6.6 B-spline surfaces and NURBS surfaces

- *Curves and Surfaces for CAGD: A Practical Guide*, 5th Edition by Gerald Farin, 2002.
- *Bezier and B-spline Techniques*, H. Prautzsch, W. Boehm, and M. Paluszny, 2002.

6.6.1 B-spline surfaces

A tensor-product B-spline surface can be represented as

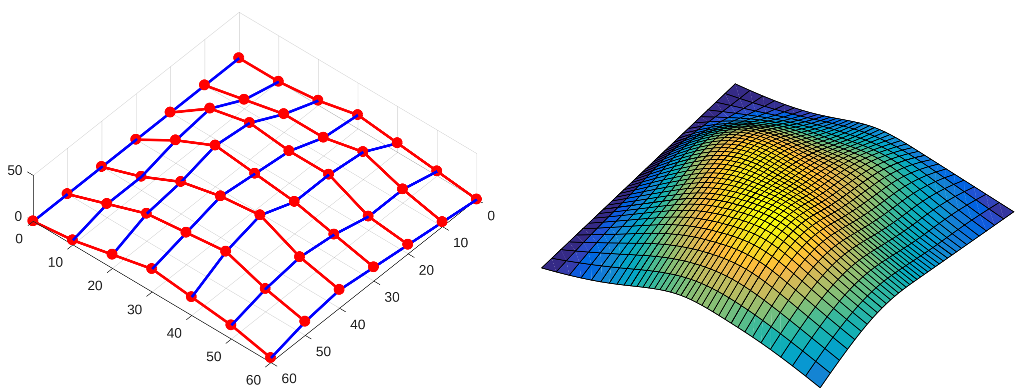
$$\mathbf{s}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{i,j} N_{i,p}(u) N_{j,q}(v), \quad (u,v) \in [u_{p-1}, u_m] \times [v_{q-1}, v_n]$$

where $N_{i.p}(u)$ and $N_{j,q}(v)$ are B-spline blending functions of degree p and q along u and v directions, $\mathbf{p}_{i,j}$, $i = 0, \cdots, m$, $j = 0, \cdots, n$ are $(m+1)(n+1)$ control points. The B-splines are defined from knots $u_0, u_1, \cdots, u_{m+p-2}$ in u direction, and knots $v_0, v_1, \cdots, v_{n+q-2}$ in v direction. Here we assume there is no superfluous knot in either u or v direction.

B-spline surfaces can also be cast into a matrix form as

$$\mathbf{s}(u,v) = \begin{bmatrix} N_{0,p}(u) & N_{1,p}(u) & \cdots & N_{m,p}(u) \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \cdots & \mathbf{p}_{0,n} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \cdots & \mathbf{p}_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{p}_{m,0} & \mathbf{p}_{m,1} & \cdots & \mathbf{p}_{m,n} \end{bmatrix} \begin{bmatrix} N_{0,q}(v) \\ N_{1,q}(v) \\ \cdots \\ N_{n,q}(v) \end{bmatrix}$$

The tensor-product nature makes it possible to evaluate a B-spline surface through a sequence of B-spline curve evaluations.



(a) Control net (b) Rendered surface

Figure 6.19: B-spline control net and its rendered surface.

Example 8. ***B-spline surface evaluation***

Assuming a bi-cubic B-spline surface with control points as follows

```
0 0 0; 0 10 0; 0 20 5; 0 30 15; 0 40 10; 0 50 5; 0 60 0;
10,0,0; 10 10 10; 10 20 20; 10 30 20; 10 40 30; 10 50 15; 10 60 5;
20 0 0; 20 10 30; 20 20 40; 20 30 35; 20 40 35; 20 50 15; 20 60 10;
```

```
30 0 0; 30 10 25; 30 20 45; 30 30 40; 30 40 35; 30 50 25; 30 60 15;
40 0 0; 40 10 15; 40 20 35; 40 30 45; 40 40 50; 40 50 30; 40 60 20;
50 0 0; 50 10 15; 50 20 30; 50 30 35; 50 40 40; 50 50 25; 50 60 15;
60 0 0; 60 10 5; 60 20 15; 60 30 25; 60 40 20; 60 50 15; 60 60 5;
```

The control net is shown in Fig. 6.19 with knot vectors $[0, 0, 0, 1, 2, 3, 4, 4, 4]$ *in both u and v directions.*

a. *Compute* $\boldsymbol{s}(2.5, 1.5)$

b. *Compute their partial derivatives* $\dfrac{\partial \boldsymbol{s}}{\partial u}, \dfrac{\partial \boldsymbol{s}}{\partial v}$ *and* $\dfrac{\partial^2 \boldsymbol{s}}{\partial u \partial v}$ *at* $u = 2.5$ *and* $v = 1.5$.

Solution: *Using tensor-product nature of B-spline surfaces, we can write the B-spline surface evaluation as a product of univariate functions as follows.*

$$\begin{aligned}
\mathbf{s}(2.5, 1.5) &= [\mathbf{N}(u)]^T \widehat{\mathbf{P}} \mathbf{N}(v), \\
\mathbf{s}_u(2.5, 1.5) &= [\mathbf{N}'(u)]^T \widehat{\mathbf{P}} \mathbf{N}(v), \\
\mathbf{s}_v(2.5, 1.5) &= [\mathbf{N}(u)]^T \widehat{\mathbf{P}} \mathbf{N}'(v), \\
\mathbf{s}_{uv}(2.5, 1.5) &= [\mathbf{N}'(u)]^T \widehat{\mathbf{P}} \mathbf{N}'(v).
\end{aligned}$$

Using pyramid diagrams to obtain the blossoms, we can find the following B-spline functions and their derivatives at $u = 2.5$ *and* $v = 1.5$.

For u direction with knot vector $[0, 0, 0, 1, 2, 3, 4, 4, 4]$, *based on figure 6.20*

$$\begin{aligned}
\mathbf{b}[2.5^{<3>}] = \;& \tfrac{0.5}{3}\tfrac{0.5}{2}\tfrac{1}{2}\mathbf{b}[0,1,2] \\
&+ \left(\tfrac{2.5}{3}\tfrac{0.5}{2}\tfrac{1}{2} + \tfrac{1.5}{3}\tfrac{1.5}{2}\tfrac{1}{2} + \tfrac{1.5}{3}\tfrac{1.5}{2}\tfrac{1}{2}\right)\mathbf{b}[1,2,3] \\
&+ \left(\tfrac{1.5}{4-1}\tfrac{1.5}{2}\tfrac{1}{2} + \tfrac{1.5}{3}\tfrac{1.5}{2}\tfrac{1}{2} + \tfrac{1.5}{2}\tfrac{0.5}{2}\tfrac{1}{2}\right)\mathbf{b}[2,3,4] \\
&+ \tfrac{0.5}{2}\tfrac{0.5}{2}\tfrac{1}{2}\mathbf{b}[3,4,4]
\end{aligned}.$$

$$\mathbf{b}[2.5^{<3>}] = \begin{bmatrix} \frac{1}{48} & \frac{23}{48} & \frac{15}{32} & \frac{1}{32} \end{bmatrix} \begin{bmatrix} \mathbf{b}[0,1,2] \\ \mathbf{b}[1,2,3] \\ \mathbf{b}[2,3,4] \\ \mathbf{b}[3,4,4] \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{b}[2.5^{<2>}, \vec{1}] = \;& \tfrac{0.5}{3}\tfrac{0.5}{2}(-1)\mathbf{b}[0,1,2] \\
&+ \left(\tfrac{2.5}{3}\tfrac{0.5}{2}(-1) + \tfrac{1.5}{3}\tfrac{1.5}{2}(-1) + \tfrac{1.5}{3}\tfrac{1.5}{2}(1)\right)\mathbf{b}[1,2,3] \\
&+ \left(\tfrac{1.5}{4-1}\tfrac{1.5}{2}(-1) + \tfrac{1.5}{3}\tfrac{1.5}{2}(1) + \tfrac{1.5}{2}\tfrac{0.5}{2}(1)\right)\mathbf{b}[2,3,4] \\
&+ \tfrac{0.5}{2}\tfrac{0.5}{2}(1)\mathbf{b}[3,4,4]
\end{aligned}.$$

$$\mathbf{b}[2.5^{<2>}, \vec{1}] = \begin{bmatrix} -\frac{1}{24} & -\frac{5}{24} & \frac{3}{16} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} \mathbf{b}[0,1,2] \\ \mathbf{b}[1,2,3] \\ \mathbf{b}[2,3,4] \\ \mathbf{b}[3,4,4] \end{bmatrix}.$$

So we have

$$[\mathbf{N}(u)]^T = \mathbf{b}[2.5^{<3>}] = \begin{bmatrix} \frac{1}{48} & \frac{23}{48} & \frac{15}{32} & \frac{1}{32} \end{bmatrix},$$

$$[\mathbf{N}'(u)]^T = 3\mathbf{b}[2.5^{<2>}, \vec{1}] = 3\begin{bmatrix} -\frac{1}{24} & -\frac{5}{25} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

For v direction, based on figure 6.21, we have

$$\mathbf{N}(v) = \begin{bmatrix} \frac{1}{32} & \frac{15}{32} & \frac{23}{48} & \frac{1}{48} \end{bmatrix}^T.$$

$$\mathbf{N}'(v) = 3\begin{bmatrix} -\frac{1}{24} & -\frac{5}{25} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}^T.$$

We can identify 4×4 control points $\widehat{\mathbf{P}}$ for defining the knot interval $[2,3] \times [1,2]$ from Fig. 6.19 and they are shown below

$$\widehat{\mathbf{P}} = \begin{bmatrix} (20,10,30) & (20,20,40) & (20,30,35) & (20,40,35) \\ (30,10,25) & (30,20,45) & (30,30,40) & (30,40,35) \\ (40,10,15) & (40,20,35) & (40,30,45) & (40,40,50) \\ (50,10,15) & (50,20,30) & (50,30,35) & (50,40,40) \end{bmatrix}$$

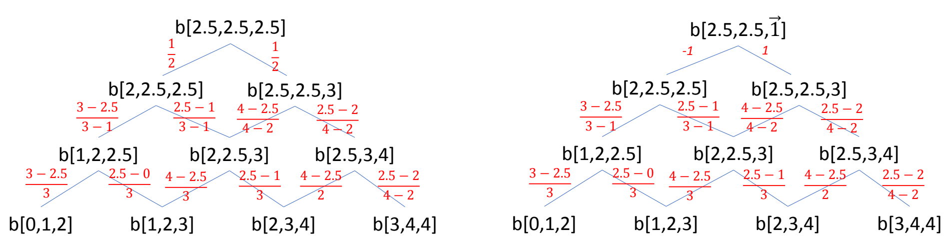
So we have

$$\mathbf{s}(2.5, 1.5) = (35.1042, 24.8958, 40.3),$$

$$\mathbf{s}_u(2.5, 1.5) = (10.625, 0, -2.5228),$$

$$\mathbf{s}_v(2.5, 1.5) = (0, 10.6250, 4.2318),$$

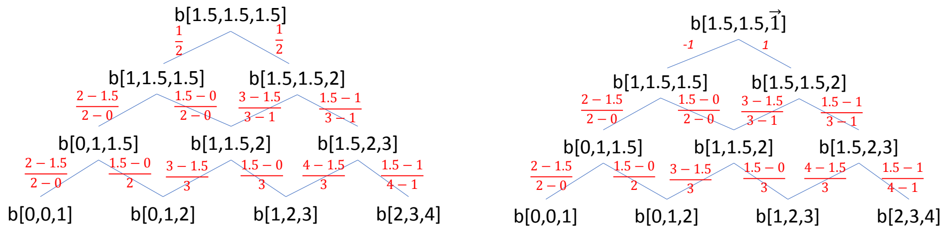
$$\mathbf{s}_{uv}(2.5, 1.5) = (0, 0, 9.0625).$$



(a) $\mathbf{b}[2.5, 2.5, 2.5]$ (b) $\mathbf{b}[2.5, 2.5, \vec{1}]$

Figure 6.20: Extracting B-spline functions and derivatives for u direction

Example 9. *Bézer extraction*



(a) $\mathbf{b}[1.5, 1.5, 1.5]$ (b) $\mathbf{b}[1.5, 1.5, \vec{1}]$

Figure 6.21: Extracting B-spline functions and derivatives for v direction

A B-spline surface of degree 3×2 with a control point list as follows

$$\begin{bmatrix} (50,0,10) & (65,0,10) & (65,25,10) & (50,25,10) \\ (40,0,20) & (55,0,20) & (55,25,20) & (40,25,20) \\ (40,0,45) & (55,0,45) & (55,25,45) & (40,25,45) \\ (50,0,55) & (65,0,55) & (65,25,55) & (50,25,55) \\ (50,0,70) & (65,0,70) & (65,25,70) & (50,25,70) \\ (34,0,105) & (49,0,105) & (49,25,105) & (34,25,105) \end{bmatrix}$$

where the v direction of points are listed first. The knot vector along u is $\{0,0,0,1,2,3,3,3\}$ and the knot vector along v is $\{0,0,1,2,2\}$. Develop Bézier representation over the parametric interval $[1,2] \times [0,1]$.

Solution

In u direction, knot interval $[1,2]$ corresponds to $[u_l, u_{l+1}]$ with $l = 3$ so the control points for this interval are $\mathbf{p}_{l-p+1}, \mathbf{p}_{l-p+2}, \cdots, \mathbf{p}_{l+1}$, i.e. $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. Their blossom labels include $\mathbf{b}[0,0,1], \mathbf{b}[0,1,2], \mathbf{b}[1,2,3]$ and $\mathbf{b}[2,3,3]$.

In v direction, knot interval $[0,1]$ corresponds to $[v_l, v_{l+1}]$ with $l = 1$. The control points for this interval are $\mathbf{p}_{l-p+1}, \mathbf{p}_{l-p+2}, \cdots, \mathbf{p}_{l+1}$, i.e. $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$. Their blossom labels are $\mathbf{b}[0,0], \mathbf{b}[0,1]$ and $\mathbf{b}[1,2]$.

So we have the following 4×3 control points for the B-spline surface over the interval,

$$\widehat{\mathbf{P}} = \begin{bmatrix} (40,0,20) & (55,0,20) & (55,25,20) \\ (40,0,45) & (55,0,45) & (55,25,45) \\ (50,0,55) & (65,0,55) & (65,25,55) \\ (50,0,70) & (65,0,70) & (65,25,70) \end{bmatrix}.$$

The Bézier extraction matrix in u direction, $\mathbf{M}^u$, is obtained through knot insertion as shown in Fig. 6.22 where

$$\begin{bmatrix} \mathbf{b}[1,1,1] \\ \mathbf{b}[1,1,2] \\ \mathbf{b}[1,2,2] \\ \mathbf{b}[2,2,2] \end{bmatrix} = \begin{bmatrix} 1/4 & 7/12 & 1/6 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 1/6 & 7/12 & 1/4 \end{bmatrix} \begin{bmatrix} \mathbf{b}[0,0,1] \\ \mathbf{b}[0,1,2] \\ \mathbf{b}[1,2,3] \\ \mathbf{b}[2,3,3] \end{bmatrix}.$$

$$\mathbf{M}^u = \begin{bmatrix} 1/4 & 7/12 & 1/6 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 1/6 & 7/12 & 1/4 \end{bmatrix}.$$

The Bézier extraction matrix in v direction, $\mathbf{M}^v$, is obtained through knot insertion as shown in Fig. 6.23. We have

$$\begin{bmatrix} \mathbf{b}[0,0] \\ \mathbf{b}[0,1] \\ \mathbf{b}[1,1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \mathbf{b}[0,0] \\ \mathbf{b}[0,1] \\ \mathbf{b}[1,2] \end{bmatrix}.$$

$$\mathbf{M}^v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Therefore, the Bézier control points for the interval are

$$\widetilde{\mathbf{P}} = \mathbf{M}^u \widehat{\mathbf{P}} [\mathbf{M}^v]^T = \begin{bmatrix} (41.6667, 0, 40.4167) & (56.6667, 0, 40.4167) & (56.6667, 12.5000, 40.4167) \\ (43.3333, 0, 48.3333) & (58.3333, 0, 48.3333) & (58.3333, 12.5000, 48.3333) \\ (46.6667, 0, 51.6667) & (61.6667, 0, 51.6667) & (61.6667, 12.5000, 51.6667) \\ (48.3333, 0, 57.0833) & (63.3333, 0, 57.0833) & (63.3333, 12.5000, 57.0833) \end{bmatrix}.$$

Figure 6.24 shows the B-spline surface and the extracted 4×3 control points for the Bézier patch over the knot interval $[1, 2] \times [0, 1]$.

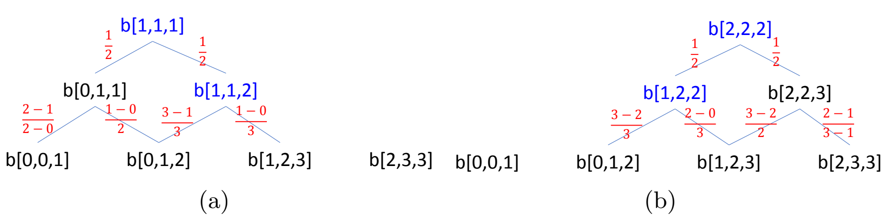


Figure 6.22: Extracting Bézier control points for u direction

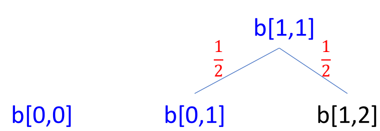


Figure 6.23: Extracting Bézier control points for v direction

6.6.2 NURBS surfaces

A NURBS surface is, strictly speaking, not a tensor-product surface.

$$\mathbf{s}(u, v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} \mathbf{p}_{i,j} N_{i,p}(u) N_{j,q}(v)}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} N_{i,p}(u) N_{j,q}(v)}, \quad (u, v) \in [u_{p-1}, u_m] \times [v_{q-1}, v_n]$$

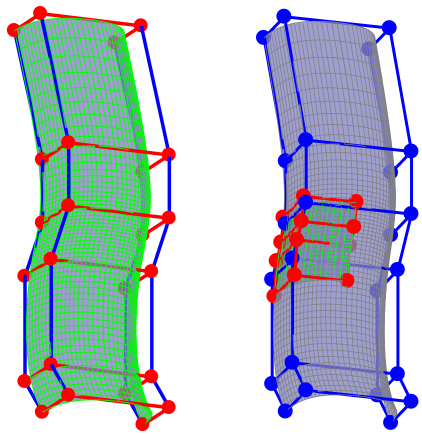


Figure 6.24: Bézier extraction

The basis can be represented as

$$R_{i,j}(u,v) = \frac{w_{i,j}N_{i,p}(u)N_{j,q}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}N_{i,p}(u)N_{j,q}(v)}, \quad (u,v) \in [u_{p-1}, u_m] \times [v_{q-1}, v_n]$$

It cannot be represented as a product of two univariate functions.

A NURBS surface in homogeneous coordinates (4D) $\mathbf{s}^w(u,v)$ can be represented as a tensor-product form.

$$\mathbf{s}^w(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{p}_{i,j}^w N_{i,p}(u)N_{j,q}(v).$$

Its tensor product form is

$$\mathbf{s}^w(u,v) = \begin{bmatrix} N_{0,p}(u) & N_{1,p}(u) & \cdots & N_{m,p}(u) \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0,0}^w & \mathbf{p}_{0,1}^w & \cdots & \mathbf{p}_{0,n}^w \\ \mathbf{p}_{1,0}^w & \mathbf{p}_{1,1}^w & \cdots & \mathbf{p}_{1,n}^w \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{p}_{m,0}^w & \mathbf{p}_{m,1}^w & \cdots & \mathbf{p}_{m,n}^w \end{bmatrix} \begin{bmatrix} N_{0,q}(v) \\ N_{1,q}(v) \\ \cdots \\ N_{n,q}(v) \end{bmatrix} \qquad (6.36)$$

Example 10. *NURBS surface evaluation (Piegl)*

There are 8×4 *control points* $\mathbf{p}_{i,j}, i = 0 \cdots 7, j = 0 \cdots 3$, *biquadratic NURBS surface with*

- *knots* $\{U\} = \{0, 0, 1, 2, 3, 4, 4, 5, 5\}$ *(no superfluous knots),*
- *knots* $\{V\} = \{0, 0, 1, 2, 3, 3\}$ *(no superfluous knots),*
- *control points* $[\mathbf{p}_{i,j}^w] = (x_{i,j}w_{i,j}, y_{i,j}w_{i,j}, z_{i,j}w_{i,j}, w_{i,j}) = \begin{bmatrix} (0,2,4,1) & (0,6,4,2) & (0,2,0,1) \\ (4,6,8,2) & (12,24,12,6) & (4,6,0,2) \\ (4,2,4,1) & (8,6,4,2) & (4,2,0,1) \end{bmatrix}$,

  $i = 2, 3, 4; j = 1, 2, 3$

Compute $\mathbf{s}(2.5, 1)$.

Solution:

- *Use blossom to find along u direction the following B-spline basis,* $N_{2,2}(2.5) = 1/8$*,* $N_{3,2}(2.5) = 6/8$*,* $N_{4,2}(2.5) = 1/8$*.*

  Linear interpolation wrt 2.5 twice among $\boldsymbol{b}[1,2], \boldsymbol{b}[2,3], \boldsymbol{b}[3,4]$

- *Use blossom to find along v direction,* $N_{1,2}(1) = 1/2$*,* $N_{2,2}(1) = 1/2$*,* $N_{3,2}(1) = 0$*.*

  Linear interpolation wrt 1 twice among $\boldsymbol{b}[0,1], \boldsymbol{b}[1,2], \boldsymbol{b}[2,3]$

Note, the indicies $i = 2, 3, 4$ *and* $j = 1, 2, 3$ *can be obtained by examining knot sequence.* $2.5 \in [u_3, u_4]$*, for this interval, we need blossom* $\boldsymbol{b}[u_2, u_3], \cdots \boldsymbol{b}[u_4, u_5]$*, corresponding to* $\mathbf{p}_2, \cdots, \mathbf{p}_4$*, i.e* $i = 2$ *to* 4*.*

- $\mathbf{s}^w(2.5, 1) = \begin{bmatrix} 1/8 & 6/8 & 1/8 \end{bmatrix} \begin{bmatrix} (0,2,4,1) & (0,6,4,2) & (0,2,0,1) \\ (4,6,8,2) & (12,24,12,6) & (4,6,0,2) \\ (4,2,4,1) & (8,6,4,2) & (4,2,0,1) \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \left(\frac{54}{8}, \frac{98}{8}, \frac{68}{8}, \frac{27}{8} \right)$

- *Projection yields* $\mathbf{s}(2.5, 1) = (x^w(2.5,1), y^w(2.5,1), z^w(2.5,1))/w(2.5,1) = (2, \frac{98}{27}, \frac{68}{27})$

6.6.3 Exercises

Ex. 6.7 — B-spline surface

Assuming a B-spline surface of degree 3×2 with control points as follows

```
0 0 0; 0 10 0; 0 20 5; 0 30 15; 0 40 10; 0 50 5; 0 60 0;
10,0,0; 10 10 10; 10 20 20; 10 30 20; 10 40 30; 10 50 15; 10 60 5;
20 0 0; 20 10 30; 20 20 40; 20 30 35; 20 40 35; 20 50 15; 20 60 10;
30 0 0; 30 10 25; 30 20 45; 30 30 40; 30 40 35; 30 50 25; 30 60 15;
40 0 0; 40 10 15; 40 20 35; 40 30 45; 40 40 50; 40 50 30; 40 60 20;
50 0 0; 50 10 15; 50 20 30; 50 30 35; 50 40 40; 50 50 25; 50 60 15;
60 0 0; 60 10 5; 60 20 15; 60 30 25; 60 40 20; 60 50 15; 60 60 5;
```

The knot vectors are $[0, 0, 0, 1, 2, 3, 4, 4, 4]$ in u direction and $[0, 0, 1, 2, 3, 4, 5, 5]$ in v direction.

a. Compute $\boldsymbol{S}(1.5, 2.5)$ and their partial derivatives $\frac{\partial \boldsymbol{S}}{\partial u}, \frac{\partial \boldsymbol{S}}{\partial v}$ and $\frac{\partial^2 \boldsymbol{S}}{\partial u \partial v}$.

Ex. 6.8 — Bézier extraction

For the B-spline surface in Ex. 6.7,

a. Develop Bézier representation of the patch over parametric domain $[1, 2] \times [2, 3]$.

Ex. 6.9 — NURBS surface

A NURBS surface has the same control points, knot vectors and degrees as the B-spline surface in Ex. 6.7. All control points have weight 1 except the control point $(30, 30, 40)$, which has weight 4.

a. Compute $\boldsymbol{S}(1.5, 2.5)$.

Ex. 6.10 — Bézier extraction

Using a CAD software to model a NURBS surface that has at least 2 knot intervals in u and 2 knot interval in v,

a. Extract Bézier representation for at least one knot interval.
Hint: CAD models can be accessed and queried via IGES file format.

b. Modify the Bézier control points from (a)

c. Import the modified Bézier patch back to the CAD system and compare the models before and after the change.