

A diffuse interface framework for modelling the evolution of multi-cell aggregates as a soft packing problem driven by the growth and division of cells

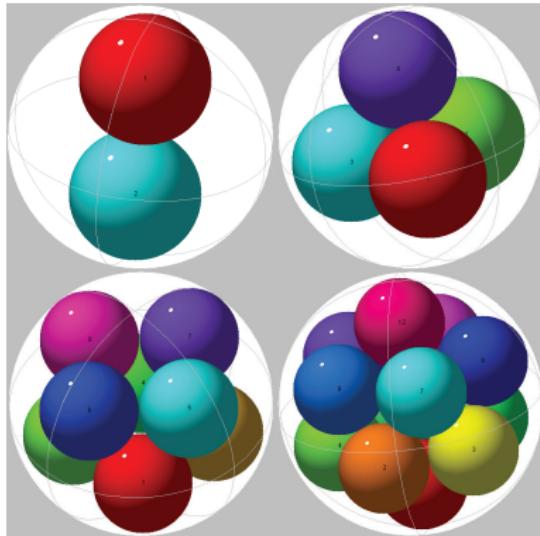
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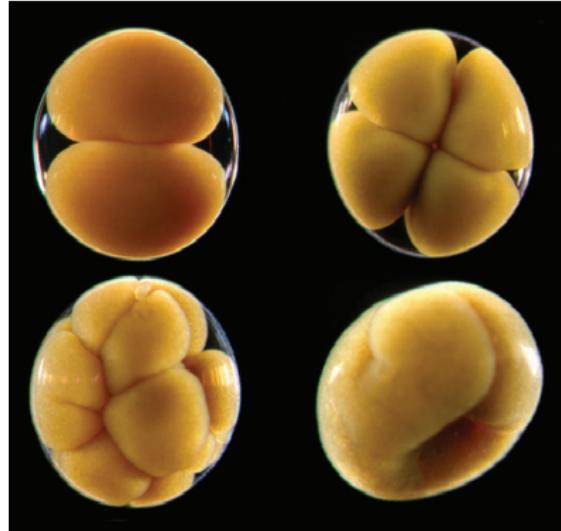
² University of Michigan Ann Arbor

Midwest Mechanics Graduate Symposium 2019
UW Madison, March 10, 2019

Soft packing of cells in cellular aggregates



Hard Packing of Spheres¹



Soft Packing of Cells²

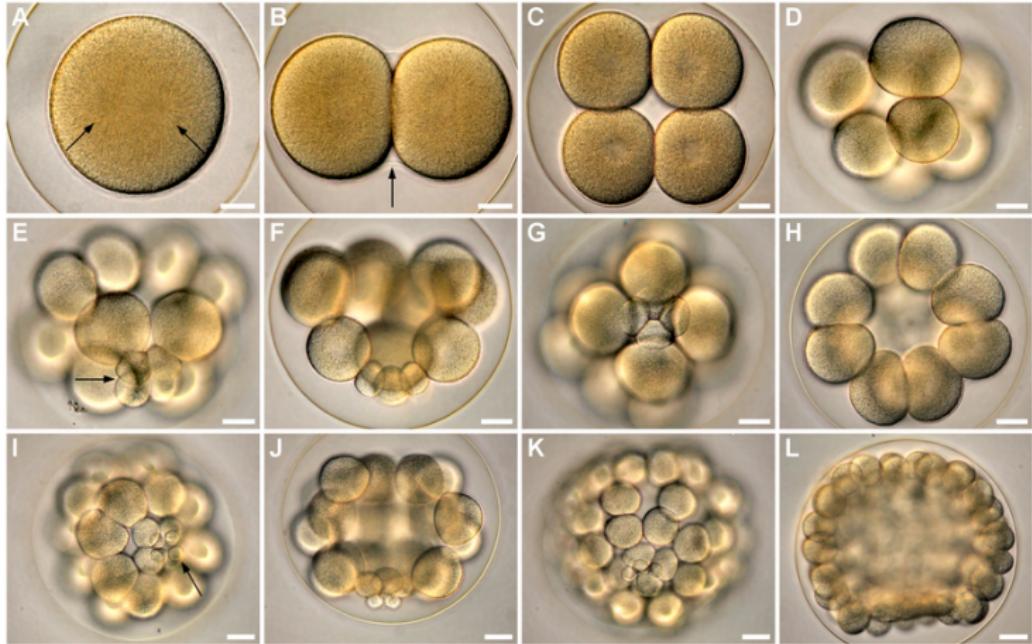
¹ <https://commons.wikimedia.org/w/index.php?curid=29251495>

² Embryo of *Echinaster brasiliensis* (A. E Migotto, Universidade de Sao Paulo)
<https://www.cell.com/pictureshow/embryogenesis>

Soft packing problem: Overview

- ▶ Motivation
 - ▶ Embroyogenesis
 - ▶ Tumor growth
- ▶ Relevant numerical models
 - ▶ Lattice (Cellular automata) and Off-Lattice (Vertex and cell based) models.
- ▶ Phase field formulation of soft packing
- ▶ Mechanics of soft packing
- ▶ Results
- ▶ Highlight: Scutoids
- ▶ Summary and Look Ahead

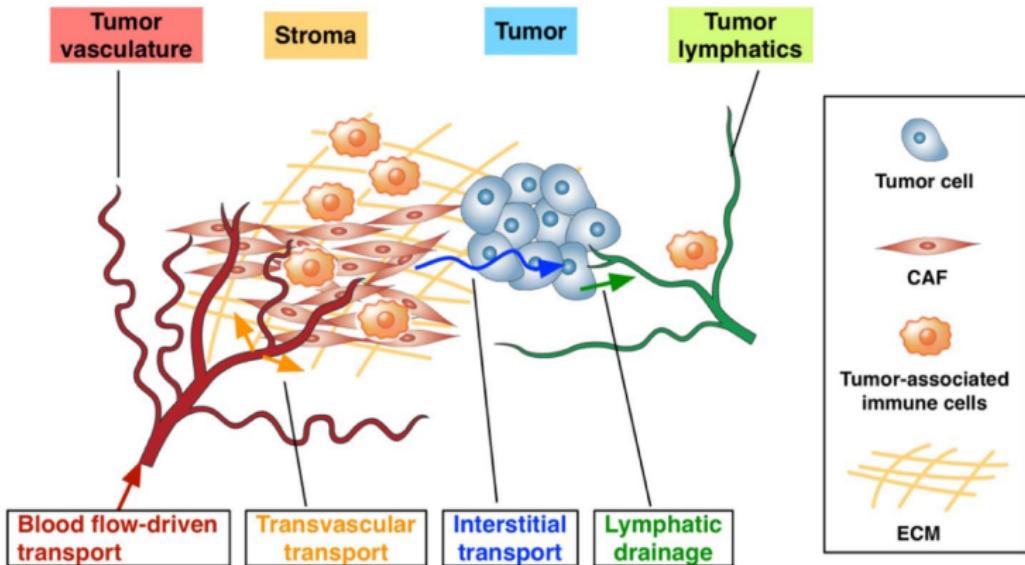
Motivation: Embryogenesis



Early cleavages of *C. subdepressus* under light microscopy [Reference: B. C. Vellutini and A. E. Migotto, PLOS One, 2010]

Embryogenesis in *C. subdepressus*

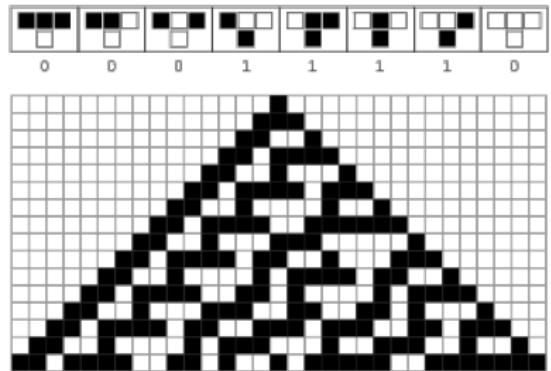
Motivation: Tumor growth



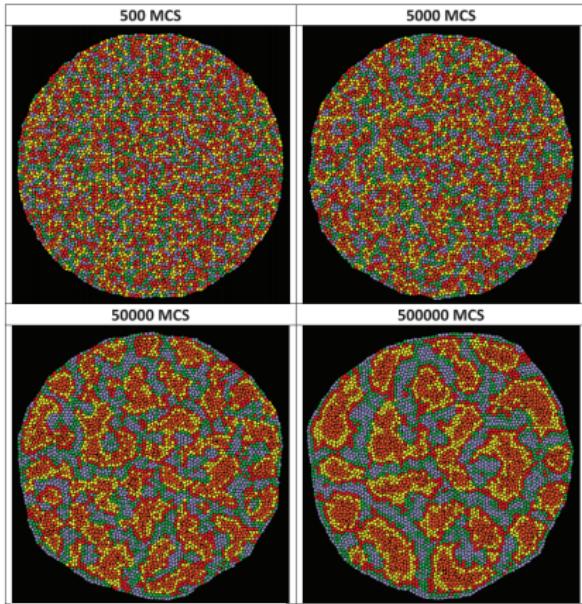
Complexity of the tumor microenvironment [Reference: Bumsoo Han et al., Cancer Letters, Vol. 380: 1, 2016]

Cell packing in growing tumors [Reference: Kristen Mills Lab, RPI]

Relevant numerical models: On-Lattice (Cellular automata / High-Q Potts) models



Cellular automata rules¹

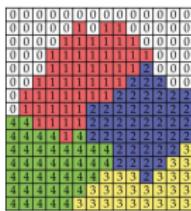


Clustering dynamics using CA models²

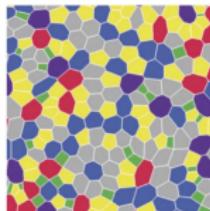
¹ <http://mathworld.wolfram.com/CellularAutomaton.html>

² Y. Zhang et al., PLoS ONE 6(10): e24999. doi:10.1371/journal.pone.0024999, 2011

Relevant numerical models: Off-lattice (Vertex and Cell based) models



Comparison of lattice based and off-lattice models¹



A hexagonal lattice structure composed of black hexagons. The boundaries between the hexagons are highlighted with a combination of red and blue lines, creating a pattern of alternating colored segments.

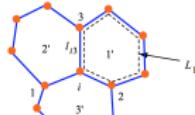
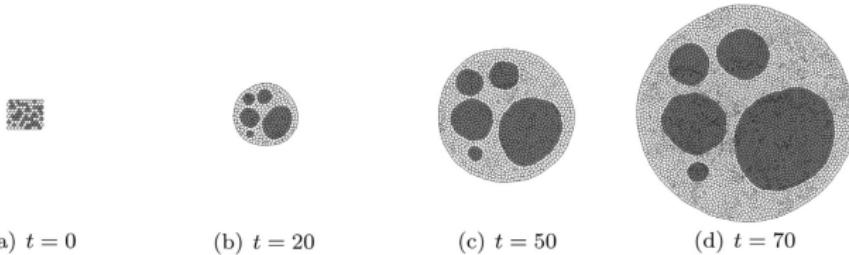


Illustration of vertex dynamics models²



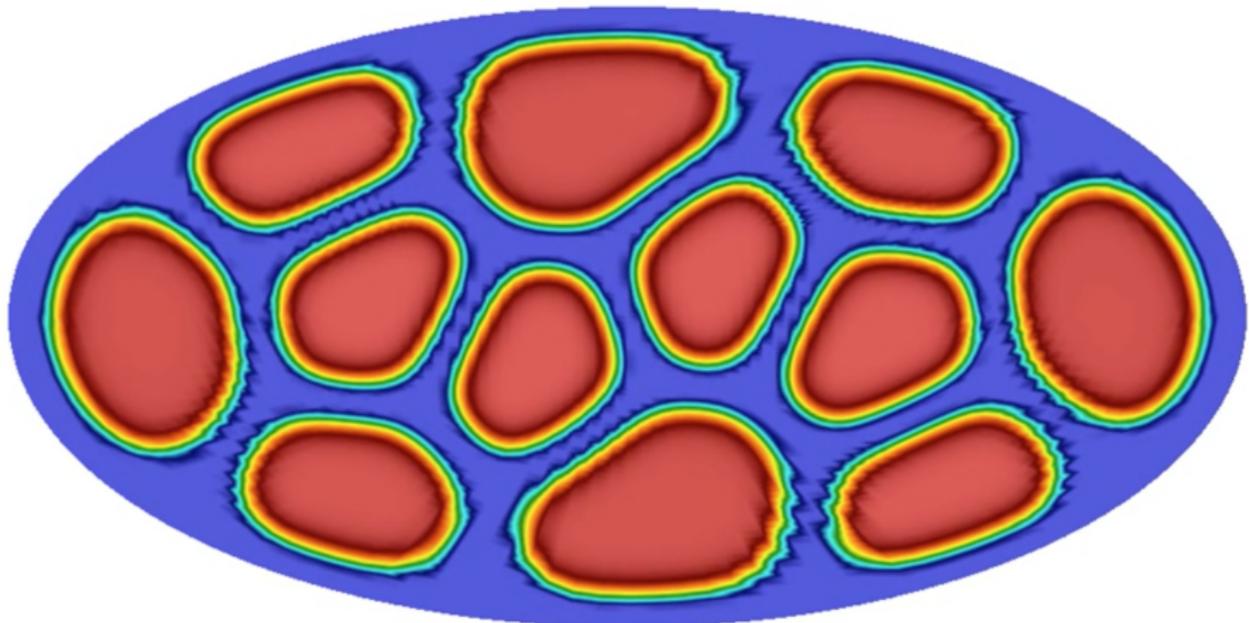
Simulation of cell sorting using the Nagai-Honda vertex dynamics model³

¹ P.J. Albert, Cell Adhesion & Migration, 10, 2016

² GK. Xu et al., Journal of Biomechanics, 49, 2016

³ A. G. Fletcher et al., Progress in Biophysics and Molecular Biology, 113, 2013

Soft packing: A novel phase field approach



Phase field simulation of soft packing
<https://arxiv.org/abs/1806.01410>

Phase field modeling

Cahn-Hilliard dynamics

$$\Pi(c, \nabla c) = \int_{\Omega} [f(c) + \nabla c \cdot \kappa(\nabla c) \nabla c] dV$$

Chemical potential:

$$\mu = \delta_c \Pi(c, \nabla c)$$

Kinetics:

$$\frac{\partial c}{\partial t} = \nabla \cdot (-L(\nabla c) \nabla \mu)$$

- ▶ Models evolution of conserved fields like composition.
- ▶ Fourth order PDE with complex anisotropic dependencies.

van der Waals, Verhandel. Konink. Akad. Wetsten. Amsterdam, 1893
Cahn & Hilliard, J. Chem. Phys., 1958

Allen-Cahn dynamics

$$\Pi(\eta_i, \nabla \eta_i) = \int_{\Omega} [f(\eta_i) + \nabla \eta_i \cdot \kappa(\nabla \eta_i) \nabla \eta_i] dV$$

Chemical potential:

$$\mu = \delta_{\eta_i} \Pi(\eta_i, \nabla \eta_i)$$

Kinetics:

$$\frac{\partial \eta_i}{\partial t} = -(L(\nabla \eta_i) \mu)$$

- ▶ Models evolution of non-conserved fields like structural order parameters.
- ▶ System of highly coupled second order PDE's.

Soft packing: A novel phase field approach

Let $\Omega \in \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Scalar fields c_k , $k = 1, \dots, N$ with $c_k \in [0, 1]$ serve to delineate the interior and exterior of the cell numbered k . Here, the interior of cell k is $\omega_k \subset \Omega$, where $\omega_k = \{\mathbf{X} \in \Omega | c_k(\mathbf{X}) = 1\}$. The exterior is $\Omega \setminus \omega_k$. The free energy density function is built up beginning with the following form:

$$\psi_1(c_k) = \alpha c_k^2 (c_k - 1)^2 + \frac{\kappa}{2} |\nabla c_k|^2$$

The total free energy of the multi-cell aggregate is a functional $\Pi[\mathbf{c}]$, defined as

$$\begin{aligned}\Pi[\mathbf{c}] &:= \int_{\Omega} \psi(\mathbf{c}, \nabla \mathbf{c}) \, dV \\ &= \int_{\Omega} \left(\sum_{k=1}^N f(c_k) + \sum_{k=1}^N \frac{\kappa}{2} |\nabla c_k|^2 + \sum_{l \neq k} \sum_{k=1}^N \lambda c_k^2 c_l^2 \right) \, dV.\end{aligned}$$

Variational formulation

Taking the variational derivative with respect to c_k yields

$$\begin{aligned}\delta\Pi_k[\mathbf{c}; \mathbf{w}] &= \frac{d}{d\epsilon} \int_{\Omega} \sum_{k=1}^N \left(f(c_k + \epsilon w) + \frac{\kappa}{2} |\nabla(c_k + \epsilon w)|^2 + \sum_{l \neq k} \lambda(c_k + \epsilon w)^2 c_l^2 \right) dV \Big|_{\epsilon=0} \\ &= \int_{\Omega} w \left(f'(c_k) - \kappa \Delta c_k + \sum_{l \neq k} 2\lambda c_k c_l^2 \right) dV + \int_{\partial\Omega} w \kappa \nabla c_k \cdot \mathbf{n} dS\end{aligned}$$

The chemical potential of the k^{th} cell is identified as,

$$\mu_k = f'(c_k) - \kappa \Delta c_k + \sum_{l \neq k} 2\lambda c_k c_l^2$$

Resulting kinetics:

$$\frac{\partial c_k}{\partial t} = - \nabla \cdot (-M \nabla \mu_k) + s_k$$

Variational formulation

Time discretization:

$$c_k^{n+1} = c_k^n + \Delta t(M \nabla \cdot (\nabla \mu_k^{n+1}) + s_k)$$

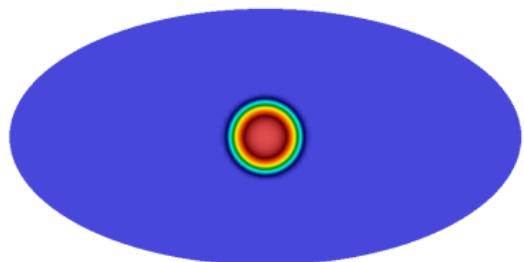
where $\mu_k^{n+1} = f'^{n+1}(c_k) - \kappa \Delta c_k^{n+1} + \sum_{l \neq k} 2\lambda c_k^{n+1} c_l^{n+1^2}$

Weak formulation:

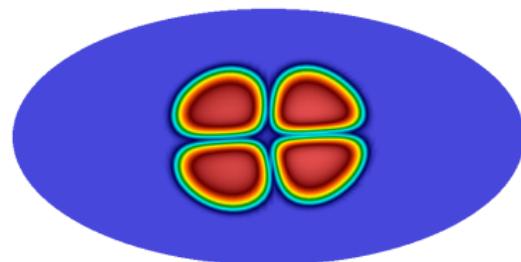
$$\int_{\Omega} w c_k^{n+1} \, dV = \int_{\Omega} (w c_k^n - \nabla w \cdot \Delta t M \nabla \mu_k^{n+1} + w \Delta t s_k) \, dV$$
$$\int_{\Omega} w \mu_k^{n+1} \, dV = \int_{\Omega} (w f'^{n+1}(c_k) + \nabla w \cdot \kappa \nabla c_k^{n+1}) \, dV + \int_{\Omega} w \sum_{l \neq k} 2\lambda c_k^{n+1} c_l^{n+1^2} \, dV$$

Implemented in the *deal.II* finite element framework.

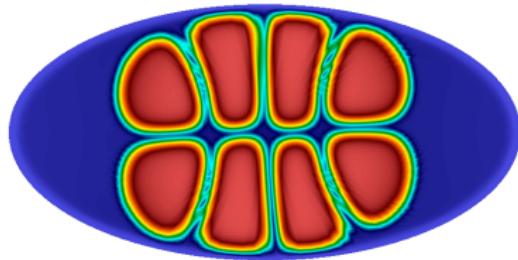
Results: Cell divisions and soft packing



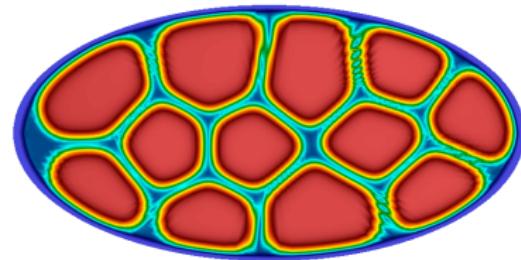
Initial single circular cell



Progression to four cells



Progression to eight cells



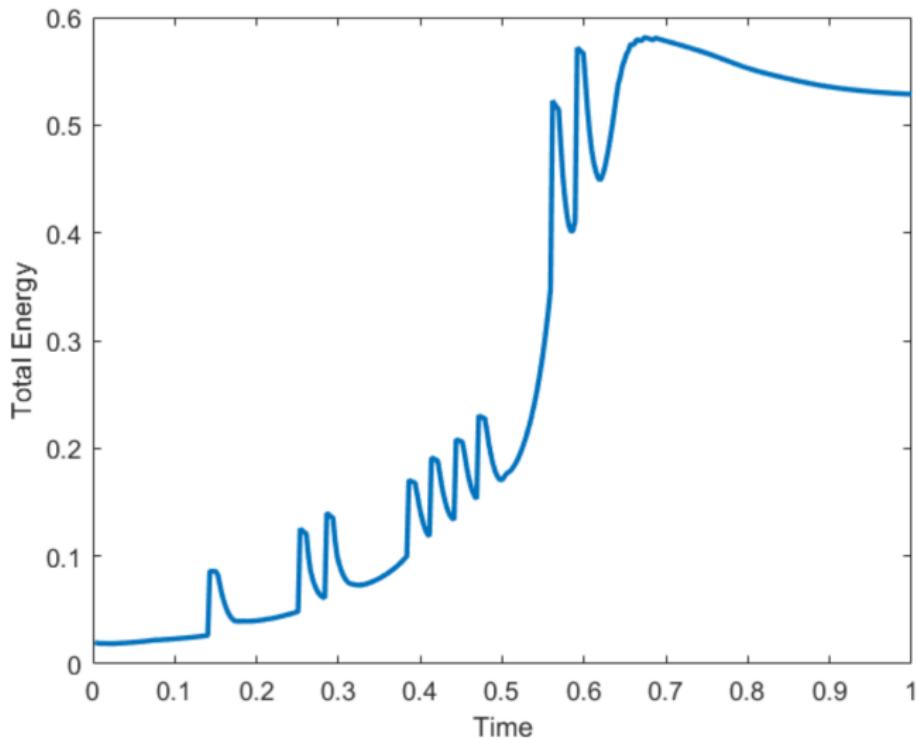
Progression to twelve cells.

Second division of cells.

Progression of cell division and packing with 12 cells.

Results: Cell divisions and soft packing

Evolution of the phase field energy due to the division and packing processes.



Scutoids as a possible geometric solution to three dimensional epithelial cell packing

- ▶ Scutoids develop because the apical and basal surface have different neighbours. Changes in cellular aspect ratio promote cellular rearrangements, and coupling between geometrical changes due to bending and line tension energy drives scutoid formation
- ▶ Appearance of scutoids is a general feature of the epithelial architecture and is directly affected by surface ratio anisotropy.
- ▶ It is significant in its study as experiments reveal a considerable minimization in tissue energy and stabilize soft tissue packing.
- ▶ Mathematical modeling and analysis of these complex geometrical shapes is executed using the Voronoi diagram on the plane of the epithelium.
- ▶ It is often assumed that this cellular architecture is driven by the cell organization in apical cells.

Scutoids are a geometrical solution to three-dimensional packing of epithelia, Escurado et al., 2018

Summary and look ahead

- ▶ Developed a diffused interface based numerical framework for modeling growth and packing of cell aggregates. Salient features:
 - ▶ No discrete interface evolving mechanisms like those employed in lattice, cell-centric and vertex dynamics models needed.
 - ▶ Time evolution occurs at realistic time scales controlled by the growth rate, or doubling time of the cells, and does not need equilibrations needed by other discrete models.
 - ▶ Any arbitrary cell shape can be represented without being limited to polygonal shapes or a jagged representation of the cell boundary.
 - ▶ Basic model for incorporating mechanics and material models of the underlying cytoskeletal network.
- ▶ Ongoing numerical work on incorporating (1) A material model in place of the shape model, (2) Active parameter tracking to allow for scaling independent of the number of cells.
- ▶ Key applications: (1) Soft packing in cells, (2) Patterning in embryogenesis, (3) Other problems involving cell aggregates like collective cell motion.