Math 10A Fall 2024 Worksheet 5

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September 12 2024

1 Finding inverses

1. When is it possible to find the inverse of a 2×2 matrix?

2. For the following matrices, calculate the inverse or explain why it's not possible to do so.

   a) $\begin{bmatrix} 7 & 9 \\ 5 & 6 \end{bmatrix}$; b) $\begin{bmatrix} 14 & 6 \\ 7 & 3 \end{bmatrix}$; c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

3. Suppose A and B are both invertible matrices of the same dimensions. Which of the following statements are ALWAYS true? Explain your reasoning for the false statements (e.g., by giving a counterexample).

   a) $A = A^{-1}$
   b) $AA^{-1} = A$
   c) $A^{-1}A = I$.
   d) A^{-1} is unique.
   e) $(AB)(A^{-1}B^{-1}) = I$.
   f) AB is invertible.
   g) B is singular.

4. Show that for any arbitrary nonsingular matrix A, its inverse A^{-1} is unique. *Hint: Show that if there are two inverses, they must be equal.*

2 Determinants

5. Write the formula for the determinant of a 1×1 matrix A.

6. Write the formula for the determinant of a 2×2 matrix B.

7. Write the formula for the determinant of a 3×3 matrix C. Drawing this out might help!

8. How do determinants inform the invertibility of a matrix?

9. Use the determinant to determine if the matrix $\begin{bmatrix} 9 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 2 & 0 \end{bmatrix}$ is singular.

10. Use the determinant to determine if the matrix $\begin{bmatrix} x & x+x^2 \\ 3x & 0 \end{bmatrix}$ is singular, assuming $x \neq 0$ and $x \neq -1$.

3 Systems of equations

11. Write the following system of equations in the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} 3x_1 - x_2 &= 9 \\ 2x_1 + 9x_2 &= -10. \end{aligned}$$

12. Write the following system of equations in the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} 2x_1 - x_2 + 9x_3 &= 12 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 1 \\ x_2 &= 2. \end{aligned}$$

Solutions

1. The inverse only exists if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

2. a) $7(6) - 9(5) = -3$ so we can find an inverse. The inverse is

$$\frac{-1}{3}\begin{bmatrix} 6 & -9 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ \frac{5}{3} & -\frac{7}{3} \end{bmatrix}.$$

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   b) $14(3) - 6(7) = 0$, so there is no inverse.

   c) $2(-2) - 3(-1) = -1$, so an inverse exists. The inverse is

$$\frac{1}{-1}\begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

3. a) False, A is not always its own inverse (see part 2a).

   b) False, $AA^{-1} = I$.

   c) True.

   d) True.

   e) False, the inverse of AB is generally $B^{-1}A^{-1}$.

   f) True.

   g) False, singular matrices are not invertible.

4. Suppose that there are two inverses, denoted B and C, such that

$$A^{-1} = B \quad \text{and} \quad A^{-1} = C.$$

   We want to show that $B = C$. Since B is the inverse of A, we have $AB = I$. But multiplying both sides by C, we get

$$CAB = CI = C.$$

   Since $CA = I$, we have

$$(CA)B = C \implies IB = C \implies B = C.$$

5. $\det A = a_{11}$.

6. $\det B = b_{11}b_{22} - b_{12}b_{21}$.

7. $\det C = c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33}$.

8. Matrices are invertible if and only the determinant is nonzero.

9. The determinant is -21, so the matrix is nonsingular.

10. The determinant is $-3x\left(x^2 + x\right)$, which is nonzero for $x \neq 0, -1$. So the matrix is nonsingular.

11. $\begin{bmatrix} 3 & -1 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \end{bmatrix}$.

12. $\begin{bmatrix} 2 & -1 & 9 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ 2 \end{bmatrix}$.