ME 535 Assignment 5 - Fall 2018

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Question 5.5: Plot the cubic B-spline curve defined with the control points. Evaluate it at u = 1.5

Ans:

The Knot vector is : {0 0 0 1 2 3 4 5 6 6 6}

Points are:

$P_0 = (10, 15, 20)$
$P_1 = (20, 25, 5)$
$P_2 = (40, 25, 0)$
$P_3 = (60, 5, 0)$
$P_4 = (80, 15, -5)$
$P_5 = (80, 30, -10)$
$P_6 = (90, 45, -10)$
$P_7 = (115, 40, -5)$
$P_8 = (125, 15, 0)$

```
%define control points
%P=[0 0; 1 2; 3 5; 5 0; 7 -1];
%define order
%define knot vector
%knots = [0 0 0 0 0.5 1 1 1 1];
%u=0.2;
%L=4; % u=0.2 is between u4=0 and u5=0.5

% P=[0 0;1 0;2 0;4 1;5 2;8 2;9 3];
% order = 4;
% knots=[0 0 0 0 .25 .5 .75 1 1 1 1];
% u=0.6;
% L= 6;

P=[10,15,20; 20,25,5; 40,25,0; 60,5,0; 80,15,-5;
    80,30,-10; 90,45,-10; 115,40,-5; 125,15,0];
order = 4;
knots = [0 0 0 1 2 3 4 5 6 6 6];
u = 1.5
```

```
u = 1.5000
```

```
%L = 5;
L = findspan(size(P,1),order-1,u,knots)
```

```
L = 4
```

```
disp("Value of the curve at u=1.5")
```

```
Value of the curve at u=1.5
```

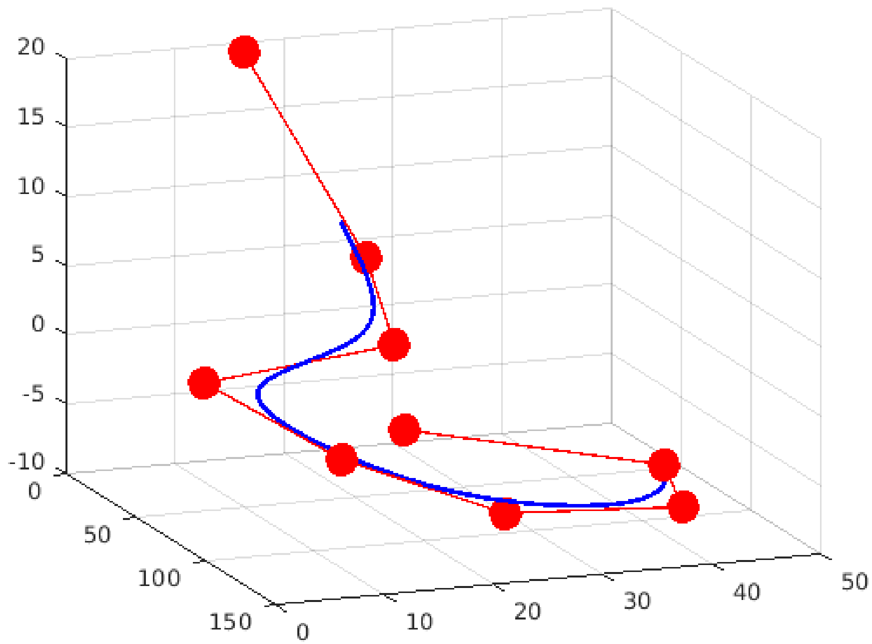
```
Q = deBoor (order, knots, P, u, L)
```

```
Q =
   30.1042   24.2708    2.9688
```

```
%define display configuration
```

```
n = 40;
%do calculation
Q = bsplineCurve(P, order, knots, n);
%do plot of control polygon
plot3(P(:,1),P(:,2),P(:,3),'r-o', 'linewidth',1,'MarkerFaceColor', 'r', 'MarkerSize',14);
hold on;
%do plot of b-spline curve
plot3(Q(:,1),Q(:,2),Q(:,3),'-b', 'linewidth',2);
grid on
hold off

view([68.9 18.8])
```



Question 5.6: Blossom for B Splines

```
P=[0, 0; 1, 0; 1, 1; 0, 1; 0, 2; 2, 2];
order = 4;
knots = [-2 -2 -1 0 2 4 5 6 6 6];
u = 3
```

```
u = 3
```

```
%L = 5;
L = findspan(size(P,1),order-1,u,knots)
```

```
L = 5
```

```
Q = deBoor(order, knots, P, u, L)
```

```
Q =
    0.4083    1.0167
```

```
%define display configuration
n = 40;
%do calculation
Q = bsplineCurve(P, order, knots, n);
```

```
%do plot of control polygon
plot(P(:,1),P(:,2),'r-o', 'linewidth',1,'MarkerFaceColor', 'r', 'MarkerSize',14);
hold on;
%do plot of b-spline curve
plot(Q(:,1),Q(:,2),'-b', 'linewidth',2);
hold on
```

Question 5.6(d): Bezier extraction and original curve

Obtain the Bézier representation of the curve segment for knot interval [2, 4]. Draw the original curve and the new Bézier curve and their control points on the same plot.

(Extracted Bezier shown as green segment**)**

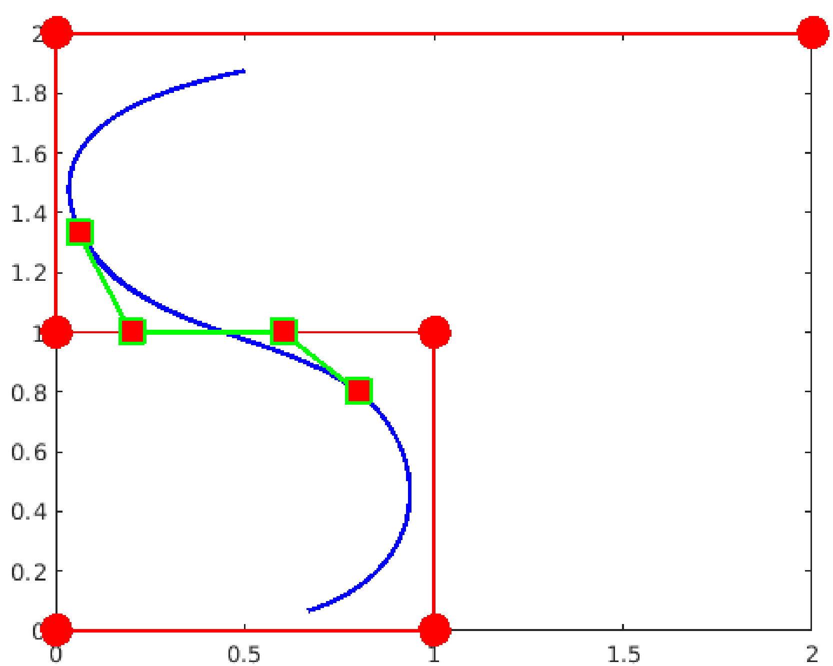
```
n = 100;
disp("Control Points of Extracted Bezier")
```

```
Control Points of Extracted Bezier
```

```
P1 = [0.8 0.8;0.6 1;0.2 1;0.06 1.33]
```

```
P1 =
    0.8000    0.8000
    0.6000    1.0000
    0.2000    1.0000
    0.0600    1.3300
```

```
Q = bezierCurve(P1, n);
bezierCurvePlot(P1, Q, '-gs','b');
```



Question 5.7: NURBS curve evaluation

Value of the curve at c(1.25) = (-0.36809,0.9297), Detailed solution attached at the end of this script.

```
clear all
close all
P=[1, 0; 1, 1; 0, 1; -1, 1; -1, 0];
P_w = [1, 0, 1; 0.7071, 0.7071, 0.7071; 0, 1, 1; -0.7071, 0.7071, 0.7071; -1, 0, 1];
```

```
order = 3;
knots = [0 0 0 1 1 2 2 2];
u = 1.25
```

```
u = 1.2500
```

```
%L = 5;
L_w = findspan(size(P_w,1),order-1,u,knots)
```

```
L_w = 5
```

```

Q_w = deBoor(order, knots, P_w, u, L_w)
```

```
Q_w =
   -0.3277    0.8277    0.8902
```

```
disp('The value of the curve at c=1.25 is')
```

```
The value of the curve at c=1.25 is
```

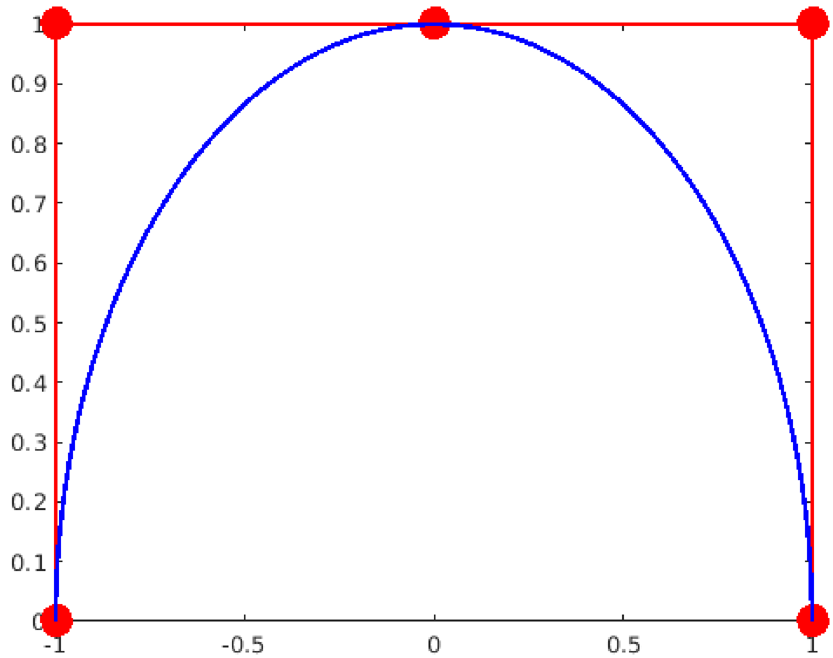
```
Q = Q_w(:,:)/Q_w(:,3)
```

```
Q =
   -0.3681    0.9298    1.0000
```

```

n = 40;
%do calculation
Qw = bsplineCurve(P_w, order, knots, n);
p =size(Qw,1);

Q = zeros(p,2);
for i=1:p
    Q(i,1) = Qw(i,1)/Qw(i,3);
    Q(i,2) = Qw(i,2)/Qw(i,3);
end
%do plot of control polygon
plot(P(:,1),P(:,2),'r-o', 'linewidth',1,'MarkerFaceColor', 'r', 'MarkerSize',14);
hold on;
%do plot of b-spline curve
plot(Q(:,1),Q(:,2),'-b', 'linewidth',2);
hold off
```



Additional functions implemented:

deBoor Algorithm

```
function [R] = deBoor(k, t, P, u, L)
%k order of b-spline
%t knot vector
%P control points with format that every row is a point
%u parameter value
%L  index of knot such as t(L) <= u < t(L+1)

for j=1:k
    A(j,:) = P(L-k+j,:);  %the control points that affect the computation of point on curve
end

for r=1:(k-1)   %time to do recursive compuation
    for j=(k):(-1):(r+1)   %do one time computation to get next level control points
        i = L-k+j;
        d1 = u - t(i);       %for left term in recursive format
        d2 = t(i+k-r) - u;   %for right term in recursive format
        A(j,:) = (d1*A(j,:) + d2*A(j-1,:))/(d1 + d2);  %carry out computation
    end
end
R = A(k,:);  %return the computed point value
end
```

finding Knot Span

```
%%% =================== find knot span =================
% U: knots
% n: number of CP minus 1; that is, p0, p1, ..., p_n
% p: degree
% u: u value
% return the span, starting from u_0.
% Date: Oct 14, 2018
```

```
function s = findspan(n,p,u,U)
if u < U(p) || u > U(n+p-1)
    print "error in u value wrt knots"
    u
    U
    return;
end
if (u==U(n+p-1))  % XQ
    s=n;
    return,
end
low = p;
high = n + 1;
mid = floor((low + high) / 2);
while (u < U(mid+1) || u >= U(mid+2))
    if (u < U(mid+1))
        high = mid;
    else
        low = mid;
    end
    mid = floor((low + high) / 2);
end
s = mid;

s= s+1; % XQ. For Matlab, we should add one to the return value.
end
```

bSplineCurve

```
function [Q] = bsplineCurve(P, k, t, n)
%P control points of b-spline
%k order of b-spline
%t knot vector of b-spline
%n for display, namely how many points to be computed on every segment

[m,d] = size(P);   %get number of control points m
L = 1;             %index to computed point

for i=(k):(m)       %b-splinbe parameter domain  is t(k) - t(m+1)
    step = (t(i+1)-t(i))/(n-1);   %parameter increment step
    for u=t(i):step:(t(i+1))      %do calculation for every segment
        Q(L,:) = deBoor(k, t, P, u, i); %P, degree, u, knots, i, 0);
        L = L+1;
    end
end
end
```

Functions for BSpline

```
function bezierCurvePlot(P, Q, sP, sQ)
% plot the given control points P and points Q on beizer curve
% P control points
% Q points on bezier curve
% style for P and style for Q: sP, sQ

dim = length(P(1,:));
if dim == 2;
    plot(Q(:,1),Q(:,2),sQ, 'linewidth',2);
    hold on;
    plot(P(:,1),P(:,2),sP, 'linewidth',2,'MarkerFaceColor', 'r', 'MarkerSize',14);
    %hold off;
else
```

```
    if dim == 3
        plot(P(:,1),P(:,2),sP, 'linewidth',2);
        hold on;
        plot(Q(:,1),Q(:,2),sQ, 'linewidth',2);
        %hold off;
    end
end
end
function [Q] = deCasteljau(P, u)
% computer point with parameter value u on bezier curve defined by control points P
% P control points, in matrix format: {size of P} * {dimension of P}
% dimension of P is 2 or 3
% u parameter with value [0 1]
% Q point lying on the bezier curve

% input: P: control points P; u:  parameter value
% output: Q: the Bezier curve point at u

% example: >> P = [0 0; 1 2; 3 5; 4,4; 5 0];
% >> Q=deCasteljau(P, 0.5)
% output:Q =
%   2.6875    3.3750

% m: # of control points; m = the degree of the curve +1
[m, n] = size(P);
if m <= 1
    err('please specify at least 2 control points');
end

if u < 0 | u > 1
    err('u must be in range from 0 to 1');
end

d = m-1; % degree

for r=1:d
    for i=1:(d+1-r) % the array index in Matlab starts with 1, not 0.
        P(i,:) = (1-u)*P(i,:) + u*P(i+1,:);
        % ':' operator on all columns: x, y, z
    end
end
Q=P(1,:);
end
function [Q] = bezierCurve(P, n)
% computer points on bezier curve defined by control points P
% P control points, in matrix format: {size of P} * {dimension of P}
% n the number of points need to be computed
% isPlot  1 for plot, 0 for non plot
% Q points lying on the bezier curve
i = 1;
for u=0:(1/(n-1)):1
    Q(i,:) = deCasteljau(P, u);
    i = i + 1;
end
end
```

EXERCISE 5.6

KNOT : $\{-2, -1, 0, 2, 4, 5, 6, 6\}$.

Control points :
Written in Blossom labels :

$P_0 = b[-2, -1, 0] = (0, 0)$

$P_1 = b[-1, 0, 2] = (1, 0)$

$P_2 = b[0, 2, 4] = (1, 1)$

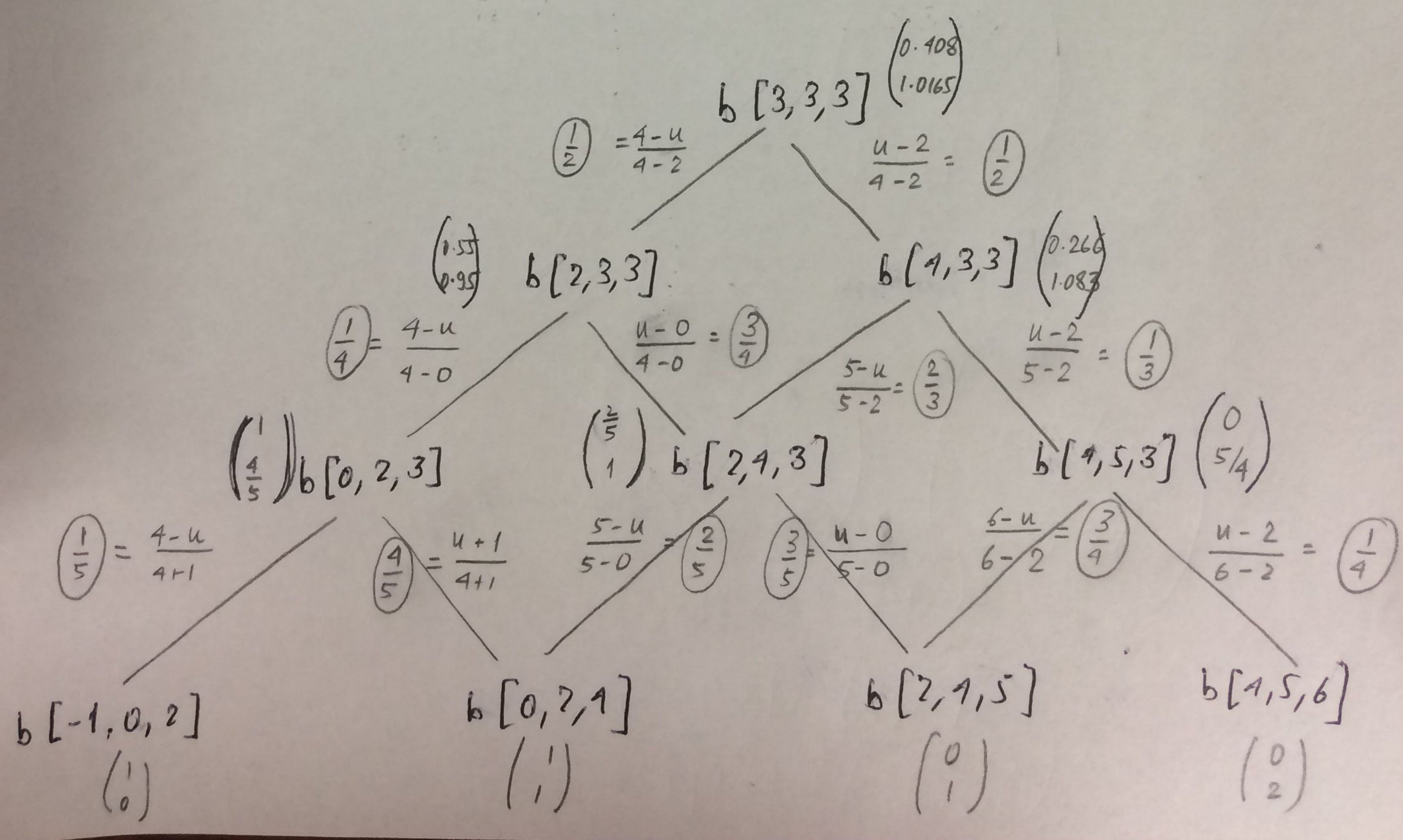
$P_3 = b[2, 4, 5] = (0, 1)$

$P_4 = b[4, 5, 6] = (0, 2)$

$P_5 = b[5, 6, 6] = (2, 2)$

Using deBoor pyramid algorithm to calculate value at $u = 3$, we have: $u \in [u_3, u_4]$ $\in [2, 4]$

For $u = 3$:



$$b[0,2,3] = \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{4}{5}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\ \frac{4}{5}\end{pmatrix}$$

$$b[2,4,3] = \frac{2}{5}\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{5}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{2}{5}\\ 1\end{pmatrix}$$

$$b[4,5,3] = \frac{3}{4}\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\ \frac{5}{4}\end{pmatrix}$$

$$b[2,3,3] = \frac{1}{4}\begin{pmatrix}1\\4/5\end{pmatrix} + \frac{3}{4}\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} = \begin{pmatrix}0.55\\0.95\end{pmatrix}$$

$$b[4,3,3] = \frac{2}{3}\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\5/4\end{pmatrix} = \begin{pmatrix}0.266\\1.083\end{pmatrix}$$

$$b[3,3,3] = \frac{1}{2}\begin{pmatrix}0.55\\0.95\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0.266\\1.083\end{pmatrix} = \begin{pmatrix}0.408\\1.0165\end{pmatrix}$$

$$\therefore c(3) = (0.408,\ 1.0165).$$

b) The derivative. @ $u = 3$.

$$c'(u) = db[u^{\langle d-1\rangle}, \vec{1}]$$

$$= 3\left(\frac{b[3,3,4] - b[3,3,2]}{4-2}\right)$$

$$= \frac{3}{2}\left(\begin{pmatrix}0.266\\1.083\end{pmatrix} - \begin{pmatrix}0.55\\0.95\end{pmatrix}\right) = \frac{3}{2}\begin{pmatrix}-0.284\\0.133\end{pmatrix} = \begin{pmatrix}-0.426\\0.1995\end{pmatrix}$$

$$\therefore c'(3) = \begin{pmatrix}-0.426\\0.1995\end{pmatrix}$$

(c) Polynomial form of the curve segment for knot interval [2,4]

$b[-1,0,2]$

$b[0,2,4]$

$b[2,4,5]$

$b[4,5,6]$

$$
\begin{array}{ccccc}
b[0,2,u] & \xrightarrow{\frac{4-u}{4}} & & & \\
 & & b[2,u,u] & \xrightarrow{\frac{4-u}{2}} & \\
b[2,4,u] & \xrightarrow{\frac{u}{4}} \quad \xrightarrow{\frac{5-u}{3}} & & & b[u,u,u] \\
 & & b[4,u,u] & \xrightarrow{\frac{u-2}{2}} & \\
b[4,5,u] & \xrightarrow{\frac{u-2}{3}} & & &
\end{array}
$$

$$b[0,2,u] = \begin{pmatrix}1\\0\end{pmatrix}\left(\frac{4-u}{5}\right) + \begin{pmatrix}1\\1\end{pmatrix}\left(\frac{u+1}{5}\right)$$

$$b[2,4,u] = \begin{pmatrix}1\\1\end{pmatrix}\left(\frac{5-u}{5}\right) + \begin{pmatrix}0\\1\end{pmatrix}\left(\frac{u}{5}\right)$$

$$b[4,5,u] = \begin{pmatrix}0\\1\end{pmatrix}\left(\frac{6-u}{4}\right) + \begin{pmatrix}0\\2\end{pmatrix}\left(\frac{u-2}{4}\right)$$

$$b[2,u,u] = \frac{4-u}{4}\left[\left(\frac{4-u}{5}\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(\frac{u+1}{5}\right)\begin{pmatrix}1\\1\end{pmatrix}\right] + \frac{u}{4}\left[\left(\frac{5-u}{5}\right)\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{u}{5}\right)\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$b[4,u,u] = \frac{5-u}{3}\left[\left(\frac{5-u}{5}\right)\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{u}{5}\right)\begin{pmatrix}0\\1\end{pmatrix}\right] + \frac{u-2}{3}\left[\left(\frac{6-u}{4}\right)\begin{pmatrix}0\\1\end{pmatrix} + \left(\frac{u-2}{4}\right)\begin{pmatrix}0\\2\end{pmatrix}\right]$$

$$c(u)\ \cancel{b[3,3,3]} = \left(\frac{4-u}{2}\right)\left(\left(\frac{4-u}{4}\right)\left[\left(\frac{4-u}{5}\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(\frac{u+1}{5}\right)\begin{pmatrix}1\\1\end{pmatrix}\right] + \frac{u}{4}\left[\left(\frac{5-u}{5}\right)\begin{pmatrix}1\\1\end{pmatrix} + \frac{u}{5}\begin{pmatrix}0\\1\end{pmatrix}\right]\right)$$

$$+ \left(\frac{u-2}{2}\right)\left(\left(\frac{5-u}{3}\right)\left[\left(\frac{5-u}{5}\right)\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{u}{5}\right)\begin{pmatrix}0\\1\end{pmatrix}\right] + \left(\frac{u-2}{3}\right)\left[\left(\frac{6-u}{4}\right)\begin{pmatrix}0\\1\end{pmatrix} + \left(\frac{u-2}{4}\right)\begin{pmatrix}0\\2\end{pmatrix}\right]\right)$$

$$\cancel{b[3,3,3]}^{c(u)} = \left(\frac{(4-u)(4-u)(4-u)}{40}\right)\begin{pmatrix}1\\0\end{pmatrix} +$$

$$\left(\frac{4-u}{2}\right)\left(\left(\frac{4-u}{4}\right)\left(\frac{u+1}{5}\right) + \left(\frac{u}{4}\right)\left(\frac{5-u}{5}\right)\right) + \left(\frac{u-2}{2}\right)\left(\left(\frac{5-u}{3}\right)\left(\frac{5-u}{5}\right)\right)\begin{pmatrix}1\\1\end{pmatrix} +$$

$$\left(\frac{4-u}{2}\right)\left(\frac{u}{4}\right)\left(\frac{u}{5}\right) + \left(\frac{u-2}{2}\right)\left(\frac{5-u}{3}\right)\left(\frac{u}{5}\right) + \frac{(u-2)(u-2)(6-u)}{24}\begin{pmatrix}0\\1\end{pmatrix} +$$

$$\left(\frac{u-2}{2}\right)\left(\frac{u-2}{3}\right)\left(\frac{u-2}{4}\right)\begin{pmatrix}0\\2\end{pmatrix}$$

$$\cancel{b[3,3,3]}^{c(u)} = \left(\frac{(4-u)^3}{40}\right)\begin{pmatrix}1\\0\end{pmatrix} +$$

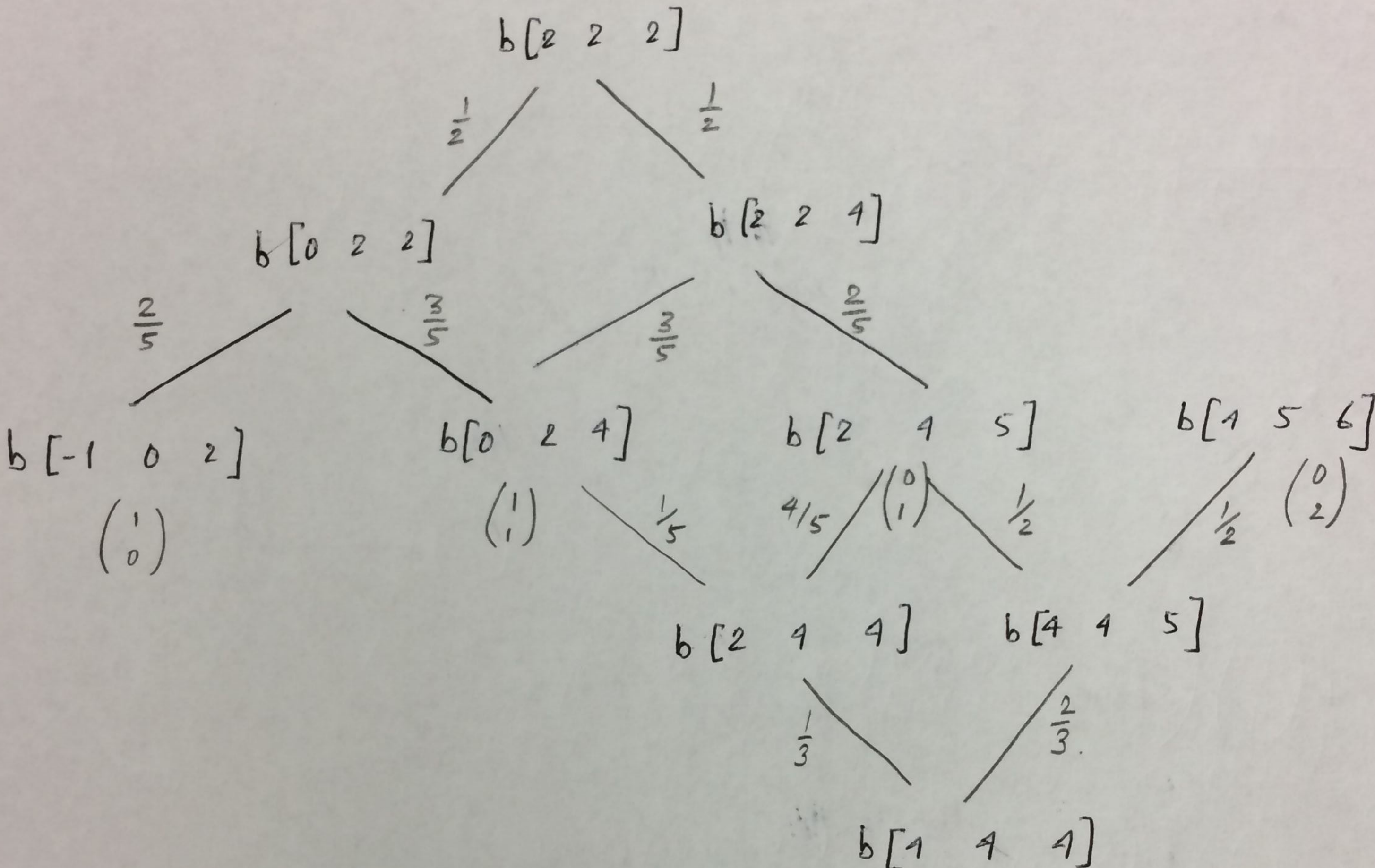
$$\left(\frac{(4-u)^2(u+1)}{40} + \left(\frac{(4-u)(5u-u^2)}{20}\right) + \frac{(u-2)(5-u)^2}{30}\right)\begin{pmatrix}1\\1\end{pmatrix} +$$

$$\left(\frac{(4-u)u^2}{40} + \frac{(u-2)(5u-u^2)}{30} + \frac{(u-2)^2(6-u)}{24}\right)\begin{pmatrix}0\\1\end{pmatrix} +$$

$$\left(\frac{(u-2)^3}{24}\right)\begin{pmatrix}0\\2\end{pmatrix}$$

(d.) For the knot interval [2,4] the Bezier extraction looks like:

Following de Boor's algorithm:



$$b[0\ 2\ 2] = \frac{2}{5}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\frac{3}{5} = \begin{pmatrix}1\\3/5\end{pmatrix}$$

$$\bullet\ b[2\ 2\ 4] = \frac{3}{5}\begin{pmatrix}1\\1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3/5\\1\end{pmatrix}$$

$$\bullet\ b[2\ 4\ 4] = \frac{1}{5}\begin{pmatrix}1\\1\end{pmatrix} + \frac{4}{5}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/5\\1\end{pmatrix}$$

$$b[4\ 4\ 5] = \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\3/2\end{pmatrix}$$

$$\bullet\ b[2\ 2\ 2] = \frac{1}{2}\begin{pmatrix}1 + \frac{3}{5}\\ \frac{3}{5} + 1\end{pmatrix} = \begin{pmatrix}4/5\\4/5\end{pmatrix}$$

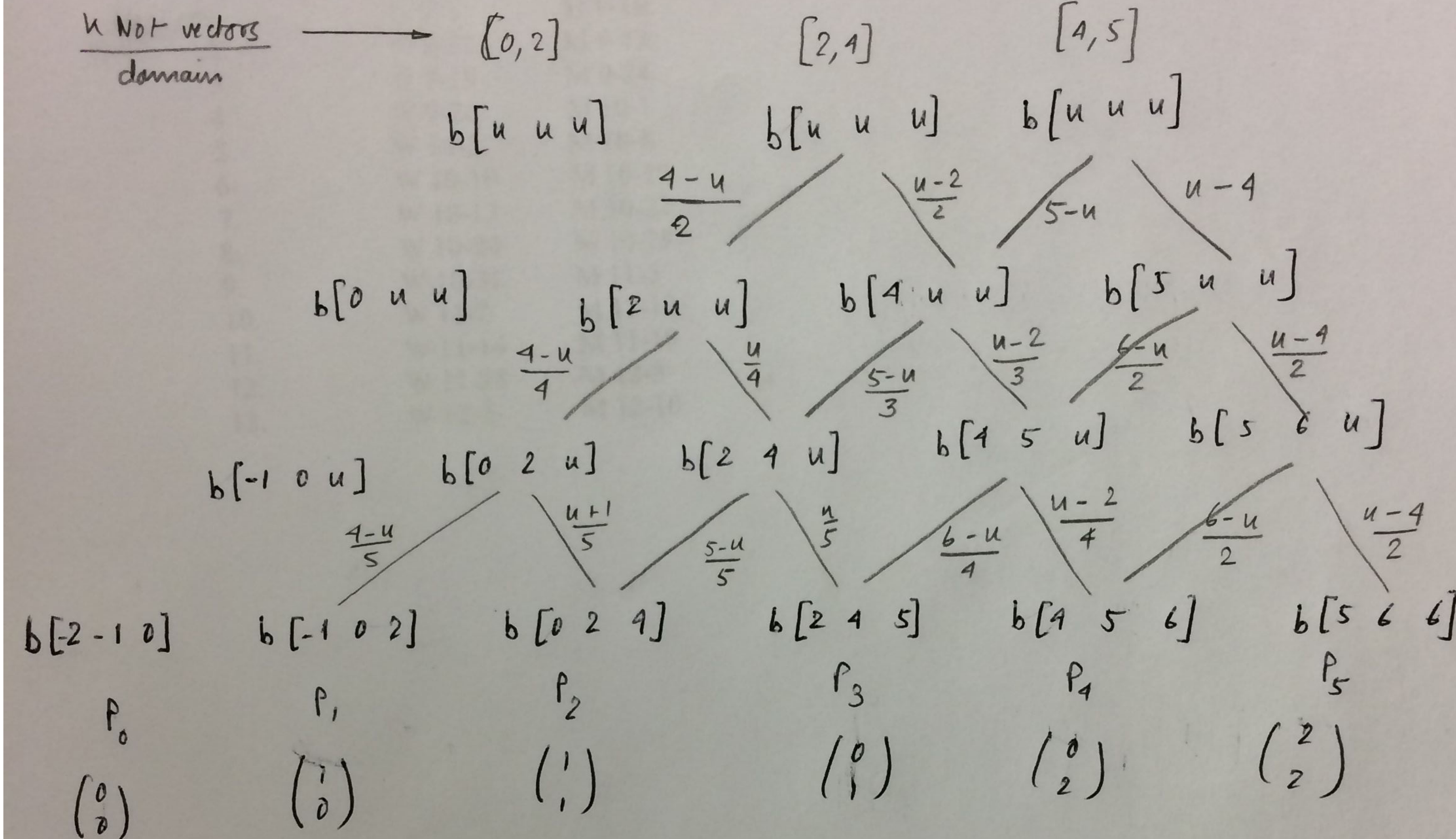
$$\bullet\ b[4\ 4\ 4] = \frac{1}{3}\begin{pmatrix}1/5\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\3/2\end{pmatrix} = \begin{pmatrix}1/15\\4/3\end{pmatrix}.$$

The control points for the extracted Bezier curve is:

$$b[2\ 2\ 2] = \begin{pmatrix}4/5\\4/5\end{pmatrix};\ b[2\ 2\ 4] = \begin{pmatrix}3/5\\1\end{pmatrix};\ b[2\ 4\ 4] = \begin{pmatrix}1/5\\1\end{pmatrix};\ b[4\ 4\ 4] = \begin{pmatrix}1/15\\4/3\end{pmatrix}$$

5.6

(e)



For control point $\begin{pmatrix}0\\2\end{pmatrix}$

* For domain $[2,4]$: piecewise polynomial: $\left(\frac{u-2}{2}\right)\left(\frac{u-2}{3}\right)\left(\frac{u-2}{4}\right) = \boxed{\frac{(u-2)^3}{24}}$

* For domain $[4,5]$: " " :

$$= \frac{u-2}{4}\left(\left(\frac{u-2}{3}\right)(5-u) + \left(\frac{6-u}{2}\right)(u-4)\right) + \left(\frac{6-u}{2}\right)\left(\left(\frac{u-4}{2}\right)(u-4)\right)$$

$$= \boxed{\frac{(u-2)^2(5-u)}{12} + \frac{(u-2)(6-u)(u-4)}{8} + \frac{(6-u)(u-4)^2}{4}}$$

Exercise 5.7 – NURBS CURVE Evaluation.

Knots : $\{0, 0, 0, 1, 1, 2, 2, 2\}$.

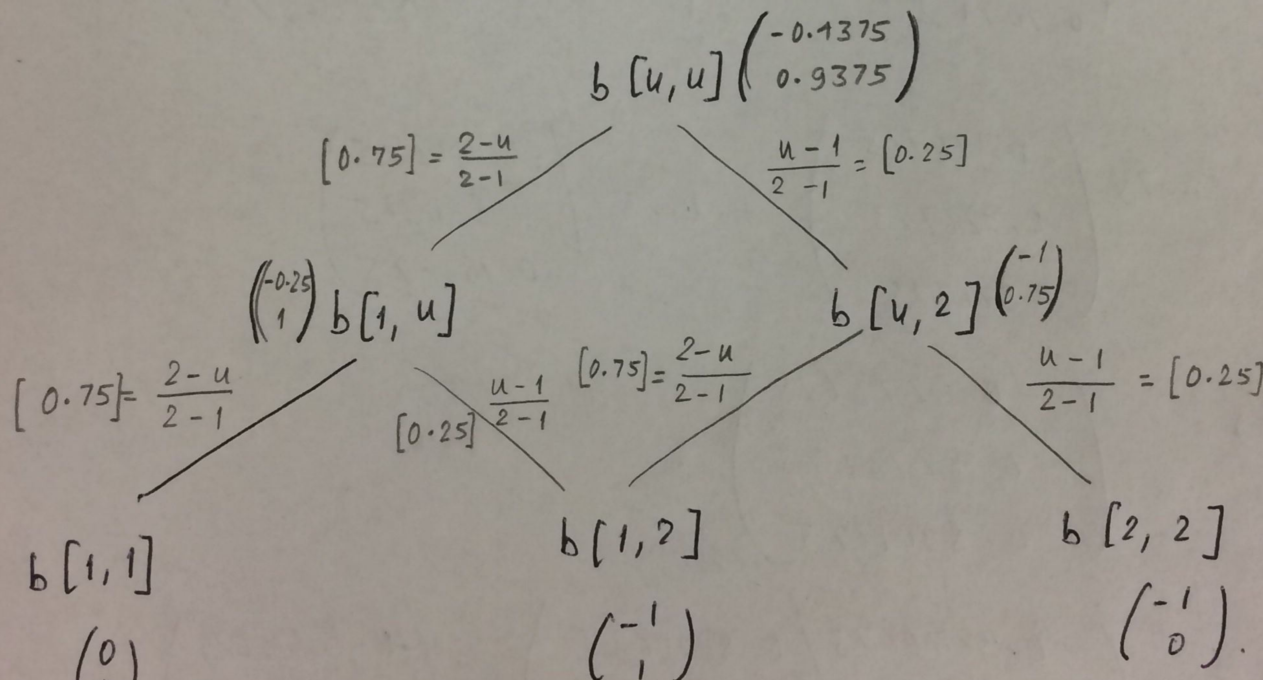
Ignoring superfluous knots we have : $\{0, 0, 1, 1, 2, 2\}$

			Weights
$b[0,0]$	$=$	$P_0(1,0)$	: 1
$b[0,1]$	$=$	$P_1(1,1)$	: 0.7071
$b[1,1]$	$=$	$P_2(0,1)$	: 1
$b[1,2]$	$=$	$P_3(-1,1)$	: 0.7071
$b[2,2]$	$=$	$P_4(-1,0)$	: 1

Required $u = 1.25$

$u \in [u_3, u_4] = [1, 2]$

The blossom pyramid looks like:
(for $u = 1.25$)



$$b[1,u] = 0.75\ b[1,1] + 0.25\ b[1,2] = \begin{pmatrix} -0.25 \\ 1 \end{pmatrix}$$

$$b[u,2] = 0.75\ b[1,2] + 0.25\ b[2,2] = \begin{pmatrix} -1 \\ 0.75 \end{pmatrix}$$

$$b[u,u] = 0.75\ b[1,u] + 0.25\ b[u,2] = \begin{pmatrix} -0.4375 \\ 0.9375 \end{pmatrix}$$

Multiplying by weights and evaluating (expressiong) the curve :

$P_i^w = (x_i w_i,\ y_i w_i,\ w_i)$

$P_2^w = (0,\ 1,\ 1)$ [Modified control points]

$P_3^w = (-0.7071, 0.7071, 0.7071)$ [Modified " "]

$P_4^w = (-1\ ,\ 0,\ 1)$

$$b^w[1,u] = 0.75 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0.25 \begin{pmatrix} -0.7071 \\ 0.7071 \\ 0.7071 \end{pmatrix} = \begin{pmatrix} -0.176775 \\ 0.926775 \\ 0.926775 \end{pmatrix}$$

$$b^w[u,2] = 0.75 \begin{pmatrix} -0.7071 \\ 0.7071 \\ 0.7071 \end{pmatrix} + 0.25 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.780325 \\ 0.530325 \\ 0.780325 \end{pmatrix}$$

$$b^w[u,u] = 0.75 \begin{pmatrix} -0.176775 \\ 0.926775 \\ 0.926775 \end{pmatrix} + 0.25 \begin{pmatrix} -0.780325 \\ 0.530325 \\ 0.780325 \end{pmatrix}$$

$$C^w(1.25) = \begin{pmatrix} -0.3276625 \\ 0.8276625 \\ 0.8901625 \end{pmatrix};$$

$$C(1.25) = \begin{pmatrix} -0.3276625/0.8901625 \\ 0.8276625/0.8901625 \end{pmatrix} = \begin{pmatrix} -0.36809 \\ 0.9297 \end{pmatrix}$$