Math 10A Fall 2024 Worksheet 11

October 4, 2024

This worksheet is a guided sketch of where the **derivative** of a function comes from. The derivative gives a way to compute the slope of a tangent line to, i.e. a line that "just barely touches," a curve at a point.

Consider any function $y = f(x)$, and let x_1 and x_2 be any two *different* real numbers.

1. Write a formula for the slope of the line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

These lines, for different x_1, x_2, are known as a **secant line** of $y = f(x)$. The geometric intuition for a derivative is that as x_1 and x_2 move closer and closer together, the secant lines move closer and closer to a tangent line.

2. Use the following Desmos applet: `https://www.desmos.com/calculator/auixylf12z` to see what secant lines are geometrically; drag the sliders around.

3. This second Desmos applet shows a tangent line: `https://www.desmos.com/calculator/ygafn5ubiy`. By dragging the sliders in the first applet and comparing with the second, convince yourself that as the two secant line points move close together, the secant line begins to look like a tangent line.

We want to take this geometry, and turn it into a formula for the derivative. To do this, we express x_2 as $x_1 + h$, then as $h \to 0$, $x_2 = x_1 + h \to x_1$, and we can express this as a limit. In other words, the slope of the tangent line to $y = f(x)$ at x_1 is

$$\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

In other words, **the slope of the tangent line is the limit of the slopes of secant lines**.

4. Using the definition of the derivative, compute that the slope of the tangent line to $y = x^2$ at any point (x_1, x_1^2) on the graph is $2x_1$.

   Hint: This is asking you to compute the limit

   $$\lim_{h \to 0} \frac{(x_1 + h)^2 - x_1^2}{h},$$

   and show that you get $2x_1$. Expand the binomial square in the top of the fraction, and show that you get out an h that cancels with the h in the denominator. By cancelling the denominator, you get something whose limit you can compute by plugging in.

5. The function in the Desmos applets is $y = x^3 - 2x^2 + 3$. Show, similarly to (4), that the slope of its tangent line at any point x_1 is

   $$3x_1^2 - 4x_1.$$

6. Look back at the second Desmos applet, `https://www.desmos.com/calculator/ygafn5ubiy`. What do you notice at the points where the graph of $f'(x)$ crosses the x-axis?

Solutions

1. The slope is just the change in the y-coordinate divided by the change in x-coordinate, so it's

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

2. The point of (2) and (3) is just to understand the picture. Ask if you have questions!

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4.
$$\lim_{h\to 0} \frac{(x_1+h)^2 - x_1^2}{h} = \lim_{h\to 0} \frac{x_1^2 + 2x_1h + h^2 - x_1^2}{h} = \lim_{h\to 0} 2x_1 + h = 2x_1.$$

5.
$$\begin{aligned}
\lim_{h\to} \frac{f(x_1+h) - f(x_1)}{h} &= \lim_{h\to 0} \frac{(x_1+h)^3 - 2(x_1+h)^2 + 3 - [x_1^3 - 2x_1^2 + 3]}{h} \\
&= \lim_{h\to 0} \frac{x_1^3 + 3x_1^2h + 3x_1h^2 + h^3 - 2x_1^2 - 4x_1h - h^2 + 3 - x_1^3 + 2x_1^2 - 3}{h} \\
&= \lim_{h\to 0} \frac{3x_1^2h + 3x_1h^2 + h^3 - 4x_1h - h^2}{h} = \lim_{h\to 0} 3x_1^2 + 3x_1h + h^2 - 4x_1 - h \\
&= 3x_1^2 - 4x_1.
\end{aligned}$$

6. The derivative $f'(x)$ (slope of the tangent line to $f(x)$) equals 0 when the graph of $f(x)$ has a "peak" or "trough".