Quiz 11 study guide

December 1st, 2024

General information

Quiz 11 covers sections 5.6 and 5.8. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Decomposing partial fractions by solving for coefficients (5.6)
- ☐ Decomposing fractions using long division (5.6)
- ☐ Integrating partial fractions to get $\ln$ terms (5.6)
- ☐ The definition of an improper integral (5.8)
- ☐ What it means for an improper integral to converge/diverge

Help! I'm stuck on....

- ...partial fraction decomposition: check out this 14 minute video
- ...integration by partial fractions: check out this 41 minute video with lots of examples
- ...long division with partial fractions: check out this 9 minute video
- ...understanding improper integrals: check out this 12 minute video
- ...solving problems with improper integrals: check out this 14 minute video

Practice problems

1. Determine if the following integrals are convergent or divergent. If convergent, find the value of the integral.

   (a) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

   (b) $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$

   (c) $\displaystyle\int_{-\infty}^{-1} e^{-2t}\,dt$

   (d) $\displaystyle\int_{-\infty}^{\infty} \cos \pi t\,dt$

   (e) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

   (f) $\displaystyle\int_{-\infty}^{6} re^{r/3}\,dt$

2. Suppose $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. Find $\displaystyle\int_0^\infty e^{-x^2/2}\,dx$.

3. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

   (b) $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

   (c) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

Solutions

1. (a) We have

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t (1+x)^{-1/4}\,dx.$$

Make the substitution $u = 1+x$, $du = dx$ to get

$$\lim_{t\to\infty}\left[\frac{4}{3}(1+x)^{3/4}\right]_0^t = \lim_{t\to\infty}\left[\frac{4}{3}(1+t)^{3/4}-\frac{4}{3}\right] = \infty.$$

Divergent.

(b)

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2}+\frac{1}{2}\right) = \frac{1}{2}\left(0+\frac{1}{2}\right) = \frac{1}{4}.$$

Convergent.

(c)

$$\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty}\left[-\frac{1}{2}e^2+\frac{1}{2}e^{-2x}\right] = \infty.$$

Divergent.

(d) Splitting this integral into two, we have

$$I = \int_{-\infty}^\infty \cos\pi t\,dt = I_1 + I_2 = \int_{-\infty}^0 \cos\pi t\,dt + \int_0^\infty \cos\pi t\,dt,$$

but

$$I_1 = \lim_{s\to-\infty}\left[\frac{1}{\pi}\sin\pi t\right]_s^0 = \lim_{s\to-\infty}\left(-\frac{1}{\pi}\sin\pi t\right),$$

and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent.

(e) Make the substitution $u = \ln x$, $du = \dfrac{dx}{x}$ to see that

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$$

Divergent.

(f) Integrate by parts by setting $u = r$ and $dv = e^{r/3}dr$. Then

$$\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3}-9e^{r/3}\right]_t^6 = \lim_{t\to-\infty}\left(18e^2-9e^2-3te^{t/3}+9e^{t/3}\right) = 9e^2-0+0,$$

where the limits of the last two terms can be evaluated with l'Hôpital. So we get $9e^2$. Convergent.

2. We have

$$\int_0^\infty e^{-x^2/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x^2/2}\,dx.$$

Make the substitution $u = x/\sqrt{2}$, $du = dx/\sqrt{2}$ to get

$$\sqrt{2}\lim_{b\to\infty}\int_0^{b/\sqrt{2}} e^{-u^2}\,du = \sqrt{2}\int_0^\infty e^{-u^2}\,du = \sqrt{2}\left(\frac{1}{2}\sqrt{\pi}\right) = \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}.$$

3. (a) If $a \neq b$, we can decompose using partial fractions to show that

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right),$$

so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C.$$

Otherwise, we have $a = b$, so

$$\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C.$$

(b) Using long division, we see that

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{(x-3)(x+2)}.$$

Begin partial fractions decomposition:

$$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}.$$

Then

$$3x-4 = A(x+2) + B(x-3).$$

Solve to get $A = 1, B = 2$. Then

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1,$$

so we get

$$\left(\frac{1}{2}+1+\ln 2 + 2\ln 3\right) - (0+0+\ln 3 + 2\ln 2) = \frac{3}{2} + \ln 3 - \ln 2 = \frac{3}{2} + \ln\frac{3}{2}.$$

(c) Begin partial fractions decomposition to get

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

After lots of algebraic manipulation, you should get $A = 1, B = -1, C = 1$. Thus

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$