Quiz 8 study guide

November 10th, 2024

General information

Quiz 8 covers sections 4.4 and 4.6. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Working with optimization word problems; here's a process you can use:
  1. List the unknowns and conditions given in the problem.
  2. If possible, draw a diagram and label variables and constants.
  3. Express your target variable in terms of other variables.
  4. Use relationships between variables to write your target variable in terms of exactly one other variable.
  5. Find the required absolute extremum over the correct domain using the first derivative test (or another relevant one).
- ☐ The definition of an antiderivative
- ☐ How to write the most general antiderivative $(+C)$
- ☐ The antiderivatives in Table 2 (page 308)
- ☐ How to solve for the equation of a function given its derivative
- ☐ How to solve for the equation of a function given its second derivative

Help! I'm stuck on....

- ...how to use **solve optimization problems**: this is a 1 hour video with lots and lots of examples, and this is a nice 10 minute overview video
- ...the **definition** of an antiderivative : check out this 3 minute video
- ...**finding antiderivatives**: check out this 33 minute video

Practice problems

1. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$. For this question, you can use a calculator only to find roots (you will not be allowed a calculator on the quiz).

2. Find the most general antiderivatives of the following functions.

   a) $f(x) = \sin x + \sec x \tan x$, $0 \leq x < \pi/2$;

   b) $g(t) = (1 + t)/\sqrt{t}$;

   c) $q(t) = 2 + (t + 1)(t^2 - 1)$;

   d) $w(\theta) = 2\theta - 3\cos\theta$;

3. Let $f''(x) = 1 - 6x + 48x^2$.

   a) Find the most general formula for $f(x)$.

   b) Now suppose $f(0) = 1$ and $f'(0) = 2$. Find $f(x)$.

4. Let $f''(x) = 2x^3 + 3x^2 - 4x + 5$.

   a) Find the most general formula for $f(x)$.

   b) Now suppose $f(0) = 2$ and $f(1) = 0$. Find $f(x)$.

Solutions

1. Since the point (x, y) lies on the hyperbola, we can substitute $y = \frac{8}{x}$ into the distance formula. This gives

$$d^2 = (x-3)^2 + \left(\frac{8}{x} - 0\right)^2.$$

Expanding $(x-3)^2$ and simplifying, we have

$$d^2 = x^2 - 6x + 9 + \frac{64}{x^2},$$

and we can simply optimize d^2 since minimizing the distance also means minimizing the distance squared. To minimize d^2, we take its derivative with respect to x. The derivative is

$$\frac{d}{dx}\left(x^2 - 6x + 9 + \frac{64}{x^2}\right) = 2x - 6 - \frac{128}{x^3}.$$

We set the derivative equal to zero to find the critical points:

$$2x - 6 - \frac{128}{x^3} = 0.$$

Multiply through by x^3 to eliminate the fraction to get

$$2x^4 - 6x^3 - 128 = 0,$$

which means

$$x^4 - 3x^3 - 64 = 0.$$

Then $x = 4$ and $x \approx -2.9$ are the roots. Using the first derivative test (ommitted here, see previous worksheets/study guides), we can see that the derivative changes from negative to positive at $x = 4$, and thus there is a minimum at $x = 4$. The corresponding y-value is $8/4 = 2$, so the point is $(4, 2)$.

2. In this problem, I'm using the shorthand $\int$ to mean "antiderivative of."

a) $f(x) = \sin x + \sec x \tan x$, $0 \le x < \pi/2$

$$\int f(x)\,dx = \int \sin x\,dx + \int \sec x \tan x\,dx$$
$$= -\cos x + \sec x + C.$$

b) $g(t) = \frac{1+t}{\sqrt{t}}$

$$\int g(t)\,dt = \int \left(\frac{1}{\sqrt{t}} + \frac{t}{\sqrt{t}}\right)\,dt$$
$$= \int t^{-1/2}\,dt + \int t^{1/2}\,dt$$
$$= 2\sqrt{t} + \frac{2}{3}t^{3/2} + C.$$

c) $q(t) = 2 + (t+1)(t^2-1)$

$$q(t) = 2 + t^3 - t + t^2 - 1 = t^3 + t^2 - t + 1.$$
$$\int q(t)\,dt = \int \left(t^3 + t^2 - t + 1\right)\,dt$$
$$= \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t + C.$$

d) $w(\theta) = 2\theta - 3\cos\theta$

$$\int w(\theta)\,d\theta = \int 2\theta\,d\theta - \int 3\cos\theta\,d\theta = \theta^2 - 3\sin\theta + C.$$

3. a) To find $f(x)$, find the antiderivative of $f''(x)$ and then find the antiderivative again.

$$f'(x) = \int f''(x)\,dx = \int (1 - 6x + 48x^2)\,dx$$

$$f'(x) = x - 3x^2 + 16x^3 + C_1$$

where C_1 is a constant of integration. Then integrate $f'(x)$ to find $f(x)$:

$$f(x) = \int f'(x)\,dx = \int (x - 3x^2 + 16x^3 + C_1)\,dx$$

$$f(x) = \frac{x^2}{2} - x^3 + 4x^4 + C_1 x + C_2$$

where C_2 is another constant of integration.

b) Use the initial condition $f'(0) = 2$ to find C_1.

$$f'(x) = x - 3x^2 + 16x^3 + C_1, \quad \text{so } f'(0) = C_1 = 2.$$

Substitute $C_1 = 2$ into $f(x)$:

$$f(x) = \frac{x^2}{2} - x^3 + 4x^4 + 2x + C_2.$$

Use the initial condition $f(0) = 1$ to find C_2:

$$f(0) = \frac{0^2}{2} - 0^3 + 4(0^4) + 2(0) + C_2 = 1 \implies C_2 = 1.$$

Therefore, $f(x)$ is:

$$f(x) = \frac{x^2}{2} - x^3 + 4x^4 + 2x + 1.$$

4. a) To find $f(x)$, find the antiderivative twice. First, find $f'(x)$:

$$f'(x) = \int f''(x)\,dx = \int \left(2x^3 + 3x^2 - 4x + 5\right)\,dx$$

$$f'(x) = \frac{2x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + 5x + C_1$$

$$f'(x) = \frac{x^4}{2} + x^3 - 2x^2 + 5x + C_1$$

where C_1 is a constant of integration. Next, find $f(x)$:

$$f(x) = \int f'(x)\,dx = \int \left(\frac{x^4}{2} + x^3 - 2x^2 + 5x + C_1\right)\,dx$$

$$f(x) = \frac{x^5}{10} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + C_1 x + C_2$$

where C_2 is another constant of integration.

b) Use the initial condition $f(0) = 2$ to find C_2:

$$f(0) = \frac{0^5}{10} + \frac{0^4}{4} - \frac{2(0^3)}{3} + \frac{5(0^2)}{2} + C_1(0) + C_2 = 2 \implies C_2 = 2.$$

Use the initial condition $f(1) = 0$ to find C_1:

$$f(1) = \frac{1^5}{10} + \frac{1^4}{4} - \frac{2(1^3)}{3} + \frac{5(1^2)}{2} + C_1(1) + 2 = 0$$

$$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C_1 + 2 = 0.$$

Simplify:

$$\frac{1}{10} + \frac{5}{20} - \frac{40}{60} + \frac{150}{60} + C_1 + 2 = 0.$$

Combine terms:

$$\frac{6}{20} + \frac{110}{60} + C_1 + 2 = 0.$$

Convert to common denominator and solve for C_1:

$$C_1 = -\left(\frac{6}{20} + \frac{110}{60} + 2\right).$$

Substitute C_1 and C_2 back into the formula for $f(x)$.