PHYS 357 Pset 1. Due 11:59 PM Thursday Sep. 12

1. Starting with the expressions for the $|\pm Z\rangle$, $|\pm X\rangle$, and $|\pm Y\rangle$ states, show that the amplitude for a state to be in the opposite direction along the same axis is zero (*i.e.* $\langle +X| -X\rangle = 0$ *etc.*) for all three directions. Similarly, show that the *magnitude* of the amplitude for any state to be in any state along a different axis is $\frac{1}{\sqrt{2}}$ (*i.e.* $|\langle +Y| \pm X\rangle| = \frac{1}{\sqrt{2}}$ *etc.*). Feel free to express this as compactly as possible - I don't need 30 separate answers. Also, you are allowed to rely on the fact that $\langle \pm Z| \pm Z\rangle = 1$ and $\langle \mp Z| \pm Z\rangle = 0$.

2. Townsend problem 1.3

3. Townsend problem 1.6

4. a) If I have a unit vector pointing in the (θ, ϕ) direction, what are θ and ϕ for the unit vector pointing in the opposite direction? By opposite, if I expressed the vector in Cartesian coordinates $v = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ then the vector in the opposite direction is $-v_x\hat{x} - v_y\hat{y} - v_z\hat{z}$.
   b) If your original unit vector gives you the $|+n\rangle$ state from problem 1.3, show that when you plug in the opposite direction, you get the $|-n\rangle$ state from problem 1.6.
   c) Show that a state (again as defined in problem 1.3) in the (θ', ϕ') direction can be expressed as the sum of the $|+n\rangle$ and $|-n\rangle$ states in the (θ, ϕ) direction (note the lack of primes on on these states), *i.e.*$|n'\rangle = c_+ |+n\rangle + c_- |-n\rangle$. You should get a 2x2 matrix equation for c_+ and c_-, feel free to leave your answer in this form.
   d) What is the determinant of the matrix from part c)? As a reminder, the determinant of a 2x2 matrix is $ad - bc$. You should get something with magnitude 1. That tells us that we can always solve the system of equations in part c), which in turn tells us that we can pick any arbitrary $|+n\rangle$ and $|n'\rangle$, and express $|n'\rangle$ as a sum of $|+n\rangle$ and $|-n\rangle$.

5. Townsend problem 1.13.

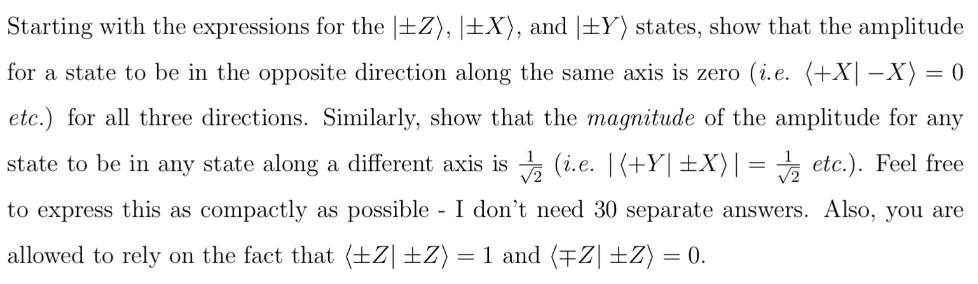
6. Townsend problem 1.15

7. Townsend problem 1.9

Audréanne Bernier (261100643)

Pset 1

Q1



Starting with the expressions for the $|\pm Z\rangle$, $|\pm X\rangle$, and $|\pm Y\rangle$ states, show that the amplitude for a state to be in the opposite direction along the same axis is zero (*i.e.* $\langle +X|-X\rangle = 0$ *etc.*) for all three directions. Similarly, show that the *magnitude* of the amplitude for any state to be in any state along a different axis is $\frac{1}{\sqrt{2}}$ (*i.e.* $|\langle +Y|\pm X\rangle| = \frac{1}{\sqrt{2}}$ *etc.*). Feel free to express this as compactly as possible - I don't need 30 separate answers. Also, you are allowed to rely on the fact that $\langle \pm Z|\pm Z\rangle = 1$ and $\langle \mp Z|\pm Z\rangle = 0$.

① we know $|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$, $|\pm y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm i|-z\rangle)$

for both we have $|\pm\psi\rangle = C_+|+z\rangle \pm C_-|-z\rangle$, and we know C_+^* & C_-^* will be the coeffs. for $\langle\pm\psi|$ so we get

$$\begin{aligned}\langle +\psi|-\psi\rangle &= (C_+^*\langle +z| + C_-^*\langle -z|)(C_+|+z\rangle - C_-|-z\rangle)\\ &= C_+^*C_+ + 0 + 0 - C_-^*C_-\\ &= \frac{1}{2} - \frac{1}{2}\\ &= 0\end{aligned}$$

from $|\pm x\rangle$ & $|\pm y\rangle$, $C_+ = \frac{1}{\sqrt{2}}$ in both cases & $C_- = \frac{1}{\sqrt{2}}\alpha$ with $|C_-| = \frac{1}{\sqrt{2}}$

$\alpha = 1$ or i depending on if we look at x or y

(similar for $\langle -\psi|+\psi\rangle$)

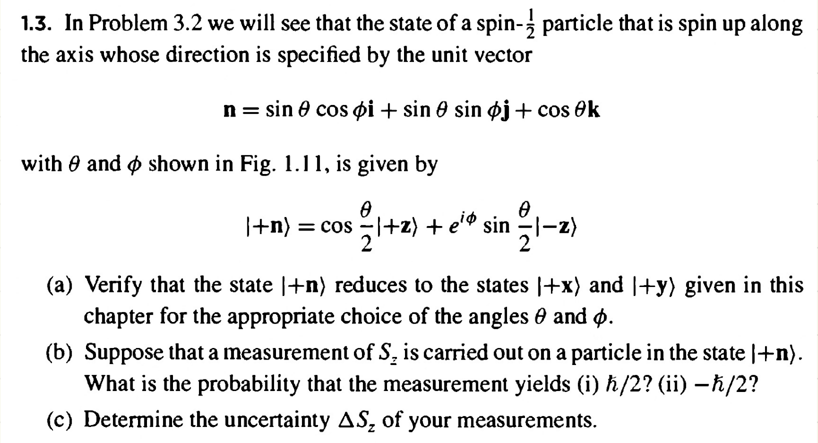
② $\langle\varphi|\psi\rangle$ with φ & ψ $\pm x, \pm y$ then,

$$\begin{aligned}|\langle\varphi|\psi\rangle| &= \left|\frac{1}{\sqrt{2}}(C_{+\varphi}\langle +z| + C_{-\varphi}\langle -z|)\frac{1}{\sqrt{2}}(C_{+\psi}|+z\rangle + C_{-\psi}|-z\rangle)\right|\\ &= \frac{1}{2}|\underbrace{C_{+\varphi}C_{+\psi} + C_{-\varphi}C_{-\psi}}|\end{aligned}$$

from the expressions above, we have $C_+ = 1$ & $C_- = \pm 1$ or $\pm i$
since we are picking states along diff. axes, we have $C_{-\varphi}C_{-\psi} = (\pm 1)(\pm i) = \pm i$

$$\begin{aligned}&= \frac{1}{2}|1 \pm i|\\ &= \frac{1}{2}\sqrt{2}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Q2



1.3. In Problem 3.2 we will see that the state of a spin-$\frac{1}{2}$ particle that is spin up along the axis whose direction is specified by the unit vector

$$\mathbf{n} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

with θ and ϕ shown in Fig. 1.11, is given by

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

(a) Verify that the state $|+\mathbf{n}\rangle$ reduces to the states $|+\mathbf{x}\rangle$ and $|+\mathbf{y}\rangle$ given in this chapter for the appropriate choice of the angles θ and ϕ.

(b) Suppose that a measurement of S_z is carried out on a particle in the state $|+\mathbf{n}\rangle$. What is the probability that the measurement yields (i) $\hbar/2$? (ii) $-\hbar/2$?

(c) Determine the uncertainty ΔS_z of your measurements.

a) $|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$

we know $|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$ & $|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}i|-z\rangle$

so we need $\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$ & $\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$ $\Rightarrow \frac{\theta}{2} = \frac{\pi}{4}$

$\theta = \pi/2$ (for both $|+x\rangle$ & $|+y\rangle$)

for $|+x\rangle$, we have $\varphi = 0$ and for $|+y\rangle$ we have $e^{i\varphi} = i \rightarrow \varphi = \pi/2$

b) ① $P(\hbar/2) = |\langle +z|+n\rangle|^2 = |\cos\theta/2|^2 = \cos^2(\theta/2)$

② $P(-\hbar/2) = |\langle -z|+n\rangle|^2 = |e^{i\varphi}\sin\theta/2|^2 = \sin^2(\theta/2)$

c) $\Delta S_z = (\langle S_z^2\rangle - \langle S_z\rangle^2)^{1/2}$

$$= \Big[\underbrace{\cos^2\theta/2(\hbar/2)^2 + \sin^2\theta/2(-\hbar/2)^2}_{=(\hbar/2)^2} - (\underbrace{\cos^2\theta/2\cdot\hbar/2 - \sin^2\theta/2\cdot\hbar/2}_{=(\frac{\hbar}{2}\cdot\cos\theta)^2})^2\Big]^{1/2}$$

$$= \Big[(\hbar/2)^2 - (\tfrac{\hbar}{2}\cos\theta)^2\Big]^{1/2}$$

$$= \left(\frac{\hbar}{2}\right)(1-\cos^2\theta)^{1/2}$$

$$= \frac{\hbar}{2}\sin\theta$$

Q3

1.6. Show that the state

$$|-\mathbf{n}\rangle = \sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle$$

satisfies $\langle+\mathbf{n}|-\mathbf{n}\rangle = 0$, where the state $|+\mathbf{n}\rangle$ is given in Problem 1.3. Verify that $\langle-\mathbf{n}|-\mathbf{n}\rangle = 1$.

① $|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\theta/2|-z\rangle$

$\langle+n| = \cos\theta/2\langle+z| + e^{-i\varphi}\sin\theta/2\langle-z|$

and $\langle-n| = \sin\theta/2\langle+z| - e^{-i\varphi}\cos\theta/2\langle-z|$

② $\langle+n|-n\rangle = (\cos\theta/2\langle+z| + e^{-i\varphi}\sin\theta/2\langle-z|)(\sin\theta/2|+z\rangle - e^{i\varphi}\cos\theta/2|-z\rangle)$

$= \cos\cancel{\frac{\theta}{2}}\sin\frac{\theta}{2} + 0 + 0 - \underbrace{e^{i\varphi}e^{-i\varphi}}_{1}\sin\cancel{\frac{\theta}{2}\cos}\frac{\theta}{2}$

$= 0$

③ $\langle-n|-n\rangle = (\sin\theta/2\langle+z| - e^{-i\varphi}\cos\theta/2\langle-z|)(\sin\theta/2|+z\rangle - e^{i\varphi}\cos\theta/2|-z\rangle)$

$= \sin^2\theta/2 + 0 + 0 + e^{i\varphi}e^{-i\varphi}\cos^2\theta/2$

$= 1$

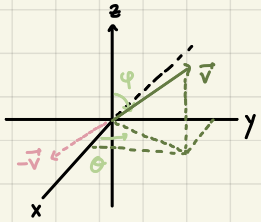
Q4 a) If I have a unit vector pointing in the (θ, ϕ) direction, what are θ and ϕ for the unit vector pointing in the opposite direction? By opposite, if I expressed the vector in Cartesian coordinates $v = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ then the vector in the opposite direction is $-v_x\hat{x} - v_y\hat{y} - v_z\hat{z}$.

b) If your original unit vector gives you the $|+n\rangle$ state from problem 1.3, show that when you plug in the opposite direction, you get the $|-n\rangle$ state from problem 1.6.

c) Show that a state (again as defined in problem 1.3) in the (θ', ϕ') direction can be expressed as the sum of the $|+n\rangle$ and $|-n\rangle$ states in the (θ, ϕ) direction (note the lack of primes on on these states), *i.e.*$|n'\rangle = c_+|+n\rangle + c_-|-n\rangle$. You should get a 2x2 matrix equation for c_+ and c_-, feel free to leave your answer in this form.

d) What is the determinant of the matrix from part c)? As a reminder, the determinant of a 2x2 matrix is $ad - bc$. You should get something with magnitude 1. That tells us that we can always solve the system of equations in part c), which in turn tells us that we can pick any arbitrary $|+n\rangle$ and $|n'\rangle$, and express $|n'\rangle$ as a sum of $|+n\rangle$ and $|-n\rangle$.

a)



so we have $\varphi \to \varphi + \pi$ and $\theta \to \pi - \theta$

b) $|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$

plug opposite: $\cos(\frac{\pi-\theta}{2})|+z\rangle + e^{i(\varphi+\pi)}\sin(\frac{\pi-\theta}{2})|-z\rangle$

$= \cos(\frac{\pi}{2} - \frac{\theta}{2})|+z\rangle - e^{i\varphi}\sin(\frac{\pi}{2} - \frac{\theta}{2})|-z\rangle$

$= \sin(\frac{\theta}{2})|+z\rangle - e^{i\varphi}\cos(\frac{\theta}{2})|-z\rangle$

$= |-n\rangle$

c) $|n'\rangle = \cos\frac{\theta'}{2}|+z\rangle + e^{i\varphi'}\sin\frac{\theta'}{2}|-z\rangle = C_+|+n\rangle + C_-|-n\rangle$

$= C_+(\cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle) + C_-(\sin(\frac{\theta}{2})|+z\rangle - e^{i\varphi}\cos(\frac{\theta}{2})|-z\rangle)$

so we have: $\begin{cases} \cos\frac{\theta'}{2}|+z\rangle = (C_+\cos\frac{\theta}{2} + C_-\sin\frac{\theta}{2})|+z\rangle \\ e^{i\varphi'}\sin\frac{\theta'}{2}|-z\rangle = (C_+e^{i\varphi}\sin\frac{\theta}{2} - C_-e^{i\varphi}\cos\frac{\theta}{2})|-z\rangle \end{cases}$

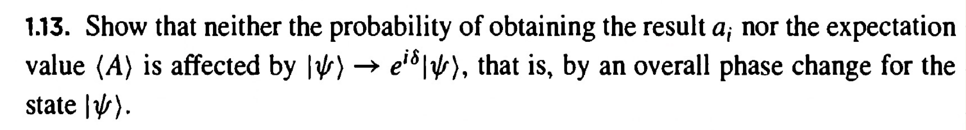
$\Rightarrow \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ e^{i\varphi}\sin\theta/2 & -e^{i\varphi}\cos\theta/2 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} \cos\theta'/2 \\ e^{i\varphi'}\sin\theta'/2 \end{pmatrix}$

d) $\det = \cos\theta/2 \cdot (-e^{i\varphi}\cos\theta/2) - \sin\theta/2\, e^{i\varphi}\sin\theta/2$

$= -e^{i\varphi}$

$\Rightarrow |\det| = |-e^{i\varphi}| = 1$

Q5



1.13. Show that neither the probability of obtaining the result a_i nor the expectation value $\langle A\rangle$ is affected by $|\psi\rangle \to e^{i\delta}|\psi\rangle$, that is, by an overall phase change for the state $|\psi\rangle$.

① $\text{Prob.} = |\langle a_i|\psi\rangle|^2 \rightarrow |e^{i\delta}\langle a_i|\psi\rangle|^2 = |\cancelto{1}{e^{i\delta}}|^2 |\langle a_i|\psi\rangle|^2$

$= |\langle a_i|\psi\rangle|^2$

$= \text{prob of } \langle a_i|\psi\rangle$

② $\langle A\rangle = \sum A_i p_i = \sum |\langle a_i|\psi\rangle|^2 \cdot A_i \longrightarrow \sum |e^{i\delta}\langle a_i|\psi\rangle|^2 A_i = \sum |\cancelto{1}{e^{i\delta}}|^2 \cdot |\langle a_i|\psi\rangle|^2 \cdot A_i$

$= \sum |\langle a_i|\psi\rangle|^2 \cdot A_i$

$= \langle A\rangle$

Q6

1.15. It is known that there is a 90% probability of obtaining $S_z = \hbar/2$ if a measurement of S_z is carried out on a spin-$\frac{1}{2}$ particle. In addition, it is known that there is a 20% probability of obtaining $S_y = \hbar/2$ if a measurement of S_y is carried out. Determine the spin state of the particle as completely as possible from this information. What is the probability of obtaining $S_x = \hbar/2$ if a measurement of S_x is carried out?

① $P(S_z=\hbar/2)=0.9=|\langle +z|\psi\rangle|^2$ and $P(S_y=\hbar/2)=0.2=|\langle +y|\psi\rangle|^2$

$\Rightarrow$ find $|\psi\rangle$, $P(S_x=\hbar/2)=?$

only relative phase matters!

② so we know $|\psi\rangle=\sqrt{0.9}\,|+z\rangle+\sqrt{0.1}\,e^{i\varphi}|-z\rangle$

③ we can compute $P(S_y=\hbar/2)$ in the z basis using $\langle +y|=\frac{1}{\sqrt{2}}(\langle +z|-i\langle -z|)$

$$|\langle +y|\psi\rangle|^2=\left|\frac{1}{\sqrt{2}}(\langle +z|-i\langle -z|)(\sqrt{0.9}\,|+z\rangle+\sqrt{0.1}\,e^{i\varphi}|-z\rangle)\right|^2$$

$$0.2=\frac{1}{2}|\sqrt{0.9}-i\sqrt{0.1}\,e^{i\varphi}|^2$$

$$0.4=|\sqrt{0.9}-i\sqrt{0.1}\,e^{i\varphi}|^2$$

$$|a-ib|^2=|a|^2+|b|^2+iab^*-iba^*$$

$$0.4=0.9+0.1+i\sqrt{0.9}\sqrt{0.1}\,e^{-i\varphi}-i\sqrt{0.1}\,e^{i\varphi}\sqrt{0.9}$$

$$0.4=1+i\sqrt{0.9}\sqrt{0.1}\,\underbrace{(e^{-i\varphi}-e^{i\varphi})}$$

$\sin\varphi=\frac{e^{i\varphi}-e^{-i\varphi}}{2i}$ so $(\cdots)=-2i\sin\varphi$

$$0.4=1+i\sqrt{0.9}\sqrt{0.1}\cdot(-2i\sin\varphi)$$

$$-1.4=2\sqrt{0.9}\sqrt{0.1}\sin\varphi$$

$$0.3=-0.3\sin\varphi$$

$$-1=\sin\varphi$$

$$\varphi=-\pi/2$$

④ so $|\psi\rangle=\sqrt{0.9}\,|+z\rangle+\sqrt{0.1}\,e^{-i\pi/2}|-z\rangle$

$$|\psi\rangle=\sqrt{0.9}\,|+z\rangle-i\sqrt{0.1}\,|-z\rangle$$

⑤ $P(S_x=\hbar/2)=|\langle +x|\psi\rangle|^2$

$$=\left|\frac{1}{\sqrt{2}}(\langle +z|+\langle -z|)(\sqrt{0.9}\,|+z\rangle-i\sqrt{0.1}\,|-z\rangle)\right|^2$$

$$=\left|\frac{1}{\sqrt{2}}\cdot\sqrt{0.9}-\frac{1}{\sqrt{2}}i\sqrt{0.1}\right|^2$$

$$=\frac{1}{2}|\sqrt{0.9}-i\sqrt{0.1}|^2$$

$$=\frac{1}{2}(0.9+0.1)$$

$$=\frac{1}{2}$$

Q7 **1.9.** Verify that $\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = 0$ for the state $|+\mathbf{x}\rangle$.

① $\langle S_x \rangle = |\langle +x|\psi\rangle|^2\left(\frac{\hbar}{2}\right) + |\langle -x|\psi\rangle|^2\left(\frac{-\hbar}{2}\right)$

$\downarrow$ we have $|\psi\rangle = |+x\rangle$

$= |\underbrace{\langle +x|+x\rangle}_{1}|^2\left(\frac{\hbar}{2}\right) + |\underbrace{\langle -x|+x\rangle}_{0}|^2\left(\frac{-\hbar}{2}\right)$

$= \frac{\hbar}{2}$

so $\langle S_x\rangle^2 = \left(\frac{\hbar}{2}\right)^2$

② $\langle S_x^2\rangle = |\underbrace{\langle +x|+x\rangle}_{1}|^2\left(\frac{\hbar}{2}\right)^2 + |\overset{0}{\cancel{\langle -x|+x\rangle}}|^2\left(\frac{-\hbar}{2}\right)^2$

$= \left(\frac{\hbar}{2}\right)^2$

③ so $\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{(\hbar/2)^2 - (\hbar/2)^2} = 0$