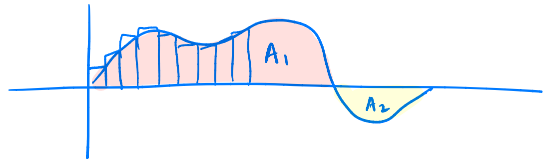
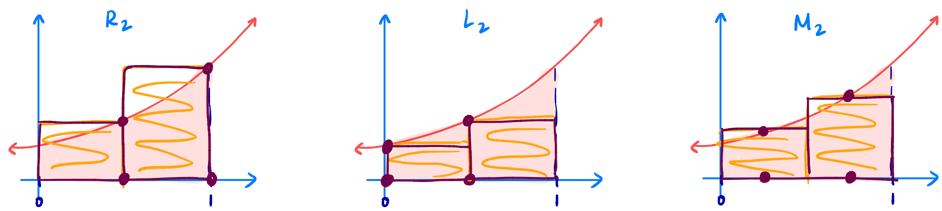
Motivating Riemann sums

November ~~11th~~ 12th, 2024

Here are some key ideas from section 5.1.



- The area problem asks us to find the area of the region S that lies under the curve $y = f(x)$ from a to b.
- We will use the convention that regions under the x-axis have negative area. So in the diagram above, the area of the region is $A_1 - A_2$.
- A Riemann sum is a way to approximate areas under curves. There are left, right, and midpoint Riemann sums. Here are some sketches:



The problems below will walk you through the process of calculating Riemann sums.

. . . .

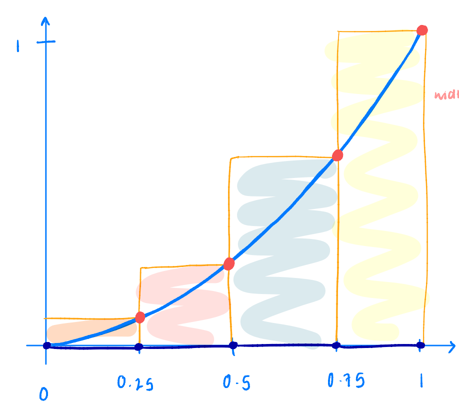
Trig practice: Sketch the graphs of $y = 2\cos x$ and $y = -3\cot x$.

Problem 1: (Stewart 5.1) Let $y = x^2$ on the interval $0 \leq x \leq 1$. In this problem, we will find R_4, a right Riemann sum with four rectangles.

1. Sketch the graph of y on the interval $[0, 1]$. Shade the area under the curve.
2. Divide the interval $[0, 1]$ on the x-axis into four equal segments. What is the width of each segment?
3. Each segment has a left and right endpoint. For each right endpoint, draw a point at the corresponding y-value.
4. Each point you drew represents the height of the rectangle whose base is the corresponding segment. Draw the four rectangles.
5. Find the area of each rectangle (width $\times$ height). Add the four areas to approximate the area under the curve.

My Attempt:

Solution:



Area of 1st rectangle:

width 0.25×0.25^2 height

Area of 2nd rectangle:

0.25×0.5^2

Area of 3rd rectangle:

0.25×0.75^2

Area of 4th rectangle:

0.25×1^2

R_4 = sum of areas of rectangles

$= 0.25 \times 0.25^2 + 0.25 \times 0.5^2 + 0.25 \times 0.75^2 + 0.25 \times 1^2$

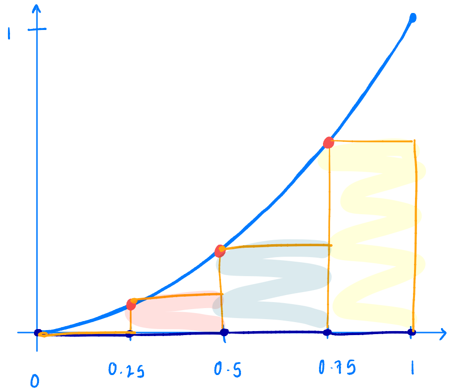
$= 0.25(0.25^2 + 0.5^2 + 0.75^2 + 1^2)$

$= \frac{30}{64}$

Problem 2: (Stewart 5.1) Repeat Problem 1, but find a left Riemann sum (L_4) instead of a right Riemann sum. In this case, the heights of the rectangles are given by the y-values of the left endpoints.

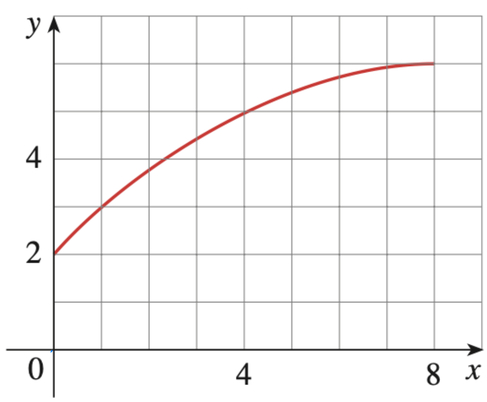
My Attempt:

Solution:



Area $= 0.25\times 0 + 0.25\times 0.25^2 + 0.25\times 0.5^2 + 0.25\times 0.75^2 = \boxed{\frac{7}{32}}$

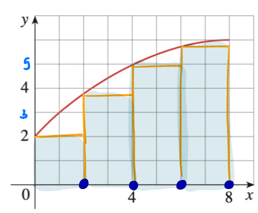
Problem 3: (Stewart 5.1) Using the diagram below, find L_4, R_4, and M_4 to estimate the area under the curve from $x = 0$ to $x = 8$. This problem is calculator friendly.



My Attempt:

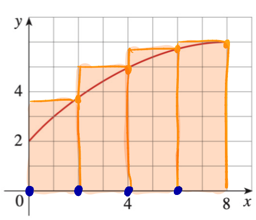
Solution:

L_4



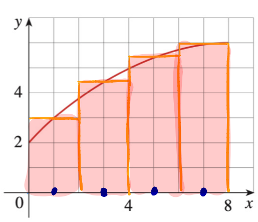
Area $\approx 2\times 2 + 2\times 3.7 + 2\times 5 + 2\times 5.7$
$\approx \boxed{32.8}$

R_4



Area $\approx 2\times 3.5 + 2\times 5 + 2\times 5.7 + 2\times 6$
$\approx \boxed{40.4}$

M_4



Area $\approx 2\times 3 + 2\times 4.5 + 2\times 5.5 + 2\times 6$
$\approx \boxed{38}$

Problem 4: (Stewart 5.1) Let $f(x) = x^2$.

a) Use the formula $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ to find R_n and L_n for arbitrary n. from $x=0$ to $x=1$.

b) Find $\lim_{n\to\infty} R_n$ and $\lim_{n\to\infty} L_n$. What do you notice? What do these value represent?

My Attempt:

Solution:

a) $R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots \frac{1}{n}(1)^2$

$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$

$= \boxed{\frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}}$

$L_n = 0 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2$

$= \frac{1}{n^3}\left(0 + 1^2 + 2^2 + \cdots + (n-1)^2\right)$

$= \boxed{\frac{1}{n^3} \frac{(n-1)(n)(2(n-1)-1)}{6}}$

b) $\lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \boxed{\frac{1}{3}}$ This is the <u>exact</u> area!

Problem 5: (Stewart 5.1) Suppose we want to find the area under the curve $f(x)$ on a given interval. We split our interval into n segments, and we let $x_1, x_2, \cdots, x_n$ represent the right endpoints of each of our n segments. Let Δx be the width of each segment.

a) Find an expression for R_n using that information (the notation $\sum_{i=1}^{n}$ might be helpful).

b) What does $\lim_{n\to\infty} R_n$ represent, and how should it compare to $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} M_n$?

My Attempt:

Solution:

a) $R_n = \boxed{\sum_{i=1}^{n} \Delta x\, f(x_i) = \Delta x \sum_{i=1}^{n} f(x_i)}$

b) $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n$, which is the exact area under the curve.

Problem 6: (Stewart 5.1) Derive an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit. Then use the sum of cubes formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ to evaluate the limit.

My Attempt:

Solution:

$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3$

$= \lim_{n\to\infty} \frac{1}{n^4}\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)$

$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$

$= \boxed{\frac{1}{4}}$