Math 10A Fall 2024 Worksheet 12

October 9, 2024

1 Continuity

1. The *Intermediate Value Theorem* (IVT) states that if $f(x)$ is a continuous function on $[a, b]$ with $f(a) \neq f(b)$, and if y_0 is any number between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = y_0$. This is often helpful in showing that a solution to some equation exists, although it is not necessarily useful for finding the solution.

   (a) Draw a picture illustrating what the IVT is saying.

   (b) Give a counterexample to the IVT if f is discontinuous somewhere on $[a, b]$. That is, come up with some discontinuous function that "misses" some value between the two endpoints.

   (c) Show that $e^x = x+1.5$ has a solution. (*Hint*: this is the same as showing that $f(x) = e^x - x - 1.5 = 0$ has a solution. Consider $f(0)$ and $f(1)$. Note that $e = 2.718\ldots$. The same strategy will apply in a lot of problems like this one.)

   (d) Show that if $f(x)$ is continuous on $(-\infty, \infty)$, and that if $\lim_{x\to-\infty} = -\infty$ and $\lim_{x\to\infty} = \infty$, then the range of f is $(-\infty, \infty)$.

   (e) A monk leaves a monastery at 7AM and takes his usual path up to the top of a mountain, arriving to the top at 7PM. The following morning, he starts at 7AM at the top and takes the same path back, arriving back at the monastery at 7PM. Show that there is some point on the path that the monk crossed at exactly the same time each day.

2. Find a constant c such that the given piecewise-defined function is everywhere continuous.

   (a) $f(x) = \begin{cases} x + c : & x \leq -3 \\ cx^2 : & x > -3 \end{cases}$

   (b) $g(x) = \begin{cases} e^{cx} : & x < 0 \\ c\cos(x) : & x \geq 0 \end{cases}$

   (c) $h(x) = \begin{cases} c^x : & x < 2 \\ 2cx - 4 : & x \geq 2 \end{cases}$

3. Let M denote the mass of the Earth and let R denote its radius. Let G denote the gravitational constant. The net gravitational forced exerted by the Earth on a unit mass at distance r from the center of the planet is given by the formula

$$F(r) = \begin{cases} GMr/R^3 : & r < R \\ GM/r^2 : & r \geq R \end{cases}$$

   (a) Is F a continuous function of r?

   (b) How strong is gravity at the center of the Earth? Does this make sense?

2 Derivatives

1. Write down the equation of the tangent line to the function $y = f(x)$ at $x = a$ based on the given information.

   (a) $a = 1,\ f(1) = 5,\ f'(1) = 1$

   (b) $a = 0,\ f(0) = 0,\ f'(0) = 10$

   (c) $a = -3, f(-3) = 2, f'(-3) = -2$

2. Sketch a function $f(x)$ with the following properties:

   - $f(1) = f(-1) = f(3) = 0$
   - $f'(2) = f'(-2) = 0$
   - $f'(-1) = 1,\ f'(1) = -2$
   - $\lim_{x\to-\infty} f(x) = -3$
   - $\lim_{x\to\infty} f(x) = \infty$

   If you want more practice, replace the numbers above with random (smallish) numbers and do this exercise again.

3. Compute the derivatives the following functions directly using the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. (Don't use the power rule if you know it already.)

   (a) $f(x) = 20231010$

   (b) $f(x) = 2x + 3$

   (c) $f(x) = \sqrt{x}$ for $x > 0$ (*Hint*: multiply the difference quotient by $1 = (\sqrt{x+h}+\sqrt{x})/(\sqrt{x+h}+\sqrt{x})$)

   (d) $f(x) = 1/x$ with $x \neq 0$

4. Find the equations of the tangent lines to each of the functions in the previous question at $x = 1$.

Solutions

1 Continuity

1. (c) We have $f(0) = e^0 - 0 - 1.5 = -0.5$ and $f(1) = e^1 - 1 - 1.5 = e - 2.5 > 0$. Since $-0.5 < 0 < e - 2.5$, we may apply the IVT to conclude that there is some $c \in (0, 1)$ such that $f(c) = 0$, which is what we wanted to show.

   (d) Let y_0 be any real number. Since $\lim_{x\to-\infty} f(x) = -\infty$, if we go far enough left there exists some a such that $f(a) < y_0$. Likewise $\lim_{x\to\infty} f(x) = \infty$ tells us that if we go far enough right there exists some b such that $f(b) > y_0$. Applying the IVT on the interval $[a, b]$ shows that that there exists some $c \in (a, b)$ with $f(c) = y_0$. Since we can choose any number for y_0, this means that the range of $f(x)$ is all real numbers.

   (e) Let $f(x)$ denote the total distance (in kilometers, say) that the monk has travelled along the path at time x on the first day, with x ranging from 7 to 19 corresponding to hours in the day. Similarly, let $g(x)$ denote the remaining distance of the monk from the monastery at time x on the second day.

   What we want to show is that $f(c) = g(c)$ for some $c \in [7, 17]$. Since the monk (hopefully) cannot teleport, both f and g are continuous functions. Letting $L > 0$ be the total length of the path, we have $f(7) = 0, f(19) = L, g(7) = L, g(17) = 0$. Since f and g are continuous, so is the function $f - g$. We have $(f - g)(7) = -L$ and $(f - g)(19) = L$. Since $-L < 0 < L$, by the IVT, this means that there exists some $c \in (7, 19)$ such that $(f - g)(c) = 0$, or equivalently $f(c) = g(c)$, which is what we wanted to show.

   Note that it doesn't matter whether the monk has a steady pace or not. In fact, it doesn't even matter if the monk stops or turns around sometimes!

2. (a) Plugging in $x = -3$, we need to solve $-3 + c = 9c$, which gives $c = -3/8$.

   (b) Plugging in $x = 0$, we need to solve $e^{0c} = c\cos(0)$, which simplifies to $1 = c$.

   (c) Plugging in $x = 2$, we need to solve $c^2 = 4c - 4$. Equivalently, we need to find the roots of the quadratic $c^2 - 4c + 4 = (c - 2)^2$. The only root is $c = 2$. (A previous version of this problem used $-2cx - 4$ instead of $2cx - 4$, but this leads to the answer $c = -2$, and the function $(-2)^x$ doesn't make sense.)

3. (a) Yes, because $\lim_{r\to R^-} F(r) = \lim_{r\to R^+} F(r) = GM/R^2$, and each of the pieces of $F(r)$ are given by continuous functions on their respective domains. Note that GM/r^2 is continuous on $r \geq R$, even though it wouldn't be continuous or even defined if we allowed $r = 0$.

   (b) At the center of the Earth, the net gravitational force is 0—you are weightless there. The reason this makes sense is that the mass of the Earth is arranged symmetrically around its center, so the contributions to gravity end up cancelling each other out there.

2 Derivatives

1. (a) $y - 5 = x - 1$, or $y = x + 4$

   (b) $y = 10x$

   (c) $y - 2 = -2(x + 3)$, or $y = -2x - 4$

3. (a) $\lim_{h\to0} \frac{20231010-20231010}{h} = 0$

   (b) $\lim_{h\to0} \frac{2(x+h)+3-(2x+3)}{h} = \lim_{h\to0} \frac{2h}{h} = 2$

   (c) We need to evaluate $\lim_{h\to0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$. To do this, we "rationalize the numerator" by multiplying by $\frac{\sqrt{x+h}+\sqrt{x}}{\cdot}$ We end up with

$$\lim_{h\to0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h\to0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

so the derivative is $f'(x) = \frac{1}{2\sqrt{x}}$.

(d)

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{-h}{x(x+h)}\right)}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -1/x^2 = f'(x)
\end{aligned}$$

4. (a) $f(1) = 20231010, f'(1) = 0$, so the tangent line at $x = 1$ is $y = 20231010$.

   (b) $f(1) = 5, f'(1) = 2$, so the tangent line at $x = 1$ is $y = 2(x - 1) + 5$, or $y = 2x + 3$. Notice that in this part and in part (a), the graph of $f(x)$ is a straight line, and the tangent line at a point is just the same line again.

   (c) $f(1) = 1, f'(1) = 1/2$. Tangent line at $x = 1$ is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

   (d) $f(1) = 1, f'(1) = -1$. Tangent line at $x = 1$ is $y - 1 = -(x - 1)$, or $y = -x + 2$.