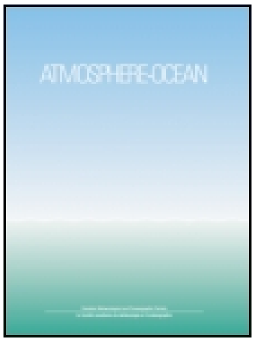

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A Review of Analytical Models of Sea-Ice Growth

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ABSTRACT *The modelling of sea-ice growth is a classical problem in geophysics, which has been traditionally treated as one-dimensional, considering only the vertical heat transfer. The modelling work commenced in the 1800s with analytic methods. These are very effective tools for examining the sea-ice growth problem, providing a clear insight into the physical mechanisms and producing simple first-order approximations for the ice thickness in various conditions. This paper describes the physical problem of sea-ice growth, presents an analytical modelling framework for the problem and provides analytic solutions for different environmental conditions.*

RÉSUMÉ *La modélisation de la formation de la glace de mer est un problème classique, par tradition ayant une seule dimension, de la géophysique ne considèrant que le transfert vertical de chaleur. Les méthodes d'analyse des années 1800 ont été les premiers efforts de modélisation. Ces méthodes sont très efficaces pour examiner le problème de la formation de la glace de mer car elles fournissent un bon aperçu des mécanismes physiques et produisent des approximations simples de premier ordre de l'épaisseur de la glace dans différentes conditions. On décrit le problème physique de la formation de la glace de mer, présente un cadre analytique de modélisation du problème et fournit des solutions analytiques pour différentes conditions environnementales.*

1 Introduction

Sea ice covers some 7% of the world ocean. It introduces significant modifications to the exchange of heat, momentum and material between the atmosphere and the ocean. The modification to heat exchange is qualitatively well understood, the ice being recognized as a thin, well insulating film at the air-sea interface. Understanding the evolution of the thickness of this film is a key problem in the physics of ice-covered seas. Sea ice may grow mechanically owing to a converging flow field or thermally owing to heat loss from the ice. The thermodynamic growth is the subject of the present work.

The thermal growth of sea ice is a classical problem in geophysics. It has been

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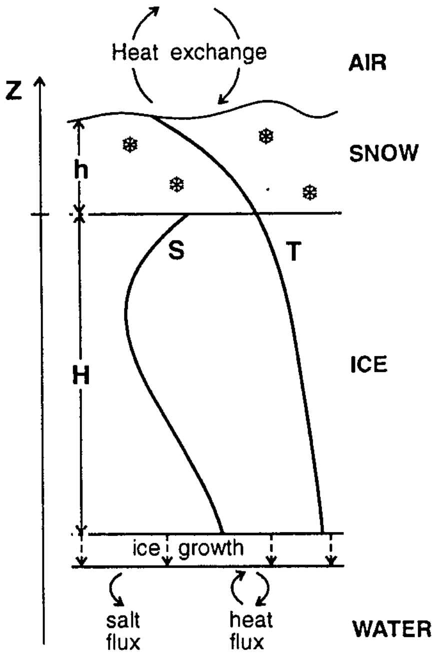


Fig. 1 Schematic illustration of thermodynamic sea-ice growth.

traditionally treated as one-dimensional and vertical, motivated by the large vertical temperature gradients that represent much greater vertical than horizontal heat fluxes. Figure 1 shows a schematic illustration of the ice-growth problem. Ice grows mainly at the bottom; the latent heat released due to freezing is conducted through the ice and snow and released to the atmosphere by radiation and turbulent heat fluxes. Occasionally ice may grow from the top when there is a slush layer present.

The thermodynamic growth of sea ice was already examined during the 1800s, and Stefan (1891) presented an elegant analytic solution for an idealised case of bare ice growth. A thorough investigation based on Stefan's model and empirical data was given by Zubov (1945); further, some theoretical and semi-empirical analytic models were developed. The next real advance was, however, made only with numerical models in the 1960s (Untersteiner, 1964; Maykut and Untersteiner, 1971). The Maykut and Untersteiner (1971) model has been the basic advanced model until now. The next generation sea-ice growth model, still on its way, will describe sea-ice thermodynamics as a fully coupled temperature-salinity system.

In this paper the sea-ice growth problem is examined using analytic methods. The melting season is not considered. First the physical problem of sea-ice growth is discussed. Then basic analytical model variants and their solutions are presented for examining particular physical questions and for obtaining first-order approximations for the ice thickness in different environmental conditions.

2 The basic equations

a *Heat Conduction in Ice*

The thermodynamics of sea ice is described with the classical heat conduction equation. This is written

$$\partial/\partial t(\rho_i c_i T) = \nabla \cdot (\kappa_i \nabla T) + q \tag{1}$$

where t is the time, ρ_i is the ice density, c_i is the specific heat of ice, T is the ice temperature, κ_i is the heat conductivity of ice, and q is an internal source term. The source term consists of solar radiation penetrating into the ice. For modelling ice growth, (1) can be much simplified. Vertical temperature gradients are usually much larger than horizontal ones, i.e. heat flows mainly in the vertical direction. Therefore the horizontal derivatives are neglected and the heat conduction equation reduces to

$$\partial/\partial t(\rho_i c_i T) = \partial/\partial z(\kappa_i \partial T/\partial z) + q \tag{2}$$

The boundary conditions are determined by the bottom temperature fixed to the freezing point of sea water T_f and by the heat flux at the top surface Q_T, which is due to heat loss to (or gain from) the atmosphere and phase changes:

$$\text{top:} \qquad \kappa_i \partial T/\partial z = Q_T \tag{3a}$$
$$\text{bottom:} \qquad T = T_f \tag{3b}$$

The lower boundary level is not fixed but changes owing to melting and freezing:

$$\rho_i L dH/dt = \kappa_i \cdot \partial T/\partial z|_{\text{bottom}} - Q_w \tag{4}$$

where L is the latent heat of freezing and Q_w is the heat flux from the water to the ice. The ice thickness may also change at the top, a process that is included in Q_T as a phase change.

Now, (2)–(4) describe the heat conduction through the ice, and the evolution of ice thickness. The problem is then to specify the thermodynamic properties of ice and determine the external heat fluxes. These properties are thoroughly considered in Weeks and Ackley (1982).

Ice formation in sea water is different in kind from what it is in fresh water. The salinity of the sea water depresses the freezing down to −2°C and produces salt inclusions in the ice. The freezing point of sea water is a well known function of salinity and pressure (e.g. Millero, 1978). The salts in sea ice are in the form of liquid brine and solid salt crystals; their relative amounts are dictated by the temperature of the ice. The phase composition is in equilibrium with the temperature (Schwerdtfeger, 1963). The salinity of the liquid brine is determined by the freezing point dependency, and therefore the salinity increases with decreasing temperature and vice versa. The salinity changes are possible owing to the melting and freezing of ice at the boundaries of the brine pockets. Consequently, the volume of liquid brine changes with temperature. This volume change increases rapidly for temperatures higher than about −5°C. The solid salts, on the other hand, exist only at low temperatures, and have a negligible effect on the thermal properties of sea ice.

The presence of brine and the associated phase changes mean that sea ice in fact has no fixed melting point. A temperature change always causes melting or freezing at the brine pocket boundaries. The latent heat of freezing is 335 J g^{-1} for pure ice. As sea ice grows brine is trapped within the ice. The more brine is trapped, the less pure ice must be produced for a unit sea-ice volume, and consequently the effective latent heat release becomes less (Cox and Weeks, 1988).

The thermodynamic properties of sea ice depend on the temperature and salinity (S) of the ice (Assur, 1958; Schwerdtfeger, 1963; Ono, 1968). The density is practically a constant for growth modelling purposes, $\rho_i = 0.9$ g cm^{-3}. Owing to the presence of brine, the specific heat and thermal conductivity are very sensitive to the temperature at high temperatures (Fig. 2). More simple analytic expressions for these parameters are given in Malmgren (1927).

The specific heat of a medium gives the amount of heat needed for a temperature increase of 1°C for a unit mass. For a temperature increase in sea ice one needs heat to increase the temperature of solid ice and brine and to melt a fraction of ice to decrease the brine salinity. Mainly because of the phase change effect, the specific heat of sea ice increases with temperature; this increase becomes drastic when the temperature approaches the freezing temperature (Fig. 2).

The thermal conductivity describes the ability of a medium to transport heat. For sea ice, heat is conducted through solid ice and through brine pockets. As the temperature increases, the brine volume increases. Convection in the brine cells has a minor effect, and therefore the thermal conductivity of sea ice decreases as the temperature increases (Fig. 2). This variation is, however, not so drastic as for the specific heat because phase changes are not involved.

The heat flux to the air/ice or air/snow interface from above consists of the turbulent heat exchange with the atmosphere, and the radiation balance (e.g. Maykut, 1986). It depends on the atmospheric conditions, characteristics of the surface type, and the surface temperature of the ice/snow system. Thus this heat flux is highly coupled with the ice/snow system and should be determined simultaneously when solving the heat conduction equation within the ice and snow.

The sensible and latent heat fluxes can be determined using the normal turbulent heat exchange laws. Semi-empirical equations have been generally used for the radiation components. These methods may sometimes lead to erroneous results; in particular, the turbulent heat exchange coefficients are highly sensitive to the atmospheric stability, and the net radiation is highly sensitive to cloudiness. The solar radiation part plays a small role since ice growth mainly concerns the fall and winter seasons.

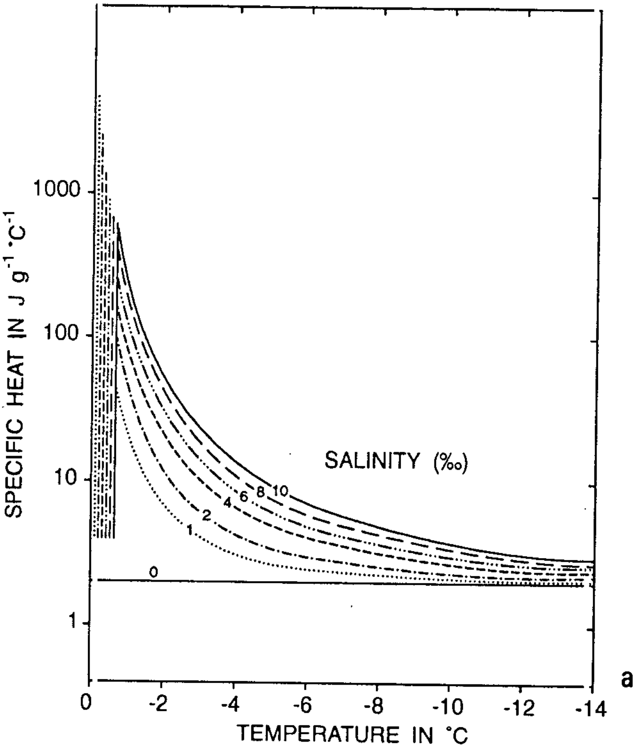
The oceanic heat flux Q_w is a purely external source. It must be described as given or determined from an oceanic model. Presently a lot of effort is being made to examine the role of the oceanic heat flux using ice-ocean coupled models (e.g. Mellor et al., 1986; Lemke, 1987; Houssais, 1988; Svensson and Omstedt, 1990).

b *Role of Snow*

If there is a snow cover on the ice, an equation for the heat conduction through the snow must be introduced. This is formally similar as for sea ice,

$$\partial/\partial t(\rho_s c_s T) = \partial/\partial z(\kappa_s \partial T/\partial z) + q \tag{5}$$

where ρ_s, c_s and κ_s are the density, specific heat and thermal conductivity of snow,



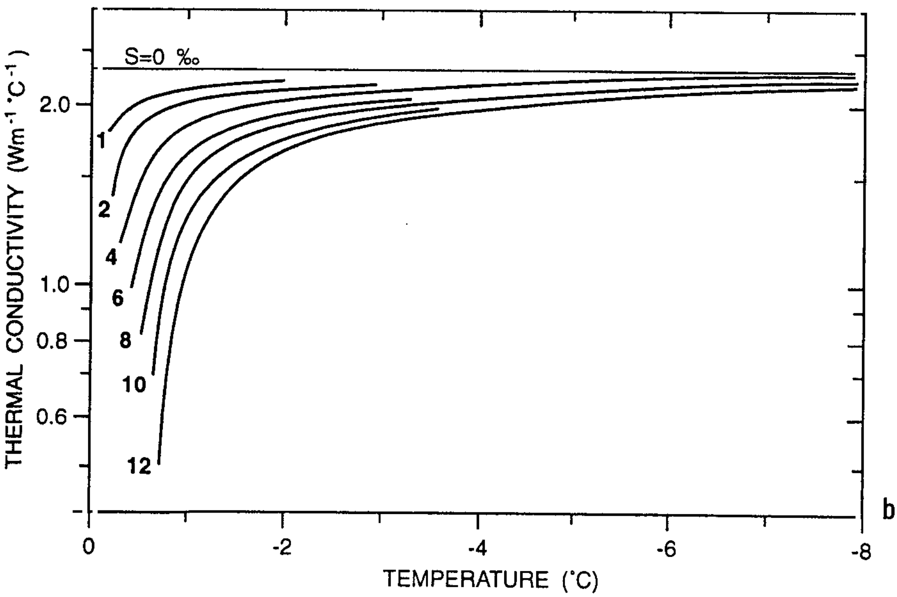


Fig. 2 (a) Specific heat (Schwerdtfeger, 1963) and (b) thermal conductivity (Ono, 1968) of sea ice. The horizontal axis is for the temperature; the various curves describe the effect of salinity.

respectively. The boundary condition at the ice/snow interface is then specified by the continuity of heat flux,

$$\kappa_i \partial T/\partial z|_{\text{ice}} = \kappa_s \partial T/\partial z|_{\text{snow}} \tag{6a}$$

and at the top of the snow cover we have

$$\kappa_s \partial T/\partial z = Q_T \tag{6b}$$

The total rate of change of snow thickness is determined by prescribed precipitation, predicted melt rate, and possibly a specified law for the evolution of snow density.

To a first approximation the specific heat of snow is constant and the thermal conductivity of snow depends only on density. The thermal conductivity plays a major role and is highly sensitive to the density of snow (Langham, 1981). A typical value would be about 10% of the thermal conductivity of ice; e.g. a 10-cm snow layer is as good an insulator as a 100-cm sea-ice layer. The thermal conductivity of snow, however, ranges over one order of magnitude for the snow density range 0.1–0.4 g cm^{-3}.

What complicates the snow-effect problem considerably is that the density of snow changes significantly with time (Leppäranta, 1983). Fresh snow may have a density as low as 0.1 g cm^{-3}, whereas during a week's time or more the density increases to 0.3–0.4 g cm^{-3}; 0.3 g cm^{-3} can be considered a typical density for snow on sea ice. Compaction of snow cover, e.g. due to the wind, has therefore a pronounced effect on the insulating strength of snow.

Snow is not only a passive insulator between the ice and air. In regions where the amount of snowfall is large, snow-ice formation occurs, which introduces a new component to the ice thermodynamics problem. Good examples of such regions are the Baltic Sea and the Antarctic seas. An attempt to include snow-ice in ice growth modelling has been made in Leppäranta (1983).

Archimedes' principle states that the water level coincides with the ice-snow interface when

$$\rho_s h = (\rho_w - \rho_i)H \tag{7}$$

where h is the snow thickness and ρ_w is the density of water. If the mass of snow is greater, the lower part of the snow submerges and a slush layer will eventually develop. The heat conduction problem has then changed to a three-layer system. Immediately after flooding, ice crystals may form or melt in the slush so that the slush temperature may equal the freezing point. The slush then freezes from the top and thus transforms into snow-ice.

We must make a few basic assumptions and specify the thermodynamic properties of slush and snow-ice. To a first approximation, assume that

i) the initial slush thickness equals the flooding height
ii) no compression occurs in the snow during the transformation

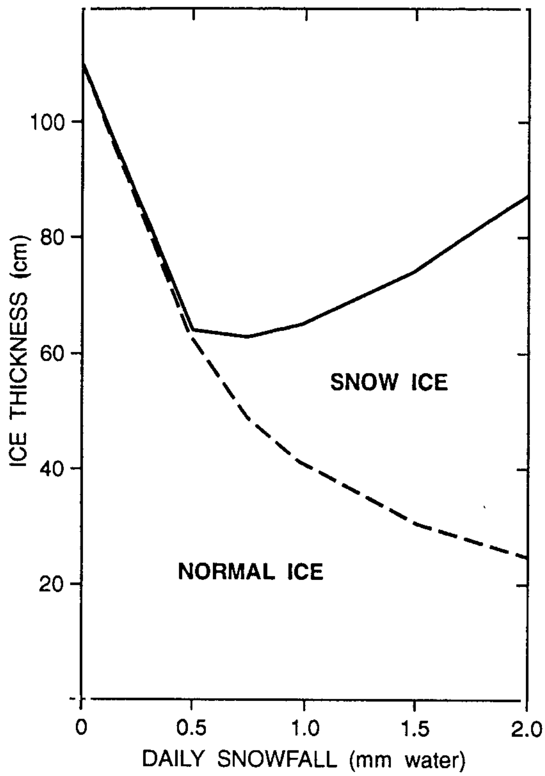


Fig. 3 Model-produced ice thickness as a function of assumed constant daily snowfall for average climatological air temperature conditions in Oulu, Finland (Leppäranta, 1983).

iii) slush properties are taken as the ice/water combination weighted linearly with their masses
iv) snow-ice properties are the same as for normal sea ice

The flooding phenomenon also modifies the ice salinity profile. The total amount of salt input is determined by the amount of water flooding onto the ice. During the course of snow-ice growth, brine rejection occurs and the lower parts of the slush may then become very salty (e.g. Leppäranta et al., 1992). This also depresses the freezing point allowing a wet layer to exist even if the air temperature becomes low.

An attempt to model the growth of normal sea ice and snow-ice was made by Leppäranta (1983). The resulting growth rates were realistic for both ice types. The relative proportions of normal sea ice and snow-ice depended heavily on the rate of snowfall (Fig. 3). Also included in the model was the dependency of the thermal conductivity of snow on snow density. As expected the ice growth was quite sensitive to the snow properties.

3 Analytic models

In spite of a long list of poorly known and highly sensitive factors in the sea-ice growth problem, it is astonishing that simple analytic models have provided rather good results. This is mainly due to a negative feedback feature for errors in modelling: underestimation in model thickness leads to less insulation so that the model ice may then grow faster to correct the error and vice versa. In analytic models the thermal properties of ice are generally taken as constants. This can result from the ice being cold during the growth season and therefore the thermal properties do not vary much (see Fig. 2). In addition the growth season involves the dark fall and winter seasons when the solar radiation is rather insignificant and the diurnal temperature variations are generally small.

a *Stefan's Law*

The basic analytic model is the classical Stefan's law (Stefan, 1891). It is based on the simple idea that the heat released by freezing at the ice bottom is conducted away through the ice by a constant temperature gradient. More precisely, Stefan's law is based on the assumptions

i) no thermal inertia
ii) no internal heat source
iii) a known temperature at the top, $T_0 = T_0(t)$
iv) no heat flux from the water

Assumptions (i)–(ii) simplify the heat conduction Eq. (2) into

$$\partial T/\partial z = \text{constant} \tag{8}$$

that is, the temperature profile within the ice is linear. Assumption (iii) simplifies the upper boundary condition (3a) into

$$T_0 = T_0(t) \quad \text{(given)} \tag{9}$$

Using assumption (iv), the ice-growth equation (4) then becomes

$$\rho_i L dH/dt = \kappa_i (T_f - T_0)/H \tag{10}$$

The analytic solution, with the initial condition $H = H_0$ for $t = 0$, is

$$H^2 = H_0^2 + a^2 S \tag{11a}$$

where

$$a = \sqrt{2\kappa_i/\rho_i L} \tag{11b}$$

$$S = \int_0^t [T_f - T_0(\tau)]d\tau \tag{11c}$$

The solution (11) is Stefan's law. The unit of time is commonly taken as one day; then S is called the sum of negative degree-days, and the value of a is about 3.3

(cm °C^{-1} d^{-1})$^{1/2}$. Example 1 presents a few cases of ice growth for different air temperatures.

The simplifications made to derive Stefan's law all tend to bias the model ice thickness upward. Therefore the Stefan estimate can be taken as an upper boundary limit for favourable ice-growth conditions; e.g. in the Baltic Sea the maximum observed thickness of undeformed ice is 122 cm whereas the Stefan estimate is 135 cm for the coldest winter observed.

Example 1 Ice thickness predicted by Stefan's law (S_0) and by Stefan's law with the correction for atmospheric coupling (S_A). Temperature is given with respect to the freezing point.

Time (d)	Temperature (°C)	Thickness (cm)	
		S_0	S_A
1	−10	10	4
5	−10	23	15
10	−10	33	24
50	−10	74	65
100	−10	104	95
100	−20	148	138
200	−20	209	199

Stefan's law can also be used to estimate the equilibrium thickness H_e of multi-year ice. Starting at the end of the growth season, during the next year the ice first melts by ΔH and then grows according to (11) from $H_e - \Delta H$ to H_e. Therefore

$$H_e^2 = (H_e - \Delta H)^2 + a^2 S \tag{12}$$

which yields

$$H_e = \tfrac{1}{2} a^2 S / \Delta H + \tfrac{1}{2} \Delta H \tag{13}$$

For $\sqrt{a^2 S} = 200$ cm and $\Delta H = 50$ cm we have $H_e = 425$ cm. The solution is rather sensitive to $\sqrt{a^2 S}$ and ΔH: e.g. for $\Delta H = 40$ or 60 cm, H_e is 520 or 363 cm, and for $\sqrt{a^2 S} = 190$ or 210 cm, H_0 is 386 or 466 cm. In light of the approximate values of these input data and the model sensitivity, the output shows reasonable agreement with the typical equilibrium thickness of 3–4 m in Maykut and Untersteiner (1971).

The main problem with Stefan's law is the poor knowledge about the top boundary condition. Usually T_0 must be estimated from the air temperature, which is very difficult to do when the ice is thin or when there is a snow cover on the ice. In addition neglecting the heat flux from the water may sometimes lead to badly wrong results. These limitations have given rise to empirical modifications of Stefan's law, produced simply by using an empirical coefficient a_* instead of a such that $0.5 \leqslant a_*/a \leqslant 1$.

Stefan's law assumes that the thermodynamic properties of sea ice are constants. The specific heat has no role here since the thermal inertia is neglected. The heat conductivity depends on the temperature and would generally be slightly lower in the lower layers. This has, however, a minor effect on the results. Full models can be solved only numerically as in Maykut and Untersteiner (1971). However, it is possible to extend analytic models from Stefan's law to various kinds of interesting cases using certain simplifying assumptions. These analytic models are most helpful for a qualitative understanding of the ice-growth problem and are the subject of the following sections.

b *Ice-Atmosphere Coupling*

Usually one knows the atmospheric surface temperature at some altitude but not the ice surface temperature T_0 itself. Assume there is no snow on the ice. Then an ice/air heat exchange formula must be established to estimate T_0. For simplicity, we take

$$Q_a = \kappa_a(T_0 - T_a) \tag{14}$$

where κ_a is a heat exchange coefficient. The continuity condition for the heat flux at the ice-air interface states that

$$\kappa_a(T_0 - T_a) = \kappa_i(T_f - T_a)/H \tag{15}$$

Solving now for T_0, the ice-growth equation (11) transforms into

$$\rho_i L dH/dt = \kappa_i(T_f - T_a)/(H + \kappa_i/\kappa_a) \tag{16}$$

and the solution is

$$H = \sqrt{H_0^2 + a^2 S + (\kappa_i/\kappa_a)^2} - (\kappa_i/\kappa_a) \tag{17}$$

The value of κ_i/κ_a is about 10 cm. This growth law was presented in Anderson (1961). The ice-atmosphere coupling has a major affect on ice growth in its early stage. The pure Stefan's law case largely overestimates the speed of ice growth when the ice is still thin.

Example 2 The ice thickness predicted with Stefan's law that has been modified to include the ice-air heat exchange has been calculated and the results are shown in the table of Example 1. The ratio κ_i/κ_a was 10 cm. In the early stage of growth the difference is relatively large, and as time increases the coupling effect reduces the ice thickness by κ_i/κ_a.

c *Snow: Insulation*

The snow cover introduces to the ice-growth law a modification similar to that of the ice/atmosphere coupling effect. However, the snow problem is much more complex owing to large temporal variations of the snow properties.

Extension of the ice-growth differential equation to an ice/snow two-layer system involves adding (5)–(6) to the ice model. The assumptions (i)–(iii) made for the ice in Section 3a are also applied for the snow cover. The snow temperature profile becomes linear, and the system can be solved similarly as for the atmospheric coupling case (Zubov, 1945; Simpson, 1958). The continuity condition for the heat flow at the ice/snow interface then gives

$$\rho_i L dH/dt = \kappa_i(T_f - T_s)/[H + (\kappa_i/\kappa_s)h] \tag{18}$$

where T_s is the snow surface temperature. Since h (and also κ_s) depends on time this equation cannot be solved analytically in general form.

Assume now simply that the snow thickness is highly correlated with ice thickness so that

$$h \approx \lambda H \tag{19}$$

and that the thermal conductivity of snow is constant. With these assumptions (18) gives for $H_0 = 0$

$$H^2 = 2\kappa_i S/[\rho_i L(1 + \lambda\kappa_i/\kappa_s)] \tag{20}$$

A comparison with Stefan's law solution (11) shows that the snow effect reduces the ice thickness by a factor of $(1 + \lambda\kappa_i/\kappa_s)^{-1/2}$. This solution is valid as long as all the snow is above the sea surface level. If the snowfall is so large that the ice becomes submerged, slush formation in the bottom layer of snow introduces a new problem. Archimedes' law states that for no submergence $\lambda \leqslant (\rho_w - \rho_i)/\rho_s \approx 0.3$. For a typical situation $\kappa_i \approx 10\kappa_s$ the snow reduction factor lies between $1/2$ and 1. No specific example is given for the present section, but if the ice thicknesses of examples 1 and 2 are halved the maximum snow reduction effect is obtained.

d *Snow-Ice*

The other snow effect is the snow-ice formation. When the snow weight is large enough the ice surface submerges and slush is formed. This slush freezes easily: less heat is released in slush freezing since the mixture already contains frozen water (snow), and this heat needs to be conducted basically through the snow only. The growth equation of snow-ice thickness H_{si} is therefore

$$\rho_i L(1 - \rho_s/\rho_i) dH_{si}/dt = \kappa_s(T_f - T_s)/h \tag{21}$$

This equation is an approximation in the sense that the freezing front proceeds downward and a thin layer of snow-ice may already lie on top of the slush. However, the heat conduction through the snow is here the determining factor. An additional question in solving (21) is the availability of slush. Only more or less continuous snowfall may produce new slush while the snow-ice layer thickens.

In an extreme case, only snow-ice is formed and the rate of snowfall is just enough to produce new slush continuously. Then $h = [(\rho_w - \rho_i)/\rho_s]H_{si}$ and (21) give the solution

$$H_{si}^2 = 2\rho_s\kappa_s S[\rho_i L(1 - \rho_s/\rho_i)(\rho_w - \rho_i)]^{-1} \tag{22}$$

Comparison with Stefan's law shows that snow-ice also grows proportional to $\sqrt{S}$ but the proportionality factor is modified by $\{(\kappa_s/\kappa_i)[\rho_s/(\rho_w - \rho_i)]/(1 - \rho_s/\rho_i)\}^{1/2}$. For $\kappa_s/\kappa_i = 1/10$ and $(\rho_w, \rho_i, \rho_s) = (1.0, 0.9, 0.3)$ g cm^{-3} this factor is 0.71. Thus, a suitably large snowfall rate gives the maximum snow-ice thickness, which lies in the middle between the thicknesses produced by bare ice growth and by a maximum snow insulation effect, when the freezing degree-days are equal.

e *Heat Flux From the Water*

The heat flux from the water reduces the ice growth at the bottom. This heat flux may melt ice, or to prevent melting it must be conducted away through the ice. The ice-growth equation is then

$$\rho_i L dH/dt = \kappa_i (T_f - T_0)/H - Q_w \tag{23}$$

This equation cannot in general be solved analytically. Typical Arctic values for Q_w are 1–5 W m^{-2} (Badgley, 1966; Maykut, 1986), and therefore the reduction effect to differential ice growth $Q_w/\rho_i L$ is typically 0.1 to 1 cm d^{-1}. There is a steady-state equilibrium thickness H_* where the oceanic heat flux equals the heat conduction through the ice, i.e. $H_* = \kappa_i(T_f - T_0)/Q_w$. In typical Arctic conditions the value of H_* is of the order of 10 m; this is never really reached because the growth season is too short. In regions with high values of the oceanic heat flux the steady-state value may be a very useful guide for understanding the ice thickness evolution.

Assume now that the surface temperature T_0 is constant. Then (23) can be solved but the solution becomes a non-linear algebraic equation for H:

$$-H/H_* - \log(1 - H/H_*) = Q_w t/\rho_i L H_* \tag{24}$$

Example 3 The solution of ice growth in the presence of oceanic heat flux Q_w. The air temperature is 10°C below the freezing point.

Time (d)	Ice Thickness (cm)	
	$Q_w = 5$ W m^{-2}	$Q_w = 0$
5	15	23
10	21	33
100	64	104
∞	400	∞

In general the oceanic heat flux problem is quite poorly understood. Usually this heat flux is taken as externally specified. Advanced coupled ice-ocean thermodynamic models are developed to examine this question further (e.g. Mellor et al., 1986; Lemke, 1987; Houssais, 1988; Svensson and Omstedt, 1990). In reverse form, ice-growth measurements can in fact be used to measure the oceanic heat flux (Uusitalo, 1973; McPhee and Untersteiner, 1982).

f *Ice Rubble*

One particular question in ice growth is the consolidation of ice rubble. Here also analytic modelling gives first-order approximations (Leppäranta and Hakala, 1992). A rubble field is composed of ice blocks and voids filled with sea water. In the consolidation process one needs to freeze the voids only, and therefore the latent heat released in freezing must be reduced by a factor of ν, the porosity of the rubble. Then Stefan's law applies; the coefficient a in Stefan's law was proportional to $L^{-1/2}$, and therefore

$$H_r = a\nu^{-1/2}\sqrt{S} \tag{25}$$

where H_r is the thickness of the consolidated layer. It is assumed that the conductivity of the rubble blocks is the same as that of undeformed ice.

Since $\nu \approx 0.3$, rubble consolidates about twice as fast as a normal ice sheet grows. This result is strongly supported by observations (Leppäranta and Hakala, 1992). The consolidated layer in first-year ice ridges is found to be up to twice as thick as the surrounding undeformed ice. Similarly as in Section 3a, it is easy to see that the equilibrium consolidation thickness of multi-year ridges is proportional to ν^{-1}, i.e. 3–4 times the undeformed ice thickness.

4 Discussion of numerical models

Analytic models of the thermal growth of sea ice provide a powerful method for examining the physics of the problem and for obtaining first-order approximations for the ice thickness in different environmental conditions. Quantitative analysis of sea-ice growth, however, must be made using numerical models. The full (vertical) sea-ice heat conduction equation was numerically solved first by Maykut and Untersteiner (1971), and this model has been the basic advanced model until now.

In numerical modelling the treatment of the heat flow within the ice and snow is relatively easy but major questions arise regarding the boundary conditions. The general features such as the equilibrium cycle are very much directly dictated by the external conditions at the boundaries. Owing to the dependency of the heat flux through the upper boundary from the underlying ice and snow, this boundary condition needs to be solved simultaneously with the heat conduction equations.

In the Maykut and Untersteiner (1971) model the parameters are constant except for the heat capacity (density times specific heat) and the thermal conductivity of ice. For these, semi-empirical equations as functions of temperature and salinity are used. The salinity distribution within the ice is prescribed. The resulting annual equilibrium cycle of ice and snow thickness was quite well representative of typical Arctic multi-year ice. The major uncertainties in the model results were caused by the uncertainties in specifying the surface albedo, snowfall and oceanic heat flux. The sensitivity of the ice growth to these quantities was clearly illustrated by the analytic analysis in Section 3.

Considering the major role of the external input in the ice-growth problem one may regard the heat flow within the ice and snow as being "too finely" resolved in the Maykut-Untersteiner model. This is in fact very much true for large-scale and long-term investigations. A highly simplified version of Semtner (1976) for the heat flow through the ice and snow gave results very similar to those of Maykut and Untersteiner (1971) and has been widely used in climatological models.

To investigate detailed short-term thermodynamic processes within the ice, the Maykut-Untersteiner model needs to be improved to include the ice salinity as a model variable. Sea ice is a three-phase medium with pure ice, brine inclusions and solid salt crystals. The thermodynamics of this medium is fully described using the heat conduction equation, the phase diagram and the salinity conservation law. Both these equations are necessary since the temperature and salinity distributions are highly coupled. Such a full thermodynamic model, not yet presented and solved, is to be regarded as the next-generation thermodynamic sea-ice model. The full model can be specified by the Maykut-Untersteiner heat conduction equation coupled with the salinity conservation law.

The first trial for a temperature-salinity coupled model was made by Cox and Weeks (1988). The resulting salinity profiles showed realistic properties and illustrated the large variability of the evolution of the salinity distribution. The ice salinity changes by vertical advection in the ice interior and by losses and gains at the boundaries. The advection is caused by brine drainage and expulsion but is not yet well understood. At the bottom, salt is released to sea water, and as the result of freezing, new salt is added to the ice. If flooding occurs, the slush layer on top is salty. Then during the process of snow-ice growth, the slush layer becomes thinner and its salinity increases as a result of brine rejection from the growing snow-ice. Developing the parametrization of a salinity model needs much more work, both modelling trials and field observations. In addition to allowing more detailed examinations of sea-ice thermodynamic processes, the use of the salinity conservation law provides better possibilities for the development of ice-ocean coupled models.

5 Concluding remarks

In this review the thermodynamic problem of sea-ice growth has been examined using analytic models. Such models are classical tools in ice-growth modelling and are still very important for understanding the physics of this phenomenon. Numerical methods are, however, necessary so that the quantitative results may overcome the complications introduced by the variable thermal properties of ice, by the snow cover and by the boundary conditions.

Starting from the basic Stefan's law for bare ice growth, the addition of ice-atmosphere coupling, snow cover and oceanic heat flux was considered. The snow effect is very complicated, since snow is an insulator but may also enhance ice growth because of snow-ice formation. In an average sense, a moderate snowfall rate reduces the ice growth by a factor down to $^1/_2$, but then for larger snowfall rates the factor increases up to $^3/_4$. The presence of oceanic heat flux introduces an asymptotic stationary ice thickness which, however, is rarely reached. Stefan's law may also be modified for analysing ice rubble and ridges. The results show that rubble grows twice as fast as a normal ice sheet, and the equilibrium thickness of multi-year ridges is 3–4 times that of undeformed ice, i.e. 10–15 m.

Analytic models could be further extended to examine additional interesting questions such as the effects of the variability of the thermal properties of ice and snow and ice-ocean coupling.

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