Average values, volumes

December 5th, 2024

Here are some key ideas from sections 6.2 and 6.4.

- We define the average value of a function on the interval $[a, b]$ to be $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.
- The Mean Value Theorem for integrals says there must be some c in $[a,b]$ such that $f(c) = f_{ave}$.
- Suppose we have a solid whose cross-sectional area at x is given by $A(x)$. Then the volume of the solid between $x = a$ and $x = b$ is

$$V = \lim_{n\to\infty} A(x^*)\,\Delta x \qquad = \int_a^b A(x)\,dx$$

. . . .

Trig practice: (Stewart) If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.

Problem 1: (Stewart 6.2) Find the average value of $f(x) = 4x - x^2$ on the interval $[0, 4]$.

My Attempt:

Solution:

$$f_{ave} = \frac{1}{4-0}\int_0^4 (4x - x^2)\,dx$$
$$= \frac{1}{4}\left[2x^2 - \frac{x^3}{3}\right]_0^4$$
$$= \frac{1}{4}\left(32 - \frac{64}{3}\right)$$
$$= \boxed{8 - \frac{16}{3}}$$

Problem 2: (Stewart 6.2) Find the average value of $\cos^4 x \sin x$ on $[0, \pi]$.

My Attempt:

Solution:

let $u = \cos x \Rightarrow du = -\sin x\,dx$

Then $\int \underbrace{\cos^4 x}_{u^4}\,\underbrace{\sin x\,dx}_{-du} = \int u^4(-du) = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C$

Thus $f_{ave} = \frac{1}{\pi - 0}\int_0^\pi \cos^4 x \sin x\,dx$

$$= \frac{1}{\pi}\left[\frac{-\cos^5 x}{5}\right]_0^\pi$$
$$= \frac{1}{\pi}\left[\frac{-\cos^5 \pi}{5} - \left(\frac{-\cos^5 0}{5}\right)\right]$$
$$= \frac{1}{\pi}\left[\frac{-(-1)^5}{5} - \left(\frac{-(1)^5}{5}\right)\right]$$
$$= \frac{1}{\pi}\left[\frac{1}{5} + \frac{1}{5}\right]$$
$$= \boxed{\frac{2}{5\pi}}$$

Problem 3: (Stewart 6.2) Find the average value of $(3 - 2u)^{-1}$ on the interval $[-1, 1]$.

My Attempt:

Solution:

$$f_{ave} = \frac{1}{1-(-1)}\int_{-1}^1 \frac{1}{3-2u}\,du$$
$$= \frac{1}{2}\left[\frac{\ln|3-2u|}{-2}\right]_{-1}^1$$
$$= \frac{1}{2}\left[\frac{\ln 1}{-2} - \left(\frac{\ln 5}{-2}\right)\right]$$
$$= \frac{1}{2}\left(0 + \frac{\ln 5}{2}\right) = \boxed{\frac{\ln 5}{4}}$$

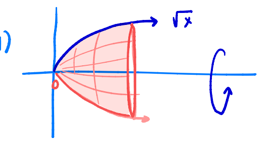
Problem 4: Consider the region under the curve $y = \sqrt{x}$ from 0 to 1. This problem will walk you through the process of finding the volume of the solid generating by rotating this region about the x axis.

1. Sketch the solid of revolution. What shape is the cross-section?
2. Find the area of a cross-section for a fixed x. This will be your function $A(x)$.
3. Integrate $A(x)$ between the given bounds to find the volume.
4. You did it! Celebrate!

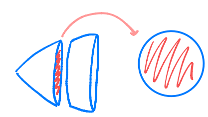
My Attempt:

Solution:

1)



Each slice (cross-section) is a circle.



2) $A(x) = \pi r^2 = \pi(\sqrt{x})^2 = \pi x$

3) $\int_0^1 A(x)\,dx = \int_0^1 \pi x \, dx = \left[\frac{\pi x^2}{2}\right]_0^1 = \boxed{\frac{\pi}{2}}$

Problem 5: (Stewart 6.4) Find the volume of the solid obtained by rotating the region $y = 1 - x^2$ and $y = 0$ about the x-axis.

My Attempt:

Solution:

$y = 1-x^2$ intersects $y=0$ at $x = 1$, $x=-1$. Then

$V = \int_{-1}^{1} \pi(1-x^2)^2\,dx$

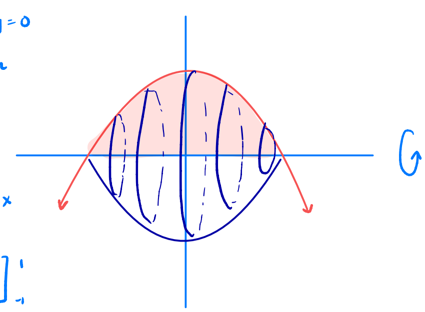
$= \int_{-1}^{1} \pi(1+x^4-2x^2)\,dx$

$= \left[\pi\left(x + \frac{x^5}{5} - \frac{2x^3}{3}\right)\right]_{-1}^{1}$

$= \pi\left(1+\frac{1}{5}-\frac{2}{3} - \left(-1-\frac{1}{5}+\frac{2}{3}\right)\right)$

$= \pi\left(2+\frac{2}{5}-\frac{4}{3}\right)$

$= \boxed{\frac{16\pi}{15}}$



Problem 6: (Stewart 6.4) Find the volume of the solid obtained by rotating the region bound by $y = \sqrt{25 - x^2}$, $y = 0$, $x = 2$, and $x = 4$ about the x-axis.

My Attempt:

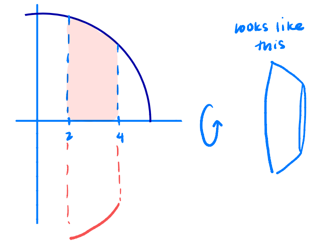
Solution:

$$V = \int_2^4 \pi \left(\sqrt{25 - x^2}\right)^2 dx$$

$$= \pi \int_2^4 (25 - x^2)\, dx$$

$$= \pi \left[25x - \frac{x^3}{3}\right]_2^4$$

$$= \boxed{\frac{94\pi}{3}}$$

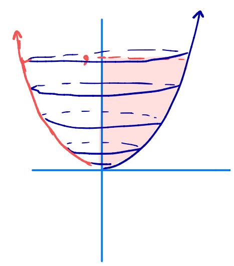


Problem 7: (Stewart 6.4) Consider the solid obtained by rotating the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y-axis. Sketch the solid, and then find its volume. *Hint: try integrating with respect to y instead of x.*

My Attempt:

Solution:

$X = 2\sqrt{y} \iff y = \frac{x^2}{4}$ (might help w/sketching)



$$V = \int_0^9 \pi (2\sqrt{y})^2\, dy$$

$$= \pi \int_0^9 4y\, dy$$

$$= \pi \cdot \left[2y^2\right]_0^9$$

$$= \boxed{162\pi}$$

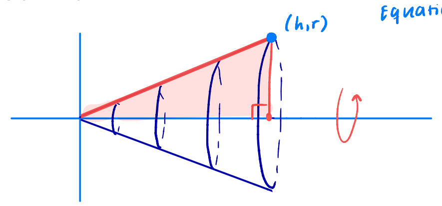
Problem 8: (Stewart 6.4) Use a definite integral to find the volume of a right circular cone with height h and base radius r.

My Attempt:

Solution:

Rotate triangle about the x-axis

Equation of line: $y = \frac{r}{h}x$



$$\text{volume} = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$$

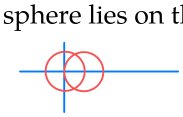
$$= \int_0^h \pi \frac{r^2x^2}{h^2}\, dx$$

$$= \pi \cdot \left[\frac{r^2x^3}{3h^2}\right]_0^h$$

$$= \frac{\pi r^2 h^3}{3h^2}$$

$$= \boxed{\frac{\pi r^2 h}{3}}$$

Challenge problem: (Stewart 6.4) Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.



$$V = 2\pi \int_{r/2}^{r} (r^2 - x^2)\, dx = \cdots = \frac{5\pi r^3}{12}$$

. . . .

You made it to the very end! Thanks for coming to discussion this semester :)

Visit tinyurl.com/sections10a for my discussion resources.