Math 10A Fall 2024 Worksheet 14

October 15, 2024

1. Find derivatives of the following functions.
   (a) $f(x) = \cos(\ln x)$
   (b) $f(x) = x^2 \ln(x)$
   (c) $f(x) = e^{e^x}$
   (d) $f(x) = x^{\sin(x)}$
   (e) $f(x) = \tan^{-1}(x^2)$
   (f) $\frac{dy}{dx}$, where $y = \ln(x^2 + y^2)$

2. Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) What is $f'(0)$?
   (b) What is $f'(x)$ for $x \neq 0$?
   (c) Check that $f'(x)$ is not continuous at $x = 0$. So this function is differentiable but the derivative is not continuous.

3. Consider the equation $y^2 = x^3 + 1$.

   (a) Find $\frac{dy}{dx}$.

   (b) Find the equation of the tangent line at $(2, 3)$.

   (c) Find the point where the line (b) and the curve $y^2 = x^3 + 1$ intersects.

   (d) Find another points on the curve other than $(-1, 0)$ where the tangent line at those points pass $(-1, 0)$.

1 Solutions

1. (a) $f'(x) = -\frac{\sin(\ln x)}{x}$.
   (b) $f'(x) = 2x\ln(x) + x$.
   (c) $f'(x) = e^x e^{e^x} = e^{x+e^x}$.
   (d) Use logarithmic differentiation. $\ln f(x) = \sin(x)\ln(x)$ and $(\ln f(x))' = \frac{f'(x)}{f(x)} = \cos(x)\ln(x) + \frac{\sin(x)}{x}$. Hence $f'(x) = x^{\sin(x)}(\cos(x)\ln(x) + \frac{\sin(x)}{x})$.
   (e) $f'(x) = \frac{2x}{1+x^4}$.
   (f) Use implicit differentiation. $\frac{dy}{dx} = \frac{1}{x^2+y^2} \cdot (2x + 2y\frac{dy}{dx})$, and solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y}$.

2. Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) By definition,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2\cos(1/h)}{h} = \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$$

   where the last equality can be justified using squeeze theorem.

   (b) By the product rule,

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

   (Be careful about the sign.)

   (c) As $x \to 0$, the term $2x\cos(1/x)$ converges to 0 by the squeeze theorem. However, the other term $\sin(1/x)$ oscillates and does not converge, hence $\lim_{x\to 0} f'(x)$ does not exist.

3. Consider the equation $y^2 = x^3 + 1$.

   (a) By using implicit differentiation, we have $2y\frac{dy}{dx} = 3x^2$ and $\frac{dy}{dx} = \frac{3x^2}{2y}$.
   (b) From (a), the slope is 2 and the tangent line is $y = 2(x-2) + 3 = 2x - 1$.
   (c) By solving the equation $(2x-1)^2 = y^2 = x^3 + 1 \Leftrightarrow x^3 - 4x^2 + 4x = x(x-2)^2 = 0$, we get $x = 2$ or $x = 0$. $x = 2$ corresponds to the alreay known point $(2, 3)$, and the other point is $(0, -1)$.
   (d) Let (a, b) be the point we are looking for. Then the tangent line at the point is

$$y = \frac{3a^2}{2b}(x - a) + b$$

   and since this line passes $(-1, 0)$, we have

$$0 = \frac{3a^2}{2b}(-1 - a) + b \Leftrightarrow 3a^3 + 3a^2 = 2b^2.$$

   Since (a, b) is on the curve, $b^2 = a^3 + 1$ and substituting b^2 by $a^3 + 1$ gives $3a^3 + 3a^2 = 2(a^3 + 1) \Leftrightarrow a^3 + 3a^2 - 2 = (a+1)(a^2 + 2a - 2) = 0$, so $a = -1 \pm \sqrt{3}$. By the way, we should have $a^3 = b^2 - 1 \geq -1$, so the only possibility is $a = -1 + \sqrt{3}$ and this gives two points $(a, b) = (-1 + \sqrt{3}, \pm\sqrt{5 - 2\sqrt{3}})$.

What you just have done are *doubling and halving points on an elliptic curve*, although you don't need to know about what these words mean.