Math 10A Fall 2024 Worksheet 20

November 14, 2024

1. Evaluate following integrals.
   (a) $\int_0^1 x dx$
   (b) $\int_0^1 x^3 dx$
   (c) $\int_0^1 e^{2x} dx$
   (d) $\int_0^{\pi/4} \sin(x) dx$
   (e) $\int_0^1 \frac{e^x}{e^x+1} dx$
   (f) $\int_{-1}^1 \arctan(x) dx$ (Hint: $\arctan(x)$ is an o...)

2. Assume that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$. Evaluate the following integrals:
   (a)
   $$\int_1^2 f'(x) dx.$$
   (b)
   $$\int_0^1 f'(x) e^{f(x)} dx.$$
   (c)
   $$\int_0^{\ln 2} e^x f'(e^x) dx.$$

3. Let
   $$F(x) = \int_0^{e^x} \sqrt{1+u^2} du.$$
   Find $F'(0)$.
   ($\star$) Can you also find $F(0)$?

1 Solutions

1. (a) $[\frac{1}{2}x^2]_0^1 = \frac{1}{2}$
   (b) $[\frac{1}{4}x^4]_0^1 = \frac{1}{4}$
   (c) $[\frac{1}{2}e^{2x}]_0^1 = \frac{1}{2}(e^2 - 1)$
   (d) $[-\cos x]_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}}$
   (e) Use substitution $u = e^x$ (or you can directly integrate if you can see the antiderivative). $\int_1^e \frac{1}{u+1} du = [\ln(u+1)]_1^e = \ln(e+1) - \ln(2)$
   (f) $f(x) = \arctan(x)$ is an odd function, so the integral over the symmetric domain $[-1, 1]$ is 0. But actually, it is possible to find antiderivative using integration by parts

$$\int \arctan(x)dx = x\arctan(x) - \int x\frac{1}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2).$$

2. (a) By the fundamental theorem of calculus, it is $f(2) - f(1) = 1$.
   (b) Use substitution $u = f(x)$ gives $\int_{f(0)}^{f(1)} e^u du = e - 1$.
   (c) Use substitution $u = e^x$ gives $\int_1^2 f'(u)du = f(2) - f(1) = 1$.

3. You can view $F(x)$ as a composition of e^x and $G(x) = \int_0^x \sqrt{1+u^2}du$, $F(x) = G(e^x)$. By the chain rule and the fundamental theorem of calculus, we get $F'(x) = G'(e^x)e^x = \sqrt{1+e^{2x}}e^x$ and $F'(0) = \sqrt{2}$.

   Computing $F(0) = \int_0^1 \sqrt{1+u^2}du$ is actually not easy - let me know if you find it (without cheating) then I may buy you coffee.