PHYS 357 Pset 4. Due 11:59 PM Thursday Oct. 3

1. Townsend 3.1

2. Townsend 3.2. If you choose, you may just verify that the states shown are eigenvectors rather than solve the full eigenvector problem by hand.

3. Townsend 3.7

4. For a 3-state spin-1 system, we know the raising/lowering operators need to look like

$$J_+ \propto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   and J_- is the conjugate-transpose. We know that for J_z, $J_\pm = J_x \pm iJ_y$. Use these forms to solve for J_x and J_y in terms of J_+ and J_-. For a spin-1 system, the eigenvalues of $J_{x,y,z}$ must be $\hbar(1, 0, -1)$. Use this fact to find the coefficient of proportionality for J_x, J_y and write the properly weighted forms of J_x and J_y. If all has gone well, they should agree with Equation 3.28 in Townsend.

5. Show that the commutation relations we expect for angular momentum hold for the spin-1 basis you've just worked out. You may do this on a computer if you choose.

6. What are the eigenstates of J_x and J_y in the J_z basis? What are the raising and lowering operators? Show that the raising and lowering operators for J_x behave as expected on the eigenstates of J_x. Do the same for J_y. Once again, you may do this on a computer.

pset 4

Q1

3.1. Verify for the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ that

(a) $[\hat{A}, \hat{B}+\hat{C}]=[\hat{A}, \hat{B}]+[\hat{A}, \hat{C}]$

(b) $[\hat{A}, \hat{B}\hat{C}]=\hat{B}[\hat{A}, \hat{C}]+[\hat{A}, \hat{B}]\hat{C}$

Similarly, you can show that

(c) $[\hat{A}\hat{B}, \hat{C}]=\hat{A}[\hat{B}, \hat{C}]+[\hat{A}, \hat{C}]\hat{B}$

a) $[A, B+C] = A(B+C) - (B+C)A$
$= AB + AC - BA - CA$
$= (AB - BA) + (AC - CA)$
$= [A, B] + [A, C]$

b) $[A, BC] = A(BC) - (BC)A$
$= ABC - BCA \underbrace{+ BAC - BAC}_{\text{adding } 0}$
$= B(AC - CA) + (AB - BA)C$
$= B[A, C] + [A, B]C$

c) $[AB, C] = ABC - CAB + ACB - ACB$
$= A(BC - CB) + (AC - CA)B$
$= A[B, C] + [A, C]B$

Q2 **3.2.** Using the $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ states of a spin-$\frac{1}{2}$ particle as a basis, set up and solve as a problem in matrix mechanics the eigenvalue problem for $\hat{S}_n = \hat{\mathbf{S}} \cdot \mathbf{n}$, where the spin operator $\hat{\mathbf{S}} = \hat{S}_x\mathbf{i} + \hat{S}_y\mathbf{j} + \hat{S}_z\mathbf{k}$ and $\mathbf{n} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$. Show that the eigenstates may be written as

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$|-\mathbf{n}\rangle = \sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle$$

Rather than simply verifying that these are eigenstates by substituting into the eigenvalue equation, obtain these states by directly solving the eigenvalue problem, as in Section 3.6.

① $\hat{S}_n = \hat{S}\cdot\hat{n} = \hat{S}_x \sin\theta\cos\varphi + \hat{S}_y \sin\theta\sin\varphi + \hat{S}_z\cos\theta$

② $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

③ $S_n = \frac{\hbar}{2}\left(\sin\theta\cos\varphi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$

$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi\\ \underbrace{\sin\theta\cos\varphi + i\sin\theta\sin\varphi}_{\sin\theta e^{i\varphi}} & -\cos\theta\end{pmatrix}$ with $\sin\theta\cos\varphi - i\sin\theta\sin\varphi \to \sin\theta e^{-i\varphi}$

$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$

④ $S_n|+n\rangle = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix}$

$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta\cos\theta/2 + \sin\theta\sin\theta/2\\ \cos\theta/2\sin\theta e^{i\varphi} - \cos\theta\sin\theta/2\, e^{i\varphi}\end{pmatrix}$

① $\cos\theta\cos\theta/2 = \frac{1}{2}[\cos(\theta-\theta/2) + \cancel{\cos(\theta+\theta/2)}]$
$\sin\theta\sin\theta/2 = \frac{1}{2}[\cos(\theta-\theta/2) - \cancel{\cos(\theta+\theta/2)}]$
$+ \; = \frac{1}{2}\cdot 2\cos(\theta-\theta/2) = \cos\theta/2$

② $\cos\theta/2\sin\theta = \frac{1}{2}[\cancel{\sin(\frac{\theta}{2}+\theta)} - \sin(\frac{\theta}{2}-\theta)]$
$\cos\theta\sin\theta/2 = \frac{1}{2}[\cancel{\sin(\theta+\frac{\theta}{2})} - \sin(\theta-\frac{\theta}{2})]$
$- \; = \frac{1}{2}(\sin\theta/2 + \sin\theta/2) = \sin\theta/2$

$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix}$

$S_n|+n\rangle = \frac{\hbar}{2}|+n\rangle$

⑤ $S_n|-n\rangle = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta/2\\ -e^{i\varphi}\cos\theta/2\end{pmatrix}$

$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta\sin\theta/2 - \sin\theta\cos\theta/2\\ \sin\theta\sin\theta/2\, e^{i\varphi} + \cos\theta\cos\theta/2\, e^{i\varphi}\end{pmatrix}$

$= \frac{\hbar}{2}\begin{pmatrix}-\sin\theta/2\\ e^{i\varphi}\cos\theta/2\end{pmatrix}$

$S_n|-n\rangle = -\frac{\hbar}{2}|-n\rangle$

Q3

3.7. Derive the Schwarz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Suggestion: Use the fact that

$$(\langle\alpha|+\lambda^*\langle\beta|)(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

and determine the value of λ that minimizes the left-hand side of the equation.

$$(\langle\alpha|+\lambda^*\langle\beta|)(|\alpha\rangle+\lambda|\beta\rangle) \geq 0$$

$$\langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda^*\lambda\langle\beta|\beta\rangle \geq 0$$

we need to end up with $|\langle\alpha|\beta\rangle|^2$ and $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$

so choose $\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$, $\lambda^* = \frac{-\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}$

$$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\langle\beta|\alpha\rangle + \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle\cancel{\langle\beta|\beta\rangle}}\cancel{\langle\beta|\beta\rangle} \geq 0$$

$= |\langle\alpha|\beta\rangle|^2$

$$\langle\alpha|\alpha\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} \geq 0$$

$$\langle\alpha|\alpha\rangle \geq \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle}$$

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

4. For a 3-state spin-1 system, we know the raising/lowering operators need to look like

$$J_+ \propto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and J_- is the conjugate-transpose. We know that for J_z, $J_\pm = J_x \pm iJ_y$. Use these forms to solve for J_x and J_y in terms of J_+ and J_-. ✓ For a spin-1 system, the eigenvalues of $J_{x,y,z}$ must be $\hbar(1, 0, -1)$. Use this fact to find the coefficient of proportionality for J_x, J_y and write the properly weighted forms of J_x and J_y. If all has gone well, they should agree with Equation 3.28 in Townsend.

① we know $J_\pm = J_x \pm iJ_y$

$\rightarrow J_+ + J_- = J_x + iJ_y + J_x - iJ_y$

$J_+ + J_- = 2J_x$

$$J_x = \frac{J_+ + J_-}{2}$$

$\rightarrow J_+ - J_- = J_x + iJ_y - J_x + iJ_y$

$J_+ - J_- = 2iJ_y$

$$J_y = \frac{J_+ - J_-}{2i}$$

② spin-1 basis: $\overset{0}{|1,1\rangle}, \overset{1}{|1,0\rangle}, \overset{2}{|1,-1\rangle}$

we have $J_+|l,m\rangle = \sqrt{j(j+1) - m(m+1)}\,\hbar\,|j,m+1\rangle$

so $J_+|1,1\rangle = 0$

$J_+|1,0\rangle = \sqrt{2}\hbar|1,1\rangle$

$J_+|1,-1\rangle = \sqrt{2}\hbar|1,0\rangle$

we can form J_+ by calculating $\langle j, \overset{\text{row } i}{m'}|J_+|j, \overset{\text{col } k}{m}\rangle = \sqrt{j(j+1) - m(m+1)}\,\hbar\,\delta_{m',m+1}$

Nonzero for $m' = m+1$

The only valid $m' = m+1$ are $\begin{cases} m' = 0, m = -1 \\ m' = 1, m = 0 \end{cases}$

so entry $(1,2)$: $\langle 1,0|J_+|1,-1\rangle = \sqrt{2}\hbar$

" $(0,1)$: $\langle 1,1|J_+|1,0\rangle = \sqrt{2}\hbar$

$$\Rightarrow J_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

③ for J_-, we have $J_- = J_+^\dagger = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

④ we get $J_x = \frac{1}{2}(J_+ + J_-)$ & $J_y = \frac{1}{2i}(J_x - J_y)$

$= \frac{1}{2} \cdot \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

5. Show that the commutation relations we expect for angular momentum hold for the spin-1 basis you've just worked out. You may do this on a computer if you choose.

```python
# Function
def commutator(A,B):
    return A@B - B@A

# Define the matrices for spin-1 basis
Jx = 1/np.sqrt(2)*np.array([[0,1,0],[1,0,1],[0,1,0]])
Jy = 1/np.sqrt(2)*np.array([[0,-1j,0],[1j,0,-1j],[0,1j,0]])
Jz = np.array([[1,0,0],[0,0,0],[0,0,-1]])
J_squared = Jx@Jx + Jy@Jy + Jz@Jz

# Calculate the commutators
print(f"[Jx,Jy]\n{commutator(Jx,Jy)}")
print(f"[Jy,Jz]\n{commutator(Jy,Jz)}")
print(f"[Jx,Jz]\n{commutator(Jx,Jz)}")

print(f"[Jx,J^2]\n{commutator(Jx,J_squared)}")
print(f"[Jy,J^2]\n{commutator(Jy,J_squared)}")
print(f"[Jz,J^2]\n{commutator(Jz,J_squared)}")
```

Not using the $\hbar$ factors

```
(phys357) audrey@Audreys-MacBook-Pro pset4 % python question5.py
[Jx,Jy]
[[0.+1.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.-1.j]]
```

$= iJ_z$

```
[Jy,Jz]
[[0.+0.j         0.+0.70710678j 0.+0.j        ]
 [0.+0.70710678j 0.+0.j         0.+0.70710678j]
 [0.+0.j         0.+0.70710678j 0.+0.j        ]]
```

$= iJ_x$

```
[Jx,Jz]
[[ 0.         -0.70710678  0.        ]
 [ 0.70710678  0.         -0.70710678]
 [ 0.          0.70710678  0.        ]]
```

$= iJ_y$

```
[Jx,J^2]
[[0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]]
[Jy,J^2]
[[0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]]
[Jz,J^2]
[[0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]
 [0.+0.j 0.+0.j 0.+0.j]]
```

J^2 commutes w/ all of the J_n so all 0 ✓

6. What are the eigenstates of J_x and J_y in the J_z basis? What are the raising and lowering operators? Show that the raising and lowering operators for J_x behave as expected on the eigenstates of J_x. Do the same for J_y. Once again, you may do this on a computer.

```python
# Question 6
# Get the eigenstates and eigenvalues
# *eigenstates are the columns of the eigenvector matrix
eigvals_Jx, eigvec_Jx = np.linalg.eigh(Jx)
eigvals_Jy, eigvec_Jy = np.linalg.eigh(Jy)
eigvals_Jx, eigvec_Jx = np.flipud(eigvals_Jx), np.fliplr(eigvec_Jx)
eigvals_Jy, eigvec_Jy = np.flipud(eigvals_Jy), np.fliplr(eigvec_Jy)
print(f"Eigenvalues of Jx: {eigvals_Jx}")
print(f"Eigenstates of Jx:\n{eigvec_Jx}")
print(f"Eigenvalues of Jy: {eigvals_Jy}")
print(f"Eigenstates of Jy:\n{eigvec_Jy}")

# Raising & lowering operators in the Jz basis
J_plus = Jx + 1j*Jy
J_minus = Jx - 1j*Jy
print(f"J_plus:\n{J_plus}")
print(f"J_minus:\n{J_minus}")

# Apply the raising operator to the eigenstates
Jx_eigvec_raised = J_plus@eigvec_Jx
Jy_eigvec_raised = J_plus@eigvec_Jy
# Apply the lowering operator to the eigenstates
Jx_eigvec_lowered = J_minus@eigvec_Jx
Jy_eigvec_lowered = J_minus@eigvec_Jy

print(f"Raised Jx eigenstates:\n{Jx_eigvec_raised}")
print(f"Raised Jy eigenstates:\n{Jy_eigvec_raised}")
print(f"Lowered Jx eigenstates:\n{Jx_eigvec_lowered}")
print(f"Lowered Jy eigenstates:\n{Jy_eigvec_lowered}")
```

*Neglecting the $\hbar$ factors

```
Eigenvalues of Jx: [ 1.00000000e+00 -1.36388639e-16 -1.00000000e+00]
Eigenstates of Jx:
[[ 5.00000000e-01 -7.07106781e-01  5.00000000e-01]
 [ 7.07106781e-01  1.02175612e-16 -7.07106781e-01]
 [ 5.00000000e-01  7.07106781e-01  5.00000000e-01]]
Eigenvalues of Jy: [ 1.00000000e+00  1.36388639e-16 -1.00000000e+00]
Eigenstates of Jy:
[[ 0.5        +0.00000000e+00j  0.70710678-0.00000000e+00j
   0.5        +0.00000000e+00j]
 [ 0.         +7.07106781e-01j  0.         +8.32667268e-17j
   0.         -7.07106781e-01j]
 [-0.5        +0.00000000e+00j  0.70710678-0.00000000e+00j
  -0.5        +0.00000000e+00j]]
J_plus:
[[0.        +0.j 1.41421356+0.j 0.        +0.j]
 [0.        +0.j 0.        +0.j 1.41421356+0.j]
 [0.        +0.j 0.        +0.j 0.        +0.j]]
J_minus:
[[0.        +0.j 0.        +0.j 0.        +0.j]
 [1.41421356+0.j 0.        +0.j 0.        +0.j]
 [0.        +0.j 1.41421356+0.j 0.        +0.j]]
Raised Jx eigenstates:
[[ 1.00000000e+00+0.j  1.44498136e-16+0.j -1.00000000e+00+0.j]
 [ 7.07106781e-01+0.j  1.00000000e+00+0.j  7.07106781e-01+0.j]
 [ 0.00000000e+00+0.j  0.00000000e+00+0.j  0.00000000e+00+0.j]]
```

columns are an eigenstate of J_x (expressed in the z basis)

← $|1,1\rangle_z$ coeffs.
← $|1,0\rangle_z$ coeffs
← $|1,-1\rangle_z$ coeffs

```
Raised Jy eigenstates:
[[ 0.         +1.00000000e+00j  0.         +1.17756934e-16j
   0.         -1.00000000e+00j]
 [-0.70710678+0.00000000e+00j  1.         +0.00000000e+00j
  -0.70710678+0.00000000e+00j]
 [ 0.         +0.00000000e+00j  0.         +0.00000000e+00j
   0.         +0.00000000e+00j]]
Lowered Jx eigenstates:
[[ 0.00000000e+00+0.j  0.00000000e+00+0.j  0.00000000e+00+0.j]
 [ 7.07106781e-01+0.j -1.00000000e+00+0.j  7.07106781e-01+0.j]
 [ 1.00000000e+00+0.j  1.44498136e-16+0.j -1.00000000e+00+0.j]]
Lowered Jy eigenstates:
[[0.        +0.00000000e+00j 0.        +0.00000000e+00j
  0.        +0.00000000e+00j]
 [0.70710678+0.00000000e+00j 1.        +0.00000000e+00j
  0.70710678+0.00000000e+00j]
 [0.        +1.00000000e+00j 0.        +1.17756934e-16j
  0.        -1.00000000e+00j]]
(phys357) audrey@Audreys-MacBook-Pro pset4 %
```

$|\psi\rangle_x = c_1|1,1\rangle_z + c_2|1,0\rangle_z + c_3|1,-1\rangle_z \longrightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ matrix column

$J_+|\psi\rangle_x = \underbrace{c_1 J_+|1,1\rangle_z}_{0,\text{ can't be raised}} + \underbrace{c_2 J_+|1,0\rangle_z}_{\text{Raised to } |1,1\rangle_z} + \underbrace{c_3 J_+|1,-1\rangle_z}_{\text{Raised to } |1,0\rangle_z} \longrightarrow \begin{pmatrix}c_2\\c_3\\0\end{pmatrix} \times \text{cst factor}$

Nothing gets raised to $|1,-1\rangle_z$ so we end up w/ a row of 0s in the eigenstate matrix

*Similar for J_y, and also for J_- (but 0 row is the first row of the matrix)