Dynamics, inverses, determinants

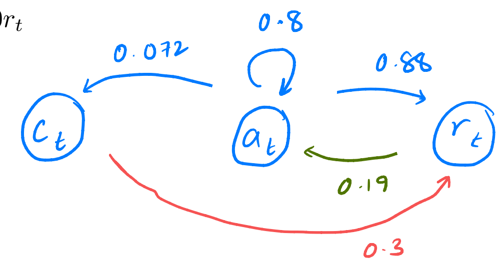
September 12th, 2024

Here are some key ideas from sections 8.5 and 8.6.

- For the given system

$$\begin{aligned} c_{t+1} &= 0c_t + 0a_t + 0.3r_t \\ a_{t+1} &= 0.072c_t + 0.8a_t + 0.88r_t \\ r_{t+1} &= 0c_t + 0.19a_t + 0r_t \end{aligned}$$

  The corresponding **Leslie diagram** is



- The identity matrix has 1s along the diagonal and 0s everywhere else. Any matrix multiplied by the identity matrix is itself! Mathematically,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = A, \quad IA = A$$

- If A is a square matrix and $AB = BA = I$, then B is the inverse of A. If A has an inverse, we say it is **nonsingular**. Otherwise, it is singular.
- The **determinant** of a 2×2 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det A = a_{11}a_{22} - a_{12}a_{21}$$

- If A is an $n \times n$ matrix, then A is invertible if and only if $\det A \neq 0$.
- The inverse of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

Problem 1: (Stewart & Day 8.6) Are $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ inverses of each other?

My Attempt:

Solution:

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

since $\begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

they are not inverses.

Problem 2: (Stewart & Day 8.6) For the following matrices, calculate the inverse or explain why it's not possible to do so.

a) $\begin{bmatrix} 7 & 9 \\ 5 & 6 \end{bmatrix}$; b) $\begin{bmatrix} 14 & 6 \\ 7 & 3 \end{bmatrix}$; c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

My Attempt:

Solution:

$\det \begin{bmatrix} 7 & 9 \\ 5 & 6 \end{bmatrix} = 42 - 45 = -3$, so it is possible.

$\frac{1}{-3} \begin{bmatrix} 6 & -9 \\ -5 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 3 \\ 5/3 & -7/3 \end{bmatrix}}$

$\det \begin{bmatrix} 14 & 6 \\ 7 & 3 \end{bmatrix} = 0$, so impossible.

$\det \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = -1$.

$\frac{1}{-1} \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

Problem 3: (Stewart & Day 8.6) Find the inverse of $\begin{bmatrix} x^2 & 2x \\ x^3 & x \end{bmatrix}$.

My Attempt:

Solution:

$\det = x^3 - 2x^4 = x^3(1-2x)$

Assume $x \neq 0$, $x \neq \frac{1}{2}$

$\frac{1}{\det} \begin{bmatrix} x & -2x \\ -x^3 & x^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{x^2(1-2x)} & \frac{-2}{x^2(1-2x)} \\ \frac{-1}{(1-2x)} & \frac{1}{x(1-2x)} \end{bmatrix}$

Problem 4: (Stewart & Day 8.6) Use the determinant to decide when $\begin{bmatrix} x & x+x^2 \\ 3x & 0 \end{bmatrix}$. is invertible.

My Attempt:

Solution:

$\det = 0 - 3x(x+x^2)$

$-3x(x+x^2) = 0 \Rightarrow -3x^2(1+x) = 0 \Rightarrow x = 0, x = -1$

NOT invertible only for $x = 0$, $x = -1$

invertible for $x \neq 0, x \neq -1$

Problem 5: (Stewart & Day 8.6) Find all 2×2 matrices A such that $\det A = 1$ and $A = A^{-1}$.

My Attempt:

Solution:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = 1 \Rightarrow ad - bc = 1$$

$$A = A^{-1} = \frac{1}{1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so $d = a$

$-b = b \Rightarrow b = 0$

$-c = c \Rightarrow c = 0$

$ad - 0 = 1$

$a^2 = 1$

$a = 1, -1$

so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Problem 6: (Stewart & Day 8.6) Verify that $\det(AB) = \det A \det B$ for arbitrary 2×2 matrices.

My Attempt:

Solution:

$$\det(AB) = \det \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22})$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21}$$

$$\det(A)\det(B) = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21}$$

$$= \det(AB) \checkmark$$

Problem 7: (Stewart & Day 8.6) Show that for any arbitrary nonsingular matrix A, its inverse A^{-1} is unique. *Hint: Show that if there are two inverses, they must be equal.*

My Attempt:

Solution:

Suppose B and C are inverses. Then

$AB = BA = I$, $AC = CA = I$. So

$$AB = I \Rightarrow CAB = CI \Rightarrow (CA)B = C \Rightarrow IB = C \Rightarrow B = C$$

A & B are inverses

left multiply by C

associative property

since $CA = I$

$I \cdot B = B$

Challenge Problem: (Stewart & Day 8.4) Suppose that A is an $n \times n$ diagonal matrix with entries d_{ii}. Show that A^{-1} is an $n \times n$ diagonal matrix with entries $1/d_{ii}$.

$$\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix} = \begin{bmatrix} 1+0+\cdots+0 & 0+0\cdots+0 & \cdots & 0+0+\cdots+0 \\ 0+0+\cdots+0 & 1+0+\cdots+0 & & 0+0+\cdots+0 \\ \vdots & \vdots & \ddots & \vdots \\ 0+0+\cdots+0 & 0+0+\cdots+0 & & 1+0+\cdots+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$