# QUANTUM I

## Chapter 1: STERN-GERLACH EXPERIMENTS

### BLACKBODY RADIATION & QUANTIZATION OF LIGHT

see stat mech. p.42

- **Classical prediction**: light bulb peak power shifting towards blue as we ↑ energy

  Relative probability to measure 2 different energies is $e^{-\delta E/KT}$ ($\delta E$ = diff. bw two energies $(E_1 - E_2)$) (stat. mech. $= P_1/P_2$)

  → for small $\delta E$ ($\delta E \ll KT$), $P_1/P_2 \sim 1$

  → for large $\delta E$, high energy state is highly disfavored, ie, $P = e^{\beta/E}$ so high $E \Rightarrow$ low $P$

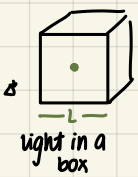
  **Average energy** (continuous): $\langle E \rangle = \int_0^\infty E_i p_i dE_i = \dfrac{\int_0^\infty E_i e^{-E_i/KT} dE_i}{\int_0^\infty e^{-E_i/KT} dE_i} = KT$ → for any mode, avg energy

  ($i$ = mode; partition fxn!)

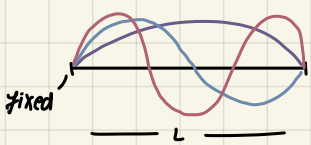
  ultraviolet catastrophe: $E \to \infty$

  planck fixed this by saying that EM wave modes could only be exciting in discrete quanta

- How many ways can we put energy into the light?




boundary conditions → can make string vibrate in specific modes




$n = \dfrac{2L}{\lambda}$ (n = modes (int); λ = allowed wavelengths)

classical physics tells us we can expect each of these vibrations to have energy ~KT

for the box, we have $\left(\dfrac{2L}{\lambda}\right)^3 = \left(\dfrac{2L\nu}{c}\right)^3$ ways to pick a wave (x, y, z directions)

($\lambda = c/\nu$; → total # of modes)

per unit volume: $8\left(\dfrac{L\nu}{c}\right)^3 \cdot \dfrac{1}{L^3} = \dfrac{8\nu^3}{c^3}$

# modes with freq. bw $\nu$ & $\nu + \delta\nu$ (diff. wrt $\nu$): $8 \cdot 3\nu^2 \left(\dfrac{L}{c}\right)^3 = 24\nu^2\left(\dfrac{L}{c}\right)^3$ or $24\nu^2/c^3$ by unit vol.

*These aren't fully accurate... actual are $\dfrac{8\pi L^3 \nu^2}{c^3}$ total modes/freq.

$\dfrac{8\pi\nu^2}{c^3}$ total modes/freq./vol.

$\rho_E = \dfrac{8\pi\nu^2}{c^3} KT$ energy/freq./vol. (energy density by freq.)

works well at low frequencies (Rayleigh-Jeans law), but keeps increasing with frequency ☹

- Instead, $E = nh\nu$ so $\int \to \sum$ bc energy is discrete

so $\langle E \rangle = \dfrac{\sum_n nh\nu e^{-nh\nu/KT}}{\sum_n e^{-nh\nu/KT}}$ → $\sum n x^n = \dfrac{1}{(1-x)^2}$

($\langle E \rangle = \dfrac{\sum E_n P_n}{\sum P_n}$)

forcing energy to be discrete

↳ $\sum (e^{-h\nu/KT})^n = \dfrac{1}{1 - e^{-h\nu/KT}}$

so $\langle E \rangle = \frac{h\nu}{e^{h\nu/kT}-1}$ energy/mode

for $h\nu \ll kT$, this reduces to the classical limit $kT$
for $h\nu \gg kT$, harder to excite a mode

$\Rightarrow \rho_E = \frac{8\pi h\nu^3/c^3}{e^{-h\nu/kT}-1}$ energy/unit vol (E density)

$I_\nu = \frac{c}{4\pi} E_\nu = \frac{2h\nu^3}{c^2(e^{h\nu/kT}-1)}$ energy is quantized

**Summary**

$\langle f \rangle = \int f(x)p(x) = \frac{\int f(x)P_{rel}(x)}{\int P_{rel}(x)}$ → Normalize

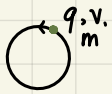
↓ discrete values of $f(x)$

$= \sum f(x)p(x) = \frac{\sum f(x)P_{rel}(x)}{\sum P_{rel}(x)} \rightarrow \mathbb{Z}$

## 4 EXPERIMENTS

· **Magnetic moments & angular momentum:**

consider a point particle with charge $q$, mass $m$, moving in a circular orbit of radius $r$ with speed $v$




magnetic moment: $\mu = IA$ → area $= \pi r^2$

→ current: particle goes around the loop $v/2\pi r$ times/s

current = charge/s $= \frac{\text{charge}}{\text{particle}} \cdot \frac{\text{particle}}{s} = q \cdot \frac{v}{2\pi r}$

$\Downarrow$

$\boxed{\mu = IA = \frac{qvr}{2}}$

angular momentum: $L = r \times p$ → orbital ang. mom.

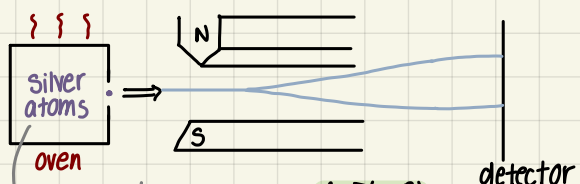
$L = mr \times v = mrv$ since $v$ always $\perp r$

$\Rightarrow \boxed{\frac{\mu}{L} = \frac{qvr/2}{mrv} = \frac{q}{2m}}$ so only the ratio matters

↳ if I have a spinning ball of $e^-$ & I change the distribution of the $e^-$ or mess with how they're moving, I'll get diff. values for $L$ & $\mu$ but the ratio would always be the same

In a classical world, if I had a ball with fixed total charge & mass, the only way to change $\mu/L$ is to move some of the charge or mass around relative to each other
eg. move all charge (but not mass) to center of ball

· **Stern-Gerlach:**




· No preferred direction
· Neutral (no electric stuff going on)

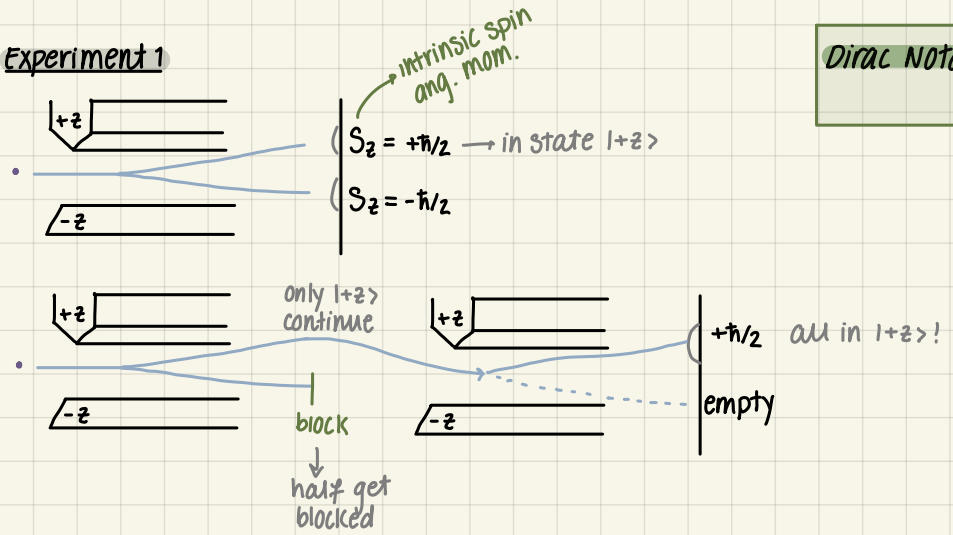
$f = \nabla(\mu \cdot B)$

inhomogeneous magnetic field
⇒ strong gradient

atom feels diff. forces depending on magnetic moment
⇒ separate beam in 2

**Dirac Notation:** $|+z\rangle$, $|-z\rangle$ ket vectors
$\langle +z|$, $\langle -z|$ bra vectors

## Experiment 1



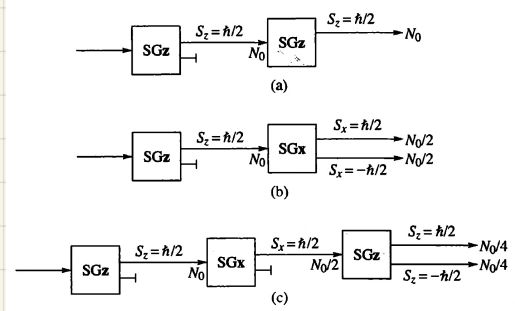
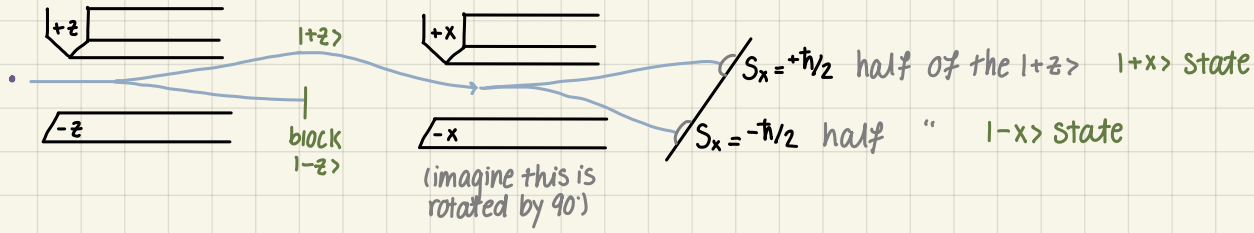



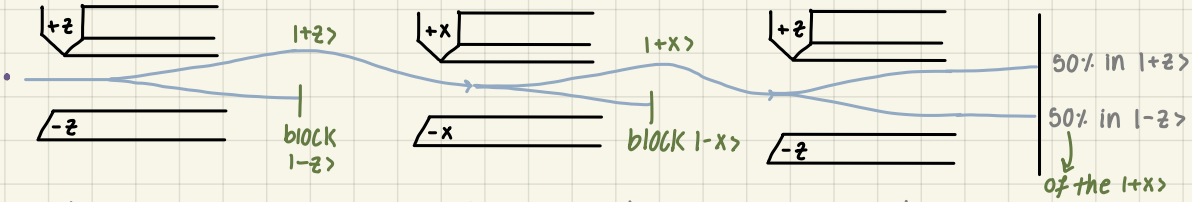

Figure 1.3 A block diagram of (a) Experiment 1, (b) Experiment 2, and (c) Experiment 3. $N_0$ is the number of particles in the beam exiting the first SG device with $S_z = \hbar/2$.

## Experiment 2




*We get the same result if we block $|+z\rangle$ & let $|-z\rangle$ pass

## Experiment 3

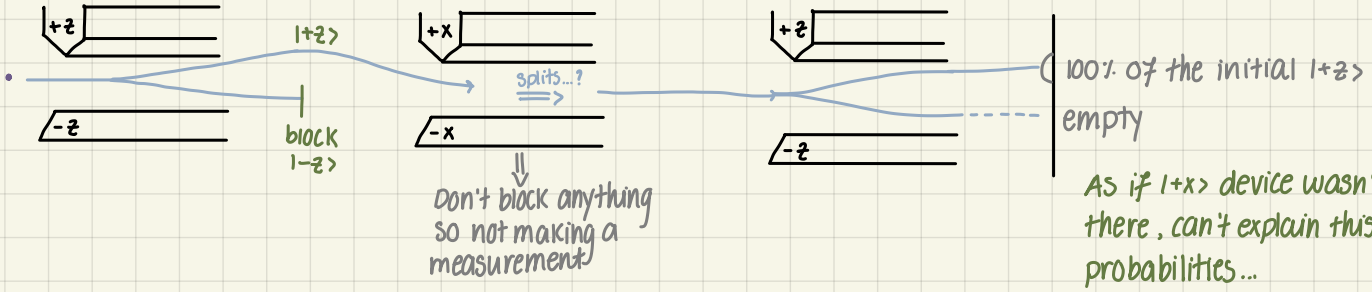



even though 1st $SG_z$ device blocks the $|-z\rangle$...

So the $|+z\rangle$ $|+x\rangle$ $|-z\rangle$ sequence lets 25% of what comes out of the initial $|+z\rangle$
→ if we didn't have the $|+x\rangle$ machine, nothing would come through as $|-z\rangle$ *Think abt the 3 polarizers!

photon decides it's vertically polarized ($|+Q\rangle$) => 45° polarizer; "photon flips a coin" to decide if it's +45° or -45°; if it's +45° ($|+U\rangle$), fully goes through (no energy lost); if it's -45° then all energy is lost
*if a photon makes it through the 45° polarizer, it has lost all memory that it used to be a $|+Q\rangle$ photon when it hits a $|-Q\rangle$ polarizer, all the photon knows is that right now it's $|+U\rangle$, & will flip another coin to choose b/w $|+Q\rangle$ & $|-Q\rangle$
=> There's no way to explain this classically, instead of losing $\sqrt{2}$ in field strength at each polarizer, the photon either makes it through at full strength or gets absorbed.

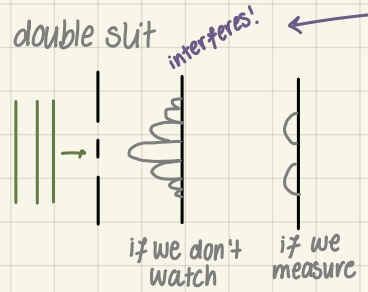
## Experiment 4




As if $|+x\rangle$ device wasn't there, can't explain this with probabilities...

- Somehow in Experiment 4 we must eliminate the probability that the particle is in the state $|-z\rangle$ when it enters the last $SG_z$ device. What we need is some sort of "interference" that can cancel out the $|-z\rangle$ state

- **In E&M:** double slit




energy $\propto$ field$^2$
(density)

- for Stern-Gerlach, we can introduce a **probability amplitude** that we'll "square" to get the probability
  $\Rightarrow$ when we don't make a measurement, we need to add the amplitudes, not the probabilities

## QUANTUM STATES

- The ket $|+x\rangle$ is just a shorthand way of saying that the spin state of the particle is st. if we were to make a measurement of $S_x$ (intrinsic spin angular momentum in the x direction), we would obtain the value $\hbar/2$

- **In E&M:** write down E field vector in i,j,k basis $\rightarrow$ $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} = (E_x, E_y, E_z)$
  get a component by taking inner product $\rightarrow$ $E_x = \vec{E}\cdot\hat{x}$ project $\vec{E}$ onto x axis

- **In QM:** if we send a spin-$\frac{1}{2}$ particle into a $SG_z$ device, we get $\hbar/2$ & $-\hbar/2$ $\Rightarrow$ particle in $|+z\rangle$ or $|-z\rangle$ state
  These two states can be considered as vectors that form a basis for our QM vector space

$$\boxed{|\psi\rangle = C_+|+z\rangle + C_-|-z\rangle}$$

The two opposite directions, forms our basis

Total state is some combination of +z & -z

Coming out of box initially $\rightarrow$ superposition of $|+z\rangle$ & $|-z\rangle$
each of these states have an amplitude associated with it $\rightarrow$ $C_+, C_-$

- **what's the amplitude to be in a state?**

| | |
|---|---|
| amplitude $= \langle\varphi|\psi\rangle =$ | probability amplitude for a particle in the state $|\psi\rangle$ to be found in the state $|\varphi\rangle$ |

Amplitude of $|\psi\rangle$ to be in $|\psi\rangle$ is 1 $\Rightarrow$ $\boxed{\underbrace{\langle\psi|\psi\rangle}_{\text{amplitude}} = 1}$ "inner product", equiv. to $\hat{x}\cdot\hat{x} = 1$

$\rightarrow$ $\langle +z|+z\rangle = 1$
$\rightarrow$ experiment 1: if particle in $|+z\rangle$, there's zero amplitude for it to be found in $|-z\rangle$
$\Rightarrow$ $\boxed{\langle -z|+z\rangle = 0}$
- orthogonal states
- equiv. to $\hat{x}\cdot\hat{y} = 0$
- "if I block $|+z\rangle$ and then block $|-z\rangle$, then nothing goes through"

· **Probability**: equiv. to energy density of EM field $\propto E^2 + B^2$
$\Rightarrow$ Amplitude square

$\boxed{\text{Prob} = |\langle\psi|\psi\rangle|^2}$ = amplitude* · amplitude  eg. $|\langle +z|\psi\rangle|^2 = P(\hbar/2) = C_+^* C_+$

prob that you measure $|\psi\rangle$ for a particle in state $|\psi\rangle$

has to be 1

$|\psi^2| = 1$ (length of state is 1)

Not prob.

* it's the amplitudes that add when we combine states without measuring along the way
↳ probabilities lose information, add amplitudes then do prob.
$\Rightarrow$ why SG changes when we measure → gets rid of amplitude info

· **How can we find $C_+$ & $C_-$?**

**EM Analogy**

EM wave propagating along +z, in vacuum → $\vec{E}(\vec{x},t) = \text{Re}[(C_x\hat{x} + C_y\hat{y})e^{i\omega(t-z/c)}]$
if we set $C_x = 1$, $C_y = 0$, we get $\vec{E}(\vec{x},t) = \cos(\omega(t-z/c))\hat{x}$ $\Rightarrow$ pure +Q EM wave
→ $C_x$ complex just adds a phase shift but still a +Q wave
setting $C_x = 0$, $C_y = 1$ $\Rightarrow$ −Q EM wave (purely in y direction)

we can get more interesting things if both $C_x, C_y \neq 0$
eg. $C_x = C_y = \frac{1}{\sqrt{2}}$ $\Rightarrow$ +U wave (45°)
$\Rightarrow$ flipping one sign $\Rightarrow$ +Q to −Q (or +U to −U), etc

**Summary**: specify $C_x, C_y$ $\Rightarrow$ specify behavior of EM wave
overall phase doesn't matter, only relative phase does  b/w $C_x$ & $C_y$

I get complete information about my wave by specifying the relative amplitudes & phases of $C_{+Q}, C_{-Q}$
Could also specify $C_{+U} = (C_{+Q} + C_{-Q})/\sqrt{2}$ & $C_{-U} = (C_{+Q} - C_{-Q})/\sqrt{2}$
$\Rightarrow$ "if I tell you relative amplitude & phase in ±U basis, we can work out what it is in ±Q basis"

"Job of the 'bra' is to meet up with a 'ket' to form an amplitude"

$\langle +z|\psi\rangle = \langle +z|(\underbrace{C_+}|+z\rangle + \underbrace{C_-}|-z\rangle)$  **order matters!** (can move csts)

constants!

$= C_+ \underbrace{\langle +z|+z\rangle}_{1} + C_- \underbrace{\langle +z|-z\rangle}_{0}$

$\boxed{\langle +z|\psi\rangle = C_+}$

Similarly, $\boxed{\langle -z|\psi\rangle = C_-}$

So we can express $|\psi\rangle$ as: $|\psi\rangle = \overbrace{\langle +z|\psi\rangle}^{C_+}|+z\rangle + \overbrace{\langle -z|\psi\rangle}^{C_-}|-z\rangle$  → csts, can move around

$$\boxed{|\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle}$$

equiv. to $\vec{E} = E_x i + E_y j + E_z k = i(i\cdot\vec{E}) + j(j\cdot\vec{E}) + k(k\cdot\vec{E})$

· Eg. spin state S → $|S\rangle = |+z\rangle\langle +z|S\rangle + |-z\rangle\langle -z|S\rangle$

- For the 'bra', we have $\boxed{\langle\psi| = C_+^*\langle +z| + C_-^*\langle -z|}$

Using same technique as prev. page...

$\Rightarrow \boxed{C_+^* = \langle\psi|+z\rangle}$ & $\boxed{C_-^* = \langle\psi|-z\rangle}$

$\Rightarrow \boxed{\langle\psi| = \langle\psi|+z\rangle\langle +z| + \langle\psi|-z\rangle\langle -z|} = C_+^*\langle +z| + C_-^*\langle -z|$

- How are $\langle +z|\psi\rangle$ & $\langle\psi|+z\rangle$ related?

We require $\langle\psi|\psi\rangle = 1 = (C_+^*\langle +z| + C_-^*\langle -z|)(|+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle)$

$$1 = (C_+^*\langle +z| + C_-^*\langle -z|)(C_+|+z\rangle + C_-|-z\rangle)$$

$$1 = C_+^*C_+\underbrace{\langle +z|+z\rangle}_{1} + C_+^*C_-\underbrace{\langle +z|-z\rangle}_{0} + C_-^*C_+\underbrace{\langle -z|+z\rangle}_{0} + C_-^*C_-\underbrace{\langle -z|-z\rangle}_{1}$$

$$1 = \underbrace{C_+^*C_+}_{\text{conj.}} + \underbrace{C_-^*C_-}_{\text{conj.}}$$

Real! → since $C_s$ are complex

$\Rightarrow \boxed{\langle\psi|+z\rangle = \langle +z|\psi\rangle^*}$ (same for $-z$)

This means that the amplitude for a particle in the state $|\psi\rangle$ to be found in the states $|\pm z\rangle$ is the complex conjugate of the amplitude for a particle in the states $|\pm z\rangle$ to be found in the state $|\psi\rangle$

- So we get

$$\boxed{\begin{aligned}\langle\psi|\psi\rangle &= \langle +z|\psi\rangle^*\langle +z|\psi\rangle + \langle -z|\psi\rangle^*\langle -z|\psi\rangle \\ &= |\langle +z|\psi\rangle|^2 + |\langle -z|\psi\rangle|^2 \\ &= C_+^*C_+ + C_-^*C_- \\ &= 1\end{aligned}}$$

→ The probability that a particle in the state $|\psi\rangle$ will be found to be in the state $|+z\rangle$ $= C_+^*C_+$

- **Absolute phase**: $C_+ \to C_+e^{i\varphi}$, $C_- \to C_-e^{i\varphi}$ } unchanged state, $C_+^*C_+ \Rightarrow C_+^*\cancel{e^{-i\varphi}}C_+\cancel{e^{i\varphi}} = C_+^*C_+ = |C_+|^2$

- When both of the probability amplitudes $\langle +z|\psi\rangle = C_+$ & $\langle -z|\psi\rangle = C_-$ are nonzero, then a particle in the state $|\psi\rangle$ is really in a **superposition** of the states $|+z\rangle$ & $|-z\rangle$
→ There are probabilities of obtaining both $S_z = \hbar/2$ and $S_z = -\hbar/2$ if a measurement of $S_z$ is made

- Ex: A measurement of $S_z$ is made on particle in state $|\psi\rangle = \frac{1}{2}|+z\rangle + \frac{i\sqrt{3}}{2}|-z\rangle$. What are the possible results of measurement & with what prob?

① $\langle +z|\psi\rangle = 1/2$ so $|\langle +z|\psi\rangle|^2 = 1/4$ $\Rightarrow$ 25% prob. of getting $S_z = \hbar/2$

② $\underbrace{\langle -z|\psi\rangle = \frac{i\sqrt{3}}{2}}_{\text{amplitude}}$ so $\underbrace{|\langle -z|\psi\rangle|^2 = \left(\frac{-i\sqrt{3}}{2}\right)\left(\frac{i\sqrt{3}}{2}\right) = 3/4}_{\text{prob.}}$ $\Rightarrow$ 75% prob of getting $\underbrace{S_z = -\hbar/2}_{\text{measurement of } \langle -z|S_z|\psi\rangle}$

· what does the $|+x\rangle$ state look like? → in the $|+z\rangle$ basis

we have $|+x\rangle = C_+|+z\rangle + C_-|-z\rangle$ with $|C_+| = \frac{1}{\sqrt{2}}$ (want $|C_+|^2 = \frac{1}{2}$ → prob = 50% = amp.², particle in $|+x\rangle$ could be either in $|+z\rangle$ or $|-z\rangle$, see SG experiments)

so $\langle +z|+x\rangle = C_+$

& $|\langle -z|+x\rangle|^2 = \cdots = \frac{1}{2} = |C_-|^2 \rightarrow |C_-| = \frac{1}{\sqrt{2}}$

One solution is to choose $C_+$ & $C_-$ to be real, $C_+ = C_- = 1/\sqrt{2}$ $\Rightarrow$ $\boxed{|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle}$

General solution: $\boxed{C_+ = \frac{e^{i\delta_+}}{\sqrt{2}} \quad C_- = \frac{e^{i\delta_-}}{\sqrt{2}}}$ — Real phases

pick a convenient phase! like having z & picking the x (picking easy phase ($\varphi = 0$))
→ once we pick, everything is set

Notice! The probabilities themselves ($C^*C$) don't give info about values of the phases $\delta_+, \delta_-$, since they cancel out in $C^*C$

Take * of coeffs → $\boxed{\langle +x| = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)}$

· finding $|-x\rangle$: We know $\langle +x|-x\rangle = 0$ & $\langle +x| = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)$

write $|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + \alpha|-z\rangle)$ Need $|\alpha| = 1$ (equiv to $|\alpha|^2 = \alpha^*\alpha = 1$) to have $\langle -x|-x\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\alpha}{\sqrt{2}}\right)^2 = 1$

only relative phase matters

so $\langle +x|-x\rangle = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)\frac{1}{\sqrt{2}}(|+z\rangle + \alpha|-z\rangle) = 0$

$\frac{1}{2} + 0 + 0 + \frac{1}{2}\alpha = 0$

$\alpha = -1$

$\Rightarrow \boxed{|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)}$ & $\boxed{\langle -x| = \frac{1}{\sqrt{2}}(\langle +z| - \langle -z|)}$

* we are always free to pick the absolute phase (doesn't change anything abt amplitudes)
→ only relative phase matters
so only need an $\alpha$ on $|-z\rangle$ (or $|+z\rangle$)
can always do $|+x\rangle \Rightarrow e^{i\varphi}|-x\rangle$

$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + e^{i\varphi}|-z\rangle)$

$\langle +x| = \frac{1}{\sqrt{2}}(\langle +z| + e^{-i\varphi}\langle -z|)$

so $\langle +x|+x\rangle = \frac{1}{\sqrt{2}}(\langle +z| + e^{-i\varphi}\langle -z|) \cdot \frac{1}{\sqrt{2}}(|+z\rangle + e^{i\varphi}|-z\rangle)$

$= \frac{1}{2}\langle +z|+z\rangle + \frac{1}{2}\underbrace{e^{i\varphi}e^{-i\varphi}}_{\text{cancel!}}\langle -z|-z\rangle$

$= 1$

· Ex: $|\psi\rangle = 0.6|+z\rangle + 0.4|-z\rangle$ NOT a valid state

since $\langle\psi|\psi\rangle = 0.6^2 + 0.4^2$

$= 0.36 + 0.16$

$= 0.52 \neq 1$

· Ex: $|\psi\rangle = 0.8|+z\rangle + 0.6|-z\rangle$

① amplitude to be in $|+z\rangle \rightarrow \langle +z|\psi\rangle = 0.8$

$|-z\rangle \rightarrow \langle -z|\psi\rangle = 0.6$

② prob. to be in $|+z\rangle \rightarrow |\langle +z|\psi\rangle|^2 = 0.64$

$|-z\rangle \rightarrow |\langle -z|\psi\rangle|^2 = 0.36$

③ Recall $\langle f(x)\rangle = \sum_{x_i} f(x_i)P(x_i)$

So $\boxed{\langle S_z\rangle = \underbrace{|\langle +z|\psi\rangle|^2}_{P_+}\underbrace{\left(\frac{\hbar}{2}\right)}_{S_+} + \underbrace{|\langle -z|\psi\rangle|^2}_{P_-}\underbrace{\left(\frac{-\hbar}{2}\right)}_{S_-}}$

expectation value

$= 0.64(\hbar/2) + 0.36(-\hbar/2)$

$= 0.28\frac{\hbar}{2}$

④ What's the uncertainty on that measurement?

$\boxed{\sigma(f) = \left(\langle f(x)^2\rangle + \langle f(x)\rangle^2\right)^{1/2}}$

so $\sigma_{S_z}^2 = \langle (f - \langle f\rangle)^2\rangle$

$= \langle f^2\rangle - \langle f\rangle^2$

$= \underbrace{\langle S_z^2\rangle}_{\hbar^2/4} - \underbrace{\langle S_z\rangle^2}_{\left(\frac{0.28\hbar}{2}\right)^2}$

$= \frac{\hbar^2}{4}(1 - 0.28^2)$

$\sigma^2 = \frac{\hbar^2}{4}(0.9216)$

$\rightarrow \sigma = 0.96 \cdot \hbar/2$

* $P_+\left(\frac{\hbar}{2}\right)^2 + P_-\left(\frac{-\hbar}{2}\right)^2$

$\left(\frac{\hbar}{2}\right)^2\underbrace{(P_+ + P_-)}_{1}$

for $\langle S_z\rangle^2$, the ^2 isn't there so we still have the −

⑤ How would we write this in the x basis?

$|\psi\rangle = ?|+x\rangle + ?|-x\rangle$

$= C_{+x}|+x\rangle + C_{-x}|-x\rangle$

we know $C_{+x} = \langle +x|\psi\rangle$ & $\langle +x| = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)$

$C_{-x} = \langle -x|\psi\rangle$ $\quad \langle -x| = \frac{1}{\sqrt{2}}(\langle +z| - \langle -z|)$

so $\langle +x|\psi\rangle = C_{+x} = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)(0.8|+z\rangle + 0.6|-z\rangle)$

$\vdots$

$C_{+x} = 0.99$ amplitude of $|\psi\rangle$ in $\langle +x|$ direction

$\Rightarrow |C_{+x}|^2 = 0.98$ prob.

and $\langle -x|\psi\rangle = C_{-x} = \frac{1}{\sqrt{2}}(\langle +z - \langle -z|)(0.8|+z\rangle + 0.6|-z\rangle)$

$\vdots$

$C_{-x} = \frac{0.2}{\sqrt{2}}$ amplitude of $|\psi\rangle$ in $\langle -x|$ direction

$\Rightarrow |C_{-x}|^2 = \frac{0.4}{2} = 0.02$ prob

⑥ Expectation value & uncertainty for $S_x$

$\langle S_x^2\rangle = \hbar^2/4$

$\langle S_x\rangle = 0.98\left(\frac{\hbar}{2}\right) + 0.02\left(\frac{-\hbar}{2}\right) = 0.96\frac{\hbar}{2}$

$\sigma^2 = \frac{\hbar^2}{4}(1 - 0.98^2)$

$\sigma = \frac{\hbar}{2}\sqrt{1-0.98^2}$

$= 0.28\,\hbar/2$

- **What does the $|+y\rangle$ state look like?**

we know the prob. of measuring the $|+y\rangle$ state along the z-axis will give us $+\hbar/2$ 50% of the time & $-\hbar/2$ 50% of the time. Same 50-50 split along x-axis

we also know amplitude of $|+y\rangle$ in the $|+z\rangle$ direction has to be $\frac{1}{\sqrt{2}} \times$ phase
Overall phase doesn't matter so we have

$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{\alpha}{\sqrt{2}}|-z\rangle$ with $|\alpha| = 1$ (equiv. $|\alpha|^2 = 1$) since we need $\langle +y|+y\rangle = 1$

if we do: $\langle +z|+y\rangle = \frac{1}{\sqrt{2}}$ & $\langle -z|+y\rangle = \frac{1}{\sqrt{2}}\alpha$ → fine since $|\alpha| = 1$

what about x: amplitude squared to be in $|+x\rangle$ is $1/2$ ← equiv. to this, just missing the square

$|\langle +x|+y\rangle| = \frac{1}{\sqrt{2}}(\langle +z| + \langle -z|)\frac{1}{\sqrt{2}}(|+z\rangle + \alpha|-z\rangle)$

↓ amp.²

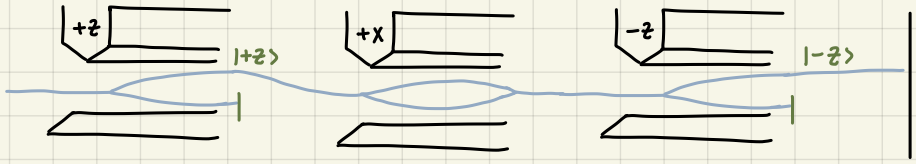
$|\langle +x|+y\rangle|^2 = \frac{1}{2} = \frac{1}{4}(2 + \alpha + \alpha^*)$

- Need $Re(\alpha) = 0$ b/c otherwise $\alpha + \alpha^* = 2Re(\alpha)$ & above $\neq \frac{1}{2}$
- Need $Im(\alpha) = \pm i$ (need $|\alpha| = 1$)

So we have:

$$\boxed{\begin{aligned} |+y\rangle &= \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle) \\ |-y\rangle &= \frac{1}{\sqrt{2}}(|+z\rangle - i|-z\rangle) \end{aligned}} \qquad \boxed{\begin{aligned} \langle +y| &= \frac{1}{\sqrt{2}}(\langle +z| - i\langle -z|) \\ \langle -y| &= \frac{1}{\sqrt{2}}(\langle +z| + i\langle -z|) \end{aligned}}$$

get in similar way... (see $|-x\rangle$)




Not observing!
Add amplitudes

So if we don't measure, we have $|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \underbrace{+|-z\rangle}$

& $|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \underbrace{-|-z\rangle}$)

These cancel out if we don't measure

=> adding amplitudes → Not measuring: $|+z\rangle$

→ measuring, ie. if we block eg. $|+x\rangle$, then we get some $|+z\rangle$ & some $|-z\rangle$

Q? Can I write a state where $\langle +z|\psi\rangle = 1$ and $\langle +x|\psi\rangle = 1$? No...

# Chapter 2 : MATRIX MECHANICS

## INTRO

- **Recall**: we can express an arbitrary spin state $|\psi\rangle$ of a spin-$\frac{1}{2}$ particle as

$$|\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$$
$$= c_+|+z\rangle + c_-|-z\rangle$$

more general:

$$\boxed{|\psi\rangle = |+n\rangle\underbrace{\langle +n|\psi\rangle}_{\text{component of } \psi \text{ along } |-n\rangle} + |-n\rangle\langle -n|\psi\rangle}$$

The complex numbers $c_\pm = \langle \pm z|\psi\rangle$ tell us how our state $|\psi\rangle$ is oriented in our quantum mechanical vector space, ie, how much of $|\psi\rangle$ is projected onto each of the states $|+z\rangle$ & $|-z\rangle$

- In EM... we can write $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ as $\vec{E} = (E_x, E_y, E_z)$ → position tells us which component we are looking at

- **Representing kets**: $|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle \xrightarrow[z\text{ basis}]{} \boxed{|\psi\rangle = \begin{pmatrix}\langle +z|\psi\rangle \\ \langle -z|\psi\rangle\end{pmatrix} = \begin{pmatrix} c_+ \\ c_-\end{pmatrix}}$ → $|+z\rangle$ coeff, → $|-z\rangle$ coeff

  In z basis: $|+z\rangle = \begin{pmatrix}\langle +z|+z\rangle \\ \langle -z|+z\rangle\end{pmatrix}$ and $|-z\rangle = \begin{pmatrix}\langle +z|-z\rangle \\ \langle -z|-z\rangle\end{pmatrix}$

  $|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ $|-z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

  $|+x\rangle = \begin{pmatrix}\langle +z|+x\rangle \\ \langle -z|+x\rangle\end{pmatrix}$ $\qquad$ $|-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

  $|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\qquad$ See textbook p. 31

- **Representing bras**: we know the bra vector corresponding to the ket is

$$\langle\psi| = \langle\psi|+z\rangle\langle +z| + \langle\psi|-z\rangle\langle -z|$$
$$= c_+^*\langle +z| + c_-^*\langle -z|$$

so we have

$$\langle\psi|\psi\rangle = c_+^*c_+ + c_-^*c_-$$
$$= \underbrace{(c_+^* \;\; c_-^*)}_{=(c_+\;c_-)^\dagger\ ?}\begin{pmatrix}c_+\\c_-\end{pmatrix}$$

$$\Rightarrow \boxed{\langle\psi| = (c_+^* \;\; c_-^*)}$$

  In z basis: take † of kets...

  $\langle +z| = (1 \;\; 0)$ $\qquad$ $\langle -z| = (0 \;\; 1)$

- **Recall**: $\langle\psi_i|\psi_j\rangle = 0$ for $\psi$ corresponding to pure states

> lin. alg
>
> $\left|\begin{pmatrix}a & b\\c & d\end{pmatrix}\right| = ad - bc = \det$
>
> & $\begin{pmatrix}a & b\\c & d\end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$

# ROTATION MATRICES

- we have: $|\psi\rangle = |+x\rangle\langle +x|\psi\rangle + |-x\rangle\langle -x|\psi\rangle = \begin{pmatrix} |+x\rangle\langle +x| \\ |-x\rangle\langle -x| \end{pmatrix}|\psi\rangle$

**projection**: $|+n\rangle\langle +n|\psi\rangle \Rightarrow \psi$ in $\hat{n}$ direction
can write bra, ket in different basis

eg. in z basis $\rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\psi$

- **for z basis**: $|+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle = \underbrace{\begin{pmatrix} |+z\rangle\langle +z| \\ +|-z\rangle\langle -z| \end{pmatrix}}_{\text{all in } z \text{ basis}}|\psi\rangle$

and $|+z\rangle\langle +z| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$|-z\rangle\langle -z| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightarrow$ add them: $\begin{pmatrix} |+z\rangle\langle +z| \\ +|-z\rangle\langle -z| \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$\boxed{\underbrace{\begin{pmatrix} |+n\rangle\langle +n| \\ +|-n\rangle\langle -n| \end{pmatrix}}_{\text{all in same basis}} = I}$$

so we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \rightarrow \psi$ in z basis, ie. $\psi_z$

- **Change of basis**:

$\psi_z \Rightarrow \psi_x$: $\boxed{|\psi\rangle_x = \underbrace{\begin{pmatrix} |+x\rangle_x\langle +x|_z \\ +|-x\rangle_x\langle -x|_z \end{pmatrix}}_{\substack{\text{Rotation matrix} = R_{zx} \\ z \rightarrow x}}|\psi\rangle_z}$ same!

basis I pick for $\langle x|$ needs to be the same basis I use for $\psi$

$|+x\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . $|-x\rangle_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

projection operators

so we have $R_{zx} = \begin{pmatrix} |+x\rangle_x\langle +x|_z \\ +|-x\rangle_x\langle -x|_z \end{pmatrix} = \begin{matrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \\ +\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} \end{matrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\right] = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\substack{\text{matrix to go from z basis to x basis} \\ \text{(only need to multiply to the ket } |\psi\rangle\text{!)}}}$

doesn't matter if we use $\pm x$ or another $\pm n$, gives same matrix

- **Eg**: $|+x\rangle_z \rightarrow |+x\rangle_x$

$|+x\rangle_x = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{R_{zx}}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{|+x\rangle_z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- **Eg**: $|+z\rangle_z \rightarrow |+z\rangle_x$

$|+z\rangle_x = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{R_{zx}}\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|+z\rangle_z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\psi_x = R_{zx}\psi_z$

so $(R_{zx})^{-1}\psi_x = \psi_z = R_{xz}\psi_x$

$\Rightarrow \boxed{R_{xz} = R_{zx}^{-1}}$ so $\psi_z = R_{xz}R_{zx}\psi_z$

$\Leftarrow$ z to x to z

· observe that a 90 deg. rotation about the z-axis rotates $|+x\rangle$ to $|+y\rangle$, and $|+y\rangle$ to $|-x\rangle$.
In the $|\pm z\rangle$ basis, we have

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

The rotation matrix has to take ① $|+x\rangle$ to $|+y\rangle$ so we have $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = R\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ *all in z basis

$|+y\rangle = R\,|+x\rangle$

② $|+y\rangle$ to $|-x\rangle$ so we have $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = R\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$|-x\rangle = R\,|+y\rangle$

combine equations: $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -1\end{pmatrix} = R\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & i\end{pmatrix}$

$$R = \begin{pmatrix}1 & 1\\ i & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & i\end{pmatrix}^{-1}$$

$$R_z(\pi/2) = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$$

check: $R(\pi/2)|+x\rangle = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}} = \begin{pmatrix}1\\i\end{pmatrix}\frac{1}{\sqrt{2}} = |+y\rangle$

we can write $R$ as $R_z(\pi/2) = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/2}\end{pmatrix}$

or rotate states by an overall phase: $R_z(\pi/2) = \begin{pmatrix}e^{-i\pi/4} & 0\\ 0 & e^{i\pi/4}\end{pmatrix}$

↳ Rotate by $\pi/4$ twice

so $R_z(\pi/2) = R_z(\pi/4)\,R_z(\pi/4)$ $\qquad R(2\cdot\frac{\pi}{4}) = R(\frac{\pi}{4})^2$

$= R_z(\pi/6)\cdots$

· **General**: $\boxed{R(\theta) = \begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}}$ Rotates around $|n\rangle$ in the $|n\rangle$ basis (works for any basis)

## OPERATORS

· **Angular momentum**: we know that the angular momentum of a $|+z\rangle$ state in the $|\pm z\rangle$ basis is $\hbar/2$

Recall: $\psi = \begin{pmatrix}c_+\\ c_-\end{pmatrix}$

so $\langle S_z\rangle = \frac{\hbar}{2}\underbrace{c_+^* c_+}_{=P(\hbar/2) = |\langle +z|\psi\rangle|^2} - \frac{\hbar}{2}c_-^* c_-$

prob. that I measure $-\frac{\hbar}{2}$

$$= (c_+^* \; c_-^*)\underbrace{\begin{pmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{pmatrix}}_{\text{call this } \hat{J}_{zz}}\begin{pmatrix}c_+\\ c_-\end{pmatrix} \quad \Rightarrow \boxed{\langle\psi_z|J_{zz}|\psi_z\rangle = \langle S_z\rangle}$$

→ ang. mom operator

$$= (c_+^* \; c_-^*)\begin{pmatrix}\hbar/2\cdot c_+\\ -\hbar/2\cdot c_-\end{pmatrix}$$

$$= \frac{\hbar}{2}c_+^* c_+ - \frac{\hbar}{2}c_-^* c_-$$

$$= \langle S_z\rangle\,!$$

always this for our momentum along |n> in |n> basis

- We have $\psi_x = R_{z\to x}\psi_z$ & $\hat{J}_{xx} = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$

$J_x$ in z basis

So $\langle \hat{J}_{xz} \rangle = \langle \psi_z | \underbrace{\hat{J}_{xz}} | \psi_z \rangle$

$\langle S_{xz} \rangle$

but what is this...?

↓ rewrite in x basis to be able to use $J_{xx}$...

$\langle \hat{J}_x \rangle = \langle \psi_x | J_{xx} | \psi_x \rangle$ → Need $\psi_x$ to act on $J_{xx}$ (x basis)

$|\psi_x\rangle = |R_{z\to x}\psi_z\rangle$

$= (R_{z\to x}|\psi_z\rangle)^\dagger \underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}}_{J_{xx}} (R_{z\to x}|\psi_z\rangle)$

$= \langle \psi_z | \underbrace{(R_{z\to x}^\dagger \hat{J}_{xx} R_{z\to x})}_{J_{xz}} | \psi_z \rangle$

$\Rightarrow \boxed{\hat{J}_{xz} = R_{z\to x}^\dagger \hat{J}_{xx} R_{z\to x}}$

↳ see p. 11

$\hat{J}^\dagger = \hat{J}$

**Intuition**
- can only 'act' when in same basis
- we can see this as...
  1) switching $\psi_z$ to x basis, acting on $J_{xx}$
  2) switching $J_{xx}$ to z basis, acting on $\psi_z$

- **General**: $\boxed{\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle = \psi^\dagger \hat{A} \psi}$

- **Momentum operators**: $\boxed{\hat{J}_{zz} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ $\begin{matrix} +z \\ -z \end{matrix}$ (same for eg. $J_y$ in y basis) $\boxed{\hat{J}_{xz} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\boxed{\hat{J}_{yz} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$

see pset 2.Q6

- **Rotation**: we had $R(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$

if $\theta$ small: $R(d\theta) = \begin{pmatrix} 1 - id\theta/2 & 0 \\ 0 & 1 + id\theta/2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\frac{d\theta}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=2J_z/\hbar}$

since $J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (z basis)

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i\,d\theta\,\hat{J}_z}{\hbar}$

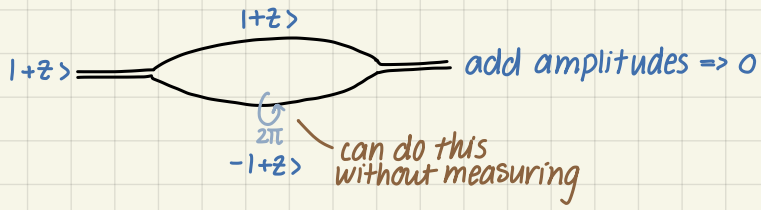
$\Rightarrow \boxed{\hat{R}_z(\theta_{small}) = I - \frac{i\theta}{\hbar}\hat{J}_z}$ → True in any basis

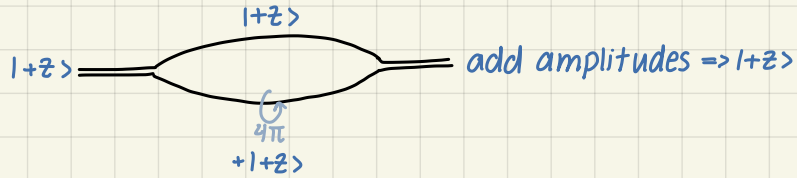
we derived this in the z basis but we could've used another basis
once I've shown that it can be written w/ $J_z$ in one basis, it's true for any basis

- $R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \xrightarrow[\text{Rotate by } 2\pi]{} \begin{pmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

So $R_z(2\pi)|\psi\rangle = -|\psi\rangle$

*Rotating by a full circle ($2\pi$) doesn't bring you back to the same state




· **Matrix representation**: we can write $\boxed{\hat{A}|\psi\rangle = |\varphi\rangle}$

in general, different kets

operator

In the z basis: $|\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$

$|\varphi\rangle = |+z\rangle\langle +z|\varphi\rangle + |-z\rangle\langle -z|\varphi\rangle$

so we have $\hat{A}(|+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle) = |+z\rangle\langle +z|\varphi\rangle + |-z\rangle\langle -z|\varphi\rangle$

just like $\vec{F} = m\vec{a}$ is 3 equations that we can obtain by taking the dot product w/ the basis vectors...

① $\langle +z|\hat{A}|+z\rangle\langle +z|\psi\rangle + \langle +z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle +z|\varphi\rangle$

② $\langle -z|\hat{A}|+z\rangle\langle +z|\psi\rangle + \langle -z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle -z|\varphi\rangle$

In matrix form: $\underbrace{\begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}}_{\text{Representation of } \hat{A} \text{ in the } z \text{ basis}} \underbrace{\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}}_{\text{coeffs of } \psi} = \underbrace{\begin{pmatrix} \langle +z|\varphi\rangle \\ \langle -z|\varphi\rangle \end{pmatrix}}_{\text{coeffs. of } \varphi}$

$\hat{A} \xrightarrow[S_z]{} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ in textbook

if we label the basis vectors by $|i\rangle, |j\rangle$, we can express the matrix representation of $\hat{A}$ as $A_{ij} = \langle i|\hat{A}|j\rangle$

· **Note**: Knowing the 4 matrix elements of ( ) allows us to determine the action of the operator $\hat{A}$ on any state $|\psi\rangle$

· **Adjoint operator**: $\hat{A}|\psi\rangle = |\varphi\rangle \longrightarrow \boxed{\langle\psi|A^\dagger = \langle\varphi|}$

① Apply $\langle\chi|$: $\langle\chi|\hat{A}|\psi\rangle = \langle\chi|\varphi\rangle$

② Apply $|\chi\rangle$: $\langle\psi|\hat{A}^\dagger|\chi\rangle = \langle\varphi|\chi\rangle$

we also know $\langle\chi|\varphi\rangle = \langle\varphi|\chi\rangle^*$ so $\boxed{\langle\psi|\hat{A}^\dagger|\chi\rangle = \langle\chi|\hat{A}|\psi\rangle^*}$

using basis states: $\langle i|\hat{A}^\dagger|j\rangle = \langle j|\hat{A}|i\rangle^* \longrightarrow A^\dagger_{ij} = A^*_{ji}$

$\Rightarrow$ The matrix representing $\hat{A}^\dagger$ is the transpose conj. of $\hat{A}$

# OBSERVABLES & EIGENSTATES

- **Notation**: $\boxed{\langle qty \rangle = \langle \psi | \hat{A} | \psi \rangle}$ where $\hat{A}$ is the operator for the quantity

- **Observables**: $\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle = (c_+^* \; c_-^*)(\hat{A}) \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$ = observable $\Rightarrow$ number has to be real

  $\langle A \rangle$ ↓ observable; $\hat{A}$ ↓ operator producing an observable

  so we need $\left( (c_+^* \; c_-^*) A \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \right)^\dagger = (c_+^* \; c_-^*) A^\dagger \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \overset{\text{need}}{=} (c_+^* \; c_-^*) A \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$

  for this to be true, we need $A^\dagger = A$ **Hermitian operator**

- **Fact**: All observable operators must be hermitian, ie $A^\dagger = A$

- **Eg**: $J_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$ hermitian

- **Eg**: **Rotation matrices**

  $\boxed{(AB)^\dagger = B^\dagger A^\dagger}$

  $|\psi\rangle = R|\varphi\rangle \xrightarrow{\dagger} \langle\psi| = \langle\varphi| R^\dagger$

  we also know $\langle\psi|\psi\rangle = 1$, $\langle\varphi|\varphi\rangle = 1$

  so $\langle\psi|\psi\rangle = \langle\varphi| R^\dagger R |\varphi\rangle = 1$

  $\Rightarrow R^\dagger R = I$ **unitary**

  † of rotation is the opposite rotation

  so inverses, not equal $\Rightarrow$ Not Hermitian, not observable

  → In S-G, we measure relative prob., we don't get to ask "what way is this spin pointing"

- **Expectation value**: if a spin-$\frac{1}{2}$ particle is in a state $|\psi\rangle = |+z\rangle\langle+z|\psi\rangle + |-z\rangle\langle-z|\psi\rangle$

  the expectation value of $S_z$ is

  $$\langle S_z \rangle = \left(\frac{\hbar}{2}\right)|\langle+z|\psi\rangle|^2 + \left(\frac{-\hbar}{2}\right)|\langle-z|\psi\rangle|^2$$

  $$= (\langle\psi|+z\rangle, \langle\psi|-z\rangle) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \langle+z|\psi\rangle \\ \langle-z|\psi\rangle \end{pmatrix}$$

  $$\boxed{\langle S_z \rangle = \langle\psi|\hat{J}_z|\psi\rangle}$$

- **Eigenvalues**: for an observable, $\boxed{\hat{A}|a_n\rangle = a_n|a_n\rangle}$ and $\boxed{\langle a_n|\hat{A} = \langle a_n|a_n^*}$

  $\hat{A}$ ↓ Hermitian $A = A^\dagger$; $a_n$ ↓ eigenvalue of $\hat{A}$; $|a_n\rangle$ ↓ eigenstate of $\hat{A}$

- eg. $\hat{J}_z|R\rangle = \hbar|R\rangle$

  ↳ state, Right G pol

- eigenvalues of a Hermitian operator are real

- We can write the inner product: $\langle a_k|\hat{A}|a_n\rangle = a_n\langle a_k|a_n\rangle$

Rewrite as

$$a_k^*\langle a_k|a_n\rangle = a_n\langle a_k|a_n\rangle \quad \text{or} \quad (a_k^* - a_n)\langle a_k|a_n\rangle = 0$$

↳ if $k=n$, $a_n^* - a_n = 0$

so $a_n^* = a_n$, ie $a_n$ is real

also $\langle a_k|a_n\rangle = 0$ if $a_k \neq a_n$

$\Rightarrow$ Eigenstates of a Hermitian operator corresponding to distinct eigenvalues are orthogonal

- Consider 2 eigenstates $|a_1\rangle$ & $|a_2\rangle$ with $a_1 \neq a_2$, we have

① $|\psi\rangle = c_1|a_1\rangle + c_2|a_2\rangle = |a_1\rangle\langle a_1|\psi\rangle + |a_2\rangle\langle a_2|\psi\rangle$

② $\langle\psi|\hat{A}|\psi\rangle = (c_1^*\langle a_1| + c_2\langle a_2|)\hat{A}(c_1|a_1\rangle + c_2|a_2\rangle)$

$= (\quad '' \quad)(c_1a_1|a_1\rangle + c_2a_2|a_2\rangle)$

$= |c_1|^2a_1 + |c_2|^2a_2$ ↙ exp val = sum of eigenvals weighted by their prob.

$= \langle A\rangle$

- **Convenient notation**: use eigenstates $|a_1\rangle$ & $|a_2\rangle$ as a basis

$$\boxed{|\psi\rangle \xrightarrow[\text{basis}]{|a_1\rangle - |a_2\rangle} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \end{pmatrix}}$$

$$\boxed{\langle\psi| \xrightarrow[\text{basis}]{|a_1\rangle - |a_2\rangle} (c_1^* \; c_2^*) = (\langle\psi|a_1\rangle \; \langle\psi|a_2\rangle)}$$

$$\boxed{\hat{B} \xrightarrow[\text{basis}]{|a_1\rangle - |a_2\rangle} \begin{pmatrix} \langle a_1|\hat{B}|a_1\rangle & \langle a_1|\hat{B}|a_2\rangle \\ \langle a_2|\hat{B}|a_1\rangle & \langle a_2|\hat{B}|a_2\rangle \end{pmatrix}}$$

In this notation, an equation such as

$$\hat{B}|\psi\rangle = |\varphi\rangle \tag{2.147}$$

becomes

$$\begin{pmatrix} \langle a_1|\hat{B}|a_1\rangle & \langle a_1|\hat{B}|a_2\rangle \\ \langle a_2|\hat{B}|a_1\rangle & \langle a_2|\hat{B}|a_2\rangle \end{pmatrix} \begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle a_1|\varphi\rangle \\ \langle a_2|\varphi\rangle \end{pmatrix} \tag{2.148}$$

Knowing the matrix elements $\langle a_i|\hat{B}|a_j\rangle$ permits us to evaluate the action of the operator $\hat{B}$ on any state $|\psi\rangle$. As an example, we can use matrix mechanics to evaluate the expectation value of $B$ in the state $|\psi\rangle$:

$$\boxed{\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle = (\langle\psi|a_1\rangle, \langle\psi|a_2\rangle) \begin{pmatrix} \langle a_1|\hat{B}|a_1\rangle & \langle a_1|\hat{B}|a_2\rangle \\ \langle a_2|\hat{B}|a_1\rangle & \langle a_2|\hat{B}|a_2\rangle \end{pmatrix} \begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \end{pmatrix}} \tag{2.149}$$

- **Note**: if basis states are eigenstates of the operator, the matrix representation is diagonal with the eigenvalues forming the diagonal matrix elements

$$\boxed{\hat{A} \xrightarrow[\text{basis}]{|a_1\rangle - |a_2\rangle} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}}$$

- **Eigenvalues**: $AV = \lambda V \xrightarrow[\Lambda]{\text{isolate}} A = V\Lambda V^{-1}$ (so $V\Lambda V^{-1}v_i = \lambda_i v_i$) ← eigenvector

↳ diagonal, eigenvalues

- if $A = V\Lambda_A V^{-1}$ & $B = V\Lambda_B V^{-1}$ (they share eigenvectors)

① $AB = V\Lambda_A V^{-1}V\Lambda_B V^{-1} = V\Lambda_A\Lambda_B V^{-1}$

② $BA = V\Lambda_B V^{-1}V\Lambda_A V^{-1} = V\Lambda_B\Lambda_A V^{-1}$

$\Rightarrow AB = BA$ if they share eigenvectors

$\Rightarrow AB - BA = 0 \longrightarrow \boxed{[A,B] \equiv AB - BA} = 0$

commutator (see ch3)

· If A is Hermitian: $AV = \lambda V$

$\downarrow$ take conj. transpose

$(AV)^\dagger = (\lambda V)^\dagger = \lambda^* V^\dagger$

$V^\dagger \underbrace{A^\dagger}_{A^\dagger = A} V = \lambda^* V^\dagger V$

$V^\dagger A V = \lambda^* V^\dagger V$

$V^\dagger \lambda V = \lambda^* V^\dagger V \longrightarrow \underbrace{V^\dagger V = 1}_{V \text{ unitary}}$ & $\underbrace{\lambda = \lambda^*}_{\text{Real!}}$ so

so eigenvectors are orthogonal

$\Longrightarrow$ eigenvalues of operators that produce an observable must be real

we also have $A^\dagger = A = V \Lambda V^{-1}$ and $A^\dagger = (V \Lambda V^{-1})^\dagger$

$= (V^\dagger)^{-1} \Lambda V^\dagger$

so $V = V^{\dagger -1}$ & $V^\dagger = V^{-1}$

· let's take $\hat{A} = \begin{pmatrix} & \end{pmatrix}$, with eigenvectors $V_1, V_2$, eigenvalues $\lambda_1, \lambda_2$

① $V_2^\dagger \hat{A} V_1 = V_2^\dagger \lambda_1 V_1$ and $A^\dagger = A$

② take †: $(V_2^\dagger A V_1)^\dagger = (\lambda_1 V_2^\dagger V_1)^\dagger$

$V_1^\dagger \underbrace{A^\dagger}_{=A} V_2 =$ "

$V_1^\dagger A V_2 =$ "

$\lambda_2 \underbrace{V_1^\dagger V_2} = (\lambda_1 \underbrace{V_2^\dagger V_1})^\dagger$

lengths are the same

(phase doesn't matter)

$\rightarrow$ for equality to hold, we need $V_1^\dagger V_2 = V_2^\dagger V_1 = 0$

$\Rightarrow$ for any pair of eigenvectors, $\boxed{V_1^\dagger V_2 = 0}$

eigenvectors of Hermitian matrix are orthogonal

· **Expectation & variance:**

Recall: $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \psi^\dagger A\psi$ (p.13)

What if $\psi = v$ (eigenvector of $\hat{A}$)

Remember: uncertainty is

$$\begin{aligned}
\sigma(A)^2 &= \langle A^2\rangle - \langle A\rangle^2 \qquad \langle A\rangle = (\underbrace{v^\dagger A v}_{v^{-1}Av=\lambda})^2 = (\lambda)^2 \\
&= \langle \underbrace{\psi|\hat{A}\hat{A}|\psi}_{\psi^\dagger A A \psi}\rangle - (\lambda)^2 \\
&= v^\dagger A A v - \lambda^2 \\
&= \lambda v^\dagger A v - \lambda^2 \\
&= \lambda^2 \underbrace{v^\dagger v}_{1} - \lambda^2 \\
&= \lambda^2 - \lambda^2 \\
&= 0
\end{aligned}$$

No uncertainty if $|\psi\rangle$ is an eigenvector of $\hat{A}$

· **Summary:** if we have an observable operator $\hat{A}$, then the eigenvectors of $\hat{A}$ are pure states. Whenever we measure the value associated with $\hat{A}$, a pure state always returns its corresponding eigenvalue. Since the eigenvectors are orthogonal, we can always describe an arbitrary state as a combination of eigenvectors

## UNITARY MATRICES

· **def:** $\boxed{v^\dagger v = I}$

· consider $(1+\frac{x}{N})^N$, then $\lim_{N\to\infty} \longrightarrow e^x$

· **what happens if these are matrices?**

$$\begin{aligned}
\left(I+\frac{iA}{N}\right)^N &= \left(I+\frac{iA}{N}\right)\left(I+\frac{iA}{N}\right)\cdots \\
&= I + N\left(I^{A-1} + \frac{iA}{N}\right) + \frac{N(N-1)}{2!}\left(I^{A-2} + \left(\frac{iA}{N}\right)^2\right) + \cdots \\
&= \underbrace{I + iA + \frac{(iA)^2}{2} + \frac{(iA)^3}{3!} + \cdots}_{\text{expansion of } e^{iA}!}
\end{aligned}$$

$$(a+b)^n = \sum a^m b^{m-n}\binom{n}{m}$$
$$= a^n + a^{n-1}b\cdot n + a^{n-2}b^2\binom{n}{2} + \cdots$$

n ways to pick a

$$e^{iA} = \sum \frac{(iA)^N}{N!} = \left(I + \frac{iA}{N}\right)^N$$

let's drop the $i$: $\sum \frac{A^N}{N!} = ?$ $\longrightarrow$

$$\begin{aligned}
A &= V\Lambda V^{-1} \\
A^2 &= V\Lambda V^{-1}V\Lambda V^{-1} = V\Lambda^2 V^{-1} \\
&\vdots \\
A^N &= V\Lambda^N V^{-1} = V\begin{pmatrix}\lambda_1^N & & \\ & \lambda_2^N & \\ & & \ddots\end{pmatrix}V^{-1}
\end{aligned}$$

so $e^A = \sum \frac{A^N}{N!} = \sum \frac{V\Lambda^N V^{-1}}{N!} = V e^{\Lambda} V^{-1}$

$e^x = \sum \frac{x^N}{N!}$

· let's say I have A hermitian.
we can write A as $A = V\Lambda V^\dagger$. Then $e^{iA} = \sum \frac{(iA)^N}{N!} = \sum \frac{(iV\Lambda V^\dagger)^N}{N!} = \sum \frac{V(i\Lambda)^N V^\dagger}{N!} = \underbrace{Ve^{i\Lambda}V^\dagger}$

eigenvals are now $e^{i\lambda_n}$
magnitudes are 1

Multiply by †: $(Ve^{i\Lambda}V^\dagger)^\dagger(Ve^{i\Lambda}V^\dagger) = Ve^{-i\Lambda}V^\dagger Ve^{i\Lambda}V^\dagger = I$

If $U = Ve^{i\Lambda}V^\dagger = e^{iA}$, then $U^\dagger U = I$ unitary matrix

$\Longrightarrow$ if A Hermitian, then $U = e^{iA}$ is unitary

· fact: Every Hermitian operator has a corresponding unitary matrix

· Recall: $R_z(\theta) = I - \underbrace{\frac{iJ_z\theta}{\hbar}}$ for small $\theta$ (p.13)

$J_z$ Hermitian

$\rightarrow \boxed{R_z(\theta) = e^{-iJ_z\theta/\hbar}}$ is unitary $= Ve^{i\Lambda}V^\dagger$ for $\Lambda, V$ = eig val & vec of $J_z$

· $\boxed{\hat{R}_z(\theta)|\psi\rangle = e^{-i\hat{J}_z\varphi/\hbar}|\psi\rangle}$ $\boxed{\langle\psi|\hat{R}_z^\dagger(\theta) = \langle\psi|e^{i\hat{J}_z^\dagger\varphi/\hbar}}$

· In order for probability to be conserved under rotation,

$$\langle\psi|\hat{R}_z^\dagger(\theta)\hat{R}_z(\theta)|\psi\rangle = \langle\psi|e^{iJ_z^\dagger\varphi/\hbar}e^{-iJ_z\varphi/\hbar}|\psi\rangle = \langle\psi|\psi\rangle$$

$\rightarrow$ Requires $J_z^\dagger = J_z$

· Consider $\langle\psi|A|\psi\rangle = \psi^\dagger U\underbrace{U^\dagger A U}_{=I \text{ since } U \text{ unitary}}U^\dagger\psi$

$= (U^\dagger\psi)^\dagger(U^\dagger A U)(U^\dagger\psi)$

$\varphi = U^\dagger\psi$, $\tilde{A} = UAU^\dagger$

$= \varphi^\dagger\tilde{A}\varphi$

· for A diagonal, eigenvalues are the diagonal entries, eigenvectors = I

## SUMMARY

- The operator that rotates states ccw by angle $\theta$ around the $z$ axis is $\hat{R}_z(\theta) = e^{-i\hat{J}_z\theta/\hbar}$

- In general, $\hat{A}|\psi\rangle = |\varphi\rangle$ & $\langle\psi|\hat{A}^\dagger = \langle\varphi|$ for an arbitrary $\hat{A}$

- Rotated state: $\hat{R}_z(\theta)|\psi\rangle = e^{-iJ_z\theta/\hbar}|\psi\rangle$
  $\langle\psi|R_z^\dagger(\theta) = \langle\psi|e^{i\hat{J}_z^\dagger\theta/\hbar}$

- for probability to be conserved under rotation, $\langle\psi|R^\dagger(\theta)R(\theta)|\psi\rangle = \langle\psi|\psi\rangle$ so $\hat{J}_z = \hat{J}_z^\dagger$ (Hermitian)

- Unitary operator $\Rightarrow \hat{R}^\dagger\hat{R} = 1$

- for an observable: $\hat{A}|a_n\rangle = a_n|a_n\rangle$ and $\langle a_n|\hat{A} = \langle a_n|a_n^*$

  $\hat{A}$: Hermitian $A = A^\dagger$; $a_n$: eigenvalue of $\hat{A}$; $|a_n\rangle$: eigenstate of $\hat{A}$

  eg. $\hat{J}_z|R\rangle = \hbar|R\rangle$
  ↳ state, Right G pol

- eigenvalues of a Hermitian operator are real

- $\langle a_k|\hat{A}|a_n\rangle = a_n\langle a_k|a_n\rangle \longrightarrow a_k^*\langle a_k|a_n\rangle = a_n\langle a_k|a_n\rangle$
  or $(a_k^* - a_n)\langle a_k|a_n\rangle = 0$
  ↳ if $k=n$, $a_n^* - a_n = 0$ so $a_n^* = a_n$, ie $a_n$ is real

also $\langle a_k|a_n\rangle = 0$ if $a_k \neq a_n$

# Chapter 3: ANGULAR MOMENTUM

## REMINDERS

- $|a+ib| = \sqrt{(a+ib)^*(a+ib)} - \sqrt{a^2+b^2}$
- opposite $\varphi \longrightarrow \varphi + \pi$
- $C_+, C_-$ are complex!
- $R_n^\dagger(\theta) = R_n(-\theta)$
- **Uncertainty**: $\sigma^2(\langle\psi|\hat{A}|\psi\rangle) = 0$ if $\psi$ is an eigenvector of $\hat{A}$

  $\sigma^2(\langle\psi|\hat{A}|\psi\rangle) = \sigma^2(\langle\psi|\hat{B}|\psi\rangle)$ if $\psi$ is an eigenvector of both

eg. ① $\hat{J}_{zz} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

eigenvectors: $\rightarrow \begin{pmatrix}1\\0\end{pmatrix}$ w/ $\lambda = 1$

$\rightarrow \begin{pmatrix}0\\1\end{pmatrix}$ w/ $\lambda = -1$

so $V = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\Lambda = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

② $\hat{J}_{xz} = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

eigenvectors: $\rightarrow \begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}$ w/ $\lambda = 1$

$\rightarrow \begin{pmatrix}-1\\1\end{pmatrix}\frac{1}{\sqrt{2}}$ w/ $\lambda = -1$

so $V = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}$ and $\Lambda = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}$

③ If we pick $|\psi\rangle$ to be an eigenvector from ①, $\sigma^2(\langle\psi|J_{zz}|\psi\rangle) = 0$, but not an eigenvector of ②
so $\sigma^2(\langle\psi|J_{xz}|\psi\rangle) \neq 0$
The only way to get $\sigma^2 = 0$ is to be an eigenvector of the operator. So 2 operators need to share eigenvectors if we want to measure with 0 uncertainty in both

## MATRIX PROPERTIES

- **If A, B are hermitian, is AB also hermitian?** No

Think about real matrices first (ie, symmetric)

$C = AB \longrightarrow C_{ij} = \sum_k A_{ik}B_{kj} = \sum A_{ik}B_{jk}$

No reason for these to be equal

$C_{ji} = \sum_k A_{jk}B_{ki} = \sum A_{jk}B_{ik}$

- **What about AA?**

$A = V\Lambda V^\dagger$ so $A^2 = V\Lambda \underbrace{V^\dagger V}_{I \text{ since } V \text{ unitary}}\Lambda V^\dagger$

$A^2 = V\Lambda^2 V^\dagger$

Hermitian!

positive definite?

- **If eigenvalues are all 0, is matrix 0?** NO

$$\left|\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}\right| = abc$$

a, b, c are eigenvalues (diagonal entries of triangular matrix = eigenvals)
so we can have

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$ eigenvals are 0, but matrix isn't 0

not hermitian

=> **defective matrix**
=> can't write as $V\Lambda V^{-1}$ (eigenval. decomp.)
=> If it was hermitian, then yes

- **Does a matrix commute w/ itself?** yes. $AA = AA$

- **What about powers?** $AA^N = A^N A$

- **If A hermitian, what about iA?** NO

$(iA)^\dagger = -iA^\dagger = -iA$

picked up a - sign

- **General matrix A, start w/ real?**

$$A = \frac{(A + A + A^T - A^T)}{2}$$

$$= \underbrace{\frac{(A+A^T)}{2}}_{\left[\frac{(A+A^T)}{2}\right]^T} + \underbrace{\frac{(A-A^T)}{2}}_{\left[\frac{(A-A^T)}{2}\right]^T}$$

$= \frac{(A^T + A)}{2}$ ie itself => symmetric

$= \frac{(A^T - A)}{2} = -\frac{(A - A^T)}{2}$ picks up a -

**for complex**: use †, we get the same thing

$$A = \underbrace{\frac{(A+A^\dagger)}{2}}_{\text{Hermitian}} + \underbrace{\frac{(A-A^\dagger)}{2}}_{\text{antihermitian}}$$

- **If A is antihermitian, ie, $A^\dagger = -A$, what about iA?**

$(iA)^\dagger = (-i)(A^\dagger) = -i(-A) = iA$ → iA is Hermitian

- $\boxed{[A, BC] = B[A,C] + [A,B]C}$

$ABC - BCA = B(AC - CA) + (AB - BA)C$
$= \cancel{BAC} - BCA + ABC - \cancel{BAC}$
$= ABC - BCA$

- **If $[A,B] = 0$ & $[B,C] = 0$, does $[A,C] = 0$?** No

# ROTATION OPERATORS

- In pset 2...

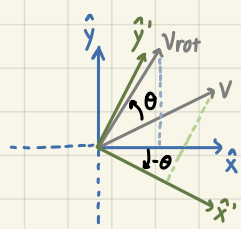
$$R_y(\tfrac{\pi}{2})|-x\rangle \longrightarrow -|+z\rangle \quad \text{Not } |+z\rangle\,!$$

this is done by using $|-x\rangle \to |+z\rangle$ instead of $-|+z\rangle$

* It takes 2 circles to go back to $|+z\rangle$. If we try to pick $R_y$ to have it go around in 1 circle, it becomes not unitary → invalid

- **fact**: Rotations don't commute (eg. doing $R_x$ then $R_y$ $\neq$ $R_y$ then $R_x$)

- **finding Rotation matrix**: Rotating a vector in a basis n is equiv. to rotating the basis in the opposite direction




If we have $x\hat{x}+y\hat{y}+z\hat{z}$, want to rotate a vector by $\theta$ around $\hat{z}$...

getting matrix by rotating axes, equiv. to -rot of the vector

$$\begin{cases} \hat{x}' = \cos\theta\,\hat{x} - \sin\theta\,\hat{y} \\ \hat{y}' = \cos\theta\,\hat{y} + \sin\theta\,\hat{x} \\ \hat{z}' = \hat{z} \end{cases}$$

→ Rotate vector by $\theta$ = Rotate basis axes by $-\theta$

we can write this as a matrix:

$$\underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\substack{\text{New rotated vector in n basis} \\ \text{equiv.: v in n' basis}}} = \underbrace{\begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}}_{\substack{\text{Rotation matrix} \\ R_{n\to n'}}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\substack{\text{v in} \\ \text{n basis}}}$$

So if we do this for eg. $\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we get $\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ ⇒ This is $\hat{x}$ in the n' basis, ie opposite rotation of $\hat{x}' = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$ → $\hat{x}'$ in n basis

- **Matrices**:

$$\boxed{R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}} \qquad \boxed{R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

can only use in some cases (pset 4)

$$\boxed{R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}}$$

$\hat{y}$ component doesn't change!

* in the x,y,z basis (since $R_{x,y,z}$)

→ We can combine versions of this (see pset 3 Q2-4)

eg. $V_{rot} = R_{z'} R_{n\to n'} = R_z R_{n\to n'}$ since $z = z'$

If $n' = (x', y', z)$ is rotated by $\varphi$ relative to $n = (x, y, z)$, $R_z(-\varphi) = R_{n\to n'}$ & $R_z(\theta)$ = Rotate by $\theta$ around $z$

$R_z(\varphi) = R_{n'\to n}$

⇒ Rotations about the same axis commute → $R_z(\alpha)R_z(\beta) = R_z(\alpha+\beta)$

· If the rotation axis is different, then we have:

$$[R_z, R_x] = R_z R_x - R_x R_z$$

$$= \begin{pmatrix} \cos & -\sin & 0 \\ \sin & \cos & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \\ 0 & \cos & -\sin \\ & \sin & \cos \end{pmatrix} - \begin{pmatrix} 1 & 0 & \\ 0 & \cos & -\sin \\ & \sin & \cos \end{pmatrix} \begin{pmatrix} \cos & -\sin & 0 \\ \sin & \cos & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos & -\sin\cos & \sin^2 \\ \sin & \cos^2 & -\sin\cos \\ 0 & \sin & \cos \end{pmatrix} - \begin{pmatrix} \cos & -\sin & 0 \\ \sin\cos & \cos^2 & -\sin \\ \sin^2 & \sin\cos & \cos \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\sin(1+\cos) & \sin^2 \\ \sin(1-\cos) & 0 & -\sin(\cos-1) \\ -\sin^2 & \sin(1-\cos) & 0 \end{pmatrix}$$

$\downarrow$ $\theta$ small, $\cos \sim 1 - \frac{\theta^2}{2}$, $\sin \sim \theta$

$$= \begin{pmatrix} 0 & -\theta^3/2 & \theta^2 \\ \theta^3/2 & 0 & \theta^3/2 \\ \theta^2 & \theta^3/2 & 0 \end{pmatrix} \quad \theta^3 \to 0$$

$$= \begin{pmatrix} 0 & 0 & \theta^2 \\ 0 & 0 & 0 \\ \theta^2 & 0 & 0 \end{pmatrix}$$

· **Rotation by small angle**: Recall $R_n(\theta) = e^{-i\hat{J}_n\theta/2\hbar} \xrightarrow[\theta \text{ small}]{} = 1 - \frac{iJ_n d\theta}{2\hbar}$ (p.19)

WRONG?

So we have... $[R_z, R_x] = \left[1 - \frac{iJ_z d\theta}{2\hbar}, 1 - \frac{iJ_x d\theta}{2\hbar}\right]$

$$= \underbrace{1 - \frac{iJ_z d\theta}{2\hbar} - \frac{iJ_x d\theta}{2\hbar} + \frac{i^2 J_z J_x d\theta^2}{4\hbar^2}}_{R_z R_x} - \underbrace{\left(1 - \frac{iJ_x d\theta}{2\hbar} - \frac{iJ_z d\theta}{2\hbar} + \frac{i^2 J_x J_z d\theta^2}{4\hbar^2}\right)}_{R_x R_z}$$

$$= \frac{i^2 J_z J_x d\theta^2}{4\hbar^2} - \frac{i^2 J_x J_z d\theta^2}{4\hbar^2}$$

$$= \frac{-d\theta^2}{4\hbar^2}[J_z, J_x]$$

$$= \frac{-d\theta^2}{4\hbar^2} \cdot i\hbar J_y$$

$$= \frac{-iJ_y}{\hbar}\left(\frac{d\theta}{2}\right)^2$$

$$\frac{-iJ_y}{\hbar}(d\theta^2) = (R_y(d\theta^2) - 1) \times 2 \quad ?$$

$$= \frac{1}{4} \cdot 2(R_y(d\theta^2) - 1)$$

$$= \frac{1}{2}(R_y(d\theta^2) - 1)$$

$\downarrow$ jon didn't have this...

# COMMUTATORS

· **Reminder**: $[A,B] = AB - BA$ $\boxed{[A,BC] = B[A,C] + [A,B]C}$ see p.22

· How is $[A,B]$ related to $[B,A]$? *careful with order!

$AB - BA$ $BA - AB$

→ $[J_z, J_x]$ or $[J_x, J_z]$ to get $J_y$? use RHR!

$$[J_z, J_x] = \left(\frac{\hbar}{2}\right)^2 \left[\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{J_z}\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{J_x} - \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{J_x}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{J_z}\right]$$

$$= \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}0&2\\-2&0\end{pmatrix}$$

$$= \frac{\hbar^2}{2}\underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{\sim J_y!}$$

$$= i\hbar \cdot \underbrace{\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{J_y}$$

$$[J_z, J_x] = i\hbar J_y$$

· **General**: $\boxed{[J_a, J_b] = i\hbar\, \varepsilon_{abc} J_c}$

$\pm 1$, RHR

eg. $a = z$, $b = x$, so $\varepsilon = +1$

$$\boxed{R_n(d\theta) = I - \frac{iJ_n d\theta}{\hbar}}$$ → $[R_z(d\theta), R_x(d\theta)] = I - R_y(d\theta^2)$

· Suppose we have $\hat{A}|a\rangle = \alpha|a\rangle$

eigenval. — The single eigenstate of $\hat{A}$

Apply $\hat{B}$: $\hat{B}\hat{A}|a\rangle = \hat{B}\alpha|a\rangle$

↓ assume $AB = BA$

$$\hat{A}(\hat{B}|a\rangle) = \alpha(\hat{B}|a\rangle)$$

→ eigenstate of $\hat{A}$ since $\hat{A}(\ ) = \text{scalar} \cdot (\ )$

so $\hat{B}|a\rangle$ has to be a scalar multiple of $|a\rangle$ since that's the only eigenstate

$\Rightarrow \hat{B}|a\rangle = \beta|a\rangle$

$\Rightarrow |a\rangle$ is also an eigenstate of $\hat{B}$

**Relabel** $|a\rangle$ as $|\alpha, \beta\rangle$ to show both eigenvalues, and say $\hat{A}$ & $\hat{B}$ both have the eigenstate $|\alpha, \beta\rangle$

- If there is more than one eigenstate of the operator $\hat{A}$ with eigenvalue $\alpha$, we say that there is **degeneracy**

- We've established that each eigenstate of $\hat{A}$ is also an eigenstate of $\hat{B}$
- $\Rightarrow$ two Hermitian operators that commute have a complete set of eigenstates in common.

## ANGULAR MOMENTUM

- **Reminder**: In the z basis, $J_z, J_x, J_y = \frac{\hbar}{2}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right]$
  p.13

- $\boxed{\hat{J}^2 = \hat{J}\cdot\hat{J} = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2}$ commutes with each $\hat{J}_n$

eg. showing $[\hat{J}_z, \hat{J}^2] = 0$

$$
\begin{aligned}
[\hat{J}_z, \hat{J}^2] &= [J_z, J_x^2 + J_y^2 + J_z^2] \\
&= [J_z, J_x^2] + [J_z, J_y^2] + \cancelto{0}{[J_z, J_z^2]} \\
&= J_x[J_z, J_x] + [J_z, J_x]J_x + J_y[J_z, J_y] + [J_z, J_y]J_y \\
&= J_x i\hbar J_y + i\hbar J_y J_x + J_y(-i\hbar J_x) + (-i\hbar J_x)J_y \\
&= i\hbar(\cancel{J_x J_y} + \cancel{J_y J_x} - \cancel{J_y J_x} - \cancel{J_x J_y}) \\
&= 0
\end{aligned}
$$

- Let's take $\hat{J}$ & $\hat{J}_z$. Because they commute, they have eigenstates in common.
  $\rightarrow$ we can label the kets with the eigenvalues : $|l, m\rangle$

$$\boxed{\hat{J}^2|l,m\rangle = l\hbar^2|l,m\rangle} \quad \& \quad \boxed{\hat{J}_z|l,m\rangle = m\hbar|l,m\rangle}$$

The dimensions of the operator are explicitely included in the $\hbar$ $\Rightarrow$ $l$ and $m$ are dimensionless

$\Rightarrow$ $|l,m\rangle$ is a state for which a measurement of the z component of the angular momentum yields $m\hbar$ and the magnitude squared of the ang. mom. is $l\hbar^2$

- $l \geq 0$ $\rightarrow$ expected physically since $l$ specifies the magnitude squared of the ang. mom. in the state $|l,m\rangle$

- $\boxed{\langle l,m|\hat{J}^2|l,m\rangle = l\hbar^2\langle l,m|l,m\rangle}$

eg. $\langle l,m|\hat{J}_z^2|l,m\rangle = \underbrace{\langle\psi|\psi\rangle}_{\text{Not normalized,}}$ where we've defined $\hat{J}_z|l,m\rangle = |\psi\rangle$ & $\langle\psi| = \langle l,m|\hat{J}_z$ since $\hat{J}_z$ is Hermitian

term in

let $|\psi\rangle = c|\varphi\rangle$

complex constant, dimensions of $\hbar$

$= c^*c\underbrace{\langle\varphi|\varphi\rangle}_{=1} \geq 0$

· Eg: Spin-1

$$\boxed{\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}}, \quad \boxed{\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}}, \quad \boxed{\hat{J}_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}}$$

↳ $J_z$'s eigenvals

① check: $J^2 = J_x^2 + J_y^2 + J_z^2$

$= 2\hbar^2 \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ → commutes with each $\hat{J}_n$ component

② $J_z$ is diagonal → suggests that matrix representations given for $J_x, J_y, J_z$ are formed using the eigenstates of $J_z$ & $\hat{J}^2$ as a basis

③ eigenstates: $\begin{pmatrix}1\\0\\0\end{pmatrix}$ for $\lambda = \hbar$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ for $\lambda = 0$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$ for $\lambda = -\hbar$

verify: $J_z\begin{pmatrix}1\\0\\0\end{pmatrix} = \hbar\begin{pmatrix}1\\0\\0\end{pmatrix}$, $J_z\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$ ...

each is also an eigenstate of $\hat{J}^2$ with eigenvalue $2\hbar^2$

④ Since $J_x, J_y$ aren't diagonal, the states in ③ aren't their eigenstates

⑤ $J_x + iJ_y = \sqrt{2}\hbar\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

$\rightarrow \sqrt{2}\hbar\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$ shifted up!

$\lambda = -\hbar$ turned into state w/ $\lambda = 0$ $\times\sqrt{2}\hbar$

→ Similarly, when the operator acts on the state w/ $\lambda = 0$ for $\hat{J}_z$, it turns it into a state with $\lambda = \hbar$ $\times\sqrt{2}\hbar$

⑥ This raising action terminates when the operator $\hat{J}_x + i\hat{J}_y$ acts on the state w/ $\lambda = \hbar$ (max eig. val.)

⑦ lowering operator: $\hat{J}_x - i\hat{J}_y = \sqrt{2}\hbar\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

· **Raising & lowering operators:** $\boxed{\hat{J}_{\pm z} = \hat{J}_x \pm i\hat{J}_y}$ Not hermitian, $J_+^\dagger = J_-$

for $J_z$ — can write in any basis

whatever basis we choose for $J_x$ & $J_y$ will give us $J_\pm$ for $J_z$, in that basis

we can also permute the Js to get $J_\pm$ for different operators:

$\boxed{\hat{J}_{\pm x} = \hat{J}_y \pm i\hat{J}_z}$ $\boxed{\hat{J}_{\pm y} = \hat{J}_z \pm i\hat{J}_x}$

eg. in z basis

Raises x states written in the z basis

*$J_\pm$ needs to be in the same basis as the state it's operating on!

*$J_{\pm x}$ written in the x basis is the same as $J_{\pm z}$ written in the z basis → There's nothing special abt z!

· Eg: See pset 4, Q6 → spin-1, ignoring factors of $\hbar$ (working in $z$ basis)

{see pset 5! (code)}

① $J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $J_x = \cdots$ $J_y = \cdots$

② Get $J_\pm$ for $J_z$: $J_\pm = J_x \pm i J_y$

③ Eigenstates of $J_z$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ → how much of $|1,1\rangle$
→ " $|1,0\rangle$
→ " $|1,-1\rangle$

$|\psi_1\rangle = |1,1\rangle$
$|\psi_2\rangle = |1,0\rangle$
$|\psi_3\rangle = |1,-1\rangle$

If we raise them: $J_+ \begin{pmatrix} | & | & | \end{pmatrix} = \begin{pmatrix} 1 & | & | \\ 0 & | & | \\ 1 & | & | \end{pmatrix}$

↳ $|1,1\rangle$ can't be raised

④ for $J_x$: $J_x$ written in $z$ basis

$J_{\pm,x} = \underbrace{J_y \pm i J_z}_{\text{written in } z \text{ basis}}$

$\Rightarrow J_{\pm x} \begin{pmatrix} J_x \text{ eig. states} \\ \text{as columns} \end{pmatrix}$ = Raised/lowered $J_x$

see oct7 code

OR $\begin{pmatrix} | & | & | & | \end{pmatrix}$ = kets & $\begin{pmatrix} | & | & | & | \end{pmatrix}^\dagger$ = bras $\Rightarrow$ kets @ $J_{+,n\,in\,n}$ @ bras @ state

Rot. mat: $x$ to $z$ — $J_x$ eigenstates in $z$ basis

Rot. mat. $z$ to $x$

$x$ to $z$ basis

so $J_{+,x}$ in $x$ basis = $J_{+,z}$ in $z$ basis

Raise in $x$ basis

$z$ to $x$ basis

to raise the state by a unit of $\hbar$ along the $J_x$ momentum

expressed in $z$ basis

*might pick up an overall phase (depending on method)

· Eg: spin-$\frac{1}{2}$

$$J_+ = J_x + iJ_y = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$
$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$
$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$
$$= \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so $J_x|-z\rangle = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $J_+|+z\rangle = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$|-z\rangle \longrightarrow \hbar|+z\rangle$

can't raise $|+z\rangle$!

· Any matrix that looks like $\begin{pmatrix} \ddots & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}\begin{pmatrix} a \\ b \\ \vdots \\ e \end{pmatrix} = \begin{pmatrix} b \\ c \\ \vdots \\ e \\ 0 \end{pmatrix}$ ↑ shifted them up

and $\begin{pmatrix} \ddots & & 0 \\ & 0 & \\ 0 & 1 & \ddots \end{pmatrix}\begin{pmatrix} a \\ \vdots \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ \vdots \\ d \end{pmatrix}$ ↓ shifted down

· To see the effect of $\hat{J}_+$ on the eigenstates, we evaluate $J_z J_+ |l,m\rangle$

① $J^2$ & $J_z$ commute $\rightarrow [J^2, J_z] = 0$
so we can label states $|l,m\rangle$ so that $J_z|l,m\rangle = m\hbar|l,m\rangle$
$J^2|l,m\rangle = l\hbar^2|l,m\rangle$

(see p. 26)
*can 'ask' at the same time since $J^2$ and $J_z$ commute

② $J^2$ also commutes with $J_\pm \rightarrow [J^2, J_\pm] = 0$

$$J^2 J_+ |l,m\rangle = J_+ J^2 |l,m\rangle$$
$$= J_+ l\hbar^2 |l,m\rangle$$
$$J^2(J_+|l,m\rangle) = \hbar^2 l \underbrace{(J_+|l,m\rangle)}$$

so $J^2$ on this still gets out an $\hbar^2 l$
$\Rightarrow$ eigenval doesn't change, still $l$

③ What about $J_z$ & $J_\pm$?

$$[J_z, J_\pm] = [J_z, J_x] \pm i[J_z, J_y]$$
$$= i\hbar J_y \pm i(-i\hbar J_x)$$
$$= \pm\hbar(J_x \pm iJ_y)$$
$$[J_z, J_\pm] = \pm\hbar J_\pm$$

$\Rightarrow$ so $J_z$ & $J_\pm$ don't commute!
we have $J_z J_\pm - J_\pm J_z = \pm\hbar J_\pm$

let's take $J_+$ ... $J_z J_+ |l,m\rangle = (J_+ J_z + \hbar J_+)|l,m\rangle$
$= (\hbar m J_+ + \hbar J_+)|l,m\rangle$

$$J_z(\underbrace{J_+|l,m\rangle}_{\text{eigenstate of } J_z}) = \hbar\underbrace{(m+1)}_{\text{changed eigenval by 1}}(\underbrace{J_+|l,m\rangle}_{\text{still an eigenstate}})$$

(raised, units of $\hbar$)

ie $J_z(\text{state}) = (m+1)\cdots$ so state is raised by 1 from $|l,m\rangle$
$\Rightarrow$ The action of $J_+$ on the state $|l,m\rangle$ is to produce a new state with eigenvalue $\hbar(m+1)$

Similar for $J_-$ : $J_z(J_-|l,m\rangle) = \hbar(m-1)(J_-|l,m\rangle)$

· **Determining the eigenvalues $l$ and $m$**: There are bounds on how far we can raise or lower

① physically: we expect that the square of the projection of the angular momentum on any axis shouldn't exceed the magnitude of $J^2$

$\Rightarrow m^2 \leq l$

formally: $\langle\psi|J_x^2 + J_y^2|\psi\rangle \geq 0$
$\langle\psi|J^2 - J_z^2|\psi\rangle \geq 0$
$|\psi\rangle = |l,m\rangle$
$\langle l,m|(J^2 - J_z^2)|l,m\rangle \geq 0$

$J^2|l,m\rangle = \hbar^2 l|l,m\rangle$
$J_z^2|l,m\rangle = \hbar^2 m^2|l,m\rangle$

$$\hbar^2(l - m^2)\underbrace{\langle l,m|l,m\rangle}_{\neq 0} \geq 0$$

$$l \geq m^2$$

② Let's call the maximum m value j. Then we have $\boxed{J_+|\ell, j\rangle = 0}$ since otherwise $J_+$ would create a state $|\ell, j+1\rangle$, violating our assumption that j is the ———— max eigenvalue of $J_z$

③ We have
$$\begin{aligned} J_-J_+ &= (J_x - iJ_y)(J_x + iJ_y) \\ &= J_x^2 + J_y^2 + iJ_xJ_y - iJ_yJ_x \\ &= J_x^2 + J_y^2 + i[J_x, J_y] \\ &= J_x^2 + J_y^2 - \hbar J_z \end{aligned}$$
and $J_-(J_+|\ell, j\rangle) = 0$

$$J_-J_+ = J^2 - J_z^2 - \hbar J_z$$

so $\langle \ell, j|(J_-J_+)|\ell, j\rangle = 0$ → only $=0$ bc $m=j$

$$\langle \ell, j|(\underbrace{J^2}_{\hbar^2\ell} - \underbrace{J_z^2}_{\hbar^2 j^2} - \underbrace{\hbar J_z}_{\hbar j})|\ell, j\rangle = 0$$

$$\langle \ell, j|(\hbar^2\ell - \hbar^2 j^2 - \hbar^2 j)|\ell, j\rangle = 0$$

$$\hbar^2(\ell - j^2 - j)\langle \ell, j|\ell, j\rangle = 0$$

$$\ell - j^2 - j = 0$$

$$\boxed{\ell = j(j+1)}$$

eg. for spin-$\frac{1}{2}$, $J_z = \pm\frac{\hbar}{2}$ so $J_z = \hbar j \rightarrow j = \frac{1}{2}$

$\rightarrow J^2 = \hbar^2\ell = \hbar^2 j(j+1) = \hbar^2(\frac{1}{2})(\frac{3}{2}) = \hbar\frac{3}{4}$

$\Rightarrow \sqrt{|J^2|} = \frac{\hbar}{2}\sqrt{3}$

④ Similarly, if we call the minimum m value j', then $\boxed{J_-|\ell, j'\rangle = 0}$ and we have

$$\langle \ell, j'|J_+J_-|\ell, j'\rangle = 0$$

$$\langle \ell, j'|(J^2 - J_z^2 + \hbar J_z)|\ell, j'\rangle = 0$$

$$\hbar^2(\ell - j'^2 + j')\langle \ell, j'|\ell, j'\rangle = 0$$

$$\ell - j'^2 + j' = 0$$

$$\boxed{\ell = j'(j'-1)}$$

$J_+J_- = J^2 - J_z^2 + \hbar J_z$

⑤ Combining the two results: $\ell = j(j+1) = j'(j'-1)$

$\downarrow$

$\boxed{j' = -j}$ OR $\underbrace{j' = j+1}_{\text{violates our assumption that } j = \max m}$

· If we start at the $m=j$ state and apply $J_-$ a sufficient number of times, we must reach the state with $m=-j$. If this wasn't the case, we would either reach a state with $m \neq -j$ which satisfies $J_-|\ell, j'\rangle = 0$ or we would violate the bound on the m values

→ Since we lower an int. # of times... $j - j' = j - (-j) = 2j = \text{int}$ (# times we decrease by 1 from $j \rightarrow j'$)

$\Rightarrow$ $j = \text{int or } \frac{1}{2}\text{int}$ with $m = \underbrace{j, j-1, \cdots, -j+1, -j}_{2j+1 \text{ states}}$

· **Change of notation**: we denote the eigenstates of both $J^2$ and $J_z$ by $|j, m\rangle$ instead of $|\ell, m\rangle = |j(j+1), m\rangle$

with this notation, we have

$$\boxed{\begin{aligned} J^2|j, m\rangle &= j(j+1)\hbar^2|j, m\rangle \\ J_z|j, m\rangle &= m\hbar|j, m\rangle \end{aligned}}$$

so $J_+|j, m\rangle = 0$ if $m = j$, ie $|j, j\rangle$

· Eg: spin-$\frac{1}{2}$

$j=\frac{1}{2}$ and for $|+z\rangle$ we have $m=\frac{1}{2}$ $(J_z=+\frac{\hbar}{2})$ $\Rightarrow |+z\rangle = |\frac{1}{2},\frac{1}{2}\rangle$

· **Matrix elements of $J_\pm$**: We've seen that the action of $J_+$ on a state of angular momentum $j$ (related to $\ell$ ie $J^2$) is to create a state with the same magnitude of angular momentum ($\ell$ unchanged) but with the $z$ component ($m$) increased by one unit of $\hbar$

$$\boxed{J_+|j,m\rangle = C_+\hbar|j,m+1\rangle}$$

and for $J_-$...

$$\boxed{J_-|j,m\rangle = C_-\hbar|j,m-1\rangle}$$

finding $C_+$ & $C_-$: ① $\langle j,m|J_-J_+|j,m\rangle = \langle j,m|J_+^\dagger J_+|j,m\rangle$

$$= (J_+|j,m\rangle)^\dagger(J_+|j,m\rangle)$$
$$= C_+^* C_+ \hbar^2 \langle j,m+1|j,m+1\rangle$$

② also, using $J_-J_+ = J^2 - J_z^2 - \hbar J_z$ (see prev. page), we have

$$\langle j,m|(\underbrace{J^2}_{\hbar^2\ell=\hbar^2 j(j+1)} - \underbrace{J_z^2}_{\hbar^2 m^2} - \underbrace{\hbar J_z}_{\hbar m})|j,m\rangle = (\hbar^2 j(j+1) - \hbar^2 m^2 - \hbar^2 m)\langle j,m|j,m\rangle$$
$$= \hbar^2(j(j+1) - m^2 - m)\langle j,m|j,m\rangle$$

③ Equating: $C_+^* C_+ \hbar^2 \langle j,m+1|j,m+1\rangle = \hbar^2(j(j+1)-m^2-m)\langle j,m|j,m\rangle$

↓ Assuming the ang. mom. states satisfy $\langle j,m|j,m\rangle = \langle j,m+1|j,m+1\rangle$

$$C_+^* C_+ = j(j+1) - m^2 - m$$
$$C_+ = [j(j+1) - m^2 - m]^{1/2}$$

$$\Rightarrow \boxed{J_+|j,m\rangle = \sqrt{j(j+1)-m(m+1)}\,\hbar\,|j,m+1\rangle}$$ ↳ vanishes when $m=j$ → Raising action terminates!

Similarly for $C_-$...

$$\Rightarrow \boxed{J_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\,\hbar\,|j,m-1\rangle}$$ ↳ vanishes when $m=-j$

So as a matrix, using $|j,m\rangle$ as a basis:

$$\boxed{\langle j,m'|J_+|j,m\rangle = \sqrt{j(j+1)-m(m+1)}\,\hbar\,\delta_{m',m+1}}$$

$$\boxed{\langle j,m'|J_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\,\hbar\,\delta_{m',m-1}}$$

$m' \to$ row, $m \to$ col

· from $J_\pm = J_x \pm iJ_y$, we can get $\boxed{J_x = \frac{J_+ + J_-}{2}}$ and $\boxed{J_y = \frac{J_+ - J_-}{2i}}$

↳ for $J_z$

·Eg:

**EXAMPLE 3.3** Obtain the matrix representation of the raising and lowering operators using the $j = 1$ states as a basis.

**SOLUTION** The three $j = 1$ basis states are $|1\rangle = |1, 1\rangle$, $|2\rangle = |1, 0\rangle$, and $|3\rangle = |1, -1\rangle$. Using (3.61), we see that $\hat{J}_+|1, 1\rangle = 0$, $\hat{J}_+|1, 0\rangle = \sqrt{2}\,\hbar|1, 1\rangle$, and $\hat{J}_+|1, -1\rangle = \sqrt{2}\,\hbar|1, 0\rangle$. Thus the only nonzero matrix elements are $\langle 1|\hat{J}_+|2\rangle = \langle 1, 1|\hat{J}_+|1, 0\rangle = \sqrt{2}\,\hbar$ in the first row, second column and $\langle 2|\hat{J}_+|3\rangle = \langle 1, 0|\hat{J}_+|1, -1\rangle = \sqrt{2}\,\hbar$ in the second row, third column:

$$\hat{J}_+ \xrightarrow[J_z \text{ basis}]{} \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $\hat{J}_- = \hat{J}_+^\dagger$, the matrix representation of $\hat{J}_-$ is the transpose, complex conjugate of the matrix representation for $\hat{J}_+$:

$$\hat{J}_- \xrightarrow[J_z \text{ basis}]{} \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

These results are in agreement with (3.32) and (3.36), showing that the $3 \times 3$ matrix representations in Section 3.3 are indeed those for $j = 1$.

·Eg:

**EXAMPLE 3.4** Determine the matrix representation for $\hat{S}_x$ using the spin-$\frac{3}{2}$ states as a basis.

**SOLUTION** For $s = \frac{3}{2}$, there are four basis states, namely $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$, and $|\frac{3}{2}, -\frac{3}{2}\rangle$. These four states are eigenstates of $\hat{\mathbf{S}}^2$ with eigenvalue $\frac{3}{2}(\frac{3}{2} + 1)\hbar^2$ as well as being eigenstates of $\hat{S}_z$ with eigenvalues $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, and $-\frac{3}{2}\hbar$, respectively.

Using

$$\hat{S}_+|s, m\rangle = \sqrt{s(s+1) - m(m+1)}\,\hbar|s, m+1\rangle$$

we see that

$$\hat{S}_+|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{3}\,\hbar|\tfrac{3}{2}, \tfrac{3}{2}\rangle$$
$$\hat{S}_+|\tfrac{3}{2} - \tfrac{1}{2}\rangle = 2\hbar|\tfrac{3}{2}, \tfrac{1}{2}\rangle$$
$$\hat{S}_+|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = \sqrt{3}\,\hbar|\tfrac{3}{2}, -\tfrac{1}{2}\rangle$$

Thus the matrix representation for $\hat{S}_+$ is given by

$$\hat{S}_+ \to \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix representation of $\hat{S}_-$ is the transpose, complex conjugate of this matrix, namely

$$\hat{S}_- \to \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Thus the matrix representation of $\hat{S}_x$ is given by

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \to \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

- **Classical limit**: let $j$ be a large number, $m=j$ so $|j,m\rangle \to |j,j\rangle$

  for this state, $J^2 = \hbar^2 j(j+1)$ (see p.26 & bottom p.30)

  and $J_z|j,j\rangle = j\hbar|j,j\rangle \longrightarrow J_z(j\hbar|j,j\rangle) = j^2\hbar^2|j,j\rangle$
  $\Longrightarrow J_z^2 = j^2\hbar^2$

  so we have $J_x^2 + J_y^2 = J^2 - J_z^2$
  $= \hbar^2[j(j+1) - j^2]$
  $= \hbar^2 j$ square of ang. mom. in direction $\perp$ to $J_z$

## UNCERTAINTY RELATIONS

- So far, we've used the fact that $[J^2, J_z] = 0$ to form simultaneous eigenstates of $J^2$ and $J_z$. Since we also have $[J^2, J_x] = 0$, we can also form simultaneous eigenstates of $J^2$ and $J_x$ (for spin-$\frac{1}{2}$, these would be $|+x\rangle$ & $|-x\rangle$)

- However, we didn't try to form simultaneous eigenstates of $J^2$, $J_z$ and $J_x$ $\Longrightarrow$ we can't ☹
  $\to$ This is why we only used one component of $J$ to write the eigenstates
  ↳ eg. $J_z$

- Two operators that don't commute and whose commutator can be expressed in the form $[\hat{A},\hat{B}] = i\hat{C}$ (an hermitian) implies a fundamental uncertainty relation

- **Schwarz inequality**: $\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2}$ *Analogous to $(a\cdot a)(b\cdot b) \geq (a\cdot b)^2$
  see pset 4 for proof $\qquad |a|^2|b|^2 \geq |a|^2|b|^2|\cos\theta|^2$

- **var(f(x))** $= \sigma(f) = \langle(f(x) - \langle f(x)\rangle)^2\rangle = \boxed{\langle f(x)^2\rangle - \langle f(x)\rangle^2}$

- **Uncertainty Relation**: ① Substitute $|\alpha\rangle = (\hat{A} - \langle A\rangle)|\psi\rangle = \hat{A}|\psi\rangle$ and $|\beta\rangle = \hat{B}|\psi\rangle$
  can change variables st. $\langle A\rangle = 0$
  ie. $\tilde{A} = A - I\langle A\rangle$ so $\langle\tilde{A}\rangle = 0$

  ② Expectation values: $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$ and $\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle$
  $\to$ Real numbers since operators are Hermitian (observable)

  ③ Variance: $\sigma(A)^2 = \langle A^2\rangle - \cancel{\langle A\rangle}^{\,0\,2} = \langle\psi|A^2|\psi\rangle = \langle\alpha|\alpha\rangle$
  $\sigma(B)^2 = \langle\beta|\beta\rangle$
  Hermitian so $A = A^\dagger$
  ie $\langle\psi|A = \langle\psi|A^\dagger = \langle\alpha|$

  ④ Schwarz inequality: $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$
  $\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \geq |\langle\psi|AB|\psi\rangle|^2$

⑤ We can write $AB$ as $AB = \frac{AB + (AB)^\dagger}{2} + \frac{AB - (AB)^\dagger}{2}$

$= \frac{\hat{\mathcal{F}}}{2} + \frac{i\hat{\mathcal{G}}}{2}$ where $\hat{\mathcal{F}} = AB + (AB)^\dagger$ and $\hat{\mathcal{G}} = -i(AB - (AB)^\dagger)$ are Hermitian

so we have $\hat{\mathcal{G}} = -i(AB - BA)$

$= -i[A, B]$

$[A, B] = i\hat{C}$

$= \hat{C}$

⑥ $|\langle\alpha|\beta\rangle|^2 = |\langle\psi|AB|\psi\rangle|^2$

$AB = \frac{1}{2}\hat{\mathcal{F}} + \frac{i}{2}\hat{C}$

$= \left|\frac{1}{2}\langle\psi|\hat{\mathcal{F}}|\psi\rangle + \frac{i}{2}\langle\psi|\hat{C}|\psi\rangle\right|^2$ $|z|^2 = Re^2 + Im^2$

$= \frac{1}{4}|\langle\psi|\hat{\mathcal{F}}|\psi\rangle|^2 + \frac{1}{4}\underbrace{|\langle\psi|\hat{C}|\psi\rangle|^2}_{\langle C\rangle} \geq \frac{1}{4}|\langle C\rangle|^2$

bc $a + b \geq a$ if all positive & real

($\langle\mathcal{F}\rangle$ & $\langle C\rangle$ are real so ^2 are positive)

⑦ So we get $\underset{\langle A^2\rangle}{\mathrm{var}(A)}\,\underset{\langle B^2\rangle}{\mathrm{var}(B)} = \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \geq \frac{1}{4}|\langle C\rangle|^2$

$\Rightarrow \sigma^2(A)\sigma^2(B) \geq \frac{1}{4}|\langle C\rangle|^2$

$\boxed{\sigma(A)\sigma(B) \geq \frac{1}{2}|\langle C\rangle|}$ for $[A, B] = iC$

· **Angular momentum**: we have $[J_x, J_y] = \underbrace{i\hbar J_z}_{C}$ so $\boxed{\sigma(J_x)\sigma(J_y) \geq \frac{\hbar}{2}|\langle J_z\rangle|}$

· Eg: If a spin-$\frac{1}{2}$ particle is in a state with a definite value of $J_z$, $\langle J_z\rangle$ is either $\hbar/2$ or $-\hbar/2$ (nonzero!). The eqn above says $\sigma(J_x)$ must also be nonzero, and thus the particle cannot have a definite value of $J_x$ when it has a definite value of $J_z$

# STERN-GERLACH

· **Spin-1**: Not as obvious as spin-$\frac{1}{2}$ ...

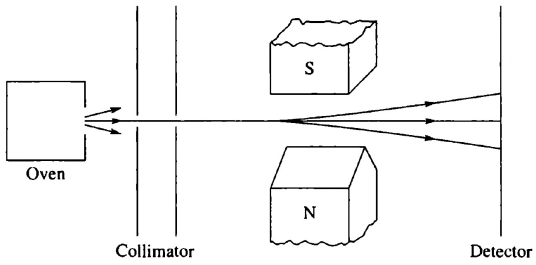



Figure 3.8 A schematic diagram indicating the paths that a spin-1 particle with $S_z$ equal to $\hbar$, 0, or $-\hbar$ would follow in a Stern–Gerlach device.

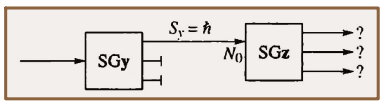



Figure 3.9 A block diagram for an experiment with spin-1 particles with two SG devices whose inhomogeneous magnetic fields are oriented at right angles to each other. What fraction of the particles exiting the SGy device with $S_y = \hbar$ exits the SGz device in each of the three channels?

What fraction of particles with $S_y = \hbar$ will be found to have $S_z = \hbar$?

$\Rightarrow |{}_z\langle 1,1|\underbrace{1,1\rangle_y}_{\text{state } |\psi\rangle, \text{ before } SG_z}| =$ amplitude of $|\psi\rangle$ to be in $|1,1\rangle_z$

$\Rightarrow P(S_z = \hbar) = \text{amplitude}^2$

To find ${}_z\langle \cdots | \cdots \rangle_y$, we can find the eigenstates of $S_y$ in the $S_z$ basis

① We know $S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ in the $S_z$ basis

② Get eigenstates of $S_y$ (in $S_z$ basis) $\rightarrow \begin{array}{c} \begin{array}{ccc} e_1 & e_1 & e_1 \end{array} \\ \begin{pmatrix} | & | & | \\ v_y & & \\ | & | & | \end{pmatrix} \end{array}$

each column = a $S_y$ eigenstate for an eigenvalue $\mu$ $= \begin{pmatrix} {}_z\langle 1,1|1,\mu\rangle_y \\ {}_z\langle 1,0|1,\mu\rangle_y \\ {}_z\langle 1,-1|1,\mu\rangle_y \end{pmatrix}$ = coeffs. of $|1,\mu\rangle_z$ expressed in terms of the $J_z$ basis

How much of $|1,1\rangle_z$ is in $|1,\mu\rangle_y$

So this is essentially how much of each $J_z$ basis states is in $|1,1\rangle_y$

→ use code but eg. $|1,1\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$ → how much of $|1,-1\rangle_z$

$= \frac{1}{2}|1,1\rangle_z + \frac{1}{2}i\sqrt{2}|1,0\rangle_z - \frac{1}{2}|1,-1\rangle_z$

③ Calculate amplitude & probability:

→ if we take the $|1,1\rangle_y$ state exiting $SG_y$ (ie, $S_y = \hbar$), it'll split when it passes through $SG_z$

→ The prob. of the particle exiting the $SG_z$ with $S_z = \hbar$ is given by $|{}_z\langle 1,1|\underbrace{1,1\rangle_y}_{|\psi\rangle, \text{ state coming out of 1st } SG_z. \text{ Expressed in } J_z \text{ basis}}|^2$

These need to be expressed in the same basis (to be able to 'apply' to each other)

opt 1: express $|1,1\rangle_y$ in $J_z$ basis

opt 2: express ${}_z\langle 1,1|$ in $J_y$ basis (more useful when ${}_x\langle 1,m|\psi\rangle_z$

* In pset 5: $|\psi\rangle_z$ & we have ${}_x\langle \cdots |$ expressed in $J_z$ basis

· Eg: pset 5 & oct 7 code

- $J_x$ in $J_z$ basis → $J_x$ eigenstates in $J_z$ basis
- xkets = eigenstates (ex) → use to convert from x to z basis
  xbras = conj(ex).T → use to convert from z to x basis
- eg. $J_{x+}$ = xkets@$J_{n+}$@xbras (see p. 28)
- Rotations: use $R_n(\theta) = e^{-i\hat{J}_n\theta/2\hbar}$
- measurement of ··· = $\langle\psi|\cdots|\psi\rangle$ *$J_x^2 = J_x@J_x$
- $Prob(J_x = 2\hbar) = |\langle 2\hbar\ J_x\ \text{eigenstate}|\psi\ \text{state}\rangle|^2$ = how much of $\psi$ is the $J_x = 2\hbar$ state
  need to be expressed in same basis (any)
  OR see as switch $\psi$ from z basis to x basis using xbras
- $Uncert(J_x) = \sigma(J_x) = (\langle J_x^2\rangle - \langle J_x\rangle^2)^{1/2}$
  $\langle\psi|J_x@J_x|\psi\rangle$ $\langle\psi|J_x|\psi\rangle$
- see pset5 Q2 & Q4 for $J^2 = \cdots j(j+1)$ stuff

· Sept 30 class:

Stern-Gerlach: only observe certain Js
state of atom set by complex amplitudes, amplitudes of $|\psi\rangle$ to be in $|\varphi\rangle$ given by $\langle\varphi|\psi\rangle$
behaviour of states follows linear algebra

Operators expressed as matrices, observables must be Hermitian, $A^\dagger = A$
operators cannot commute, $AB \neq BA$ (unless?)
once we measure in a pure state, cannot be in a pure state of new operator unless commute

Showed $[J_x, J_y] = \frac{iJ_z}{\hbar}$ for spin $\frac{1}{2}$

Given forms of $|\pm x\rangle, |\pm y\rangle, |\pm z\rangle$
Rotations look like $e^{-iJ\theta/\hbar}$

# Chapter 4 : TIME EVOLUTION

## HAMILTONIAN & SCHRÖDINGER EQUATION

- **Time-evolution operator**: $\hat{U}(t)$, translates a ket vector forward in time

$$\boxed{\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle}$$

$\psi(0)$ = initial state of system at $t=0$

state at time $t$

- In order to conserve probability, time evolution shouldn't affect the normalization of the state

$$\rightarrow \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\underbrace{\hat{U}^\dagger(t)\,\hat{U}(t)}_{\text{Need } U^\dagger(t)U(t)=1}|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$\Rightarrow$ Time-evolution operator must be unitary

- Consider an **infinitesimal time translation** $dt$: $\boxed{\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt}$ similar to $R_n(d\theta) = 1 - \frac{i}{\hbar}\hat{J}_n d\theta$ (p.13)

$\hat{H}$: Generator of time translations; is Hermitian

- **Schrödinger equation**: we can show that $\hat{U}$ satisfies a 1st order ODE in time.

Since

$$\hat{U}(t+dt) = \hat{U}(dt)\,\hat{U}(t) = \left(1 - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t)$$

then

$$\underbrace{\hat{U}(t+dt) - \hat{U}(t)}_{d_tU = \lim_{dt\to 0}\frac{U(t+dt)-U(t)}{dt}} = \left(\frac{-i}{\hbar}\hat{H}dt\right)\hat{U}(t)$$

$$\Rightarrow \boxed{i\hbar\, d_t\hat{U} = \hat{H}\hat{U}(t)}$$

$$\Rightarrow \boxed{i\hbar\, d_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$$

- **If $\hat{H}$ is time independent**, we can also get an expression for $\hat{U}$ from a series of infinitesimal time translations

$$\hat{U}(t) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\hat{H}\left(\frac{t}{N}\right)\right]^N = e^{-i\hat{H}t/\hbar}$$

Then $\boxed{|\psi(t)\rangle = \underbrace{e^{-i\hat{H}t/\hbar}}_{\hat{U}}|\psi(0)\rangle}$

**In class**: Evolution/translation operators <u>must</u> be unitary to conserve probability

Assume $|\Psi(x)\rangle = \hat{U}|\Psi(0)\rangle$ ← some unitary translation

we know $\langle\Psi(x)|\Psi(x)\rangle = 1$ for any $x$ so $\langle\Psi(x)| = \langle\Psi(0)|U^\dagger$

$\Rightarrow \langle\Psi(x)|\Psi(x)\rangle = \langle\Psi_0|U^\dagger U|\Psi_0\rangle = 1$ for any $\Psi$ so $U^\dagger U = I \Rightarrow U^\dagger = U^{-1}$

If we have $U = V\Lambda V^{-1}$, then $U^{-1} = V\Lambda^{-1}V^{-1}$ and $U^\dagger = V^{-1\dagger}\Lambda^\dagger V^\dagger$

$U^{-1} = U^\dagger$ so $\Lambda^\dagger = \Lambda^{-1}$ & $V^\dagger = V^{-1}$

**General form**: $U = V\Lambda V^\dagger$ ↳ $\Lambda = e^{i\theta}$

$= Ve^{i\theta}V^\dagger$, $\theta \in \mathbb{R}$

$= e^{iA}$ for $A = V\theta V^\dagger$

We can write $|\Psi(x)\rangle = \hat{U}|\Psi_0\rangle = e^{iA}|\Psi_0\rangle$

So $|\Psi(2dx)\rangle = U(2dx)|\Psi_0\rangle$ and $|\Psi(2dx)\rangle = U(dx)|\Psi(dx)\rangle$

$= U(dx)U(dx)|\Psi_0\rangle$

$U(2dx) = U(dx)U(dx)$

$= e^{iBdx}e^{iBdx}$ for $A = Bdx$

$= e^{iB(2dx)}$

Then $|\Psi(dx)\rangle = e^{iBdx}|\Psi_0\rangle$

$= (I + iBdx)|\Psi_0\rangle$

$|\Psi(dx)\rangle - |\Psi_0\rangle = iBdx|\Psi_0\rangle$

$\dfrac{|\Psi(dx)\rangle - |\Psi_0\rangle}{dx} = iB|\Psi_0\rangle$

$d_x|\Psi(x)\rangle = iB|\Psi(x)\rangle$

We can let $x \to t$, $B \to \frac{-H}{\hbar}$, then $d_t|\Psi\rangle = \frac{-iH}{\hbar}|\Psi\rangle$

$i\hbar d_t|\Psi\rangle = H|\Psi\rangle$ energy

J/s ↓ 1/s ↳ energy unit

let $x \to \theta$, $B \to \frac{-J}{\hbar}$, then $i\hbar d_\theta|\Psi\rangle = J|\Psi\rangle$ angular momentum

↳ ang. mom. units

let $B \to \frac{-P}{\hbar}$, then $i\hbar d_x|\Psi\rangle = P|\Psi\rangle$ position

· **what is the physical significance of the operator $\hat{H}$?**

$\hat{H}$ is a Hermitian operator, the dimensions of $\hat{H}$ are $\frac{\hbar}{\text{time}}$ = energy

When $\hat{H}$ is time independent, we have

$$\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = \langle\Psi(0)|\hat{U}^\dagger(t)\hat{H}\hat{U}(t)|\Psi(0)\rangle$$
$$= \langle\Psi(0)|\hat{H}|\Psi(0)\rangle$$

$\hat{H}$ commutes with $\hat{U}$ (operator commutes with its exp. $\hat{H}$ & $\hat{U} = e^{\cdots\hat{H}}$)

$\Rightarrow$ expectation value is also time independent (same at $t=0$ & $t$)

· We identify $\hat{H}$ as the **energy operator**, known as the **Hamiltonian**

$\Rightarrow$ $\boxed{\langle E\rangle = \langle\Psi|\hat{H}|\Psi\rangle}$

The eigenstates of the Hamiltonian (the energy eigenstates) satisfy $\hat{H}|E\rangle = \underbrace{E}_{\text{energy eigenvalue}}\underbrace{|E\rangle}_{\text{energy eigenstate}}$

· **Effect of $\hat{U}(t)$ on eigenstates**: $\hat{U}(t)|E\rangle = e^{-i\hat{H}t/\hbar}|E\rangle$

$$= \left[1 - \frac{i\hat{H}t}{\hbar} + \frac{1}{2!}\left(-\frac{i\hat{H}t}{\hbar}\right)^2 + \cdots\right]|E\rangle$$

$\hat{H}|E\rangle = E|E\rangle$ Energy eigenvalue

$$= \left[1 - \frac{iEt}{\hbar} + \frac{1}{2!}\left(-\frac{iEt}{\hbar}\right)^2 + \cdots\right]|E\rangle$$
$$= e^{-iEt/\hbar}|E\rangle$$

$\Rightarrow$ The operator $\hat{H}$ in the exponent can simply be replaced by the energy eigenvalue when $\hat{U}(t)$ acts on an eigenstate of $\hat{H}$

· If the **initial state of the system is an energy eigenstate**, ie. $|\Psi(0)\rangle = |E\rangle$, then $\boxed{|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|E\rangle = e^{-iEt/\hbar}|E\rangle}$

The state just picks up an overall phase as time progresses

$\rightarrow$ the physical state of the system doesn't change with time

$\Rightarrow$ we often call such an energy eigenstate a **stationary state** (lack of time dependence)

if $|\Psi_0\rangle \neq$ Eigenstate but we write it in terms of the eig. states, we can use this

eg. $|+x\rangle = e^{-iH}(\cdots|+z\rangle + |-z\rangle)$
$= e^{E}|+z\rangle + e^{E}|-z\rangle$
(see next page)

So we have $\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = \langle\Psi_0|U^\dagger\hat{H}U|\Psi_0\rangle$

$$= \langle\Psi_0|\hat{H}|\Psi_0\rangle$$
$$= E\langle\Psi_0|\Psi_0\rangle$$
$$= E$$

constant!

$U^\dagger\hat{H}U = e^{-i\hat{H}t/\hbar}\hat{H}e^{i\hat{H}t/\hbar}$
$= e^{-i\hat{H}t/\hbar}e^{i\hat{H}t/\hbar}\hat{H}$
$= \hat{H}$

$\rightarrow \frac{d}{dt}\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = 0$ if $|\Psi(t)\rangle = e^{-iEt/\hbar}|E\rangle$

· **If initial state $|\Psi(0)\rangle$ is a superposition of energy eigenstates w/ different energies**, the relative phase bw these energy eigenstates will change with time

eg. 2 eigenstates: $|\Psi_1(t)\rangle = e^{-iE_1t/\hbar}|\Psi_1(0)\rangle$ ← overall phase picked up by eigenstate w/ time; ← eigenstate

$|\Psi_2(t)\rangle = e^{-iE_2t/\hbar}|\Psi_2(0)\rangle$

So if $|\Psi_0\rangle = c_1|\Psi_1(0)\rangle + c_2|\Psi_2(0)\rangle$ then $|\Psi(t)\rangle = \begin{pmatrix} e^{-iE_1t/\hbar}c_1 \\ e^{-iE_2t/\hbar}c_2 \end{pmatrix} = \underbrace{e^{-iE_1t/\hbar}}_{\text{abs phase}}\begin{pmatrix} c_1 \\ e^{-(E_2-E_1)it/\hbar}c_2 \end{pmatrix}$

$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $c_1|\Psi_1\rangle + c_2|\Psi_2\rangle$ at $t=0$

relative phase

# EXPECTATION VALUES

- Consider an observable $A$, then $\langle A\rangle = \langle\Psi(t)|\hat{A}|\Psi(t)\rangle$

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\langle\Psi(t)|\hat{A}|\Psi(t)\rangle$$

$$= \underbrace{(\partial_t\langle\Psi(t)|)}_{\text{conj. T of}}\hat{A}|\Psi(t)\rangle + \langle\Psi(t)|\hat{A}\underbrace{(\partial_t|\Psi(t)\rangle)}_{\text{we know}} + \langle\Psi(t)|\partial_t\hat{A}|\Psi(t)\rangle$$

we know
$i\hbar\partial_t|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$ (p.37-38)
so $\partial_t|\Psi(t)\rangle = \frac{-i}{\hbar}\hat{H}|\Psi(t)\rangle$

$$= \left(\frac{i}{\hbar}\langle\Psi(t)|\hat{H}\right)\hat{A}|\Psi(t)\rangle + \langle\Psi(t)|\hat{A}\left(\frac{-i}{\hbar}\hat{H}|\Psi(t)\rangle\right) + \langle\Psi(t)|\partial_t\hat{A}|\Psi(t)\rangle$$

$$= \frac{i}{\hbar}\left[\langle\Psi|HA|\Psi\rangle - \langle\Psi|AH|\Psi\rangle\right] + \langle\Psi|\partial_t A|\Psi\rangle$$

$$\boxed{\partial_t\langle A\rangle = \underbrace{\frac{i}{\hbar}\langle\Psi(t)|[\hat{H},\hat{A}]|\Psi(t)\rangle}_{\text{will be a cst of the motion}} + \underbrace{\langle\Psi(t)|\partial_t\hat{A}|\Psi(t)\rangle}_{=0 \text{ if } \hat{A} \text{ is time independent (ie, } \hat{A} \text{ cst)}}}$$

# SPIN-$\frac{1}{2}$ IN A MAGNETIC FIELD

- Consider a spin-$\frac{1}{2}$ particle in a constant magnetic field $\vec{B} = B_0\hat{k}$ (along $z$). Take the charge of particle to be $q = -e$

- The **energy operator** is given by $\hat{H} = -\hat{\mu}\cdot\vec{B} = \frac{-gq}{2mc}\hat{S}\cdot\vec{B} = \frac{ge}{2mc}\hat{S}_z B_0 = \omega_0\hat{S}_z$

magnetic dipole moment

frequency $\omega_0 = \frac{ge}{2mc}B_0$

$\Rightarrow$ The eigenstates of $\hat{H}$ are the eigenstates of $\hat{S}_z$:

$$\hat{H}|+z\rangle = \omega_0\hat{S}_z|+z\rangle = \frac{\hbar\omega_0}{2}|+z\rangle = E_+|+z\rangle$$
$$\hat{H}|-z\rangle = \omega_0\hat{S}_z|-z\rangle = \frac{-\hbar\omega_0}{2}|-z\rangle = E_-|-z\rangle$$

energy eigenvalues of spin up/down

- **What happens as time progresses?**

Since the Hamiltonian is time independent, we have $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\underbrace{\omega_0\hat{S}_z}_{E}t/\hbar} = e^{-i\hat{S}_z\varphi/\hbar} = R_z(\varphi)$ with $\varphi = \omega_0 t$

$\Rightarrow$ Placing the particle in a magnetic field in the $z$ direction rotates the spin (of the particle) about the $z$ axis as time progresses, with a period $T = 2\pi/\omega_0$

- **Eg:** $\vec{B} = B_0\hat{k}$, choose $|\Psi(0)\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

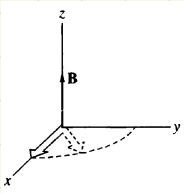



Figure 4.1 A spin-$\frac{1}{2}$ particle, initially in the state $|+x\rangle$, precesses about the magnetic field, which points in the $z$ direction.

① $|+x\rangle$ is a superposition of $S_z$ eigenstates $(|+z\rangle, |-z\rangle)$
$\Rightarrow$ superposition of energy eigenstates w/ different energies

② As a fxn of time: $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)$ $\quad |+x\rangle = |\Psi(0)\rangle$

$\hat{H} \to E$ w/ diff energies

$$= e^{-iE_+t/\hbar}\frac{1}{\sqrt{2}}|+z\rangle + e^{-iE_-t/\hbar}\frac{1}{\sqrt{2}}|-z\rangle$$

$E_+ = \frac{\hbar}{2}\omega_0$
$E_- = \frac{-\hbar}{2}\omega_0$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2}\\ e^{i\omega_0 t/2}\end{pmatrix}$$

we can write this as $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\varphi\begin{pmatrix}1\\ e^{i\omega_0 t}\end{pmatrix}$ with $\varphi = e^{-i\omega_0 t/2}$

$\Rightarrow$ The state doesn't just pick up an overall phase as time progresses $\rightarrow$ Not a stationary state

③ if $\omega t = \pi/2 \rightarrow \psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\pi/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = |+y\rangle$ so we went from $|+x\rangle$ to $|+y\rangle$

$\omega t = \pi \rightarrow \psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{-i\pi}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = |-y\rangle$

$\omega t = 2\pi \rightarrow \psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{-i2\pi}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = |+x\rangle$

④ Probabilities: $P(\hbar/2) = |\langle +z|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t/2}}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$P(-\hbar/2) = |\langle -z|\psi(t)\rangle|^2 = \left|\frac{e^{i\omega_0 t/2}}{\sqrt{2}}\right|^2 = \frac{1}{2}$

} indep. of time

so $\langle S_z\rangle = \frac{1}{2}\left(\frac{\hbar}{2}\right) + \frac{1}{2}\left(\frac{-\hbar}{2}\right) = 0$ $\rightarrow$ also a constant of the motion

⑤ Spin in xy plane: $\langle +x|\psi(t)\rangle = \frac{1}{\sqrt{2}}(1\ \ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2}\\ e^{i\omega_0 t/2}\end{pmatrix} = \cos\left(\frac{\omega_0 t}{2}\right)$

$\rightarrow |\langle +x|\psi(t)\rangle|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right)$ $= 1$ at $t=0$ ✓

$\langle -x|\psi(t)\rangle = \frac{1}{\sqrt{2}}(1\ \ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2}\\ e^{i\omega_0 t/2}\end{pmatrix} = -i\sin\left(\frac{\omega_0 t}{2}\right)$

$\rightarrow |\langle -x|\psi(t)\rangle|^2 = \sin^2\left(\frac{\omega_0 t}{2}\right)$

The sum of the probabilities to be spin up or spin down along x is 1 for all times

⑥ Expectation of $S_x$:

way 1: $\langle S_x\rangle = P(\hbar/2)\cdot\frac{\hbar}{2} + P(-\hbar/2)\cdot(-\hbar/2)$

$= \cos^2\left(\frac{\omega_0 t}{2}\right)\cdot\frac{\hbar}{2} + \sin^2\left(\frac{\omega_0 t}{2}\right)\left(-\frac{\hbar}{2}\right)$

$= \frac{\hbar}{2}\cos\omega_0 t$

way 2: $\langle S_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\omega_0 t/2} & e^{-i\omega_0 t/2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2}\\ e^{i\omega_0 t/2}\end{pmatrix}$

$= \frac{\hbar}{2}\cos\omega_0 t$

⑦ for y... $|\langle +y|\psi(t)\rangle|^2 = \frac{1+\sin\omega_0 t}{2}$, $|\langle -y|\psi(t)\rangle|^2 = \frac{1-\sin\omega_0 t}{2}$

$\rightarrow \langle S_y\rangle = \frac{\hbar}{2}\sin\omega_0 t$

⑧ So the spin is precessing ccw around the z axis w/ a period $T = 2\pi/\omega_0$

If $q = e$ (instead of $-e$), then $\omega_0$ is negative and the spin precesses cw

Note: $d_t\langle S_z\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{S}_z]|\psi\rangle$

$\rightarrow \hat{H} = \omega_0\hat{S}_z$ so $\hat{H}$ commutes w/ $\hat{S}_z$ so $\langle S_z\rangle$ is time independent

· **Symmetry**: If I rotate a state around an axis (eg. z axis, in xy plane) and I have $B = B_z$, does the rotation change my energy? No

Let's say $\langle \psi | \hat{H} | \psi \rangle = E$ and let $\psi \to \hat{U}\psi$ (Rotation), then we need

$$\langle (U\psi) | \hat{H} | U\psi \rangle = \langle \psi | \hat{H} | \psi \rangle$$

$$\psi^\dagger \underbrace{U^\dagger H U}_{\text{we need } U^\dagger H U = H} \psi = \psi^\dagger H \psi$$

$\Rightarrow$ U & H need to commute. $HU = HU + UH - UH = UH + [H, U]$

so

$$U^\dagger H U = H$$

$$U^\dagger (UH + [H,U]) = H$$

$$U^\dagger U H + U^\dagger [H,U] = H$$

$$H + \underbrace{U^\dagger [H,U]}_{\text{This needs to be 0!}} = H$$

$\Rightarrow [H,U] = 0$

$\Rightarrow$ U is unitary $\to$ has a Hermitian friend A ($U = e^{iA}$), and U commutes w/ H

$\Rightarrow$ A also commutes with H (since U & A have the same eigenvectors)

$\Rightarrow$ A is conserved with time

$U = R_z(\theta) = \cdots e^{S_z}$

$A = S_z$

$\Rightarrow$ With the external magnetic field in the z direction, rotations about the z axis leave the spin Hamiltonian unchanged. So the generator $\hat{S}_z$ of these rotations must commute with $\hat{H}$

$\Rightarrow \langle S_z \rangle$ is a constant of the motion (from eqn at bottom of prev. page)

We also know that $\langle S_x \rangle$ & $\langle S_y \rangle$ should vary with time since H isn't invariant under rotations about x and y

# MAGNETIC RESONANCE

- When a spin-$\frac{1}{2}$ particle precesses in a magnetic field in the z direction, the probability of the particle being spin up or spin down along z doesn't vary with time ($|+z\rangle, |-z\rangle$ are stationnary states of H)

If we alter the Hamiltonian by applying in addition an oscillating magnetic field transverse to the z axis, we can induce transitions between these 2 states by properly adjusting the frequency of this transverse field.

- Magnetic field includes a constant magnetic field in the z direction and an oscillating magnetic field in the x direction:

$$\vec{B} = \underbrace{B_0}_{\text{large}} \hat{z} + \underbrace{B_1}_{\text{smaller}} \cos(\omega t)\, \hat{x} \quad \& \quad \psi(t=0) = \begin{pmatrix}1\\0\end{pmatrix} = |+z\rangle \text{ (in z basis)}$$

if this was 0,
no magnetic resonance → Nothing changes
since $\psi_0$ = eigenstates of H

So we have $\hat{H} = -\hat{\mu}\cdot\vec{B} = \frac{-gq}{2mc}\hat{S}\cdot\vec{B} = \frac{-gq}{2mc}\hat{S}\cdot(B_0\hat{z} + B_1\cos\omega t\,\hat{x})$

$$\hat{H} = \underbrace{\omega_0}_{\text{large}} \hat{S}_z + \underbrace{\omega_1}_{\text{smaller}} \cos\omega t\, \hat{S}_x \Rightarrow \text{time dependent} \quad *\text{can't use } \hat{U} = e^{-i\hat{H}t/\hbar}$$

'spins fast around $\hat{z}$, spins slowly around $\hat{x}$'

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos\omega t & -\omega_0\end{pmatrix}$$

and $|\psi(t)\rangle = \begin{pmatrix}a(t)\\b(t)\end{pmatrix}$ with $|\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

So $i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle$

$$\underbrace{i\hbar\frac{d}{dt}\begin{pmatrix}a(t)\\b(t)\end{pmatrix}}_{i\hbar\begin{pmatrix}\dot{a}(t)\\\dot{b}(t)\end{pmatrix}} = \frac{\hbar}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos\omega t & -\omega_0\end{pmatrix}\begin{pmatrix}a(t)\\b(t)\end{pmatrix}$$

coupled 1st order ODEs

Guess the solution: $\begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \begin{pmatrix}c(t)e^{-i\omega_0 t/2}\\ \underbrace{d(t)e^{i\omega_0 t/2}}\end{pmatrix}$ → so $c(0) = 1$, $d(0) = 0$

we expect to have included the major part of the time dependence in the exp terms

ODEs become: $i\begin{pmatrix}\dot{c}(t)\\\dot{d}(t)\end{pmatrix} = \frac{\omega_1}{2}\cos\omega t\begin{pmatrix}d(t)e^{i\omega_0 t}\\c(t)e^{-i\omega_0 t}\end{pmatrix}$

$$= \frac{\omega_1}{4}\begin{pmatrix}(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t})d(t)\\(e^{i(\omega-\omega_0)t} + e^{-i(\omega_0+\omega)t})c(t)\end{pmatrix}$$

If $\omega$ is near $\omega_0$, the terms oscillating at $\omega_0+\omega$ can be neglected wrt those oscillating at $\omega_0-\omega$ since $\omega_0-\omega$ are oscillating sufficiently slowly that c & d vary with time

· Solving for the case $\omega = \omega_0$ (resonant condition) & neglecting the exponentials oscillating at $2\omega_0$

① $i\begin{pmatrix}\dot{c}(t)\\ \dot{d}(t)\end{pmatrix} = \frac{\omega_1}{4}\begin{pmatrix}d(t)\\ c(t)\end{pmatrix}$

↓ take $d_t$ and use to get rid of $\dot{c}, \dot{d}$

$\begin{pmatrix}\ddot{c}(t)\\ \ddot{d}(t)\end{pmatrix} = -\left(\frac{\omega_1}{4}\right)^2\begin{pmatrix}c(t)\\ d(t)\end{pmatrix}$ uncoupled, 2nd order!

② The solution satisfying the initial conditions $c(0)=1$, $d(0)=0$ is $c(t)=\cos\left(\frac{\omega_1 t}{4}\right)$ & $d(t) = -i\sin\left(\frac{\omega_1 t}{4}\right)$

③ for a spin-$\frac{1}{2}$ particle initially in the state $|+z\rangle$ $(t=0)$, the probabilities to find it in a state at time $t$ is ...

$|-z\rangle \longrightarrow P(|-z\rangle) = |\langle -z|\psi(t)\rangle|^2 = b^*(t)b(t) = d^*(t)d(t) = \sin^2\frac{\omega_1 t}{4}$

the $e^{\cdots}$ cancel in $d^*d$

$|+z\rangle \longrightarrow P(|+z\rangle) = |\langle +z|\psi(t)\rangle|^2 = a^*(t)a(t) = c^*(t)c(t) = \cos^2\frac{\omega_1 t}{4}$

with $P(|-z\rangle) + P(|+z\rangle) = 1$

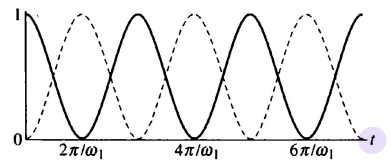



Figure 4.5 The probabilities $|\langle +z|\psi(t)\rangle|^2$ (solid line) and $|\langle -z|\psi(t)\rangle|^2$ (dashed line) for a spin-$\frac{1}{2}$ particle that is in the state $|+z\rangle$ at $t=0$ when the time-dependent magnetic field in the $x$ direction is tuned to be resonant frequency.

④ If a particle initially in the state $|+z\rangle$ makes a transition to the state $|-z\rangle$, the energy of the spin system is reduced by $E_+ - E_- = \hbar\omega_0$ (assuming $\omega_0 > 0$)
→ This energy is added to the electromagnetic energy of the oscillating field that's stimulating the transition

⑤ for $0 < t < \frac{2\pi}{\omega_1}$, the probability to transition to the lower energy state ($|+z\rangle \Rightarrow |-z\rangle$) grows until $b^*(t)b(t) = 1$ and $a^*(t)a(t) = 0$
Then the particle is in the state $|-z\rangle$

for $\frac{2\pi}{\omega_1} < t < \frac{4\pi}{\omega_1}$, $P(|-z\rangle)$ decreases and $P(|+z\rangle)$ increases as the system absorbs energy back from the EM field

· There is a probability of inducing a transition bw the 2 spin states even when $\omega \neq \omega_0$
If the system is initially in $|+z\rangle$, we have:

$$\boxed{P(|-z\rangle) = |\langle -z|\psi(t)\rangle|^2 = \frac{\omega_1^2/4}{(\omega_0-\omega)^2+\omega_1^2/4}\sin^2\frac{1}{2}\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}\,t}$$

Rabi's formula

→ prob. to be in $|-z\rangle$ at time $t$

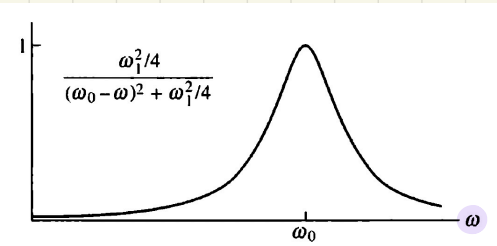



Figure 4.6 A sketch of the magnetic-resonance transition probability as a function of the frequency $\omega$ of the time-dependent magnetic field.

Max prob. of transition as a fxn of $\omega$

so basically, $A = \frac{\omega_1^2/4}{\cdots}$ & $P(|-z\rangle) = A\sin^2$ so max $= A\cdot\underbrace{\sin}_{=1} = A$
⇒ for $\omega = \omega_0$, state fully switches since $P_{max} = 1$
→ if eg $\omega$ is st. $P_{max} = 0.5$, then we have $P(|-z\rangle) = 0.5\sin(\cdots t)$

# AMMONIA MOLECULE

- Is the nitrogen atom above/below the plane formed by the 3 H atoms?

① Two states : $|1\rangle = |N \text{ above the plane}\rangle$
$|2\rangle = |N \text{ below the plane}\rangle$

→ use these as basis states to construct $\hat{H}$

② find $\hat{H}$: we know $\hat{H} = \begin{pmatrix} \langle 1|\hat{H}|1\rangle & \langle 1|\hat{H}|2\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle \end{pmatrix}$

Symmetry of system → energy should be the same for the two states
$\Rightarrow \langle 1|\hat{H}|1\rangle = \langle 2|\hat{H}|2\rangle = E_0$ (diagonal elements of $\hat{H}$)

If off-diagonal = 0 → stationary states, would mean that a molecule initially in eg. $|1\rangle$ would remain in that state
However, the potential barrier bw the two wells aren't inf. high → there's a small amplitude for a nitrogen atom to tunnel bw the two states
$\Rightarrow$ off-diagonal $\neq 0$, let them be $-A$

$$\hat{H} = \begin{pmatrix} \langle 1|\hat{H}|1\rangle & \langle 1|\hat{H}|2\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle \end{pmatrix} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$$ with A a positive constant

③ find energy eigenstates & eigenvalues of $\hat{H}$:

coeffs. : $|\psi\rangle = C_1|1\rangle + C_2|2\rangle$
$\langle 1|\psi\rangle$ $\langle 2|\psi\rangle$

$\hat{H}|\psi\rangle = E|\psi\rangle$ in the $|1\rangle$-$|2\rangle$ basis → $\begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}\begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \end{pmatrix} = E\begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \end{pmatrix}$

eigenvalues

eigenvalues : need $\begin{vmatrix} E_0 - E & -A \\ -A & E_0 - E \end{vmatrix} = 0$
$\Rightarrow E = E_0 \pm A$

denote energy eigenstates by $\begin{cases} |I\rangle \text{ for } E_I = E_0 - A \\ |II\rangle \text{ for } E_{II} = E_0 + A \end{cases}$

substituting $|\psi\rangle = \begin{cases} |I\rangle \\ |II\rangle \end{cases}$ & $E = \begin{cases} E_I \\ E_{II} \end{cases}$ in eigval. eqn & solving for the eigenstate coeffs, we get

$\begin{cases} \langle 1|I\rangle = \langle 2|I\rangle \\ \langle 1|II\rangle = -\langle 2|II\rangle \end{cases}$ ie $C_1 = C_2$ for $|I\rangle = C_1|1\rangle + C_2|2\rangle$
...

$\Rightarrow$ eigenstates : $\begin{cases} |I\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \\ |II\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle \end{cases}$

④ The existence of tunneling between the states $|1\rangle$ & $|2\rangle$ has split the energy states of the molecule into 2 (eigen)states w/ 2 different energies, ie $E_0 - A$ & $E_0 + A$ (These have the same energy $E_0$!)
$E_{II} - E_I = 2A$ shows how small the amplitude is for tunneling from $|1\rangle$ to $|2\rangle$

**Notice**: neither $|I\rangle$ or $|II\rangle$ has the nitrogen atom located above or below the plane formed by the 3 H.

we can localize the N atom above the plane by eg. superposing the energy eigenstates

$$|1\rangle = \frac{1}{\sqrt{2}}|I\rangle + \frac{1}{\sqrt{2}}|II\rangle$$

## ENERGY-TIME UNCERTAINTY RELATION

- $\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}}$

- Only $\Delta E$ is a legitimate uncertainty. It reflects the spread in energy characterizing a particular state

- **Understanding $\Delta t$**: think about ammonia molecule initially in state $|1\rangle = |\text{N atom above plane}\rangle$. This state isn't an energy eigenstate but a superposition of 2 energy eigenstates w/ different energies

→ energy uncertainty is given by: $\Delta E = (\langle E^2\rangle - \langle E\rangle^2)^{1/2}$

$$= \left[\tfrac{1}{2}(E_0+A)^2 + \tfrac{1}{2}(E_0-A)^2 - \left\{\tfrac{1}{2}(E_0+A) + \tfrac{1}{2}(E_0+A)\right\}^2\right]^{1/2}$$

$$= A$$

→ we can express the time evolution of the state in terms of $\Delta E$:

$$|\psi(t)\rangle = (\text{overall phase})\left(\frac{1}{\sqrt{2}}|I\rangle + \frac{1}{\sqrt{2}}e^{-2i\Delta E t/\hbar}|II\rangle\right)$$

→ $\Delta t$ = how long we have to wait for the state of the molecule to change
To be sure that the state has changed, we need to be sure the relative phase between the energy eigenstates $|I\rangle$ & $|II\rangle$ has changed significantly from its value of 0 at $t=0$
⇒ This requires that the time interval $\Delta t$ satisfy $2\Delta E \Delta t/\hbar \gtrsim 1$

**Notice**: If the state of the molecule had been an energy eigenstate (instead of a superposition of energy eigenstates w/ different energies), there would be a definite value for the energy, ie $\Delta E = 0$
The ket would pick up only an overall phase as time evolved so $\Delta t = \infty$ to change state

- **Ex**: let's say I have an observable A

we know $\sigma(A)\sigma(E) \geq \frac{|\langle C\rangle|}{2}$ if $[\hat{A},\hat{H}] = iC$

(quantity: $\langle C\rangle$; operator: $\hat{A}$, $\hat{H}$)

so,

$$\sigma(A)\sigma(E) \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{H}]|\psi\rangle|$$

we know $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle + \langle\psi|\cancel{\partial_t\hat{A}}^{\,0}|\psi\rangle$

$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle$$

$= -\langle\psi|[\hat{A},\hat{H}]|\psi\rangle$ OR $|[\hat{H},\hat{A}]| = |[\hat{A},\hat{H}]|$

$$\Rightarrow |\langle\psi|[\hat{H},\hat{A}]|\psi\rangle| = \hbar\left|\frac{d\langle A\rangle}{dt}\right|$$

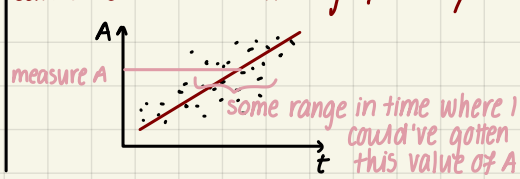
$$\sigma(A)\sigma(E) \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|$$

$$\underbrace{\left(\frac{\sigma(A)}{d\langle A\rangle/dt}\right)}_{\equiv\sigma(t)}\sigma(E) \geq \frac{\hbar}{2}$$

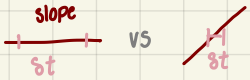
$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

or other observable

can also think abt this graphically




→ $\delta t = \frac{\sigma(A)}{d\langle A\rangle/dt} = \frac{\delta A}{\text{slope}}$




larger slope ⇒ smaller $\delta t$

lets pick $\vec{B} = B_z$

$\vec{S}_x$

spins around

$E_1, E_2 \rightarrow +\frac{\hbar}{2}\omega, -\frac{\hbar}{2}\omega$

if $\psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ then $\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-iE_1t/\hbar}\\e^{-iE_2t/\hbar}\end{pmatrix}$

$$= \frac{1}{\sqrt{2}} e^{-iE_1t/\hbar}\begin{pmatrix}1\\e^{-i(E_2-E_1)t/\hbar}\end{pmatrix}$$

if $\frac{E_2-E_1}{\hbar} t \sim 1$

ie $(E_2 - E_1) \sim \delta E$ so $\delta E t \sim \hbar$

then the change matters (in phase $\rightarrow$ out of phase)

$\delta t = t_{\text{in phase}} - t_{\text{out of phase}}$

$\delta E \delta t > \hbar$ for system to change

?

# Chapter 6 : WAVE MECHANICS IN 1D

## POSITION EIGENSTATES & WAVE FUNCTION

- **Position eigenstates**: $\boxed{\hat{x}|x\rangle = x|x\rangle}$ — position state

  position operator; $-\infty$ to $\infty$

* can't obtain a single value for the position of a particle when we try to measure it (unlike spin, where we get either $\hbar/2$ or $-\hbar/2$ when making a measurement of $S_z$)

- Think of physical states that occur in nature as a superposition of these position eigenstates

**Try**: $|\psi\rangle = \sum_i |x_i\rangle \underbrace{\langle x_i|\psi\rangle}_{\text{amplitude to be in } x_i}$ & $\langle\psi| = \sum_j \langle\psi|x_j\rangle\langle x_j|$

so $\langle\psi|\psi\rangle = \sum_{ij} \langle\psi|x_j\rangle \underbrace{\langle x_j|x_i\rangle}_{=\delta_{ij}} \langle x_i|\psi\rangle$

$= \sum_i \langle\psi|x_i\rangle\langle x_i|\psi\rangle$

$= \sum_i |\langle x_i|\psi\rangle|^2$

consider a point $x_i = a$ where $\langle a|\psi\rangle \neq 0$. We can choose a nearby point $x_i = a + \Delta x$ where $\langle a+\Delta x|\psi\rangle \neq 0$

However, there are infinitely many points between $a$ & $a+\Delta x$, no matter how small we choose $\Delta x$

$\Rightarrow \sum_i |\langle x_i|\psi\rangle|^2 = \infty$ ie we don't satisfy $\langle\psi|\psi\rangle = 1$

- **We need an integral**: $\boxed{|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi\rangle}$ → coeff. of pos. ket $|x\rangle$ is $dx\langle x|\psi\rangle$

if we $\int$ from $a$ to $a+\Delta x$, we get $\int_a^{a+\Delta x} dx\, |x\rangle\langle x|\psi\rangle$ which vanishes for small $\Delta x$

we require $\boxed{\int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = 1}$ **identity operator**

To see this, consider the case where $|\psi\rangle = |x'\rangle$ (ie a position state)

then, $|\psi\rangle = |x'\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle \underbrace{\langle x|x'\rangle}_{\langle x|x'\rangle = \delta(x-x')}$

when $x \neq x'$, the amplitude to find a position state $|x'\rangle$ at position $x$ vanishes

we have $\boxed{\langle\psi| = \int_{-\infty}^{\infty} dx\, \langle\psi|x\rangle\langle x|}$

so $\langle\psi|\psi\rangle = \iint dx'dx\, \langle\psi|x\rangle\langle x|x'\rangle\langle x'|\psi\rangle$

$= \iint dx'dx\, \langle\psi|x\rangle\, \delta(x-x')\, \langle x'|\psi\rangle$

$= \int dx\, \langle\psi|x\rangle\langle x|\psi\rangle$ $= \langle x|\psi\rangle^*\langle x|\psi\rangle$

$\boxed{\langle\psi|\psi\rangle = \int dx\, \underbrace{|\langle x|\psi\rangle|^2}_{\text{probability of finding the particle bw } x \text{ \& } x+dx}} = 1$ → since we need total prob. of finding particle in position space to be 1

- **Wave functions**:

  $\langle x|\psi\rangle$ (complex number) = amplitude to find a particle in the state $|\psi\rangle$ at position $x$

  → in general, diff. for each diff. value of $x$

  ⇒ wave function: $\boxed{\psi(x) \equiv \langle x|\psi\rangle}$

  we can rewrite the previous eqn as: $\boxed{1 = \int dx\, \psi^*(x)\psi(x) = \int dx\, |\psi(x)|^2}$

- **Expectation value**: for an observable $A$, we have $\boxed{\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle}$

  so avg position is: $\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$ $\quad |\psi\rangle = \int dx\, |x\rangle\langle x|\psi\rangle$

$$= \int dx\, \langle\psi|\hat{x}|x\rangle\langle x|\psi\rangle$$

$\hat{x}|x\rangle = x|x\rangle$

$$= \int dx\, \langle\psi|x\rangle x\langle x|\psi\rangle$$

$$= \int dx\, \psi^*(x)\, x\, \psi(x)$$

$$\boxed{\langle x\rangle = \int dx\, |\psi(x)|^2 x}$$

"The 'sum' over all positions $x$ · prob. of finding the particle bw $x$ & $x+dx$"

- **Eg**: if we know $\langle x|\varphi\rangle$ & $\langle x|\psi\rangle$, what's the amplitude $\langle\varphi|\psi\rangle$?

  use the identity operator $\int dx\, |x\rangle\langle x|$

  we get:

$$\langle\varphi|\psi\rangle = \int dx\, \langle\varphi|x\rangle\langle x|\psi\rangle$$

$$= \int dx\, \varphi^*(x)\psi(x)$$

## TRANSLATIONS

- **Translation operator**: $\boxed{\hat{T}(a)|x\rangle = |x+a\rangle}$

  ↳ changes a state w/ position $x$ to one with position $x+a$

- **Operator on an arbitrary state $|\psi\rangle$**: $|\psi'\rangle = \hat{T}(a)|\psi\rangle$ — just to denote integ. variable

$$= \hat{T}(a) \int dx'\, |x'\rangle\langle x'|\psi\rangle$$

$$= \int dx'\, |x'+a\rangle\langle x'|\psi\rangle$$

so we have: $\psi'(x) = \langle x|\psi'\rangle = \langle x|\hat{T}(a)|\psi\rangle$ — T acting on $\psi$

$$= \int dx'\, \underbrace{\langle x|x'+a\rangle}_{\delta(x-(x'+a))}\langle x'|\psi\rangle$$

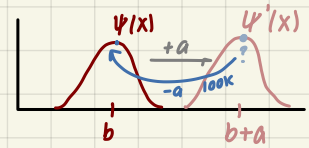
*picks out when $x' = x-a$

$$= \int dx'\, \delta[x-(x'+a)]\langle x'|\psi\rangle$$

$$= \langle x-a|\psi\rangle$$

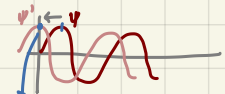
$$\psi'(x) = \psi(x-a)$$

*Not $\psi'(x) = \psi(x+a)$




so if you want to evaluate $\psi'$, "check $\psi$ at $x-a$ before"

eg.




evaluate $\psi$ $+a$ of that pos.

$\psi'(x) = \psi(x+a)$ *for a shift to left

- **$\hat{T}(a)$ must be unitary**: Translating shouldn't affect normalization

$$\langle\psi'|\psi'\rangle = \langle\psi|\hat{T}^\dagger(a)\hat{T}(a)|\psi\rangle = \langle\psi|\psi\rangle$$

Need $\hat{T}^\dagger(a)\hat{T}(a) = 1$

- We can also do: $\boxed{\langle x|\hat{T}^\dagger(a) = \langle x+a|}$

  but translation operator is unitary so $\hat{T}^\dagger$ is the inverse of $\hat{T}$

  $\Rightarrow \langle x|\hat{T}(a) = \langle x-a|$

  so $\psi'(x) = \underbrace{\langle x|\hat{T}(a)}_{\text{T acting on } \langle x|}|\psi\rangle = \langle x-a|\psi\rangle = \psi(x-a)$

- Consider a infinitesimal translation $\delta x$, we have $\boxed{\hat{T}(dx) = 1 - \frac{i}{\hbar}\hat{p}_x dx}$ with $\hat{T}(\delta x)|x\rangle = |x+\delta x\rangle$

  Generator of translations dim = linear momentum

  $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

  (see p.52)

We can generate a finite translation by a using an infinite number of small translations

$$\hat{T}(a) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\hat{p}_x\left(\frac{a}{N}\right)\right]^N$$

$\boxed{\hat{T}(a) = e^{-i\hat{p}_x a/\hbar}}$ similar to $\hat{U}$ & $\hat{H}$ (p.37)

- Fact: $\hat{T}(a)$ is unitary $\Rightarrow \hat{p}_x$ is Hermitian, $\hat{p}_x = \hat{p}_x^\dagger$

- Fact: $\hat{p}_x$ (generator of translation) doesn't commute with $\hat{x}$ (position operator)

① $\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \hat{x}\left(1 - \frac{i}{\hbar}\hat{p}_x\delta x\right) - \left(1 - \frac{i}{\hbar}\hat{p}_x\delta x\right)\hat{x}$

$= \left(\frac{-i\delta x}{\hbar}\right)[\hat{x}, \hat{p}_x]$

② $(\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x})|\psi\rangle = (\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x})\int dx|x\rangle\langle x|\psi\rangle$

$\left(\frac{-i\delta x}{\hbar}\right)[\hat{x}, \hat{p}_x] = \hat{x}\int dx|x+\delta x\rangle\langle x|\psi\rangle - \hat{T}(\delta x)\int dx\, x|x\rangle\langle x|\psi\rangle$

$= \int dx\,(x+\delta x)|x+\delta x\rangle\langle x|\psi\rangle - \int dx\, x|x+\delta x\rangle\langle x|\psi\rangle$

$= \delta x\int dx|x+\delta x\rangle\langle x|\psi\rangle$

↓ Taylor exp. & keep leading order term

$= \delta x\int dx|x\rangle\langle x|\psi\rangle$

$\left(\frac{-i\delta x}{\hbar}\right)[\hat{x}, \hat{p}_x] = \delta x|\psi\rangle$

In class:

$[x,p]\psi = (xp - px)\psi$

$= -i\hbar(x\partial_x - \partial_x x)\psi$

$= -i\hbar(x\partial_x\psi - \partial_x(x\psi))$

$= -i\hbar[x\partial_x\psi - (\psi + x\partial_x\psi)]$

$= -i\hbar(-\psi)$

$= i\hbar\psi$

$\Rightarrow \boxed{[\hat{x}, \hat{p}_x] = i\hbar}$

- We can write the Hamiltonian as $\boxed{\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})}$

  kinetic (momentum operator = generator of translations); potential energy

① $\frac{d\langle x\rangle}{dt}$ = velocity $= v = \frac{p}{m}$ (what we expect from classical)

Recall: $\frac{d\langle A\rangle}{dt}|\psi\rangle = \frac{i}{\hbar}[\hat{H}, \hat{A}]|\psi\rangle$

so $\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{x}]|\psi\rangle$

$= \frac{i}{\hbar}\langle\psi|[\frac{\hat{p}_x^2}{2m} + V(\hat{x}), \hat{x}]|\psi\rangle$

$= \frac{i}{\hbar}\langle\psi|[\frac{\hat{p}_x^2}{2m}, \hat{x}] + \underbrace{[V(\hat{x}), \hat{x}]}|\psi\rangle$

V is a fxn of $\hat{x}$ so we can Taylor exp → powers of $\hat{x}$

⇒ always commutes since all $\hat{x}$ so $[\,] = 0$

$= \frac{i}{\hbar 2m}\langle\psi|[\hat{p}_x^2, \hat{x}]|\psi\rangle$ Recall: $[AB,C] = A[B,C] + [A,C]B$

$= \frac{i}{\hbar 2m}\langle\psi|(\hat{p}_x[\hat{p}_x, \hat{x}] + [\hat{p}_x, \hat{x}]\hat{p}_x)|\psi\rangle$

$= \frac{i}{\hbar 2m}\langle\psi|(\hat{p}_x(-i\hbar) + (-i\hbar)\hat{p}_x)|\psi\rangle$

$\boxed{\frac{d\langle x\rangle}{dt} = \frac{\langle p_x\rangle}{m}}$ ↓ $\langle\psi|\hat{p}_x|\psi\rangle = \langle p_x\rangle$

② $\frac{d\langle p_x\rangle}{dt} = \text{force} = F = \frac{-\partial V}{\partial x}$ (from classical)

→ $\frac{d\langle p_x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{p}_x]|\psi\rangle$

$\boxed{\frac{d\langle p_x\rangle}{dt} = \langle\frac{-dV}{dx}\rangle}$ see pset8

· Uncertainty relation: $\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$ Heisenberg

## MOMENTUM OPERATOR IN POSITION BASIS

· Consider $\hat{T}(\delta x)$ on $|\psi\rangle$: $\hat{T}(\delta x)|\psi\rangle = \hat{T}(\delta x)\int dx|x\rangle\langle x|\psi\rangle$

$= \int dx|x+\delta x\rangle\langle x|\psi\rangle$

let $x' = x + \delta x$, $dx' = dx$

$= \int dx'|x'\rangle\underbrace{\langle x' - \delta x|\psi\rangle}$

$= \psi(x' - \delta x)$

→ Taylor expand about $x'$

$= \psi(x') - \delta x\frac{\partial}{\partial x'}\psi(x')$

$= \langle x'|\psi\rangle - \delta x\frac{\partial}{\partial x'}\langle x'|\psi\rangle$

$= \int dx'|x'\rangle(\langle x'|\psi\rangle - \delta x\frac{\partial}{\partial x'}\langle x'|\psi\rangle)$

$(1 - \frac{i}{\hbar}\hat{p}_x\delta x)|\psi\rangle = |\psi\rangle - \delta x\int dx'|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle$

$\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\int dx'|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle$

↓ take $\langle x|$

$\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\int dx'\underbrace{\langle x|x'\rangle}_{\delta(x-x')}\partial_{x'}\langle x'|\psi\rangle$

$= \frac{\hbar}{i}\int dx'\delta(x-x')\partial_{x'}\langle x'|\psi\rangle$

$\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$

- If we choose $|\psi\rangle = |x'\rangle$, we get the matrix elements of $\hat{p}_x$ in the $\hat{x}$ basis

$$\langle x|\hat{p}_x|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\delta(x-x')$$

- **Expectation value:** $\langle p_x\rangle = \langle\psi|\hat{p}_x|\psi\rangle$ see prev. page $\hat{p}_x|\psi\rangle$

$$= \int dx'\langle\psi|x'\rangle\frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\psi\rangle$$

$$= \int dx'\psi^*(x')\frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x')$$

$$\boxed{\langle p_x\rangle = \int dx\,\psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)} = \int dp\,|\langle p|\psi\rangle|^2 p$$

we don't know this yet

- **Momentum operator:** $\boxed{\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}}$
  (in $\hat{x}$ basis)

can use to get

## MOMENTUM SPACE

- **Momentum eigenstates:** $\boxed{\hat{p}_x|p\rangle = p|p\rangle}$
- **Arbitrary state:** $\boxed{|\psi\rangle = \int dp|p\rangle\langle p|\psi\rangle}$ (from $-\infty$ to $\infty$)

  → momentum is a continuous variable (like position)
  → write $|\psi\rangle$ as superposition of momentum states

- we also have $\langle p'|p\rangle = \delta(p-p')$ and $1 = \langle\psi|\psi\rangle = \int dp\,|\langle p|\psi\rangle|^2$

  probability that a particle in the state $|\psi\rangle$ has its momentum between $p$ and $p+dp$ if we make a measurement

- **Wave function in momentum space:** $\boxed{\psi(p) = \langle p|\psi\rangle}$

- **Finding $\langle x|p\rangle$:** This is the momentum eigenstate's position-space wave function

  we know $\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$

  ↓ let $|\psi\rangle = |p\rangle$

  $$\langle x|\hat{p}_x|p\rangle = p\langle x|p\rangle = \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|p\rangle}_{\text{solve ODE}}$$

  → $\langle x|p\rangle = Ne^{ipx/\hbar}$

  see townsend p. 203 (pdf p. 219) for how to find N

$$\Rightarrow \boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}$$ position-space momentum eigenfunction

and $\lambda = \frac{h}{p}$ or $p = \hbar k$

· **wave functions**:

$$\boxed{\begin{aligned}\langle p|\psi\rangle &= \int dx \langle p|x\rangle\langle x|\psi\rangle = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\langle x|\psi\rangle \\ \langle x|\psi\rangle &= \int dp \langle x|p\rangle\langle p|\psi\rangle = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\langle p|\psi\rangle\end{aligned}}$$

} fourier transform pair

## GAUSSIAN WAVE PACKET

· from prev. section, we have $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

if we had a single momentum state → $\Delta p = 0$ since definite momentum $p$
→ prob. of finding particle bw $x$ & $x+dx$ is $|\langle x|p\rangle|^2 = \frac{dx}{2\pi\hbar}$
ie. indep. of $x$ (particle has a completely indefinite position)
so $\Delta x = \text{inf.}$

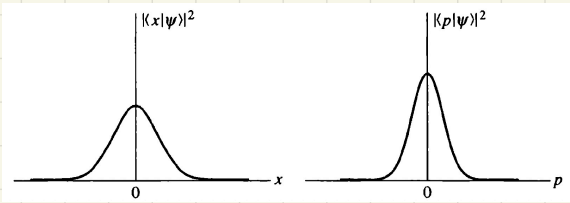
Since the particle isn't permitted to have a definite momentum, we should superpose momentum states to obtain a physically allowed state $|\psi\rangle$ satisfying $\langle\psi|\psi\rangle = 1$

· **wave packet**: superposition of the momentum states.
In position space the momentum eigenfunctions are oscillating

$$\boxed{\langle x|\psi\rangle = \psi(x) = \frac{1}{\sqrt{\sqrt{\pi}\,a}} e^{-x^2/2a^2}}$$

· **probability density**: $\boxed{\psi^*(x)\psi(x) = \frac{1}{\sqrt{\pi}a} e^{-x^2/a^2}}$

( since no dx




· **Position uncertainty**: ① $\langle x\rangle = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{\pi}a} e^{-x^2/a^2} x = 0$ (odd fxn)

② $\langle x^2\rangle = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{\pi}a} e^{-x^2/a^2} x^2 = \frac{a^2}{2}$

③ $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{a}{\sqrt{2}}$

· **momentum-space wave function**: $\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx} \underbrace{\frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x^2/2a^2}}_{\langle x|\psi\rangle}$

$$\boxed{\langle p|\psi\rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}} e^{-p^2a^2/2\hbar^2}}$$

· **probability** to have momentum bw $p$ & $p+dp$ is $\boxed{|\langle p|\psi\rangle|^2 dp = \frac{a}{\hbar\sqrt{\pi}} e^{-p^2a^2/\hbar^2} dp}$

· **momentum uncertainty**: ① $\langle p_x \rangle = \langle \Psi | \hat{p}_x | \Psi \rangle$

$$= \int dp\, |\langle p|\Psi\rangle|^2 p$$
$$= \frac{a}{\hbar\sqrt{\pi}} \int dp\, e^{-p^2a^2/\hbar^2} p$$
$$= 0$$

② $\langle p_x^2 \rangle = \langle \Psi | \hat{p}_x^2 | \Psi \rangle$

$$= \int dp\, |\langle p|\Psi\rangle|^2 p^2$$
$$= \frac{a}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} dp\, e^{-p^2a^2/\hbar^2} p^2$$
$$= \frac{\hbar^2}{2a^2}$$

③ $\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \frac{\hbar}{\sqrt{2}a}$

· **Note**: $\Delta x \Delta p = \frac{\hbar}{2}$ so the Gaussian wave function is the minimum uncertainty state

· **Time evolution**: for a free particle, we have $\hat{H} = \frac{\hat{p}_x^2}{2m}$ so the momentum states $|p\rangle$ are also energy eigenstates
see pset9, 6.4

In time, the state is: $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi(t=0)\rangle$

$$= e^{-i\hat{H}t/\hbar} \int dp\, |p\rangle\langle p|\Psi(0)\rangle$$ ↳ $\hat{H} \to \hat{p}_x$ bc $dp$ in $\int$

$$= \int dp\, e^{-i\hat{p}_x^2 t/2m\hbar} |p\rangle\langle p|\Psi(0)\rangle$$ ↳ going from $\hat{p}_x \to p$ in exp bc eigenvalues

$$= \int dp\, e^{-ip^2t/2m\hbar} |p\rangle\langle p|\Psi(0)\rangle$$

like $e^{\cdot\cdot\hat{H}} = e^{\cdot\cdot E}$

for a Gaussian wave packet, we have $\langle p|\Psi(t=0)\rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\, e^{-p^2a^2/2\hbar^2}$ (what we found on prev. page)
so we have,

$$\Psi(x,t) = \langle x|\Psi(t)\rangle$$
$$= \int dp\, e^{-ip^2t/2m\hbar} \langle x|p\rangle\langle p|\Psi(0)\rangle$$

↓ ... $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ & $\langle p|\Psi(0)\rangle =$ ↑

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}[a + (i\hbar t/ma)]}}\, e^{-x^2/[2a^2(1+(i\hbar t/ma^2))]}}$$

probability density as a fxn of time is $\Psi^*(x,t)\Psi(x,t)$
* at $t=0$, we have the one we found on prev. page, $|\Psi(x,t=0)|^2 = \frac{1}{\sqrt{\pi}a} e^{-x^2/a^2}$

position uncertainty: ① $\langle x \rangle = \int dx\, |\langle x|\Psi(x,t)\rangle|^2 x$
(as a fxn of time)
② $\langle x^2 \rangle = \int dx\, |\langle x|\Psi(x,t)\rangle|^2 x^2$
③ $\boxed{\Delta x = \frac{a}{\sqrt{2}}\left(1 + \frac{\hbar^2 t^2}{m^2 a^4}\right)^{1/2}}$

# SCHRÖDINGER EQUATION IN POSITION SPACE

- **So far**: time evolution in 1D, free particle, energy eigenstates = momentum eigenstates (ie, no $V(\hat{x})$)
- **Now**: $\boxed{\hat{H} = \frac{\hat{P}_x^2}{2m} + V(\hat{x})}$
- **Equation of motion in position space**: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ (from ch4)

↓ apply $\langle x|$ to project in position space

$$i\hbar \langle x|\frac{d}{dt}|\psi(t)\rangle = \langle x|\hat{H}|\psi(t)\rangle$$
$$= \langle x|\left(\frac{\hat{P}_x^2}{2m} + V(\hat{x})\right)|\psi(t)\rangle$$

$\langle x|\hat{P}_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$ and $\langle x|V(\hat{x}) = \langle x|V(x)$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|\psi\rangle + V(x)\langle x|\psi\rangle$$
$$= \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\langle x|\psi(t)\rangle$$

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)}$$ **time-dependent Schrödinger equation**

- Take $|\psi(t)\rangle$ to be an energy eigenstate, w/ time dependence given by $|E\rangle e^{-iEt/\hbar}$
  → wave fxn: $\boxed{\psi_E(x,t) = \langle x|E\rangle e^{-iEt/\hbar}}$

Sub. into time-dep. schrödinger eqn: $i\hbar\left(\frac{-iE}{\hbar}\right)\langle x|E\rangle e^{\cdots} = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\langle x|E\rangle e^{\cdots}$

$$\boxed{E\underbrace{\langle x|E\rangle}_{\psi_E(x)} = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\underbrace{\langle x|E\rangle}_{\psi_E(x)}}$$ **time-independent Schrödinger equation in position space**

we can also get this by doing: $\hat{H}|E\rangle = E|E\rangle$

↓ project in pos. space

$$\langle x|\hat{H}|E\rangle = E\langle x|E\rangle$$

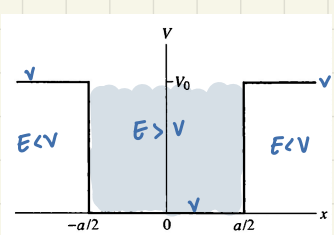
$\psi_E(x) = \langle x|E\rangle$ (no time dep.!) $= \psi(x)$ for rest of chapter

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$

$$\hookrightarrow \frac{d^2}{dx^2}\psi = \frac{-2m}{\hbar^2}(E - V(x))\psi(x)$$

- **finite square well**:

$$V(x) = \begin{cases} 0 & |x| < a/2 \\ V_0 & |x| > a/2 \end{cases}$$




→ we can solve analytically inside & outside the well
→ look at solutions with $0 < E < V_0$ (particle bound inside well)

① diff. eqn: $\frac{d^2\psi}{dx^2} = \begin{cases} \frac{-2mE}{\hbar^2}\psi = -k^2\psi & |x| < a/2 \\ \frac{-2m(E-V_0)}{\hbar^2} = q^2\psi & |x| > a/2 \end{cases}$

② solns: $\psi(x) = \begin{cases} A\sin kx + B\cos kx & |x| < a/2 \\ Ce^{qx} + De^{-qx} & |x| > a/2 \end{cases}$

These need to satisfy $\int dx |\psi(x)|^2 = 1$
→ Need to look at the two $|x| > a/2$ regions separately & discard the exp that blows up

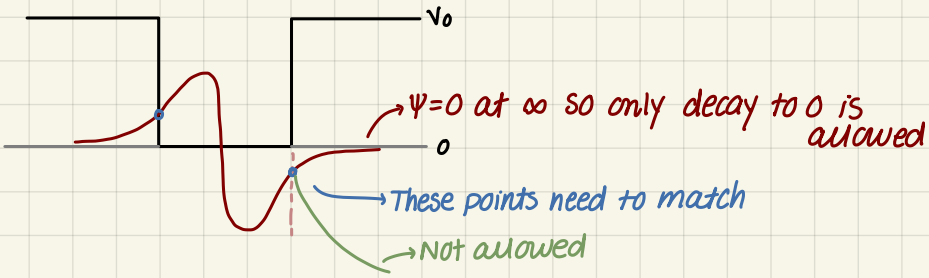
**Note**: $k^2$ & $q^2$ are positive constants

$$\boxed{k = \sqrt{\frac{2mE}{\hbar^2}}}, \quad \boxed{q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}}$$

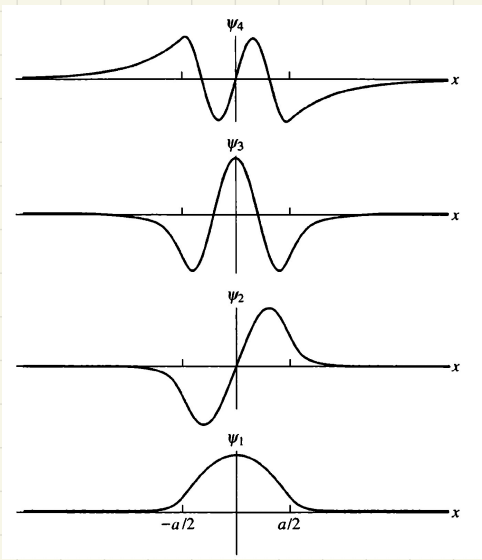
③ outside well: $\psi(x) = \begin{cases} Ce^{qx} & x < -a/2 \\ De^{-qx} & x > a/2 \end{cases}$

$\Rightarrow$ so solution oscillates inside well $(E > V)$ and is exponentially damped outside the well $(E < V)$

④ we need fxns in 3 regions to join smoothly
- $\rightarrow \psi$ continuous (1st derivatives well defined)
- $\rightarrow$ continuous 1st derivatives




So we have discrete levels of energy where this all works:




etc...

ground

- **Note**: see townsend p.218 (pdf p.234) for case where we have $V(x)$ instead of $V_0$

- **Infinite potential well**: (particle in a box) $V(x) = \begin{cases} 0 & |x| < a/2 \\ \infty & |x| > a/2 \end{cases}$

① Inside well: $\psi(x) = A\sin Kx + B\cos Kx \quad |x| < a/2$

② outside well: energy eigenfunction must vanish (lim as $V_0 \rightarrow \infty$)
also needs to be cts w/ inside well

$\rightarrow \psi(\frac{a}{2}) = A\sin(\frac{Ka}{2}) + B\cos(\frac{Ka}{2}) = 0$

$\rightarrow \psi(\frac{-a}{2}) = A\sin(\frac{-Ka}{2}) + B\cos(\frac{-Ka}{2})$

$= -A\sin(\frac{Ka}{2}) + B\cos(\frac{Ka}{2}) = 0$

matrix $\Longrightarrow$ $\begin{pmatrix} \sin(Ka/2) & \cos(Ka/2) \\ -\sin(Ka/2) & \cos(Ka/2) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$

③ Nontrivial solutions if $\begin{vmatrix} \sin(ka/2) & \cos(ka/2) \\ -\sin(ka/2) & \cos(ka/2) \end{vmatrix} = 0$

$\rightarrow 2\sin\left(\frac{ka}{2}\right)\cos\left(\frac{ka}{2}\right) = \sin(ka) = 0$

$\Rightarrow k_n a = n\pi$ for $n = 1, 2, \ldots$

$\cos\left(\frac{ka}{2}\right) = \cos\left(\frac{n\pi}{2}\right) = 0$ for $n$ odd
so $A\sin\left(\frac{n\pi}{2}\right) = 0$ from matrix
& $\sin\left(\frac{n_{odd}\pi}{2}\right) \neq 0$
$\Rightarrow A = 0$

④ for $n$-odd, we find that $A = 0$ so $\psi_n(x) = B_n \cos\left(\frac{n\pi x}{a}\right)$, $n = 1, 3, \ldots$
for $n$-even, we have that $B = 0$ so $\psi_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right)$, $n = 2, 4, \ldots$

⑤ find $A_n$ & $B_n$ by imposing normalization (Townsend p.221, pdf p.237)

$\rightarrow A_n = B_n = \sqrt{\frac{2}{a}}$

$\Rightarrow$ Inside well: $\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) & n = 1, 3, 5, \ldots \\ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & n = 2, 4, \ldots \end{cases}$ $\quad |x| < a/2$

**Note**: if $n = 0$, $\psi = 0$ (no particle)

⑥ we can label the energy levels: $k_n a = \sqrt{\frac{2mE_n}{\hbar^2}}\, a$ for $n = 1, 2, \ldots$

$$\boxed{E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}}$$

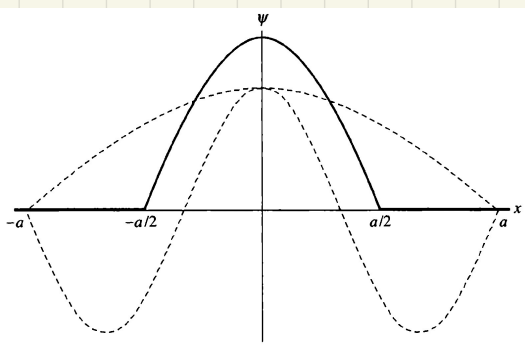



Figure 6.11 The ground-state energy eigenfunction for the small box of width $a$ (solid line) and the ground-state and second-excited-state energy eigenfunctions of the bigger box of width $2a$ (dashed lines).

## SCATTERING (in 1D)

- Look at solutions to the Schrödinger equation where particle is not bound in potential well.
  eg. for potential well, consider $E > V_0 \longrightarrow$ energy eigenvalues are continuous

- **Probability current**: Related to flux reflected & transmitted

① consider $\frac{\partial(\psi^*\psi)}{\partial t}$ = time rate of change of prob. density

② Time dep. S eqn: $\frac{\partial \psi(x,t)}{\partial t} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)\right]$

and $\frac{\partial \psi^*(x,t)}{\partial t} = \frac{-1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t)\right]$

③ $\frac{\partial \psi^*\psi}{\partial t} = \psi\frac{\partial \psi^*}{\partial t} + \psi^*\frac{\partial \psi}{\partial t}$

$= \frac{-\psi}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \cancel{V(x)\psi^*}\right] + \frac{\psi^*}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \cancel{V(x)\psi}\right]$

$= \frac{-\psi}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}\right) + \frac{\psi^*}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\right)$

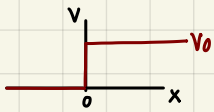
$$\boxed{\frac{\partial \psi^*\psi}{\partial t} = -\frac{\partial j_x}{\partial x}}$$

where $\boxed{j_x = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)}$ probability current

→ The probability in a region of space increases or decreases bc there's a net probability flow in/out of that region

· **Potential step**: eg of scattering from potential energy step

$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & x>0 \end{cases}$$




① find energy eigenstates for which $\psi_E(x,t) = \psi(x)e^{-iEt/\hbar}$ where $\psi(x)$ satisfies the time indep. S. eqn

② left of barrier: $\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$ $x<0$

soln: $\psi(x) = Ae^{ikx} + Be^{-ikx}$ with $k = \sqrt{\frac{2mE}{\hbar^2}}$ $x<0$

so prob. current is (for) $j_x = \frac{\hbar k}{m}(|A|^2 - |B|^2)$ $x<0$

→ incident: $j_{inc} = \frac{\hbar k}{m}|A|^2$

→ Reflected: $j_{ref} = \frac{\hbar k}{m}|B|^2$

⇒ **probability of reflection**: $\boxed{R = \frac{j_{ref}}{j_{inc}}} = \frac{|B|^2}{|A|^2}$

**probability of transmission**: $\boxed{T = \frac{j_{trans}}{j_{inc}}}$ — will get from $j_x$ found using $\psi(x)$ on right of barrier

② Need to solve for wave function for $x>0$ + boundary conditions at $x=0$ to get R & T

Right of barrier: $\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\psi(x)$ $x>0$

case 1: $E > V_0$

$\psi(x) = Ce^{ik_0x} + De^{-ik_0x}$ $x>0$ where $k_0 = \sqrt{\frac{2m}{\hbar^2}(E-V_0)}$

generates a prob. current flowing to the left, ie particles getting projected from the right

choose experiment → particles only projected from left

⇒ $D=0$

⇒ $\psi(x) = Ce^{ik_0x}$ $x>0$

so $j_{trans} = \frac{\hbar k_0}{m}|C|^2$ and we have $T = \frac{k_0}{k}\frac{|C|^2}{|A|^2}$

finding R & T in terms of $V_0$ & E (ie the Ks):

going from left → right region, $\psi(x)$ and $\partial_x \psi(x)$ must be continuous

$$\begin{cases} A+B=C \\ ik(A-B)=ik_0 C \end{cases}$$

$$\Rightarrow C=\frac{2k}{k+k_0}A \quad \text{and} \quad B=\frac{k-k_0}{k+k_0}A$$

and we get: $R=\frac{(k-k_0)^2}{(k+k_0)^2}$ $\qquad T=\frac{4kk_0}{(k+k_0)^2}$

**Note**: $\boxed{R+T=1}$

**case 2: $E<V_0$**

we have $\frac{d^2\psi(x)}{dx^2}=\frac{2m}{\hbar^2}(V_0-E)\psi(x)=q^2\psi(x) \quad x>0$

with $(V_0-E)>0$ so $q^2>0$. we have to choose $\psi(x)=Ce^{-qx}$, otherwise $\psi$ blows up as $x\to\infty$
Instead of using boundary conditions again, we can use case 1 and let $ik_0 \to -q$ in R
we have:

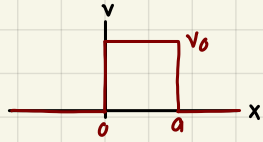
$$R=\frac{|k-iq|^2}{|k+iq|^2}=\frac{k^2+q^2}{k^2+q^2}=1$$

and T must vanish even though $C=\frac{2k}{k+iq}A \neq 0$ (ie, $j_{trans} \neq \cdots |C|^2$)

*can't do this directly in T bc R always given by $\frac{|B|^2}{|A|^2}$ but T depends on $j_{trans}$. when arg in $\psi(x)$ is real, $j_{trans}=0$

· **Tunneling**: consider particles with energy $E<V_0$ incident on a potential energy barrier of height $V_0$ but we have

$$V(x)=\begin{cases} 0 & x<0 \\ V_0 & 0<x<a \\ 0 & x>a \end{cases}$$




(square barrier)

① energy eigenfunction:

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & x<0 \\ Fe^{qx}+Ge^{-qx} & 0<x<a \\ Ce^{ikx} & x>a \end{cases}$$

with $k=\sqrt{\frac{2mE}{\hbar^2}}$

$q=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

② Boundary conditions:

$$\begin{cases} A+B=F+G \\ ik(A-B)=q(F-G) \\ Fe^{qa}+Ge^{-qa}=Ce^{ika} \\ q(Fe^{qa}-Ge^{-qa})=ikCe^{ika} \end{cases}$$

③ $T=\frac{j_{x>a}}{j_{inc}}=\frac{(\hbar k/m)|C|^2}{(\hbar k/m)|A|^2}=\frac{1}{1+\left(\frac{k^2+q^2}{2kq}\right)^2\sinh^2 qa}$ → probability the particle will tunnel through the barrier

· see townsend p.231 (pdf p.246) for eg