Quiz 9 study guide

November 17th, 2024

General information

Quiz 8 covers sections 5.1 and 5.2. Here are some general things of note about the quiz:

- You should review the hardest problems on the **homework** when you study for this quiz.
- Technically, any concept in the book in the sections we covered is fair game. However, you should focus on concepts that were covered in lecture and discussion.

Here are some things you should know for the quiz (feel free to use this as a checklist):

- ☐ Using definite integrals to find areas under curves
- ☐ Using rectangles (left, right, and midpoint) to approximate definite integrals
- ☐ The limit definition of the area of a region under the graph of a function between $x = a$ and $x = b$
- ☐ Using definite integrals to find distance and displacement
- ☐ The limit definition of a definite integral
- ☐ The definition of integrability
- ☐ Properties of the definite integral
- ☐ The definition of the indefinite integral
- ☐ Summation/difference/constant rules for indefinite integrals

Help! I'm stuck on....

- ...**summation notation**: check out this 20 minute video
- ...finding **areas under curves**: check out this 8 minute video
- ...the different types of **Riemann sums**: check out this 9 minute video
- ... the **limit definition** of areas: check out this 13 minute video
- ...**properties** of definite integrals: check out this 11 minute video

Practice problems

1. Determine a region whose area is equal to the following limit (no need to evaluate the limit):

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\frac{1}{1+(2i/n)}$$

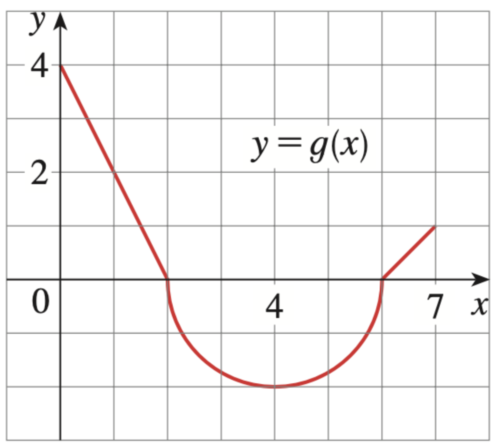
2. Consider the function $f(x) = x^2$ on the interval $[0, 2]$. Write the expression for the Riemann sum with n equal subintervals using the right endpoints.

3. Evaluate the Riemann sum for $f(x) = \sin(x)$ on the interval $[0, \pi]$ using the midpoint rule with $n = 4$.

4. Write the Riemann sum expression for $f(x) = e^x$ on $[1, 3]$ using the left endpoints for n subintervals.

5. Approximate the area under the curve $f(x) = \sqrt{x}$ on $[1, 4]$ using a Riemann sum with $n = 3$ subintervals and the right endpoints.

6. The graph of $g(x)$ is pictured below, and it is made up of two straight lines and a semicircle.



Find the following values.

a) $\int_0^2 g(x)\,dx$; b) $\int_2^6 g(x)\,dx$; c) $\int_0^7 g(x)\,dx$.

Solutions

1. It corresponds to the definite integral

$$\int_0^2 \frac{1}{1+x}\,dx.$$

We can recognize the sum as a Riemann sum. The form of the sum suggests that it is an approximation of the area under the curve $y = \frac{1}{1+x}$ on the interval $[0, 2]$ using n subintervals.

The term $\frac{2}{n}$ represents the width of each subinterval, and $\frac{1}{1+i\cdot\frac{2}{n}}$ corresponds to the height of a rectangle associated with the i-th subinterval. To see this, consider this to be a left Riemann sum. Then $x_0 = 0, x_1 = \frac{2}{n}, x_2 = \frac{4}{n}$, and so on; we can see that $x_i = i \times \frac{2}{n}$. As n approaches infinity, the width of the subintervals approaches zero, and the sum approaches the definite integral.

2. For $f(x) = x^2$ on $[0, 2]$, the Riemann sum using the right endpoints is:

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where $x_i^* = 0 + i \cdot \frac{2}{n}$ and $\Delta x = \frac{2}{n}$. Substituting, we get:

$$R_n = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n}.$$

3. For $f(x) = \sin(x)$ on $[0, \pi]$, with $n = 4$ and the midpoint rule:

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

The midpoints are $x_1^* = \frac{\pi}{8}, x_2^* = \frac{3\pi}{8}, x_3^* = \frac{5\pi}{8}, x_4^* = \frac{7\pi}{8}$. The Riemann sum is:

$$M_4 = \frac{\pi}{4}\left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right].$$

4. For $f(x) = e^x$ on $[1, 3]$, the Riemann sum using the left endpoints is:

$$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x,$$

where $x_i^* = 1 + i \cdot \frac{3-1}{n} = 1 + \frac{2i}{n}$ and $\Delta x = \frac{2}{n}$. Substituting, we get:

$$L_n = \sum_{i=0}^{n-1} e^{1+\frac{2i}{n}} \cdot \frac{2}{n}.$$

5. For $f(x) = \sqrt{x}$ on $[1, 4]$ with $n = 3$ and right endpoints:

$$\Delta x = \frac{4-1}{3} = 1.$$

The right endpoints are $x_1^* = 2, x_2^* = 3, x_3^* = 4$. The Riemann sum is:

$$R_3 = \sum_{i=1}^{3} f(x_i^*)\Delta x = \sum_{i=1}^{3} \sqrt{x_i^*} \cdot 1 = \sqrt{2} + \sqrt{3} + \sqrt{4}.$$

6. We calculate each integral by finding the area of the geometric shapes formed by the graph of $g(x)$ and the x-axis.

(a) $\int_0^2 g(x)\,dx$
From $x = 0$ to $x = 2$, the graph forms a triangle with base 2 and height 4. The area of this triangle is:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2 \times 4 = 4.$$

Since the triangle is above the x-axis, the integral is:

$$\int_0^2 g(x)\,dx = 4.$$

(b) $\int_2^6 g(x)\,dx$
From $x = 2$ to $x = 6$, the graph forms a semicircle below the x-axis. The radius of the semicircle is 2 The area of the semicircle is

$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi.$$

Since the semicircle is below the x-axis, the integral is:

$$\int_2^6 g(x)\,dx = -2\pi.$$

(c) $\int_0^7 g(x)\,dx$
To compute the total integral from $x = 0$ to $x = 7$, we sum the areas from parts (a) and (b), and add the area from $x = 6$ to $x = 7$:

- From $x = 0$ to $x = 2$, the area is 4 (from part (a)).
- From $x = 2$ to $x = 6$, the area is -2π (from part (b)).
- From $x = 6$ to $x = 7$, the graph forms a triangle with base 1 and height 1. The area of this triangle is:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$

Adding these areas together:

$$\int_0^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi.$$