# Does non-stationary spatial data always require non-stationary random fields?

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# Summary

**Real world processes have spatially varying second-order structure, but is modeling this non-stationarity worth it?**

The authors develop a novel model for non-stationary covariance structure and illustrate methods for parameterizing the model. They then apply their model to US precipitation data and compare predictions from their stationary and non-stationary models. They conclude by recommending careful consideration of the sources of non-stationarity and encourage balance between fitting complicated/flexible models and fitting simple/smarter models.

# Classical Approaches to Non-stationarity and Anisotropy

Anisotropy = Not Isotropic

$\rho(h) = \rho(\tilde{h})$ implies that $h = \tilde{h}$. This is not true in the case of anisotropy.

Non-stationarity in the covariance structure

- Global Deformation
  - Pre-compute a transformation of distance and then use a stationary model

$\rho(\tilde{h}) = \rho(L_1 ||\vec{s}_2 - \vec{s}_1||)$

- Local Deformation via SPDE

# Stationary SPDE

The following equation defines a stochastic partial differential equation (SPDE), $u(\vec{s})$, whose solution is the Matérn covariance function

$$(\kappa^2 - \nabla \cdot \nabla)u(\vec{s}) = \sigma \mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

Where $\kappa$ and $\sigma > 0$ are constants, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ and $\mathcal{W}$ is a standard Gaussian white noise process. This correlation structure is isotropic because the Laplacian, $\Delta = \nabla \cdot \nabla$ is equal to the sum of the diagonal elements of the Hessian, is invariant to a change of coordinates that involves rotation and translation. The solution to this SPDE is a class of equations that have covariance described by the Matérn covariance function.

# GMRF Approximation

Insert picture of GMRF approximation

# Model for Non-stationarity

The authors introduce a $2 \times 2$ matrix $\mathbf{H}$ into the SPDE which acts as a transformation of the grid on top of which we are measuring distance

$$(\kappa^2 - \nabla \cdot \mathbf{H}\nabla)u(\vec{s}) = \sigma\mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

This results in an updated covariance function

$$r(\vec{s}_1, \vec{s}_2) = \frac{\sigma^2}{4\pi\kappa^2\sqrt{\det(\mathbf{H})}} \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right) K_1 \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right)$$

Parameters $\kappa$ and $\mathbf{H}$ control the marginal variance and directionality of correlation, allowing $\sigma$ to fall out of the SPDE formmula. The $\sqrt{\det(\mathbf{H})}$ that appears in the denominator of the covariance function is a consequence of the change of variable.

# 2D-Random Walk Penalty

To enforce smoothness of parameters across space, the authors introduce a second-order penalty into their model for spatially-specific covariance parameters:

$$-\Delta\beta(\vec{s}) = \mathcal{W}_\beta(\vec{\int})/\sqrt{\tau_\beta}$$

where $\beta(\vec{s})$ is the location-specific value for parameter $\beta$ and

$$\log(\beta(\vec{s})) = \sum_{i=1}^{k}\sum_{j=1}^{l}\alpha_{ij}f_{ij}(\vec{s})$$

where $\{\alpha_{ij}\}$ are the parameters for real-valued basis functions $\{f_{ij}\}$.

$$\vec{\alpha} \sim \mathcal{N}_{\|\updownarrow}\left(\vec{\imath}, \mathbf{Q}_{\mathrm{RW2}}^{-\infty}/\tau_\beta\right)$$

# Full Hierachical Model

# Stationary vs. Non-stationary Model Predictions (Authors)

Insert plots

# Stationary vs. Non-stationary Model Predictions (Re-implementation)