Analytical treatment - 11/13/2018

In order to obtain an analytical solution for the problem initially we consider g = 0. The boundary limits of the 1-dimensional problem assume the length of the domain spanning form x=-L to x=+L. The stresses are taken to be zero at each of the boundaries. We begin by implementing a trial solution for u_x and w_x.

$$u_x = Ax + Bsinh\Big(\frac{x}{\sigma}\Big) \tag{16}$$

$$w_x = Cx + Dsinh\Big(\frac{x}{\sigma}\Big) \tag{17}$$

For estimating the values of A,B,C,D we assume additionally p=q=0. Rewriting (12):

$$2\nu\frac{\partial^2}{\partial x^2}\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{18}$$

$$2\nu\frac{\partial}{\partial x}\Big(A + \frac{B}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)) \tag{19}$$

$$2\nu\frac{B}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{20}$$

and (13)

$$2\mu(x)\frac{\partial^2}{\partial x^2}\Big(Cx + Dsinh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{21}$$

$$2\mu(x)\frac{\partial}{\partial x}\Big(C + \frac{D}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)) \tag{22}$$

$$2\mu(x)\frac{D}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{23}$$

From (20) and (23) we have:

$$D = -\frac{\nu}{\mu(x)}B \tag{24}$$

Comparing coefficients of hyperbolic sine terms we see that (20) and (23) is satisfied only when the coefficient of the linear term is zero. Therefore,

$$A = C \tag{25}$$

Hence the length constant[1] σ has the value[2]

$$\sigma = \sqrt{\frac{2\nu\mu}{K(\nu+\mu)}} \tag{26}$$

To obtain the values of the unknown parameters B and C, we impose boundary conditions. At the edge (x = ±L) we have, normal stresses τ_{ix} and τ_{ex} as zero. As a result we get,

$$C + \frac{D}{\sigma}cosh\Big(\frac{x}{\sigma}\Big) = 0 \tag{27}$$

$$A + \frac{B}{\sigma}cosh\Big(\frac{x}{\sigma}\Big) = -\frac{T}{2\nu} \tag{28}$$

[1] This might also be a variable since μ is a function of x for the problem

[2] $\mu(x)$ is also written as μ

Solving (27) and (28) using (24)-(26) we have:

$$A = C = -\frac{T}{2(\nu + \mu)} \tag{29}$$

$$B = -\frac{T}{2\nu}\Big(\frac{\mu}{\nu + \mu}\Big)\frac{\sigma}{cosh(\frac{L}{\sigma})}\ , D = \frac{T}{2(\mu + \nu)}\frac{\sigma}{cosh(\frac{L}{\sigma})} \tag{30}$$

Using this we get the intra and extracellular displacement in terms of T as:

$$u_x = -\frac{T}{2(\nu + \mu)}\Big(x + \frac{\mu}{\nu}\sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{31}$$

$$w_x = -\frac{T}{2(\nu + \mu)}\Big(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{32}$$

Since μ is variable for the problem we now write it as: $\mu = \mu_0 + gx$. Remaining derivation of u_x and w_x along with its derivatives and double derivatives has been attached separately.