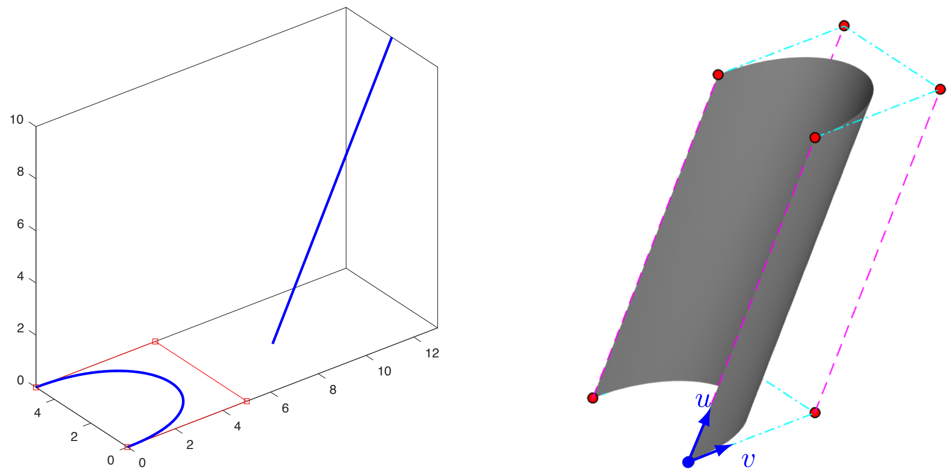
6.7 Surface construction techniques

Common surface constructions can be represented through NURBS. These constructed surfaces include extruded surfaces, surfaces of revolution, ruled surfaces, sweeping surfaces, etc.

6.7.1 Extrusion



(a) Generating curve and a line (b) Extrusion surface

Figure 6.25: Extruding a curve along a straight line.

Figure 6.25 shows a NURBS represented extrusion surface. The surface is extruded from a generating curve and a straight line. The input is a generating curve $\mathbf{c}(v)$, straight line vector $\mathbf{a}$ with its length $||\mathbf{a}|| = d$. Assume the curve is a NURBS curve and has control points $\mathbf{q}_j$, and weight w_j, j =0 to n, the degree q. The knots are $v_0, v_1, \cdots, v_{n+q-1}$.

The output is a NURBS surface $\mathbf{s}(u, v)$. The surface touches the curve when $u = 0$, i.e.

$$\mathbf{s}(0, v) = \mathbf{c}(v). \tag{6.39}$$

The curve $\mathbf{s}(1, v)$ is a translation of $\mathbf{s}(0, v)$ by $\mathbf{a}$, i.e.

$$\mathbf{s}(1, v) = \mathbf{s}(0, v) + \mathbf{a}. \tag{6.40}$$

The curve $\mathbf{s}(u, v)$ is a translation of $\mathbf{s}(0, v)$ by $u\mathbf{a}$, i.e.

$$\mathbf{s}(u, v) = \mathbf{s}(0, v) + u\mathbf{a}, \quad u \in [0, 1]. \tag{6.41}$$

The isoparametric curve for any given constant v is a straight line and it is a translation of the given line vector. The new NURBS surface is defined as follows.

- $2 \times (n + 1)$ control points $\mathbf{p}_{i,j}$

$$\mathbf{p}_{0,j} = \mathbf{q}_j, \quad \mathbf{p}_{1,j} = \mathbf{q}_j + \mathbf{a}, \quad j = 0 \cdots n. \tag{6.42}$$

  The two rows of control points are offset of each other with offset vector being the line vector $\mathbf{a}$.

- $2 \times (n+1)$ weights
$$w_{0,j} = w_j, \quad w_{1,j} = w_j, \quad j = 0 \cdots n.. \tag{6.43}$$
The new surface retains the original weights from the curve.

- degrees: $1 \times q$, i.e. degree 1 in u and degree q in v.

- knot vectors.
u knots: $\{0, 1\}$,
v knots $\{v_0, \cdots, v_{n+q-1}\}$.

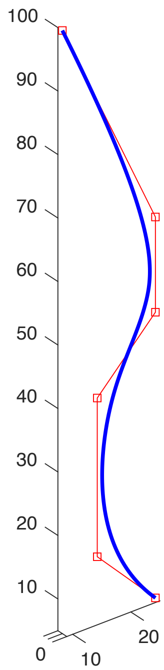
6.7.2 Revolution

- Input: a profile curve $\mathbf{c}(v)$ in the x-z plane with
  - degree q
  - knots $\{v_0, v_1, \cdots, v_{n+q-1}\}$
  - weights w_i
  - control points $\mathbf{q}_i$, $i = 0$ to n

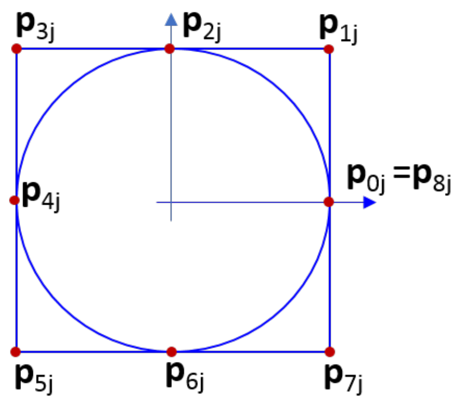
- rotation axis: z-axis, axis =(0,0,0)

- output a revolved surface
  - degrees: $2 \times q$
  - knots in u: $\{0, 0, 1, 1\}$
    knots in v: $\{v_0, v_1, \cdots, v_{n+q-1}\}$
  - control points and weights: $9 \times (n+1)$

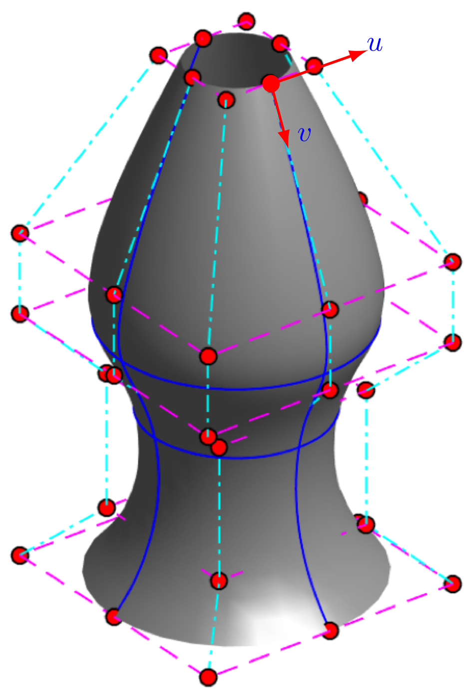
$$\begin{aligned}
\mathbf{p}_{0,j} &= \mathbf{q}_j, \quad w_{0,j} = 1 \cdot w_j, \\
\mathbf{p}_{1,j} &= \mathbf{p}_{0,j} + x_j \vec{\mathbf{e}}_y, \quad w_{1,j} = \frac{\sqrt{2}}{2} \cdot w_j, \\
\mathbf{p}_{2,j} &= \mathbf{p}_{1,j} - x_j \vec{\mathbf{e}}_x, \quad w_{2,j} = 1 \cdot w_j, \\
\mathbf{p}_{3,j} &= \mathbf{p}_{2,j} - x_j \vec{\mathbf{e}}_x, \quad w_{3,j} = \frac{\sqrt{2}}{2} \cdot w_j, \\
\mathbf{p}_{4,j} &= \mathbf{p}_{3,j} - x_j \vec{\mathbf{e}}_y, \quad w_{4,j} = 1 \cdot w_j, \\
\mathbf{p}_{5,j} &= \mathbf{p}_{4,j} - x_j \vec{\mathbf{e}}_y, \quad w_{5,j} = \frac{\sqrt{2}}{2} \cdot w_j, \\
\mathbf{p}_{6,j} &= \mathbf{p}_{5,j} + x_j \vec{\mathbf{e}}_x, \quad w_{6,j} = 1 \cdot w_j, \\
\mathbf{p}_{7,j} &= \mathbf{p}_{6,j} + x_j \vec{\mathbf{e}}_x, \quad w_{7,j} = \frac{\sqrt{2}}{2} \cdot w_j, \\
\mathbf{p}_{8,j} &= \mathbf{p}_{7,j} + x_j \vec{\mathbf{e}}_y, \quad w_{8,j} = 1 \cdot w_j.
\end{aligned}$$



(a) Revolving profile curve

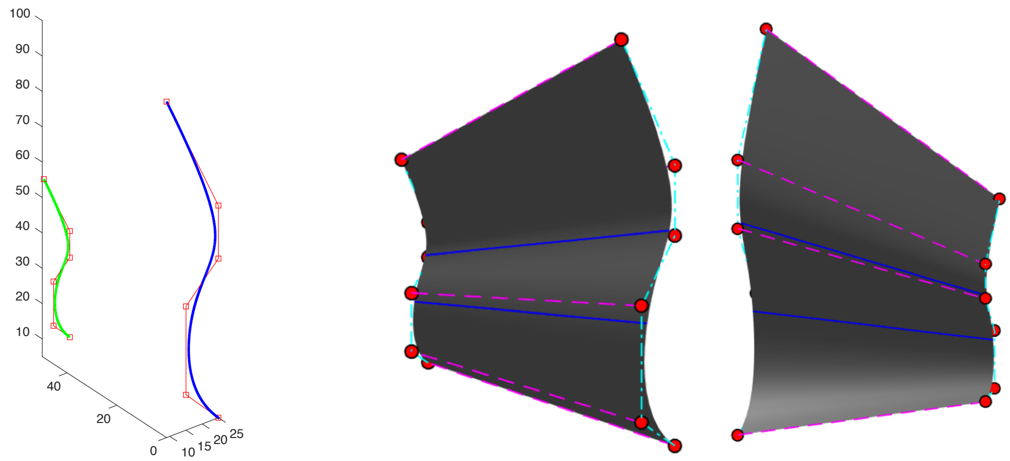


(b) A circle



(c) Extrusion surface

Figure 6.26: Revolution in NURBS



(a) Two profile curves (b) Ruled surface

Figure 6.27: Ruled surface in NURBS.

6.7.3 Ruled surfaces

- Input: two curves **a** and **b**
  - degree: q
  - knots: $\{v_0, v_1, \cdots, v_{n+q-1}\}$
  - control points: $\mathbf{q}_0^a, \cdots, \mathbf{q}_n^a$ for curve **a** and $\mathbf{q}_0^b, \cdots, \mathbf{q}_n^b$ for curve **b**
  - weights: $w_0^a, w_1^a, \cdots, w_n^a$ for curve **a** and $w_0^b, w_1^b, \cdots, w_n^b$ for curve **b**.

  The two curves are assumed to have the same knot vector and the degree. If they are not the same, the two curves can be made compatible with knot insertion and degree elevation.

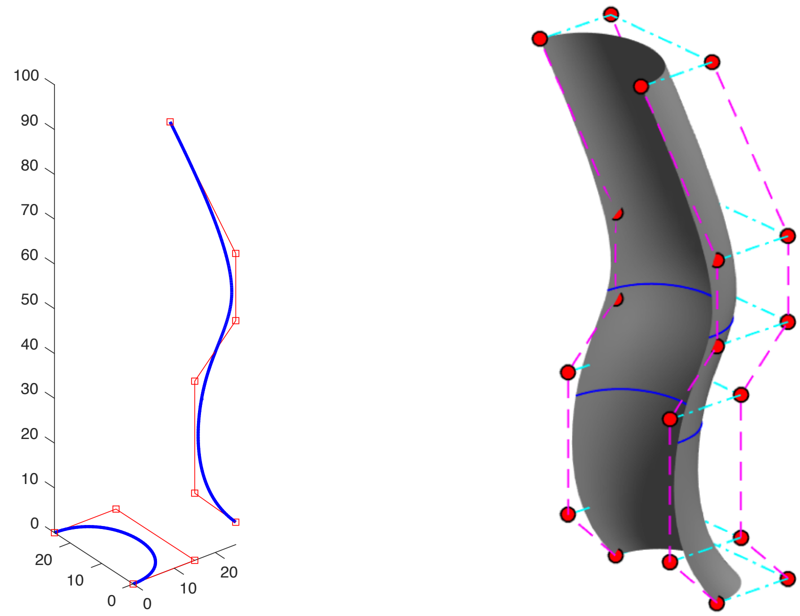
- Output: a NURBS surface
  - degree: $1 \times q$
  - knots: $\{0, 1\}$ in u and $\{v_0, v_1, \cdots, v_{n+q-1}\}$ in v
  - control points: $\mathbf{p}_{0,j} = \mathbf{q}_j^a$ and $\mathbf{p}_{1,j} = \mathbf{q}_j^b$
  - weights: $w_{0,j} = w_j^a$ and $w_{1,j} = w_j^b$.

The resulting ruled surface interpolates the two given curves.

6.7.4 Sweeping surfaces

- Input: a generating curve $\mathbf{c}_1(u)$ and a path curve $\mathbf{c}_2(v)$

  For the generating curve $\mathbf{c}_1(u)$
  - degree: p
  - knots: $\{u_0, u_1, \cdots, u_{m+p-1}\}$
  - control points: $\mathbf{q}_0^1, \mathbf{q}_1^1, \cdots, \mathbf{q}_m^1$



(a) Generating curve and sweeping path

(b) Swept surface

Figure 6.28: Translationally sweeping a surface

- weights: $w_0^1, w_1^1, \cdots, w_m^1$

For the path curve $\mathbf{c}_2(v)$

- degree: q
- knots: $\{v_0, v_1, \cdots, v_{n+q-1}\}$
- control points: $\mathbf{q}_0^2, \mathbf{q}_1^2, \cdots, \mathbf{q}_n^2$
- weights: $w_0^2, w_1^2, \cdots, w_n^2$

- output: surface
  - degrees: $p \times q$
  - knots: $\{u_0, u_1, \cdots, u_{m+p-1}\}$ in u $\{v_0, v_1, \cdots, v_{n+q-1}\}$ in v
  - control points: $(m+1) \times (n+1)$

$$\mathbf{p}_{i,j} = \mathbf{q}_i^1 + \tilde{\mathbf{q}}_j^2, \quad i = 0, \cdots, m, j = 0, \ldots, n$$

    where $\tilde{\mathbf{q}}_j^2$ is a translation of $\mathbf{q}_j^2$ such that $\mathbf{q}_0^2 = (0, 0, 0)$.
  - weights:

$$w_{i,j} = w_i^1 \times w_j^2.$$

6.7.5 Tensor-product NURBS surfaces

In all the above surface construction techniques, the weight $w_{i,j}$ for control point $\mathbf{p}_{i,j}$ is a product of two weights w_i and w_j, corresponding to weights in u and v directions. This makes the resulting NURBS surfaces to have the tensor-product structure. Note, a general form of NURBS surface does not have the tensor-product structure.

With the way the weights are obtained for the surface, we have the following representation of the resulting NURBS surfaces,

$$\mathbf{s}(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_i w_j \mathbf{p}_{i,j} N_{i,p}(u) N_{j,q}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_i w_j N_{i,p}(u) N_{j,q}(v)}. \tag{6.44}$$

It can also be represented as follow.

$$\mathbf{s}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{i,j} R_{i,j}(u,v) \tag{6.45}$$

where

$$R_{i,j}(u,v) = \frac{w_i w_j}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_i w_j N_{i,p}(u) N_{j,q}(v)} = \frac{w_i}{\sum_{i=0}^{m} w_i N_{i,p}(v)} \frac{w_j}{\sum_{j=0}^{n} w_j N_{j,q}(v)}. \tag{6.46}$$

The above basis can be noted as

$$R_{i,j}(u,v) = R_{i,p}(u) R_{j,q}(v). \tag{6.47}$$

That is, the rational two-dimensional basis is a simple product of two one-dimensional bases. So the surface can be represented as

$$\mathbf{s}(u,v) = \begin{bmatrix} R_{0,p}(u) & R_{1,p}(u) & \cdots & R_{m,p}(u) \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \cdots & \mathbf{p}_{0,n} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \cdots & \mathbf{p}_{1,n} \\ \cdots & & & \\ \mathbf{p}_{m,0} & \mathbf{p}_{m,1} & \cdots & \mathbf{p}_{m,n} \end{bmatrix} \begin{bmatrix} R_{0,q}(v) \\ R_{1,q}(v) \\ \cdots \\ R_{n,q}(v) \end{bmatrix}. \tag{6.48}$$

NURBS representation of the surface of revolution

$$\mathbf{s}^w(u,v) = \sum_i \sum_j N_{i,p}(u) N_{j,q}(v) \mathbf{p}^w_{i,j}$$

- at $u = 0$, we have

$$\mathbf{s}(0,v) = \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \cdots & \mathbf{p}_{0,n} \end{bmatrix} \begin{bmatrix} R_{0,q}(v) \\ R_{1,q}(v) \\ \cdots \\ R_{n,q}(v) \end{bmatrix}. \tag{6.49}$$

Since $\mathbf{p}_{0,j} = \mathbf{p}_j$, so at $u = 0$, the isoparametric curve retains the revolving profile.

- For a given fixed $u = \bar{u}$, we have

$$\mathbf{s}(0, v) = \begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} R_{0,q}(v) \\ R_{1,q}(v) \\ \cdots \\ R_{n,q}(v) \end{bmatrix}, \tag{6.50}$$

where $\mathbf{q}_j = \sum_{i=0}^{m} R_{i,p}(\bar{u})\mathbf{p}_{i,j}$. Referring to Fig. 6.26(b), this intermediate control point $\mathbf{q}_j$ essentially corresponds to a rotated control point $\mathbf{p}_j$. Due to the affine invariance of NURBS curve, the resulting isoparametric curve $\mathbf{s}(\bar{u}, v)$ is thus a rotated revolving profile. Therefore, the resulting surface is a surface of revolution.

6.7.6 Exercise

Ex. 6.11 — Extrusion

1. Create a NURBS represented extrusion surface.
2. Export the surface into an IGES file.
3. Import the IGES file into a CAD software and render the surface.

Ex. 6.12 — Revolution

1. Create a NURBS represented revolved surface.
2. Export the surface into an IGES file.
3. Import the IGES file into a CAD software and render the surface.

Ex. 6.13 — Sweeping

1. Create a NURBS represented translationally sweeping surface.
2. Export the surface into an IGES file.
3. Import the IGES file into a CAD software and render the surface.

Ex. 6.14 — Ruled surface

1. Create a NURBS represented ruled surface.
2. Export the surface into an IGES file.
3. Import the IGES file into a CAD software and render the surface.