The Fundamental Theorem of Calculus

November 19th, 2024

Here are some key ideas from section 5.3.

F is an antiderivative of f

- The indefinite integral is $\int f(x)\,dx$. If $\int f(x)\,dx = F(x)$ +C, then $F'(x) = f(x)$.
  - A definite integral looks like $\int_a^b f(x)\,dx$ and is a number.
  - An indefinite integral looks like $\int f(x)\,dx$ and is a family of functions (of the form $F(x) + C$).
- **FTC 1**: Also known as the Evaluation Theorem. If f is continuous on the interval $[a, b]$, then $\int_a^b f(x)\,dx =$ $F(b) - F(a)$, where F is any antiderivative of f. For example:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

- **FTC 2**: First, some motivation.

let $g(x) = \int_a^x f(t)\,dt = F(x) - F(a)$. Then $g'(x) = (F(x) - F(a))' = F'(x) - F(a)' = f(x) - 0 = f(x)$

constant

Now, the theorem. Suppose f is continuous on $[a, b]$. Let $g(x) = \int_a^x f(t)\,dt$ on $[a, b]$. Then $g'(x) =$ $f(x)$ on (a, b). In other words, g is an antiderivative of f.

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Trig practice: Find all values of $\arcsin \frac{1}{\sqrt{2}}$ and $\arccos \frac{\sqrt{3}}{2}$ on $[0, 2\pi]$.

Problem 1: (Stewart 5.3) Why is this process incorrect?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right]_{-1}^3 = -\frac{4}{3}.$$

My Attempt:

Solution:

we need $\frac{1}{x^2}$ to be continuous on $[-1, 3]$ in order to use the evaluation rule like that, but there is a discontinuity at $x = 0$.

Problem 2: (Stewart 5.3) Find $\int (1 - t)(2 + t^2)\,dt$.

My Attempt:

Solution:

$(1-t)(2+t^2) = 2 + t^2 - 2t - t^3$

Then $\int (1-t)(2+t^2)\,dt = \int (2 + t^2 - 2t - t^3)\,dt$

$= 2t + \frac{t^3}{3} - t^2 - \frac{t^4}{4} + C$

Problem 3: (Stewart 5.3) Evaluate the following integrals.

a) $\displaystyle\int_{-2}^{3}(x^2-3)\,dx$; b) $\displaystyle\int_{-5}^{5} e\,dx$; c) $\displaystyle\int_0^1 10^x\,dx$.

My Attempt:

Solution:

a) $\left[\frac{x^3}{3}-3x\right]_{-2}^{3} = \frac{27}{3}-9-\left(\frac{-8}{3}+6\right)$

b) $\left[ex\right]_{-5}^{5} = 10e$

c) $\left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$

Problem 3: (Stewart 5.3) Find the following indefinite integrals.

a) $\displaystyle\int(1+\tan^2\alpha)\,d\alpha$; b) $\displaystyle\int\frac{\sin x}{1-\sin^2 x}\,dx$; c) $\displaystyle\int v(v^2+2)^2\,dv$.

My Attempt:

Solution:

a) $1+\tan^2\alpha = \sec^2\alpha$

$\int(1+\tan^2\alpha)\,d\alpha = \int \sec^2\alpha\,d\alpha = \boxed{\tan\alpha + C}$

b) $1-\sin^2 x = \cos^2 x$

$\int \frac{\sin x}{1-\sin^2 x}\,dx = \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx = \int \tan x \sec x\,dx = \boxed{\sec x + C}$

c) use substitution (or expand)

let $u = v^2+2$. Then $du = 2v\,dv$

$\int v(v^2+2)^2\,dv = \int \frac{du}{2}\cdot u^2 = \frac{u^3}{6} + C = \boxed{\frac{(v^2+2)^3}{6}+C}$

Problem 4: (Stewart 5.3) Find the first derivatives of the following functions.

a) $g(x) = \displaystyle\int_1^x \frac{1}{t^3+1}\,dt$; b) $g(y) = \displaystyle\int_3^y e^{t^2-t}\,dt$; c) $g(r) = \displaystyle\int_0^4 \sqrt{x^2+4}\,dx$.

My Attempt:

Solution:

a) $g'(x) = \boxed{\frac{1}{x^3+1}}$

b) $g'(y) = \boxed{e^{y^2-y}}$

c) $g'(r) = \boxed{0}$, since $\int_0^4 \sqrt{x^2+4}\,dx$ is a constant

Problem 5: (Stewart 5.3) Find the first derivatives of the following functions.

a) $h(x) = \int_2^{1/x} \arctan t \, dt$; b) $h(x) = \int_0^{x^2} \sqrt{1+r^3} \, dr$; c) $y = \int_{e^x}^0 \sin^3 t \, dt$.

My Attempt:

Solution:

a) $h'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(\frac{1}{x}\right)' = \frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right) = \boxed{\frac{-1}{x^2+1}}$

b) $h'(x) = \sqrt{1+x^6} \cdot 2x = \boxed{2x\sqrt{1+x^6}}$

c) $y = -\int_0^{e^x} \sin^3 t \, dt$

$\frac{dy}{dx} = \boxed{-\sin^3(e^x) \cdot e^x}$

Problem 6: (Stewart 5.3) Find a function f and a number a such that, for all $x > 0$,

$$6 + \int_a^x \frac{f(t)}{t^2} \, dt = 2\sqrt{x}. \quad = 2x^{\frac{1}{2}}$$

My Attempt:

Solution: Take the derivative of both sides

$\frac{f(x)}{x^2} = \frac{d}{dx} 2x^{\frac{1}{2}} = x^{\frac{-1}{2}} \Rightarrow f(x) = x^2 x^{-1/2} = x^{3/2}$

Then $6 + \int_a^x \frac{t^{3/2}}{t^2} dt = 6 + \int_a^x t^{-1/2} \, dt = 6 + \left[2t^{\frac{1}{2}}\right]_a^x$
$= 6 + 2\sqrt{x} - 2\sqrt{a}$

we want $6 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x}$
thus $6 - 2\sqrt{a} = 0 \Rightarrow a = 9$
so $\boxed{f(t) = t^{3/2}}$ and $\boxed{a = 9}$

Problem 7: (Apostol 5.5) A function f is continuous everywehre and satisfies the equation

$$\int_0^x f(t) \, dt = -\frac{1}{2} + x^2 + x \sin 2x + \frac{1}{2} \cos 2x.$$

for all x. Compute $f(\pi/4)$ and $f'(\pi/4)$.

My Attempt:

Solution:

$f(x) = 2x + \cancel{\sin 2x} + 2x\cos 2x - \cancel{\sin 2x}$

$= 2x + 2x \cos 2x$

Thus $f(\pi/4) = 2\pi/4 + 2\pi/4 \cos 2\pi/4 = \boxed{\frac{\pi}{2}}$

$f'(x) = 2 + 2(\cos 2x - 2x \sin 2x)$

Thus $f'(\pi/4) = 2 + 2(\cos 2\pi/4 - 2\pi/4 \sin 2\pi/4)$

$= 2 + 2(0 - \pi/2) = \boxed{2 - \pi}$

Challenge problem: (Apostol 5.5) Show that, for all ~~a~~ real x,

oops ☺

$$\int_0^x (t + |t|)^2 \, dt = \frac{2x^2}{3}(x + |x|).$$

$x \geq 0 \Rightarrow \int_0^x 4t^2 \, dt = \frac{2}{3}x^2(2x) = \frac{2}{3}x^2(x+|x|)$ ✓

$x < 0 \Rightarrow \int_0^x (t-t)^2 = 0 = \frac{2}{3}x^2(x+|x|)$ ✓