Computer Vision & Digital Image Processing

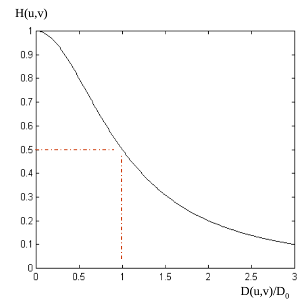
Frequency Domain Filters

Butterworth lowpass filter

- The transfer function of a Butterworth lowpass filter (BLPF) of order *n* with cutoff frequency D_0 is given by

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

- where $D(u,v)=[u^2+v^2]^{1/2}$
- For this smooth transition filter, a cutoff frequency locus is chosen such that D(u,v) is a certain percentage of its maximum
- Designed such that at $D(u,v)=D_0$ H(u,v)=0.50 (50% of its maximum value)



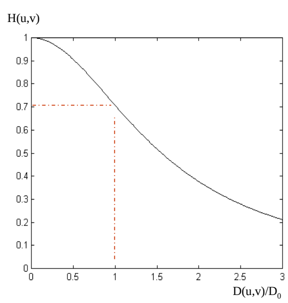
Butterworth lowpass filter (continued)

- Another transfer function of a Butterworth lowpass filter (BLPF) of order *n* with cutoff frequency D_0 is given by

$$H(u,v)=\frac{1}{1+[\sqrt{2}-1][D(u,v)/D_0]^{2n}}$$

- Designed such that at $D(u,v)=D_0$

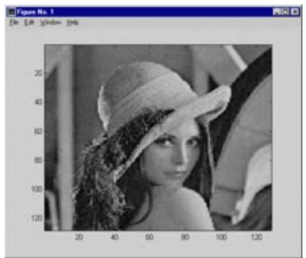
$$H(u,v)=\frac{1}{\sqrt{2}}$$



MATLAB Butterworth lowpass filter

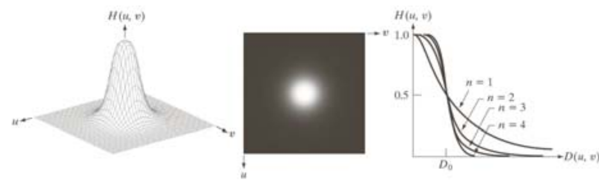
```
function [g]=blpf(f,order,cutoff);
% Usage [g]=blpf(f,order,cutoff);
F=fft2(f);
F=fftshift(F);
[umax vmax]=size(F);
for u=1:umax
   for v=1:vmax
      H(u,v)=1/(1+(sqrt(2)-1)*(sqrt(((umax/2-(u-1)).^2+(vmax/2-(v-
   1)).^2))/cutoff).^(2*order));
   end;
end;
G=H.*F;
G=ifft2(G);
g=sqrt(real(G).^2+imag(G).^2);
```

Example MATLAB output



- Cutoff frequency=10
- Cutoff frequency=100

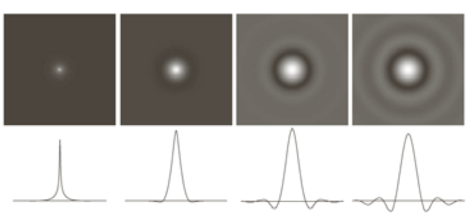
Butterworth lowpass filter function



a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Ringing in BLPF



a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

Gaussian lowpass filter

- The transfer function of a Gaussian lowpass filter (GLPF) is given by

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

- Here, σ is a measure of spread about the center
- Let $\sigma = D_0$, then

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

- where D_0 is the cutoff frequency

Gaussian lowpass filter function

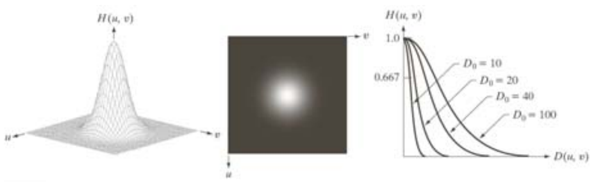


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

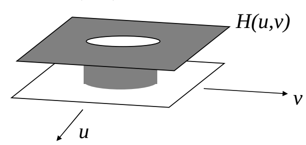
Ideal highpass filter (IHPF)

- A transfer function for a 2-D ideal highpass filter (IHPF) is given as

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- where D_0 is a stated nonnegative quantity (the cutoff frequency) and D(u,v) is the distance from the point (u,v) to the center of the frequency plane

$$D(u,v) = \sqrt{u^2 + v^2}$$

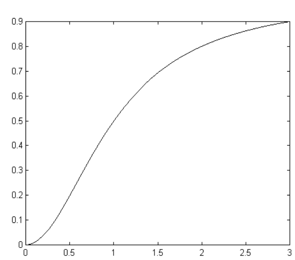


Butterworth highpass filter

- The transfer function of a Butterworth highpass filter (BHPF) of order *n* with cutoff frequency D_0 is given by

$$H(u,v) = \frac{1}{1 + \left[D_0 / D(u,v)\right]^{2n}}$$

- where $D(u,v)=[u^2+v^2]^{1/2}$
- For this smooth transition filter, a cutoff frequency locus is chosen such that D(u,v) is a certain percentage of its maximum
- Designed such that at $D(u,v)=D_0$ H(u,v)=0.50 (50% of its maximum value)



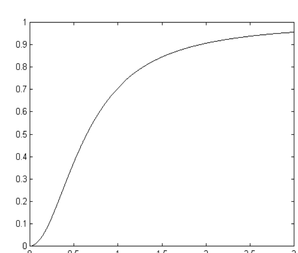
Butterworth highpass filter (continued)

- Another transfer function of a Butterworth highpass filter (BHPF) of order *n* with cutoff frequency D_0 is given by

$$H(u,v) = \frac{1}{1 + [\sqrt{2} - 1][D_0 / D(u,v)]^{2n}}$$

- Designed such that at $D(u,v)=D_0$

$$H(u,v) = \frac{1}{\sqrt{2}}$$



Gaussian highpass filter

- The transfer function of a Gaussian highpass filter (GHPF) is given by

$$H(u,v) = 1 - e^{-D^2(u,v)/2\sigma^2}$$

- Here, σ is as in the Gaussian lowpass case
- Let $\sigma = D_0$, then

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

- where D_0 is the cutoff frequency

Highpass filter functions

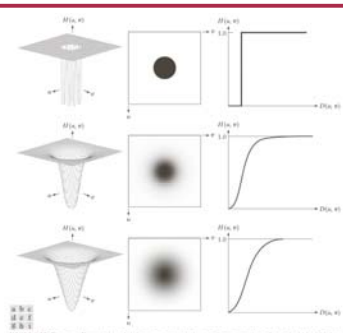


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Unsharp masking, highboost filtering, and high-frequency-emphasis filtering

- Let

$$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

- where

$$f_{LP}(x,y) = \Im^{-1}[H_{LP}(u,v)F(u,v)]$$

- Where $H_{LP}(u,v)$ is a lowpass filter and $F(u,v)$ is the Fourier transform of f(x,y)
- Then

$$g(x,y) = f(x,y) + k * g_{mask}(x,y)$$

- If k=1 this is an unsharp mask
- If k>1 this is a highboost filter

Unsharp masking, highboost filtering, and high-frequency-emphasis filtering

- In frequency domain only terms

$$g(x,y) = \Im^{-1}\{[1 + k * [1 - H_{LP}(u,v)]]F(u,v)\}$$

or in terms of a highpass filter

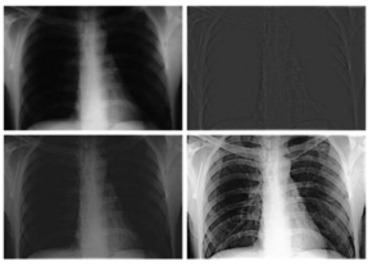
$$g(x,y) = \Im^{-1}\{[1 + k * H_{HP}(u,v)]F(u,v)\}$$

- This is a *high-frequency-emphasis* filter
- A more general form is

$$g(x,y) = \Im^{-1}\{[k_1 + k_2 * H_{HP}(u,v)]F(u,v)\}$$

- Here $k_1 \geq 0$ controls an offset from the origin and $k_2 \geq 0$ controls contributions of high frequencies

Example use of a high-frequency-emphasis filter



a b
c d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Homomorphic filtering

- Recall f(x,y) can be expressed as *f(x,y)=i(x,y)r(x,y)*
- We cannot use this directly to operate on the frequency components of i(x,y) and r(x,y) because

$$\Im\{f(x,y)\} \neq \Im\{i(x,y)\}\Im\{r(x,y)\}$$

- But if we define

$$\begin{aligned} z(x,y) &= \ln[f(x,y)] \\ &= \ln[i(x,y)] + \ln[r(x,y)] \end{aligned}$$

- then

$$\begin{aligned} \Im\{z(x,y)\} &= \Im\{\ln[f(x,y)]\} \\ &= \Im\{\ln[i(x,y)]\} + \Im\{\ln[r(x,y)]\} \end{aligned}$$

Homomorphic filtering (continued)

- Then,

$$Z(u,v) = I(u,v) + R(u,v)$$

- where I(u,v) and R(u,v) are the Fourier transforms of ln[i(x,y)] and ln[r(x,y)] respectively
- Z(u,v) can be processed by a filter function

$$\begin{aligned} S(u,v) &= H(u,v)Z(u,v) \\ &= H(u,v)I(u,v) + H(u,v)R(u,v) \end{aligned}$$

- where S(u,v) is the Fourier transform of the result
- In the spatial domain,

$$\begin{aligned} s(x,y) &= \Im^{-1}\{S(u,v)\} \\ &= \Im^{-1}\{H(u,v)I(u,v)\} + \Im^{-1}\{H(u,v)R(u,v)\} \end{aligned}$$

Homomorphic filtering (continued)

- If,

$$i'(x,y) = \Im^{-1}\{H(u,v)I(u,v)\} \text{ and}$$
$$r'(x,y) = \Im^{-1}\{H(u,v)R(u,v)\}$$

- then $s(x,y) = i'(x,y) + r'(x,y)$
- Taking the exponential yields the final result

$$\begin{aligned} g(x,y) &= \exp[s(x,y)] \\ &= \exp[i'(x,y)] * \exp[r'(x,y)] \\ &= i_0(x,y)r_0(x,y) \end{aligned}$$

Homomorphic filtering (continued)

$$f(x,y) \Rightarrow \boxed{\ln} \Rightarrow \boxed{\text{FFT}} \Rightarrow \boxed{H(u,v)} \Rightarrow \boxed{\text{FFT}^{-1}} \Rightarrow \boxed{\exp} \Rightarrow g(x,y)$$

- The process can be viewed graphically as above
- The illumination of an image is “generally” characterized by slow spatial variations (associated with the low frequencies of the Fourier transform of the logarithm)
- The reflectance of an image tends to vary abruptly, especially at the junctions of dissimilar objects (associated with the high frequencies of the Fourier transform of the logarithm)

Homomorphic filtering (continued)

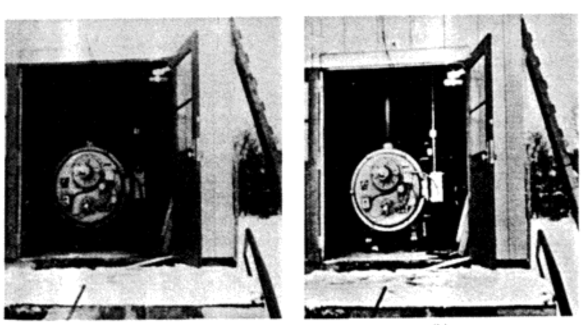
- The filter function H(u,v) should/will affect the low- and high-frequency components in different ways and can be approximated by

$$H(u,v) = (\gamma_H - \gamma_L)[1 - e^{-c[D^2(u,v)/D_0^2]}] + \gamma_L$$

where c controls the slope of the function

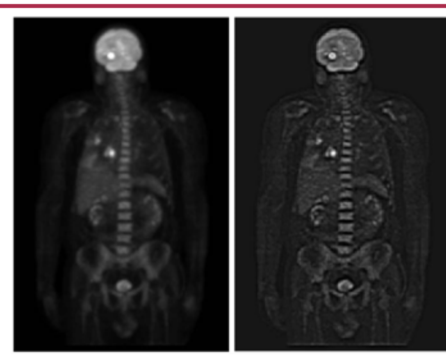
- If the filter function chosen is such that γ_L<1 and γ_H>1 then the low frequencies tend to be decreased and the high frequencies are amplified

Homomorphic filtering (example)



(a) (b)

Homomorphic filtering (example)



a b

FIGURE 4.62 (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)

Selective filtering

- Previously discussed filters operate over the entire frequency rectangle (i.e. complete representation in the frequency domain)
- Occasionally it is useful to operate on specific frequency bands or small regions of the frequency rectangle
- *Bandreject* or *Bandpass* filters operate on specific frequency bands
- *Notch* filters operate on small regions of the frequency rectangle

Bandreject filters

- Bandreject filters can, in general, be easily constructed using the same concepts as described for other filters
- Assume the following:
  - $D(u,v)$ is the distance from the center of the frequency rectangle
  - D_0 is the radial center of the band of interest
  - W is the width of the band of interest

Bandreject filters

- Ideal bandreject filter

$$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$$

- Butterworth bandreject filter

$$H(u,v) = \frac{1}{1 + \left[\frac{DW}{D^2 - D_0^2}\right]^{2n}}$$

- Gaussian bandreject filter

$$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$$

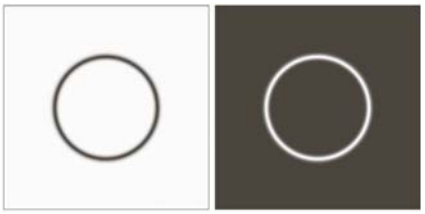
Bandpass filters

- Bandpass filters can be derived from any of the bandreject expressions as

$H_{BP}(u,v)=1- H_{BR}(u,v)$

- $H_{BR}(u,v)$ is the corresponding bandreject filter
- This formulation is exactly as in the highpass/lowpass case

Gaussian filter example



a b

FIGURE 4.63 (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Notch filters

- A *notch* filter rejects (or passes depending on its construction) frequencies in a pre-defined area (neighborhood) about the center of the frequency rectangle
- We desire that the filters be zero-phase-shift
  - Must be symmetric about the origin
  - A notch with center at (u_0,v_0) must have a corresponding notch at $(-u_0,-v_0)$

Notch reject filters

- Notch reject filters are easily constructed as products of highpass filters whose centers have been translated to the center of the notches
- The general form is:

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v)$$

- Where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filters whose centers are at (u_k,v_k) and $(-u_k,-v_k)$
- Q is the number of notches

Notch reject filters (continued)

- The centers at (u_k,v_k) and $(-u_k,-v_k)$ are specified with respect to the center of the frequency rectangle, $(M/2,N/2)$
- Distances can be calculated as:

$$D_k(u,v) = \sqrt{(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2}$$

and

$$D_{-k}(u,v) = \sqrt{(u - M/2 + u_k)^2 + (v - N/2 + v_k)^2}$$

Notch reject filters (continued)

- A general form for a Butterworth notch reject filter of order n and containing three notch pairs is:

$$H_{NR}(u,v) = \prod_{k=1}^{3}\left[\frac{1}{1+[D_{0k}/D_k(u,v)]^{2n}}\right]\left[\frac{1}{1+[D_{0k}/D_{-k}(u,v)]^{2n}}\right]$$

- The constant D_{0k} is the same for each pair of notches, but can be different for different pairs
- A *notch pass filter* can be expressed as

$$H_{NP}(u,v) = 1 - H_{NR}(u,v)$$