Does non-stationary spatial data always require non-stationary random fields?

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Summary

Real world processes have spatially varying second-order structure, but is modeling this non-stationarity worth it?

The authors develop a novel model for non-stationary covariance structure and illustrate methods for parameterizing the model. They then apply their model to US precipitation data and compare predictions from their stationary and non-stationary models. They conclude by recommending careful consideration of the sources of non-stationarity and encourage balance between fitting complicated/flexible models and fitting simple/smarter models.

Classical Approaches to Non-stationarity and Anisotropy

Anisotropy = Not Isotropic

$\rho(h) = \rho(\tilde{h})$ implies that $h = \tilde{h}$. This is not true in the case of anisotropy.

Non-stationarity in the covariance structure

- Global Deformation
  - Pre-compute a transformation of distance and then use a stationary model

$\rho(\tilde{h}) = \rho(L_1||\vec{s}_2 - \vec{s}_1||)$

- Local Deformation via SPDE

Stationary SPDE

The following equation defines a stochastic partial differential equation (SPDE), $u(\vec{s})$, whose solution is the Matérn covariance function

$$(\kappa^2 - \nabla \cdot \nabla)u(\vec{s}) = \sigma \mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

Where κ and $\sigma > 0$ are constants, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ and $\mathcal{W}$ is a standard Gaussian white noise process. This correlation structure is isotropic because the Laplacian, $\Delta = \nabla \cdot \nabla$ is equal to the sum of the diagonal elements of the Hessian, is invariant to a change of coordinates that involves rotation and translation. The solution to this SPDE is a class of equations that have covariance described by the Matérn covariance function.

Introducing Anisotropy

The authors introduce a 2×2 matrix $\mathbf{H}$ into the SPDE which acts as a transformation of the grid on top of which we are measuring distance

$$(\kappa^2 - \nabla \cdot \mathbf{H}\nabla)u(\vec{s}) = \sigma\mathcal{W}(\vec{s}), \qquad \vec{s} \in \mathbb{R}^2$$

This results in an updated covariance function

$$r(\vec{s}_1, \vec{s}_2) = \frac{\sigma^2}{4\pi\kappa^2\sqrt{\det(\mathbf{H})}} \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right) K_1 \left(\kappa||\mathbf{H}^{-1/2}(\vec{s}_2 - \vec{s}_1)||\right)$$

Parameters κ and $\mathbf{H}$ control the marginal variance and directionality of correlation, allowing σ to fall out of the SPDE formula. The $\sqrt{\det(\mathbf{H})}$ that appears in the denominator of the covariance function is a consequence of the change of variable.

Non-stationary SPDE

Finally, to construct a non-stationary SPDE, the authors allow κ^2 and $\mathbf{H}$ to vary over the domain. This results in global non-stationarity via the chaining together of processes with local covariance structures.

$$(\kappa^2(\vec{s}) - \nabla \cdot \mathbf{H}(\vec{s})\nabla)u(\vec{s}) = \mathcal{W}(\vec{s})$$

Now, the range, anisotropy and marginal variance can vary spatially.

2D-Random Walk Penalty

To enforce smoothness of parameters across space, the authors introduce a second-order penalty into their model for the spatially-specific covariance parameters:

$$-\Delta\beta(\vec{s}) = \mathcal{W}_\beta(\vec{s})/\sqrt{\tau_\beta}$$

where $\beta(\vec{s})$ is the location-specific value for parameter β and

$$\beta(\vec{s}) = \sum_{i=1}^{k}\sum_{j=1}^{l}\alpha_{ij}f_{ij}(\vec{s})$$

where $\{\alpha_{ij}\}$ are the parameters for real-valued basis functions $\{f_{ij}\}$ and i and j are indices of the $k \times l$ mesh.

$$\vec{\alpha} \sim \mathcal{N}_{kl}\left(\vec{0}, \mathbf{Q}_{\mathrm{RW2}}^{-1}/\tau_\beta\right)$$

Full Hierachical Model

This

$$
\begin{aligned}
&\text{Stage 1: } \vec{y}|\vec{\beta}, \vec{u}, \log(\tau_{\text{noise}}) \sim \mathcal{N}_N(\mathbf{X}\vec{\beta} + \mathbf{E}\vec{u}, \mathbf{I}_N/\tau_{\text{noise}}) \\
&\text{Stage 2: } \vec{u}|\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4 \sim \mathcal{N}_{nm}(\vec{0}, \mathbf{Q}^{-1}), \qquad \vec{\beta} \sim \mathcal{N}_p(\vec{0}, \mathbf{I}_p/\tau_\beta) \\
&\text{Stage 3: } \vec{\alpha}_i|\tau_i \sim \mathcal{N}_{kl}(\vec{0}, \mathbf{Q}_{\text{RW2}}^{-1}/\tau_i) \quad \text{for } i = 1, 2, 3, 4
\end{aligned}
$$

Stationary vs. Non-stationary Model Predictions (Authors)

Insert plots

Stationary vs. Non-stationary Model Predictions (Re-implementation)