# Math 10A Fall 2024 Worksheet 13

October 12, 2024

## 1 The Power Rule

1. Let $f(x) = 2 + x - x^2$. Compute

   a) $f'(0)$; b) $f'(1/2)$; c) $f'(1)$; d) $f'(-10)$.

2. Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$, find all $x$ for which

   a) $f'(x) = 0$; b) $f'(x) = -2$; c) $f'(x) = 10$.

3. For each function below, find the derivative and the domain of the derivative.

   a) $f(x) = x^2$;
   b) $f(x) = 3x$;
   c) $f(x) = x^2 + 3x + 2$;
   d) $f(x) = x^4 + \sin x$;
   e) $f(x) = x^{3/2}$;
   f) $f(x) = x^{1/2} + x^{1/3} + x^{1/4}$;
   g) $f(x) = \frac{d}{dx} \sin x$;
   h) $f(x) = \frac{d}{dx}\left(\frac{d}{dx} \sin x\right)$.

4. Find the derivative of
$$f(x) = \frac{\sqrt{x}}{x^{7/2}}.$$

5. Suppose $P(x) = ax^3 + bx^2 + cx + d$. Moreover, $P(0) = P(1) = -2$, $P'(0) = -1$, and $P''(0) = 10$. Find $a, b, c$, and $d$.

6. Suppose that the height of a projectile is given by $f(t)$ at $t$ seconds after being fired directly upward from the ground. If the initial velocity of the projectile is $v_0$, then
$$f(t) = v_0 t - 16t^2 \text{ ft/sec}.$$

   a) Show that the average velocity of the projectile during a time interval from $t$ to $t+h$ is $v_0 - 32t - 16h$ ft/sec. *Hint: the velocity is the instantaneous rate of change of the height function.*
   b) What is the velocity at the moment the projectile returns to the ground?
   c) What must the initial velocity of the projectile be for it to return to the ground after $s$ seconds?
   d) The acceleration is the rate of change of velocity. Show that the acceleration of this projectile is constant.
   e) Find a formula for a height function $g(t)$ which has a constant acceleration of $-20$ ft/sec.

## 2 Other derivative shortcuts

1. Evaluate
$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1}.$$

2. For each function below, find an equation of the tangent line to the curve at the given point.

   a) $y = 8\cos x, (\pi/3, 4)$;

   b) $y = x^2 - x^5, (1, 0)$.

3. The normal line to a function $f$ at $x = a$ is perpendicular to the tangent line at $x = a$ and intersects the curve at $(a, f(a))$. For each curve in problem $a$, find an equation of the normal line to the curve at the given point.

4. Find the first five derivatives of $f(x) = x^4 - 2x^3 + 4x^2 - 8x + 16$.

5. Find the first five derivatives of $\frac{1}{x}$.

6. Find a formula for the $n$th derivative of $\frac{1}{x}$.

7. Suppose $P = (c, d)$ is a point on the graph of $f(x) = \frac{1}{x}$.

   a) Find an equation of the line tangent to $f$ at $P$.
   Find the area of the triangle formed by the tangent line through $P$ and the coordinate axes. *Hint: sketch the triangle.*

8. Let $f(x) = |\sin x|$ and $g(x) = \sin |x|$.

   a) Where is $f$ differentiable?

   b) Where is $g$ differentiable?

9. In this exercise, we will prove that the derivative of $\cos x$ is $-\sin x$.

   a) Find a limit expression for the derivative of $\cos x$ using the form
   $$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

   b) Use the cosine sum formula
   $$\cos(a + b) = \cos a \cos b - \sin a \sin b$$
   to rewrite a term in the numerator.

   c) Simplify your expression using the following known limits:
   $$\lim_{a\to 0} \frac{\sin a}{a} = 1; \quad \lim_{a\to 0} \frac{\cos a - 1}{a} = 0.$$

10. Prove that the derivative of $\sin x$ is $\cos x$.

# Solutions

## 1 The Power Rule

1. a) $f'(0) = 1$; b) $f'(1/2) = 0$; c) $f'(1) = -1$; d) $f'(-10) = 21$.

2. a) $f'(x) = 0 \implies x = -2, 1$; b) $f'(x) = -2 \implies x = -1, 0$; c) $f'(x) = 10 \implies x = -4, 3$.

3. a) Derivative: $f'(x) = 2x$, Domain: $\mathbb{R}$
   b) Derivative: $f'(x) = 3$, Domain: $\mathbb{R}$
   c) Derivative: $f'(x) = 2x + 3$, Domain: $\mathbb{R}$
   d) Derivative: $f'(x) = 4x^3 + \cos x$, Domain: $\mathbb{R}$
   e) Derivative: $f'(x) = \frac{3}{2}x^{1/2}$, Domain: $x \geq 0$
   f) Derivative: $f'(x) = \frac{1}{2}x^{-1/2}+\frac{1}{3}x^{-2/3}+\frac{1}{4}x^{-3/4}$, Domain: $x > 0$
   g) Derivative: $f'(x) = \cos x$, Domain: $\mathbb{R}$
   h) Derivative: $f'(x) = -\cos x$, Domain: $\mathbb{R}$

4. $f(x) = x^{1/2} \cdot x^{-7/2} = x^{-3}$, $\quad f'(x) = -3x^{-4} = \frac{-3}{x^4}$.

5. $a = -4$, $\quad b = 5$, $\quad c = -1$, $\quad d = -2$.

6. a) The average velocity over the interval from $t$ to $t + h$ is:

$$\frac{f(t+h) - f(t)}{h} = \frac{\big(v_0(t+h) - 16(t+h)^2\big) - \big(v_0 t - 16t^2\big)}{h}$$

   Simplifying, we get

$$\frac{v_0 h - 16\left((t+h)^2 - t^2\right)}{h} = \frac{v_0 h - 16\left(t^2 + 2th + h^2 - t^2\right)}{h} = v_0 - 32t - 16h.$$

   Thus, the average velocity is $v_0 - 32t - 16h$ ft/sec.

   b) The velocity is the derivative of the height function, so

$$f'(t) = v_0 - 32t.$$

   At the moment the projectile returns to the ground, $f(t) = 0$:

$$v_0 t - 16t^2 = 0 \implies t(v_0 - 16t) = 0.$$

   So, $t = 0$ or $t = \frac{v_0}{16}$. The projectile begins its trajectory at $t = 0$, so it returns at $t = \frac{v_0}{16}$. The velocity is

$$f'\left(\frac{v_0}{16}\right) = v_0 - 32\left(\frac{v_0}{16}\right) = v_0 - 2v_0 = -v_0.$$

   Therefore, the velocity at the moment it returns to the ground is $-v_0$ ft/sec.

   c) When the projectile returns to the ground at time $t = s$, we have:

$$v_0 s - 16s^2 = 0 \implies s(v_0 - 16s) = 0.$$

   Solving for $v_0$:

$$v_0 = 16s.$$

   Therefore, the initial velocity must be $v_0 = 16s$ ft/sec for the projectile to return to the ground after $s$ seconds.

   d) The acceleration is the derivative of the velocity function, so

$$f'(t) = v_0 - 32t \implies f''(t) = -32.$$

   Since $f''(t) = -32$ is constant, the acceleration of the projectile is constant at $-32$ ft/sec$^2$.

   e) We want the second derivative to be $-20$. One possibility is $g(t) = -10t^2$.

## 2 Other derivative shortcuts

1. This is the limit definition of the derivative of $f(x) = x^{1000}$ evaluated at $x = 1$. Notice $f'(x) = 1000x^{999}$, so $f'(1) = 1000$.

2. a) $y = -4\sqrt{3}x + 4\sqrt{3}\frac{\pi}{3} + 4$. b) $y = -3x + 3$.

3. a) $y - 4 = \frac{1}{4\sqrt{3}}\left(x - \frac{\pi}{3}\right)$. b) $y = \frac{1}{3}x - \frac{1}{3}$.

4. $f'(x) = 4x^3 - 6x^2 + 8x - 8, f''(x) = 12x^2 - 12x + 8, f^{(3)}(x) = 24x - 12, f^{(4)}(x) = 24, f^{(5)}(x) = 0$.

5. $f'(x) = -\frac{1}{x^2}, f''(x) = \frac{2}{x^3}, f^{(3)}(x) = -\frac{6}{x^4}, f^{(4)}(x) = \frac{24}{x^5}, f^{(5)}(x) = -\frac{120}{x^6}$.

6. $$f^{(n)}(x) = (-1)^n \cdot \frac{n!}{x^{n+1}}.$$

7. a) $y = -\frac{1}{c^2}x + \frac{2}{c}$.

   b) The $x$-intercept occurs when $y = 0$, which is at $x = 2c$. The $y$-intercept occurs when $x = 0$, so $y = \frac{2}{c}$. The area of the triangle formed by the tangent line and the coordinate axes is:
   $$A = \frac{1}{2} \cdot \frac{2}{c} \cdot 2c = 2.$$

8. a) $f(x) = |\sin x|$ is differentiable everywhere except where $\sin x = 0$, i.e., at $x = n\pi$ for integers $n$.

   b) $g(x) = \sin|x|$ is differentiable everywhere except at $x = 0$.

9. a)
   $$\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

   b) Using the cosine sum formula, we have
   $$\cos(x+h) = \cos x \cos h - \sin x \sin h.$$
   The numerator becomes
   $$\lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}.$$

   c) Using the known limits, we get:
   $$\lim_{h\to 0} \frac{-\sin x \sin h}{h} = -\sin x,$$
   and thus the derivative of $\cos x$ is $-\sin x$.

10. Follow the exact same steps from Problem 9.