1 Introduction

- broad motivation
  - phase transitions in supersymmetric theories, thermal inflation
- potential
  - general form of potential

    $$V = V_0 - m_\phi^2 |\phi|^2 + \ldots \tag{1}$$

    with vacuum at $\phi = \phi_0$ where

    $$m_\phi \ll \phi_0 \ll m_{\mathrm{Pl}} \tag{2}$$

    typically $m_\phi \sim \mathrm{TeV}$ and $\ln(\phi_0/m_\phi) \sim 20$
  - realistic examples

    $$V = V_0 - m_\phi^2 \left(1 - \alpha_\phi \ln \sqrt{\frac{|\phi|^2 + m_{\mathrm{s}}^2}{m_{\mathrm{s}}^2}} \right) |\phi|^2 \tag{3}$$

    with $m_{\mathrm{s}} \sim m_\phi$, $\alpha_\phi^{-1} \simeq \ln(\phi_0/m_{\mathrm{s}}) + \frac{1}{2} \sim 20$ and $V''(\phi_0) \simeq \alpha_\phi m_\phi^2$, or more generally

    $$V = V_0 - m_\phi^2 f \left(\alpha_\phi \ln \sqrt{\frac{|\phi|^2 + m_{\mathrm{s}}^2}{m_{\mathrm{s}}^2}} \right) |\phi|^2 \tag{4}$$

    or alternatively

    $$V = V_0 - m_\phi^2 |\phi|^2 + \left(\frac{1}{4} A_\phi \lambda_\phi \phi^4 + \mathrm{c.c.} \right) + \left| \lambda_\phi \phi^3 \right|^2 \tag{5}$$

    with $|A_\phi| \sim m_\phi$ and $V''(\phi_0) \geq 4 m_\phi^2 + \frac{2}{3} |A_\phi|^2$.
- phase transition
  1. initially, finite temperature effective potential holds ϕ at origin
  2. when temperature drops to $T_{\mathrm{c}} \sim m_\phi$, bubbles nucleate at

     $$\phi_{\mathrm{c}} \sim m_\phi \tag{6}$$

  3. roll out from ϕ_{c} to ϕ_0

     $$t_{\mathrm{roll}} \sim m_\phi^{-1} \ln \frac{\phi_0}{m_\phi} \sim \alpha_\phi^{-1} m_\phi^{-1} \tag{7}$$

     For the logarithmic potential, Eq. (3), or the piecewise quadratic potential, Eq. (12), the roll out time is similar to the small amplitude oscillation period, $t_{\mathrm{roll}} \sim \alpha_\phi^{-1} m_\phi^{-1} \sim 2\pi \alpha_\phi^{-1/2} m_\phi^{-1} \sim t_{\mathrm{osc}}$, but for the polynomial potential, Eq. (5), the roll out time is greater than the small amplitude oscillation period, $t_{\mathrm{roll}} \sim \alpha_\phi^{-1} m_\phi^{-1} \gg \pi m_\phi^{-1} \geq t_{\mathrm{osc}}$.
  4. percolation

  - before roll out finishes - similar to 2nd order transition?
  - after roll out - can be many oscillations in bubble wall if percolation time $t_{\rm p} \gg t_{\rm osc}$.

Estimate tunnelling rate, and hence percolation time, and initial state of bubble using tunnelling theory? But this depends on new parameters, specifically the coupling of ϕ to other fields. Also, initial expansion of bubble through thermal bath may be complicated. Better to use tunnelling rate, or percolation time, and amplitude of deviations from $\mathrm{O}(1,3)$ symmetry as effective parameters?

- specific problem we are tackling as an exercise in tunelling dynamics

2 Model / Setup

- equations of motion

$$\Box\phi + \frac{\partial V}{\partial \phi^*} = 0 \tag{8}$$

- reduction to simple model
- simple model as a standalone problem

3 Simple analytic bubble solution

- $\mathrm{O}(1,n)$ symmetric equation of motion

$$\phi = \phi(\tau) \tag{9}$$

where

$$\tau \equiv \sqrt{t^2 - r^2} \tag{10}$$

gives

$$\frac{d^2\phi}{d\tau^2} + \frac{n}{\tau}\frac{d\phi}{d\tau} + \frac{\partial V}{\partial \phi^*} = 0 \tag{11}$$

where $n = 1, 2, 3$ is the spatial dimension.

- solution for piecewise quadratic potential

$$V(\phi) = \begin{cases} V_0 - m_\phi^2|\phi|^2 & \text{for } |\phi| \le \phi_* \\ \alpha_\phi m_\phi^2 \left(|\phi| - \phi_0\right)^2 & \text{for } |\phi| \ge \phi_* \end{cases} \tag{12}$$

with

$$V_0 = \frac{\alpha_\phi m_\phi^2 \phi_0^2}{1+\alpha_\phi} \tag{13}$$

$$\phi_* = \frac{\alpha_\phi \phi_0}{1+\alpha_\phi} \tag{14}$$

to give continuous 0th and 1st derivatives.

Bounded solution for initial condition $\phi(0) = \phi_{\rm c} \sim \phi_0 \exp(-1/\alpha_\phi)$ is

$$\phi(\tau) = \begin{cases} 2^\nu \, \Gamma(1+\nu) \, \dfrac{I_\nu(m_\phi \tau)}{(m_\phi \tau)^\nu} \phi_{\rm c} & \text{for } \phi \leq \phi_* \\ \phi_0 + A \sqrt{\dfrac{\pi}{2}} \dfrac{J_\nu\left(\sqrt{\alpha_\phi}\, m_\phi \tau\right)}{\left(\sqrt{\alpha_\phi}\, m_\phi \tau\right)^\nu} + B \sqrt{\dfrac{\pi}{2}} \dfrac{Y_\nu\left(\sqrt{\alpha_\phi}\, m_\phi \tau\right)}{\left(\sqrt{\alpha_\phi}\, m_\phi \tau\right)^\nu} & \text{for } \phi \geq \phi_* \end{cases} \tag{15}$$

where

$$\nu = \frac{n-1}{2} \tag{16}$$

and A and B are determined by the matching conditions at $\phi = \phi_*$.

- thermal inflation limit of $\phi_{\rm c} \ll \phi_0$

  For $\tau \gg m_\phi^{-1}$, Eq. (15) reduces to

$$\phi(\tau) \simeq \begin{cases} \phi_* \left(\dfrac{\tau_*}{\tau}\right)^{\frac{n}{2}} e^{m_\phi(\tau - \tau_*)} & \text{for } \tau \leq \tau_* \\ \phi_0 - \dfrac{\phi_0}{\sqrt{1+\alpha_\phi}} \left(\dfrac{\tau_*}{\tau}\right)^{\frac{n}{2}} \cos\left[\sqrt{\alpha_\phi}\, m_\phi \left(\tau - \tau_*\right) + \tan^{-1} \sqrt{\alpha_\phi}\right] & \text{for } \tau \geq \tau_* \end{cases} \tag{17}$$

where

$$\tau_* = \tau(\phi_*) \sim \alpha_\phi^{-1} m_\phi^{-1} \tag{18}$$

Figure 1: $\phi\left(\sqrt{t^2 - r^2}\right)$

- particular solution for logarithmically running potential

$$V = V_0 - m_\phi^2 \left(1 - \alpha_\phi \ln \frac{|\phi|}{m_{\rm s}}\right) |\phi|^2 \tag{19}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{\partial V}{\partial \phi^*} = 0 \tag{20}$$

has particular solution

$$\phi = m_{\rm s} \exp\left(m_\phi \tau - \frac{1}{4} \alpha_\phi m_\phi^2 \tau^2\right) \tag{21}$$

$$= m_{\rm s} \exp\left(\frac{1}{\alpha_\phi}\right) \exp\left[-\frac{1}{4} \alpha_\phi m_\phi^2 \left(\tau - \frac{2}{\alpha_\phi m_\phi}\right)^2\right] \tag{22}$$

- perturbations about $\mathrm{O}(1,n)$ symmetry

4 Numerical

- numerical solvers and result
- initial condition dependence
  - initial nucleated bubble is localised and has all dimensions $\sim m_\phi \ll \phi_0$
  - but detailed form unknown? not qualitatively important?
  - approximate $\mathrm{O}(1,n)$ symmetry? fluctuations about $\mathrm{O}(1,n)$ symmetry important?
  - correlation between amplitude of fluctuations and percolation scale from tunnelling theory?
- comparison with simple analytic bubble solution

5 Spectrum

- gives effective equation of state of flaton after transition. needed in calculation of
  - redshifting of primordial perturbations
  - modification of primordial spectrum on small scales
  - gravity waves produced by thermal inflation
  - dilution of moduli
- fourier transform of simple analytic bubble solution
  - periodize to model bubble percollation
  - free passage of high k modes ($k \gg m_\phi$) preserves spectrum
  - periodization straightforward in 1D but in 3D?

5.1 Analytic estimate

For percolation on a scale x_p

$$\sum_n \phi\left(\sqrt{t^2 - (x - 2n x_\mathrm{p})^2}\right) = \frac{1}{2} f_0(t) + \sum_{l=1}^{\infty} f_l(t) \cos\left(\frac{l\pi x}{x_p}\right) \tag{23}$$

$$f_l(t) = \frac{1}{x_\mathrm{p}} \int_{-t}^{t} dx\, \phi\left(\sqrt{t^2 - x^2}\right) \cos\left(\frac{l\pi x}{x_\mathrm{p}}\right) \tag{24}$$

where the range is taken to be $[-t, t]$ to take into account the overlap of free passing bubbles. Should be evaluated at

$$t \sim \sqrt{x_\mathrm{p}^2 + \tau_*^2} \tag{25}$$

since the individual bubble solutions undergo forced expansion, not the free expansion expected after percolation.

Defining μ, k and ω by

$$\mu = \sqrt{\alpha_\phi}\, m_\phi \tag{26}$$

$$k = \frac{l\pi}{x_{\rm p}} \tag{27}$$

$$\omega = \sqrt{k^2+\mu^2} \tag{28}$$

and α, β and γ by

$$\tan\alpha = \sqrt{\alpha_\phi} \tag{29}$$

$$\sin\beta = \frac{\tau_*}{t} \tag{30}$$

$$\tan\gamma = \frac{\mu}{k} \tag{31}$$

and Eqs. (7) and (18) giving

$$\tan\alpha\sin\beta\,\mu t \sim 1 \tag{32}$$

Appendix A.3 gives, for $m_\phi t > m_\phi\tau_* \sim \alpha_\phi^{-1} \gg 1$,

$$f_l(t) = f_l^{\rm ro}(t) + f_l^{\rm ph}(t) + f_l^{\rm os}(t) \tag{33}$$

The roll out contribution

$$f_l^{\rm ro}(t) = 2\frac{\phi_0}{kx_{\rm p}}\sin^2\alpha\sin\eta\cos\left(kt\cos\beta+\eta\right) \tag{34}$$

$$\dot{f}_l^{\rm ro}(t) = -2\frac{\phi_0}{x_{\rm p}}\sin^2\alpha\frac{\sin\eta}{\cos\beta}\left[\sin\left(kt\cos\beta+\eta\right)+\frac{\cos\eta}{kt\cos\beta}\cos\left(kt\cos\beta+2\eta\right)\right] \tag{35}$$

where

$$\eta = \tan^{-1}\frac{k\tan\beta}{m_\phi} \tag{36}$$

and we have used Eqs. (106), (107) and (113). The phase transition contribution

$$f_l^{\rm ph}(t) = 2\frac{\phi_0}{kx_{\rm p}}\sin(kt\cos\beta) \tag{37}$$

$$\dot{f}_l^{\rm ph}(t) = 2\frac{\phi_0}{x_{\rm p}}\frac{\cos(kt\cos\beta)}{\cos\beta} \tag{38}$$

The oscillation contribution

$$f_l^{\rm os}(t) = -\frac{\phi_0}{x_{\rm p}}\sqrt{\frac{2\tau_*}{\omega}}\cos\alpha\,(A+B) \tag{39}$$

with

$$\begin{aligned} A = {} & \sqrt{\sin^{n-1}\beta\sin^{2-n}\gamma} \\ & \times\left\{\cos\left(\omega t-\mu\tau_*+\alpha\right)\left[\sqrt{\frac{\pi}{8}}+{\rm C}\left(\sqrt{\frac{\omega t}{2}}\,(\gamma-\beta)\right)\right]\right. \\ & \left.+\sin\left(\omega t-\mu\tau_*+\alpha\right)\left[\sqrt{\frac{\pi}{8}}+{\rm S}\left(\sqrt{\frac{\omega t}{2}}\,(\gamma-\beta)\right)\right]\right\} \end{aligned} \tag{40}$$

$$\begin{aligned}\frac{dA}{dt} \;\sim\;& \sqrt{\sin^{n-1}\beta\sin^{2-n}\gamma} \\ &\times\left\{-\omega\sin\left(\omega t-\mu\tau_*+\alpha\right)\left[\sqrt{\frac{\pi}{8}}+\mathrm{C}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right. \\ &+\omega\cos\left(\omega t-\mu\tau_*+\alpha\right)\left[\sqrt{\frac{\pi}{8}}+\mathrm{S}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right] \\ &\left.+\sqrt{\frac{\omega}{2t}}\left[\tan\beta+\frac{1}{2}\left(\gamma-\beta\right)\right]\cos\left[\omega t-\mu\tau_*+\alpha-\frac{\omega t}{2}\left(\gamma-\beta\right)^2\right]\right\} \qquad (41)\end{aligned}$$

for $\gamma > \beta$, as $\omega t\sin^2\gamma \to \infty$, and

$$\begin{aligned}A \;=\;& \sqrt{\frac{\sin\beta}{\cos(\beta-\gamma)}} \\ &\times\left\{\cos\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+kt\cos\beta+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right. \\ &\left.+\sin\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+kt\cos\beta+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right\} \quad (42)\end{aligned}$$

$$\begin{aligned}\frac{dA}{dt} \;\sim\;& \frac{\omega}{2\cos\beta}\sqrt{\frac{\sin\beta}{\cos^5(\beta-\gamma)}}\left[2\cos\beta\cos(\beta-\gamma)-\cos\gamma\sin^2(\beta-\gamma)\right] \\ &\times\left\{-\sin\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+kt\cos\beta+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right. \\ &\left.+\cos\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+kt\cos\beta+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right\} \\ &+\frac{1}{2\cos\beta}\sqrt{\frac{\omega}{2t}}\,\frac{\sqrt{\sin\beta}}{\cos^2(\beta-\gamma)}\left[\sin\beta+\sin\gamma\cos(\beta-\gamma)\right]\cos\left(kt\cos\beta+\alpha\right) \qquad (43)\end{aligned}$$

for $\gamma < \beta$, as $\omega t\sin^2\beta \to \infty$, and

$$\begin{aligned}B \;=\;& \sqrt{\frac{\sin\beta}{\cos(\beta+\gamma)}} \\ &\times\left\{\cos\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+kt\cos\beta-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right. \\ &\left.+\sin\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+kt\cos\beta-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right\} \\ & \qquad (44)\end{aligned}$$

$$\frac{dB}{dt} \;\sim\; \frac{\omega}{2\cos\beta}\sqrt{\frac{\sin\beta}{\cos^5(\beta+\gamma)}}\left[2\cos\beta\cos(\beta+\gamma)-\cos\gamma\sin^2(\beta+\gamma)\right]$$

$$\times \left\{ -\sin\left[\frac{\omega t \sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)} + kt\cos\beta - \alpha\right]\left[\sqrt{\frac{\pi}{8}} - \mathrm{C}\left(\sqrt{\frac{\omega t \sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right.$$
$$\left. + \cos\left[\frac{\omega t \sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)} + kt\cos\beta - \alpha\right]\left[\sqrt{\frac{\pi}{8}} - \mathrm{S}\left(\sqrt{\frac{\omega t \sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right\}$$
$$+ \frac{1}{2\cos\beta}\sqrt{\frac{\omega}{2t}}\,\frac{\sqrt{\sin\beta}}{\cos^2(\beta+\gamma)}\left[\sin\beta - \sin\gamma\cos(\beta+\gamma)\right]\cos\left(kt\cos\beta - \alpha\right)$$

(46)

Figure 2: $f_l\big(\sqrt{x_{\mathrm{p}}^2 + \tau_*^2}\big)$

Figure 3: $E_l\big(\sqrt{x_{\mathrm{p}}^2 + \tau_*^2}\big)$

5.1.1 Rough analytic limits

For $t \gg \tau_*$, the above reduces to

$$f_l(t) \sim \frac{2\phi_0}{kx_{\mathrm{p}}} \times \left\{ \begin{array}{rlrcl} \sin(kt) & \text{for} & k & \ll & \sqrt{\alpha_\phi}\,\mu^2 t \\ \sqrt{\frac{\pi}{8}}\sqrt{\frac{\mu^2 t}{k}}\sin\left(kt + \frac{5\pi}{4}\right) & \text{for} & k & \sim & \sqrt{\alpha_\phi}\,\mu^2 t \\ \alpha_\phi \sin(kt) & \text{for} & \sqrt{\alpha_\phi}\,\mu^2 t \ll k & \ll & \mu^2 t \\ \alpha_\phi \frac{\mu^2 t}{k}\frac{\cos(kt)}{\alpha_\phi m_\phi \tau_*} & \text{for} & \mu^2 t & \ll & k \end{array} \right. \tag{47}$$

with the phase transition contribution dominating for $k \ll \sqrt{\alpha_\phi}\,\mu^2 t$, the oscillation contribution dominating for $k \sim \sqrt{\alpha_\phi}\,\mu^2 t$, the phase transition and oscillation contributions dominating for $\sqrt{\alpha_\phi}\,\mu^2 t \ll k \ll \mu^2 t$ and the residual roll out contribution dominating for $k \gg \mu^2 t$.

5.2 Numerical spectrum

6 Summary

A Calculations

A.1 Potential

Eq. (3)

$$V = V_0 - m_\phi^2\left(1 - \alpha_\phi \ln\sqrt{\frac{|\phi|^2 + m_{\mathrm{s}}^2}{m_{\mathrm{s}}^2}}\right)|\phi|^2 \tag{48}$$

$$+\frac{\partial V}{\partial\phi^*} = -m_\phi^2\left(1-\alpha_\phi\ln\sqrt{\frac{|\phi|^2+m_{\rm s}^2}{m_{\rm s}^2}}-\frac{1}{2}\alpha_\phi\frac{|\phi|^2}{|\phi|^2+m_{\rm s}^2}\right)\phi \tag{49}$$

$$\frac{\partial V}{\partial|\phi|} = -2m_\phi^2\left(1-\alpha_\phi\ln\sqrt{\frac{|\phi|^2+m_{\rm s}^2}{m_{\rm s}^2}}-\frac{1}{2}\alpha_\phi\frac{|\phi|^2}{|\phi|^2+m_{\rm s}^2}\right)|\phi| \tag{50}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 V}{\partial|\phi|^2} &= -m_\phi^2\left(1-\alpha_\phi\ln\sqrt{\frac{|\phi|^2+m_{\rm s}^2}{m_{\rm s}^2}}-\frac{1}{2}\alpha_\phi\frac{|\phi|^2}{|\phi|^2+m_{\rm s}^2}\right) \\ &\quad +\alpha_\phi m_\phi^2\left[1-\frac{m_{\rm s}^4}{(|\phi|^2+m_{\rm s}^2)^2}\right]\end{aligned} \tag{51}$$

A.2 Dynamics

Eq. (17)

$$\phi(\tau) = \phi_0-\frac{\phi_0}{\sqrt{1+\alpha_\phi}}\left(\frac{\tau_*}{\tau}\right)^{\frac{n}{2}}\cos\left[\sqrt{\alpha_\phi}\,m_\phi\,(\tau-\tau_*)+\tan^{-1}\sqrt{\alpha_\phi}\right] \tag{52}$$

$\phi' = 0$ when

$$\tan\left[\sqrt{\alpha_\phi}\,m_\phi\,(\tau-\tau_*)-\tan^{-1}\frac{1}{\sqrt{\alpha_\phi}}\right] = \frac{2\sqrt{\alpha_\phi}\,m_\phi\tau}{n} \tag{53}$$

Therefore, at the end of the first oscillation,

$$\tau-\tau_* \sim \frac{2\pi}{\sqrt{\alpha_\phi}\,m_\phi} \tag{54}$$

and ϕ returns to

$$\begin{aligned}\phi &\sim \phi_0-\frac{\phi_0}{\sqrt{1+\alpha_\phi}}\left(1+\frac{2\pi}{\sqrt{\alpha_\phi}\,m_\phi\tau_*}\right)^{-\frac{n}{2}}\left[1+\left(\frac{n}{2\sqrt{\alpha_\phi}\,m_\phi\tau}\right)^2\right]^{-\frac{1}{2}} && (55)\\ &\sim \frac{n\pi\phi_0}{\sqrt{\alpha_\phi}\,m_\phi\tau_*} > \phi_* && (56)\end{aligned}$$

and hence ϕ does not return to $\phi<\phi_*$.

A.3 Spectrum for $t>\tau_*$

Defining θ and β by

$$\begin{aligned} x &= t\cos\theta && (57)\\ \tau_* &= t\sin\beta && (58)\end{aligned}$$

and α, μ and k by

$$\begin{aligned}\tan\alpha &= \sqrt{\alpha_\phi} && (59)\\ \mu &= \sqrt{\alpha_\phi}\,m_\phi && (60)\\ k &= \frac{l\pi}{x_{\rm p}} && (61)\end{aligned}$$

with Eqs. (7) and (18) giving

$$\tan\alpha\sin\beta\,\mu t\sim 1 \tag{62}$$

Eq. (17) becomes

$$\phi(t\sin\theta)=\begin{cases}\phi_0\sin^2\alpha\exp\left(-\dfrac{\mu t\sin\beta}{\tan\alpha}\right)\left(\dfrac{\sin\beta}{\sin\theta}\right)^{\frac{n}{2}}\exp\left(\dfrac{\mu t\sin\theta}{\tan\alpha}\right) & \text{for } \theta<\beta \text{ or } \theta>\pi-\beta\\ \phi_0-\phi_0\cos\alpha\left(\dfrac{\sin\beta}{\sin\theta}\right)^{\frac{n}{2}}\cos\left(\mu t\sin\theta-\mu t\sin\beta+\alpha\right) & \text{for } \beta<\theta<\pi-\beta\end{cases} \tag{63}$$

Eq. (24) becomes

$$f_l(t)=\frac{t}{x_{\rm p}}\int_0^\pi d\theta\sin\theta\,\phi(t\sin\theta)\cos(kt\cos\theta) \tag{64}$$

and combining gives

$$\begin{aligned}f_l(t) &= \frac{2\phi_0 t}{x_{\rm p}}\sin^2\alpha\,(\sin\beta)^{\frac{n}{2}}\exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^\beta\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos(kt\cos\theta)\\ &\quad+\frac{\phi_0 t}{x_{\rm p}}\int_\beta^{\pi-\beta}d\theta\sin\theta\cos(kt\cos\theta)\\ &\quad-\frac{\phi_0 t}{x_{\rm p}}\cos\alpha\,(\sin\beta)^{\frac{n}{2}}\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left(\mu t\sin\theta-\mu t\sin\beta+\alpha\right)\cos(kt\cos\theta)\end{aligned} \tag{65}$$

Now

$$\frac{\phi_0 t}{x_{\rm p}}\int_\beta^{\pi-\beta}d\theta\sin\theta\cos(kt\cos\theta)=\frac{2\phi_0}{kx_{\rm p}}\sin(kt\cos\beta) \tag{66}$$

and defining ω and γ by

$$\sqrt{\alpha_\phi}\,m_\phi = \mu = \omega\sin\gamma \tag{67}$$

$$\frac{l\pi}{x_{\rm p}} = k = \omega\cos\gamma \tag{68}$$

i.e.

$$\omega = \sqrt{k^2+\mu^2} = \sqrt{\frac{l^2\pi^2}{x_{\rm p}^2}+\alpha_\phi m_\phi^2} \tag{69}$$

$$\tan\gamma = \frac{\mu}{k} = \frac{\sqrt{\alpha_\phi}\,m_\phi x_{\rm p}}{l\pi} \tag{70}$$

we have

$$\begin{aligned}&\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left(\mu t\sin\theta-\mu t\sin\beta+\alpha\right)\cos(kt\cos\theta)\\ &=\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right]\end{aligned} \tag{71}$$

Therefore

$$f_l(t) = \frac{2\phi_0 t}{x_{\rm p}} \sin^2\alpha\,(\sin\beta)^{\frac{n}{2}} \exp\left(-\frac{\mu t \sin\beta}{\tan\alpha}\right) \int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}} \exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos(kt\cos\theta)$$
$$+ \frac{2\phi_0}{kx_{\rm p}}\sin(kt\cos\beta)$$
$$- \frac{\phi_0 t}{x_{\rm p}}\cos\alpha\,(\sin\beta)^{\frac{n}{2}} \int_\beta^{\pi-\beta} \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}} \cos\left[\omega t\cos(\theta-\gamma) - \omega t\sin\beta\sin\gamma + \alpha\right] \tag{72}$$

A.3.1 Roll out

For $m_\phi \tau_* = \frac{\mu t \sin\beta}{\tan\alpha} \gg 1$, using Appendix A.5.2,

$$(\sin\beta)^{\frac{n}{2}}\exp(-m_\phi t\sin\beta)\int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp(m_\phi t\sin\theta)\cos(kt\cos\theta)$$
$$\sim \sin\beta \int_{-\infty}^{\beta} d\theta \exp\left[m_\phi t\cos\beta\,(\theta-\beta)\right]\cos\left[kt\cos\beta - kt\sin\beta\,(\theta-\beta)\right] \tag{73}$$
$$= \sin\beta\int_0^\infty d\theta\exp(-m_\phi t\cos\beta\theta)\cos(kt\cos\beta + kt\sin\beta\,\theta) \tag{74}$$
$$= \frac{\sin\beta}{\sqrt{m_\phi^2 t^2\cos^2\beta + k^2t^2\sin^2\beta}}\cos\left(kt\cos\beta + \tan^{-1}\frac{k\tan\beta}{m_\phi}\right) \tag{75}$$

A.3.2 Oscillation

For $\gamma > \beta$, as $\omega t\sin^2\gamma \to \infty$, using Appendix A.5.3, [1]

$$(\sin\beta)^{\frac{n}{2}}\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma) - \omega t\sin\beta\sin\gamma+\alpha\right]$$
$$\sim (\sin\beta)^{\frac{n}{2}}(\sin\gamma)^{\frac{2-n}{2}}\int_\beta^\infty d\theta\cos\left[\omega t - \omega t\sin\beta\sin\gamma + \alpha - \frac{1}{2}\omega t\,(\theta-\gamma)^2\right]$$
$$+\sin\beta\int_{-\infty}^{\pi-\beta} d\theta\cos\Big\{\omega t\cos(\pi-\beta-\gamma) - \omega t\sin\beta\sin\gamma+\alpha$$
$$+\frac{\omega t\sin^2(\pi-\beta-\gamma)}{2\cos(\pi-\beta-\gamma)} - \frac{1}{2}\omega t\cos(\pi-\beta-\gamma)\left[(\theta-\pi+\beta)+\tan(\pi-\beta-\gamma)\right]^2\Big\} \tag{76}$$

$$= (\sin\beta)^{\frac{n}{2}}(\sin\gamma)^{\frac{2-n}{2}}\sqrt{\frac{2}{\omega t}}$$
$$\times\left\{\cos(\omega t - \omega t\sin\beta\sin\gamma+\alpha)\left[\sqrt{\frac{\pi}{8}} + \mathrm{C}\left(\sqrt{\frac{\omega t}{2}}(\gamma-\beta)\right)\right]\right.$$
$$\left.+\sin(\omega t - \omega t\sin\beta\sin\gamma+\alpha)\left[\sqrt{\frac{\pi}{8}}+\mathrm{S}\left(\sqrt{\frac{\omega t}{2}}(\gamma-\beta)\right)\right]\right\}$$

[1] Boundary contributions can also be important, particularly for $n = 3$.

$$
\begin{aligned}
&+\sin\beta\sqrt{\frac{2}{\omega t\cos(\beta+\gamma)}} \\
&\times\left\{\cos\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+\omega t\cos\beta\cos\gamma-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right. \\
&\left.+\sin\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+\omega t\cos\beta\cos\gamma-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right\}
\end{aligned} \tag{77}
$$

For $\gamma<\beta$, as $\omega t\sin^2\beta\to\infty$, using Appendix A.5.3,

$$
\begin{aligned}
&(\sin\beta)^{\frac{n}{2}}\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right] \\
\sim\ &\sin\beta\int_\beta^\infty d\theta\cos\left\{\omega t\cos(\beta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha+\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}\right. \\
&\qquad\left.-\frac{1}{2}\omega t\cos(\beta-\gamma)\left[(\theta-\beta)+\tan(\beta-\gamma)\right]^2\right\} \\
&+\sin\beta\int_{-\infty}^{\pi-\beta}d\theta\cos\left\{\omega t\cos(\pi-\beta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right. \\
&\qquad\left.+\frac{\omega t\sin^2(\pi-\beta-\gamma)}{2\cos(\pi-\beta-\gamma)}-\frac{1}{2}\omega t\cos(\pi-\beta-\gamma)\left[(\theta-\pi+\beta)+\tan(\pi-\beta-\gamma)\right]^2\right\}
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
=\ &\sin\beta\sqrt{\frac{2}{\omega t\cos(\beta-\gamma)}} \\
&\times\left\{\cos\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+\omega t\cos\beta\cos\gamma+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right. \\
&\left.+\sin\left[\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}+\omega t\cos\beta\cos\gamma+\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta-\gamma)}{2\cos(\beta-\gamma)}}\right)\right]\right\} \\
&+\sin\beta\sqrt{\frac{2}{\omega t\cos(\beta+\gamma)}} \\
&\times\left\{\cos\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+\omega t\cos\beta\cos\gamma-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right. \\
&\left.+\sin\left[\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}+\omega t\cos\beta\cos\gamma-\alpha\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta+\gamma)}{2\cos(\beta+\gamma)}}\right)\right]\right\}
\end{aligned} \tag{79}
$$

A.3.3 Rough limits

For $k\gg\mu$ we have $\tan\gamma\ll1$ and $\omega\simeq k$.

$k\tan\beta\gg\mu/\tan\alpha$

For $k\tan\beta \gg \mu/\tan\alpha$ we have $\beta/\gamma \to \infty$, $\omega t\sin^2\beta \to \infty$ and $kt \gg \mu^2 t^2$, therefore, using Appendix A.5.3, Eqs. (75) and (79) reduce to

$$\begin{aligned}
&(\sin\beta)^{\frac{n}{2}} \exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos\left(kt\cos\theta\right) \\
&\sim \frac{1}{kt}\left[-\sin\left(kt\cos\beta\right) + \frac{\mu}{k\tan\alpha\tan\beta}\cos\left(kt\cos\beta\right)\right] \qquad (80)
\end{aligned}$$

and

$$\begin{aligned}
&(\sin\beta)^{\frac{n}{2}} \int_\beta^{\pi-\beta} \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma) - \omega t\sin\beta\sin\gamma + \alpha\right] \\
&\sim \frac{2}{kt}\cos\alpha\sin\left(kt\cos\beta\right) \qquad (81)
\end{aligned}$$

and so Eq. (72) reduces to

$$\begin{aligned}
f_l(t) &\sim -\frac{2\phi_0}{kx_{\mathrm{p}}}\sin^2\alpha\sin\left(kt\cos\beta\right) + \frac{\phi_0}{kx_{\mathrm{p}}}\frac{\mu}{k}\frac{\sin(2\alpha)}{\tan\beta}\cos(kt\cos\beta) \\
&\quad + \frac{2\phi_0}{kx_{\mathrm{p}}}\sin\left(kt\cos\beta\right) \\
&\quad - \frac{2\phi_0}{kx_{\mathrm{p}}}\cos^2\alpha\sin\left(kt\cos\beta\right) \qquad (82) \\
&= \frac{\phi_0}{kx_{\mathrm{p}}}\frac{\mu}{k}\frac{\sin(2\alpha)}{\tan\beta}\cos(kt\cos\beta) \qquad (83)
\end{aligned}$$

$\mu \ll k\tan\beta \ll \mu/\tan\alpha$

For $\mu \ll k\tan\beta \ll \mu/\tan\alpha$ we have $\beta \gg \gamma$, $\omega t\sin^2\beta \to \infty$ and $\tan\alpha\,\mu^2 t^2 \ll kt \ll \mu^2 t^2$, therefore, using Appendix A.5.3, Eqs. (75) and (79) reduce to

$$\begin{aligned}
&(\sin\beta)^{\frac{n}{2}} \exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos\left(kt\cos\theta\right) \\
&\sim \frac{\tan\alpha\tan\beta}{\mu t}\cos(kt\cos\beta) \qquad (84)
\end{aligned}$$

and

$$\begin{aligned}
&(\sin\beta)^{\frac{n}{2}} \int_\beta^{\pi-\beta} \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma) - \omega t\sin\beta\sin\gamma + \alpha\right] \\
&\sim \frac{2\cos\alpha}{kt}\sin(kt\cos\beta) \qquad (85)
\end{aligned}$$

and so Eq. (72) reduces to

$$\begin{aligned}
f_l(t) &\sim \frac{2\phi_0}{\mu x_{\mathrm{p}}}\tan\alpha\tan\beta\sin^2\alpha\cos(kt\cos\beta) \\
&\quad + \frac{2\phi_0}{kx_{\mathrm{p}}}\sin\left(kt\cos\beta\right) \\
&\quad - \frac{2\phi_0}{kx_{\mathrm{p}}}\cos^2\alpha\sin(kt\cos\beta) \qquad (86) \\
&\sim \frac{2\phi_0}{kx_{\mathrm{p}}}\sin^2\alpha\sin(kt\cos\beta) \qquad (87)
\end{aligned}$$

$k\tan\beta = \mu$

For $k\tan\beta = \mu$ we have $\beta = \gamma$ and $kt \sim \tan\alpha\,\mu^2 t^2$, therefore, using Appendix A.5.3, Eqs. (75) and (79) reduce to

$$
\begin{aligned}
&(\sin\beta)^{\frac{n}{2}}\exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos\left(kt\cos\theta\right) \\
&\sim \frac{\sin\alpha}{kt}\cos\left(kt\cos\beta+\alpha\right)
\end{aligned}
\tag{88}
$$

and

$$
\begin{aligned}
&(\sin\beta)^{\frac{n}{2}}\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right] \\
&\sim \sqrt{\frac{\pi}{2kt}}\sin\beta\cos\left(kt\cos^2\beta+\alpha-\frac{\pi}{4}\right)
\end{aligned}
\tag{89}
$$

and so Eq. (72) reduces to

$$
\begin{aligned}
f_l(t) &\sim \frac{2\phi_0}{kx_\mathrm{p}}\sin^3\alpha\cos\left(kt\cos\beta+\alpha\right) \\
&\quad+\frac{2\phi_0}{kx_\mathrm{p}}\sin\left(kt\cos\beta\right) \\
&\quad-\sqrt{\frac{\pi}{2}}\frac{\phi_0}{kx_\mathrm{p}}\sqrt{kt}\cos\alpha\sin\beta\cos\left(kt\cos^2\beta+\alpha-\frac{\pi}{4}\right) && (90)\\
&\sim -\sqrt{\frac{\pi}{2}}\frac{\phi_0}{kx_\mathrm{p}}\sqrt{kt}\cos\alpha\sin\beta\cos\left(kt\cos^2\beta+\alpha-\frac{\pi}{4}\right) && (91)
\end{aligned}
$$

$\mu\tan\beta \ll k\tan\beta \ll \mu$

For $k\tan\beta \ll \mu$ we have $k\tan\alpha\tan\beta/\mu \to 0$, $\beta \ll \gamma$, $\omega t\sin^2\gamma \to \infty$ and $kt \ll \tan\alpha\,\mu^2 t^2$, therefore, using Appendix A.5.3, Eqs. (75) and (79) reduce to

$$
\begin{aligned}
&(\sin\beta)^{\frac{n}{2}}\exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^\beta \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos\left(kt\cos\theta\right) \\
&\sim \frac{\tan\alpha\tan\beta}{\mu t}\cos(kt\cos\beta)
\end{aligned}
\tag{92}
$$

and

$$
\begin{aligned}
&(\sin\beta)^{\frac{n}{2}}\int_\beta^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right] \\
&\sim -\frac{1}{kt}\left(\frac{\sin\beta}{\sin\gamma}\right)^{\frac{n}{2}}\frac{\sin\gamma}{\gamma}\sin\left(kt-kt\sin\beta\sin\gamma-\frac{kt}{2}\gamma+\alpha\right)
\end{aligned}
\tag{93}
$$

and so Eq. (72) reduces to

$$
\begin{aligned}
f_l(t_\mathrm{p}) &\sim \frac{2\phi_0}{\mu x_\mathrm{p}}\sin^2\alpha\tan\alpha\tan\beta\cos(kt\cos\beta) \\
&\quad+\frac{2\phi_0}{kx_\mathrm{p}}\sin\left(kt\cos\beta\right)
\end{aligned}
$$

$$+\frac{\phi_0}{kx_{\rm p}}\cos\alpha\left(\frac{\sin\beta}{\sin\gamma}\right)^{\frac{n}{2}}\frac{\sin\gamma}{\gamma}\sin\left(kt-kt\sin\beta\sin\gamma-\frac{kt}{2}\gamma+\alpha\right) \tag{94}$$

$$\sim \frac{2\phi_0}{kx_{\rm p}}\sin\left(kt\cos\beta\right) \tag{95}$$

For $k \ll \mu$ we have $\tan\gamma \gg 1$ and $\omega \simeq \mu$.
Therefore, using Appendix A.5.3, Eqs. (75) and (79) reduce to

$$(\sin\beta)^{\frac{n}{2}}\exp\left(-\frac{\mu t\sin\beta}{\tan\alpha}\right)\int_0^{\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\exp\left(\frac{\mu t\sin\theta}{\tan\alpha}\right)\cos(kt\cos\theta)$$

$$\sim \frac{\tan\alpha\tan\beta}{\mu t}\cos(kt\cos\beta) \tag{96}$$

and

$$(\sin\beta)^{\frac{n}{2}}\int_{\beta}^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right]$$

$$\sim -\frac{1}{\mu t}\left(\frac{\sin\beta}{\sin\gamma}\right)^{\frac{n}{2}}\frac{\sin\gamma}{\gamma}\sin\left(\mu t-\mu t\sin\beta\sin\gamma-\frac{\mu t}{2}\gamma+\alpha\right) \tag{97}$$

and so Eq. (72) reduces to

$$f_l(t_{\rm p}) = \frac{2\phi_0}{\mu x_{\rm p}}\sin^2\alpha\tan\alpha\tan\beta\cos(kt\cos\beta)$$

$$+\frac{2\phi_0}{kx_{\rm p}}\sin(kt\cos\beta)$$

$$+\frac{\phi_0}{\mu x_{\rm p}}\cos\alpha\left(\frac{\sin\beta}{\sin\gamma}\right)^{\frac{n}{2}}\frac{\sin\gamma}{\gamma}\sin\left(\mu t-\mu t\sin\beta\sin\gamma-\frac{\mu t}{2}\gamma+\alpha\right) \tag{98}$$

$$\sim \frac{2\phi_0}{kx_{\rm p}}\sin(kt\cos\beta) \tag{99}$$

A.3.4 $n = 3$

For $n = 3$, $\gamma > \beta$, as $\omega t\sin^2(\gamma-\beta)\to\infty$, using Appendix A.5.3,

$$\int_{\beta}^{\pi-\beta}\frac{d\theta}{(\sin\theta)^{\frac{1}{2}}}\cos\left[\omega t\cos(\theta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha\right]$$

$$\sim (\sin\gamma)^{-\frac{1}{2}}\int_{\beta}^{\infty}d\theta\cos\left[\omega t-\omega t\sin\beta\sin\gamma+\alpha-\frac{1}{2}\omega t\left(\theta-\gamma\right)^2\right]$$

$$+\int_{\beta}^{\infty}\frac{d\theta}{\theta^{\frac{1}{2}}}\cos\left[\omega t\cos(\beta-\gamma)-\omega t\sin\beta\sin\gamma+\alpha-\omega t\sin(\beta-\gamma)\left(\theta-\beta\right)\right] \tag{100}$$

$$= \sqrt{\frac{2}{\omega t\sin\gamma}}\int_{-\sqrt{\frac{\omega t}{2}}(\gamma-\beta)}^{\infty}dx\cos\left[\omega t-\omega t\sin\beta\sin\gamma+\alpha-x^2\right]$$

$$+\frac{2}{\sqrt{\omega t\sin(\gamma-\beta)}}\int_{\sqrt{\omega t\beta\sin(\gamma-\beta)}}^{\infty}dx\cos\left[\omega t\cos(\gamma-\beta)-\omega t\sin\beta\sin\gamma+\alpha-\omega t\beta\sin(\gamma-\beta)+x^2\right] \tag{101}$$

$$
\begin{aligned}
= \quad & (\sin\gamma)^{\frac{2-n}{2}} \sqrt{\frac{2}{\omega t}} \cos\left(\omega t - \omega t \sin\beta \sin\gamma + \alpha\right) \left[\sqrt{\frac{\pi}{8}} + \mathrm{C}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma - \beta\right)\right)\right] \\
& + (\sin\gamma)^{\frac{2-n}{2}} \sqrt{\frac{2}{\omega t}} \sin\left(\omega t - \omega t \sin\beta \sin\gamma + \alpha\right) \left[\sqrt{\frac{\pi}{8}} + \mathrm{S}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma - \beta\right)\right)\right] \\
& + \ldots
\end{aligned} \tag{102}
$$

A.4 Kinetic calculations

$$
\begin{aligned}
& \frac{d}{dt}\Bigg(\sqrt{\sin^{n-1}\beta \sin^{2-n}\gamma} \\
& \qquad \times \left\{\cos\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{C}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right. \\
& \qquad \left. + \sin\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{S}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right\}\Bigg) \\
= \quad & -\frac{n-1}{2t}\sqrt{\sin^{n-1}\beta \sin^{2-n}\gamma} \\
& \times \left\{\cos\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{C}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right. \\
& \left. + \sin\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{S}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right\} \\
& + \omega\sqrt{\sin^{n-1}\beta \sin^{2-n}\gamma} \\
& \times \left\{-\sin\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{C}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right. \\
& \left. + \cos\left(\omega t - \mu\tau_* + \alpha\right)\left[\sqrt{\frac{\pi}{8}} + \mathrm{S}\left(\sqrt{\frac{\omega t}{2}}\left(\gamma-\beta\right)\right)\right]\right\} \\
& + \sqrt{\frac{\omega}{2t}}\left[\tan\beta + \frac{1}{2}\left(\gamma-\beta\right)\right]\sqrt{\sin^{n-1}\beta \sin^{2-n}\gamma} \\
& \times \cos\left[\omega t - \mu\tau_* + \alpha - \frac{\omega t}{2}\left(\gamma-\beta\right)^2\right]
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
& \frac{d}{dt}\Bigg(\sqrt{\frac{\sin\beta}{\cos(\beta\pm\gamma)}} \\
& \quad \times \left\{\cos\left[\frac{\omega t \sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)} + \omega t\cos(\beta\pm\gamma) \pm \left(\mu\tau_* - \alpha\right)\right]\left[\sqrt{\frac{\pi}{8}} - \mathrm{C}\left(\sqrt{\frac{\omega t \sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right. \\
& \quad \left. + \sin\left[\frac{\omega t \sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)} + \omega t\cos(\beta\pm\gamma) \pm \left(\mu\tau_* - \alpha\right)\right]\left[\sqrt{\frac{\pi}{8}} - \mathrm{S}\left(\sqrt{\frac{\omega t \sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right\}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{1}{2t}\frac{\cos\gamma}{\cos\beta}\sqrt{\frac{\sin\beta}{\cos^3(\beta\pm\gamma)}} \\
& \times\left\{\cos\left[\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}+\omega t\cos(\beta\pm\gamma)\pm(\mu\tau_*-\alpha)\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right. \\
& \left.+\sin\left[\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}+\omega t\cos(\beta\pm\gamma)\pm(\mu\tau_*-\alpha)\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right\} \\
& +\frac{\omega}{2\cos\beta}\sqrt{\frac{\sin\beta}{\cos^3(\beta\pm\gamma)}}\left[2\cos\beta-\cos\gamma\sin(\beta\pm\gamma)\tan(\beta\pm\gamma)\right] \\
& \times\left\{-\sin\left[\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}+\omega t\cos(\beta\pm\gamma)\pm(\mu\tau_*-\alpha)\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{C}\left(\sqrt{\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right. \\
& \left.+\cos\left[\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}+\omega t\cos(\beta\pm\gamma)\pm(\mu\tau_*-\alpha)\right]\left[\sqrt{\frac{\pi}{8}}-\mathrm{S}\left(\sqrt{\frac{\omega t\sin^2(\beta\pm\gamma)}{2\cos(\beta\pm\gamma)}}\right)\right]\right\} \\
& +\sqrt{\frac{\omega}{2t}}\frac{2\sin\gamma+\cos\gamma\tan(\beta\pm\gamma)}{2\cos\beta\cos(\beta\pm\gamma)}\sqrt{\sin\beta}\cos\left[\omega t\cos(\beta\pm\gamma)\pm(\mu\tau_*-\alpha)\right] \qquad (104)
\end{aligned}
$$

where we have used Eqs. (116), (117) and (118).

A.5 Mathematical formulae

A.5.1 Parameter relations

Defining

$$\sin\beta=\frac{\tau_*}{t} \tag{105}$$

gives

$$\frac{d\beta}{dt}=-\frac{\tan\beta}{t} \tag{106}$$

and

$$\frac{d}{dt}\left(t\cos\beta\right)=\frac{1}{\cos\beta} \tag{107}$$

Defining

$$
\begin{aligned}
\omega &= \sqrt{k^2+\mu^2} && (108)\\
\tan\gamma &= \frac{\mu}{k} && (109)
\end{aligned}
$$

gives

$$
\begin{aligned}
\omega\cos\gamma &= k && (110)\\
\omega\sin\gamma &= \mu && (111)
\end{aligned}
$$

Defining

$$\tan\eta=\frac{k\tan\beta}{m_\phi} \tag{112}$$

gives

$$\frac{d\eta}{dt} = -\frac{\sin\eta\cos\eta}{t\cos^2\beta} \tag{113}$$

$$\begin{aligned} \omega t\cos(\beta\pm\gamma) &= kt\cos\beta \mp \mu\tau_* && (114)\\ \omega t\sin(\beta\pm\gamma) &= k\tau_* \pm \mu t\cos\beta && (115)\end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left[t\sin(\beta\pm\gamma)\right] &= \pm\frac{\sin\gamma}{\cos\beta} && (116)\\ \frac{d}{dt}\left[t\cos(\beta\pm\gamma)\right] &= \frac{\cos\gamma}{\cos\beta} && (117)\\ \frac{d}{dt}\left[\frac{t\sin^2(\beta\pm\gamma)}{\cos(\beta\pm\gamma)}\right] &= \frac{\tan(\beta\pm\gamma)}{\cos\beta}\left[\pm 2\sin\gamma - \cos\gamma\tan(\beta\pm\gamma)\right] && (118)\end{aligned}$$

A.5.2

$$\int_0^\infty e^{-px}\cos(qx+\lambda)\,dx = \frac{1}{\sqrt{p^2+q^2}}\cos\left(\lambda + \tan^{-1}\frac{q}{p}\right) \tag{119}$$

A.5.3 Fresnel integrals

$$\begin{aligned} \mathrm{S}(x) &\equiv \int_0^x \sin t^2\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(4n+3)\,(2n+1)!} && (120)\\ \mathrm{C}(x) &\equiv \int_0^x \cos t^2\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)\,(2n)!} && (121)\end{aligned}$$

As $x \to 0$,

$$\begin{aligned} \mathrm{S}(x) &= \frac{1}{3}x^3 + O\big(x^7\big) && (122)\\ \mathrm{C}(x) &= x + O\big(x^5\big) && (123)\end{aligned}$$

As $x \to \infty$,

$$\begin{aligned} \mathrm{S}(x) &= \sqrt{\frac{\pi}{8}} - \frac{1}{2x}\cos x^2 - \frac{1}{4x^3}\sin x^2 + \frac{3}{8x^5}\cos x^2 + O\left(\frac{1}{x^7}\right) && (124)\\ \mathrm{C}(x) &= \sqrt{\frac{\pi}{8}} + \frac{1}{2x}\sin x^2 - \frac{1}{4x^3}\cos x^2 - \frac{3}{8x^5}\sin x^2 + O\left(\frac{1}{x^7}\right) && (125)\end{aligned}$$

A.5.4

$$\begin{aligned} \int_0^\pi \frac{d\theta}{(\sin\theta)^{\frac{n-2}{2}}}\cos\left(A\cos\theta+\alpha\right) &= \sqrt{\pi}\,\Gamma\left(\frac{4-n}{4}\right)\left(\frac{2}{A}\right)^{\frac{2-n}{4}}\cos\alpha\, J_{\frac{2-n}{4}}(A)\\ \sim\ \Gamma\left(\frac{4-n}{4}\right)\left(\frac{2}{A}\right)^{\frac{4-n}{4}}\cos\alpha\cos\left[A - \left(\frac{4-n}{4}\right)\frac{\pi}{2}\right] &\qquad \text{as } A\to\infty && (126)\end{aligned}$$

and

$$\int_0^\infty \frac{d\theta}{\theta^{\frac{n-2}{2}}} \cos\left(A+\alpha-\frac{1}{2}A\theta^2\right) = \frac{1}{2}\,\Gamma\left(\frac{4-n}{4}\right)\left(\frac{2}{A}\right)^{\frac{4-n}{4}} \cos\left[A+\alpha-\left(\frac{4-n}{4}\right)\frac{\pi}{2}\right] \tag{127}$$

where

$$\Gamma\left(\frac{1}{4}\right) \simeq 3.6256 \tag{128}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{129}$$

$$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)} \tag{130}$$