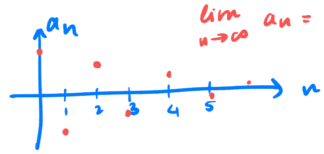
Limits of sequences, infinite limits

October 1st, 2024



$\lim_{n\to\infty} a_n = 0$

$\infty + \infty = \infty$

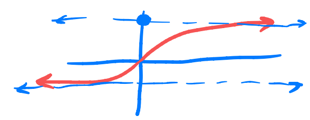
$\infty \cdot \infty = \infty$

$\infty - \infty =$ indet.

Here are some key ideas from sections 2.1 and 2.2.

- We write $\lim_{n\to\infty} a_n = L$ to mean the sequence a_n approaches <u>L</u>. If a_n becomes large as n becomes large, we write <u>$\lim_{n\to\infty} a_n = \infty$</u>
- If the limit exists, the sequence <u>converges</u>. Otherwise, it <u>diverges</u>.
- A geometric sequence has the form $a, ar, ar^2, \cdots$. If $-1 < r < 1$, the sum of the infinite geometric series is
$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$
- To evaluate limits of rational sequences, divide by <u>the highest power of n</u>.
- The expression $\lim_{x\to\infty} f(x) = L$ can be thought of as the <u>end behavior</u> of f. We say $y = L$ is a horizontal asymptote of f if

$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$



- To evaluate limits of functions with radical expressions, it may help to multiply by <u>the conjugate</u>.

$$\frac{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})} = \frac{a-b}{\sqrt{a}-\sqrt{b}}$$

Midterm practice:

(a) Diagonalize the following matrix:
$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}.$$
Explicitly write what P, D, and P^{-1} are.

(b) Consider the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Compute $A^6\mathbf{v}$.

My Attempt:

Solution:

a) Eigenvalues and eigenvectors:

$\lambda = 2$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

b) $A = PDP^{-1}$, so

$$A^6\vec{v} = PD^6P^{-1}\vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 64 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 127 & -63 \\ 126 & -62 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 65 \\ 66 \end{bmatrix}}$$

Problem 1: (Stewart 2.1) Determine if $a_n = \frac{2n^2+n-1}{n^2}$ converges. If it is convergent, find the limit.

My Attempt:

Solution:

$$a_n = \frac{2n^2+n-1}{n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \frac{2 + \frac{1}{n} - \frac{1}{n^2}}{1}$$

$$\lim_{n\to\infty} 2 + \cancel{\frac{1}{n}}^{0} - \cancel{\frac{1}{n^2}}^{0} = \boxed{0}$$

Problem 2: (Stewart 2.1) Find the limit of $a_n = \frac{n^2}{\sqrt{n^3+4n}}$.

My Attempt:

Solution:

$$a_n = \frac{n^2}{\sqrt{n^3+4n}} \cdot \frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3}}} = \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+\frac{4}{n^2}}}$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+\cancel{\frac{4}{n^2}}^{0}}} \overset{\infty}{=} \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+0}} = \lim_{n\to\infty} n^{\frac{1}{2}} = \boxed{\infty}$$

Problem 3: (Stewart 2.1) Use a series to express $0.\overline{8}$ as a ratio of integers.

My Attempt:

Solution:

$0.8 + 0.08 + 0.008 + \cdots$

$r = \frac{1}{10}$, $a = 0.8$

$$\text{sum} = \frac{0.8}{1-\frac{1}{10}} = \frac{0.8}{0.9} = \boxed{\frac{8}{9}}.$$

Problem 4: (Stewart 2.1) Use a series to express $1.53\overline{42}$ as a ratio of integers.

My Attempt:

Solution:

$$1.53\overline{42} = 1.53 + 0.00\overline{42}$$

$$= 1.53 + \underbrace{0.0042 + 0.000042 + \ldots}$$

geo series, $a = 0.0042$, 0.01

$$\text{sum} = \frac{0.0042}{1-0.01} = \frac{0.0042}{0.99} = \frac{42}{9900}$$

1.53 repeating decimal

$$\frac{153}{100} + \frac{42}{9900} = \boxed{\frac{5063}{3300}}$$

Problem 5: (Stewart 2.2) Find $\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx}$.

My Attempt:

Solution: multiply by the conjugate:

$$\lim_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right) \frac{\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right)}{\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right)}$$

$$= \lim_{x\to\infty} \frac{x^2+ax-(x^2+bx)}{\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right)}$$

$$= \lim_{x\to\infty} \frac{x(a-b)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty} \frac{a-b}{\sqrt{1+\cancel{\frac{a}{x}}^{0}}+\sqrt{1+\cancel{\frac{b}{x}}^{0}}}$$

$$= \lim_{x\to\infty} \frac{a-b}{\sqrt{1}+\sqrt{1}} = \boxed{\frac{a-b}{2}}$$

Problem 6: (Stewart 2.2) Find $\lim_{x\to\infty} e^{-1/x^2}$.

My Attempt:

Solution:

Notice $\lim_{x\to\infty} \frac{-1}{x^2} = 0$

so $\lim_{x\to\infty} e^{-1/x^2} = e^0 = \boxed{1}$

Problem 7: (Stewart 2.2) Find $\lim_{x\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$.

My Attempt:

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2} \cdot \frac{\frac{1}{t^2}}{\frac{1}{t^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{t}}{t^2}+1}{\frac{2}{t}-1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{t}{t^4}}+1}{\frac{2}{t}-1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\cancel{\frac{1}{t^3}}^{0}}+1}{\cancel{\frac{2}{t}}^{0}-1}$$

$$= \frac{1}{-1}$$

$$= \boxed{-1}$$

Problem 8: (Stewart 2.2) Find $\lim_{x\to\infty}(e^{-x}+2\cos 3x)$.

My Attempt:

Solution:

$$\lim_{x\to\infty} e^{-x} + \lim_{x\to\infty} 2\cos 3x$$

$$= 0 + \lim_{x\to\infty} 2\cos 3x$$

DNE because $\cos 3x$ oscillates.

Challenge problem: (Stewart 2.2) Let $f(x) = \left(3^x+3^{2x}\right)^{\frac{1}{x}}$. Find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} \left(9^x\left(\frac{3^x}{9^x}+\frac{9^x}{9^x}\right)\right)^{\frac{1}{x}} = \lim_{x\to\infty} \left(9^x\left(\frac{3}{9}\right)^x+1\right)^{\frac{1}{x}}$$

$$= \lim_{x\to\infty} (9^x)^{\frac{1}{x}} \left(\cancel{\left(\frac{1}{3}\right)^x}^{0}+1\right)^{\frac{1}{x}}$$

$$= 9 \cdot 1^0 = \boxed{9}$$