1 Supplementary Methods: MBDETES

Let the behavioral state variable name from MBITES denote the expected number of mosquitoes of a given chronological age (a), that are in each behavioral state; *e.g.,* the proportion in the post-prandial resting state (R) of age a at time t is $R(a, t)$. Similarly, let the waiting times to events be modeled as a rate that is dependent on the behavioral state (x), age, and time (t): $\xi_x(a, t)$. The proportion of mosquitoes that transition from state x to state y at the end of a bout is denoted P_{xy}. Death rates can be age dependent (*i.e.,* due to senescence), which affects the proportions transitioning to other states, so we write $P_{xy}(a)$. To deal with the event-driven nature of these bouts and the possibility some bouts may be repeated many times before transitioning to another state, we index mosquitoes by the i^{th} attempt to repeat the same event as a way of computing waiting times properly; for example, a mosquito repeating a blood feeding attempt bout transitions from $B_n(a)$ to $B_{n+1}(a)$. Finally, we let $\Lambda(t)$ represent the rate of emergence of adult female mosquitoes. The following system of coupled PDEs is homologous to MBITES:

$$
\begin{aligned}
F_1(0,t) = \quad & \Lambda(t) \\
\frac{\partial F_1(a,t)}{\partial t} + \frac{\partial F_1(a,t)}{\partial a} = \quad & \xi_O(a,t)P_{OF}(a)\sum_i O_i(a,t) + \xi_B(a,t)P_{BF}(a)\sum_i B_i(a,t) \\
& +\xi_R(a,t)P_{RF}(a)R(a,t) - \xi_F(a,t)F_1(a,t) \\
\frac{\partial F_i(a,t)}{\partial t} + \frac{\partial F_i(a,t)}{\partial a} = \quad & \xi_F(a,t)P_{FF}(a)F_{i-1}(a,t) - \xi_F(a,t)F_i(a,t) \\
\frac{\partial B_1(a,t)}{\partial t} + \frac{\partial B_1(a,t)}{\partial a} = \quad & \xi_O(a,t)P_{OB}(a)\sum_i O_i(a,t) + \xi_F(a,t)P_{FB}(a)\sum_i F_i(a,t) \\
& +\xi_R(a,t)P_{RB}(a)R(a,t) \\
& -\xi_B(a,t)B_1(a,t) \\
\frac{\partial B_i(a,t)}{\partial t} + \frac{\partial B_i(a,t)}{\partial a} = \quad & \xi_B(a,t)P_{BB}(a)B_{i-1}(a,t) - \xi_B(a,t)B_i(a,t) \\
\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} = \quad & \xi_B(a,t)P_{BR}(a)\sum_i B_i(a,t) - \xi_R(a,t)R(a,t) \\
\frac{\partial L_1(a,t)}{\partial t} + \frac{\partial L_1(a,t)}{\partial a} = \quad & \xi_R(a,t)P_{RL}(a)R(a,t) + \xi_O(a,t)P_{OL}(a)\sum_i O_i(a,t) \\
& -\xi_L(a,t)L_1(a,t) \\
\frac{\partial L_i(a,t)}{\partial t} + \frac{\partial L_i(a,t)}{\partial a} = \quad & \xi_L(a,t)P_{LL}(a)L_{i-1}(a,t) - \xi_L(a,t)L_i(a,t) \\
\frac{\partial O_1(a,t)}{\partial t} + \frac{\partial O_1(a,t)}{\partial a} = \quad & \xi_L(a,t)P_{LO}(a)\sum_i L_i(a,t) + \xi_R(a,t)P_{RO}(a)R(a,t) \\
& -\xi_O(a,t)O_1(a,t) \\
\frac{\partial O_i(a,t)}{\partial t} + \frac{\partial O_i(a,t)}{\partial a} = \quad & \xi_O(a,t)P_{OO}(a)O_{i-1}(a,t) - \xi_O(a,t)O_i(a,t)
\end{aligned} \tag{1}
$$

It is a nuisance to deal with an infinite set of equations, but if the state transitions are Markovian, then a change of variables to lump the the n^{th} states together: $x = \sum_i x_i$, with a rescaled rate variable, $\gamma_x(a,t) = \xi(a,t)(1 - P_{xx})$

Proof 1

To justify this summation, consider the infinite set of equations

$$\frac{dx_1}{dt} = -\lambda x_1$$

$$\frac{dx_i}{dt} = p\lambda x_{i-1} - \lambda x_i$$

with initial conditions $x_1(0) = 1$, $x_i(0) = 0$. That is, initially all of the mosquitoes are in their first attempt for an exponentially distributed length of time. A proportion p are successful or leave frustrated, and $1 - p$ attempt again. This system can be solved iteratively; x_1 has the solution

$$x_1(t) = e^{-\lambda t}$$

which, when plugged into the equation for x_2, gives

$$\frac{dx_2}{dt} = p\lambda e^{-\lambda t} - \lambda x_2$$

which can be solved using an integrating factor. This yields

$$x_2(t) = p\lambda t e^{-\lambda t}$$

This appears to be a weighted gamma distribution, which motivates the assumption for an inductive-step solution of

$$x_i(t) = \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t}$$

Assuming this, we look at the x_{i+1} equation:

$$\frac{dx_{i+1}}{dt} = p\lambda x_i - \lambda x_{i+1}$$

Plugging in the assumed solution, we get

$$\frac{dx_{i+1}}{dt} = p\lambda \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t} - \lambda x_{i+1}$$

$$= \frac{(p\lambda)^i}{i!} t^{i-1} e^{-\lambda t} - \lambda x_{i+1}$$

Which is again amenable to an integrating factor combined with i-1 integration by parts, yielding

$$x_{i+1} = \frac{(p\lambda t)^i}{i!} e^{-\lambda t}$$

which completes the induction. Because we are interested in the total amount of time spent and not the time spent in any one compartment, we add the solutions together:

$$\sum_{i=1}^{\infty} x_i(t) = \sum_{i=1}^{\infty} \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t}$$

$$= e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(p\lambda t)^i}{i!}$$

$$= e^{-\lambda t} e^{p\lambda t}$$

$$= e^{-(1-p)\lambda t}$$

Normalizing this gives us the total expected waiting time in this state for the mosquito is exponential with intensity $(1-p)\lambda$.

As a note, this convergence is uniform as it is the Taylor series representation of the exponential function - this justifies summing the infinite equations and the differential operator which gives us

$$\frac{d}{dt}\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \frac{dx_i}{dt}$$
$$= \sum_{i=1}^{\infty} -\lambda(1-p)x_i$$

setting $X = \sum_{i=1}^{\infty} x_i$, we get a very simple single differential equation

$$\frac{dX}{dt} = -(1-p)\lambda X$$

with the initial condition $X(0) = 1$. This has the same solution we found through induction.

Proof 2

Say we want to find the total waiting time T a mosquito spends in a given state. Using the law of total probability, we can condition this on the number of attempts N a mosquito will make:

$$P(T=t) = \sum_{n=1}^{\infty} P(T=t|N=n)P(N=n)$$

The number of attempts is geometrically distributed, as it will succeed or give up with probability (1-p) and therefore try again with probability p - it will try until it succeeds or gives up. Therefore

$$P(N=n) = (1-p)p^{n-1}$$

The waiting time between each attempt is iid with an exponentially distributed waiting time with intensity λ, so given it takes n trials the distribution follows a gamma distribution:

$$T|N = \sum_{i=1}^{n} Exp_i(\lambda)$$
$$\sim \Gamma(n,\lambda)$$

Therefore we have

$$P(T=t) = \sum_{i=1}^{\infty} P(T=t|N=n)P(N=n)$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}(1-p)p^{n-1}\frac{\lambda^n}{\Gamma(n)}t^{n-1}e^{-\lambda t} \\
&= (1-p)\lambda e^{-\lambda t}\sum_{i=0}^{\infty}\frac{(p\lambda t)^n}{n!} \\
&= \lambda(1-p)e^{-\lambda t}e^{p\lambda t} \\
&= (1-p)\lambda e^{-(1-p)\lambda t}
\end{aligned}
$$

which is exactly the normalized solution to the previous system of ordinary differential equations - the waiting time for the mosquito to leave the state is exponentially distributed with intensity $(1-p)\lambda$. Importantly, this proof does not depend on $(1-p)$ or λ being constant - they can be age- or time-dependent.

This means we can rewrite the infinite system of equations as a set of 5 differential equations:

$$
\begin{aligned}
F(0,t) = \quad & \Lambda(t) \\
\tfrac{\partial F(a,t)}{\partial t} + \tfrac{\partial F(a,t)}{\partial a} = \quad & \gamma_O(a,t)P_{OF}(a)O(a,t) + \gamma_B(a,t)P_{BF}(a)B(a,t) \\
& + \gamma_R(a,t)P_{RF}(a)R(a,t) - \gamma_F(a,t)(1-P_{FF}(a))F(a,t) \\
\tfrac{\partial B(a,t)}{\partial t} + \tfrac{\partial B(a,t)}{\partial a} = \quad & \gamma_O(a,t)P_{OB}(a)O(a,t) + \gamma_F(a,t)P_{FB}(a)F(a,t) \\
& + \gamma_R(a,t)P_{RB}(a)R(a,t) - \gamma_B(a,t)(1-P_{BB}(a))B(a,t) \\
\tfrac{\partial R(a,t)}{\partial t} + \tfrac{\partial R(a,t)}{\partial a} = \quad & \gamma_B(a,t)P_{BR}(a)B(a,t) - \gamma_R(a,t)R(a,t) \\
\tfrac{\partial L(a,t)}{\partial t} + \tfrac{\partial L(a,t)}{\partial a} = \quad & \gamma_R(a,t)P_{RL}(a)R(a,t) + \gamma_O(a,t)P_{OL}(a)O(a,t) \\
& - \gamma_L(a,t)(1-P_{LL}(a))L(a,t) \\
\tfrac{\partial O(a,t)}{\partial t} + \tfrac{\partial O(a,t)}{\partial a} = \quad & \gamma_L(a,t)P_{LO}(a)L(a,t) + \gamma_R(a,t)P_{RO}(a)R(a,t) \\
& - \gamma_O(a,t)(1-P_{OO}(a))O(a,t)
\end{aligned}
\tag{2}
$$

1.1 The MBITES-de for Cohorts

Finally, we want a version of these equations to model changes in a cohort of individuals with respect to age (assuming all the mosquitoes emerge from aquatic habitats at the same time of day):

$$
\begin{array}{rcl}
F(0) & = & 1 \\
\dot{F} & = & \gamma_O(a)P_{OF}(a)O(a) + \gamma_B(a)P_{BF}(a)B(a) + \gamma_R(a,t)P_{RF}(a)R(a) \\
 & & -\gamma_F(a)(1 - P_{FF}(a))F(a) \\
\dot{B} & = & \gamma_O(a)P_{OB}(a)O(a) + \gamma_F(a)P_{FB}(a)F(a) + \gamma_R(a)P_{RB}(a)R(a) \\
 & & -\gamma_B(a)(1 - P_{BB}(a))B(a) \\
\dot{R} & = & \gamma_B(a)P_{BR}(a)B(a) - \gamma_R(a)R(a) \\
\dot{L} & = & \gamma_R(a)P_{RL}(a)R(a) + \gamma_O(a)P_{OL}(a)O(a) \\
 & & -\gamma_L(a)(1 - P_{LL}(a))L_e(a) \\
\dot{O} & = & \gamma_L(a)P_{LO}(a)L(a) + \gamma_R(a)P_{RO}(a)R(a) \\
 & & -\gamma_O(a)(1 - P_{OO}(a))O(a)
\end{array}
\tag{3}
$$

1.2 Infection Dynamics in the MBITES-de Equations

To simulate infection dynamics in MBITES-de, we subdivide each variable X into new variables X_x, $x \in \{U, Y, Z\}$, to represent the fraction of mosquitoes in behavioral state X that are uninfected, U, infected, Y, or infected and infectious Z. These lead to the following systems of coupled differential equations that remain unchanged, but for the equation describing resting mosquitoes. We let $Q\kappa(t)$ the proportion of mosquitoes becoming infected after blood feeding at time t.

$$
\begin{array}{rcl}
\frac{\partial R_U(a,t)}{\partial t} + \frac{\partial R_U(a,t)}{\partial a} & = & (1 - Q\kappa(t))\,\gamma_B(a,t)P_{BR}(a)B_U(a,t) - \gamma_R(a,t)R_U(a,t) \\
\frac{\partial R_Y(a,t)}{\partial t} + \frac{\partial R_Y(a,t)}{\partial a} & = & Q\kappa(t)\gamma_B(a,t)P_{BR}(a)B_U(a,t) \\
 & & +\gamma_B(a,t)P_{BR}(a)B_Y(a,t) - \gamma_R(a,t)R_Y(a,t)
\end{array}
\tag{4}
$$

We let $\tau(t)$ denote the (possibly time-dependent) extrinsic incubation period. Because $\tau(t)$ is time dependent, we let $\hat{t}$ denote that point in the past when the mosquito became infected in order to become infectious at time t: *i.e.,* $t = \hat{t} + \tau(\hat{t})$. Let $\rho(t)$ the proportion of mosquitoes surviving through the extrinsic incubation period (*i.e.,* from $\hat{t}$ to $t = \hat{t} + \tau(\hat{t})$). An equation describing the proportion of infectious mosquitoes is:

$$
\begin{array}{rcl}
\frac{\partial R_Z(a,t)}{\partial t} + \frac{\partial R_Z(a,t)}{\partial a} & = & \rho(t)Q\kappa\left(\hat{t}\right)\gamma_B(a,t)P_{BR}(a)B_U(a,t) \\
 & & +\gamma_B(a,t)P_{BR}(a)B_Z(a,t) - \gamma_R(a,t)R_Z(a,t)
\end{array}
\tag{5}
$$

The remaining equations remain as follows:

$$
\begin{aligned}
F_x(0,t) = &\quad \Lambda(t) \\
\frac{\partial F_x(a,t)}{\partial t} + \frac{\partial F_x(a,t)}{\partial a} = &\quad \gamma_O(a,t)P_{OF}(a)O_x(a,t) + \gamma_B(a,t)P_{BF}(a)B_x(a,t) \\
&\quad +\gamma_R(a,t)P_{RF}(a)R_x(a,t) - \gamma_F(a,t)(1-P_{FF}(a))F_x(a,t) \\
\frac{\partial B_{o,x}(a,t)}{\partial t} + \frac{\partial B_x(a,t)}{\partial a} = &\quad \gamma_O(a,t)P_{OB}(a)O_x(a,t) + \gamma_F(a,t)P_{FB}(a)F(a,t) \\
&\quad +\gamma_R(a,t)P_{RB}(a)R_x(a,t) - \gamma_B(a,t)(1-P_{BB}(a))B_x(a,t) \\
\frac{\partial L_x(a,t)}{\partial t} + \frac{\partial L_x(a,t)}{\partial a} = &\quad \gamma_R(a,t)P_{RL}(a)R_x(a,t) + \gamma_O(a,t)P_{OL}(a)O_x(a,t) \\
&\quad -\gamma_L(a,t)(1-P_{LL}(a))L_x(a,t) \\
\frac{\partial O_{o,x}(a,t)}{\partial t} + \frac{\partial O_x(a,t)}{\partial a} = &\quad \gamma_L(a,t)P_{LO}(a)L(a,t) + \gamma_R(a,t)P_{RO}(a)R_x(a,t) \\
&\quad -\gamma_O(a,t)(1-P_{OO}(a))O_x(a,t)
\end{aligned}
\tag{6}
$$