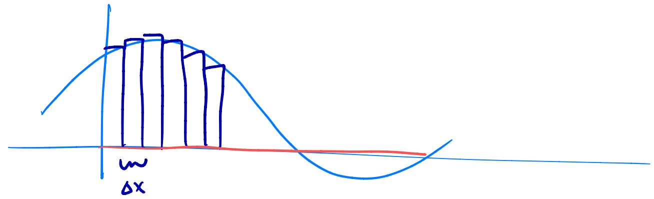
The definite ~~integral~~ squiggle!

November 14th, 2024

Here are some key ideas from section 5.2.

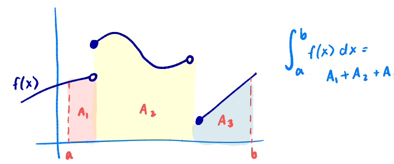


- Last time, we learned that a Riemann sum can be used to approximate the area bounded between a curve and the x-axis on the interval $a \leq x \leq b$.
  - So suppose we use n rectangles, each with width Δx.
  - Moreover, suppose we use $x_1^*, x_2^*, x_3^*, \dots, x_n^*$ to represent the *sample points* (for example, left endpoints, right endpoints, or midpoints).
  - Then the exact area under the curve is

$$\lim_{n\to\infty}\left(f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- If the limit exists, we say the function is integrable, and the definite integral is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$



- When can we integrate a function (when can we find the area under the curve)?
  - A continuous function is always integrable.
  - A function with finitely many jump discontinuities is always integrable.

. . . .

⋆ In this worksheet, you will need these summation laws to solve problems.

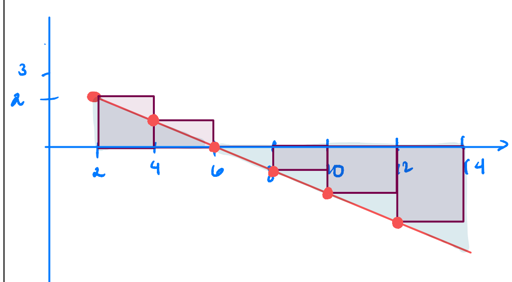
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Trig practice: Sketch the graphs of $\tan x$ and $\arctan x$. On what interval is $\arctan x$ a function?

Problem 1: (Stewart 5.2) Evaluate the left Riemann sum for $f(x) = 3 - \frac{1}{2}x$ on the interval $2 \leq x \leq 14$, with six subintervals. Use left endpoints as sample points.

My Attempt:

Solution:



$f(2) = 3 - \frac{1}{2}(2) = 2$

$f(4) = 3 - \frac{1}{2}(4) = 1$

$f(6) = 3 - \frac{1}{2}(6) = 0$

$f(8) = 3 - \frac{1}{2}(8) = -1$

$f(10) = 3 - \frac{1}{2}(10) = -2$

$f(12) = 3 - \frac{1}{2}(12) = -3$

Width of each rectangle: $\frac{14-2}{6} = \frac{12}{6} = 2$

Sum of areas of rectangles:

$2(2) + 2(1) + 2(0) + 2(-1) + 2(-2) + 2(-3)$

Here ● is a width and ● is a height.

$= 2\left(2+1+0+(-1)+(-2)+(-3)\right) = 2(-3) = -6$

Problem 2: (Stewart 5.2) Express the following limits as definite integrals.

a) $\lim_{n\to\infty}\sum_{i=1}^{n} x_i \ln(1+x_i^2)\,\Delta x$, on the interval $[2,6]$;

b) $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{\cos x_i}{x_i}\,\Delta x$, on the interval $[\pi, 2\pi]$;

c) $\lim_{n\to\infty}\sum_{i=1}^{n} \sqrt{2x_i^*(x_i^*)^2}\,\Delta x$, on the interval $[\pi, 2\pi]$.

My Attempt:

Solution:

Recall $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n} f(x_i)\,\Delta x$

a) $\int_2^6 x\ln(1+x^2)\,dx$

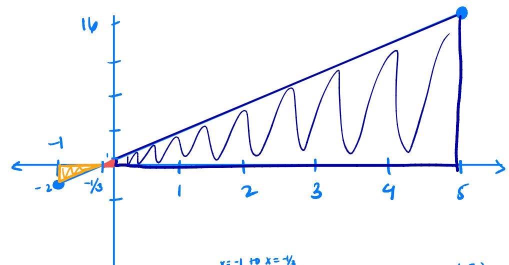
b) $\int_\pi^{2\pi} \frac{\cos x}{x}\,dx$

c) $\int_\pi^{2\pi} \sqrt{2x(x)^2}\,dx$

Problem 3: (Stewart 5.2) Geometrically find $\int_{-1}^{5}(1+3x)\,dx$ by sketching a graph of the function and dividing the area into known and friendly shapes.

My Attempt:

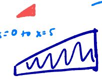
Solution:



we get three shapes:

$x=-1$ to $x=-1/3$  $\rightarrow$ Area: $\frac{(-2)(\frac{2}{3})}{2} = -\frac{2}{3}$

$x=-1/3$ to $x=0$  $\rightarrow$ Area: $\frac{(\frac{1}{3})(1)}{2} = \frac{1}{6}$

$x=0$ to $x=5$  $\rightarrow$ Area: $\left(\frac{1+16}{2}\right)5 = \frac{85}{2}$

Sum: $-\frac{4}{6}+\frac{1}{6}+\frac{85}{2}$

$= 42$

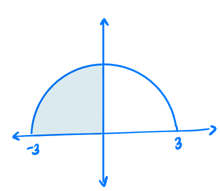
Problem 4: (Stewart 5.2) Evaluate $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)\,dx$ by interpreting in terms of areas.

My Attempt:

Solution:

we can write this integral as:

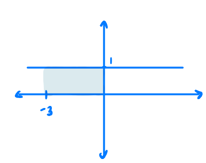
$\int_{-3}^{0} 1\,dx + \int_{-3}^{0}\sqrt{9-x^2}\,dx$



Notice $y=\sqrt{9-x^2}$ is a semicircle

Thus $\int_{-3}^{0}\sqrt{9-x^2}\,dx = \frac{1}{4}\pi\cdot 3^2 = \frac{9\pi}{4}$

Moreover $\int_{-3}^{0} 1\,dx = 1(3-0) = 3$.



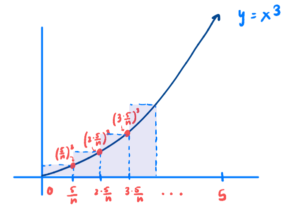
The total area is $\frac{9\pi}{4}+3$

Problem 5: (Stewart 5.2) Use the limit definition to evaluate $\int_0^5 (1+2x^3)\,dx$.

My Attempt:

Solution: Notice $\int_0^5 (1+2x^3)\,dx = \int_0^5 1\,dx + 2\int_0^5 x^3\,dx$

We will use the limit def'n for $\int_0^5 x^3\,dx$. The graph is pictured. If we have n rectangles, and since our interval has length 5, we know the width of each rectangle is $5/n$.



The height of the first rectangle is $(1\cdot\frac{5}{n})^3$
" " second rectangle is $(2\cdot\frac{5}{n})^3$
" " third rectangle is $(3\cdot\frac{5}{n})^3$
⋮
" " n^{th} rectangle is $(n\cdot\frac{5}{n})^3$

The area for n rectangles is $(\frac{5}{n})(1\cdot\frac{5}{n})^3 + (\frac{5}{n})(2\cdot\frac{5}{n})^3 + \cdots + (\frac{5}{n})(n\cdot\frac{5}{n})^3$

$$= \frac{5}{n}\cdot\frac{1^3\cdot 5^3}{n^3} + \frac{5}{n}\cdot\frac{2^3\cdot 5^3}{n^3} + \cdots + (\frac{5}{n})(\frac{n^3\cdot 5^3}{n^3})$$ (factor out stuff in pink)

$$= \frac{5^4}{n^4}(1^3+2^3+\cdots+n^3) = \frac{5^4}{n^4}\sum_{i=1}^{n} i^3 = \frac{5^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

Then $\int_0^5 x^3\,dx = \lim_{n\to\infty} \frac{5^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{5^4(n^4+2n^3+n^2)}{4n^4} = \frac{5^4}{4}$

so $\int_0^5 (1+2x^3) = \int_0^5 1\,dx + 2\int_0^5 x^3\,dx = 5 + 2\cdot\frac{5^4}{4} = \boxed{5+\frac{5^4}{2}}$

Problem 6: (Stewart 5.2) Using lower and upper bounds for $\int_0^2 \frac{1}{1+x^2}\,dx$, estimate the value of the integral.

My Attempt:

Solution:

The function $f(x) = \frac{1}{1+x^2}$ decreases.
So on $[0,2]$, the maximum is $f(0)$ and the minimum is $f(2)$. And $f(0) = 1$, $f(2) = \frac{1}{1+4} = \frac{1}{5}$

Thus

$$(2-0)\,f(2) \le \int_0^2 \frac{1}{1+x^2}\,dx \le (2-0)\,f(0)$$

so

$$\boxed{\frac{2}{5} \le \int_0^2 \frac{1}{1+x^2}\,dx \le 2}$$

Problem 7: (Stewart 5.2) Evaluate $\int_{-2}^{2}$ sin(x)x^3 dx ~~$\sin^3(x)x^3$~~.

My Attempt:

Solution:

This is an odd function, and since -2 and 2 are symmetric about the y-axis:

$$\int_{-2}^{2} \sin x\cdot x^3\,dx = \boxed{0}$$ ☺

Challenge problem: (Stewart 5.2) If

$$\int_0^4 e^{(x-2)^4}\,dx = k,$$

find the value of

$$\int_0^4 xe^{(x-2)^4}\,dx.$$

Make a substitution.

Let $a = x-2$. Then

$$\int_0^4 x e^{(x-2)^4}\,dx = \int_{-2}^{2} (a+2)e^{a}\,da$$

$$= \underbrace{\int_{-2}^{2} ae^{a}\,da}_{\text{odd function, integral goes to 0}} + \underbrace{2\int_{-2}^{2} e^{a}\,da}_{= 2\int_0^4 e^{(x-2)^4}\,dx = 2k}$$

$$= 0 + 2k = \boxed{2k}$$