Math 10A Fall 2024 Worksheet 1

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Do NOT use a calculator for any of these problems!

1 Algebra Review

1. What is $\frac{3}{5} + \frac{5}{7}$? Also do $\frac{1}{32} + \frac{3}{4}$.
2. What is $(-2)^3$? What about $-8 - (-4)$?
3. What is $\log_{12}(144)$? Also do $\log_3(27)$ and $\ln(e^7)$.
4. Compute $\sin(\pi/6)$, $\cos(5\pi/3)$, and $\tan(1001\pi/4)$.
5. Write the following expression as a single fraction.
$$\frac{x+1}{x^2} + \frac{x+3}{x+4}.$$
6. Compute $\sqrt{81}$, $\sqrt[3]{-343}$, and $\sqrt{-16}$.
7. Expand the following product of polynomials.
$$(3x^2 + 4x + 7)(y^4 - xy + 2y).$$

2 Vectors

Remember, scalar product is when you multiply one scalar and one vector to get out a vector. Dot product is when you multiply two vectors to get out a scalar.

1. *Vector arithmetic.*
   (a) Write down the zero vector in $\mathbb{R}^6$.
   (b) Compute the following vector additions and subtractions. $(2,4)+(6,5)$, $(3,-6)+(-4,2)$, $(0,0)+(2,2)$, $(1,2)-(3,4)$, $(15,6)-(9,8)$.
   (c) Compute the following multiplications of a vector by scalars. $1.5 \cdot (2,4)$, $\frac{6}{7} \cdot (7,-49)$, $-1 \cdot (3,0)$, $-2 \cdot (6,-1)$, $2024\pi \cdot (0,0)$.
   (d) Compute the following dot products.
   $$(2,5,7) \cdot (3,-4,1), \quad (4,0,-1) \cdot (-1,0,4), \quad (2,3,7) \cdot (-1,5,-13/7).$$
   (e) Let $\mathbf{v}, \mathbf{w}$ be two vectors. What does $\mathbf{v} \cdot \mathbf{w} = 0$ mean geometrically?
2. *Norm of a vector.*
   (a) What is the norm, or length, of a general vector $(a,b,c) \in \mathbb{R}^3$? Explain how this relates to the Pythagorean theorem.

(b) Compute the norms of the following vectors in $\mathbb{R}^3$: $(2,-4,3)$, $(-2,4,3)$, $(3,7,0)$, $(1,1,1)$.

(c) What does it mean for a vector to be a unit vector? Write down a specific unit vector in $\mathbb{R}^3$.

(d) Name a vector having norm 0 (in $\mathbb{R}^3$). Are there any other vectors having norm 0? Explain why or why not. What about negative norm?

(e) What do you get if you take the dot product of any vector with itself? What about the dot product of any vector and the zero vector?

3. Using norm and dot product, explain how to compute the angle between two vectors, given only the vectors. Let θ be the angle between the two vectors $(3,5,1)$ and $(0,2,4)$. What is $\cos\theta$?

3 Graphing

1. Graph the following functions in $\mathbb{R}^2$. $y = x^2$, $y = \sqrt{1-x^2}$, $y = \sin x$, $y = 4x+7$, $y = x^3 - 2x^2 + 4x$.
2. What does the graph of $x = 2$ look like in $\mathbb{R}^2$? What does the graph of the same equation look like in $\mathbb{R}^1$, and in $\mathbb{R}^3$?
3. What is the equation of a circle with radius 2 and center $(4,7)$?
4. The equation $x^2 - 6x + y^2 = 25$ is the graph of a circle in $\mathbb{R}^2$. What are its center and radius?
5. The equation $x^2 + 4x + y^2 - 2y + z^2 = 41$ defines the graph of a sphere in $\mathbb{R}^3$. What are its center and radius?
6. What is the equation of a line in $\mathbb{R}^2$ that passes through the points $(0,1)$ and $(2,3)$?
7. Explain how, starting with the graph of $y = x^2$, you can get to the graph of $y = 2(x-4)^2 + 1$.
8. Let F be the graph of the function $y = f(x)$. What function has a graph that is the exact same shape as F, but its graph is shifted up 1 and 2 to the right?
9. What is the equation of a plane in $\mathbb{R}^3$ that passes through the points $(1,0,0)$, $(0,0,0)$, and $(0,1,0)$?

Solutions

Algebra Review

1. Find a common denominator, then add.

$$\frac{3}{5}+\frac{5}{7}=\frac{3\cdot 7}{5\cdot 7}+\frac{5\cdot 5}{7\cdot 5}=\frac{21}{35}+\frac{25}{35}=\boxed{\frac{46}{35}}.$$

$$\frac{1}{32}+\frac{3}{4}=\frac{1}{32}+\frac{24}{32}=\boxed{\frac{25}{32}}.$$

2. We have

$$(-2)^3=(-2)\cdot(-2)\cdot(-2)=[(-2)\cdot(-2)]\cdot(-2)=[4]\cdot(-2)=\boxed{-8}.$$

   Remember that subtracting a negative is the same as adding a positive, so

$$-8-(-4)=-8+4=\boxed{-4}.$$

3. $\log_b(a)$ is the number x such that $b^x=a$. $12^2=144$, so $\log_{12}(144)=\boxed{2}$. Similarly, $\log_3(27)=\boxed{3}$. Finally, ln is another way to write $\log_e$, so since $e^7=e^7$, $\ln(e^7)=7$.

4. Use your unit circle; $\sin(\pi/6)=\boxed{1/2}$, and $\cos(5\pi/3)=\cos(\pi/3)=\boxed{1/2}$. For the last one, remember that tangent is periodic with period π, i.e. for any integer k and any number x, $\tan(k\pi+x)=\tan x$. So while it looks scary,

$$\tan(\frac{1001\pi}{4})=\tan(\frac{1000\pi}{4}+\frac{\pi}{4})=\tan(250\pi+\frac{\pi}{4})=\tan\pi/4=\boxed{1}.$$

5. Same deal as finding a common denominator with numbers, there's just a variable.

$$\frac{x+1}{x^2}+\frac{x+3}{x+4}=\frac{(x+1)(x+4)}{x^2(x+4)}+\frac{(x+3)(x^2)}{(x+4)(x^2)}=\frac{(x+1)(x+4)+x^2(x+3)}{x^2(x+4)}=\boxed{\frac{x^3+4x^2+5x+4}{x^3+4x^2}}.$$

6. $\sqrt{81}=\boxed{9}$, $\sqrt[3]{-343}=\boxed{-7}$, $\sqrt{-16}=\boxed{4i}$.

7. It's

$$\begin{aligned}3x^2(y^4-xy+2y)+4x(y^4-xy+2y)&+7(y^4-xy+2y)\\ &=3x^2y^4-3x^3y+6x^2y+4xy^4-4x^2y+8xy+7y^4-7xy+14y\\ &=\boxed{3x^2y^4+4xy^4-3x^3y+7y^4+2x^2y+xy+14y}.\end{aligned}$$

Vectors

1. *Vector arithmetic.*

   (a) $\boxed{(0,0,0,0,0,0)}$.

   (b) Add or subtract coordinate-wise.

$$\begin{aligned}(2,4)+(6,5)&=\boxed{(8,9)}.\\ (3,-6)+(-4,2)&=\boxed{(-1,-4)}.\\ (0,0)+(2,2)&=\boxed{(2,2)}.\\ (1,2)-(3,4)&=\boxed{(-2,-2)}.\\ (15,6)-(9,8)&=\boxed{(6,-2)}.\end{aligned}$$

(c) You multiply each coordinate by the scalar in turn.

$$\begin{aligned}
1.5 \cdot (2,4) &= \boxed{(3,6)}. \\
6/7 \cdot (7,-49) &= \boxed{(6,-42)}. \\
-1 \cdot (3,0) &= \boxed{(-3,0)}. \\
-2 \cdot (6,-1) &= \boxed{(-12,2)}. \\
2024\pi \cdot (0,0) &= \boxed{(0,0)}.
\end{aligned}$$

(d) Recall that the dot product of two general vectors (a,b,c) and (d,e,f) is the number $ad+be+cf$. In words, you go down their coordinates and multiply the corresponding coordinates, then add everything up. So,

$$\begin{aligned}
(2,5,7) \cdot (3,-4,1) = 6 + (-20) + 7 &= \boxed{-7}. \\
(4,0,-1) \cdot (-1,0,4) = (-4) + 0 + (-4) &= \boxed{-8}. \\
(2,3,7) \cdot (-1,5,-13/7) = -2 + 15 - 13 &= \boxed{0}.
\end{aligned}$$

(e) $\mathbf{v} \cdot \mathbf{w} = 0$ means $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, i.e. the angle between them is exactly $90^\circ = \pi/2$ rad.

2. *Norm of a vector.*

(a) The norm of (a,b,c) is the number $\sqrt{a^2+b^2+c^2}$. We write $\|\mathbf{v}\|$ for the norm of $\mathbf{v}$.

To explain the Pythagorean theorem connection, it is easier to consider two dimensions for simplicity. There, the Pythagorean theorem says this "norm" number is literally the distance from the point (a,b) in the coordinate plane to the origin. A slightly more complicated argument shows the same thing in 3, or any number of, dimensions.

(b)

$$\begin{aligned}
\|(2,-4,3)\| &= \sqrt{4+16+9} = \boxed{\sqrt{29}}. \\
\|(-2,4,3)\| &= \|(2,-4,3)\| = \boxed{\sqrt{29}}. \\
\|(3,7,0)\| &= \sqrt{9+49+0} = \boxed{\sqrt{58}}. \\
\|(1,1,1)\| &= \sqrt{1+1+1} = \boxed{\sqrt{3}}.
\end{aligned}$$

(c) A unit vector is a vector whose norm is 1. There are many unit vectors; the easiest to write down are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. A more interesting one is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

(d) $(0,0,0)$ is the only vector in $\mathbb{R}^3$ having norm 0. The reason is because the norm is the square root of a sum of squares, and a sum of squares is always non-negative since each square is non-negative. Moreover, the only way for a square to be 0 is to take 0^2. The same argument shows there are no vectors of negative norm.

(e) We compute that

$$(a,b,c) \cdot (a,b,c) = a^2 + b^2 + c^2 = \boxed{\|(a,b,c)\|^2}.$$

In other words, $\mathbf{v} \cdot \mathbf{v}$ is the square of $\mathbf{v}$'s norm. For any vector (a,b,c), we have

$$(a,b,c) \cdot \mathbf{0} = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0 + 0 + 0 = \boxed{0}.$$

3. The point of this problem is to remind you of the formula

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\cdot\|\mathbf{w}\|},$$

where θ is the angle between two vectors $\mathbf{v},\mathbf{w}$. This formula tells you that the size of the dot product tells you about the size of θ. If $\mathbf{v}=\mathbf{w}$, so that they lie exactly on top of each other, you get this cosine is 1, which indeed corresponds to an angle of 0.

Using this formula in the example, we find

$$\cos\theta = \frac{(3,5,1)\cdot(0,2,4)}{\|(3,5,1)\|\cdot\|(0,2,4)\|} = \boxed{\frac{14}{\sqrt{35}\cdot\sqrt{20}}}.$$

Graphing

1. Check your answers yourself using a graphing calculator. I suggest Desmos.
2. In $\mathbb{R}^2$, the graph of $x=2$ is a vertical line, in $\mathbb{R}^1$ it is the point 2 on the number line, and in $\mathbb{R}^3$ it is a plane parallel to the yz-plane.
3. It's

$$\boxed{(x-4)^2+(y-7)^2=4}.$$

4. We want to turn this equation into one that looks like $(x-a)^2+(y-b)^2=r^2$. To do this, we complete the square with x and with y. These manipulations look like

$$\begin{aligned} x^2-6x+y^2=25 &\implies \\ (x-3)^2-9+y^2=25 &\implies \\ (x-3)^2+(y-0)^2=34&. \end{aligned}$$

So this circle has center $\boxed{(3,0)}$ and radius $\boxed{\sqrt{34}}$.

5. This is in perfect analogy with the above example, you complete the square in each variable separately then read off the center and radius. This turns the equation into

$$(x+2)^2+(y-1)^2+(z-0)^2=46,$$

so the sphere has center $\boxed{(-2,1,0)}$ and radius $\boxed{\sqrt{46}}$.

6. Unless it is a vertical line $x=a$, any line in $\mathbb{R}^2$ has an equation that looks like $y=mx+b$ for some numbers m,b. Since $(0,1)$ and $(2,3)$ lie on this line, we find $1=m\cdot 0+b$ and $3=m\cdot 2+b$. The first equation shows $b=1$, then the second equation shows $m=1$ as well, so our line is $\boxed{y=x+1}$. As a check, if you plug 0 in for x you get 1, and if you plug 2 in for x you get 3.
7. It's shifting 4 units to the right, stretching vertically by 2, then moving up by 1.
8. $f(x-2)+1$.
9. Through visualizing the points in 3D space, we notice this is the plane $\boxed{z=0}$.

Method that works more generally: Set up a system of 3 equations in 3 variables. Planes have equations of the form $ax+by+cz=d$, and we need

$$\begin{aligned} a(1)+b(0)+c(0)&=d \\ a(0)+b(0)+c(0)&=d \\ a(0)+b(1)+c(0)&=d \end{aligned}$$

which tells us that $a=b=d=0$ and c can be any number.