Vectors and dot products solutions

September 3rd, 2024

Here are some key ideas from sections 8.2 and 8.3.

- When we **scale** a vector by a constant c, we multiply its coordinates by a factor of c and multiply its magnitude by a factor of c.
- For a vector $\vec{a} = [a_1, a_2, \cdots a_n]$, its **magnitude** is given by

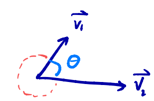
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

can be derived from the distance formula!

- For two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, their **dot product** is $\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$.
- Let θ be the angle between the two vectors lying in the interval $0 \leq \theta \leq \pi$ (in other words, the smaller of the two angles we could draw). Then we can calculate θ using the formula

divide by $|\vec{a}||\vec{b}|$ isolate θ

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \iff \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \iff \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$



always take the smaller θ!

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Problem 1: (Stewart & Day 8.2) Find a unit vector in the same direction as $[8, -1, 4]$.

My Attempt:

Solution:

The unit vector in the same direction as $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$.

Let $\vec{v} = [8, -1, 4]$. Then

$$|\vec{v}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{64 + 1 + 16} = \sqrt{81} = 9$$

So $\frac{\vec{v}}{|\vec{v}|} = \frac{[8, -1, 4]}{9} = \boxed{\left[\frac{8}{9}, \frac{-1}{9}, \frac{4}{9}\right]}$

Problem 2: (Stewart & Day 8.2) Find a vector that has the same direction as $[-2, 4, 2]$ but has length 6.

My Attempt:

Solution:

One idea is to find the unit vector (length 1) and then scale that vector by 6.

unit vector:

$$[-2, 4, 2] / \sqrt{(-2)^2 + 4^2 + 2^2} = [-2, 4, 2] / \sqrt{24}$$

scaled unit vector:

$$6\left([-2, 4, 2]/\sqrt{24}\right) = [-12, 24, 12]/\sqrt{24}$$

$$= \boxed{\left[-12/\sqrt{24},\ 24/\sqrt{24},\ 12/\sqrt{24}\right]}$$

$$= \left[-12/2\sqrt{6},\ 24/2\sqrt{6},\ 12/2\sqrt{6}\right]$$

$$= \boxed{\left[-\sqrt{6},\ 2\sqrt{6},\ \sqrt{6}\right]}$$

Problem 3: (LibreTexts) A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1,1,-1), Q(1,-1,1), R(-1,1,1)$, and $S(-1,-1,-1)$. Let O be the origin. Find the angle between vectors OS and OR (both beginning at O) that connect the carbon atom with the hydrogen atoms located at S and R. This is also called the bond angle.

My Attempt:

Solution:

Since O is the origin, $\vec{OR}$ and $\vec{OS}$ are $[-1,1,1]$ and $[-1,-1,-1]$, respectively.

Then from our "θ-formula":

$\vec{OR} \cdot \vec{OS} = |\vec{OR}| \cdot |\vec{OS}| \cdot \cos\theta$

Note $\vec{OR} \cdot \vec{OS} = [-1,1,1] \cdot [-1,-1,-1] = 1 + (-1) + (-1) = -1$

$|\vec{OR}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$|\vec{OS}| = \sqrt{3}$ (check this yourself!)

Then plug in values:

$-1 = \sqrt{3} \cdot \sqrt{3} \cdot \cos\theta$

$-1 = 3\cos\theta$

$-1/3 = \cos\theta$

$\boxed{\theta = \cos^{-1}(-1/3)} \approx \boxed{1.910 \text{ rad}, 109.471^\circ}$

Problem 4: (Stewart & Day 8.3) For what values of b are $[-6, b, 2]$ and $[b, b^2, b]$ perpendicular?

My Attempt:

Solution:

Perpendicular vectors are separated by an angle of $\pi/2$ ($90°$). Note that $\cos \pi/2 = 0$. Then

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cancel{\cos \pi/2}^{\,0} = 0.$

* So, perpendicular vectors have a dot product of 0. This problem asks for values of b such that the dot product is 0.

$[-6, b, 2] \cdot [b, b^2, b] = 0 \implies -6b + b^3 + 2b = 0 \implies b^3 - 4b = 0$

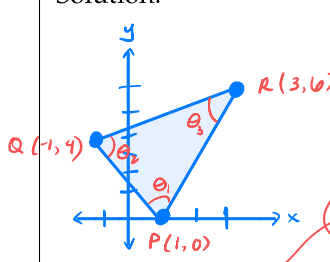
$b^3 - 4b = b(b^2 - 4) = 0 \implies \cancel{b = 0}$, $\boxed{b = 2}$, $\boxed{b = -2}$

not a solution since $b = 0$ produces the zero vector

Problem 5: (Stewart & Day 8.3) Find the three angles of the triangle with the vertices $(1,0)$, $(3,6)$, and $(-1,4)$.

My Attempt:

Solution:



I labeled the points P, Q, and R. I'll show you how to find angle θ_1. This is the angle between the vectors $\vec{PR}$ and $\vec{PQ}$, which are $[2,6]$ and $[-2,4]$, respectively. Then

$\vec{PR} \cdot \vec{PQ} = |\vec{PR}|\,|\vec{PQ}| \cos\theta_1$

We know $\vec{PR} \cdot \vec{PQ} = [2,6] \cdot [-2,4] = 20$

$|\vec{PR}| = \sqrt{4+36} = \sqrt{40}$, $|\vec{PQ}| = \sqrt{4+16} = \sqrt{20}$

Then $20 = \sqrt{40}\sqrt{20} \cos\theta_1$

$\frac{20}{\sqrt{20}\sqrt{40}} = \cos\theta_1$

$\frac{\sqrt{20}}{\sqrt{40}} = \cos\theta_1$

$\frac{1}{\sqrt{2}} = \cos\theta_1$

$\boxed{\theta_1 = \frac{\pi}{4}}$

Use $\vec{QP}$ and $\vec{QR}$ for θ_2, to get $\boxed{\theta_2 = \frac{\pi}{2}}$

Use $\vec{RQ}$ and $\vec{RP}$ for θ_3 to get $\boxed{\theta_3 = \frac{\pi}{4}}$.

Problem 5: Use a formula we discussed today to derive new ones.

1. Write a formula for the dot product of two parallel vectors.
2. Write a formula for the dot product of two vectors pointing in opposite directions.
3. Write a formula for the dot product of two perpendicular vectors.

My Attempt:

Solution:

① Parallel: $\theta = 0$, so $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{a}|\,|\vec{b}|$

$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|}$

② Opposite: $\theta = \pi$, so $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\pi = -|\vec{a}|\,|\vec{b}|$

$\boxed{\vec{a}\cdot\vec{b} = -|\vec{a}|\cdot|\vec{b}|}$

③ Perpendicular: $\theta = \frac{\pi}{2}$, so $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\frac{\pi}{2} = 0.$

$\boxed{\vec{a}\cdot\vec{b} = 0}$

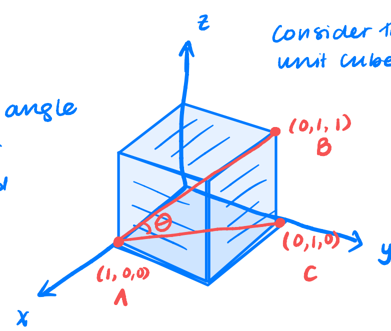
Problem 6: (Stewart & Day 8.3) Let $\vec{u}$ be a diagonal of some cube (going through the cube), and let $\vec{v}$ be a diagonal of one of its faces. Find the angle between $\vec{u}$ and $\vec{v}$. *Hint: what are the vectors?*

My Attempt:

Solution:

We want to find the angle between the vector $\vec{u}$ from A to B and $\vec{v}$ from A to C.

Consider the unit cube



$\vec{u}$ is $[-1, 1, 1]$

$\vec{v}$ is $[-1, 1, 0]$

$\vec{u}\cdot\vec{v} = 1 + 1 + 0 = 2$

$|\vec{u}| = \sqrt{3}$, $|\vec{v}| = \sqrt{2}$

$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta \Longrightarrow 2 = \sqrt{3}\sqrt{2}\cos\theta \Longrightarrow \sqrt{\frac{2}{3}} = \cos\theta$

$\boxed{\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)} \approx \boxed{0.615 \text{ rad}, 35.264^\circ}$

Problem 7: (Apostol 12.8) Prove that for two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, we have

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2.$$

(*) First, notice $(x+y)^2 + (x-y)^2 = x^2 + \cancel{2xy} + y^2 + x^2 - \cancel{2xy} + y^2 = 2x^2 + 2y^2$

My Attempt:

Solution:

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2$

$= \left(\sqrt{(a_1+b_1)^2 + \dots + (a_n+b_n^2)}\right)^2 + \left(\sqrt{(a_1-b_1)^2 + \dots + (a_n-b_n)^2}\right)^2$

$= (a_1+b_1)^2 + \dots + (a_n+b_n^2) + (a_1-b_1)^2 + \dots + (a_n-b_n)^2$

$= (a_1+b_1)^2 + (a_1-b_1)^2 + \dots + (a_n+b_n)^2 + (a_n-b_n)^2$ rearrange

$= 2a_1^2 + 2b_1^2 + \dots + 2a_n^2 + 2b_n^2$ Use the (*) from above

$= 2(a_1^2 + \dots + a_n^2) + 2(b_1^2 + \dots + b_n^2)$ rearrange

$= 2|\vec{a}|^2 + 2|\vec{b}|^2$ ✓

Challenge Problem: (Stewart & Day 8.3) Show that if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then the vectors $\vec{u}$ and $\vec{v}$ must have the same length. orthogonal $\Rightarrow$ Dot prod $= 0 \Rightarrow (u_1+v_1)(u_1-v_1) + \dots + (u_n+v_n)(u_n-v_n) = 0 \Rightarrow u_1^2 - v_1^2 + \dots + u_n^2 - v_n^2 = 0$

That means $u_1^2 + \dots + u_n^2 = v_1^2 + \dots + v_n^2$ so $\sqrt{u_1^2 + \dots + u_n^2} = \sqrt{v_1^2 + \dots + v_n^2}$. Magnitudes are equal ✓