Quiz 7 study guide

November 3rd, 2024

General information

Quiz 7 covers sections 4.1-4.3. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

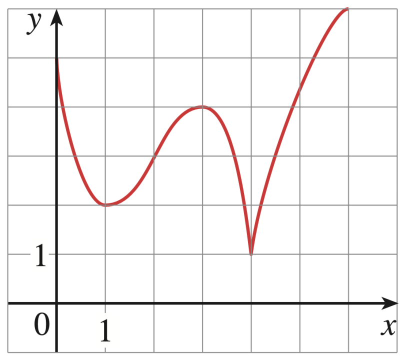
- ☐ The Extreme Value Theorem
- ☐ Absolute extrema can be at critical points or endpoints, while local extrema can only be at critical points (otherwise put, endpoints are *not* local extrema)
- ☐ Fermat's theorem
- ☐ The statement of the Mean Value Theorem
- ☐ How to find the absolute/local extrema over a closed interval
- ☐ How to find the absolute/local extrema over an open interval (in particular, how the method is different than over a closed interval)
- ☐ If $f'(x) > 0$ on an interval I, then f is increasing on I
- ☐ If $f'(x) < 0$ on an interval I, then f is decreasing on I
- ☐ Using the first derivative to find intervals where a function is increasing or decreasing
- ☐ The statement of the First Derivative Test
- ☐ How to use the First Derivative Test to find local extrema
- ☐ The definition of concavity (concave up and concave down)
- ☐ The definition of an inflection point
- ☐ The statement of the Second Derivative Test
- ☐ How to use the Second Derivative Test to find extrema
- ☐ The types of indeterminate forms (e.g. $\frac{0}{0}$, $\frac{\infty}{\infty}$), etc.
- ☐ If and when and how l'Hôpital's rule might be applied to each different indeterminate form
- ☐ The statement of l'Hôpital's rule and the conditions that must be satisfied to use it

Help! I'm stuck on....

- ...solving **absolute extrema** problems: check out this 9 minute video
- ...solving **local extrema** problems: check out 6 minute video
- ...understanding the **first derivative test**: check out this 12 minute video
- ...**solving relative extrema problems** using the first derivative test: check out this 12 minute video
- ...understanding **concavity**: check out this 12 minute video
- ...understanding the **Second Derivative Test**: check out this 6 minute video
- ...**solving problems** with the Second Derivative Test: check out this 12 minute video
- ...understanding **l'Hôpital's rule**: check out this 9 minute video
- ...**solving problems** with l'Hôpital's rule: check out this 13 minute video.

Practice problems

1. Consider the following graph. Find (or approximate) the following:
   (a) The open intervals on which f is increasing
   (b) The open intervals on which f is decreasing
   (c) The open intervals on which f is concave upward
   (d) The open intervals on which f is concave downward
   (e) The coordinates of the points of inflection



2. Find the local extrema, intervals of concavity, and the inflection points of $f(x) = \ln(x^2 + 9)$.
3. Find the local extrema, intervals of concavity, and the inflection points of $f(x) = 3x^4 - 8x^3 + 12$.
4. Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x^3}$.
5. Evaluate $\lim_{x\to -\infty}(x^2 - x^3)e^{2x}$.
6. Evaluate $\lim_{x\to(-\pi/2)^-}(\tan x)^{\cos(x)}$.
7. Evaluate $\lim_{x\to\infty}(\sqrt{x^2 + x} - \sqrt{x})$.

Solutions

1. (a) The function f is increasing on intervals where the graph goes up, so $(1, 3) \cup (4, 6)$.
   (b) The function f is decreasing on intervals where the graph goes down, so $(0, 1) \cup (3, 4)$.
   (c) The function f is concave upward where the slope is increasing (getting steeper if positive or getting less steep if negative), so $(0, 2)$.
   (d) The function f is concave downward where the slope is decreasing (getting less steep if positive or getting more steep if negative), so $(2, 4) \cup (4, 6)$.
   (e) The points of inflection are where concavity changes, so using parts (c) and (d) we have the point, *not* the interval, $(2, 3)$.

2. First, let's find the derivative $f'(x)$ and the second derivative $f''(x)$:

$$f(x) = \ln(x^2 + 9)$$

$$f'(x) = \frac{d}{dx} \ln(x^2 + 9)$$

We can use the chain rule to find $f'(x)$:

$$f'(x) = \frac{1}{x^2 + 9} \cdot \frac{d}{dx}(x^2 + 9)$$

Simplify:

$$f'(x) = \frac{2x}{x^2 + 9}$$

Now, let's find the second derivative $f''(x)$:

$$f''(x) = \frac{d}{dx}\left(\frac{2x}{x^2 + 9}\right)$$

Using the quotient rule:

$$f''(x) = \frac{(x^2 + 9) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(x^2 + 9)}{(x^2 + 9)^2}$$

Simplify:

$$f''(x) = \frac{18 - 2x^2}{(x^2 + 9)^2}$$

Now, let's find the critical points by setting $f'(x) = 0$:

$$\frac{2x}{x^2 + 9} = 0$$

This implies $2x = 0$, which leads to $x = 0$.

So, the critical point is at $x = 0$. To determine the nature of this point, we can analyze the sign of $f''(0)$ (this is the Second Derivative Test):

$$f''(0) = \frac{18 - 2 \cdot 0^2}{(0^2 + 9)^2} = \frac{18}{81} = \frac{2}{9}$$

Since $f''(0) > 0$, the function is concave up at $x = 0$, which indicates a local minimum.

To find the local minimum value, plug $x = 0$ into the original function:

$$f(0) = \ln(0^2 + 9) = \ln(9)$$

Now, let's analyze the intervals of concavity and find any inflection points. The function is concave up when $f''(x) > 0$, which occurs when $18 - 2x^2 > 0$.

Solve for $2x^2 < 18$:

$$-3 < x < 3$$

So, the function is concave up on the interval $(-3, 3)$. The function is concave down when $f''(x) < 0$, which occurs when $18 - 2x^2 < 0$.

Solve for $2x^2 > 18$:

$$x < -3 \quad \text{or} \quad x > 3$$

So, the function is concave down on the intervals $(-\infty, -3)$ and $(3, \infty)$.

Thus $x = -3$ and $x = 3$ are locations of inflection points. To find exactly what the inflection points are, note that $f(-3) = \ln(18) = f(3)$, so $(-3, \ln(18))$ and $(3, \ln(18))$ are the inflection points.

3. First, let's find the derivative $f'(x)$ and the second derivative $f''(x)$:

$$f(x) = 3x^4 - 8x^3 + 12$$

$$f'(x) = \frac{d}{dx}(3x^4) - \frac{d}{dx}(8x^3) + \frac{d}{dx}(12)$$

Using the power rule, we find $f'(x)$:

$$f'(x) = 12x^3 - 24x^2$$

Now, let's find the second derivative $f''(x)$:

$$f''(x) = \frac{d}{dx}(12x^3) - \frac{d}{dx}(24x^2)$$

Using the power rule again, we find $f''(x)$:

$$f''(x) = 36x^2 - 48x$$

Now, let's find the critical points by setting $f'(x) = 0$:

$$12x^3 - 24x^2 = 0$$

Factor out $12x^2$:

$$12x^2(x - 2) = 0$$

This gives two critical points: $x = 0$ and $x = 2$. Now, we can use the first derivative test to determine extrema. For $x < 0$, take a test point $x = -1$:

$$f'(-1) = 12(-1)^3 - 24(-1)^2 = -12 - 24 = -36$$

Since $f'(-1) < 0$, the function is decreasing in the interval $(-\infty, 0)$.

For $0 < x < 2$, take a test point $x = 1$:

$$f'(1) = 12(1)^3 - 24(1)^2 = 12 - 24 = -12$$

Since $f'(1) < 0$, the function is decreasing in the interval $(0, 2)$.

For $x > 2$, take a test point $x = 3$:

$$f'(3) = 12(3)^3 - 24(3)^2 = 108 - 216 = -108$$

Since $f'(3) < 0$, the function is decreasing in the interval $(2, \infty)$.

Based on the first derivative test, we can conclude that at $x = 2$, the function changes from decreasing to increasing. Therefore, there is a local minimum at $x = 2$. To find the local minimum value, plug $x = 2$ into the original function:

$$f(2) = 3(2)^4 - 8(2)^3 + 12 = 48 - 64 + 12 = -4$$

So, the function has a local minimum at $x = 2$ with a minimum value of -4.

Now, let's analyze the intervals of concavity. The function is concave up when $f''(x) > 0$, which occurs when $36x^2 - 48x > 0$.

Solve for x:

$$12x(3x - 4) > 0$$

So either $12x > 0$ and $3x - 4 > 0$ or $12x < 0$ and $3x - 4 < 0$. In other words, either $x > 0$ and $x > \frac{4}{3}$ or $x < 0$ and $x < \frac{4}{3}$. Thus the function is concave up on $(\frac{4}{3}, \infty)$ and $(-\infty, 0)$.
The function is concave down when $f''(x) < 0$, which occurs when $36x^2 - 48x < 0$.

Solve for x:

$$12x(3x - 4) < 0$$

So either $12x < 0$ and $3x - 4 > 0$ or $12x > 0$ and $3x - 4 < 0$. In other words, $x < 0$ and $x > \frac{4}{3}$ or $x > 0$ and $x < \frac{4}{3}$. The former is impossible, but the latter gives that the function is concave down on $(0, \frac{4}{3})$.
thus there are inflection points at $x = 0$ and $x = \frac{4}{3}$. Plugging into the function, we get that $(0, 12)$ and $(\frac{4}{3}, \frac{68}{27})$ are inflection points.

4. This limit is in the indeterminate form $\frac{0}{0}$ as $x \to 0$, so we apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{e^x - 1}{x^3} = \lim_{x\to 0} \frac{e^x}{3x^2}.$$

This is no longer indeterminate, and $\lim_{x\to 0} \frac{1}{3x^2}$ is simply ∞.

5. The expression is indeterminate, in the form $\infty \cdot \infty$. We can rewrite the expression to make it ∞/∞:

$$\lim_{x\to -\infty} (x^2 - x^3)e^{2x} = \lim_{x\to -\infty} \frac{x^2 - x^3}{e^{-2x}}.$$

This is in the form $\frac{\infty}{\infty}$, so we apply L'Hôpital's Rule:

$$\lim_{x\to -\infty} \frac{2x - 3x^2}{-2e^{-2x}} = \lim_{x\to -\infty} \frac{-3x^2 + 2x}{2e^{-2x}}.$$

This is still indeterminate, so reapplying L'Hôpital's Rule gives:

$$\lim_{x\to -\infty} \frac{-6x + 2}{4e^{-2x}}.$$

Since e^{-2x} grows faster than any polynomial term as $x \to -\infty$, this limit approaches 0. Alternatively, you could apply L'Hôpital yet again to get the same result.

$$\lim_{x\to -\infty} (x^2 - x^3)e^{2x} = 0.$$

6. This expression is indeterminate, in the form ∞^0. Let

$$L = \lim_{x \to (-\pi/2)^-} (\tan x)^{\cos(x)}.$$

Taking the natural logarithm of both sides, we have:

$$\ln L = \ln \left(\lim_{x \to (-\pi/2)^-} (\tan x)^{\cos x} \right) = \lim_{x \to (-\pi/2)^-} \cos(x) \ln(\tan(x)).$$

We now need to evaluate $\lim_{x \to (-\pi/2)^-} \cos(x) \ln(\tan(x))$, which is in the form $0 \cdot \infty$. Rewrite it as

$$\lim_{x \to (-\pi/2)^-} \frac{\ln(\tan x)}{1/\cos(x)}.$$

This is now in the form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's Rule to get

$$\lim_{x \to (-\pi/2)^-} \frac{\frac{\sec^2(x)}{\tan(x)}}{\sec(x)\tan(x)}.$$

Simplifying the fraction, we get

$$\lim_{x \to (-\pi/2)^-} \frac{\sec^2(x)}{\sec(x)\tan^2(x)}.$$

Rewriting in terms of $\cos(x)$, we have

$$\lim_{x \to (-\pi/2)^-} \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\cos(x)}\tan^2(x)} = \lim_{x \to (-\pi/2)^-} \frac{1}{\cos(x)\tan^2(x)}.$$

Using $\tan(x) = \frac{\sin(x)}{\cos(x)}$, this becomes:

$$\lim_{x \to (-\pi/2)^-} \frac{\cos(x)}{\sin^2(x)}.$$

As $x \to -\pi/2$, $\cos(x) \to 0$ and $\sin^2(x) \to 1$, so this limit is simply 0. Then $\ln L = 0$ (remember, we took the natural logarithm earlier) so $L = 1$.

7. Start by multiplying by the conjugate to get

$$\lim_{x \to \infty} \left(\sqrt{x^2 + x} - \sqrt{x} \right) = \lim_{x \to \infty} \frac{\left(\sqrt{x^2+x} - \sqrt{x}\right)\left(\sqrt{x^2+x} + \sqrt{x}\right)}{\sqrt{x^2+x} + \sqrt{x}}.$$

This simplifies to

$$\lim_{x \to \infty} \frac{x^2 + x - x}{\sqrt{x^2+x} + \sqrt{x}} = \lim_{x \to \infty} \frac{x^2}{\sqrt{x^2+x} + \sqrt{x}}.$$

Now, divide both the numerator and the denominator by x to get

$$\lim_{x \to \infty} \frac{x}{\sqrt{1 + \frac{1}{x}} + \frac{1}{\sqrt{x}}}.$$

As $x \to \infty$, the terms $\frac{1}{x}$ and $\frac{1}{\sqrt{x}}$ approach zero, so we get

$$\lim_{x \to \infty} \frac{x}{1+0} = x.$$

Thus, the limit tends to infinity:

$$\lim_{x \to \infty} \left(\sqrt{x^2 + x} - \sqrt{x} \right) = \infty.$$