Math 10A Fall 2024 Worksheet 18

November 5, 2024

1 Derivatives and the shape of a graph

1. In his campaign for reelection, Richard Nixon claimed that "the rate at which inflation is increasing is decreasing." Let $f(t)$ denote the price of a cheeseburger in the USA at time t. We assume the real value of a cheeseburger remains constant, so that its price accurately reflects the purchasing power of the dollar. Set $t = 0$ to be the time of Nixon's announcement.

   (a) Based on the announcement and real-world context, determine which of $f(0), f'(0), f''(0)$, and $f'''(0)$ are positive or negative.

   (b) Sketch a possible graph of $f(t)$ near $t = 0$.

2. Let $f(x) = \frac{2}{1+e^{-x}}$. This is an example of a *logistic* function, which can be used to model population growth in a confined environment.

   (a) Find the horizontal asymptotes, inflection points, and critical points of $f(x)$.

   (b) Determine the intervals of concavity and increase/decrease.

   (c) Sketch a graph of $f(x)$.

3. Compute the intervals of increase and decrease and the intervals of concavity of the function $f(x) = x^4 - 4x^3$.

4. Write down a quartic polynomial with exactly 2 inflection points. Is it possible to find a quartic polynomial with exactly 1 inflection point? (Quartic means degree exactly 4.)

2 L'Hopital's rule

1. Determine whether L'Hopital's rule can be used to evaluate each limit below (possibly after some algebraic manipulation). If not, explain why not. Then evaluate the limit (even if L'Hopital's rule doesn't work).

   (a) $\lim_{x\to 1} \frac{\ln(x)}{\sin(\pi x)}$

   (b) $\lim_{x\to 0} \frac{x^3+2x^2+1}{\cos(x)-1}$

   (c) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

   (d) $\lim_{x\to\infty} \ln(x)^{1/x}$

   (e) $\lim_{x\to 0^+} \ln(x)\tan(x)$

   (f) $\lim_{x\to\infty} \frac{3x+\sin(x)}{2x}$

2. Let $f(x)$ and $g(x)$ be functions on $(0,\infty)$.

   (a) Give a reasonable definition of what it means for $f(x)$ to "grow faster" than $g(x)$.

   (b) Rank the following functions from slowest- to fastest-growing:

   $$x+e^{-x}, 10\ln(x), 5\sqrt{x}, x\sqrt{x}$$

3 Optimization

1. Consider the line $f(x)=2x+3$.

   (a) Write down a function of x that computes how far the point $(x, f(x))$ is from the origin.

   (b) Find the closest point on the line to the origin.

2. Suppose that $a+b=c$.

   (a) Assume that c is a fixed constant. Write b as a function of a.

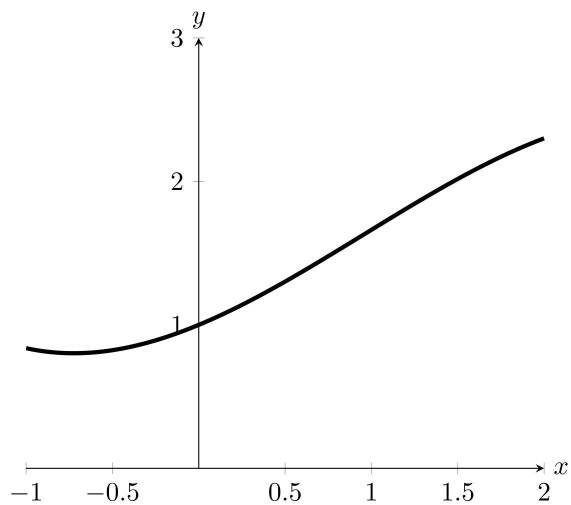
   (b) Prove that $ab \leq (c/2)^2$, and give an example where equality occurs.

Solutions

1 Derivatives and the shape of a graph

1. (a) The price of a cheeseburger is positive, so $f(0) > 0$. Inflation exists, which means that prices are going up, so $f'(0) > 0$. Nixon implies that inflation is increasing, meaning that $f''(0) > 0$. Finally, the "rate of increase of inflation" is $f''(t)$, so Nixon's claim that this quantity is decreasing means that $f'''(0) < 0$.

   (b) The following graph is consistent with the previous information. The key point is that the graph increasing and concave up at $t = 0$, but it becomes less concave up as you go to the right, eventually hitting an inflection point (which is what Nixon is hoping will happen in the future).



2. (a) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} = 2$ are the two horizontal asymptotes. For the critical points, we need the zeros of

$$f'(x) = \frac{2e^{-x}}{(1+e^{-x})^2}.$$

   However, neither the numerator nor the denominator this expression is ever 0, so there are no critical points. For the inflection points, we need the zeros of

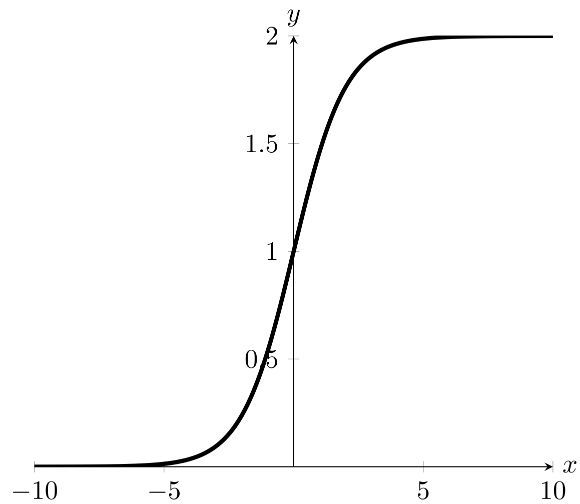
$$\begin{aligned} f''(x) &= \frac{-2e^{-x}(1+e^{-x})^2 - 2e^{-x}(2(1+e^{-x})(-e^{-x}))}{(1+e^{-x})^4} \\ &= \frac{-2e^{-x}(1+e^{-x}) + 4e^{-2x}}{(1+e^{-x})^4} \end{aligned}$$

   So we need to solve $0 = -2e^{-x}(1+e^{-x}) + 4e^{-2x}$. Dividing by $2e^{-x}$ gives $0 = -1 - e^{-x} + 2e^{-x} = -1 + e^{-x}$. This has a zero exactly when $e^{-x} = 1$, that is, when $x = 0$. So $x = 0$ is the only inflection point.

   (b) Since there are no critical points, the function is either increasing or decreasing on all of $(-\infty, \infty)$, so since $f'(0) = \frac{2}{(1+1)^2} > 0$, it must be increasing everywhere.

   For concavity, we note that the denominator in our expression for $f''(x)$ is always positive, so we need only test the sign of the numerator. The numerator of $f''(-1)$ is $-2e(1+e)+4e^2 = -2e+2e^2$, which is positive since $e^2 > e$. Hence the function is concave up on $(-\infty, 0)$. The numerator of $f''(1)$ is likewise $-2e^{-1}+2e^{-2}$, which is negative since $e^{-1} > e^{-2}$, so the function is concave down on $(0, \infty)$.

(c) Here is the graph:



3. We have $f'(x) = 4x^3 - 12x^2$, which has roots at $x = 0, 3$, so these are the two critical points. $f''(x) = 12x^2 - 24x$, which has roots at $x = 0, 2$, so these are the inflection points.

   We test $f'(-1) = -16, f'(1) = -8, f'(5) = 440$. Notice that the derivative does *not* change signs at $x = 0$: it is negative before, and negative after. We conclude that the interval of decrease is $(-\infty, 3)$ and the interval of increase is $(3, \infty)$. (It is correct to include 0 in the interval of decrease, even though $f'(0)$ is not negative, since the function is decreasing on this interval in the sense that $f(b) < f(a)$ if $a < b$.)

   For concavity, we test $f''(-1) = 36$, $f''(1) = -12$, and $f''(5) = 12 * 25 - 24 * 5 = 120$. Therefore, the graph is concave down on $(0, 2)$ and concave up on $(-\infty, 0)$ and $(2, \infty)$.

4. Quartic polynomials are generally of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Then $f''(x) = 12ax^2 + 6bx + 2c$. We want $f''(x)$ to have exactly two roots, so we just need to choose a, b, c so that this quadratic has two roots. Taking $a = 1, b = 0, c = -1$ is a perfectly fine choice for this (and take d, e to be anything).

   Likewise, we can in fact choose a quartic with exactly one inflection point by choosing a, b, c so that the quadratic $f''(x) = 12ax^2 + 6bx + 2c$ has only one root. For example, $a = 1, b = 0, c = 0$ works.

2 L'Hopital's rule

1. (a) L'Hopital is applicable.

$$\lim_{x \to 1} \frac{\ln(x)}{\sin(\pi(x)} = \lim_{x \to 1} \frac{1/x}{\pi \cos(\pi x)} = \boxed{-1/\pi}$$

   (b) L'Hopital is not applicable since the numerator goes to 1. The denominator goes to 0 from the bottom on both sides, so this is enough to conclude that the limit is $\boxed{-\infty}$.

   (c) L'Hopital is applicable, but not useful: if you apply the derivative to the top and bottom, you'll find that you get exactly the same expression back, so this doesn't make progress. Instead, we can evaluate the limit in a more basic way by dividing top and bottom by x:

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x^2}} = \boxed{1}.$$

   (d) By continuity, $\lim_{x \to \infty} \ln(x)^{1/x}$ exists if and only if $\lim_{x \to \infty} \ln(\ln(x)^{1/x})$ exists, in which case

$$\lim_{x \to \infty} \ln(\ln(x)^{1/x}) = \ln\left(\lim_{x \to \infty} \ln(x)^{1/x}\right).$$

We can write

$$\lim_{x\to\infty} \ln(\ln(x)^{1/x}) = \lim_{x\to\infty} \frac{\ln(\ln(x))}{x}$$

and now L'Hopital's rule is applicable:

$$= \lim_{x\to\infty} \frac{\ln(x)}{x}$$

We apply L'Hopital a second time:

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

Finally, we need to remember that this is ln of the actual limit we are looking for, so we conclude that $\lim_{x\to\infty} \ln(x)^{1/x} = e^0 = \boxed{1}$.

(e) Rewrite

$$\lim_{x\to 0^+} \ln(x)\tan(x) = \lim_{x\to 0^+} \frac{ln(x)}{\cot(x)}$$

Now L'Hopital is applicable:

$$= \lim_{x\to 0^+} \frac{1/x}{-\csc^2(x)}$$

$$= \lim_{x\to 0^+} \frac{\sin^2(x)}{x}.$$

From here, you can either use $\lim_{x\to 0} \sin(x)/x = 0$ or use L'Hopital's rule again to conclude that the limit is $\boxed{0}$.

(f) L'Hopital is not applicable, since the answer it gives is DNE. If we were try to apply L'Hopital, then the limit would be the limit of $\frac{3+\cos(x)}{2}$, which does not exist. However, the limit does actually exist and equals $\boxed{3/2}$, since $\lim_{x\to\infty} \sin(x)/x = 0$. The trick here is that one of the conditions for L'Hopital's rule to be applicable is that the output it gives exists (or equals $\pm\infty$). You *cannot* use L'Hopital's rule to conclude that a limit does not exist (in the sense that it neither converges nor goes to $\pm\infty$).

2. (a) A reasonable definition is to say that $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \infty$, or equivalently $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

(b) Comparing via L'Hopital's rule tells us that the order is $\boxed{10\ln(x), 5\sqrt{x}, x + e^{-x}, x\sqrt{x}}$, from slowest to fastest.

3 Optimization

1. (a) The distance of $(x, f(x))$ to the origin is

$$g(x) = \sqrt{x^2 + f(x)^2} = \sqrt{x^2 + (2x+3)^2} = \boxed{\sqrt{5x^2 + 12x + 9}}.$$

(b) We need to minimize $g(x)$ from the previous part; it's equivalent to minimize $g(x)^2 = 5x^2+12x+9$. The derivative of this is $10x + 12$, so the critical point (= vertex of the parabola) is at $x = -1.2$. The corresponding point on the line $f(x) = 2x + 3$ is $\boxed{(-1.2, 0.6)}$.

2. (a) $\boxed{b = c - a}$

(b) We need to try to maximize ab, or equivalently maximize $a(c - a)$, as a function of a. This is downward-facing parabola, so it will have a maximum at its unique critical point. The derivative with respect to a is $c-2a$, so the critical point occurs at $a = c/2$. Therefore, $(c/2)(c-c/2) = (c/2)^2$ is the maximum possible value of ab, and taking $a = c/2$ acheives equality.