Math 10A Fall 2024 Worksheet 22

November 19, 2024

1. Evaluate following integrals.
   (a) $\int x \cos 6x \; dx$
   (b) $\int x^2 \ln x \; dx$
   (c) $\int (\ln x)^2 \; dx$
   (d) $\int x \ln(1+x) \; dx$

2. Suppose $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, and $f'(4) = 3$ and f'' is continuous. Compute $\int_1^4 x f''(x) \; dx$

1 Solutions

1. Recall integration by parts:
$$\int uv' = uv - \int u'v.$$

   (a) Let $u = x$, $v' = \cos 6x$. Then $u' = 1$ and $v' = \frac{1}{6}\sin(6x)$. So
$$\begin{aligned}\int x\cos(6x) &= \frac{x}{6}\sin(6x) - \int \frac{1}{6}\sin(6x)\ dx \\ &= \frac{x}{6}\sin(6x) + \frac{1}{36}\cos(6x) + C\end{aligned}$$

   (b) Let $u = \ln(x)$, $v' = x^2$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{x^3}{3}$. So
$$\begin{aligned}\int x^2\ln(x)\ dx &= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3x}\ dx \\ &= \frac{x^3}{3}\ln(x) - \frac{x^3}{6} + C\end{aligned}$$

   (c) Let $u = (\ln(x))^2$, $v' = 1$. Then $u' = 2\ln(x)/x$ and $v = x$. So
$$\begin{aligned}\int (\ln x)^2\ dx &= x(\ln(x))^2 - \int \frac{2\ln(x)}{x}x\ dx \\ &= x(\ln(x))^2 - 2\int \ln(x)\ dx\end{aligned}$$

   So we reduce to solving $\int \ln(x)\ dx$. For this integral, let $u = \ln(x)$ and $v' = 1$. Then $u' = 1/x$ and $v = x$. Hence
$$\begin{aligned}\int \ln(x)\ dx &= x\ln(x) - \int \frac{1}{x}x\ dx \\ &= x\ln(x) - x + C.\end{aligned}$$

   Thus
$$\begin{aligned}\int (\ln x)^2\ dx &= x(\ln(x))^2 - 2\int \ln(x)\ dx \\ &= x(\ln(x))^2 - 2x\ln(x) + 2x + C\end{aligned}$$

   (d) First, substitute $t = 1 + x$. Then $dt = dx$ so
$$\begin{aligned}\int x\ln(1+x) &= \int (t-1)\ln(t)\ dt \\ &= \int t\ln(t)\ dt - \int \ln(t)\ dt\end{aligned}$$

   We compute each integral seperately. For the leftmost integral, set $w = \ln(t)$. Then $dw = 1/tdt$ hence
$$\int t\ln(t)\ dt = \int w\ dw = w^2/2 + C = \ln(t)^2/2 + C$$
   And for the rightmost integral, we already computed in the previous problem that
$$\int \ln(t)\ dt = t\ln(t) - t + C$$

So

$$\begin{aligned}
\int x\ln(1+x) &= \int (t-1)\ln(t)\ dt \\
&= \int t\ln(t)\ dt - \int \ln(t)\ dt \\
&= \frac{\ln(t)^2}{2} - t\ln(t) + t + C \\
&= \frac{\ln(1+x)^2}{2} - (1+x)\ln(1+x) + (1+x) + C
\end{aligned}$$

2. Setting $u = x$ and $v' = f''(x)$, we have $u' = 1$ and $v = f'(x)$. So

$$\begin{aligned}
\int_1^4 xf''(x)\ dx &= xf'(x)\Big|_1^4 - \int_1^4 f'(x)\ dx \\
&= xf'(x)\Big|_1^4 - (f(4) - f(1)) \\
&= 4f'(4) - f'(1) - (f(4) - f(1)) \\
&= 4(3) - 5 - (7 - 2) \\
&= 2
\end{aligned}$$