \frac{4}{3}(4)^{3/2}\right) - \left(9(0) + \frac{4}{3}(0)^{3/2}\right) = 36 + \frac{32}{3} = \frac{140}{3}.$$

Problem 4: (Stewart Section 5.4) Suppose the acceleration of a particle is given by $a(t) = t+4$, and its initial velocity is $v(0) = 5$. Find the velocity function $v(t)$ and the distance traveled from $t = 0$ to $t = 10$.

Given that $a(t) = t + 4$, we integrate it to find the velocity function:

$$v(t) = \int a(t)\,dt = \int (t+4)\,dt = \frac{1}{2}t^2 + 4t + C$$

Now, we use the initial velocity $v(0) = 5$ to solve for the constant C:

$$v(0) = \frac{1}{2}(0)^2 + 4(0) + C = C = 5$$

Therefore, the velocity function is $v(t) = \frac{1}{2}t^2 + 4t + 5$. To find the distance traveled from $t = 0$ to $t = 10$, we integrate the velocity function over this interval (notice that we do not need to take the absolute value because the velocity function is strictly positive):

$$\text{Distance} = \int_0^{10} v(t)\,dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right)\,dt = \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10} = \frac{1000 + 1500}{6} = \frac{2500}{6} = \frac{1250}{3} \approx 416.67.$$

Problem 5: (Stewart Section 5.4) Evaluate the integral below.

$$\int_{-1}^{1} t(1-t)^2\,dt$$

First, let's expand $t(1-t)^2$:

$$t(1-t)^2 = t(1 - 2t + t^2) = t - 2t^2 + t^3$$

Then

$$\int_{-1}^{1} t(1-t)^2\,dt = \int_{-1}^{1} (t - 2t^2 + t^3)\,dt = \left[\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right]_{-1}^{1}.$$

This is

$$\left(\frac{1}{2}(1)^2 - \frac{2}{3}(1)^3 + \frac{1}{4}(1)^4\right) - \left(\frac{1}{2}(-1)^2 - \frac{2}{3}(-1)^3 + \frac{1}{4}(-1)^4\right) = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) = \frac{-4}{3}.$$

Challenge problem: (Stewart Section 5.4) Find the general indefinite integral.

$$\int_{-1}^{2} (x - 2|x|)\,dx$$

The Substitution Rule

Chapter 5, Section 5

While we can usually find derivatives using some set of rules, finding antiderivatives can be a little bit harder. One strategy to evaluate antiderivatives is *substitution*.

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

We can modify this for definite integrals:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

We can also make observations about definite integrals of symmetric functions.

- If f is even, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$
- If f is odd, then
$$\int_{-a}^{a} f(x)\,dx = 0$$

Problem 1: Prove the above identities for even and odd functions.

1. Since f is even, $f(-x) = f(x)$, so
$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$
2. Since f is odd, $f(-x) = -f(x)$, so
$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

Problem 2: (Stewart Section 5.5) Evaluate the integral below by making the substitution $u = 2x$.

$$\int \cos 2x\,dx$$

We make the substitution $u = 2x$. Then, $du = 2\,dx$, and $dx = \frac{1}{2}du$. So the integral becomes

$$\int \cos(u)\cdot\frac{1}{2}\,du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(2x) + C.$$

Problem 3: (Stewart Section 5.5) If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

Let $u = x^2$. Then, $du = 2x\,dx$, or $dx = \frac{1}{2x}du$. When $x = 0$, $u = 0$, and when $x = 3$, $u = 9$. So the integral becomes

$$\begin{aligned}\int_0^3 xf(x^2)\,dx &= \int_0^9 \frac{1}{2}f(u)\,du \\ &= \frac{1}{2}\int_0^9 f(u)\,du \\ &= \frac{1}{2}\cdot 4 \\ &= 2.\end{aligned}$$

Problem 4: (Stewart Section 5.5) Evaluate the integral below.

$$\int_{-\pi/3}^{\pi/3} x^4 \sin x\; dx$$

We notice that x^4 is an even function, and $\sin(x)$ is an odd function. The product of an even function and an odd function is an odd function. Thus, $x^4\sin(x)$ is odd (we can also show this by evaluating the function at $-x$ and comparing it with the evaluation at x). The integral of an odd function over a symmetric interval about the origin is zero. Therefore,

$$\int_{-\pi/3}^{\pi/3} x^4\sin(x)\,dx = 0.$$

Problem 5: (Stewart Section 5.5) Evaluate the integral below.

$$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\;dx.$$

We can write this integral as

$$\int_{-2}^{2} x\sqrt{4-x^2}\;dx + 3\int_{-2}^{2}\sqrt{4-x^2}\;dx.$$

We integrate term by term. For the first, notice that the expression inside of the integral is an odd function, so it evaluates to 0. For the second integral, we have that $\sqrt{4-x^2}$ represents the upper-half of a circle of radius 2 centered at the origin. Thus

$$3\int_{-2}^{2}\sqrt{4-x^2}\;dx = 3\times\frac{2^2\pi}{2} = 6\pi.$$

Thus

$$\int_{-2}^{2}(x+3)\sqrt{4-x^2}\;dx = 6\pi.$$

Challenge problem: (Stewart Section 5.5) If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\;dx = \int_0^1 x^b(1-x)^a\;dx.$$

Areas Between Curves

Chapter 6, Section 1

In the previous chapter, we learned that the region bounded by the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

In this section, we find that the area of the region bounded by two curves $f(x)$ and $g(x)$ (where $f(x) \geq g(x)$) from $x = a$ to $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

If we only want to consider the positive area, we get

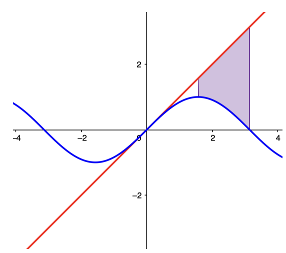
$$A = \int_a^b |f(x) - g(x)|\,dx$$

Sometimes, it might be easier to integrate with respect to y and treat x as the independent variable. Graphing the functions can help decide which is better.

Problem 1: (Stewart Section 6.1) Sketch the region enclosed by $y = \sin x$, $y = x$, $x = \pi/2$, and $x = \pi$, and find its area.

We get that the area is

$$\int_{\pi/2}^{\pi} x - \sin x\,dx.$$



The antiderivative of $x - \sin x$ is $\frac{x^2}{2} + \cos x$. We evaluate the definite integral:

$$\begin{aligned}
\int_{\pi/2}^{\pi} (x - \sin x)\,dx &= \left[\frac{x^2}{2} + \cos x\right]_{\pi/2}^{\pi} \\
&= \left(\frac{\pi^2}{2} + \cos(\pi)\right) - \left(\frac{(\pi/2)^2}{2} + \cos(\pi/2)\right) \\
&= \left(\frac{\pi^2}{2} - 1\right) - \left(\frac{\pi^2}{8} + 0\right) \\
&= \frac{3\pi^2}{8} - 1.
\end{aligned}$$

Problem 2: (Stewart Section 6.1) If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent?

We need to find

$$\text{Area} = \int_0^{10} 2200e^{0.024t} - 1460e^{0.018t}\,dt = \left(\frac{2200}{0.024}e^{0.024t} - \frac{1460}{0.018}e^{0.018t}\right)\Bigg|_0^{10} \approx 8868.$$

This area represents the net change in population over the given time interval. It's the difference between the number of people being born and the number of people dying over the period from $t = 0$ to $t = 10$ years.

Problem 3: (Stewart Section 6.1) Use integration to find the area of the triangle with the coordinates $(0,0)$, $(3,1)$, and $(1,2)$.

Graphing these points, we see that the line connecting $(0,0)$ and $(1,2)$ is $y = 2x$, the line connecting $(1,2)$ and $(3,1)$ is $y = -0.5x + 2.5$, and the line connecting $(0,0)$ and $(3,1)$ is $y = x/3$.

Then we see that the area is given by

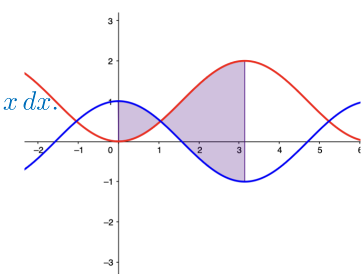
$$\int_0^1 2x - x/3\, dx + \int_1^3 2.5 - 0.5x - x/3\, dx.$$

Evaluating, we get

$$\begin{aligned}
&\int_0^1 (2x - \frac{x}{3})\, dx + \int_1^3 (2.5 - 0.5x - \frac{x}{3})\, dx \\
&= \left[x^2 - \frac{x^2}{6}\right]_0^1 + \left[2.5x - \frac{x^2}{4} - \frac{x^2}{6}\right]_1^3 \\
&= \left(1 - \frac{1}{6}\right) - (0 - 0) + \left(7.5 - \frac{9}{4} - \frac{9}{6}\right) - \left(2.5 - \frac{1}{4} - \frac{1}{6}\right) \\
&= \frac{5}{2}.
\end{aligned}$$

Problem 4: (Stewart Section 6.1) Sketch the region enclosed by $y = \cos x$, $y = 1 - \cos x$, $x = 0$, and $x = \pi$, and find its area.

First, we note that the graphs first intersect at $x = \pi/3$, since this is where $\cos x = 1 - \cos x$. From the diagram, we can see that the area is given by



$$\int_0^{\pi/3} \cos x - (1 - \cos x)\, dx + \int_{\pi/3}^{\pi} (1 - \cos x) - \cos x\, dx = \int_0^{\pi/3} 2\cos x - 1\, dx + \int_{\pi/3}^{\pi} 1 - 2\cos x\, dx.$$

Evaluating, we get

$$\begin{aligned}
\int_0^{\pi/3} (2\cos x - 1)\, dx + \int_{\pi/3}^{\pi} (1 - 2\cos x)\, dx &= [2\sin x - x]_0^{\pi/3} + [x - 2\sin x]_{\pi/3}^{\pi} \\
&= (2\sin(\pi/3) - \pi/3) - (0 - 0) + (\pi - 2\sin\pi) - (\pi/3 - 2\sin(\pi/3)) \\
&= \left(2 \cdot \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) + \left(\pi - \left(\frac{\pi}{3} - \sqrt{3}\right)\right) \\
&= \sqrt{3} - \frac{\pi}{3} + \frac{2\pi}{3}\sqrt{3} \\
&= 2\sqrt{3} + \frac{\pi}{3}.
\end{aligned}$$

Challenge problem: (Stewart Section 6.1) Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Volumes

Chapter 6, Section 2

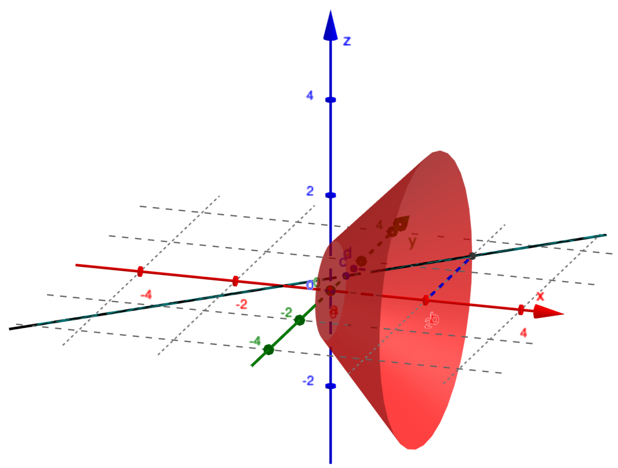
In this section, we treat volumes as sums of cross-sectional areas. Mathematically, for a a solid S lying between $x = a$ and $x = b$, if the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is continuous, then the volume of S is

$$V = \int_a^b A(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x$$

If we want to rotate about the y-axis, we may treat y as the dependent variable and rotate accordingly. Solids that are generated by rotation about axes are called solids of revolution .

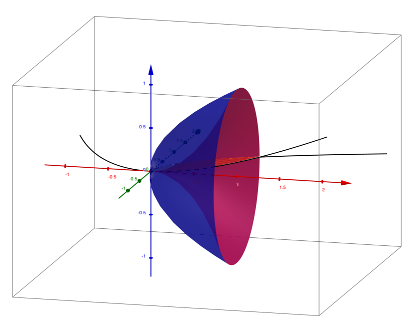
. . . .

Problem 1: (Stewart Section 6.2) Sketch the solid obtained by rotating the region bound by $y = x + 1$, $y = 0$, $x = 0$, and $x = 2$ about the x-axis.



Problem 2: (Stewart Section 6.2) Sketch the solid obtained by rotating the region bound by $y = x^2$, $y^2 = x$, and $x \geq 0$ about the x-axis. Then find its volume.

First, we see that the curves (2-dimensional) will intersect at $x = 1$, since this is where $x^2 = \sqrt{x}$. From the diagram, we can see that the volume will be given by



$$V = \int_0^1 \pi\sqrt{x}^2\,dx - \int_0^1 \pi x^2\,dx = \int_0^1 \pi x\,dx - \int_0^1 \pi x^2\,dx.$$

Now we evaluate the integrals:

$$\begin{aligned}
\int_0^1 \pi x\,dx - \int_0^1 \pi x^2\,dx &= \left[\frac{\pi}{2}x^2\right]_0^1 - \left[\frac{\pi}{3}x^3\right]_0^1 \\
&= \frac{\pi}{2}(1)^2 - \frac{\pi}{2}(0)^2 - \left(\frac{\pi}{3}(1)^3 - \frac{\pi}{3}(0)^3\right) \\
&= \frac{\pi}{2} - \frac{\pi}{3} \\
&= \frac{3\pi}{6} - \frac{2\pi}{6} \\
&= \frac{\pi}{6}.
\end{aligned}$$

Problem 3: (Stewart Section 6.2) Find the volume of a pyramid with height h and rectangular base with dimensions b and $2b$.

Observe that any horizontal cross-section will also be a rectangle with dimensions c and $2c$ for some c. Consider any given cross section; let y be the distance from the base to this "slice." Then above the slice, we get a pyramid of height $h - y$ and with dimensions c and $2c$. Drawing similar triangles with the vertical axis of the pyramid, we see that we may write $c = \frac{b(h-y)}{h}$.

At this point, we may integrate over each horizontal "slice" of the pyramid. A slice of thickness Δy has volume $c \times 2c \times \Delta x = 2c^2 \Delta y = 2\left(\frac{b(h-y)}{h}\right)^2 \Delta y$.

Then our volume is

$$V = \int_0^h 2\left(\frac{b(h-y)}{h}\right)^2 \, dy.$$

Simplifying, we get

$$\begin{aligned}
\int_0^h 2\left(\frac{b(h-y)}{h}\right)^2 \, dy &= 2\int_0^h \left(\frac{b(h-y)}{h}\right)^2 \, dy \\
&= 2\int_0^h \left(\frac{b^2(h-y)^2}{h^2}\right) \, dy \\
&= 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h \\
&= 2b^2 \left[h - h + \frac{h}{3}\right] \\
&= \frac{2b^2 h}{3}.
\end{aligned}$$

Problem 4: (Stewart Section 6.2) Use integration to find the volume of a right circular cone with height h and base radius r.

We can integrate over the cross-sectional area as we move along the height of the cone. At any given height y, the radius of the cone is $r_y = \frac{r}{h}y$, where y ranges from 0 to h (we can see this using similar triangles). The cross-sectional area of the cone at height y is given by the area of the circle with radius r_y, which is $A_y = \pi r_y^2 = \pi\left(\frac{r}{h}y\right)^2$. We can now integrate the cross-sectional area over the height of the cone to find the volume:

$$V = \int_0^h A_y \, dy = \int_0^h \pi\left(\frac{r}{h}y\right)^2 \, dy.$$

Simplifying:

$$V = \pi\int_0^h \left(\frac{r}{h}\right)^2 y^2 \, dy = \pi\left(\frac{r}{h}\right)^2 \int_0^h y^2 \, dy = \pi\left(\frac{r}{h}\right)^2 \left[\frac{y^3}{3}\right]_0^h = \pi\left(\frac{r}{h}\right)^2 \left(\frac{h^3}{3} - 0\right) = \frac{\pi r^2 h}{3}.$$

So, the volume of the right circular cone is $\frac{\pi r^2 h}{3}$.

Challenge problem: (Stewart Section 6.2) Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

Volumes by Cylindrical Shells

Chapter 6, Section 3

The *method of cylindrical shells* can help us find volumes of solids that have other solids removed from them.

For a thickness of Δr, average radius of r, and height of h, the volume of a single cylindrical shell is

$$V = 2\pi rh\Delta x$$

Then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b, using cylindrical shells, is

$$V = \int_a^b 2\pi x f(x)\, dx \quad \text{where } 0 \le a < b$$

Problem 1: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = \sqrt[3]{x}$, $y = 0$, and $x = 1$ about the y-axis.

This is a direct application of the formula above. Here we are integrating from $x = 0$ to $x = 1$. Our function is $f(x) = \sqrt[3]{x}$. Thus the volume is given by

$$\begin{aligned} V &= \int_0^1 2\pi x \sqrt[3]{x}\, dx \\ &= \int_0^1 2\pi x^{4/3}\, dx \\ &= \left[\frac{6\pi}{7} x^{7/3}\right]_0^1 = \frac{6\pi}{7}(1)^{7/3} - \frac{6\pi}{7}(0)^{7/3} \\ &= \frac{6\pi}{7}. \end{aligned}$$

Problem 2: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = x^3$, $y = 0$, $x = 1$, and $x = 2$ about the y-axis.

Once again, we use the formula from above. Here we are integrating from $x = 1$ to $x = 2$. Our function is $f(x) = x^{1/3}$. Thus the volume is given by

$$\begin{aligned} V &= \int_1^2 2\pi x \cdot x^3\, dx \\ &= \int_1^2 2\pi x^4\, dx \\ &= \left[2\pi \cdot \frac{x^5}{5}\right]_1^2 \\ &= \frac{2\pi}{5}(2)^5 - \frac{2\pi}{5}(1)^5 \\ &= \frac{64\pi}{5} - \frac{2\pi}{5} \\ &= \frac{62\pi}{5}. \end{aligned}$$

Problem 3: (Stewart Section 6.3) Use cylindrical shells to find the volume of a sphere of radius r.

We can use the method of cylindrical shells to find the volume of a hemisphere, and then double it. In particular, we get that the function representing this hemisphere is $y = \sqrt{r^2 - x^2}$, which means the volume is precisely

$$V = \int_0^r 2\pi x \sqrt{r^2 - x^2}\, dx.$$

Evaluating, we use the substitution $u = r^2 - x^2$ to get

$$\begin{aligned}
\int_0^r 2\pi x \sqrt{r^2 - x^2}\, dx &= -\pi \int_{r^2}^0 \sqrt{u}\, du \\
&= -\pi \left[\frac{2}{3}u^{3/2}\right]_{r^2}^0 \\
&= -\pi \left(\frac{2}{3} \cdot 0 - \frac{2}{3}r^3\right) \\
&= \frac{2}{3}\pi r^3.
\end{aligned}$$

We double this to get the volume of the whole sphere, which is $\frac{4}{3}\pi r^3$ as expected.

Problem 4: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = 4x - x^2$ and $y = x$ about the y-axis.

First, we find where these curves intersect. We solve $4x - x^2 = x$ to get

$$4x - x^2 - x = 0 \implies 3x - x^2 = 0 \implies x(3 - x) = 0 \implies x = 0,\ x = 3.$$

Testing points, we see that on the interval $(0, 3)$ the curve $y = 4x - x^2$ is above $y = x$. To find the volume generated by the bounded region, we may subtract the volume bounded by $y = x$ from the volume bounded by $y = 4x - x^2$. So we have

$$\begin{aligned}
V &= \int_0^3 2\pi x \cdot (4x - x^2)\, dx - \int_0^3 2\pi x \cdot x\, dx \\
&= \int_0^3 2\pi(4x^2 - x^3)\, dx - \int_0^3 2\pi x^2\, dx \\
&= 2\pi \int_0^3 (3x^2 - x^3)\, dx \\
&= 2\pi \left[x^3 - \frac{1}{4}x^4\right]_0^3 \\
&= 2\pi \left(3^3 - \frac{1}{4}(3^4)\right) - \left(0 - \frac{1}{4}(0^4)\right) \\
&= 2\pi \left(27 - \frac{81}{4}\right) - 0 \\
&= 2\pi \left(27 - \frac{81}{4}\right) \\
&= 2\pi \left(\frac{108}{4} - \frac{81}{4}\right) \\
&= \frac{54\pi}{4}.
\end{aligned}$$

Challenge problem: (Stewart Section 6.3) Use cylindrical shells to find the volume of a right circular cone with height h and base radius r.

Average Value of a Function

Chapter 6, Section 5

We define the *average value of f* on the interval $[a, b]$ to be

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The *Mean Value Theorem for Integrals* says that if f is continuous on $[a, b]$, then there must be some number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Equivalently, there must be some c such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

....

Problem 1: (Stewart Section 6.5) Find the average value of $f(x) = 3x^2 + 8x$ on $[-1, 2]$.

We directly apply the formula:

$$f_{\text{ave}} = \frac{1}{2-(-1)}\int_{-1}^2 (3x^2+8x)\,dx = \frac{1}{3}\left(\int_{-1}^2 3x^2\,dx + \int_{-1}^2 8x\,dx\right) = \frac{1}{3}\left(\left[x^3\right]_{-1}^2 + \left[4x^2\right]_{-1}^2\right) = \frac{9+12}{3} = 7.$$

Problem 2: (Stewart Section 6.5) Find the average value of $f(x) = \sqrt{x}$ on $[0, 4]$.

Once again, we have

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx \\ &= \frac{1}{4}\int_0^4 \sqrt{x}\,dx \end{aligned}$$

Now, we compute the integral:

$$\int_0^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(4^{3/2}-0) = \frac{2}{3}\cdot 8 = \frac{16}{3}$$

Therefore, the average value of $f(x) = \sqrt{x}$ on $[0, 4]$ is $\frac{1}{4} \times \frac{16}{3} = \frac{4}{3}$.

Problem 3: (Stewart Section 6.5) Find the average value of $f(u) = (\ln u)/u$ on $[1, 5]$.

We apply the formula to get

$$f_{\text{ave}} = \frac{1}{5-1}\int_1^5 (\ln u)/u\,du.$$

Let $v = \ln u$. Then $dv = 1/u\,du$. Then

$$\int (\ln u)/u\,du = \int v dv = \frac{v^2}{2} = \frac{(\ln u)^2}{2},$$

and

$$\frac{1}{5-1}\int_1^5 (\ln u)/u\,du = \frac{1}{4}\left(\frac{(\ln 5)^2}{2} - \frac{(\ln 1)^2}{2}\right) = \frac{(\ln 5)^2}{8}.$$

Problem 4: (Stewart Section 6.5) If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

The Mean Value Theorem for Integrals states that there exists at least one value c in $[1, 3]$ such that:

$$\int_1^3 f(x)\,dx = f(c) \cdot (3 - 1).$$

We can rewrite the equation as:

$$8 = f(c) \cdot (3 - 1)$$

Solving for $f(c)$, we have:

$$f(c) = \frac{8}{2} = 4.$$

Thus f takes on the value 4 at least once on the interval $[1, 3]$.

Problem 5: (Stewart Section 6.5) The linear density in a rod 8 meters long is $12/\sqrt{x+1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

We directly compute the average of the density function. Let $\rho(x) = \frac{12}{\sqrt{x+1}}$ be the linear density function. Then the average density $\bar{\rho}$ of the rod is given by:

$$\bar{\rho} = \frac{1}{8} \int_0^8 \frac{12}{\sqrt{x+1}}\,dx.$$

Now, we compute the integral. Using the substitution $u = x + 1$, we have $du = dx$ and:

$$\begin{aligned}\bar{\rho} &= \frac{12}{8} \int_1^9 u^{-1/2}\,du \\ &= \frac{3}{2}\left[2u^{1/2}\right]_1^9 \\ &= 3(3 - 1) \\ &= 3 \cdot 2 \\ &= 6 \text{ kg/m}.\end{aligned}$$

Problem 6: (Stewart Section 6.5) Find a c such that $f_{\text{ave}} = f(c)$ for $f(x) = 1/x$ on $[1, 2]$.

To find a value c such that $f_{\text{ave}} = f(c)$ for $f(x) = \frac{1}{x}$ on $[1, 2]$, we first need to compute the average value of $f(x)$ over the interval $[1, 2]$ and then find the corresponding value c where $f(c)$ equals the average value. For $f(x) = \frac{1}{x}$, we have:

$$f_{\text{ave}} = \int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2 = \ln(2) - \ln(1) = \ln(2).$$

Therefore, $f_{\text{ave}} = \ln(2)$. Now

$$\ln(2) = \frac{1}{c} \implies c = \frac{1}{\ln(2)}.$$

Challenge problem: (Stewart Section 6.5) Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives.

References

[1] Charles Schulz. *The Complete Peanuts Vol. 10: 1969–1970*. Vol. 10. Fantagraphics Books, 2008.

[2] James Stewart. *Calculus: Early Transcendentals*. Cengage Learning, 2012.

