Quiz 5 study guide

October 6th, 2024

General information

Quiz 5 covers sections 2.2-2.5. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ The definition of a limit at infinity, and how limits at infinity relate to horizontal asymptotes (section 2.2)
- ☐ Using limit laws with limits at infinity (section 2.2)
- ☐ Strategies for evaluating limits (combining fractions, dividing the numerator/denominator by the highest power of x, multiplying/dividing by the conjugate) (section 2.2, 2.4)
- ☐ The limit laws for sums, differences, products, quotients, and exponentiation (section 2.3)
- ☐ The direct substitution property for polynomials: If f is a polynomial, then $\lim_{x\to a} f(x) = f(a)$ (section 2.4)
- ☐ If direct substitution gives us an indeterminate form (i.e. $\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0$, etc), we must modify the function somehow using the limit laws or other properties (section 2.4)
- ☐ The definition of a finite limit, and how finite limits connect to vertical asymptotes (section 2.3)
- ☐ Solving limits using the squeeze theorem (section 2.4)
- ☐ The limit definition of continuity, and how to find where functions are continuous (by first finding all of their discontinuities) (section 2.5)
- ☐ The limit definition of continuity from the left, right, and on an interval (section 2.5)
- ☐ If f and g are continuous at a, then so are $f+g$, $f-g$, fg, and f/g (assuming $g(a) \neq 0$). (section 2.5)
- ☐ Where polynomial, rational, root, exponential, power, logarithmic, trigonometric, and inverse trigonometric functions are continuous (section 2.5)
- ☐ Continuity of compositions of functions (section 2.5)
- ☐ The statement of the Intermediate Value Theorem (IVT) (section 2.5)

Help! I'm stuck on....

- ...solving problems about **limits at infinity**: check out this 13 minute video (tons of examples)
- ...solving problems with the **limit laws**: check out this 12 minute video
- ...intution for the **Squeeze Theorem**: check out this 11 minute video
- ...intuition for **continuity**: check out this 13 minute video
- ...how to tell **where a function is continuous**: check out this 10 minute video
- ...what continuity from the **left/right** means: check out this 3 minute video
- ...what the limits of **compositions** of functions are: check out this 5 minute video
- ...what the **Intermediate Value Theorem** (IVT) says: check out this 8 minute video
- ...how to use IVT to find **roots**: check out this 13 minute video
- ...how to evaluate **limits at infinity**/horizontal asymptotes: check out this 19 minute video

Practice problems

1. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.
2. Find $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)}\right)$.
3. Find $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$.
4. Find $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.
5. Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.
6. Find $\lim_{x\to 0} x^2 \sin\left(\frac{3}{x}\right) + 2$.
7. Determine the where the following rational function is continuous:
$$f(x) = \frac{3x^2 - 2x - 1}{x^2 - 4}.$$
8. Determine the values of x where the following piecewise function is continuous from the right:
$$f(x) = \begin{cases} 2x+3, & \text{if } x < 1 \\ x^2 - 1, & \text{if } x \geq 1. \end{cases}$$
9. Determine the value of a that makes the following piecewise function continuous at $x = a$:
$$f(x) = \begin{cases} 2x+3, & \text{if } x < a \\ x^2 - 1, & \text{if } x \geq a. \end{cases}$$
10. Determine the points of discontinuity of the following piecewise function:
$$f(x) = \begin{cases} \sin(x), & \text{if } x < \pi \\ \cos(x), & \text{if } x \geq \pi \end{cases}$$
11. Verify that the following function is continuous:
$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$
12. Find $\lim_{x\to\infty} \arctan(x)$.
13. Find $\lim_{x\to\infty} \sqrt{\frac{3}{x} + 7}$.
14. Find $\lim_{x\to\infty} \frac{\sqrt{4x^2+1}-2x}{3x}$.
15. Find $\lim_{x\to 0} \left(x^2 + \sin^2(x)\right)$.
16. Find $\lim_{x\to -\infty} \tan\left(\frac{\pi}{2} + \frac{1}{x}\right)$.

Solutions

1. If we plug in the value of 1, we get an indeterminate form. This means we need to modify the function somehow. To find the limit of the function
$$\lim_{x\to 1} \frac{x^2-1}{x-1},$$
we can simplify the expression by factoring the numerator:
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1}.$$
Now, we can cancel out the common factor of $(x-1)$ in the numerator and denominator:
$$\lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1}(x+1).$$
Finally, we can directly evaluate the limit:
$$\lim_{x\to 1}(x+1) = 1+1 = 2.$$
So, the limit of the given function as x approaches 1 is 2.

2. If we plug in the value of 0, we get an indeterminate form. This means we need to modify the function somehow. To find the limit of the function
$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)}\right),$$
we can start by simplifying the expression. First, find a common denominator for the two terms in the expression:
$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)}\right) = \lim_{t\to 0} \frac{(t+1)-1}{t(t+1)}.$$
Now, simplify the numerator:
$$t+1-1 = t.$$
So, our expression becomes:
$$\lim_{t\to 0} \frac{t}{t(t+1)}.$$
Now, we can cancel the common factor of t in the numerator and denominator:
$$\lim_{t\to 0} \frac{\cancel{t}}{\cancel{t}(t+1)} = \lim_{t\to 0} \frac{1}{t+1}.$$
Next, we can directly evaluate the limit:
$$\lim_{t\to 0} \frac{1}{t+1} = \frac{1}{0+1} = 1.$$

3. If we plug in the value of 2, we get an indeterminate form. This means we need to modify the function somehow. To find the limit of the function
$$\lim_{x\to 2} \frac{x^3-8}{x^2-4},$$
we can simplify the expression by polynomial division. First, factor the numerator and denominator:
$$x^3-8 = (x-2)(x^2+2x+4),$$
$$x^2-4 = (x-2)(x+2).$$

Now, we can rewrite the expression as follows:

$$\lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}.$$

Next, cancel the common factor of $(x-2)$ in the numerator and denominator:

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)}.$$

Now, we can directly evaluate the limit:

$$\lim_{x \to 2} \frac{x^2+2x+4}{x+2} = \frac{2^2+2(2)+4}{2+2} = \frac{4+4+4}{4} = \frac{12}{4} = 3.$$

So, the limit of the given function as x approaches 2 is 3.

4. If we plug in the value of 1, we get an indeterminate form. This means we need to modify the function somehow. To find the limit of the function

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1},$$

we can multiply both the numerator and denominator by the conjugate of the numerator, which is $\sqrt{x}+1$:

$$\begin{aligned} \lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} &= \lim_{x \to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} \\ &= \lim_{x \to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)}. \end{aligned}$$

Now, we can cancel the common factor of $(x-1)$ in the numerator and denominator:

$$\lim_{x \to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1}.$$

Now, we can directly evaluate the limit:

$$\lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

So, the limit of the given function as x approaches 1 is $\frac{1}{2}$.

5. If we plug in the value of 0, we get an indeterminate form. This means we need to use a different strategy. We can use the Squeeze Theorem by setting $f(x) = -x^2$, and $h(x) = x^2$. These were calculated by noting that $\sin(\frac{1}{x})$ will always be between -1 and 1.

Now, let's compute the limits of the lower and upper bounds as x approaches 0:

$$\lim_{x \to 0} -x^2 = 0 \quad \text{(Lower Bound)}$$

$$\lim_{x \to 0} x^2 = 0 \quad \text{(Upper Bound)}$$

Both the lower and upper bounds have limits of 0 as x approaches 0.

Let $g(x) = x^2 \sin(\frac{1}{x})$. Since $f(x) \leq g(x) \leq h(x)$ for all x in some interval except possibly at $x = 0$, and the limits of both the lower and upper bounds are 0, by the Squeeze Theorem, the limit of the given function as x approaches 0 is also 0:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

6. If we plug in the value of 0, we get an indeterminate form. This means we need to use a different strategy. To find
$$\lim_{x\to 0} x^2 \sin\left(\frac{3}{x}\right) + 2,$$
we can use the Squeeze Theorem by establishing bounds. Set $f(x) = 2 - x^2$ and $h(x) = 2 + x^2$. We know the given function $g(x) = x^2 \sin\left(\frac{3}{x}\right) + 2$ is bounded by the lower and upper bounds, since $\sin(\frac{3}{x})$ will always be between -1 and 1. For $x \neq 0$, we have:
$$2 - x^2 \leq h(x) \leq 2 + x^2 \quad \text{(Given Function)}$$
Now, let's compute the limits of the lower and upper bounds as x approaches 0:
$$\lim_{x\to 0}(2 - x^2) = 2 \quad \text{(Lower Bound)}$$
$$\lim_{x\to 0}(2 + x^2) = 2 \quad \text{(Upper Bound)}$$
Both the lower and upper bounds have limits of 2 as x approaches 0.

Since $2 - x^2 \leq h(x) \leq 2 + x^2$ for all x in some interval except possibly at $x = 0$, and the limits of both the lower and upper bounds are 2, by the Squeeze Theorem, the limit of the given function as x approaches 0 is also 2:
$$\lim_{x\to 0} x^2 \sin\left(\frac{3}{x}\right) + 2 = 2.$$

7. To find where the rational function $f(x)$ is continuous, we use the fact that rational functions (i.e. divisions of polynomials) are continuous wherever their denominator is nonzero. Note that this rule does not necessarily hold when we do not have a polynomial in the numerator, denominator, or both.

First, let's consider the denominator of the function, which is $x^2 - 4$. It is a quadratic polynomial, and it is equal to zero when $x = \pm 2$. Therefore, the rational function is discontinuous at $x = 2$ and $x = -2$.

So, the rational function $f(x)$ is continuous for all values of x except $x = 2$ and $x = -2$. In interval notation, this can be written as:
$$\text{Continuous on } (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

8. To find where the piecewise function $f(x)$ is continuous from the right, first note that linear functions are continuous (both on the left and on the right) on their domains. So we only really need to consider the points where the two pieces of the function meet, which is $x = 1$. We'll analyze the continuity of $f(x)$ at this point.

We can only say $f(x)$ is continuous from the right at $x = 1$ if $\lim_{x\to 1^+} f(x) = f(1)$. Note that $f(1) = 1^2 - 1 = 0$, because the second expression in our piecewise function defines the functional value for $x = 1$. Also, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1} x^2 - 1 = 1$, because the second part of the piecewise function defines the function for values bigger than 1.

Since $\lim_{x\to 1^+} f(x) = f(1)$, the function is continuous from the right at $x = 1$. Since the function must be continuous from the right everywhere else, we can conclude that the function is continuous on $(-\infty, \infty)$.

9. To make the piecewise function $f(x)$ continuous at a specific value of $x = a$, we need to ensure that the values of the two pieces of the function match at this point. In other words, we need to make sure that the limit of the function as x approaches a from the left and the right are equal.

As we approach the value of a from the left, we use the first expression in our piecewise function to find the limit. For the first piece of the function $(2x + 3)$, as x approaches a (i.e., as x gets closer to a without actually equaling it), the value of $2x + 3$ is $2a + 3$.

As we approach the value of a from the right, we use the second expression in our piecewise function to find the limit. For the second piece of the function $(x^2 - 1)$, as x approaches a from the right (i.e., as x gets closer to a without actually equaling it), the value of $x^2 - 1$ is $a^2 - 1$.

To make $f(x)$ continuous at $x = a$, we need to set these two values equal to each other:

$$2a + 3 = a^2 - 1$$

Now, let's solve this equation for a:

$$\begin{aligned} 2a + 3 &= a^2 - 1 \\ 0 &= a^2 - 2a - 4 \end{aligned}$$

To find the values of a that satisfy this quadratic equation, we can use the quadratic formula. In this case, a can be found as follows:

$$a = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

Simplify further:

$$a = \frac{2 \pm \sqrt{4 + 16}}{2}$$

$$a = \frac{2 \pm \sqrt{20}}{2}$$

Now, simplify the square root:

$$a = \frac{2 \pm 2\sqrt{5}}{2}$$

We can simplify this further by factoring out a common factor of 2:

$$a = \frac{2(1 \pm \sqrt{5})}{2}$$

Finally, cancel the common factor of 2:

$$a = 1 \pm \sqrt{5}$$

Therefore, the piecewise function $f(x)$ can be made continuous at either $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$. These are the values of a that make $f(x)$ continuous.

10. To find the points of discontinuity of the piecewise function $f(x)$, we need to identify where the function changes from one piece to another. In this case, the function switches from $\sin(x)$ to $\cos(x)$ at $x = \pi$.

    The first piece of the function($\sin(x)$) is defined and continuous for all real numbers.The second piece of the function $(\cos(x))$ is also defined and continuous for all real numbers. Therefore, both pieces of the function are individually continuous on their respective domains.

    However, the point of discontinuity occurs at $x = \pi$, where the function transitions from $\sin(x)$ to $\cos(x)$. At this point, the function is not continuous because the values of $\sin(x)$ and $\cos(x)$ do not match at $x = \pi$ (verify this yourself).

    In summary, the point of discontinuity for the piecewise function $f(x)$ is $x = \pi$. The function is continuous for all values of x except at $x = \pi$.

11. Note that each of the individual functions in the piecewise function are continuous on their own domains (do you see why?), so we only really care about $x = 0$. To prove that the piecewise function $f(x)$ is continuous at $x = 0$, we need to show that the following three conditions are met:

    (a) $f(0)$ is defined.

    (b) The limit of $f(x)$ as x approaches 0 exists.

(c) The limit of $f(x)$ as x approaches 0 is equal to $f(0)$.

Let's first evaluate the first condition. We evaluate $f(0)$ directly: $f(0) = 0$. Therefore, $f(0)$ is defined and the first condition is met.

For the second condition, we need to show the limit of $f(x)$ as x approaches 0 exists. To find the limit of $f(x)$ as x approaches 0, we use the definition of the piecewise function:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(x \sin\left(\frac{\pi}{x}\right)\right)$$

Now, we need to evaluate this limit. We can use the Squeeze Theorem here. Recall that $-1 \leq \sin(\theta) \leq 1$ for any real number θ. Therefore, we can establish the following inequalities:

$$-x \leq x \sin\left(\frac{\pi}{x}\right) \leq x$$

Now, we have the expression $x \sin\left(\frac{\pi}{x}\right)$ bounded by $-x$ and x. We can take the limit as x approaches 0 for each of these inequalities separately:

$$\lim_{x\to 0}(-x) = 0 \quad \text{and} \quad \lim_{x\to 0}(x) = 0$$

By the Squeeze Theorem, if we have a sequence of functions $g(x) \leq f(x) \leq h(x)$, and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$. In our case, $g(x) = -x$, $f(x) = x\sin\left(\frac{\pi}{x}\right)$, and $h(x) = x$, and we've shown that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$. Therefore, by the Squeeze Theorem:

$$\lim_{x\to 0} x \sin\left(\frac{\pi}{x}\right) = 0$$

So, $\lim_{x\to 0} x\sin\left(\frac{\pi}{x}\right) = 0$, which means $\lim_{x\to 0}$ exists and the second condition is met.

As for the third condition, we have $0 = \lim_{x\to 0} f(x)$ and $0 = f(0)$, $\lim_{x\to 0} f(x) = f(0)$ and the third condition is met.

Since all three conditions are met, the function is continuous at 0. We can conclude that the function is indeed continuous on its entire domain.

12. The arctan function represents the angle whose tangent is x. As x becomes very large, the tangent of the angle approaches infinity, which means the angle itself approaches $\frac{\pi}{2}$ radians or 90 degrees. Therefore, we have:

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

So, the limit of $\arctan(x)$ as x approaches infinity is $\frac{\pi}{2}$, indicating that the arctan function approaches $\frac{\pi}{2}$ radians or 90 degrees as x becomes infinitely large.

13. As x approaches infinity, $\frac{3}{x}$ approaches 0. So, the expression inside the square root approaches 7.

$$\lim_{x\to\infty} \sqrt{\frac{3}{x} + 7} = \sqrt{0+7}$$

Now, as x approaches infinity, the limit is simply:

$$\lim_{x\to\infty} \sqrt{\frac{3}{x} + 7} = \sqrt{7}$$

Therefore, the limit as x approaches infinity of $\sqrt{\frac{3}{x} + 7}$ is $\sqrt{7}$.

14. We want to find the limit as x approaches infinity of the expression:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}-2x}{3x}$$

We can simplify the expression by multiplying and dividing by the conjugate of the numerator:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}-2x}{3x} \cdot \frac{\sqrt{4x^2+1}+2x}{\sqrt{4x^2+1}+2x}$$

Now, let's multiply the numerators and denominators:

$$\lim_{x\to\infty} \frac{(\sqrt{4x^2+1}-2x)(\sqrt{4x^2+1}+2x)}{3x(\sqrt{4x^2+1}+2x)}$$

Next, we can expand the numerators:

$$\begin{aligned}(\sqrt{4x^2+1}-2x)(\sqrt{4x^2+1}+2x) &= (4x^2+1)-(2x)^2 \\ &= 4x^2+1-4x^2 \\ &= 1\end{aligned}$$

Now, our expression becomes:

$$\lim_{x\to\infty} \frac{1}{3x(\sqrt{4x^2+1}+2x)}$$

Clearly as x approaches infinity, the denominator grows without bound. This means that the limit as x approaches infinity of the given expression is 0.

15. To find the limit as x approaches 0 of the expression:

$$\lim_{x\to 0} \left(x^2+\sin^2(x)\right)$$

we can evaluate this limit using properties of limits.

First, let's consider the behavior of the terms:

$$\lim_{x\to 0} x^2 = 0$$

$$\lim_{x\to 0} \sin^2(x) = \sin^2(0) = 0$$

Now, we can use the limit laws to find the limit of the sum:

$$\lim_{x\to 0} \left(x^2+\sin^2(x)\right) = \lim_{x\to 0}(0+0) = 0$$

So, the limit as x approaches 0 of the given expression is 0.

16. Let's evaluate this as a composition of functions. The function whose limit we're trying to evaluate can be written as $f(g(x))$ where $g(x) = \frac{\pi}{2} + \frac{1}{2}$ and $f(x) = \tan(x)$.

Let's first find the limit as x approaches negative infinity of $\frac{\pi}{2} + \frac{1}{x}$:

$$\lim_{x\to -\infty} \left(\frac{\pi}{2} + \frac{1}{x}\right)$$

As x approaches negative infinity, the second term in the sum approaches 0:

$$\lim_{x \to -\infty} \frac{1}{x} = 0,$$

but note that it will always be slightly *less* than 0 because we are dividing by a very large *negative* number. This means $\left(\frac{\pi}{2} + \frac{1}{x}\right)$ will approach $\frac{\pi}{2}$ but will always be *less* than $\frac{\pi}{2}$. Now, we can evaluate the limit $\lim_{x \to -\infty} \tan\left(\frac{\pi}{2} - \frac{1}{2}\right)$ as

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x),$$

because (as we established) the value inside of the tangent will be approaching $\frac{\pi}{2}$ from the negative side.

Using the graph of $\tan(x)$ (which you should know!!), we evaluate this limit to be ∞.