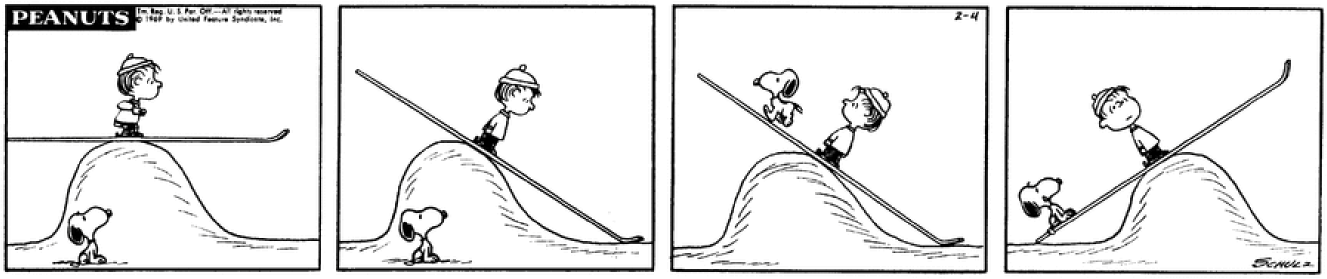

Math 1A: Calculus

Discussion Workbook

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Contents

Introduction

Welcome to Math 1A!

What are Discussion Sections?

According to Berkeley's GSI Teaching and Resource Center:

> *Discussion sections provide opportunities for collaboration and active learning that do not always take place in a traditional lecture context. The role of section goes beyond clearing up any confusion remaining after the course material has been presented in lecture. Section also provides students with the opportunity to discuss, ask questions, and apply course content, resulting in deeper learning.*

During remote summer sessions, section looks a little bit different than during regular, in-person sessions—rather than working on problems in classrooms, you have been provided with the worksheets in this book to be completed remotely and in tandem with lecture. But all discussion sections have one thing in common: solving problems to further understanding. We that you can take advantage of collaborative platforms, such as Ed discussion, to work through these problems with your peers, increase your confidence with course content, and prepare yourself for exams.

How to Use This Workbook

This workbook is made up of worksheets, one for each section. The worksheets have three major parts:

1. <u>Lecture recap</u>: The material above the dotted line on the first page of each worksheet will ask you to recall major theorems and results from lecture.

2. <u>Mastery problems</u>: Each worksheet has between three and six problems designed to test your understanding of the section.

3. <u>Challenge problems</u>: These problems are usually more difficult than what you can expect on an exam, but can be solved using similar techniques as the mastery problems.

Each worksheet is accompanied with video explanations and written solutions for the mastery problems. It is most effective to attempt the problems on your own or with peers first, and then check your work after.

Strategies for Challenge Problems

Challenge problems are supplements to mastery problems—they may be more difficult than homework and exams. Many challenge problems will ask you to prove statements. Here, we provide some information on a few types of mathematical proofs.

A *direct* proof is perhaps the most straightforward kind of proof: we begin with the information provided, and use facts and inference to arrive at the statement to be proven.

For example, suppose we want to show that the sum of any two odd integers is even. We start with what we know: the two odd integers can be written as $2k+1$ and $2l+1$ where k and l are integers. Then the sum of the integers is $2k+1+2l+1 = 2(k+l)+2 = 2(k+l+1)$, which must be even. Thus the sum of any two odd numbers is even.

A proof by *contradiction* is a way to prove a statement is true by showing that the statement cannot be false. For example, suppose someone claims that they cannot eat an entire pizza in one sitting without getting full. To prove this by contradiction, we suppose on the contrary that they can eat an entire pizza in one sitting without getting full. Then at some point, while eating a pizza, they will get full, which contradicts our hypothesis.

Here's a mathematical example—let's say we want to prove that if x^2 is odd, then x is also odd. To do this, we suppose on the contrary that if x^2 is odd then x can be even. Then we may write $x = 2k$ for some integer k, such that $x^2 = (2k)^2 = 2(2k^2)$, which is necessarily even. But this contradicts our premise: we assumed that x^2 is odd. Thus we can conclude that x must be odd (otherwise, we arrive at a contradiction).

A proof by *mathematical induction* is a way to show that something is true for every natural number. They often have the same general format:

1. Base case: Show that the statement is true for the first or first few cases.
2. Induction hypothesis: Suppose that the statement is true for some arbitrary-numbered case—for example, the kth case.
3. Induction step: Show that the statement is true for the $(k+1)$st case.

To see how this is used in practice, let's do a worked example. We'll prove a familiar identity $1+2+3+...+n = \frac{n(n+1)}{2}$.

1. Base case: We'll show that the statement is true for the first case: for $n = 1$. Notice that $1 = \frac{1(1+1)}{2}$, so we are done.
2. Induction hypothesis: We suppose that the statement is true for the kth case; in other words, we suppose that $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$.
3. Induction step: We show that $1 + 2 + 3 + ... + k + (k+1) = \frac{(k+1)(k+2)}{2}$, assuming the induction hypothesis. Notice that
$$1 + 2 + 3 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$
Then we can rewrite the right hand side:
$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$
as desired.

Then we can conclude, by the principle of mathematical induction, that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$.

Additional Resources

Here is a short list of resources available to Berkeley students.

- If you are in need of laptops, Wi-Fi hotspots, or other required technologies, you can visit the Student Technology Equity Program (STEP) at https://studenttech.berkeley.edu/step.
- If your name differs from your legal name, you may indicate a preferred name by following the directions provided by the registrar, which can be found at https://registrar.berkeley.edu/academic-records/your-name-on-records-rosters/.
- For free and confidential mental health services, Counseling and Psychological Services (CAPS) is here for you. University Health Services also lists some resources.
- For books, magazines, movies, printing services, museum passes, state park passes, and more, you can apply for a Berkeley Public Library card at https://catalog.berkeleypubliclibrary.org/obr/.

Feedback and Errata

To report any errors in discussion resources, visit the form at https://tinyurl.com/math1a-errata. This form is *not* anonymous in case we need to have some dialogue before making changes.

To provide anonymous feedback, visit the form at https://tinyurl.com/math1a-feedback. This form will verify you are a Berkeley student by checking your email address, but will not collect any information about you.

Four Ways to Represent a Function

Chapter 1, Section 1

Here are some important ideas from lecture:

- **Circle one**: A function is a rule that assigns to each element x from its domain [more than one / exactly one / less than one] element in its range.
- We can represent functions verbally (words), numerically (table of values), visually (graph), or algebraically (explicit formula).
- The Vertical Line Test is a way to tell whether or not a graph in the xy-plane is a function.

Vertical Line Test

An xy-curve is the graph of a function *if and only if* no vertical line intersects the curve more than once.

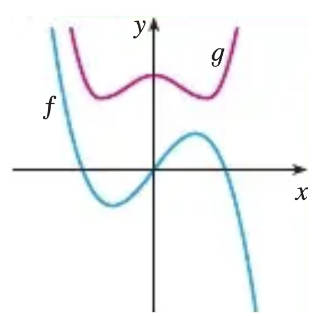
- **Fill in the blanks**: A function f is *even* if ________________. A function f is *odd* if __________________. These rules must hold for all x.

Mnemonic for even and odd functions

Symmetry has to do with the behavior of the function for input values of $-x$ as opposed x. One way to remember what even and odd functions do is **E**ven **E**ats the negative while **O**dd spits it **O**ut.

- A function f is called *increasing* on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
- A function f is called *decreasing* on an interval I if ____.

Problem 1: (Stewart Section 1.1) Consider the following graph, which depicts the functions f and g. If f even, odd, or neither? Why? Is g even, odd, or neither? Why?



Problem 2: (Borcherds '05 Midterm 1) Find the domain of the function $g(u) = \sqrt{u} + \sqrt{2-u}$.

Problem 3: (Stewart Section 1.1) Recall that a *piecewise function* splits its domain into pieces and is defined by different formulas for each piece. Sketch the graph of the following piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$

Problem 4: (Stewart Section 1.1) Determine whether $f(x) = x|x|$ is even, odd, or neither.

Problem 5: (Stewart Section 1.1) Does $x^2 + (y-3)^2 = 5$ define a function? Explain why or why not.

Problem 6: (Stewart Section 1.1) An open rectangular box with volume 2m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

Challenge problem: Consider the function $f(x) = 4 + 3x - x^2$. Evaluate the difference quotient given by

$$\frac{f(3+h) - f(3)}{h}.$$

Mathematical Models: a Catalog of Essential Functions

Chapter 1, Section 2

We use *mathematical models* to describe and predict real world phenomena. Below are some of the mathematical models we covered. **Refer to the table at the end of the section for graphs of these functions.**

Type of function	**Function definition**
Linear	Expression:
Polynomial	General expression: Quadratic: Cubic:
Power	General expression: $n > 0$: Root function: Reciprocal function: Inverse square law:
Rational	Expression:
Algebraic	Definition:
Transcendental	Definition:
Trigonometric	Three main examples:
Exponential	Expression:
Logarithmic	Expression:

Problem 1: (Stewart Section 1.2) A landlord knows that charging x dollars for a rental space at the market will cause y spaces to be rented. The landlord creates a mathematical model for y which is the equation $y = 200 - 4x$.

a. What type of function is y?

b. Sketch a graph of the function. *Hint:* what values of x make sense?

c. Practically, what do the slope, y-intercept, and x-intercept represent?

Problem 2: (Stewart Section 1.2) Find the domain of the function $f(x) = \frac{\cos x}{1-\sin x}$.

Problem 3: (Stewart Section 1.2) Review the inverse square law from the table on the previous page. Suppose we create a light source and model its illumination from the source using an inverse square law (i.e., our independent variable is distance and our dependent variable is illumination). You are at a distance x from the light source. Then, you move to a distance $x/2$ from the light source. How much brighter is the light?

Problem 4: (Stewart Section 1.2) Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

Problem 5: (Stewart Section 1.3) At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent. Express the water pressure as a function of the depth below the ocean surface.

Challenge problem: (Bamler Spring '18 Final Exam) Find the range of $f(x) = \ln(x^2 + 3)$.

New Functions From Old Functions

Chapter 1, Section 3

We can get new functions by modifying familiar forms. Here are some examples:

Transformation	**Equation of transformation**
c units upward	
c units downward	
c units to the right	
c units to the left	
Vertical stretch by a factor of c	
Vertical shrink by a factor of c	
Horizontal shrink by a factor of c	
Horizontal stretch by a factor of c	
Reflection across the x-axis	
Reflection across the y-axis	

Finally, we can combine functions to get other functions. When we modify functions, whether by transformation or by combination, it is important to **consider how the domain and range changes accordingly**.

Function combination	**Equation of function combination**
Composition of functions	
Sum of functions	
Difference of functions	
Quotient of functions	

Problem 1: (Stewart Section 1.3) Relative to $f(x) = \sin(x)$, how have the following graphs been changed?

- $a(x) = \sin(\frac{x}{6})$
- $b(x) = \frac{\sin(x)}{6}$
- $c(x) = \sin(x + 6)$.

Problem 2: (Stewart Section 1.3) Under ideal conditions, a certain bacteria population is known to double every 2 hours. Suppose there are initially 700 bacteria. What is the size of the population after t hours?

Problem 3: (Stewart Section 1.4) Find the domain of $f(x) = \frac{1+x}{e^{\cos x}}$.

Problem 4: (Stewart Section 1.3) Express $R(x) = \sqrt{\sqrt{x} - 1}$ in the form $f \circ g \circ h$.

Problem 5: (Borcherds '05 Midterm 1) Sketch the graph of $y = |x^2 - 2x|$.

Problem 6: (Stewart Section 1.4) Describe the symmetry of $f(x) = \frac{1-e^{1/x}}{1+e^{1/x}}$. Is it even, odd, or neither?

Challenge problem: (Bamler Fall '18 Final Exam)

Exponential Functions

Chapter 1, Section 4

Exponential functions take the form ____________. These are the important laws of exponents:

1. $b^{x+y} =$ ____________.
2. $b^{x-y} =$ ____________.
3. $(b^x)^y =$____________.
4. $(ab)^x =$ ____________.

The constant e satisfies that the condition that slope of the tangent line to $y = e^x$ at ____________ is _____. We call the function e^x the **natural exponential function**.

. . . .

Problem 1: (Stewart Section 1.4) Simplify $27^{2/3}$.

Problem 2: (Stewart Section 1.4) Rewrite and simplify the expression below.

$$\frac{x^3 \cdot x^n}{x^{n+1}}$$

Problem 3: (Stewart Section 1.4) Use transformation laws to sketch the graph of $h(x) = 2\left(\frac{1}{2}\right)^x - 3$.

Problem 4: (Stewart Section 1.4) Find the domain of the function below.

$$f(x) = \frac{1 - e^{x^2}}{1 - e^{1-x^2}}$$

Problem 5: (Stewart Section 1.4) Find the domain of $g(t) = \sin(e^t - 1)$.

Problem 6: (Stewart Section 1.4) If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right).$$

Challenge problem: (Stewart Chapter 1) Prove that if n is a positive integer, $7^n - 1$ is divisible by 6.

Inverse Functions and Logarithms

Chapter 1, Section 5

- A function f is called *one-to-one* if it never takes the same value twice: ________________________.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

- Suppose f is a one-to-one function with domain A and range B. Then the *inverse* of f, denoted f^{-1}, has domain ____________ and range ____________. If $f(x) = y$, then $f^{-1}(y) =$____________.
- For an inverse function, the cancellation rules are the following: $f^{-1}(f(x)) =$ _____ and $f(f^{-1}(x)) =$ _____.
- The following are the steps to find the inverse of a one-to-one function f.
  1.
  2.
  3.
- In order to make trig functions one-to-one, we have to apply domain restrictions. The restricted domain of $\sin(x)$ is $[\frac{-\pi}{2}, \frac{\pi}{2}]$. The restricted domain of $\cos(x)$ is $[0, \pi]$. The restricted domain of $\tan(x)$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Use the space below to think about what the domain and range of the inverse trig functions are.

- The inverse of the exponential function b^x is the logarithmic function $\log_b x$. The following are the laws of logarithms: $\log_b(xy) =$________________, $\log_b(\frac{x}{y}) =$ ________________, $\log_b(x^r) =$________________.
- If the base of the logarithm is e (i.e., if $\log_e x$), we can write $\ln x$ instead.
- The change of of base formula says ________________________________.

Problem 1: (Stewart Section 1.5) Find the inverse of $y = 3\ln(x-2)$.

Problem 2: (Bamler Fall '18 Final Exam) If a function f has an inverse function f^{-1}, how many roots can it have?

Problem 3: (Bamler Spring '18 Final Exam) Compute and simplify

$$\log_3\left(\frac{\sqrt[5]{27}}{16}\right) + \frac{4}{\log_2 3}.$$

Problem 4: (Bamler Fall '18 Final Exam) Simplify $\sin(\tan^{-1}(x))$.

Problem 5: (Bamler Fall '18 Final Exam) Find the inverse function f^{-1} of $f(x) = \frac{1}{3}\sqrt{7+e^{5x}}$.

Problem 6: (Bamler Spring '18 Final Exam) Simplify the following expression as much as possible:

$$\frac{1}{2}\ln\left(\frac{12}{e^5}\right) - \ln(\sqrt{3}) - \frac{1}{\log_2 e}$$

Challenge problem: (Stewart Section 1.5) Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

The Tangent and Velocity Problems

Chapter 2, Section 1

Here are some important ideas from lecture:

- **Circle one:** A line is *tangent* to a curve if it touches the curve (once / more than once) and follows the direction of the curve at the point of contact. Use the space below to draw a line l that is tangent to a circle:

- **Circle one:** A line is *secant* to a curve if it touches the curve (once / more than once).
- Draw a series of secant lines that get closer and closer to the tangent line in your drawing above. We show that we can use ____________ lines to approximate ____________ lines.

- The ____________________________ velocity is defined to be a measure of how fast an object is moving at a point along its path. The velocity problem is an application of tangent line approximations.

The Velocity Problem

The velocity problem asks to find the instantaneous velocity of an object at *any* time, provided the position at a *particular* time is given.

- We may sometimes use the ____________ velocity to approximate the instantaneous velocity, which is given by change in positition divided by time elapsed.

. . . .

Problem 1: (Stewart Section 2.1) Suppose the position of an object is given by $y = 10t - 1.86t^2$. Find its average velocity in the interval $[1, 2]$.

Problem 2: (Stewart Section 2.1) We are given that the point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$. Find the slope of the secant line PQ for $x = 0$.

Problem 3: Do the same as in problem 2, but for following values of x:

- $x = 0.4$
- $x = 1$
- $x = 0.6$

Problem 4: (Stewart Section 2.1) The position of a particle moving back and forth along a straight line is $s = 2 \sin \pi t + 3 \cos \pi t$. Find the average velocity during the interval $[1, 1.1]$.

Challenge problem: Consider a function $f(x)$ defined for $x \geq 0$ such that $f(x)$ represents the height of a staircase at position x. The staircase is formed by stacking unit cubes. The height of the staircase at $x = 0$ is $f(0) = 1$ unit. At each integer x, the height of the staircase increases by one unit, i.e., $f(x) = f(x - 1) + 1$.

Determine the equation of the secant line connecting two points $P(a, f(a))$ and $Q(b, f(b))$ on the staircase.
Explain how the secant line can be used to approximate the slope of the tangent line at the point $P(a, f(a))$. Does this approximation become more accurate as a and b get closer? Why or why not?

The Limit of a Function

Chapter 2, Section 2

In this section, we introduce the idea of limits.

The Limit of a Function

We say the limit of of $f(x)$, as x approaches a, equals L if we can make the values of $f(x)$ as close to L as we like by making x sufficiently close to a but not equal to a.

- One way to estimate the limit of a function is to __________, and making a table might help.
- A _____________________ limit only approaches an x-value from one side, either the left or the right.
- We may write
$$\lim_{x\to} f(x) = L \quad \text{if and only if} \quad \lim_{x\to} f(x) = \lim_{x\to} f(x) = L$$
- An **infinite limit** is written as $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinity is *not* a number; instead, it represents ____.
- We say the line $x = a$ is a **vertical asymptote** if:

Problem 1: Sketch a graph of $f(x) = \frac{1}{x}$. Use your graph to find $\lim_{x\to 0} f(x)$, if it exists.

Problem 2: Draw the graph of a function $f(x)$ satisfying the limits below.

- $\lim_{x\to 0^-} f(x) = -2$
- $\lim_{x\to 0^+} f(x) = -1$
- $\lim_{x\to 2^-} f(x) = 2$
- $\lim_{x\to 2^+} f(x) = 1$
- $\lim_{x\to 4^-} f(x) = -\infty$
- $\lim_{x\to 4^+} f(x) = \infty$

Problem 3: (Stewart Section 2.2) Consider the function given by

$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2-x & \text{if } x \geq 1. \end{cases}$$

Sketch the graph of f, and use your graph to find the x values where the limit does not exist.

Problem 4: (Stewart Section 2.2) Evaluate the following limit.

$$\lim_{x \to 3^-} \frac{\sqrt{x}}{(x-3)^2}$$

Problem 5: (Stewart Section 2.2) Let m represent the mass of a particle with velocity v, let m_0 represent its mass at rest, and let c be the speed of light. The theory of relativity is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

What happens as $v \to c^-$?

Challenge problem: (Stewart Section 2.2) Consider the function $f(x) = \tan \frac{1}{x}$.

1. Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$
2. Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \ldots$
3. Make a conclusion about $f(x) = \tan \frac{1}{x}$.

Calculating Limits Using the Limit Laws

Chapter 2, Section 3

The limit laws help us solve limits. They **only** apply when the limits exist and are not $\pm\infty$. Here are some of them, and be sure to refer to the book for a complete list.

- $\lim_{x\to a}\big(f(x)+g(x)\big) =$ ______________________ and $\lim_{x\to a}\big(f(x)-g(x)\big) =$ ______________________
- $\lim_{x\to a}\big(f(x)g(x)\big) =$ ______________________ and $\lim_{x\to a}\big(f(x)/g(x)\big) =$ ______________________
- $\lim_{x\to a}\big(cf(x)\big) =$ _________________ and $\lim_{x\to a}\big(f(x)\big)^n =$ ___________________

Some other properties of limits might help us better understand how to evaluate them.

- _________________ $f(x)$ assuming $x \neq a$, then take the limit with the limit laws.
- __________________________ to get a rational function. Apply polynomial division if necessary.
- If there is a radical expression $(\sqrt{a}+b)$in the denominator, rationalize by _____________________________.

The Squeeze Theorem says that if $f(x) \leq g(x) \leq h(x)$ when x is near a (can exclude a) **and** if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then ____________________________________.

Problem 1: (Stewart Section 2.3) Find $\lim_{x\to\infty}\frac{\sin x}{x}$ using the Squeeze Theorem.

Problem 2: (Stewart Section 2.3) Find the following limit in terms of a.

$$\lim_{h\to 0}\frac{\sqrt{7(a+h)}-\sqrt{7a}}{h}.$$

Problem 3: (Stewart Section 2.3) Is there a value a such that $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exists? Prove it.

Problem 4: (Stewart Section 2.3) Let $[x]$ denote the greatest integer function, where $[x]$ is the largest integer less than or equal to x. Graph $f(x) = [\cos(x)]$ and find the values of a such that $\lim_{x \to a} f(x)$ exists.

Problem 5: (Stewart Section 2.3) If p is a polynomial, show that the following equation holds (*Hint: recall that any polynomial can be written as* $a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$.)

$$\lim_{x \to a} p(x) = p(a).$$

Problem 6: (Stewart Section 2.3) If the limit below exists, find it. If not, explain why it doesn't exist.

$$\lim_{x \to 0} \frac{2 - |x|}{2 + x}$$

Challenge problem: (Stewart Section 2.3) Show that the following equation holds.

$$\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0.$$

The Precise Definition of a Limit

Chapter 2, Section 4

In this section, we formalize some concepts about limits that we learned in previous sections.

The Precise Definition of a Limit

If L is the limit at a, the distance between $f(x)$ and L can be made arbitrarily small by letting the distance from x to a be sufficiently small (but nonzero).

Let's write this mathematically. We use ε to represent the distance between the y-values, and δ to represent the distance between the x-values. If $\lim_{x\to a} f(x) = L$, then:

For solving problems, it might be helpful to know that $|x - a| < \delta$ means $a - \delta < x < a + \delta$. Also, $|f(x) - L| < \varepsilon$ means $L - \varepsilon < f(x) < L + \varepsilon$.

Let's do an epsilon-delta proof on a linear function.

Problem 1: (Stewart Section 2.4) Prove that $\lim_{x\to 3}(4x - 5) = 7$.

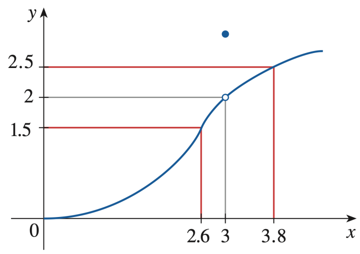
The idea of an epsilon-delta proof is the following: if you give me $\varepsilon > 0$ such that $|f(x) - L| < \varepsilon$, then I will give you some $\delta > 0$ such that ____ (*Think: if I tell you the y-values must be a distance of ε from L, then you give me a distance δ from a for the corresponding x-values.*)

Here are the steps for an epsilon-delta proof:

1. First find a possible expression for δ. Suppose you give me $|f(x) - L| < \varepsilon$. Since we know what $f(x)$ is, we can write this as: ____. Now we will modify this to get our δ:

2. Check that this value of δ works! Given some arbitrary $\epsilon > 0$, we choose $\delta =$ ____________________. If $0 < |x - a| < \delta$, then:

Problem 2: (Stewart Section 2.4) Consider the following graph of a function f. Use the graph to find a number δ such that $0 < |x-3| < \delta$ implies $|f(x)-2| < 0.5$.



Recall the definition of an *infinite limit* from previous sections. Now, to define infinite limits rigorously, consider these definitions:

- $\lim_{x\to a} f(x) = \infty$ when for every positive number M there is a positive number δ such that if $0 < |x-a| < \delta$ then $f(x) > M$.
- $\lim_{x\to a} f(x) = -\infty$ when for every negative number M there is a positive number δ such that if $0 < |x-a| < \delta$ then $f(x) < M$.

Problem 3: Prove $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Problem 4: (Stewart Section 2.4) Find a number δ such that if $|x-2| < \delta$, then $|4x-8| < \epsilon$, where $\epsilon = 0.1$.

Continuity

Chapter 2, Section 5

One way to think about continuity is the following: can we draw the graph without picking up our pencil?

- We say f is continuous at a if $\lim_{x\to a} f(x) =$ __________. (*Think: if the function approaches $f(a)$ as x gets closer to a, then we can draw the graph without picking up our pencil!*)
- We say f is continuous from the right at a if ____.
- We say f is continuous from the left at a if ____.
- A function is continuous on an interval if it is continuous at ____________________ point in the interval.
- If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. Another way to write this is:
- The Intermediate Value Theorem says that if f is continuous on $[a, b]$, then if $f(a) \neq f(b)$ and if N is between $f(a)$ and $f(b)$, there exists some c in (a, b) such that ________________________________.

. . . .

Problem 1: (Stewart Section 2.5) Use interval notation to indicate where the function below is continuous.

$$f(x) = \frac{x^2 - 1}{x^2 - 7x + 6}$$

Problem 2: (Stewart Section 2.5) Find $\lim_{x\to 1} e^{x^2-5x+4}$, and explain how your solution works.

Problem 3: (Stewart Section 2.5) Suppose $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Find $(f \circ g)(x)$. Is $f \circ g$ continuous everywhere?

Problem 4: (Stewart Section 2.5) Prove that $\cos x = x^3$ has at least one real root.

Problem 5: (Stewart Section 2.5) For the function f below, construct a function g that is continuous and "removes the discontinuities" of f.

$$f(x) = \frac{x^2 - 7x + 12}{x - 3}$$

Challenge Problem: Is there a number that is exactly 1 more than its cube?

Limits at Infinity, Horizontal Asymptotes

Chapter 2, Section 6

In a previous section, we discussed infinite limits. Now, we discuss *limits at infinity*.

- "The limit of $f(x)$, as x approaches infinity, is L" is written as:____.
- "The limit of $f(x)$, as x approaches *negative* infinity, is L" is written as:_________________________________.
- If we can find L for any infinite limit (i.e. if we can find a value L such that $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a _____________________________________.

When finding limits at infinity, it often helps to manipulate the function.

Problem 1: (Stewart Section 2.6) Find the limit below using at least two limit laws. *Hint: divide by the highest power of x in the denominator.*

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Problem 2: (Stewart Section 2.6) Find the limit below. *Hint: how can we make this look like an infinite limit?*

$$\lim_{x\to 2^+} e^{\frac{3}{2-x}}$$

Problem 3: (Stewart Section 2.6) Sketch the graph of a function satisfying the following conditions.

- $\lim_{x\to 2} f(x) = -\infty$
- $\lim_{x\to\infty} f(x) = \infty$
- $\lim_{x\to-\infty} f(x) = 0$
- $\lim_{x\to 0^+} f(x) = \infty$

Problem 4: (Stewart Section 2.6) Find $\lim_{x\to\infty}(\sqrt{x^2+5}-\sqrt{x^2-5})$. *Hint: do something with the conjugate!*

Problem 5: (Stewart Section 2.6) Suppose we have two polynomials P and Q. Find the limit below first when the degree of P is less than the degree of Q, and second when the degree of P is greater than the degree of Q.

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}$$

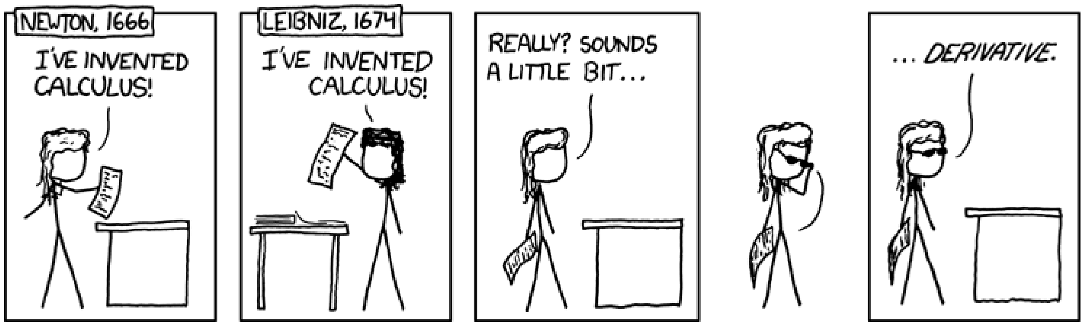
Problem 6: (Stewart Section 2.6) Find a *formula* for a function f that satisfies the following conditions:

- $\lim_{x\to\pm\infty} f(x) = 0$
- $\lim_{x\to 0} f(x) = -\infty$
- $\lim_{x\to 3^-} f(x) = \infty$
- $\lim_{x\to 3^+} f(x) = -\infty$
- $f(2) = 0$.

Challenge problem: (Stewart Section 2.6) Find $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

Derivatives and Rates of Change

Chapter 2, Section 7



It's helpful to think of the derivative of a function at a to be *the slope of the tangent line* at a.

- The limit definition of a derivative at a point is:

- We can write the derivative of $f(x)$ as:

- A function $f(x)$ is called differentiable at a if:

- If f is differentiable at a, it is also continuous at a. **Circle one:** The reverse (does / does not) hold.

Problem 1: Use the limit definition to find the derivative of $f(x) = \sqrt{x}$.

Problem 2: For $f(x) = \sqrt{x}$, sketch the graph of $f'(x)$.

Problem 3: (Stewart Section 2.7) Find an equation of the tangent line to the curve $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

Problem 4: (Stewart Section 2.7) For each of the following limits, find a function f and a value a such that the given expression represents $f'(a)$.

- $\lim_{x \to 2} \frac{x^6 - 64}{x - 2}$
- $\lim_{\theta \to \pi/6} \frac{\sin\theta - 1/2}{\theta - \pi/6}$
- $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

Problem 5: If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

Challenge Problem: (Stewart Chapter 2) Suppose f is function satisfying $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$, and then show that $f'(0) = 0$.

The Derivative as a Function

Chapter 2, Section 8

In the previous section, we considered derivatives at a particular point. In this section, we consider the derivative as a *function*, defined on a domain of points.

- The limit definition of the derivative *as a function* is:

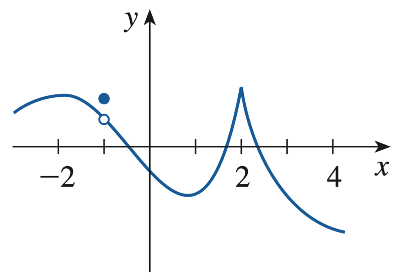
- We say a function f is ________________________________ if $f'(a)$ exists. We can also find ___________________ of differentiability.
- A function can fail to be continuous if it has a ____________, ____________________________, or ____________. Draw these three possibilities in the space below.

- We may take derivatives of derivatives, and so on. The derivatives of a position function f are:

Problem 1: (Stewart Section 2.8) Find the derivative of $f(x) = mx + b$ using the limit definition of a derivative. State the domain of the function and the domain of its derivative.

Problem 2: (Stewart Section 2.8) Make a graph of $f(x) = \sin x$, and then sketch its derivative. Use your graph to guess the equation of $f'(x)$.

Problem 3: (Stewart Section 2.8) Below is the graph of a function $f(x)$. State where (in numbers) the function is not differentiable, and explain why.



Problem 4: (Stewart Section 2.8) Prove that the derivative of an even function is an odd function.

Challenge Problem: (Stewart Section 2.8) Let l be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The angle of inclination of l is the angle that ϕ that l makes with the positive direction of the x-axis. Calculate ϕ to the nearest degree.

Differentiation Rules

Chapter 3, Section 1

Now that we know what a derivative is, we can use certain rules to find derivatives efficiently. All these rules can be derived from the limit definition of a derivative.

- For a constant a, the derivative of a with respect to x is $\frac{d}{dx}(a) =$ ____________________________.
- The power rule: x is $\frac{d}{dx}(x^n) =$ ____________________________.
- The sum/difference rule: $\frac{d}{dx}\left(f(x) \pm g(x)\right) =$ ____________________________.
- The derivative of a constant times a function: $\frac{d}{dx}(c \cdot f(x)) =$ ____________________________.
- The derivative of the natural exponential function: $\frac{d}{dx}e^x =$ ____________________________.

Problem 1: (Stewart Section 3.1) Differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$.

Problem 2: (Stewart Section 3.1) Find the derivative of $k(r) = e^r + r^e$.

Problem 3: (Stewart Section 3.1) Find the derivative of $f(x) = \frac{x^2-5x^3-\sqrt{x}}{x^2}$ without using the product rule or the quotient rule.

Problem 4: (Stewart Section 3.1) Let $y = x + e^x$. Find the equation of the line tangent to this function at $x = 0$.

Problem 5: (Stewart Section 3.1) Find equations of the tangent line and normal line to $y = x^4 + 2e^x$ at $(0, 2)$. Recall that the normal line is perpendicular to the tangent line.

Challenge problem: Find the second derivative of $\frac{2}{(6+2v-v^2)^4}$.

The Product and Quotient Rules

Chapter 3, Section 2

There are two more derivatives rules that will show up often in this course.

- The product rule: $\frac{d}{dx}\left(f(x) \cdot g(x)\right) =$ ______________________________.
- The quotient rule: $\frac{d}{dx}\left(f(x)/g(x)\right) =$ ______________________________.

Problem 1: (Stewart Section 3.2) Differentiate $y = (10x^2 + 7x - 2)(2 - x^2)$ using the product rule.

Problem 2: (Stewart Section 3.2) If $g(x)$ is a differentiable funciton, find an expression fo the derivative of

$$y = \frac{g(x)}{x}.$$

Problem 3: (Stewart Section 3.2) Suppose $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x. Find $f''(2)$.

Problem 4: (Stewart Section 3.2) For f defined as $f(x) = \sqrt{x}e^x$, find $f'(x)$ and $f''(x)$.

Problem 5: (Stewart Section 3.2) Find equations of the tangent lines to the curve f, defined below, that are parallel to the line $x - 2y = 2$.

$$f(x) = \frac{x-1}{x+1}.$$

Problem 6: (Stewart Section 3.2) Find equations of the tangent line and normal line to the curve $y = 2xe^x$ at $(0, 0)$.

Challenge problem: (Stewart Section 3.2) Consider $f(x) = x^2e^x$. Find the first 5 derivatives of f, and guess an expression for the nth derivative.

Derivatives of Trigonometric functions

Chapter 3, Section 3

In this section, we learned the derivatives of the trigonometric functions.

- $\frac{d}{dx}\sin x =$ ______________________________
- $\frac{d}{dx}\cos x =$ ______________________________
- $\frac{d}{dx}\tan x =$ ______________________________
- $\frac{d}{dx}\csc x =$ ______________________________
- $\frac{d}{dx}\sec x =$ ______________________________
- $\frac{d}{dx}\cot x =$ ______________________________

We also learned about two important limits.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \qquad\qquad \text{and} \qquad\qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} =$$

Problem 1: (Stewart Section 3.3) Use derivative rules to find $\frac{d}{dx}\csc x$ and $\frac{d}{dx}\sec x$, and check that your answer matches the above.

Problem 2: (Stewart Section 3.3) Find the values of x such that $f(x) = x + 2\sin x$ has a horizontal tangent.

Problem 3: (Stewart Section 3.3) Find the equation of the tangent line to $y = e^x \cos x$ at $(0, 1)$.

Problem 4: (Stewart Section 3.3) Find $\lim_{x\to 0} \csc x \sin(\sin x)$.

Problem 5: (Stewart Section 3.3) If $H(\theta) = \theta \sin\theta$, find $H'(\theta)$ and $H''(\theta)$.

Challenge problem: (Stewart Section 3.3) Find constants A and B such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

The Chain Rule

Chapter 3, Section 4

The Chain Rule tells us how to differentiate ____ of functions.

The Chain Rule

Suppose g is differentiable at x and f is differentiable at $g(x)$. If $F = f(g(x))$, then $F'(x) = f'(g(x)) \cdot g'(x)$.

- We can write this in Leibniz notation. If $y = f(u)$ and $u = g(x)$, then:

- We can also combine the power rule with the chain rule. For any real number n and differentiable function $u = g(x)$, then

$$\frac{d}{dx}(u^n) =$$

We can also write this as

$$\frac{d}{dx}[g(x)]^n =$$

- One important differentiation rule that comes from the chain rule is

$$\frac{d}{dx}b^x =$$

. . . .

Problem 1: (Stewart Section 3.4) Find the derivative of $y = e^{\sin(2x)} + \sin(e^{2x})$.

Problem 2: (Stewart Section 3.4) Find the derivative of $J(\theta) = \tan^2(n\theta)$.

Problem 3: (Stewart Section 3.4) Find the 1000th derivative of $f(x) = xe^{-x}$.

Problem 4: (Stewart Section 3.4) Find an equation of the tangent line to the curve below at the point $(1, 1)$.

$$y = \frac{|x|}{\sqrt{2 - x^2}}$$

Problem 5: (Stewart Section 3.4) Suppose f is differentiable everywhere. Let $F(x) = f(e^x)$ and let $G(x) = e^{f(x)}$. Find expressions for $F'(x)$ and $G'(x)$.

Challenge problem: (Stewart Section 3.4) One of the reasons for using radian measure in calculus is because it makes differentiation much easier. To see this, use the chain rule to show that if θ is measured in degrees, we may have

$$\frac{d}{d\theta}(\sin\theta) \neq \cos\theta.$$

Implicit Differentiation

Chapter 3, Section 5

Implicit differentiation concerns functions in which it is hard to separate variables.

For example, if $y^2 = x$, then $y = \sqrt{x}$ and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}.$$

Note that the ____________________ variable is a function of the ____________________ variable. Then we can use the __________________ rule:

Implicit differentiation can be used to find these big results:

- $\frac{d}{dx}\sin^{-1}(x) =$ ____________________ $\frac{d}{dx}\csc^{-1}(x) =$ ____________________
- $\frac{d}{dx}\cos^{-1}(x) =$ ____________________ $\frac{d}{dx}\sec^{-1}(x) =$ ____________________
- $\frac{d}{dx}\tan^{-1}(x) =$ ____________________ $\frac{d}{dx}\cot^{-1}(x) =$ ____________________

. . . .

Problem 1: (Stewart Section 3.5) Use implicit differentiation to find an equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$. *Fun fact: this curve is called a hyperbola!*

Problem 2: (Stewart Section 3.5) Find the second derivative of the following function by implicit differentiation: $x^2 + 4y^2 = 4$.

Problem 3: (Stewart Section 3.5) Suppose $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$. Find $f'(1)$.

Problem 4: (Stewart Section 3.5) Consider the equation below. Use implicit differentiation to find $\frac{dx}{dy}$ (i.e. treat y as the independent variable).

$$x^4y^2 - x^3y + 2xy^3 = 0.$$

Problem 5: (Stewart Section 3.5) Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is c.

Challenge problem: (Stewart Section 3.5) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to find an expression for $(f^{-1})'(x)$.

Derivatives of Logarithmic Functions

Chapter 3, Section 6

In this section, we first discuss some differentiation rules for logarithms.

- $\frac{d}{dx}(\log_b x) =$ __________________________, and specifically $\frac{d}{dx}(\ln x) =$ __________________________
- $\frac{d}{dx}(\ln u) =$ __________________________, and more generally $\frac{d}{dx}[\ln g(x)] =$ __________________________
- $\frac{d}{dx}(b^x) =$ __________________________, and more generally $\frac{d}{dx}\left(b^{g(x)}\right) =$ __________________________
- $\frac{d}{dx} \ln |x| =$ __________________________

We also learn two different ways to write the number e as a limit:

$$e = \lim_{x \to 0} \qquad\qquad \text{and} \qquad e = \lim_{x \to \infty}$$

Finally, we are introduced to *logarithmic differentiation* as a way to take the derivative of complex expressions. It involves taking the natural logarithm of both sides of an equation $y = f(x)$, differentiating __________________________, and finally solving for y'.

. . . .

Problem 1: (Stewart Section 3.6) Use logarithmic differentiation to find the derivative of $y = x^x$.

Problem 2: (Stewart Section 3.6) Differentiate $f(x) = \ln \frac{1}{x}$.

Problem 3: (Stewart Section 3.6) Differentiate $f(x) = \ln \ln x$.

Problem 4: (Stewart Section 3.6) Use logarithmic differentiation to find the derivative of $y = (x^2 + 2)^2(x^4 + 4)^4$.

Problem 5: (Stewart Section 3.6) Find y' if $x^y = y^x$.

Problem 6: (Stewart Section 3.6) Find

$$\frac{d^9}{dx^9}(x^8 \ln x).$$

Challenge problem: (Stewart Section 3.6) Use the definition of the derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1.$$

Exponential Growth and Decay

Chapter 3, Section 8

Sometimes the rate of change of something depends on the amount of it we have. Then we write

$$\frac{dy}{dt} = \qquad\qquad .$$

One example of exponential growth/decay is ____.

- The *differential equation* above expresses ____.
- The solutions to this differential equation are exactly given by ____.

One example of exponential growth is *continuous compounding interest*. For an interest rate r and an initial amount A_0 invested in an account, the amount of money after t years is given by

$$A(t) = \qquad\qquad\qquad .$$

Differentiating, we get

$$\frac{dA}{dt} = \qquad\qquad\qquad .$$

One example of exponential decay is Newton's Law of Cooling. For a surrounding temperature of T_s, a dependent temperature variable T, and a constant k, we get

$$\frac{dT}{dt} = \qquad\qquad\qquad .$$

To find an equation for the temperature function $T(t)$, we may then use

$$T(t) = T_s + (T_0 - T_s)e^{-kt}.$$

. . . .

Problem 1: (Stewart Section 3.8) Suppose we can write $g'(t) = 13g(t)$ and $g(10) = 13$. Find an equation for $g(t)$.

Problem 2: (Stewart Section 3.8) A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

Problem 3: (Stewart Section 3.8) The half-life of cesium-137 is 30 years. Suppose we have a 100mg sample.

(a) Find an expression for the mass that remains after t years.

(b) Find an expression for the amount that will remain after 100 years.

(c) Find an expression for the amount of time it will take for 1mg to remain.

Problem 4: (Stewart Section 3.8) When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room, its temperature has increased to 10°C. What is the temperature of the drink after 50 minutes? When will its temperature be 15°C?

Challenge problem: (Stewart Section 3.8) Show with proof what the solutions to the differential equation $\frac{dy}{dx} = ky$ are. *Hint: you already know and used the end result, now prove that it's valid!*

Related Rates

Chapter 3, Section 9

Related rates problems deal with finding a relation between two variables in order to find how the change in one variable affects the change in another variable. Here are the general steps:

1. Find ____ (if possible, draw a diagram).
2. Differentiate using ____.
3. Plug in known values and solve for your target variable.

The best way to master related rates problems is practice!

Problem 1: (Stewart Section 3.9) The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80mm?

Problem 2: (Stewart Section 3.9) A particle is moving along a hyperbolic trajectory given by $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

Problem 3: (Stewart Section 3.9) Suppose $y = \sqrt{2x+1}$, where x and y are functions of t. If $dx/dt = 3$, find dy/dt when $x = 4$.

Problem 4: (Stewart Section 3.9) If the minute hand of a clock has length r (in cm), find the rate at which it sweeps out area as a function of r.

Problem 5: (Stewart Section 3.9) A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

Challenge problem: (Stewart Section 3.9) The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Linear Approximations and Differentials

Chapter 3, Section 10

Here are some big results from the previous lectures:

- The linear approximation, or tangent line approximation, for $f(x)$ is given by:

  $$f(x) \approx \qquad\qquad\qquad\qquad\qquad\qquad .$$

  When finding this value, choose a value of a that is easy to calculate.

- The differential is given by ______________________________________.

Problem 1: (Stewart Section 3.10) Estimate $(1.999)^4$ using a linear approximation or differentials.

Problem 2: (Stewart Section 3.10) Find the linearization $L(x)$ of $f(x) = \sin x$ at $a = \pi/6$.

Problem 3: (Stewart Section 3.10) Find the differential of $y = xe^{-4x}$.

Problem 4: (Stewart Section 3.10) Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2+5}$ for all x. Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

Problem 5: (Stewart Section 3.10) Are the estimates in the previous problem too large or too small? Explain.

Problem 6: (Stewart Section 3.10) The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk.

Challenge problem: (Stewart Section 3.10) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Maximum and Minimum Values

Chapter 4, Section 1

Today we'll be discussing extrema, which are maximum or minimum values. We learn some theorems that help us find locations and values of extrema.

- The Extreme Value Theorem says that if f is continuous on $[a, b]$, then f must have a ________________ and a ________________.
- Fermat's Theorem says that if f has a local minimum or maximum at c, then c is a __________________________.
- A local maximum is bigger than the values near it. An absolute maximum is bigger than all the other values (likewise for minima).
- Circle one: endpoints (can/cannot) be local minima or maxima.

Absolute extrema occur at _____________________ points, which are where the derivative is _____ or ___________________________. On a closed interval, we must also consider the ___________________________.

Problem 1: (Stewart Section 4.1) Find the absolute maximum and absolute minimum values of $f(t) = (t^2 - 4)^3$ on the interval $[-2, 3]$.

Problem 2: (Stewart Section 4.1) Find the absolute maximum and absolute minimum values of $f(x) = x - \sqrt[3]{x}$ on the interval $[-1, 4]$.

Problem 3: (Stewart Section 4.1) Prove that the function $f(x) = x^{101} + x^{51} + x + 1$ has neither a local maximum nor a local minimum.

Problem 4: (Stewart Section 4.1) Find the critical numbers of $f(x) = x^3 + 6x^2 - 15x$.

Problem 5: (Stewart Section 4.1) Find the absolute maximum and absolute minimum of $f(x) = \ln x$ on $[1, 2]$.

Challenge problem: (Stewart Section 4.1) If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, where $0 \leq x \leq 1$.

The Mean Value Theorem

Chapter 4, Section 2

In this section, we learn two important theorems about values of derivatives.

- **Rolle's theorem**: if f is continuous on $[a, b]$ and differentiable on (a, b) and if ____________ = ____________, then there is some $c \in (a, b)$ such that ________________________________.
- **The Mean Value theorem:** if f is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ such that ________________________________.

We can also characterize graphs based on their derivatives.

- If the derivative is always 0 for a continuous function, then that function is a __________________ function.
- If $f'(x) = g'(x)$ for all $x \in (a, b)$, then ________________________.

. . . .

Problem 1: (Stewart Section 4.2) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Explain why this does *not* contradict Rolle's theorem.

Problem 2: (Stewart Section 4.2) Show that $2x + \cos x = 0$ has exactly one real solution.

Problem 3: (Stewart Section 4.2) Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

Problem 4: (Stewart Section 4.2) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Problem 5: (Stewart Section 4.2) Show that $\sin x < x$ if $0 < x < 2\pi$.

Challenge problem: (Stewart Section 4.2) Show that a polynomial of degree n has at most n real roots using the results from this section.

How Derivatives Affect the Shape of a Graph

Chapter 4, Section 3

The first derivative helps us decide if a function is increasing or decreasing.

$$\text{On a given interval, we say } f \text{ is } \begin{cases} \rule{6cm}{0.4pt} & \text{if } f'(x) > 0 \\ \rule{6cm}{0.4pt} & \text{if } f'(x) < 0 \end{cases} \quad \text{on that interval.}$$

The *first derivative test* is used to find extrema. Recall that c is a critical number if $f'(c)$ is 0 or undefined.

$$\text{We say } f \text{ has a } \begin{cases} \rule{6cm}{0.4pt} & \text{at } c \text{ if } f' \text{ changes from positive to negative} \\ \rule{6cm}{0.4pt} & \text{at } c \text{ if } f' \text{ changes from negative to positive} \end{cases} \quad \text{at } c.$$

But if f' does not change sign at c, then ____.

We can also define concavity in terms of second derivatives.

$$\text{On a given interval, we say } f \text{ is } \begin{cases} \rule{6cm}{0.4pt} & \text{if } f''(x) > 0 \\ \rule{6cm}{0.4pt} & \text{if } f''(x) < 0 \end{cases} \quad \text{on that interval.}$$

The second-derivative analogue for critical points is *inflection points*. A point P on a curve $f(x)$ is an inflection point if f is continuous at P and ______________________________.

Here is the *second derivative test* for a function f continuous near c:

$$\text{We say } f \text{ has a } \begin{cases} \rule{6cm}{0.4pt} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) > 0 \\ \rule{6cm}{0.4pt} & \text{at } c \text{ if } f'(c) = 0 \text{ and } f''(c) < 0 \end{cases} \quad \text{at } c.$$

Problem 1: (Stewart Section 4.3) Sketch the graph of a function that satisfies $f'(x) < 0$ and $f''(x) < 0$ for all x. Then sketch the graph of another function that satisfies $g'(x) > 0$ and $g''(x) < 0$.

Problem 2: (Stewart Section 4.2) Find the critical numbers of $f(x) = x^4(x-1)^3$. What does the first derivative test tell you?

Problem 3: (Stewart Section 4.3) Find the intervals of concavity of $f(x) = x^3 - 3x^2 - 9x + 4$.

Problem 4: (Stewart Section 4.3) Find the intervals of concavity of $f(x) = \sin x + \cos x$ defined on $0 \leq x \leq 2\pi$.

Challenge problem: (Stewart Section 4.2) Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number x. Prove that f is increasing on the entire interval I.

Indeterminate Forms and L'Hôpital's Rule

Chapter 4, Section 4

L'Hôpital's rule helps us calculate limits having certain types of *indeterminate forms*. We need the following conditions in order to use l'Hôpital's rule:

1. f and g are ____________________________ near a
2. $g'(x) \neq 0$ if x is close to a
3. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) =$ ___________ *or* $\lim_{x\to a} f(x) =$ ___________ and $\lim_{x\to a} g(x) =$ ___________

Then l'Hôpital's rule's says

$$\lim_{x\to a} \frac{f(x)}{g(x)} =$$

For other types of indeterminate forms, we may come up with strategies to determine whether or not a limit exists.

- Indeterminate products (______ $\cdot$ ______): write the product as a quotient, and use l'Hôpital's rule.
- Indeterminate differences (______ $-$ ______): convert the difference into a quotient by rationalizing or factoring
- Indeterminate powders (______, ______, or ______): take the natural logarithm and write the function as an exponential

. . . .

Problem 1: (Stewart Section 4.4) Evaluate $\lim_{x\to 0^+} x\ln(x)$ using l'Hôpital's rule, and then use your result to find $\lim_{x\to 0^+} x^x$.

Problem 2: (Stewart Section 4.4) Evaluate

$$\lim_{x\to 0} \frac{x-\sin(x)}{x-\tan(x)}.$$

Problem 3: (Stewart Section 4.4) Evaluate
$$\lim_{x\to 0^+} x^{\sqrt{x}}.$$

Problem 4: (Stewart Section 4.4) Evaluate
$$\lim_{x\to 0}(\csc x - \cot x).$$

Problem 5: (Stewart Section 4.4) Evaluate
$$\lim_{x\to \infty} x^3 e^{-x^2}.$$

Challenge problem: (Stewart Section 4.4) Show that 0^∞ is not an indeterminate form using L'Hôpital's rule.

Summary of Curve Sketching

Chapter 4, Section 5

Curve sketching pieces together everything we learned in this chapter so far. Here are the tools that will help you graph a function $f(x)$:

- **Domain** This is where the function is defined.
- **Intercepts** The y-intercept can be found by setting ____________________ and the x-intercept can be found by setting ____________________.
- **Symmetry** If a function is even, then ________________, and if a function is odd, then ________________.
- **Asymptotes**: If $\lim_{x\to\infty} f(x) = L$ or if $\lim_{x\to-\infty} f(x) = L$, then we call $y = L$ a __________________________.
  If $\lim_{x\to a^\pm} f(x) = \pm\infty$ or if $\lim_{x\to a} f(x) = \pm\infty$, we have a ____ at $x = a$.
- **Intervals of increase or decrease**: Find the __________________________ and evaluate the sign of the derivative at the surrounding intervals.
- **Local extrema**: This follows from intervals of increase and decrease, or from using the second derivative test.
- **Concavity**: Find the second derivative and evaluate its sign at different intervals.

. . . .

Problem 1: (Stewart Section 4.5) Sketch the curve $y = 2 - 2x - x^3$.

Problem 2: (Stewart Section 4.5) Sketch the curve $y = \frac{x^3}{x^3+1}$.

Problem 3: (Stewart Section 4.5) Sketch the curve $y = \arctan(e^x)$.

Optimization Problems

Chapter 4, Section 7

Optimization is about finding absolute extrema. Here are some pointers for optimization problems.

- Draw a diagram and identify given quantities on the diagram.
- Assign a variable to the value that is to be maximized or minimized, and express it in terms of ____.
- Find the domain of the independent variable. If the domain is an **open** interval or **half-open** interval, then you must compute the value of f at open endpoints as well. If the absolute extremum appears at the value of an open endpoint, then ______________________________.

. . . .

Problem 1: (Stewart Section 4.7) Find an equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant.

Problem 2: (Stewart Section 4.7) Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

Problem 3: (Stewart Section 4.7) What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

Problem 4: (Stewart Section 4.7) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

Challenge problem: (Stewart Section 4.7) Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

Newton's Method

Chapter 4, Section 8

In this section we learned about Newton's method, which is an algorithm for approximating roots of equations.

- First make a guess, call it x_1.
- Then find a formula for the x-intercept of the tangent line at x_1:

- Then keep going! Given the nth approximation, the $(n+1)$st approximation is:

If a suitable x_1 is chosen, then the approximation should get ________________ as n increases. Otherwise, the approximations will diverge, and a new value of x_1 should be chosen.

. . . .

Problem 1: (Stewart Section 4.8) Starting with $x_1 = 2$, find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$.

Problem 2: (Stewart Section 4.8) Explain why Newton's method doesn't work for finding the solution of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_1 = 1$.

Problem 3: (Stewart Section 4.8) Find, correct to six decimal places, the solution of the equation $\cos(x) = x$. You may use a calculator.

Challenge problem: (Stewart Section 4.8) Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$ defined on $0 \leq x \leq \pi$, correct to six decimal places. You may use a calculator.

Antiderivatives

Chapter 4, Section 9

Antiderivatives can be thought of as the opposite of derivatives, reversing differentiation. We call a function an antiderivative if $F'(x) = f(x)$. Also, if $F(x)$ is an antiderivative, then so is __________________.

You should know the antiderivatives for common functions. Let $F'(x) = f(x)$. Here are some of them.

- For the function $cf(x)$, the general antiderivative is ____.
- For the function x^n, the general antiderivative is ____.
- For the function $\cos(x)$, the general antiderivative is ____.
- For the function $\sin(x)$, the general antiderivative is ____.

⋆ Be sure to check out page 358 of the textbook for a complete list.

Problem 1: (Stewart Section 4.9) Find the most general antiderivative of $f(x) = 7x^{2/5} + 8x^{-4/5}$, and check your answer by differentiating.

Problem 2: (Stewart Section 4.9) Let $f''(x) = 2x + 3e^x$. Find the most general formula for $f(x)$.

Problem 3: (Stewart Section 4.9) Find the most general antiderivative of the function below.

$$f(x) = 1 + 2\sin x + \frac{3}{\sqrt{x}}$$

Problem 4: (Stewart Section 4.9) For $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$, find f.

Problem 5: (Stewart Section 4.9) A particle is moving at velocity $v(t) = \sin t - \cos t$. Suppose the position function is given as $s(t)$, where $s(0) = 0$. Find a formula for $s(t)$.

Challenge problem: (Stewart Section 4.9) What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 seconds?

Areas and Distances

Chapter 5, Section 1

At the end of chapter 4, we defined antiderivatives. In this chapter, we will see that antiderivatives are closely connected to areas under curves. The *area problem* is about finding the area of the region that lies under the curve of any function $y = f(x)$ from $x = a$ to $x = b$.

We can approximate areas under curves by drawing ______________________________ and adding up their areas. We may use __________________ endpoints, __________________ endpoints, or midpoints to find the height of each rectangle.

We say that the *area* A of the region S that lies under the graph of the continuous function f is the __________________ of the sum of the areas of approximating rectangles. Mathematically, we can write this as:

The *distance problem* asks for the distance traveled by an object in a given time interval if the __________________ is known at all times. We can calculate displacement by finding the area under the ________________________ curve, and we can calculate distance by finding the area under the ____ curve.

Problem 1: (Stewart Section 5.1) Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is the calculated area an overestimate or an underestimate?

Problem 2: (Stewart Section 5.1) Determine a region whose area is equal to the limit below.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$$

Problem 3: (Stewart Section 5.1) Use the limit definition to find an expression for the area under the graph of $f(x)=\sqrt{\sin x}$ from 0 to π.

Problem 4: (Stewart Section 5.1) Estimate the area under the graph of $f(x)=\sin x$ from $x=0$ to $x=\pi/2$ using four approximating rectangles and left endpoints. Sketch the graph and the rectangles. Is the calculated area an overestimate or an underestimate?

Challenge problem: (Stewart Section 5.1) Find an expression for the area under the curve $y=x^2$ as a limit. Then use a formula for the sum of squares of natural numbers to evaluate the limit.

The Definite Integral

Chapter 5, Section 2

In the previous section, we learned about how rectangles can be used to find sums. In this section, we learn that these approximations are called *Riemann sums*.

The *definite integral* is defined as:

$$\int_a^b f(x)\,dx =$$

We can only say a function can be integrated if it meets certain conditions.

Conditions for integrability

If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$. In other words, the definite integral $\int_a^b f(x)\,dx$ exists.

In this case, for $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$, we may write:

$$\int_a^b f(x)\,dx =$$

Certain summation rules can help us evaluate definite integrals from limits. we have

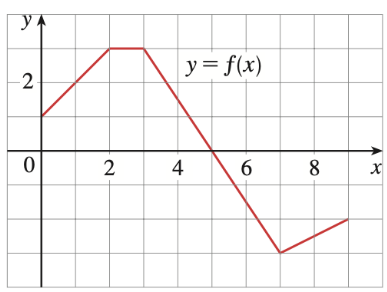
$$\sum_{i=1}^{n} i = \qquad \sum_{i=1}^{n} i^2 = \qquad \sum_{i=1}^{n} i^3 =$$

Definite integrals respect many of the same properties as summations. For example, we may split up the bounds of integration:

$$\int_a^c f(x)\,dx =$$

Be sure to check out section 5.2 in the book for more of these properties!

Problem 1: (Stewart Section 5.2) Use the graph below to find $\int_0^2 f(x)\,dx$, $\int_0^5 f(x)\,dx$, and $\int_5^7 f(x)\,dx$.



Problem 2: (Stewart Section 5.2) Prove that

$$\int_a^b x\, dx = \frac{b^2 - a^2}{2}.$$

Problem 3: (Stewart Section 5.2) Evaluate

$$\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx.$$

Problem 4: (Stewart Section 5.2) Find $\int_0^5 f(x)\, dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3. \end{cases}$$

Challenge problem: (Stewart Section 5.2) Express the limit below as a definite integral.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$$

The Fundamental Theorem of Calculus

Chapter 5, Section 3

The first part of this section's big theorem tells us about derivatives of definite integrals.

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function $g(x)$ below is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$$g(x) = \int_a^x f(t)\, dt \qquad \text{defined on } a \leq x \leq b$$

We can rewrite this first part in Leibniz notation as follows:

The second part relates definite integrals to antiderivatives.

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is any antiderivative of f.

Recall that if $F(x)$ is an antiderivative of $f(x)$, so is ____________________________. We can rewrite the second part as follows:

Problem 1: (Stewart Section 5.3) Find the derivative of the function below.

$$F(x) = \int_x^0 \sqrt{1 + \sec t}\, dt$$

Problem 2: (Stewart Section 5.3) Find the derivative of the function below.

$$F(x) = \int_1^{3x+2} \frac{t}{1 + t^3}\, dt$$

Problem 3: (Stewart Section 5.3) Sketch the area enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Find its area.

Problem 4: (Stewart Section 5.3) Find the error in the calculation below, and fix it.

$$\int_{-1}^{2} \frac{4}{x^3}\, dx = -\frac{2}{x^2}\bigg]_{-1}^{2} = \frac{3}{2}$$

Problem 5: (Stewart Section 5.3) On what interval is the function below increasing?

$$f(x) = \int_0^x (t - t^2)e^{t^2}\, dt$$

Challenge problem: (Stewart Section 5.3) Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x}.$$

Indefinite Integrals and the Net Change Theorem

Chapter 5, Section 4

As we learned, antiderivatives and derivatives are closely related:

$$\int f(x)\,dx = F(x) \quad \text{means}$$

We call $\int f(x)\,dx$ an *indefinite integral*, which represents a family of functions. We may find the most general antiderivative by adding a ______________________ to a particular antiderivative.

In this section, we find out that second part of the Fundamental Theorem of Calculus can tell us something about net change.

Net Change Theorem

The integral of a rate of change is the net change.

Mathematically, we can write this as:

....

Problem 1: (Stewart Section 5.4) Verify by differentiation that the identity below holds.

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Problem 2: (Stewart Section 5.4) Evaluate the integral below.

$$\int_0^{\pi} (5e^x + 3\sin x)\,dx$$

Problem 3: (Stewart Section 5.4) The linear density of a rod of length 4m is given by $\rho(x) = 9 + 2\sqrt{x}$, measured in kilograms per meter, where x is measured in meters from one end of the rod. What is the total mass of the rod?

Problem 4: (Stewart Section 5.4) Suppose the acceleration of a particle is given by $a(t) = t + 4$, and its initial velocity is $v(0) = 5$. Find the velocity function $v(t)$ and the distance traveled from $t = 0$ to $t = 10$.

Problem 5: (Stewart Section 5.4) Evaluate the integral below.

$$\int_{-1}^{1} t(1-t)^2 \, dt$$

Challenge problem: (Stewart Section 5.4) Find the general indefinite integral.

$$\int_{-1}^{2} (x - 2|x|) \, dx$$

The Substitution Rule

Chapter 5, Section 5

While we can usually find derivatives using some set of rules, finding antiderivatives can be a little bit harder. One strategy to evaluate antiderivatives is *substitution*.

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

We can modify this for definite integrals:

$$\int_a^b f(g(x))g'(x)\,dx =$$

We can also make observations about definite integrals of symmetric functions.

- If f is even, then
$$\int_{-a}^{a} f(x)\,dx =$$
- If f is odd, then
$$\int_{-a}^{a} f(x)\,dx =$$

Problem 1: Prove the above identities for even and odd functions.

Problem 2: (Stewart Section 5.5) Evaluate the integral below by making the substitution $u = 2x$.

$$\int \cos 2x\,dx$$

Problem 3: (Stewart Section 5.5) If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$.

Problem 4: (Stewart Section 5.5) Evaluate the integral below.

$$\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx$$

Problem 5: (Stewart Section 5.5) Evaluate the integral below.

$$\int_{-2}^{2} (x+3)\sqrt{4-x^2}\, dx$$

Challenge problem: (Stewart Section 5.5) If a and b are positive numbers, show that

$$\int_0^1 x^a (1-x)^b \, dx = \int_0^1 x^b (1-x)^a \, dx.$$

Areas Between Curves

Chapter 6, Section 1

In the previous chapter, we learned that the region bounded by the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x.$$

In this section, we find that the area of the region bounded by two curves $f(x)$ and $g(x)$ (where $f(x) \geq g(x)$) from $x = a$ to $x = b$ is

$$A = \int_a^b$$

If we only want to consider the positive area, we get

$$A = \int_a^b$$

Sometimes, it might be easier to integrate with respect to y and treat x as the independent variable. ____________________ ______________________________ can help decide which is better.

Problem 1: (Stewart Section 6.1) Sketch the region enclosed by $y = \sin x$, $y = x$, $x = \pi/2$, and $x = \pi$, and find its area.

Problem 2: (Stewart Section 6.1) If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent?

Problem 3: (Stewart Section 6.1) Use integration to find the area of the triangle with the coordinates $(0, 0)$, $(3, 1)$, and $(1, 2)$.

Problem 4: (Stewart Section 6.1) Sketch the region enclosed by $y = \cos x$, $y = 1 - \cos x$, $x = 0$, and $x = \pi$, and find its area.

Challenge problem: (Stewart Section 6.1) Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Volumes

Chapter 6, Section 2

In this section, we treat volumes as sums of ____________________________________. Mathematically, for a a solid S lying between $x = a$ and $x = b$, if the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is continuous, then the volume of S is

$$V = \int_a^b A(x)\, dx = \lim_{n\to\infty}$$

If we want to rotate about the y-axis, we may treat y as the dependent variable and rotate accordingly. Solids that are generated by rotation about axes are called ____.

Problem 1: (Stewart Section 6.2) Sketch the solid obtained by rotating the region bound by $y = x + 1$, $y = 0$, $x = 0$, and $x = 2$ about the x-axis.

Problem 2: (Stewart Section 6.2) Sketch the solid obtained by rotating the region bound by $y = x^2$, $y^2 = x$, and $x \geq 0$ about the x-axis. Then find its volume.

Problem 3: (Stewart Section 6.2) Find the volume of a pyramid with height h and rectangular base with dimensions b and $2b$.

Problem 4: (Stewart Section 6.2) Use integration to find the volume of a right circular cone with height h and base radius r.

Challenge problem: (Stewart Section 6.2) Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

Volumes by Cylindrical Shells

Chapter 6, Section 3

The *method of cylindrical shells* can help us find volumes of solids that have other solids removed from them.

For a thickness of Δr, average radius of r, and height of h, the volume of a single cylindrical shell is

$$V =$$

Then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b, using cylindrical shells, is

$$V = \int_a^b \qquad\qquad \text{where } 0 \leq a < b$$

....

Problem 1: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = \sqrt[3]{x}$, $y = 0$, and $x = 1$ about the y-axis.

Problem 2: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = x^3$, $y = 0$, $x = 1$, and $x = 2$ about the y-axis.

Problem 3: (Stewart Section 6.3) Use cylindrical shells to find the volume of a sphere of radius r.

Problem 4: (Stewart Section 6.3) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = 4x - x^2$ and $y = x$ about the y-axis.

Challenge problem: (Stewart Section 6.3) Use cylindrical shells to find the volume of a right circular cone with height h and base radius r.

Average Value of a Function

Chapter 6, Section 5

We define the *average value of f* on the interval $[a, b]$ to be

$$f_{\text{ave}} =$$

The *Mean Value Theorem for Integrals* says that if f is continuous on $[a, b]$, then there must be some number c in $[a, b]$ such that

$$f(c) =$$

Equivalently, there must be some c such that

$$\int_a^b f(x)\,dx =$$

Problem 1: (Stewart Section 6.5) Find the average value of $f(x) = 3x^2 + 8x$ on $[-1, 2]$.

Problem 2: (Stewart Section 6.5) Find the average value of $f(x) = \sqrt{x}$ on $[0, 4]$.

Problem 3: (Stewart Section 6.5) Find the average value of $f(u) = (\ln u)/u$ on $[1, 5]$.

Problem 4: (Stewart Section 6.5) If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

Problem 5: (Stewart Section 6.5) The linear density in a rod 8 meters long is $12/\sqrt{x+1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

Problem 6: (Stewart Section 6.5) Find a c such that $f_{\text{ave}} = f(c)$ for $f(x) = 1/x$ on $[1, 2]$.

Challenge problem: (Stewart Section 6.5) Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives.

References

[1] Charles Schulz. *The Complete Peanuts Vol. 10: 1969–1970*. Vol. 10. Fantagraphics Books, 2008.

[2] James Stewart. *Calculus: Early Transcendentals*. Cengage Learning, 2012.

