Math 10A Fall 2024 Worksheet 16

October 29, 2024

1. Let $f(x) = x\sqrt{1-x}$.
   (a) Find the domain of $f(x)$ and $f'(x)$.
   (b) Find all critical points of $f(x)$.
   (c) Find the intervals of increase or decrease.
   (d) Find the intervals of concavity and the inflection points.
   (e) Find the absolute maximum and minimum values on the interval $[-1, 1]$.

2. Compute the following limits.
   (a) $\lim_{x\to\infty} \frac{\ln x}{x}$
   (b) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$
   (c) $\lim_{x\to\infty} xe^{-x}$
   (d) $\lim_{x\to 0} xe^{-x}$
   (e) $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$
   (f) $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

1 Solutions

1. (a) By the product rule, $f'(x) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$. Domain $f(x)$ is $x \leq 1$, but of $f'(x)$ is $x < 1$.

   (b) If we solve $f'(x) = 0$, we get:

   $$\sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = 0 \Leftrightarrow \sqrt{1-x} = \frac{x}{2\sqrt{1-x}} \Leftrightarrow 2(1-x) = x \Leftrightarrow x = \frac{2}{3}.$$

   So the unique critical point is $(\frac{2}{3}, f\left(\frac{2}{3}\right)) = (\frac{2}{3}, \frac{2}{3\sqrt{3}})$. Note that $x = 1$ is not a critical point since it is not in a domain of $f(x)$.

   (c) We have

   $$f'(x) = \frac{2(1-x) - x}{2\sqrt{1-x}} = \frac{2-3x}{\sqrt{1-x}},$$

   and this is positive (resp. negative) if $x < 2/3$ (resp. $2/3 < x < 1$). Hence $f(x)$ increases on $(-\infty, 2/3)$ and decreases on $(2/3, 1)$.

   (d) The second derivative of $f(x)$ is

   $$f''(x) = \frac{3x-4}{4(1-x)^{3/2}},$$

   and this is always negative on the domain $x < 1$. Hence $f(x)$ is concave downward for $x < 1$. There's no inflection point since $f''(x)$ is always negative on its domain.

   (e) Compare the boundary values $f(-1) = -\sqrt{2}$, $f(1) = 0$ with critical value $f(2/3) = 2/(3\sqrt{3})$. The absolute maximum is $f(2/3) = 2/(3\sqrt{3})$ and the absolute minimum is $f(-1) = -\sqrt{2}$.

2. Compute the following limits.

   (a) Since it has a form of ∞/∞, we can apply l'Hospital's rule:

   $$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

   (b) Since it has a form of $0/0$, we can apply l'Hospital's rule:

   $$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{4/(1+(4x)^2)} = \frac{1}{4}.$$

   (c) Since the limit $\lim_{x\to\infty} \frac{x}{e^x}$ has a form of ∞/∞, we can apply l'Hospital's rule:

   $$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

   (d) We don't need l'Hospital's rule (and actually we can't). The limit is $0 \cdot e^{-0} = 0$.

   (e) Since it has a form of $0/0$, we can apply l'Hospital's rule:

   $$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2} = \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\cos(2x)}{1} = 2$$

   where we use l'Hospital's rule again for the last equality (actually we don't have to - can you interpret it as a derivative of some function?).

   (f) We can write limit as

   $$\lim_{x\to 1^+} \frac{x}{x-1} - \frac{1}{\ln x} = \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x}$$

   which is the case where we can use l'Hospital's rule (0/0). By applying it twice,

   $$\lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\ln x + 1 - 1}{\ln x + (x-1)/x} = \lim_{x\to 1^+} \frac{\ln x}{\ln x + 1 - 1/x}$$
   $$= \lim_{x\to 1^+} \frac{1/x}{1/x + 1/x^2} = \frac{1}{2}.$$