Modeling Durotaxis using Mechanical Bidomain model

Prof Brad Roth, Debabrata Auddya

January 22, 2019

Analytical treatment - 10/03/2018

The equations formulated below consider a gradient in the extracellular stiffness in a 1-d strand of tissue using the mechanical bidomain model.

Consider u_x and w_x as the intra- and extra-cellular displacements of the bidomain layer. x being the principal direction of the one-dimensional tissue strand, T being the tension, p and q being the intra- and extra-cellular pressure contributions arising mostly from hydrostatic forces.

The standard equations of the bidomain model are as follows [1]:

$$-\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}) + \gamma\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial T}{\partial x} = K(u_x - w_x) \tag{1}$$

$$-\frac{\partial q}{\partial x} + \mu(\frac{\partial^2 w_x}{\partial x^2} + \frac{\partial^2 w_x}{\partial y^2}) = -K(u_x - w_x) \tag{2}$$

For this problem we consider:

- $\mu = \mu_0 + gx$
- No contributions along the y axis
- p,q and T are constant and hence their gradients are zero along the x axis
- Slope of μ is a constant
- K is same for intra- and extra-cellular displacements

$$\nu\frac{\partial^2 u_x}{\partial x^2} + \gamma\frac{\partial^2 u_x}{\partial x^2} = K(u_x - w_x) \tag{3}$$

$$\mu_0\frac{\partial^2 w_x}{\partial x^2} + gx\frac{\partial^2 w_x}{\partial x^2} = -K(u_x - w_x) \tag{4}$$

Simplifying the above expression, replacing u_x in equation (2) from (1):

$$u_x = w_x + \frac{\nu}{K}\frac{\partial^2 u_x}{\partial x^2} + \frac{\gamma}{K}\frac{\partial^2 u_x}{\partial x^2} + \gamma\frac{\partial^2 u_x}{\partial x^2} \tag{5}$$

$$\therefore \frac{\mu_0}{K}\frac{\partial^2 w_x}{\partial x^2} + x\frac{g}{k}\frac{\partial^2 w_x}{\partial x^2} = -(\frac{\nu}{K}\frac{\partial^2 u_x}{\partial x^2} + \frac{\gamma}{K}\frac{\partial^2 u_x}{\partial x^2}) \tag{6}$$

Final form the equation can be written as:

$$\frac{\partial^2 w_x}{\partial x^2}(\frac{\mu_0}{K} + \frac{gx}{K}) + \frac{\partial^2 u_x}{\partial x^2}(\frac{\nu}{K} + \frac{\gamma}{K}) = 0 \tag{7}$$

Analytical treatment - 10/04/2018

The equations formulated below consider a gradient in the extracellular stiffness in a 1-d strand of tissue using the mechanical bidomain model.

Consider u_x and w_x as the intra- and extra-cellular displacements of the bidomain layer. x being the principal direction of the one-dimensional tissue strand, T being the tension, p and q being the intra- and extra-cellular pressure contributions arising mostly from hydrostatic forces.

The intra and extra-cellular stresses arising in the 1-d strand of tissue can be expressed as [2]:

$$\tau_{ix} = -p + 2\nu\epsilon_{ix} + T \tag{8}$$
$$\tau_{ex} = -q + 2\mu(x)\epsilon_{ex} \tag{9}$$

Relationship between the strains and displacement can be written as:

$$\frac{\partial \tau_{ix}}{\partial x} = K(u_x - w_x) \tag{10}$$
$$\frac{\partial \tau_{ex}}{\partial x} = -K(u_x - w_x) \tag{11}$$

Using equation (8) and (9) in (10) and (11) the resulting intra- and extra-cellular equations are:

$$\frac{\partial}{\partial x}(-p + 2\nu\frac{\partial u_x}{\partial x}) = K(u_x - w_x) \tag{12}$$
$$\frac{\partial}{\partial x}(-q + 2\mu(x)\frac{\partial w_x}{\partial x}) = -K(u_x - w_x) \tag{13}$$

For the problem we have assumed $\mu = \mu_0 + gx$, where g is a constant. Rewriting (12) and (13) we have:

$$-\frac{\partial p}{\partial x} + 2\nu\frac{\partial^2 u_x}{\partial x^2} = K(u_x - w_x) \tag{14}$$
$$-\frac{\partial q}{\partial x} + 2(g\frac{\partial w_x}{\partial x} + \frac{\partial^2 w_x}{\partial x^2}\mu(x)) = -K(u_x - w_x) \tag{15}$$

Analytical treatment - 11/13/2018

In order to obtain an analytical solution for the problem initially we consider g = 0. The boundary limits of the 1-dimensional problem assume the length of the domain spanning form x=-L to x=+L. The stresses are taken to be zero at each of the boundaries. We begin by implementing a trial solution for u_x and w_x.

$$u_x = Ax + Bsinh\Big(\frac{x}{\sigma}\Big) \tag{16}$$

$$w_x = Cx + Dsinh\Big(\frac{x}{\sigma}\Big) \tag{17}$$

For estimating the values of A,B,C,D we assume additionally p=q=0. Rewriting (12):

$$2\nu\frac{\partial^2}{\partial x^2}\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{18}$$

$$2\nu\frac{\partial}{\partial x}\Big(A + \frac{B}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)) \tag{19}$$

$$2\nu\frac{B}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{20}$$

and (13)

$$2\mu(x)\frac{\partial^2}{\partial x^2}\Big(Cx + Dsinh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{21}$$

$$2\mu(x)\frac{\partial}{\partial x}\Big(C + \frac{D}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)) \tag{22}$$

$$2\mu(x)\frac{D}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{23}$$

From (20) and (23) we have:

$$D = -\frac{\nu}{\mu(x)}B \tag{24}$$

Comparing coefficients of hyperbolic sine terms we see that (20) and (23) is satisfied only when the coefficient of the linear term is zero. Therefore,

$$A = C \tag{25}$$

Hence the length constant[1] σ has the value[2]

$$\sigma = \sqrt{\frac{2\nu\mu}{K(\nu+\mu)}} \tag{26}$$

To obtain the values of the unknown parameters B and C, we impose boundary conditions. At the edge (x = ±L) we have, normal stresses τ_{ix} and τ_{ex} as zero. As a result we get,

$$C + \frac{D}{\sigma}cosh\Big(\frac{L}{\sigma}\Big) = 0 \tag{27}$$

$$A + \frac{B}{\sigma}cosh\Big(\frac{L}{\sigma}\Big) = -\frac{T}{2\nu} \tag{28}$$

[1] This might also be a variable since μ is a function of x for the problem

[2] $\mu(x)$ is also written as μ

Solving (27) and (28) using (24)-(26) we have:

$$A = C = -\frac{T}{2(\nu + \mu)} \tag{29}$$

$$B = -\frac{T}{2\nu}\Big(\frac{\mu}{\nu + \mu}\Big)\frac{\sigma}{cosh(\frac{L}{\sigma})} \ , D = \frac{T}{2(\mu + \nu)}\frac{\sigma}{cosh(\frac{L}{\sigma})} \tag{30}$$

Using this we get the intra and extracellular displacement in terms of T as:

$$u_x = -\frac{T}{2(\nu + \mu)}\Big(x + \frac{\mu}{\nu}\sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{31}$$

$$w_x = -\frac{T}{2(\nu + \mu)}\Big(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{32}$$

Since μ is variable for the problem we now write it as: $\mu = \mu_0 + gx$. Remaining derivation of u_x and w_x along with its derivatives and double derivatives has been attached separately.

Analytical treatment - 12/22/2018

We are considering plane stress conditions for the model. Nothing depends on y.

$$\tau_{exx} = -q + 2\mu\epsilon_{exx}$$
$$\tau_{eyy} = -q$$
$$\tau_{ezz} = -q + 2\mu\epsilon_{ezz}$$

Since $\tau_{ezz} = 0$ so

$$-q + 2\mu\epsilon_{ezz} = 0$$

or, $q = 2\mu\epsilon_{ezz}$, but, $\epsilon_{exx} + \epsilon_{ezz} = 0$, so

$$q = -2\mu\epsilon_{exx}$$

Therefore the stresses along the principal directions can be written as:

$$\tau_{exx} = 4\mu\epsilon_{exx} \quad \tau_{eyy} = 2\mu\epsilon_{exx} \quad \tau_{ezz} = 0 \quad \tau_{exy} = 0$$

The final partial differential form of the equations look like:

$$\frac{\partial \tau_{exx}}{\partial x} + \frac{\partial \tau_{exy}}{\partial y} = -K(u_x - w_x)$$
$$or, \quad 4\Big(\frac{\partial \mu}{\partial x}\epsilon_{exx} + \mu\frac{\partial \epsilon_{exx}}{\partial x}\Big) = -K(u_x - w_x)$$

For the intracellular layer the principal stress is the same except that Tension T along the principal direction (x) is also taken into account. Thus the final set of working equations are, considering μ has a gradient along x and ν is a constant for the problem.

$$\boxed{\begin{aligned} &4\mu\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial w}{\partial x}\frac{\partial \mu}{\partial x} = -K(u_x - w_x) \\ &4\nu\frac{\partial^2 u}{\partial x^2} = K(u_x - w_x) \end{aligned}}$$

μ = constant

$$\boxed{\begin{aligned} &4\mu\frac{\partial^2 w}{\partial x^2} = -K(u_x - w_x) \\ &4\nu\frac{\partial^2 u}{\partial x^2} = K(u_x - w_x) \end{aligned}}$$

The stresses are taken to be zero at each of the boundaries. We begin by implementing a trial solution for u_x and w_x.

$$u_x = Ax + Bsinh\Big(\frac{x}{\sigma}\Big)$$
$$w_x = Cx + Dsinh\Big(\frac{x}{\sigma}\Big)$$

For estimating the values of A,B,C,D we assume: Rewriting (12):

$$4\nu\frac{\partial^2}{\partial x^2}\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big)$$
$$4\nu\frac{\partial}{\partial x}\Big(A + \frac{B}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big)$$
$$4\nu\frac{B}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big)$$

and (13)

$$4\mu(x)\frac{\partial^2}{\partial x^2}\Big(Cx + Dsinh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big(Ax + Bsinh\Big(\frac{x}{\sigma}\Big) - Cx - Dsinh\Big(\frac{x}{\sigma}\Big)\Big)$$
$$4\mu(x)\frac{\partial}{\partial x}\Big(C + \frac{D}{\sigma}cosh\Big(\frac{x}{\sigma}\Big)\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big)$$
$$4\mu(x)\frac{D}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big)$$

From (20) and (23) we have:

$$D = -\frac{\nu}{\mu}B \tag{33}$$

Comparing coefficients of hyperbolic sine terms we see that (20) and (23) is satisfied only when the coefficient of the linear term is zero. Therefore,

$$A = C \tag{34}$$

Length constant σ remains as:

$$\sigma = \sqrt{\frac{4\nu\mu}{K(\nu+\mu)}} \tag{35}$$

To obtain the values of the unknown parameters B and C, we impose boundary conditions. At the edge (x = ±L) we have, normal stresses τ_{ix} and τ_{ex} as zero. As a result we get,

$$C + \frac{D}{\sigma}cosh\Big(\frac{L}{\sigma}\Big) = 0 \tag{36}$$
$$A + \frac{B}{\sigma}cosh\Big(\frac{L}{\sigma}\Big) = -\frac{T}{4\nu} \tag{37}$$

Solving the equations using given by (33) - (35) and using them in (36) and (37) we have: Solving (27) and (28) using (24)-(26) we have:

$$A = C = -\frac{T}{4(\nu+\mu)} \tag{38}$$
$$B = -\frac{T}{4\nu}\Big(\frac{\mu}{\nu+\mu}\Big)\frac{\sigma}{cosh(\frac{L}{\sigma})}\ , D = \frac{T}{4(\mu+\nu)}\frac{\sigma}{cosh(\frac{L}{\sigma})} \tag{39}$$

Using this we get the intra and extracellular displacement in terms of T as:

$$u_x = -\frac{T}{4(\nu+\mu)}\Big(x + \frac{\mu}{\nu}\sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{40}$$
$$w_x = -\frac{T}{4(\nu+\mu)}\Big(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big) \tag{41}$$

$\mu = \mu_0 + \mathbf{gx}$

For this condition the set of equations which we need to consider for obtaining an analytical solution are:

$$4\mu\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial w}{\partial x}\frac{\partial \mu}{\partial x} = -K(u_x - w_x) \tag{42}$$

$$4\nu\frac{\partial^2 u}{\partial x^2} = K(u_x - w_x) \tag{43}$$

Replacing unknowns term by term from the previous solutions and given data we have an expression for the difference between u_x and w_x:

$$\boxed{\frac{-4}{K}(\mu_0 + gx)\frac{\partial^2}{\partial x^2}\left(-\frac{T}{4(\nu+\mu)}\left(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\right)\right) - \frac{4g}{K}\frac{\partial}{\partial x}\left(-\frac{T}{4(\nu+\mu)}\left(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\right)\right)}$$

Mathematica Code

The code below returns the value of the above expression. f(x) returns the value of $u_x - w_x$

```
sigma[x_] = ((4*(muz + gx)*nu)/(K*(nu+muz+gx)))^0.5
g[x_] = -(Tx/(4*(nu+muz+gx))) +
    ((T*sigma[x_]*sinh[x/sigma[x]])/(4*(nu+muz+gx)*cosh[L/sigma[x]]))
g'[x_]
g''[x_]
f[x_] = (-4/K)*(muz+gx)*g''[x] + (-4*g/K)*g'[x]
```

Analytical treatment - 12/28/2018

We assume the trial solution for u_x and w_x for the given problem as:

$$u_x = Ax + Bsinh\Big(\frac{x}{\sigma}\Big) \tag{44}$$

$$w_x = Cx + Dsinh\Big(\frac{x}{\sigma}\Big) \tag{45}$$

Additionally the extracellular gradient is taken as $\mu = \mu_0 + gx$. The final set of working equations are taken as:

$$\boxed{\begin{aligned}&4\mu\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial w}{\partial x}\frac{\partial \mu}{\partial x} = -K(u_x - w_x)\\ &4\nu\frac{\partial^2 u}{\partial x^2} = K(u_x - w_x)\end{aligned}}$$

Replacing the values for u and w with that of (44)-(45) we have

$$4(\mu_0 + gx)\frac{D}{\sigma^2}sinh(\frac{x}{\sigma}) + 4g(C + \frac{D}{\sigma}cosh(\frac{x}{\sigma})) = -K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{46}$$

$$4\nu\frac{B}{\sigma^2}sinh(\frac{x}{\sigma}) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big) \tag{47}$$

Using boundary conditions on $\epsilon_{exx} = 0$ at x=L (Since we have $\tau_{exx} = 0$ at x=±L)

$$\frac{\partial w}{\partial x} = 0$$

$$\therefore \quad C + \frac{D}{\sigma}cosh(\frac{x}{\sigma}) = 0$$

Plugging the above result in equation (46) and (47) we have:

$$(\mu_0 + gx)\frac{D}{\sigma^2}sinh(\frac{L}{\sigma}) = -\nu\frac{B}{\sigma^2}sinh(\frac{L}{\sigma})$$

$$\boxed{D = -\frac{\nu}{\mu_0 + gL}B}$$

For finding relation between A and C we manipulate equation (47) with the relationship between B and D

$$\begin{aligned}&4\nu\frac{B}{\sigma^2}sinh(\frac{x}{\sigma}) = K\Big((A-C)x + (B-D)sinh\Big(\frac{x}{\sigma}\Big)\Big)\\ or,\quad &4\nu\frac{B}{\sigma^2}sinh\Big(\frac{x}{\sigma}\Big) - K(B-D)sinh\Big(\frac{x}{\sigma}\Big) = K(A-C)x\\ or,\quad &sinh\Big(\frac{x}{\sigma}\Big)\Big(\frac{4\nu B}{\sigma^2} - K(B + \frac{B\nu}{\mu_0 + gL})\Big) = K(A-C)x\end{aligned}$$

$$\boxed{A - C = \frac{B}{Kx}sinh\Big(\frac{x}{\sigma}\Big)\Big(\frac{4\nu}{\sigma^2} - K(1 + \frac{\nu}{\mu_0 + gL})\Big)}$$

To obtain the values of the unknown parameters B and C, we impose boundary conditions. At the edge (x = ±L) we have, normal stresses τ_{ix} and τ_{ex} as zero. Rewriting it, we have,

$$C + \frac{D}{\sigma} cosh\Big(\frac{L}{\sigma}\Big) = 0 \tag{48}$$

$$A + \frac{B}{\sigma} cosh\Big(\frac{L}{\sigma}\Big) = -\frac{T}{4\nu} \tag{49}$$

Subtracting (49) - (48) we have:

$$(A - C) + (\frac{B - D}{\sigma}) cosh\Big(\frac{L}{\sigma}\Big) = -\frac{T}{4\nu} \tag{50}$$

Plugging in A - C and B,D relationship in the equation we have:

$$\frac{B}{Kx} sinh\Big(\frac{x}{\sigma}\Big)\Big(\frac{4\nu}{\sigma^2} - K(1 + \frac{\nu}{\mu_0 + gL})\Big) + \frac{1}{\sigma}\Big(B + \frac{B\nu}{\mu_0 + gL}\Big) cosh\Big(\frac{L}{\sigma}\Big) = -\frac{T}{4\nu}$$

However we have to replace x in the above equation with L as we have been using boundary conditions in (48) - (49). The final form is: (taking out B common in the expression)

$$\mathbf{B} = -\frac{T}{4\nu}\left(\frac{1}{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma^2 L} - \frac{K}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(\frac{1}{\sigma}(1 + \frac{\nu}{\mu_0+gL})\Big)}\right) \tag{51}$$

The value of D can be computed using its relationship with B which is $D = -\frac{\nu}{\mu_0+gL} B$

$$\mathbf{D} = \frac{T}{4(\mu_0 + gL)}\left(\frac{1}{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma^2 L} - \frac{K}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(\frac{1}{\sigma}(1 + \frac{\nu}{\mu_0+gL})\Big)}\right) \tag{52}$$

The value of C can be computed from (48) as $C = -\frac{D}{\sigma} cosh\Big(\frac{L}{\sigma}\Big)$

$$\mathbf{C} = -\frac{T}{4(\mu_0 + gL)}\left(\frac{cosh\Big(\frac{L}{\sigma}\Big)}{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(1 + \frac{\nu}{\mu_0+gL}\Big)}\right) \tag{53}$$

The value of A can be obtained from (49) as $A = -\frac{B}{\sigma} cosh\Big(\frac{L}{\sigma}\Big) - \frac{T}{4\nu}$

$$\mathbf{A} = -\frac{T}{4(\nu)}\left(1 - \frac{cosh\Big(\frac{L}{\sigma}\Big)}{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(1 + \frac{\nu}{\mu_0+gL}\Big)}\right)$$

$$\therefore \quad \mathbf{A} = -\frac{T}{4\nu}\left(\frac{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(\frac{\nu}{\mu_0+gL}\Big)}{sinh\Big(\frac{L}{\sigma}\Big)\Big(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1 + \frac{\nu}{\mu_0+gL})\Big) + cosh\Big(\frac{L}{\sigma}\Big)\Big(1 + \frac{\nu}{\mu_0+gL}\Big)}\right) \tag{54}$$

Thus the value of of u_x is:

$$u_x = -\frac{T}{4(\nu)}\left(\frac{sinh\left(\frac{L}{\sigma}\right)\left(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1+\frac{\nu}{\mu_0+gL})\right) + cosh\left(\frac{L}{\sigma}\right)\left(\frac{\nu}{\mu_0+gL}\right)}{sinh\left(\frac{L}{\sigma}\right)\left(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1+\frac{\nu}{\mu_0+gL})\right) + cosh\left(\frac{L}{\sigma}\right)\left(1+\frac{\nu}{\mu_0+gL}\right)}\right)\mathbf{x}$$
$$-\frac{T}{4\nu}\left(\frac{1}{sinh\left(\frac{L}{\sigma}\right)\left(\frac{4\nu}{\sigma^2 L} - \frac{K}{L}(1+\frac{\nu}{\mu_0+gL})\right) + cosh\left(\frac{L}{\sigma}\right)\left(\frac{1}{\sigma}(1+\frac{\nu}{\mu_0+gL})\right)}\right)\mathbf{sinh}\left(\frac{x}{\sigma}\right)$$

The value of w_x is:

$$w_x = -\frac{T}{4(\mu_0+gL)}\left(\frac{cosh\left(\frac{L}{\sigma}\right)}{sinh\left(\frac{L}{\sigma}\right)\left(\frac{4\nu}{\sigma L} - \frac{K\sigma}{L}(1+\frac{\nu}{\mu_0+gL})\right) + cosh\left(\frac{L}{\sigma}\right)\left(1+\frac{\nu}{\mu_0+gL}\right)}\right)\mathbf{x}+$$
$$\frac{T}{4(\mu_0+gL)}\left(\frac{1}{sinh\left(\frac{L}{\sigma}\right)\left(\frac{4\nu}{\sigma^2 L} - \frac{K}{L}(1+\frac{\nu}{\mu_0+gL})\right) + cosh\left(\frac{L}{\sigma}\right)\left(\frac{1}{\sigma}(1+\frac{\nu}{\mu_0+gL})\right)}\right)\mathbf{sinh}\left(\frac{x}{\sigma}\right)$$

Issues

Need some insights for calculating the length constant σ

Perturbation Theory - 1/22/2019

The following result has been obtained using first order perturbation to obtain approximate analytical solutions.

$$u_0 = -\frac{T}{4(\nu+\mu)}\Big(x + \frac{\mu}{\nu}\sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big)$$

$$w_0 = -\frac{T}{4(\nu+\mu)}\Big(x - \sigma\frac{sinh(\frac{x}{\sigma})}{cosh(\frac{L}{\sigma})}\Big)$$

$$u_1 = c_1 e^{\sqrt{\phi_1}x} + c_2 e^{-\sqrt{\phi_1}x} - \frac{\phi_2 e^{-\sqrt{\phi_1}x}(e^{2\sqrt{\phi_1}x}Ei(-\sqrt{\phi_1 x}) + Ei(\sqrt{\phi_1 x}) - 2e^{\sqrt{\phi_1}x}ln(x))}{2\phi_1}$$

$$w_1 = c_3 ln(x)$$

$$Ei(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dx$$

where Ei(x) is the Exponential Integral Function. However this is different from the proposed guess for u_1 which involved consisting of sinh and cosh terms only.

Bibliography

[1] Bradley J Roth *The Mechanical Bidomain Model: A Review* ISRN Tissue Eng. 2013

[2] Sharma, Kharananda; Al-Asuoad, Nofe; Shillor, Meir; Roth, Bradley J. *Intracellular, extracellular, and membrane forces in remodeling and mechanotransduction: The mechanical bidomain model* Journal of Coupled Systems and Multiscale Dynamics, Volume 3, Number 3, September 2015, pp. 200-207(8)