Math 10A Fall 2024 Worksheet 10

October 1 2024

1 Limits of sequences

1. Find a sequence a_n satisfying the given property. Then, sketch the sequence on a graph.

   a) $\lim_{n\to\infty} a_n = 0$; b) $\lim_{n\to\infty} a_n = \infty$; c) $\lim_{n\to\infty} a_n$ is undefined.

2. For each sequence in Problem 1, decide if it converges or diverges.

3. For each of the following cases, decide whether it is possible or impossible. If it is possible, find an example. If it is impossible, explain why.

   (a) A sequence a_n that has infinitely many ones, but $\lim_{n\to\infty} a_n \neq 1$;

   (b) Sequences a_n and b_n such that a_n converges, b_n diverges, but $a_n + b_n$ converges;

   (c) Sequences a_n and b_n such that $a_n b_n$ and a_n both converge but b_n does not.

4. For each of the following sequences, determine whether it is convergent or divergent. If it is convergent, find the limit.

   a) $a_n = \ln(2n^2+1) - \ln(n^2+1)$

   b) $a_n = 1 - (0.2)^n$

   c) $a_n = \frac{10^n}{1+9^n}$

   d) $a_n = \frac{\pi^n}{3^n}$

Limits at infinity

1. Find the following limits.

   a) $\lim_{x\to\infty} \frac{1}{2x+3}$

   b) $\lim_{x\to\infty} \frac{3x+5}{x-4}$

   c) $\lim_{t\to-\infty} 0.6^t$

   d) $\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$

   e) $\lim_{x\to-\infty} \frac{x^4-3x^2+x}{x^3-x+2}$

   f) $\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6}$.

2. Find a function $f(x)$ such that $\lim_{x\to\infty} = 2$ and $\lim_{x\to-\infty} = -2$.

Solutions

1 Limits of sequences

1. **Solution:**

   a) One possible sequence is $a_n = \frac{1}{n}$. As n grows, the terms of the sequence approach 0.

   b) Consider $a_n = n$. As n grows, the terms of the sequence grow without bound.

   c) One possible sequence is $a_n = (-1)^n$. This sequence oscillates between 1 and -1 and does not converge to a single value.

2. **Solution:**

   (a) $a_n = \frac{1}{n}$ converges to 0, so it **converges**.

   (b) $a_n = n$ diverges to infinity, so it **diverges**.

   (c) $a_n = (-1)^n$ does not converge to a limit, so it **diverges**.

3. **Solution:**

   (a) It is **possible**. An example is $a_n = 1$ for even n and $a_n = 0$ for odd n. The sequence does not converge to 1, but it has infinitely many ones.

   (b) It is **impossible**. If $a_n + b_n$ converges and a_n converges, then $a_n + b_n - a_n = b_n$ must also converge.

   (c) It is **possible**. For example, let $a_n = \frac{1}{n}$ (which converges to 0) and $b_n = (-1)^n$ (which diverges). Then, the product $a_n b_n = \frac{(-1)^n}{n}$ converges to 0.

4. **Solution:**

   a) $\lim_{n\to\infty} \ln(2n^2 + 1) - \ln(n^2 + 1) = \ln 2$.

   b) $\lim_{n\to\infty} 1 - (0.2)^n = 1$.

   c) $\lim_{n\to\infty} \frac{10^n}{1+9^n} = \infty$.

   d) $\lim_{n\to\infty} \frac{\pi^n}{3^n} = \infty$ since $\pi > 3$.

2 Limits at infinity

1. **Solution:**

   a) $\lim_{x\to\infty} \frac{1}{2x+3} = 0$.

   b) $\lim_{x\to\infty} \frac{3x+5}{x-4} = 3$.

   c) $\lim_{t\to-\infty} 0.6^t = 0$ since $0.6 < 1$.

   d) $\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2} = -1$.

   e) $\lim_{x\to-\infty} \frac{x^4-3x^2+x}{x^3-x+2} = \infty$ since the degree of the numerator is higher than the degree of the denominator.

   f) $\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6} = \frac{1}{2}$.

2. **Solution:** One possible function is $f(x) = \frac{4}{\pi}\arctan(x)$. As $x \to \infty$, $\arctan(x) \to \pi/2$, so $\lim_{x\to\infty} f(x) = 2$, and as $x \to -\infty$, $\arctan(x) \to -\pi/2$, so $\lim_{x\to-\infty} f(x) = -2$.