PHYS 357 Pset 6. Due 11:59 PM Thursday Oct. 31

1. Townsend 4.3
2. Townsend 4.11
3. Let a spin-1/2 electron in the $|1/2, 1/2\rangle_x$ state be placed in a time-varying magnetic field in the $z-$direction: $B = B(t)\hat{z}$. Solve for the spin "direction" as a function of time. By direction I mean the direction in which we always measure $+\hbar/2$ (equivalently, that are θ, ϕ if we write the state as $|+n\rangle$). You may leave your answer in the form of an integral, and assume that the constants work out so that the Hamiltonian can be written $\omega(t)\mathrm{S}_z$ where $\omega(t)$ is just $B(t)$ times constants.

   Now let the applied field be such that $H(t) = (\omega_0 + \omega_1 \cos(\omega_f t))\mathrm{S}_z$. Work out the solution analytically, and compare to the numerical code I have helpfully worked out for you in magres_z.py in the codes directory on github.

4. Consider a 2x2 matrix

   $$E_0 \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

   Work out the eigenvalues and eigenvectors of this matrix.

5. Take the previous matrix to be the Hamiltonian of a 2-state system. If we start in the (1,0) state, what is the probability as a function of time to be in the (1,0)/(0,1) states? Ammonia is seen to emit radiation at 24 GHz, when it transitions between the higher and lower-energy eigenstates, so $2E_0\epsilon = h\nu$ (note that we're using $h = 2\pi\hbar$ here). Given this, what is the length of time it takes for an ammonia molecule that starts in the in the (1,0) state to transition to the (0,1) state?

   Note - in the interests of time and clarity, I'm going to skip the book's section on ammonia for now, and revisit it after we've done more work with continuous wave functions in chapters 6 and 7. Once we're there, we'll be able to work out why the Hamiltonian looks the way it does instead of it being plucked from thin air as it seems here. Even if we don't yet know how to work out the Hamiltonian, undestanding how to work with a split-energy state is still useful for understanding the energy-time uncertainty relationship.

Q1

4.3. Use (4.16) to verify that the expectation value of an observable A does not change with time if the system is in an energy eigenstate (a stationary state) and $\hat{A}$ does not depend explicitly on time.

we know $\partial_t \hat{A} = 0$ and $|\psi(t)\rangle = |E\rangle \longrightarrow \hat{H}|\psi(t)\rangle = E|E\rangle$

so,

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle + \langle\psi(t)|\cancel{\partial_t \hat{A}}^{\,0}|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle E|\hat{H}\hat{A}-\hat{A}\hat{H}|E\rangle$$

$$= \frac{i}{\hbar}\left(\langle E|\hat{H}\hat{A}|E\rangle - \langle E|\hat{A}\hat{H}|E\rangle\right)$$

$$= \frac{i}{\hbar}\left(\langle E|E\hat{A}|E\rangle - \langle E|\hat{A}E|E\rangle\right)$$

$$= \frac{iE}{\hbar}\left(\langle E|A|E\rangle - \langle E|A|E\rangle\right)$$

$$= 0$$

$\Rightarrow \langle A\rangle$ is independent of time

Q2

4.11. A spin-1 particle with a magnetic moment $\boldsymbol{\mu} = (gq/2mc)\mathbf{S}$ is situated in a magnetic field $\mathbf{B} = B_0\mathbf{k}$ in the z direction. At time $t = 0$ the particle is in a state with $S_y = \hbar$ [see (3.115)]. Determine the state of the particle at time t. Calculate how the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ vary in time.

$$\hat{H} = -\hat{\mu}\cdot\vec{B} = \frac{-gq}{2mc}\hat{S}\cdot\vec{B} = \frac{-gq}{2mc}S_z B_0 = \omega_0 \hat{S}_z$$

at $t=0$, $S_y = \hbar$ so $|\psi(t=0)\rangle = |1,1\rangle_y = \frac{1}{2}\begin{pmatrix}1\\ i\sqrt{2}\\ -1\end{pmatrix}$ (in z basis)

so $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle$

$$= \underbrace{e^{-iHt/\hbar}}_{\hat{H}=\omega_0\hat{S}_z}\cdot\frac{1}{2}\begin{pmatrix}1\\ i\sqrt{2}\\ -1\end{pmatrix}$$

so $e^{-i\omega_0 t\hat{S}_z/\hbar}$

$$= e^{-i\omega_0 t\hat{S}_z/\hbar}\,\frac{1}{2}\begin{pmatrix}1\\ i\sqrt{2}\\ -1\end{pmatrix} \quad \hat{S}_z = \hbar, 0, -\hbar$$

$$|\psi(t)\rangle = \frac{1}{2}\begin{pmatrix}e^{-i\omega_0 t}\\ i\sqrt{2}\\ -e^{i\omega_0 t}\end{pmatrix}$$

We have $\langle S_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle = \frac{1}{2}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}e^{-i\omega_0 t}\\ i\sqrt{2}\\ -e^{i\omega_0 t}\end{pmatrix}$

$$= \frac{1}{4}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\begin{pmatrix}i\sqrt{2}\\ e^{-i\omega_0 t}-e^{i\omega_0 t}\\ i\sqrt{2}\end{pmatrix}$$

$$= \frac{\hbar}{4\cancel{\sqrt{2}}}\, i\cancel{\sqrt{2}}\left(e^{i\omega_0 t} - e^{-i\omega_0 t} + e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$= \frac{i\hbar}{\cancel{4}2}\cdot\cancel{2}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$= \frac{i\hbar}{2}\cdot 2i\sin(\omega_0 t)$$

$\langle S_x\rangle = -\hbar\sin(\omega_0 t)$ → $=0$ at $t=0$ ✓

$\langle S_y\rangle = \langle\psi(t)|\hat{S}_y|\psi(t)\rangle = \frac{1}{2}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}e^{-i\omega_0 t}\\ i\sqrt{2}\\ -e^{i\omega_0 t}\end{pmatrix}$

$$= \frac{\hbar}{4\sqrt{2}}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ ie^{-i\omega_0 t}+ie^{i\omega_0 t}\\ -\sqrt{2}\end{pmatrix}$$

$$= \frac{\hbar}{4\cancel{\sqrt{2}}}\cancel{\sqrt{2}}\left(e^{i\omega_0 t} - i(ie^{-i\omega_0 t} + ie^{i\omega_0 t}) + e^{-i\omega_0 t}\right)$$

$$= \frac{\hbar}{4}\left(e^{i\omega_0 t} + e^{-i\omega_0 t} + e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

$$= \frac{\hbar}{4}\cdot 2\underbrace{\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)}_{2\cos}$$

$\langle S_y\rangle = \hbar\cos(\omega_0 t)$ → $=\hbar$ at $t=0$ ✓

$\langle S_z\rangle = \frac{1}{2}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}e^{-i\omega_0 t}\\ i\sqrt{2}\\ -e^{i\omega_0 t}\end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix}e^{i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t}\end{pmatrix}\begin{pmatrix}e^{-i\omega_0 t}\\ 0\\ e^{i\omega_0 t}\end{pmatrix}$

$$= \frac{\hbar}{4}(1 + 0 - 1)$$

$\langle S_z\rangle = 0$ → constant with time ✓

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Now let the applied field be such that $H(t) = (\omega_0 + \omega_1 \cos(\omega_f t))\mathrm{S}_z$. Work out the solution analytically, and compare to the numerical code I have helpfully worked out for you in magres_z.py in the codes directory on github.

① $B = B(t)\hat{z}$

$|\psi_0\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ (in z basis)

$\hat{H}(t) = \omega(t) S_z$, $\omega(t) = c\,B(t)$

② $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

we know $\hat{H}(t) = \omega(t)S_z$, and we want the direction so $\varphi(t) = \int_0^t \omega(t')dt'$

$$= e^{-i\hat{H}(t)dt/\hbar} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= e^{-\frac{i}{\hbar}\int_0^t \omega(t')S_z dt'} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= e^{-\frac{i}{\hbar}S_z \cdot \varphi(t)} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i\cancel{\hbar}}{\cancel{\hbar}2}\varphi(t)}|+z\rangle + e^{\frac{i\cancel{\hbar}}{\cancel{\hbar}2}\varphi(t)}|-z\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\varphi(t)/2}\left(|+z\rangle + e^{i\varphi(t)}|-z\rangle\right)$$

with $\varphi(t) = \int_0^t \omega(t')dt'$

③ $H(t) = (\omega_0 + \omega_1 \cos(\omega_f t))S_z$

then $\varphi(t) = \int_0^t \omega(t')dt$

$$= \int_0^t \omega_0 + \omega_1\cos(\omega_f t)\,dt$$

$$= \omega_0 t + \frac{\omega_1}{\omega_f}\sin(\omega_f t)$$

so $|\psi(t)\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}e^{i(\omega_0 t + \frac{\omega_1}{\omega_f}\sin(\omega_f t))}|-z\rangle$

*dropping abs phase since it doesn't matter

This solution is consistent with the plot from the code, where we see the oscillations introduced by the $\sin(\omega_f t)$ term.

4. Consider a 2x2 matrix

$$E_0 \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

Work out the eigenvalues and eigenvectors of this matrix.

$A = E_0 \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \longrightarrow Av = \lambda v$

$(A - \lambda I)v = 0$

$v=0 \rightarrow$ trivial solution

we need $(A-\lambda I)$ not invertible

$\Rightarrow \det(A - \lambda I) = 0$

$\left| E_0 \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\left| \begin{pmatrix} E_0 - \lambda & E_0\varepsilon \\ E_0\varepsilon & E_0 - \lambda \end{pmatrix} \right| = 0$

$(E_0 - \lambda)^2 - (E_0\varepsilon)^2 = 0$

$(E_0 - \lambda - E_0\varepsilon)(E_0 - \lambda + E_0\varepsilon) = 0$

① $E_0 - \lambda - E_0\varepsilon = 0 \longrightarrow E_0 \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = E_0(1-\varepsilon)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\lambda_1 = E_0(1-\varepsilon)$

$\begin{cases} E_0 x_1 + E_0\varepsilon x_2 = E_0(1-\varepsilon)x_1 \\ E_0\varepsilon x_1 + E_0 x_2 = E_0(1-\varepsilon)x_2 \end{cases} \rightarrow (\cancel{E_0 - E_0} - E_0\varepsilon)x_1 = E_0\varepsilon x_2$

$x_1 = -x_2$

so $v_1 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (for any scalar multiple)

② $E_0 - \lambda + E_0\varepsilon = 0 \longrightarrow E_0 \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = E_0(1+\varepsilon)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\lambda_2 = E_0(1+\varepsilon)$

$\begin{cases} E_0 x_1 + E_0\varepsilon x_2 = E_0(1+\varepsilon)x_1 \\ E_0\varepsilon x_1 + E_0 x_2 = E_0(1+\varepsilon)x_2 \end{cases} \rightarrow (\cancel{E_0 - E_0} - E_0\varepsilon)x_1 = -E_0\varepsilon x_2$

$x_1 = x_2$

so $v_2 = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (for any scalar multiple)

5. Take the previous matrix to be the Hamiltonian of a 2-state system. If we start in the (1,0) state, what is the probability as a function of time to be in the (1,0)/(0,1) states? ✓ Ammonia is seen to emit radiation at 24 GHz, when it transitions between the higher and lower-energy eigenstates, so $2E_0\epsilon = h\nu$ (note that we're using $h = 2\pi\hbar$ here). Given this, what is the length of time it takes for an ammonia molecule that starts in the in the (1,0) state to transition to the (0,1) state?

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in z basis

① $\hat{H} = E_0\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$ with energy eigenstates $|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $E_1 = E_0(1+\varepsilon)$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $E_2 = E_0(1-\varepsilon)$

$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+z\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-z\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$)

② $$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi_0\rangle = e^{-i\hat{H}t/\hbar}\cdot\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$

$$= \frac{1}{2}e^{-iE_1t/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-iE_2t/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}e^{-iE_0(1+\varepsilon)t/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-iE_0(1-\varepsilon)t/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}e^{-iE_0t/\hbar}\left[e^{-iE_0\varepsilon t/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{iE_0\varepsilon t/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$= \frac{1}{2}e^{-iE_0t/\hbar}\begin{pmatrix} e^{-iE_0\varepsilon t/\hbar} + e^{iE_0\varepsilon t/\hbar} \\ e^{-iE_0\varepsilon t/\hbar} - e^{iE_0\varepsilon t/\hbar} \end{pmatrix}$$

$$= \frac{1}{2}e^{-iE_0t/\hbar}\begin{pmatrix} 2\cos(E_0\varepsilon t/\hbar) \\ -2i\sin(E_0\varepsilon t/\hbar) \end{pmatrix}$$

③ $$\text{Prob}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\langle(1,0)|\psi(t)\rangle|^2$$

$$= \left|(1\ 0)\frac{1}{2}e^{-iE_0t/\hbar}\begin{pmatrix} 2\cos(E_0\varepsilon t/\hbar) \\ -2i\sin(E_0\varepsilon t/\hbar) \end{pmatrix}\right|^2$$

$$= \frac{1}{4}|2\cos(E_0\varepsilon t/\hbar)|^2$$

$$= \cos^2(E_0\varepsilon t/\hbar)$$

$$\text{Prob}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\langle(0,1)|\psi(t)\rangle|^2$$

$$= \left|(0\ 1)\frac{1}{2}e^{-iE_0t/\hbar}\begin{pmatrix} 2\cos(E_0\varepsilon t/\hbar) \\ -2i\sin(E_0\varepsilon t/\hbar) \end{pmatrix}\right|^2$$

$$= \frac{1}{4}|-2i\sin(E_0\varepsilon t/\hbar)|^2$$

$$= \sin^2(E_0\varepsilon t/\hbar)$$

Total prob = 1 ✓

④ time to go from $|1\,0\rangle$ to $|0\,1\rangle$?

$2E_0\varepsilon = h\nu = 2\pi\hbar\nu$ with $\nu = 24\,GHz$

$E_0\varepsilon = \pi\hbar\nu$

start in $\left|\begin{smallmatrix}1\\0\end{smallmatrix}\right\rangle$, after $t=?$ will we get Prob $\left|\begin{smallmatrix}0\\1\end{smallmatrix}\right\rangle = 1$?

$$\rightarrow \sin^2\left(\frac{E_0\varepsilon t}{\hbar}\right) = 1$$

$$\frac{E_0\varepsilon t}{\hbar} = \frac{\pi}{2}$$

$$\frac{\cancel{\pi}\cancel{\hbar}\nu t}{\cancel{\hbar}} = \frac{\cancel{\pi}}{2}$$

$$t = \frac{1}{2\nu}$$

$$t = \frac{1}{48GHz} = 2.083 \times 10^{-11}\ s$$