PHYS 357 Pset 9. Due 11:59 PM Thursday Nov. 21

1. Townsend 6.4.

2. Balancing a pencil. What's the longest can you balance a pencil on its tip? The Heisenberg uncertainty principle means I can't start with both $x = 0$ and $p = 0$. I'm after a ballpark answer here, so you can e.g. assume all the mass in the pencil is at the end away from the tip. You can use 10g and 20cm for the pencil weight/length. NB - a pencil balanced on its tip looks very much like a pendulum, but with a sign change in the equation of motion.

3. Townsend 6.6

4. Townsend 6.12

5. Repeat Townsend 6.4, but this time do it on a computer. We can approximate a free particle by having a long stretch of space with zero potential. Find the eigenvalues/eigenvectors of this free space (this is fastest using scipy.linalg.eigh_tridiagonal, but you could certainly use numpy.linalg.eigh if you wanted), and describe a Gaussian well away from the boundary region as the sum of these eigenmodes. Make a movie showing the evolution of the Gaussian as it spreads out. Does your time for the width to double agree with your calculation from 6.4? Now make the same movie, but using a boxcar initial wave function ($\Psi(x) = 1$ for $0 < x < 1$, and zero otherwise).

6. Bonus - repeat the previous problem, but now use the Fourier transforms built into numpy (numpy.fft.rfft and irfft will be easiest to use). The discrete Fourier transform is defined to be

$$\sum_{0}^{N-1} f(x) \exp(-2\pi i k x/N)$$

for integer x, k, and $0 \leq x, k < N$. You'll need to take care with numerical factors and tune the spacings of your points, but you'll see that you can handle these free-space questions much, much faster with FFTs than with direct matrix inversions.

6.4.

(a) Show for a free particle of mass m initially in the state

$$\psi(x) = \langle x|\psi\rangle = \frac{1}{\sqrt{\sqrt{\pi}\,a}}\, e^{-x^2/2a^2}$$

that

$$\psi(x,t) = \langle x|\psi(t)\rangle = \frac{1}{\sqrt{\sqrt{\pi}\,[a+(i\hbar t/ma)]}}\, e^{-x^2/\{2a^2[1+(i\hbar t/ma^2)]\}}$$

and therefore

$$\Delta x = \frac{a}{\sqrt{2}}\sqrt{1+\left(\frac{\hbar t}{ma^2}\right)^2}$$

Suggestion: Start with (6.75) and take advantage of the Gaussian integral (D.7), but in momentum space instead of position space.

(b) What is Δp_x at time t? *Suggestion:* Use the momentum-space wave function to evaluate Δp_x.

a) ① $\psi(x) = \langle x|\psi\rangle = \frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x^2/2a^2} \longrightarrow \langle p|\psi\rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\, e^{-p^2a^2/2\hbar^2}$ initial state

and $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

② $\psi(x,t) = \langle x|\psi(t)\rangle = \int dp\, e^{-ip^2t/2m\hbar} \langle x|p\rangle\langle p|\psi(0)\rangle$

$$= \int dp\, e^{-ip^2t/2m\hbar} \cdot \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\, e^{-p^2a^2/2\hbar^2}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\int_{-\infty}^{\infty} dp\, \exp\left[-\left(\frac{it}{2m\hbar}+\frac{a^2}{2\hbar^2}\right)p^2 + \left(\frac{ix}{\hbar}\right)p\right]$$

Gaussian integral: $I(\alpha,b) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2+bx} = e^{b^2/4\alpha}\sqrt{\frac{\pi}{\alpha}}$

so $\alpha = \frac{it}{2m\hbar}+\frac{a^2}{2\hbar^2} = \frac{it\hbar+a^2m}{2m\hbar^2}$ and $b = \frac{ix}{\hbar}$

$$= \frac{1}{\sqrt{\cancel{2\pi\hbar}}}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\exp\left[\left(\frac{ix}{\hbar}\right)^2\cdot\frac{1}{\cancel{4}_{\,2}}\cdot\frac{\cancel{2}m\cancel{\hbar^2}}{it\hbar+a^2m}\right]\sqrt{\frac{\pi\,\cancel{2}m\hbar^2}{it\hbar+a^2m}}$$

$$= \sqrt{\frac{am}{\sqrt{\pi}(it\hbar+a^2m)}}\,\exp\left[\frac{-x^2}{2}\,\frac{\cancel{m}}{a^2\cancel{m}(it\hbar/a^2m+1)}\right]$$

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}(a+it\hbar/am)}}\exp\left[\frac{-x^2}{2a^2(1+it\hbar/a^2m)}\right]}$$

③ $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle}$, $\langle x\rangle = 0$, with $\langle x^2\rangle = \int dx\, \underbrace{|\psi(x)|^2}_{=\psi^*\psi} x^2$

$$\psi(x,t)^*\psi(x,t) = \frac{1}{\sqrt{\pi}}\underbrace{\frac{1}{\sqrt{a+it\hbar/am}}\cdot\frac{1}{\sqrt{a+it\hbar/am}}}_{zz^* = |z|^2 = Re^2+Im^2}\cdot\exp\left[\frac{-x^2}{2a^2}\cdot\underbrace{\left(\frac{1}{1+it\hbar/a^2m}+\frac{1}{1-it\hbar/a^2m}\right)}_{=\frac{2}{1+(t\hbar/a^2m)^2}}\right]$$

conj.

$$= \frac{1}{\sqrt{\pi(a^2+(t\hbar/am)^2)}}\exp\left[\frac{-x^2}{a^2}\,\frac{1}{1+(t\hbar/a^2m)^2}\right]$$

so $\langle x^2\rangle = \frac{1}{\sqrt{\pi(a^2+(t\hbar/am)^2)}}\int dx\, \exp\left[\frac{-x^2}{a^2}\cdot\frac{1}{1+(t\hbar/a^2m)^2}\right]x^2$

$\int dx\, \frac{1}{\sqrt{\pi}A} e^{-x^2/A^2} x^2 = \frac{A^2}{2}$, here $A^2 = a^2(1+(t\hbar/a^2m)^2)$

$\langle x^2\rangle = \frac{1}{2}a^2(1+(t\hbar/a^2m)^2)$

$$\Rightarrow \Delta x = \sqrt{\langle x^2 \rangle} = \frac{a}{\sqrt{2}}\left[1+\left(\frac{t\hbar}{a^2 m}\right)^2\right]^{1/2}$$

b) ① $$\langle p|\psi\rangle = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \underbrace{\langle x|\psi\rangle}$$

$$= \frac{1}{\sqrt{\sqrt{\pi}\,(a+it\hbar/am)}} \exp\left[\frac{-x^2}{2a^2(1+it\hbar/a^2m)}\right]$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{\pi}\,(a+it\hbar/am)}} \int dx \exp\left[\underbrace{\frac{-x^2}{2a^2(1+it\hbar/a^2m)}}_{A} + \underbrace{\left(\frac{-ip}{\hbar}\right)}_{B} x\right]$$

$$\int dx\, e^{-Ax^2+Bx} = e^{B^2/4A}\sqrt{\frac{\pi}{A}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{\pi}\,(a+it\hbar/am)}} \exp\left[\left(\frac{-ip}{\hbar}\right)^2 \cdot \frac{1}{4} \cdot 2a^2(1+it\hbar/a^2m)\right] \sqrt{\pi}\sqrt{2a^2(1+it\hbar/a^2m)}$$

$$\psi(p,t) = \sqrt{\frac{a}{\hbar\sqrt{\pi}}} \exp\left[-\frac{p^2}{2\hbar^2}(a^2+i\hbar t/m)\right]$$

② $$|\psi(p,t)|^2 = \psi(p,t)^* \psi(p,t)$$

$$= \frac{a}{\hbar\sqrt{\pi}} \cdot \exp\left[\frac{-p^2}{2\hbar^2}(a^2-i\hbar t/m) - \frac{p^2}{2\hbar^2}(a^2+i\hbar t/m)\right]$$

$$= \frac{a}{\hbar\sqrt{\pi}} \exp\left[-\left(\frac{pa}{\hbar}\right)^2\right]$$

③ $$\Delta p_x = \sqrt{\langle p_x^2\rangle - \underbrace{\langle p_x\rangle}_{0}^2}$$

$$\rightarrow \langle p_x^2 \rangle = \int dp\, |\psi(p,t)|^2 p^2$$

$$= \int dp \frac{a}{\hbar\sqrt{\pi}} e^{-p^2a^2/\hbar^2} p^2$$

$$= \frac{\hbar^2}{2a^2}$$

$$\Rightarrow \Delta p_x = \frac{\hbar}{\sqrt{2}\,a}$$

2. Balancing a pencil. What's the longest can you balance a pencil on its tip? The Heisenberg uncertainty principle means I can't start with both $x = 0$ and $p = 0$. I'm after a ballpark answer here, so you can e.g. assume all the mass in the pencil is at the end away from the tip. You can use 10g and 20cm for the pencil weight/length. NB - a pencil balanced on its tip looks very much like a pendulum, but with a sign change in the equation of motion.

① pendulum: $\ddot{\theta} = \frac{g}{l}\theta$ (assuming small angle)

solution: $\theta(t) = Ae^{t\sqrt{g/l}} + Be^{-t\sqrt{g/l}}$

Initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = \omega_0$ $\to A + B = \theta_0$

$\to A\sqrt{g/l} - B\sqrt{g/l} = \omega_0$ $\Rightarrow$ $A = \frac{1}{2}(\sqrt{l/g}\,\omega_0 + \theta_0)$, $B = \frac{1}{2}(\theta_0 - \sqrt{l/g}\,\omega_0)$

so we have $\theta(t) = \frac{1}{2}(\theta_0 + \sqrt{l/g}\,\omega_0)e^{t\sqrt{g/l}} + \frac{1}{2}(\theta_0 - \sqrt{l/g}\,\omega_0)\underbrace{e^{-t\sqrt{g/l}}}$

as t increases, this term doesn't contribute a lot $\to$ Neglect

$\Rightarrow \theta(t) = \frac{1}{2}(\theta_0 + \sqrt{l/g}\,\omega_0)e^{t\sqrt{g/l}}$

② uncertainty principle: $\Delta x \Delta p \geq \frac{\hbar}{2}$

$\downarrow \Delta x = \theta_0 l$, $\Delta p = ml\omega_0$

$\theta_0 l \cdot ml\omega_0 \geq \frac{\hbar}{2}$

min uncertainty

$\omega_0 = \frac{\hbar}{2\theta_0 m l^2}$

$\to \theta(t) = \frac{1}{2}(\theta_0 + \sqrt{\frac{l}{g}} \cdot \frac{\hbar}{2\theta_0 m l^2})e^{t\sqrt{g/l}}$

③ maximize balance time $\Rightarrow$ minimize the initial

$\to$ minimize $\theta_0 + \sqrt{l/g}\,\frac{\hbar}{2\theta_0 m l^2}$

$\downarrow$

$0 = 1 + \sqrt{l/g}\,\frac{\hbar}{2ml^2} \cdot (-1)\theta_0^{-2}$

$-1 = -\sqrt{l/g}\,\frac{\hbar}{2ml^2} \cdot \frac{1}{\theta_0^2}$

$\theta_0^2 = \omega_0 \frac{\hbar}{2ml^2}$

$\theta_0 = \sqrt{\frac{\omega_0 \hbar}{2ml^2}}$

④ critical time is given by $t = \omega_0 \ln(\frac{1}{\theta_0})$

$= \sqrt{l/g}\,\ln\left(\sqrt{\frac{2ml^2}{\hbar\sqrt{l/g}}}\right)$

$\downarrow$ wolfram with $m = 10g$, $l = 20cm$

$t = 5.266\ s$

6.6.

(a) Show that $\langle p_x \rangle = 0$ for a state with a *real* wave function $\langle x|\psi\rangle$.

(b) Show that if the wave function $\langle x|\psi\rangle$ is modified by a position-dependent phase

$$\langle x|\psi\rangle \to e^{ip_0 x/\hbar}\langle x|\psi\rangle$$

then

$$\langle x\rangle \to \langle x\rangle \quad \text{and} \quad \langle p_x\rangle \to \langle p_x\rangle + p_0$$

a) ① Real wave function $\langle x|\psi\rangle = \psi(x) = \psi(x)^* = \langle \psi | x\rangle$

② so we have $\langle p_x\rangle = \int dx \langle \psi|x\rangle \frac{\hbar}{i}\partial_x \langle x|\psi\rangle$

$$= \frac{\hbar}{i}\underbrace{\int dx\, \psi(x)\partial_x\psi(x)}$$

integ. by parts:
$u = \psi \;.\; dv = \partial_x\psi\, dx$
$du = \psi dx \;,\; v = \partial_x\psi$
$\to \int = \underbrace{\psi^2|_{-\infty}^{\infty}}_{=0,\ \psi \text{ vanishes at inf}} - \int \partial_x\psi \cdot \psi dx$

$$\langle p_x\rangle = \underbrace{\frac{\hbar}{i}\Big(-}\underbrace{\int dx\, \psi\, \partial_x\psi}_{\langle p_x\rangle}\Big)$$

$$\langle p_x\rangle = -\langle p_x\rangle$$

so we must have $\langle p_x\rangle = 0$

b) $\langle x|\psi\rangle \to e^{ip_0x/\hbar}\langle x|\psi\rangle = \langle x|\psi\rangle' = \psi'$

① $|\psi'|^2 = \psi'^*\psi' = e^{-ip_0x/\hbar}e^{ip_0x/\hbar}\psi^*\psi = \psi^*\psi = |\psi|^2$

so $\langle x\rangle' = \int dx\, |\psi'|^2 x^2 = \int dx |\psi|^2 x^2 = \langle x\rangle$ (unchanged)

② $\langle p_x\rangle' = \int dx \langle\psi|x\rangle'^* \frac{\hbar}{i}\partial_x\langle x|\psi\rangle'$

$$= \int dx\, e^{-ip_0x/\hbar}\psi^* \frac{\hbar}{i}\partial_x(e^{ip_0x/\hbar}\psi)$$

$$= \frac{\hbar}{i}\int dx e^{-ip_0x/\hbar}\psi^*\Big[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi + e^{ip_0x/\hbar}\partial_x\psi\Big]$$

$$= p_0\underbrace{\int dx\, \psi^*\psi}_{=\int|\psi|^2dx \\ =1} + \underbrace{\frac{\hbar}{i}\int dx\, \psi^*\partial_x\psi}_{\langle p_x\rangle}$$

$\langle p_x\rangle' = p_0 + \langle p_x\rangle$

6.12. The normalized wave function for a free particle is given by

$$\langle x|\psi\rangle = \begin{cases} \sqrt{\dfrac{2}{a}}\cos\dfrac{\pi x}{a} & |x| \le a/2 \\ 0 & |x| > a/2 \end{cases}$$

Such a state might be created by putting the particle into the ground state of the potential energy box discussed in Section 6.9 and then instantaneously removing the potential. What is the probability that a measurement of the momentum yields a value between p and $p + dp$? Your final answer should not involve any complex numbers, since the probability of having momentum between p and $p + dp$ is a real quantity. Simplify your answer as much as possible. Suggest a strategy for measuring this probability.

① $$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \langle x|\psi\rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-a/2}^{a/2} dx\, e^{-ipx/\hbar} \cdot \sqrt{\frac{2}{a}} \underbrace{\cos\left(\frac{\pi x}{a}\right)}_{\cos\left(\frac{\pi x}{a}\right) = \frac{1}{2}\left(e^{i\pi x/a} + e^{-i\pi x/a}\right)}$$

$$= \sqrt{\frac{1}{a\pi\hbar}} \int dx\, e^{-ipx/\hbar} \frac{1}{2}\left(e^{i\pi x/a} + e^{-i\pi x/a}\right)$$

$$= \sqrt{\frac{1}{4\pi\hbar a}} \Big[\underbrace{\int dx\, e^{ix(\pi/a - P/\hbar)}}_{= \frac{-i e^{ix(\pi/a - P/\hbar)}}{(\pi/a - P/\hbar)}} + \underbrace{\int dx\, e^{-ix(\pi/a + P/\hbar)}}_{= \frac{i e^{-ix(\pi/a + P/\hbar)}}{(\pi/a + P/\hbar)}} \Big]$$

$$= \sqrt{\frac{1}{4\pi\hbar a}} \left[\frac{-i e^{ix(\pi/a - P/\hbar)}}{(\pi/a - P/\hbar)} + \frac{i e^{-ix(\pi/a + P/\hbar)}}{(\pi/a + P/\hbar)} \right] \Bigg|_{x=-a/2}^{x=a/2}$$

$$\frac{-i}{\pi/a - P/\hbar} \underbrace{\left(e^{i(\pi/a - P/\hbar)a/2} - e^{-i(\pi/a - P/\hbar)a/2}\right)}_{= 2i\sin(\pi/2 - Pa/2\hbar)}$$

$$\frac{i}{(\pi/a + P/\hbar)} \underbrace{\left(e^{-i(\pi/a + P/\hbar)a/2} - e^{i(\pi/a + P/\hbar)a/2}\right)}_{= -2i\sin(\pi/2 + Pa/2\hbar)}$$

$$= \sqrt{\frac{1}{4\pi\hbar a}} \left[\frac{-i \cdot 2i\sin(\pi/2 - Pa/2\hbar)}{\pi/a - P/\hbar} - \frac{i \cdot 2i\sin(\pi/2 + Pa/2\hbar)}{\pi/a + P/\hbar} \right]$$

$$= \frac{1}{\sqrt{\pi\hbar a}} \left[\frac{\sin(\pi/2 - Pa/2\hbar)}{\pi/a - P/\hbar} + \frac{\sin(\pi/2 + Pa/2\hbar)}{\pi/a + P/\hbar} \right]$$

② prob of momentum to be between p & $p+dp$ is

$$\text{Prob}(P) = |\langle p|\psi\rangle|^2 dp$$

$$= dp \frac{1}{\pi\hbar a} \left[\frac{\sin(\pi/2 - Pa/2\hbar)}{\pi/a - P/\hbar} + \frac{\sin(\pi/2 + Pa/2\hbar)}{\pi/a + P/\hbar} \right]^2$$

5. Repeat Townsend 6.4, but this time do it on a computer. We can approximate a free particle by having a long stretch of space with zero potential. Find the eigenvalues/eigenvectors of this free space (this is fastest using scipy.linalg.eigh_tridiagonal, but you could certainly use numpy.linalg.eigh if you wanted), and describe a Gaussian well away from the boundary region as the sum of these eigenmodes. Make a movie showing the evolution of the Gaussian as it spreads out. Does your time for the width to double agree with your calculation from 6.4? Now make the same movie, but using a boxcar initial wave function ($\Psi(x) = 1$ for $0 < x < 1$, and zero otherwise).

```
import numpy as np
from scipy.linalg import eigh_tridiagonal
from matplotlib import pyplot as plt
```

✓ 0.0s

Setup

```
# Define the stretch of space with zero potential
x = np.linspace(-10,10,3001)
dx = x[1] - x[0]
a = 1  # length of box
V0 = 0  # potential inside box
V = 0 * np.ones((len(x)))
V[np.abs(x)>a/2] = V0  # set potential to zero inside the box

# Define Hamiltonian
vec = np.ones(len(x))/dx**2
H_diag = vec + V
H_offdiag = -0.5*vec[:-1]
E,psi = eigh_tridiagonal(H_diag,H_offdiag)  # eigenvalues (energies) & eigenstates (wavefunctions)
```

✓ 0.4s

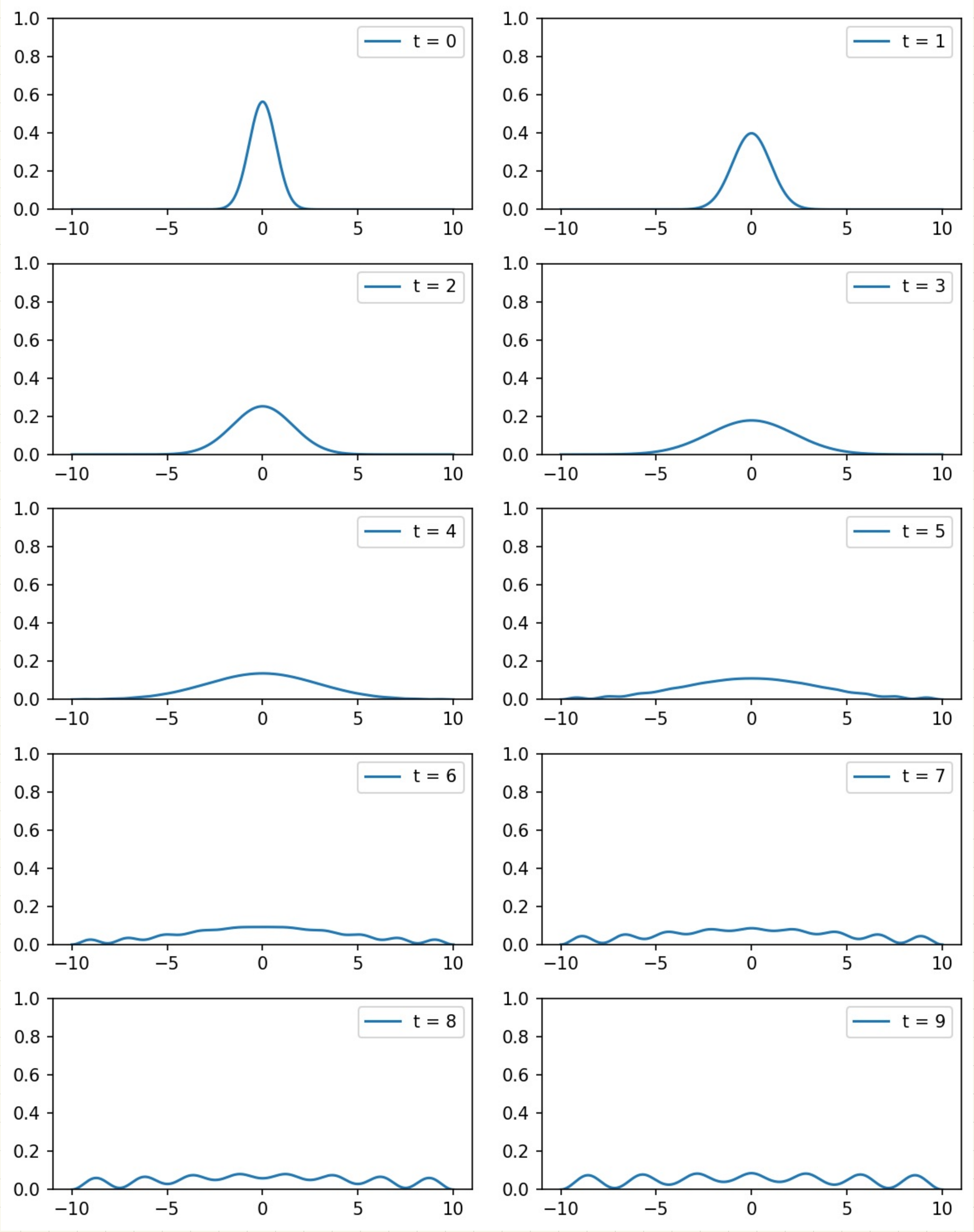
Gaussian (Redoing 6.4)

```
# Gaussian wave packet given by 6.4
psi0 = np.exp(-0.5*x**2 / a**2) / np.sqrt(np.sqrt(np.pi) * a)
amps0 = psi.T@psi0  # amplitudes

# Time evolution of the wavefunction
t_values = np.arange(0,10,1)
fig, axes = plt.subplots(int(len(t_values)/2),2, figsize=(8,10), dpi=150)
for i,t in enumerate(t_values):
    amps = amps0*np.exp(-1j*t*E)
    psi_cur = psi@amps
    ax = axes[i//2,i%2]
    ax.plot(x, np.abs(psi_cur)**2, label = f't = {t}')
    ax.set_ylim(0,1)
    ax.legend()

plt.tight_layout()
plt.show()
```

✓ 2.9s


t = 0
t = 1
t = 2
t = 3
t = 4
t = 5
t = 6
t = 7
t = 8
t = 9

from 6.4, we have $\Delta x = \sqrt{\langle x^2 \rangle} = \frac{a}{\sqrt{2}}\left[1 + \left(\frac{t\hbar}{a^2 m}\right)^2\right]^{1/2}$ so we are looking for t_{double} when

$$\Delta x(t_{double}) = 2\Delta x(0) \longrightarrow \frac{a}{\sqrt{2}}\left[1 + \left(\frac{t_{double}\hbar}{a^2 m}\right)^2\right]^{1/2} = 2 \cdot \frac{a}{\sqrt{2}}$$

set $a=1$, $m=1$, $\hbar=1$

$$(1+t^2)^{1/2} = 2$$
$$1 + t^2 = 4$$
$$t = \sqrt{3}$$

```python
# Checking half width
def fwhm(t, psi=psi, amps0=amps0, x=x):
    # Get wave function at time t
    amps = amps0*np.exp(-1j*t*E)
    psi_cur = psi@amps
    prob = np.abs(psi_cur)**2

    # Find the half width
    half_pos = len(prob)//2
    x_pos1 = np.argmin(np.abs(prob[:half_pos+1] - np.max(prob)/2))
    x_pos2 = np.argmin(np.abs(prob[half_pos:] - np.max(prob)/2)) + half_pos
    fwhm = np.abs(x[x_pos1]-x[x_pos2])  # half width
    print(f'At t={t:.2f}: {fwhm:.3f}')

    return fwhm

print(fwhm(np.sqrt(3))/fwhm(0))
```

✓ 0.3s

```
At t=1.73: 3.333
At t=0.00: 1.667
2.0000000000000001
```

As expected, we get $2 = \frac{\Delta x(t=\sqrt{3})}{\Delta x(t=0)}$

Boxcar

```python
# Boxcar wave function: 1 for 0 < x < 1, 0 elsewhere
psi0 = np.zeros(len(x))
psi0[(x > 0) & (x < 1)] = 1
amps0 = psi.T@psi0  # amplitudes

# Time evolution of the wavefunction
t_values = np.arange(0,0.9,0.1)
fig, axes = plt.subplots(int(len(t_values)/3),3, figsize=(10,10), dpi=150)
for i,t in enumerate(t_values):
    amps = amps0*np.exp(-1j*t*E)
    psi_cur = psi@amps
    ax = axes[i//3,i%3]
    ax.plot(x, np.abs(psi_cur)**2, label = f't = {t:.1f}')
    ax.set_ylim(0,1.5)
    ax.legend()

plt.tight_layout()
plt.show()
```

✓ 3.1s