Implicit differentiation

October 17th, 2024

Here are some key ideas from section 3.5.

- The chain rule says

  composition

  Given $F(x) = f(g(x))$, then

  $$F'(x) = f'(g(x)) \cdot g'(x)$$

- The chain rule helps us with implicit differentiation, which is used when we can't isolate the dependent variable.

  Let $y = 2x+3$. We can solve for dy/dx in two ways:

  Intuition: $\frac{d}{dx} = \frac{d}{\cancel{dy}} \cdot \frac{\cancel{dy}}{dx}$

  ① Expanding y^2:

  $$\frac{d}{dx} y^2 = \frac{d}{dx}(2x+3)^2$$
  $$= \frac{d}{dx}(4x^2+12x+9)$$
  $$= \underline{8x+12}$$

  ② Using implicit diff.

  $$\frac{d}{dx} y^2 = \frac{d}{dy} \cdot y^2 \cdot \frac{dy}{dx}$$
  $$= 2y \frac{dy}{dx}$$
  $$= 2(2x+3)(2) = 4(2x+3) = \underline{8x+12}$$

  We get the same answer!

- Here's an example:

  Let $x^3 + y^3 = 6x$. Solve for $\frac{dy}{dx}$.

  $$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx} 6x$$
  $$3x^2 + \frac{d}{dx} y^3 = 6$$
  $$3x^2 + 3y^2 \frac{dy}{dx} = 6$$
  $$3y^2 \frac{dy}{dx} = 6 - 3x^2$$
  $$\boxed{\frac{dy}{dx} = \frac{6-3x^2}{3y^2}}$$

  More examples

  $\frac{d}{dx} y^2 = 2y \frac{dy}{dx}$ $\quad$ $\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$

  $\frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}$ $\quad$ $\frac{d}{dx} \cos y = -\sin y \frac{dy}{dx}$

  $\frac{d}{dx} y^4 = 4y^3 \frac{dy}{dx}$ $\quad$ $\frac{d}{dx} e^y = e^y \frac{dy}{dx}$

Midterm practice (Vojta MT2 '22): Find the following derivatives.

a) $\frac{d}{dx} e^x \cos x$

b) $\frac{d}{dx} \frac{\sin x}{x^2+1}$.

My Attempt:

Solution:

a) $\left(\frac{d}{dx} e^x\right) \cdot \cos x + e^x \frac{d}{dx}(\cos x)$

$e^x \cdot \cos x + e^x(-\sin x)$ Leave it like this!

$e^x \cos x - e^x \sin x$

b) $\dfrac{(x^2+1)\cdot(\sin x)' - \sin x\,(x^2+1)'}{(x^2+1)^2}$

$\dfrac{(x^2+1)\cdot \cos x - \sin x\,(2x)}{(x^2+1)^2}$ Leave it like this!

Problem 1: Suppose $x^3 + y^3 = 1$. In this problem, we will find $\frac{dy}{dx}$ by implicit differentiation.

a) Write $\frac{d}{dx}$ on both sides.
Simplify the right hand side, and expand the left hand side with the sum rule.

b) Use a technique from the example to simplify $\frac{d}{dx}y^3$.

c) Solve for $\frac{dy}{dx}$.

My Attempt:

Solution:

a) $\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}1$

$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 0$

b) $3x^2 + 3y^2\frac{dy}{dx} = 0$

c) $3y^2\frac{dy}{dx} = -3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \boxed{\frac{-x^2}{y^2}}$

Problem 2: (Stewart 3.5) Suppose $2\sqrt{x} + \sqrt{y} = 3$. Use the same steps as in the previous problem to find $\frac{dy}{dx}$ by implicit differentiation.

My Attempt:

Solution:

Start with $2x^{1/2} + y^{1/2} = 3$

Then $\frac{d}{dx}(2x^{1/2} + y^{1/2}) = \frac{d}{dx}3$

$2\cdot\frac{1}{2}x^{-1/2} + \frac{d}{dx}y^{1/2} = 0$

$x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$

$\frac{1}{2}y^{-1/2}\frac{dy}{dx} = -x^{-1/2}$

$\frac{dy}{dx} = -2x^{-1/2}\cdot y^{1/2} = \boxed{\frac{-2\sqrt{y}}{\sqrt{x}}}$

Problem 3: (Stewart 3.5) Suppose $e^{x/y} = x - y$. Find $\frac{dy}{dx}$ by implicit differentiation.

My Attempt:

Solution:

$\frac{d}{dx}e^{x/y} = \frac{d}{dx}(x-y)$

$e^{x/y}(x/y)' = 1 - \frac{dy}{dx}$

$e^{x/y}\cdot\frac{y(1) - x\frac{dy}{dx}}{y^2} = 1 - \frac{dy}{dx}$

$e^{x/y}\left(y - x\frac{dy}{dx}\right) = y^2 - y^2\frac{dy}{dx}$

$ye^{x/y} - xe^{x/y}\frac{dy}{dx} = y^2 - y^2\frac{dy}{dx}$

$ye^{x/y} - y^2 = xe^{x/y}\frac{dy}{dx} - y^2\frac{dy}{dx}$

$= \frac{dy}{dx}(xe^{x/y} - y^2)$

$\frac{dy}{dx} = \boxed{\frac{ye^{x/y} - y^2}{xe^{x/y} - y^2}}$

Problem 4: (Stewart 3.7) Two important properties are

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b} \qquad \frac{d}{dx}\ln x = \frac{1}{x}.$$

Find

$$\frac{d}{dz}\ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}.$$

Here a is just a constant ☺

My Attempt:

Solution:

$$\frac{d}{dz}\ln\sqrt{\frac{a^2-z^2}{a^2+z^2}} = \frac{d}{dz}\ln\frac{\sqrt{a^2-z^2}}{\sqrt{a^2+z^2}}$$

$$= \frac{d}{dz}\left(\ln\sqrt{a^2-z^2} - \ln\sqrt{a^2+z^2}\right)$$

$$= \frac{d}{dz}\left(\ln(a^2-z^2)^{\frac{1}{2}} - \ln(a^2+z^2)^{\frac{1}{2}}\right)$$

$$= \frac{d}{dz}\left(\frac{1}{2}\ln(a^2-z^2) - \frac{1}{2}\ln(a^2+z^2)\right)$$

$$= \frac{1}{2}\cdot\frac{1}{a^2-z^2}\cdot -2z - \frac{1}{2}\cdot\frac{1}{a^2-z^2}\cdot 2z$$

$$= \boxed{\frac{-z}{a^2-z^2} - \frac{z}{a^2-z^2}}$$

Problem 5: (Stewart 3.7) Logarithmic differentiation is helpful when want to take the derivative of a complicated function (i.e. lots of products, fractions, exponents, etc.). Here are the steps we can use:

1. Take natural logarithms of both sides and use logarithm laws to simplify.
2. Implicitly differentiate with respect to x.
3. Solve for y'.

Fancy log properties of yore:

- $\log(ab) = \log a + \log b$
- $\log(a/b) = \log a - \log b$
- $\log(a^b) = b\log a$

Use these steps to differentiate

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^2}.$$

My Attempt:

Solution:

① $$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^2}\right)$$

$$= \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln\left((3x+2)^2\right)$$

$$= \ln x^{3/4} + \ln(x^2+1)^{\frac{1}{2}} - \ln\left((3x+2)^2\right)$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 2\ln(3x+2)$$

use log rules

② $$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 2\ln(3x+2)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}(\ln x)' + \frac{1}{2}\left(\ln(x^2+1)\right)' - 2\left(\ln(3x+2)\right)'$$

③ $$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 2\cdot\frac{1}{3x+2}\cdot 3$$

$$\boxed{\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{6}{3x+2}\right)}$$

Problem 6: If $y = \tan^{-1} x$, then $\tan y = x$. Use implicit differentiation to find the derivative of the inverse tangent function.

My Attempt:

Solution:

we want to find $\frac{dy}{dx}$.

$\frac{d}{dx}(\tan y) = \frac{d}{dx} x$

$\sec^2 y \frac{dy}{dx} = 1$

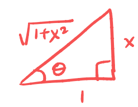
$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\tan^{-1}x)}$

simplify $\frac{1}{\sec^2(\tan^{-1}x)}$

let $\tan^{-1}x = \theta$

then $\tan\theta = x$

so $\frac{opp}{adj} = \frac{x}{1}$



Thus $\frac{1}{\sec^2(\tan^{-1}x)} = \frac{1}{\sec^2\theta}$

$= \frac{1}{\frac{1}{\cos^2\theta}}$

$= \cos^2\theta$ use the triangle!

$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \boxed{\frac{1}{1+x^2}}$

Problem 7: (Stewart 3.7) Find $\frac{d}{dx} x \arctan\sqrt{x}$.

My Attempt:

Solution:

$\frac{d}{dx} x \arctan\sqrt{x} = \left(\frac{d}{dx} x\right)\arctan\sqrt{x} + x\left(\frac{d}{dx}\arctan\sqrt{x}\right)$ chain rule

$= 1 \arctan\sqrt{x} + x \cdot \underbrace{\frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})'}$

$= \boxed{\arctan\sqrt{x} + \frac{x}{1+x} \cdot \frac{1}{2\sqrt{x}}}$

Challenge problem: Show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

for any $x > 0$.

We use the limit in the book:

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e.$$

Then let $m = n/x$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n/x\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}\cdot x} = \lim_{n/x\to\infty}\left[\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x = e^x.$$