PHYS 357 Midterm. You have 3 hours from when you start, due by 11:59 PM on Wednesday October 23

You may use 3 pages of notes. The maximum score is 100 points, even though there are 101 (plus up to 5 bonus) points available.

1. For matrices A and B, would you expect that $\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A}+\mathrm{B})$? Please answer for the following three cases.

   A: If A and B are general matrices.

   B: If A and B are Hermitian.

   C: if A and B commute.

   For each case, please justify your answer mathematically. (7 points each)

2. Some operator A has an eigenstate $|\Psi\rangle$ with eigenvalue λ, so $\mathrm{A}\,|\Psi\rangle = \lambda\,|\Psi\rangle$. Let's assume there is another operator B that anti-commutes with A: $\{\mathrm{A},\mathrm{B}\} \equiv \mathrm{AB}+\mathrm{BA} = 0$. Show that if $\mathrm{B}\,|\Psi\rangle$ is non-zero, it is also an eigenstate of A (10 points), and determine its eigenvalue (10 points).

3. Your professor promised to show you at some point in time why a rotation matrix has the form $\exp(-iJ\theta/\hbar)$ and is not allowed to have an overall phase. That time has now come. We will do this for a spin-1/2 particle, where we saw the rotation matrix about an arbitrary axis must have the form

$$R_n(\theta) = \exp(i\delta)\begin{bmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{bmatrix} \tag{1}$$

   as long as we are working in the basis defined by that axis. *i.e.* if we are rotating about the $n-$axis, then the two states in R_n are $|{+n}\rangle$ and $|{-n}\rangle$.

   A: Show that δ must be proportional to θ (reminder - a rotation by $n\theta$ is the same as n rotations by θ about the same axis). (5 points)

1. For matrices A and B, would you expect that $\exp(A)\exp(B) = \exp(A+B)$? Please answer for the following three cases.

A: If A and B are general matrices.

B: If A and B are Hermitian.

C: if A and B commute.

For each case, please justify your answer mathematically. (7 points each)

$e^A = \sum \frac{A^N}{N!}$ so $e^A e^B = \sum_N \frac{A^N}{N!} \sum_N \frac{B^N}{N!}$ and $e^{A+B} = \sum_n \frac{(A+B)^n}{n!} = \underbrace{\sum_n \frac{1}{n!} \sum_m^n \binom{n}{m} A^n B^{m-n}}_{\text{binomial expansion}}$

$= \underbrace{\sum_{N,M} \frac{A^N B^M}{N!\,M!}}_{\text{The only way to rearrange this to match } e^{A+B} \text{ is if } A \text{ and } B \text{ commute ie } AB = BA}$

if they commute, we have

$= \sum \frac{A^N B^{M-N}}{N!(M-N)!}$

$= \sum \binom{N}{M} A^N B^{M-N}$

so $e^A e^B = e^{A+B}$ only holds if A & B commute (not true for Hermitian or general matrices)

2. Some operator A has an eigenstate $|\Psi\rangle$ with eigenvalue λ, so $A|\Psi\rangle = \lambda|\Psi\rangle$. Let's assume there is another operator B that anti-commutes with A: $\{A, B\} \equiv AB + BA = 0$. Show that if $B|\Psi\rangle$ is non-zero, it is also an eigenstate of A (10 points), and determine its eigenvalue (10 points).

$A(B|\psi\rangle) = \underbrace{AB}_{AB=-BA}|\psi\rangle$ with $B|\psi\rangle \neq 0$

$= -BA|\psi\rangle$

$= -B(\lambda|\psi\rangle)$

$= -\lambda\underbrace{(B|\psi\rangle)}$

so $(B|\psi\rangle)$ is an eigenstate of A
with eigenvalue $-\lambda$

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as long as we are working in the basis defined by that axis. *i.e.* if we are rotating about the $n-$axis, then the two states in R_n are $|+n\rangle$ and $|-n\rangle$.

~~A~~: Show that δ must be proportional to θ (reminder - a rotation by $n\theta$ is the same as n rotations by θ about the same axis). (5 points)

a) we need $R_n(a\theta) = R_n(\theta)^a$

$$e^{i\delta'}\begin{pmatrix} 1 & 0 \\ 0 & e^{ia\theta} \end{pmatrix} = \left[e^{i\delta}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\right]^a$$

$$e^{i\delta'}\begin{pmatrix} 1 & 0 \\ 0 & e^{ia\theta} \end{pmatrix} = e^{ia\delta}\begin{pmatrix} 1 & 0 \\ 0 & e^{ia\theta} \end{pmatrix}$$

for this to hold, we require $e^{i\delta'} = e^{ia\delta} \Rightarrow \delta' = a\delta$

δ = overall phase of $R_n(\theta)$

δ' = overall phase of $R_n(a\theta)$

ie, δ must be proportional to θ

~~B~~: Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the $|+z\rangle, |-z\rangle$-basis swaps $+z$ and $-z$. (5 points)

$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|+z\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-z\rangle$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|-z\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+z\rangle$

C: The key step in getting the correct overall phase is noting that a rotation by θ about the $|+z\rangle$-axis is the same as a rotation by $-\theta$ about the $-z$-axis[1]. Show that fact, along with the results from parts a) and b), requires $\delta = -\theta/2$. (10 points)

if we have $|\psi\rangle = C_+|+z\rangle + C_-|-z\rangle$, then we can write:

→ Rotating by θ about $|+z\rangle$: $R_{+z}(\theta) = e^{i\delta}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

→ Rotating by θ about $|-z\rangle$: $R_{-z}(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\delta}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

$$= e^{i\delta}\begin{pmatrix} 0 & e^{i\theta} \\ 1 & 0 \end{pmatrix}$$

so we have $|\psi\rangle_{rot+} = e^{i\delta}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$

$$= e^{i\delta}\begin{pmatrix} C_+ \\ e^{i\theta}C_- \end{pmatrix}$$

$R_{-z}(-\theta)$

and

$$|\psi\rangle_{rot-} = e^{i\delta}\begin{pmatrix} 0 & e^{-i\theta} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

$$= e^{i\delta}\begin{pmatrix} e^{-i\theta}C_- \\ C_+ \end{pmatrix}$$

B: Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the $|+z\rangle, |-z\rangle$-basis swaps $+z$ and $-z$. (5 points)

C: The key step in getting the correct overall phase is noting that a rotation by θ about the $|+z\rangle$-axis is the same as a rotation by $-\theta$ about the $-z$-axis[1]. Show that fact, along with the results from parts a) and b), requires $\delta = -\theta/2$. (10 points)

4. Consider a two-state system with operator A that has eigenvalues λ_1 and λ_2. Consider a state (c_1, c_2) with amplitudes c_1 and c_2 to be in the eigenstates with eigenvalues λ_1 and λ_2.
   A: What is the expectation value of the operator A? Please express in terms of $p_1 = c_1^* c_1$, $p_2 = c_2^* c_2$, λ_1, and λ_2? (5 points)

   B: What is the uncertainty in a measurement of A, again in terms of the same variables. (5 points)

   C: Show the three conditions under which the uncertainty is zero, and show that it is otherwise greater than zero. Hint: you might want to convince yourself that $p_1 - p_1^2 = p_2 - p_2^2$. (10 points)

5. **A:** Working in the $x-$basis, sketch out where the non-zero elements of the J_x raising and lowering operators are for some modest value of j (you may assume that, as usual, the m's are in decreasing order). I don't need to see the actual numbers, just where the non-zero entries are. If you feel the need to be concrete, you may write out the $j = 2$ case. (5 points)

   B: Continuing to work in the $x-$basis, show where the non-zero entries of J_y and J_z are. (5 points)

   C: Given the form of the operators from part B, explain why $< J_y >=< J_z >= 0$ for *any* pure state of J_x. (10 points)

[1] You can see this by bringing your hands together with your thumbs pointing in the opposite direction. The fingers on both your right-hand and left-hand curl in the same direction, but your left hand is the opposite sign from your right hand.

4. Consider a two-state system with operator A that has eigenvalues λ_1 and λ_2. Consider a state (c_1, c_2) with amplitudes c_1 and c_2 to be in the eigenstates with eigenvalues λ_1 and λ_2.

~~A~~: What is the expectation value of the operator A? Please express in terms of $p_1 = c_1^* c_1$, $p_2 = c_2^* c_2$, λ_1, and λ_2? (5 points)

~~B~~: What is the uncertainty in a measurement of A, again in terms of the same variables. (5 points)

~~C~~: Show the three conditions under which the uncertainty is zero, and show that it is otherwise greater than zero. Hint: you might want to convince yourself that $p_1 - p_1^2 = p_2 - p_2^2$. (10 points)

$|\psi\rangle = C_1|\varphi_1\rangle + C_2|\varphi_2\rangle = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, $A|\varphi_1\rangle = \lambda_1|\varphi_1\rangle$, $A|\varphi_2\rangle = \lambda_2|\varphi_2\rangle$

a) $\langle A\rangle = \langle\psi|A|\psi\rangle = (C_1^* \; C_2^*)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

from $\langle\psi| = |\psi\rangle^\dagger$

$= (C_1^* \; C_2^*)\begin{pmatrix} \lambda_1 C_1 \\ \lambda_2 C_2 \end{pmatrix}$

$= C_1^* C_1 \lambda_1 + C_2^* C_2 \lambda_2$

$= P_1\lambda_1 + P_2\lambda_2$

b) $\sigma(A) = [\langle A^2\rangle - \langle A\rangle^2]^{1/2}$

$\langle A\rangle^2 = (p_1\lambda_1 + p_2\lambda_2)^2$

same as above but with λ^2

$\langle A^2\rangle = P_1\lambda_1^2 + P_2\lambda_2^2$

$= [P_1\lambda_1^2 + P_2\lambda_2^2 - (p_1\lambda_1 + P_2\lambda_2)^2]^{1/2}$

$= [p_1\lambda_1^2 + P_2\lambda_2^2 - p_1^2\lambda_1^2 - p_2^2\lambda_2^2 - 2p_1p_2\lambda_1\lambda_2)^{1/2}$

$= (\lambda_1^2 p_1(1-p_1) + \lambda_2^2 p_2(1-p_2) - 2p_1p_2\lambda_1\lambda_2)^{1/2}$

$P_1 + P_2 = 1$

$= [\lambda_1^2 p_1 p_2 + \lambda_2^2 p_2 p_1 - 2p_1p_2\lambda_1\lambda_2)^{1/2}$

$= \sqrt{p_1p_2}\sqrt{(\lambda_1 - \lambda_2)^2}$

c) we have $\sigma(A) = 0$ if ① $\lambda_1 = \lambda_2 = \lambda$

② $P_1 = 0 \quad (P_2 \neq 0)$

③ $P_2 = 0 \quad (P_1 \neq 0)$

since $A=0$ is trivial

otherwise, p_1 & p_2 are both >0 since $p = c^*c = |c|^2$, and $\sqrt{(\lambda_1-\lambda_2)^2} > 0$ since $\lambda \in \mathbb{R}$

(measuring A so it represents an observable, ie. eigenvalues are real)

5. ~~A~~: Working in the $x-$basis, sketch out where the non-zero elements of the J_x raising and lowering operators are for some modest value of j (you may assume that, as usual, the m's are in decreasing order). I don't need to see the actual numbers, just where the non-zero entries are. If you feel the need to be concrete, you may write out the $j = 2$ case. (5 points)

~~B~~: Continuing to work in the $x-$basis, show where the non-zero entries of J_y and J_z are. (5 points)

~~C~~: Given the form of the operators from part B, explain why $< J_y >=< J_z >= 0$ for *any* pure state of J_x. (10 points)

a) $J_{\pm,x}$ in the J_x basis so we have $J_+ = \begin{pmatrix} \ddots & \ddots & 0 \\ & 0 & \ddots \\ 0 & & \ddots \end{pmatrix}$ → nonzero (right above diagonal)

diagonal $=0$

$J_- = \begin{pmatrix} \ddots & & 0 \\ \ddots & 0 & \\ 0 & \ddots & \ddots \end{pmatrix}$ diagonal $=0$

Nonzero (right below diag.)

b) we have $J_{\pm,x} = J_y \pm iJ_z$ so $J_+ + J_- = 2J_y \rightarrow J_y = \frac{1}{2}(J_+ + J_-)$

and $J_+ - J_- = -2iJ_z \rightarrow J_z = \frac{1}{2i}(J_+ - J_-)$

so both J_y and J_z will have the form $\begin{pmatrix} \ddots & \ddots & 0 \\ \ddots & 0 & \ddots \\ 0 & \ddots & \ddots \end{pmatrix}$ Nonzero

c) pure state of J_x will look like $|\psi\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ie. a single 1 in a single entry (in the x basis)

This means we have $\langle J_{y,z} \rangle = \langle \psi |^{\dagger} J_{y,z} | \psi \rangle$

$= \underbrace{(0 \cdots 1 \cdots 0)}_{n^{th}\text{ entry}} \underbrace{\begin{pmatrix} \ddots & \ddots & 0 \\ \ddots & 0 & \ddots \\ 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}$ → let's say n^{th} entry $= 1$

new vector with
- n^{th} entry $=0$ since diag of J is 0
- any nonzero entry will be in a row $\neq n$ ie when the 1 matches up with an off diag. nonzero entry of J

so this will give 0 since the 1 in $(0 \cdots 1 \cdots 0)$ is in n^{th} column, ie where we for sure have a 0 in

$\Rightarrow \langle J_{y,z} \rangle = 0$

eg. $(0\,1\,0\,0\,0) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (0\,1\,0\,0\,0) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0$

Bonus: Again, looking at the form of the angular momentum operators, what is the minimal condition on a general wave function expressed in the $x-$basis to have a non-zero mean value of J_y or J_z? (up to 5 points missed elsewhere)