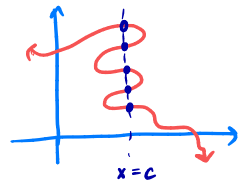
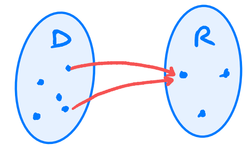
# $3 \times 3$ eigenstuff, representing functions

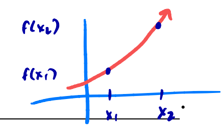



***September 24th, 2024***

Here are some key ideas from sections 1.1 and 1.3.

- A **function** is a rule that assigns each element $x$ in its domain to [more than one / exactly one / less than one] element in its range.
- The vertical line test is a way to tell whether or not a graph in the $xy$-plane is a function. It says that an $xy$-curve is the graph of a function **if and only if** no vertical line intersects the curve more than once.
- parity
  A function $f$ is **even** if $f(-x) = f(x)$. A function $f$ is **odd** if $f(-x) = -f(x)$. These must hold for all $x$.
- A function $f$ is called *increasing* on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in $I$.
- A function $f$ is called *decreasing* on an interval $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$




**Problem 1:** (Stewart & Day 8.8) But first...two more matrix problems. This time, we'll work with $3 \times 3$ ones.

(a) Let $A$ be the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Write an expression for the determinant of $A$.

(b) Let $B$ be the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Find the eigenvalues of $B$.

(c) Find an eigenvector for each eigenvalue of $B$.

My Attempt:

Solution:

a) $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

b) $\det(B - \lambda I) =$

some calculations later...

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 2 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)\left[(1-\lambda)^2 - 3\right]$$
$$= (1-\lambda)(1-\lambda+\sqrt{3})(1-\lambda-\sqrt{3})$$
$$= 0, \text{ so } \boxed{\lambda = 1,\ 1+\sqrt{3},\ 1-\sqrt{3}}$$

c) For $\lambda = 1$, we get $\begin{bmatrix} 1-1 & 0 & 1 \\ 2 & 1-1 & 0 \\ 3 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_3 = 0 \\ 2v_1 = 0 \\ 3v_1 = 0 \end{matrix}$. Pick $\boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

For $\lambda = 1+\sqrt{3}$, repeat to get $\begin{matrix} -\sqrt{3}v_1 + v_3 = 0 \\ 2v_1 - \sqrt{3}v_2 = 0 \\ 3v_1 - \sqrt{3}v_3 = 0 \end{matrix} \Rightarrow v_3 = \sqrt{3}v_1$. Pick $\boxed{\begin{bmatrix} \sqrt{3} \\ 2 \\ 3 \end{bmatrix}}$

For $\lambda = 1-\sqrt{3}$, repeat to get $\begin{matrix} \sqrt{3}v_1 + v_3 = 0 \\ 2v_1 + \sqrt{3}v_2 = 0 \\ 3v_1 + \sqrt{3}v_3 = 0 \end{matrix} \Rightarrow v_3 = -\sqrt{3}v_1$. Pick $\boxed{\begin{bmatrix} -\sqrt{3} \\ 2 \\ 3 \end{bmatrix}}$.

**Problem 2:** (Stewart & Day 8.8) Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 2 & 1 \end{bmatrix}$, and find one eigenvector for each eigenvalue.

My Attempt:

Solution:

This problem is similar to the previous.

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 7 \\ 0 & 2 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(1-\lambda+\sqrt{14})(1-\lambda-\sqrt{14})$$

so $\boxed{\lambda = 1,\ \lambda = 1+\sqrt{14},\ \lambda = 1-\sqrt{14}}$

$\lambda = 1$: $\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & 1-1 & 7 \\ 0 & 2 & 1-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2v_2 + 3v_3 = 0 \\ 7v_3 = 0 \\ 2v_2 = 0 \end{matrix}$ $\boxed{\text{Pick } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$\lambda = 1+\sqrt{14}$ $\longrightarrow$ $\left.\begin{matrix} \sqrt{14}\,v_1 + 2v_2 + 3v_3 = 0 \\ -\sqrt{14}\,v_2 + 7v_3 = 0 \\ 2v_2 - \sqrt{14}\,v_3 = 0 \end{matrix}\right\}$ these eq. all say $v_3 = \frac{\sqrt{14}}{7}v_2$, $v_1 = \frac{2v_2+3v_3}{\sqrt{14}}$ $\boxed{\text{Pick } \begin{bmatrix} 3+\sqrt{14} \\ 7 \\ \sqrt{14} \end{bmatrix}}$

$\lambda = 1-\sqrt{14}$: $\left.\begin{matrix} \sqrt{14}\,v_1 + 2v_2 + 3v_3 = 0 \\ \sqrt{14}\,v_2 + 7v_3 = 0 \\ 2v_2 + \sqrt{14}\,v_3 = 0 \end{matrix}\right\}$ $\boxed{\text{Pick } \begin{bmatrix} -3+\sqrt{14} \\ -7 \\ \sqrt{14} \end{bmatrix}}$

**Problem 3:** (Borcherds '05 Midterm 1) Find the domain of the function $g(u) = \sqrt{u} + \sqrt{2-u}$.

My Attempt:

Solution:

Stuff inside radical must be $\geq 0$.

$\sqrt{u}$ : $u \geq 0$

$\sqrt{2-u}$ : $2-u \geq 0 \Rightarrow u \leq 2$

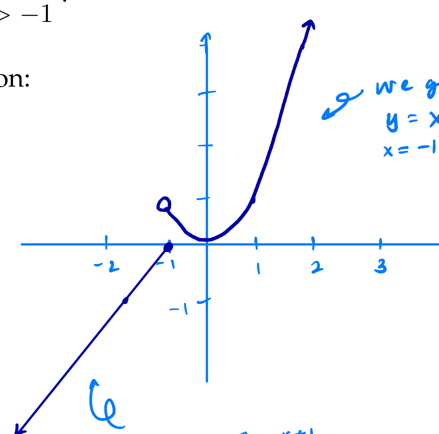
so $u \geq 0$ and $u \leq 2 \Rightarrow \boxed{[0,2]}$

**Problem 4:** (Stewart 1.1) Recall that a *piecewise function* splits its domain into pieces and is defined by different formulas for each piece. Sketch the graph of the following piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$

My Attempt:

Solution:




we graph $y = x^2$ after $x = -1$

we graph $x+1$ until and including $x=-1$

**Problem 5:** (Stewart 1.1) Determine whether $f(x) = x|x|$ is even, odd, or neither.

My Attempt:

Solution:

$f(-x) = -x\,|-x|$
$= -x\,|x|$ (since $|-x| = |x|$)
$= -(x|x|)$
$= -f(x)$

so the function is $\boxed{\text{odd}}$.

**Problem 6**: (Stewart 1.1) Does $x^2 + (y-3)^2 = 5$ define a function? Explain why or why not.

My Attempt:

Solution: Nope! Two ways to see this:

1. This is a circle, does not pass the vertical line test
2. Notice $(1, 5)$ and $(1, 1)$ are two outputs for one input, which can't define a function.

**Problem 7:** Consider the function $f(x) = 4 + 3x - x^2$. Evaluate the difference quotient given by

$$\frac{f(3+h) - f(3)}{h}.$$

My Attempt:

Solution:

$f(3+h) = 4 + 3(3+h) - (3+h)^2$
$= 4 + \cancel{9} + 3h - \cancel{9} - 6h - h^2$
$= 4 - 3h - h^2$

$f(3) = 4 + 3(3) - 3^2 = 4$

Then $\dfrac{f(3+h) - f(3)}{h} = \dfrac{\cancel{4} - 3h - h^2 - \cancel{4}}{h} = \boxed{-3-h}$

$-x+3 -x-2$

**Problem 8**: (Stewart 1.1) Solve $|x-3| + |x+2| < 11$ mathematically (don't guess and check values).

My Attempt:

Solution:

$x < -2$ : $-(x-3) + -(x+2) < 11 \Rightarrow -2x+1 < 11 \Rightarrow 2x > -10 \Rightarrow x > -5$

$-2 \leq x < 3$ : $-(x-3) + x + 2 < 11 \Rightarrow 5 < 11$ ✓ so all $x$ work here

$x > 3$  $x - 3 + x + 2 < 11 \Rightarrow 2x - 1 < 11 \Rightarrow x < 6$

so $\boxed{(-5, 6)}$ works

**Challenge Problem:** (Stewart 1.1) Sketch the region in the plane consisting of all points $(x, y)$ such that $|x-y| + |x| - |y| \leq 2$.

Split into quadrants and approach as in problem 8.