

1 Supplementary Methods: MBDETES

Let the behavioral state variable name from MBITES denote the expected number of mosquitoes of a given chronological age (a), that are in each behavioral state; *e.g.*, the proportion in the post-prandial resting state (R) of age a at time t is $R(a, t)$. Similarly, let the waiting times to events be modeled as a rate that is dependent on the behavioral state (x), age, and time (t): $\xi_x(a, t)$. The proportion of mosquitoes that transition from state x to state y at the end of a bout is denoted P_{xy} . Death rates can be age dependent (*i.e.*, due to senescence), which affects the proportions transitioning to other states, so we write $P_{xy}(a)$. To deal with the event-driven nature of these bouts and the possibility some bouts may be repeated many times before transitioning to another state, we index mosquitoes by the i^{th} attempt to repeat the same event as a way of computing waiting times properly; for example, a mosquito repeating a blood feeding attempt bout transitions from $B_n(a)$ to $B_{n+1}(a)$. Finally, we let $\Lambda(t)$ represent the rate of emergence of adult female mosquitoes. The following system of coupled PDEs is homologous to MBITES:

$$F_1(0, t) = \Lambda(t)$$

$$\begin{aligned}
\frac{\partial F_1(a, t)}{\partial t} + \frac{\partial F_1(a, t)}{\partial a} &= \xi_O(a, t)P_{OF}(a) \sum_i O_i(a, t) + \xi_B(a, t)P_{BF}(a) \sum_i B_i(a, t) \\
&\quad + \xi_R(a, t)P_{RF}(a)R(a, t) - \xi_F(a, t)F_1(a, t) \\
\frac{\partial F_i(a, t)}{\partial t} + \frac{\partial F_i(a, t)}{\partial a} &= \xi_F(a, t)P_{FF}(a)F_{i-1}(a, t) - \xi_F(a, t)F_i(a, t) \\
\frac{\partial B_1(a, t)}{\partial t} + \frac{\partial B_1(a, t)}{\partial a} &= \xi_O(a, t)P_{OB}(a) \sum_i O_i(a, t) + \xi_F(a, t)P_{FB}(a) \sum_i F_i(a, t) \\
&\quad + \xi_R(a, t)P_{RB}(a)R(a, t) \\
&\quad - \xi_B(a, t)B_1(a, t) \\
\frac{\partial B_i(a, t)}{\partial t} + \frac{\partial B_i(a, t)}{\partial a} &= \xi_B(a, t)P_{BB}(a)B_{i-1}(a, t) - \xi_B(a, t)B_i(a, t) \\
\frac{\partial R(a, t)}{\partial t} + \frac{\partial R(a, t)}{\partial a} &= \xi_B(a, t)P_{BR}(a) \sum_i B_i(a, t) - \xi_R(a, t)R(a, t) \\
\frac{\partial L_1(a, t)}{\partial t} + \frac{\partial L_1(a, t)}{\partial a} &= \xi_R(a, t)P_{RL}(a)R(a, t) + \xi_O(a, t)P_{OL}(a) \sum_i O_i(a, t) \\
&\quad - \xi_L(a, t)L_1(a, t) \\
\frac{\partial L_i(a, t)}{\partial t} + \frac{\partial L_i(a, t)}{\partial a} &= \xi_L(a, t)P_{LL}(a)L_{i-1}(a, t) - \xi_L(a, t)L_i(a, t) \\
\frac{\partial O_1(a, t)}{\partial t} + \frac{\partial O_1(a, t)}{\partial a} &= \xi_L(a, t)P_{LO}(a) \sum_i L_i(a, t) + \xi_R(a, t)P_{RO}(a)R(a, t) \\
&\quad - \xi_O(a, t)O_1(a, t) \\
\frac{\partial O_i(a, t)}{\partial t} + \frac{\partial O_i(a, t)}{\partial a} &= \xi_O(a, t)P_{OO}(a)O_{i-1}(a, t) - \xi_O(a, t)O_i(a, t)
\end{aligned} \tag{1}$$

It is a nuisance to deal with an infinite set of equations, but if the state transitions are Markovian, then a change of variables to lump the the n^{th} states together: $x = \sum_i x_i$, with a rescaled rate variable, $\gamma_x(a, t) = \xi(a, t)(1 - P_{xx})$

Proof 1

To justify this summation, consider the infinite set of equations

$$\begin{aligned}
\frac{dx_1}{dt} &= -\lambda x_1 \\
\frac{dx_i}{dt} &= (1 - p)\lambda x_{i-1} - \lambda x_i
\end{aligned} \tag{2}$$

with initial conditions $x_1(0) = 1$, $x_i(0) = 0$. That is, initially all of the mosquitoes are in their first attempt for an exponentially distributed length of time. A proportion p are successful or leave frustrated, and $1 - p$ attempt

again. This system can be solved iteratively; x_1 has the solution

$$x_1(t) = e^{-\lambda t}$$

which, when plugged into the equation for x_2 , gives

$$\frac{dx_2}{dt} = p\lambda e^{-\lambda t} - \lambda x_2$$

which can be solved using an integrating factor. This yields

$$x_2(t) = p\lambda t e^{-\lambda t}$$

This appears to be a weighted gamma distribution, which motivates the assumption for an inductive-step solution of

$$x_i(t) = \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t}$$

Assuming this, we look at the x_{i+1} equation:

$$\frac{dx_{i+1}}{dt} = p\lambda x_i - \lambda x_{i+1}$$

Plugging in the assumed solution, we get

$$\begin{aligned} \frac{dx_{i+1}}{dt} &= p\lambda \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t} - \lambda x_{i+1} \\ &= \frac{(p\lambda)^i}{i!} t^{i-1} e^{-\lambda t} - \lambda x_{i+1} \end{aligned}$$

Which is again amenable to an integrating factor combined with i-1 integration by parts, yielding

$$x_{i+1} = \frac{(p\lambda t)^i}{i!} e^{-\lambda t}$$

which completes the induction. Because we are interested in the total amount of time spent and not the time spent in any one compartment, we add the solutions together:

$$\begin{aligned} \sum_{i=1}^{\infty} x_i(t) &= \sum_{i=1}^{\infty} \frac{(p\lambda t)^{i-1}}{(i-1)!} e^{-\lambda t} \\ &= e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(p\lambda t)^i}{i!} \\ &= e^{-\lambda t} e^{p\lambda t} \\ &= e^{-(1-p)\lambda t} \end{aligned}$$

Normalizing this gives us the total expected waiting time in this state for the mosquito is exponential with intensity $(1 - p)\lambda$.

As a note, this convergence is uniform as it is the Taylor series representation of the exponential function - this justifies interchanging the infinite summation and the differential operator which gives us

$$\begin{aligned}\frac{d}{dt} \sum_{i=1}^{\infty} x_i &= \sum_{i=1}^{\infty} \frac{dx_i}{dt} \\ &= \sum_{i=1}^{\infty} -\lambda(1 - p)x_i\end{aligned}$$

setting $X = \sum_{i=1}^{\infty} x_i$, we get a very simple single differential equation

$$\frac{dX}{dt} = -(1 - p)\lambda X$$

with the initial condition $X(0) = 1$. This has the same solution we found through induction.

Proof 2

Say we want to find the total waiting time T a mosquito spends in a given state. Using the law of total probability, we can condition this on the number of attempts N a mosquito will make:

$$P(T = t) = \sum_{n=1}^{\infty} P(T = t | N = n) P(N = n)$$

The number of attempts is geometrically distributed, as it will succeed or give up with probability $(1-p)$ and therefore try again with probability p - it will try until it succeeds or gives up. Therefore

$$P(N = n) = (1 - p)p^{n-1}$$

The waiting time between each attempt is iid with an exponentially distributed waiting time with intensity λ , so given it takes n trials the distribution follows a gamma distribution:

$$T | N = n = \sum_{i=1}^n \text{Exp}_i(\lambda) \sim \text{Gamma}(n, \lambda)$$

Therefore we have

$$P(T = t) = \sum_{i=1}^{\infty} P(T = t | N = n) P(N = n)$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} (1-p)p^{n-1} \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} \\
&= (1-p)\lambda e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(p\lambda t)^n}{n!} \\
&= \lambda(1-p)e^{-\lambda t} e^{p\lambda t} \\
&= (1-p)\lambda e^{-(1-p)\lambda t}
\end{aligned}$$

which is exactly the normalized solution to the previous system of ordinary differential equations - the waiting time for the mosquito to leave the state is exponentially distributed with intensity $(1-p)\lambda$. Importantly, the second proof does not depend on $(1-p)$ or λ being constant - they can be age- or time-dependent.

This means we can rewrite the infinite system of equations as a set of 5 differential equations:

$$\begin{aligned}
F(0, t) &= \Lambda(t) \\
\frac{\partial F(a, t)}{\partial t} + \frac{\partial F(a, t)}{\partial a} &= \xi_O(a, t)P_{OF}(a)O(a, t) + \xi_B(a, t)P_{BF}(a)B(a, t) \\
&\quad + \xi_R(a, t)P_{RF}(a)R(a, t) - \gamma_F(a, t)F(a, t) \\
\frac{\partial B(a, t)}{\partial t} + \frac{\partial B(a, t)}{\partial a} &= \xi_O(a, t)P_{OB}(a)O(a, t) + \xi_F(a, t)P_{FB}(a)F(a, t) \\
&\quad + \xi_R(a, t)P_{RB}(a)R(a, t) - \gamma_B(a, t)B(a, t) \\
\frac{\partial R(a, t)}{\partial t} + \frac{\partial R(a, t)}{\partial a} &= \xi_B(a, t)P_{BR}(a)B(a, t) - \xi_R(a, t)R(a, t) \\
\frac{\partial L(a, t)}{\partial t} + \frac{\partial L(a, t)}{\partial a} &= \xi_R(a, t)P_{RL}(a)R(a, t) + \xi_O(a, t)P_{OL}(a)O(a, t) \\
&\quad - \gamma_L(a, t)L(a, t) \\
\frac{\partial O(a, t)}{\partial t} + \frac{\partial O(a, t)}{\partial a} &= \xi_L(a, t)P_{LO}(a)L(a, t) + \xi_R(a, t)P_{RO}(a)R(a, t) \\
&\quad - \gamma_O(a, t)O(a, t)
\end{aligned} \tag{3}$$

1.1 The MBITES-de for Cohorts

Finally, we want a version of these equations to model changes in a cohort of individuals with respect to age (assuming all the mosquitoes emerge from

aquatic habitats at the same time of day):

$$\begin{aligned}
F(0) &= 1 \\
\dot{F} &= \xi_O(a)P_{OF}(a)O(a) + \xi_B(a)P_{BF}(a)B(a) + \xi_R(a)P_{RF}(a)R(a) \\
&\quad - \gamma_F(a)F(a) \\
\dot{B} &= \xi_O(a)P_{OB}(a)O(a) + \xi_F(a)P_{FB}(a)F(a) + \xi_R(a)P_{RB}(a)R(a) \\
&\quad - \gamma_B(a)B(a) \\
\dot{R} &= \xi_B(a)P_{BR}(a)B(a) - \xi_R(a)R(a) \\
\dot{L} &= \xi_R(a)P_{RL}(a)R(a) + \xi_O(a)P_{OL}(a)O(a) \\
&\quad - \gamma_L(a)L(a) \\
\dot{O} &= \xi_L(a)P_{LO}(a)L(a) + \xi_R(a)P_{RO}(a)R(a) \\
&\quad - \gamma_O(a)O(a)
\end{aligned} \tag{4}$$

1.2 Infection Dynamics in the MBITES-de Equations

To simulate infection dynamics in MBITES-de, we subdivide each variable X into new variables X_x , $x \in \{U, Y, Z\}$, to represent the fraction of mosquitoes in behavioral state X that are uninfected, U , infected, Y , or infected and infectious Z . These lead to the following systems of coupled differential equations that remain unchanged, but for the equation describing resting mosquitoes. We let $Q\kappa(t)$ the proportion of mosquitoes becoming infected after blood feeding at time t .

$$\begin{aligned}
\frac{\partial R_U(a,t)}{\partial t} + \frac{\partial R_U(a,t)}{\partial a} &= (1 - Q\kappa(t)) \xi_B(a,t)P_{BR}(a)B_U(a,t) - \xi_R(a,t)R_U(a,t) \\
\frac{\partial R_Y(a,t)}{\partial t} + \frac{\partial R_Y(a,t)}{\partial a} &= Q\kappa(t)\xi_B(a,t)P_{BR}(a)B_U(a,t) \\
&\quad + \xi_B(a,t)P_{BR}(a)B_Y(a,t) - \xi_R(a,t)R_Y(a,t)
\end{aligned} \tag{5}$$

We let $\tau(t)$ denote the (possibly time-dependent) extrinsic incubation period. Because $\tau(t)$ is time dependent, we let \hat{t} denote that point in the past when the mosquito became infected in order to become infectious at time t : *i.e.*, $t = \hat{t} + \tau(\hat{t})$. Let $\rho(t)$ the proportion of mosquitoes surviving through the extrinsic incubation period (*i.e.*, from \hat{t} to $t = \hat{t} + \tau(\hat{t})$). An equation describing the proportion of infectious mosquitoes is:

$$\begin{aligned}
\frac{\partial R_Z(a,t)}{\partial t} + \frac{\partial R_Z(a,t)}{\partial a} &= \rho(t)Q\kappa(\hat{t}) \xi_B(a,t)P_{BR}(a)B_U(a,t) \\
&\quad + \xi_B(a,t)P_{BR}(a)B_Z(a,t) - \xi_R(a,t)R_Z(a,t)
\end{aligned} \tag{6}$$

The remaining equations remain as follows:

$$\begin{aligned}
F_x(0, t) &= \Lambda(t) \\
\frac{\partial F_x(a, t)}{\partial t} + \frac{\partial F_x(a, t)}{\partial a} &= \xi_O(a, t)P_{OF}(a)O_x(a, t) + \xi_B(a, t)P_{BF}(a)B_x(a, t) \\
&\quad + \xi_R(a, t)P_{RF}(a)R_x(a, t) - \gamma_F(a, t)F_x(a, t) \\
\frac{\partial B_{O,x}(a, t)}{\partial t} + \frac{\partial B_x(a, t)}{\partial a} &= \xi_O(a, t)P_{OB}(a)O_x(a, t) + \xi_F(a, t)P_{FB}(a)F_x(a, t) \\
&\quad + \xi_R(a, t)P_{RB}(a)R_x(a, t) - \gamma_B(a, t)B_x(a, t) \\
\frac{\partial L_x(a, t)}{\partial t} + \frac{\partial L_x(a, t)}{\partial a} &= \xi_R(a, t)P_{RL}(a)R_x(a, t) + \xi_O(a, t)P_{OL}(a)O_x(a, t) \\
&\quad - \gamma_L(a, t)L_x(a, t) \\
\frac{\partial O_{O,x}(a, t)}{\partial t} + \frac{\partial O_x(a, t)}{\partial a} &= \xi_L(a, t)P_{LO}(a)L_x(a, t) + \xi_R(a, t)P_{RO}(a)R_x(a, t) \\
&\quad - \gamma_O(a, t)O_x(a, t)
\end{aligned} \tag{7}$$