Quiz 6 study guide

October 13th, 2024

General information

Quiz 5 covers sections 3.1-3.4. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Finding the instantaneous rate of change (e.g. velocity given the change in position function)
- ☐ The **two different** limit definitions of a derivative: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
- ☐ The definition of differentiability (the limit must exist and cannot be $\pm\infty$)
- ☐ If something is differentiable at a, it is continuous at a, and the reverse **does not** necessarily hold!
- ☐ The power rule (derivative of x^n)
- ☐ The derivative of $c \cdot f(x)$, $f(x)+g(x)$, and $f(x)-g(x)$
- ☐ The derivative of $\sin x$, $\cos, x$, and e^x
- ☐ The product rule (derivative of $f(x) \cdot g(x)$)

Help! I'm stuck on....

- ...how to find the **instantaneous rate of change**: this 4 minute video
- ...the **limit definition** of a derivative: check out this 23 minute video (watch at 2x)
- ...the **other limit definition** of a derivative: check out this 8 minute video
- ...what **differentiability** means and how it relates to continuity: check out this 9 minute video
- ...the **product rule** and how to use it: check out this 11 minute video

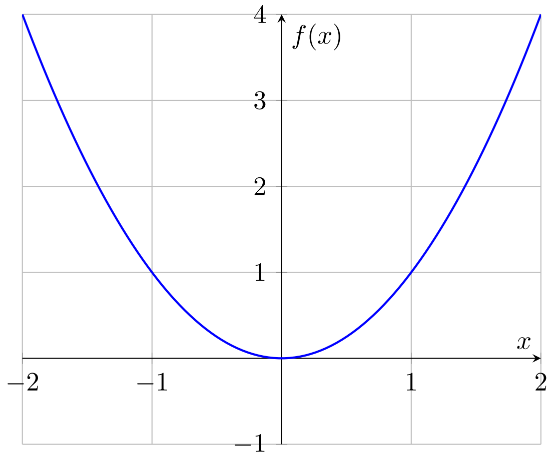
Practice problems

1. Consider a car's position s (in meters) as a function of time t (in seconds), given by:

$$s(t) = 2t^2 + 3t + 1$$

   Find the instantaneous velocity of the car at time $t = 2$ seconds using a limit definition of the derivative.

2. Use the graph of the function below to sketch the graph of the derivative of $f(x)$.



3. Determine whether the function $f(x) = |x|$ is differentiable at the point $x = 0$. If it is, find the derivative $f'(0)$. If it's not, explain why.

4. Determine if the function $g(x) = |x^{2/3}|$ is differentiable at $x = 0$. If it is differentiable, find its derivative at $x = 0$. If it is not differentiable, explain why.

5. Find the derivative of the following function with respect to x:

$$f(x) = 3x^2 - 2x + 1$$

6. Differentiate $r(t) = \frac{a}{t^2} + \frac{b}{t^4}$.

7. Differentiate the following function using the product rule:

$$f(x) = x^2 \cdot e^x$$

8. Find the derivative of the following function with respect to x:

$$\frac{d}{dx}\left((4x + 2)x^2 e^x\right)$$

Solutions

1. To find the instantaneous velocity, we need to compute the derivative of the position function $s(t)$ with respect to time t:

$$\begin{aligned} s(t) &= \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} \\ &= \lim_{h\to 0} \frac{(2(t+h)^2+3(t+h)+1)-(2t^2+3t+1)}{h} \\ &= \lim_{h\to 0} \frac{2(t^2+2th+h^2)+3(t+h)+1-(2t^2+3t+1)}{h} \\ &= \lim_{h\to 0} \frac{2t^2+4th+2h^2+3t+3h+1-2t^2-3t-1}{h} \\ &= \lim_{h\to 0} \frac{4th+2h^2+3h}{h} \\ &= \lim_{h\to 0} \frac{h(4t+2h+3)}{h} \\ &= \lim_{h\to 0} (4t+2h+3) \\ &= 4t+3 \end{aligned}$$

Now, we can find the instantaneous velocity at $t = 2$ seconds:

$$\begin{aligned} s(2) &= 4\cdot 2+3 \\ &= 8+3 \\ &= 11 \text{ m/s} \end{aligned}$$

You may also do this problem by substituting $t = 2$ in our original limit (instead of finding the derivative and evaluating at 2), which will give us the derivative at 2 directly.

2. Here are the steps you might have taken and observations you might have made to procure this graph.

   (a) First, ensure you understand the original function, $f(x) = x^2$. It's a simple quadratic function with a vertex at the origin $(0, 0)$, opening upward.

   (b) Identify key features of the original function, which will help you sketch the derivative. For $f(x) = x^2$, some important features are:

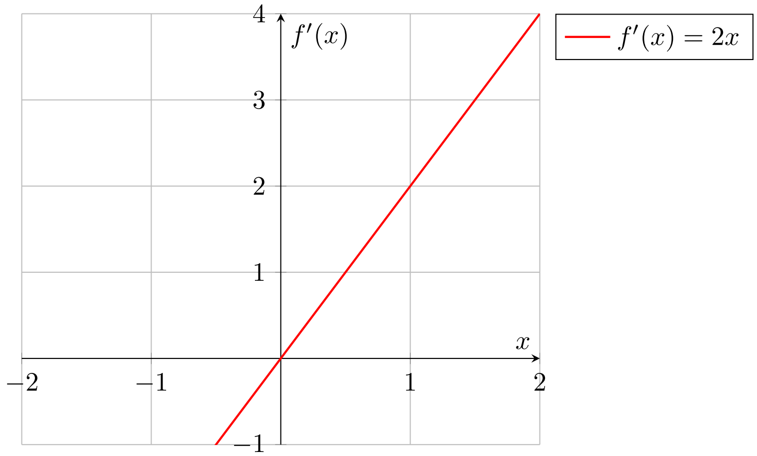
   - The slope of the tangent line is steeper for positive values of x.
   - The slope of the tangent line is less steep for negative values of x.
   - The slope is zero at the origin (the vertex).

   (c) Remember that the derivative represents the slope of the tangent line at each point on the original function's graph. In this case, you're sketching the derivative, $f'(x)$.

   (d) At each point on the graph of $f(x) = x^2$, determine the slope of the tangent line. The slope of the tangent line at any point x on the graph of $f(x) = x^2$ is $2x$. This is because the derivative $f'(x)$ represents the slope at that point.

   (e) Now, create a new graph and mark the x-axis and y-axis. For each x value on the original graph, calculate the corresponding $f'(x)$ value using the derivative formula $f'(x) = 2x$. Plot these points on the new graph.

   (f) The derivative graph will pass through the origin (0, 0) because the slope at the vertex of $f(x) = x^2$ is zero. The slope of the derivative graph increases as you move right from the origin (positive x). The slope of the derivative graph decreases as you move left from the origin (negative x). The derivative graph will be symmetric about the y-axis.



3. Note that a function is differentiable at a if and only if the limit (from our limit definition) exists and is not $\pm\infty$. In order for the limit to exist, we need that the left and right hand limits are equal. So, to determine the differentiability of the function $f(x) = |x|$ at the point $x = 0$, we need to examine the behavior of the function's derivative as x approaches 0 from both the left and right sides.

   (a) Let's calculate the derivative $f'(0)$ by considering the right-hand limit:

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{h}$$
$$= \lim_{h\to 0^+} 1$$

   Since this limit exists and is equal to 1, the right-hand derivative exists.

   (b) Now, let's consider the left-hand limit:

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h}$$
$$= \lim_{h\to 0^-} \frac{-h}{h}$$
$$= \lim_{h\to 0^-} -1$$

   Since this limit exists and is equal to -1, the left-hand derivative exists.

   (c) The function $f(x) = |x|$ has differentiable left-hand and right-hand limits at $x = 0$. However, since the left-hand derivative is -1, and the right-hand derivative is 1, they are not equal.
   Therefore, $f(x) = |x|$ is not differentiable at $x = 0$. The function has a corner at this point.

4. Use the definition of the derivative:

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

Substituting $g(x)$ into the limit:

$$g'(0) = \lim_{h \to 0} \frac{|(0+h)^{2/3}| - |0^{2/3}|}{h}$$

Simplify the expression:

$$g'(0) = \lim_{h \to 0} \frac{|h^{2/3}|}{h}$$

Notice that for $h > 0$, $|h^{2/3}| = h^{2/3}$, and for $h < 0$, $|h^{2/3}| = (-h)^{2/3} = h^{2/3}$. So, $|h^{2/3}| = h^{2/3}$ for all h. Therefore:

$$g'(0) = \lim_{h \to 0} \frac{h^{2/3}}{h}$$

Now, evaluate the limit:

$$g'(0) = \lim_{h \to 0} h^{-1/3}$$

As h approaches 0, $h^{-1/3}$ becomes undefined. This is because taking the reciprocal of zero (h^{-1}) and then taking the cube root ($h^{-1/3}$) results in division by zero. Therefore, the derivative $g'(0)$ does not exist.

Hence, the function $g(x) = |x^{2/3}|$ is not differentiable at $x = 0$ — the limit does not exist!

5. To find the derivative of $f(x)$, we'll apply the sum and power rules for differentiation:

$$f'(x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(2x) + \frac{d}{dx}(1)$$

Using the power rule ($\frac{d}{dx}(x^n) = nx^{n-1}$), we get:

$$f'(x) = 3 \cdot 2x^{2-1} - 2 \cdot 1x^{1-1} + 0$$

Simplifying further:

$$f'(x) = 6x - 2$$

So, the derivative of $f(x)$ is $f'(x) = 6x - 2$.

6.

$$\begin{aligned}
r(t) &= \frac{a}{t^2} + \frac{b}{t^4} \\
\frac{dr}{dt} &= \frac{d}{dt}\left(\frac{a}{t^2}\right) + \frac{d}{dt}\left(\frac{b}{t^4}\right) \\
\frac{dr}{dt} &= a\frac{d}{dt}\left(t^{-2}\right) + b\frac{d}{dt}\left(t^{-4}\right) \\
\frac{dr}{dt} &= a(-2t^{-2-1}) + b(-4t^{-4-1}) \\
\frac{dr}{dt} &= -2at^{-3} - 4bt^{-5}
\end{aligned}$$

7. Use the product rule. In this case, we have $u(x) = x^2$ and $v(x) = e^x$, so let's find the derivatives of each function: te derivative of $u(x) = x^2$ is $u'(x) = 2x$ and the derivative of $v(x) = e^x$ is $v'(x) = e^x$.
Now, we can apply the product rule:

$$\begin{aligned}
f(x) &= x^2 \cdot e^x \\
f'(x) &= (x^2)' \cdot e^x + x^2 \cdot (e^x)' \\
f'(x) &= (2x) \cdot e^x + x^2 \cdot e^x \\
f'(x) &= 2xe^x + x^2e^x
\end{aligned}$$

So, the derivative of $f(x) = x^2 \cdot e^x$ with respect to x is:

$$f'(x) = 2xe^x + x^2e^x$$

8. To find the derivative of the given function, we can use the product rule. The product rule states that if we have two functions $u(x)$ and $v(x)$, then the derivative of their product is given by:

$$(uv)' = u'v + uv'$$

In this case, we have $u(x) = (4x + 2)x^2$ and $v(x) = e^x$. Let's find the derivatives of each function:

The derivative of $u(x) = (4x + 2)x^2$ can be found using the product rule:

$$u'(x) = (4x + 2)'x^2 + (4x + 2)(x^2)'$$

$$u'(x) = 4x^2 + (4x + 2)(2x)$$

$$u'(x) = 12x^2 + 4x$$

The derivative of $v(x) = e^x$ is simply $v'(x) = e^x$.

Now, we can apply the product rule to find the derivative of the original function:

$$\frac{d}{dx}\left((4x + 2)x^2e^x\right) = u'v + uv'$$

$$\frac{d}{dx}\left((4x + 2)x^2e^x\right) = (12x^2 + 4x)e^x + (4x + 2)x^2(e^x)$$

Simplifying further:

$$\frac{d}{dx}\left((4x + 2)x^2e^x\right) = e^x(12x^2 + 4x + (4x + 2)x^2)$$

$$\frac{d}{dx}\left((4x + 2)x^2e^x\right) = e^x(4x^3 + 14x^2 + 4x)$$

So, the derivative of $\frac{d}{dx}\left((4x + 2)x^2e^x\right)$ is $e^x(4x^3 + 14x^2 + 4x)$.