PHYS 457 - Homework 2

Audréanne Bernier (261100643)

1. For this problem, we'll need to distinguish the covariant components (subscripted) and the contravariant components (superscripted). The physical quantities are *the superscripted components and subscripted unit vectors*. For instance, the position vector is given by

$$\mathbf{r} = \underbrace{x^i}_{\text{component}}\underbrace{\mathbf{e}_i}_{\text{unit vector}} \tag{1}$$

where the repeated index is summed.

Suppose one does a coordinate transformation such that

$$\begin{aligned} x^1 &= f^1(\boldsymbol{\xi}) \\ x^2 &= f^2(\boldsymbol{\xi}) \\ x^3 &= f^3(\boldsymbol{\xi}) \end{aligned} \tag{2}$$

where $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$ are the new coordinates. Define

$$\mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial \xi^j} = \frac{\partial x^i}{\partial \xi^j}\mathbf{e}_i \tag{3}$$

and $h_j = |\mathbf{g}_j|$. The vector $\mathbf{g}_j$ points to the direction of the change in ξ^j, but it is not necessarily normalized.

This means that as $\boldsymbol{\xi}$ changes by $d\boldsymbol{\xi}$, the physical position $\mathbf{r}$ changes by

$$\begin{aligned} d\mathbf{r} &= dx^i \mathbf{e}_i \\ &= \frac{\partial x^i}{\partial \xi^j} d\xi^j \mathbf{e}_i \\ &= \mathbf{g}_i d\xi^i \end{aligned} \tag{4}$$

(a) Show that the volume of the parallelepiped spanned by 3 vectors $(\mathbf{g}_1 d\xi^1, \mathbf{g}_2 d\xi^2, \mathbf{g}_3 d\xi^3)$ is

$$dV = |\mathrm{Det}\,(J)| d\xi^1 d\xi^2 d\xi^3 \tag{5}$$

where

$$J^i_j = \frac{\partial x^i}{\partial \xi^j} \tag{6}$$

is the Jacobian matrix.

Show or argue that under this coordinate transformation, the volume element transforms

$$dx^1 dx^2 dx^3 = |\mathrm{Det}\,(J)| d\xi^1 d\xi^2 d\xi^3 \tag{7}$$

You may use the fact that the volume of the parallelepiped spanned by three non-parallel vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is given by

$$V_{abc} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \tag{8}$$

which is the determinant of the matrix

$$J = \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix} \tag{9}$$

for the parallelepiped spanned by $(g_1 d\xi^1, g_2 d\xi^2, g_3 d\xi^3)$, the volume is given by

$$\begin{aligned} dV &= |g_1 d\xi^1 \cdot (g_2 d\xi^2 \times g_3 d\xi^3)| \\ &= d\xi^1 d\xi^2 d\xi^3 \underbrace{|g_1 \cdot (g_2 \times g_3)|} \end{aligned}$$

$$= \det\begin{pmatrix} g_1^T \\ g_2^T \\ g_3^T \end{pmatrix} = \det\underbrace{\begin{pmatrix} \partial x^1/\partial\xi^1 & \partial x^2/\partial\xi^1 & \partial x^3/\partial\xi^1 \\ \partial x^1/\partial\xi^2 & \partial x^2/\partial\xi^2 & \partial x^3/\partial\xi^2 \\ \partial x^1/\partial\xi^3 & \partial x^2/\partial\xi^3 & \partial x^3/\partial\xi^3 \end{pmatrix}}_{=J \text{ since } J_j^i = \partial x^i/\partial \xi^j} = \det J$$

$$dV = d\xi^1 d\xi^2 d\xi^3 |\det J|$$

Using the coordinates before transformation, the volume is defined as $dV = dx^1 dx^2 dx^3$ so we have

$$dV = dx^1 dx^2 dx^3 = d\xi^1 d\xi^2 d\xi^3 |\det J|$$

(the volume dV doesn't change under coordinate transformation)

(b) By definition, a delta function in any coordinate system satisfies

$$\int d^3\xi\, \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0) f(\mathbf{x}(\boldsymbol{\xi})) = f(\mathbf{x}_0) = \int d^3x\, \delta(\mathbf{x} - \mathbf{x}_0) f(\mathbf{x}) \tag{10}$$

where the correspondence between $\mathbf{x}_0$ and $\boldsymbol{\xi}_0$ such that $\mathbf{x}_0 = \mathbf{x}(\boldsymbol{\xi}_0)$ is assumed to be unique. We have shown above that

$$dx^1 dx^2 dx^3 = |\mathrm{Det}\,(J)| d\xi^1 d\xi^2 d\xi^3 \tag{11}$$

Show or argue that this means

$$\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0) = |\mathrm{Det}\,(J)| \delta(\mathbf{x} - \mathbf{x}_0) \tag{12}$$

from a), we can write $d^3x = |\mathrm{Det}(J)| d^3\xi$. we then have

$$\begin{aligned} \int d^3x\, \delta(x - x_0) f(x) &= \int d^3\xi\, \delta(\xi - \xi_0) f(x(\xi)) \\ &= \int \frac{d^3x}{|\mathrm{Det}(J)|} \delta(\xi - \xi_0) f(x(\xi)) \end{aligned}$$

$f(x)$ & $f(x(\xi))$ will both be evaluated at x_0 bc of δ and this mapping is unique so these terms match up together

To match both sides, we need $\delta(x - x_0) = \dfrac{\delta(\xi - \xi_0)}{|\mathrm{Det}(J)|}$ $\Rightarrow \delta(\xi - \xi_0) = |\mathrm{Det}(J)|\, \delta(x - x_0)$

2. Rotation operators

A rotation operator $\hat{R}(\boldsymbol{\phi})$ is defined by

$$\hat{R}(\boldsymbol{\phi})|\mathbf{x}\rangle = |\mathsf{R}(\boldsymbol{\phi})\mathbf{x}\rangle \tag{13}$$

where $\phi = |\boldsymbol{\phi}|$ is the angle of rotation (counter-clock wise), and $\mathbf{n} = \boldsymbol{\phi}/\phi$ is the direction of the rotation axis. $\mathsf{R}(\boldsymbol{\phi})$ is the corresponding rotation matrix in 3-D. The 3-D rotation matrices are known to form an orthogonal group. This means that $\mathsf{R}^T(\boldsymbol{\phi}) = \mathsf{R}^{-1}(\boldsymbol{\phi}) = \mathsf{R}(-\boldsymbol{\phi})$ and $|\text{Det}\,(\mathsf{R}(\phi))| = 1$.

(a) Using the fact that the product of two rotation matrices, $\mathsf{R}(\boldsymbol{\phi}_2)\mathsf{R}(\boldsymbol{\phi}_1)$, is itself a rotation matrix, show that the product of two rotation operators

$$\hat{R}(\boldsymbol{\psi}) = \hat{R}(\boldsymbol{\phi}_2)\hat{R}(\boldsymbol{\phi}_1) \tag{14}$$

is itself a rotation operator.

Make an argument that this means the rotation operators satisfy the closure property of a group.

operator   Rotation matrix

$$\hat{R}(\phi_2)\hat{R}(\phi_1)|x\rangle = \hat{R}(\phi_2)|R(\phi_1)x\rangle$$
$$= |\underbrace{R(\phi_2)R(\phi_1)}_{=R(\psi)}x\rangle$$

$= R(\psi)$ since product of 2 rotation matrices is another rotation matrix

$$= |R(\psi)x\rangle$$
$$= \hat{R}(\psi)|x\rangle$$

$\Rightarrow \hat{R}(\phi_2)\hat{R}(\phi_1) = \hat{R}(\psi) \in$ Rotation operators

So the group of rotation operators is closed

(b) Let

$$\begin{aligned}\hat{R}(\boldsymbol{\phi}_{21}) &= \hat{R}(\boldsymbol{\phi}_2)\hat{R}(\boldsymbol{\phi}_1)\\ \hat{R}(\boldsymbol{\phi}_{32}) &= \hat{R}(\boldsymbol{\phi}_3)\hat{R}(\boldsymbol{\phi}_2)\end{aligned} \tag{15}$$

Show that

$$\hat{R}(\boldsymbol{\phi}_{32})\hat{R}(\boldsymbol{\phi}_1)|\mathbf{x}\rangle = \hat{R}(\boldsymbol{\phi}_3)\hat{R}(\boldsymbol{\phi}_{21})|\mathbf{x}\rangle \tag{16}$$

Argue that this means rotation operators satisfy associativity.

$$\underbrace{\hat{R}(\phi_{32})}_{\hat{R}(\phi_3)\hat{R}(\phi_2)}\hat{R}(\phi_1)|x\rangle = (\hat{R}(\phi_3)\hat{R}(\phi_2))\hat{R}(\phi_1)|x\rangle = |R(\phi_3)R(\phi_2)R(\phi_1)x\rangle$$

and $\hat{R}(\phi_3)\underbrace{\hat{R}(\phi_{21})}_{\hat{R}(\phi_2)\hat{R}(\phi_1)}|x\rangle = \hat{R}(\phi_3)(\hat{R}(\phi_2)\hat{R}(\phi_1))|x\rangle = |R(\phi_3)R(\phi_2)R(\phi_1)x\rangle$

$\Rightarrow \hat{R}(\phi_{32})\hat{R}(\phi_1)|x\rangle = \hat{R}(\phi_3)\hat{R}(\phi_{21})|x\rangle$

since $(\hat{R}(\phi_3)\hat{R}(\phi_2))\hat{R}(\phi_1) = \hat{R}(\phi_3)(\hat{R}(\phi_2)\hat{R}(\phi_1))$, rotation operators satisfy associativity

(c) We know that

$$\langle \mathbf{x}'|\hat{R}(\boldsymbol{\phi})|\mathbf{x}\rangle = \langle \mathbf{x}'|\mathsf{R}(\boldsymbol{\phi})\mathbf{x}\rangle = \delta(\mathbf{x}' - \mathsf{R}(\boldsymbol{\phi})\mathbf{x}) \tag{17}$$

Show that this is equal to

$$\langle \mathbf{x}'|\hat{R}(\boldsymbol{\phi})|\mathbf{x}\rangle = \frac{1}{|\mathrm{Det}\,(\mathsf{R}(\boldsymbol{\phi}))|}\delta(\mathsf{R}^T(\boldsymbol{\phi})\mathbf{x}' - \mathbf{x}) \tag{18}$$

Then show or argue that

$$\langle \mathbf{x}'|\hat{R}(\boldsymbol{\phi})|\mathbf{x}\rangle = \langle \mathsf{R}^T(\boldsymbol{\phi})\mathbf{x}'|\mathbf{x}\rangle \tag{19}$$

$$\begin{aligned}
\langle x'|\hat{R}(\phi)|x\rangle &= \langle x'|R(\phi)x\rangle \\
&= \delta(x' - R(\phi)x) \\
&= \delta(R(\phi)(R^{-1}(\phi)x' - x)) \\
&= \frac{1}{|Det(R(\phi))|}\delta(R^{-1}(\phi)x' - x)
\end{aligned}$$

$R^{-1}(\phi) = R^T(\phi)$

$$\langle x'|\hat{R}(\phi)|x\rangle = \frac{1}{|Det(R(\phi))|}\delta(R^T(\phi)x' - x)$$

$$= \frac{1}{|Det(R(\phi))|}\langle R^T(\phi)x'|x\rangle$$

1 since $\det R = \pm 1$

$$\langle x'|\hat{R}(\phi)|x\rangle = \langle R^T(\phi)x'|x\rangle$$

(d) Show or argue that this means

$$\langle \mathbf{x}'|\hat{R}(\boldsymbol{\phi}) = \langle \mathsf{R}^T(\boldsymbol{\phi})\mathbf{x}'| \tag{20}$$

and show or argue that that means

$$\hat{R}^\dagger(\boldsymbol{\phi})|\mathbf{x}\rangle = |\mathsf{R}^T(\boldsymbol{\phi})\mathbf{x}\rangle \tag{21}$$

There are two ways we can interpret $\langle x'|\hat{R}(\phi)|x\rangle$

① $\hat{R}(\phi)$ acting on $|x\rangle$. This gives $\langle x'|R(\phi)x\rangle$
② $\hat{R}(\phi)$ acting on $\langle x'|$. This gives the result proved in c) $\langle R^T(\phi)x'|x\rangle$

Interpretation ② therefore implies $\langle x'|\hat{R}(\phi) = \langle R^T(\phi)x'|$

In the same way, we can look at $\langle x'|\hat{R}^\dagger(\phi)|x\rangle$ and interpret it as...

① $\hat{R}^\dagger(\phi)$ acting on $\langle x'| \longrightarrow \langle x'|\hat{R}^\dagger(\phi)|x\rangle = \langle R(\phi)x'|x\rangle$
② $\hat{R}^\dagger(\phi)$ acting on $|x\rangle \longrightarrow \langle x'|\hat{R}^\dagger(\phi)|x\rangle = \langle x'|R^\dagger(\phi)x\rangle$

The 2nd interpretation gives us $\hat{R}^\dagger(\phi)|x\rangle = |R^\dagger(\phi)x\rangle$. We also know $R^{-1}(\phi) = R^T(\phi) = R^\dagger(\phi)$
so we can rewrite this as $\hat{R}^\dagger(\phi)|x\rangle = |R^T(\phi)x\rangle$

(e) Show that

$$\hat{R}(\phi)\hat{R}^\dagger(\phi)|\mathbf{x}\rangle = |\mathbf{x}\rangle \tag{22}$$

$\hat{R}(\phi)\hat{R}^\dagger(\phi)|x\rangle = \hat{R}(\phi)|R^\dagger(\phi)x\rangle$

$= |\underbrace{R(\phi)R^\dagger(\phi)}_{=1}x\rangle$

$\hat{R}(\phi)\hat{R}^\dagger(\phi)|x\rangle = |x\rangle$

(f) Show or argue that the rotation operators $\hat{R}(\phi)$ form a unitary group.

To form a unitary group, the rotation operators need to satisfy the 4 group properties and $\hat{R}^\dagger(\phi) = \hat{R}^{-1}(\phi)$.

1. Associativity → shown in b)
2. closure → shown in a)
3. Identity element → $\hat{R}(0)|x\rangle = |R(0)x\rangle = |x\rangle$ ← Rotate by 0
4. Inverse element → $\hat{R}(\phi)\hat{R}(-\phi)|x\rangle = |R(\phi)R(-\phi)x\rangle = |R(\phi)R^{-1}(\phi)x\rangle = |x\rangle$
   so $\hat{R}(-\phi) = \hat{R}^{-1}(\phi)$
5. $\hat{R}^\dagger(\phi) = \hat{R}^{-1}(\phi)$ → shown in e)

3. We know that a rotation matrix can be always represented by an exponential form:

$$\mathsf{R}(\phi) = e^{\phi\cdot\mathbf{G}} \tag{23}$$

where $\mathbf{G} = (\mathsf{G}_1, \mathsf{G}_2, \mathsf{G}_3) = -\mathbf{G}^T$ is a matrix vector, and the component matrices satisfy

$$[\mathsf{G}_i, \mathsf{G}_j] = \epsilon_{ijk}\mathsf{G}_k \tag{24}$$

Here the commutator is the matrix commutator.

Suppose that the rotation operator can be also represented by a corresponding form

$$\hat{R}(\phi) = e^{\phi\cdot\hat{\mathbf{G}}} \tag{25}$$

where $\hat{\mathbf{G}} = (\hat{G}_1, \hat{G}_2, \hat{G}_3)$ is an operator vector. Let's show that the component operators also satisfy

$$[\hat{G}_i, \hat{G}_j] = \epsilon_{ijk}\hat{G}_k \tag{26}$$

Be careful in distinguishing the operators $\hat{G}_i, \hat{R}(\phi)$ and the matrices $\mathsf{G}_i, \mathsf{R}(\phi)$.

(a) Using the unitarity, show that

$$\hat{\mathbf{G}}^\dagger = -\hat{\mathbf{G}} \tag{27}$$

Unitarity: $\hat{R}^\dagger(\phi) = \hat{R}^{-1}(\phi)$

$(e^{\phi\cdot\hat{G}})^\dagger = (e^{\phi\cdot\hat{G}})^{-1}$

$e^{\phi\cdot\hat{G}^\dagger} = e^{-\phi\cdot\hat{G}}$

$\hat{G}^\dagger = -\hat{G}$

(b) We know that for any vector operator $\hat{\mathbf{V}}$,

$$\hat{R}^\dagger(\boldsymbol{\phi})\hat{\mathbf{V}}\hat{R}(\boldsymbol{\phi}) = \mathsf{R}(\boldsymbol{\phi})\hat{\mathbf{V}} \tag{28}$$

Letting $\boldsymbol{\phi} = \phi\mathbf{e}_x$ with $\phi \ll 1$, show that

$$-[\hat{G}_x, \hat{\mathbf{V}}] = \mathsf{G}_x\hat{\mathbf{V}} \tag{29}$$

Note here that $\hat{G}_x$ is an operator while G_x is a 3×3 matrix.
Make an argument without performing explicit calculations that

$$-[\hat{G}_i, \hat{\mathbf{V}}] = \mathsf{G}_i\hat{\mathbf{V}} \tag{30}$$

for all $i = x, y, z$.

$$\hat{R}^\dagger(\varphi)\hat{V}R(\varphi) = R(\varphi)\hat{V} \quad \hat{R}(\varphi) = e^{\varphi\cdot\hat{G}} \;\&\; R(\varphi) = e^{\varphi\cdot G}$$

$$e^{\varphi\cdot\hat{G}_x^\dagger}\hat{V}e^{\varphi\cdot\hat{G}_x} = e^{\varphi\cdot G_x}\hat{V}$$

$$\Downarrow \varphi \ll 1$$

$$(1\!\!1+\varphi\cdot\hat{G}_x^\dagger)\hat{V}(1\!\!1+\varphi\cdot\hat{G}_x) = (1+\varphi\cdot G_x)\hat{V}$$

$$(1\!\!1+\varphi\cdot\hat{G}_x^\dagger)(\hat{V}+\hat{V}\varphi\cdot\hat{G}_x) = \hat{V}+\varphi\cdot G_x\hat{V}$$

$$\cancel{\hat{V}}+\hat{V}\cancel{\varphi}\cdot\hat{G}_x+\cancel{\varphi}\cdot\hat{G}_x^\dagger\hat{V}+\underbrace{\varphi\cdot\hat{G}_x^\dagger\hat{V}\varphi\cdot\hat{G}_x}_{\varphi^2 \to \text{neglect since } \varphi\ll 1} = \cancel{\hat{V}}+\cancel{\varphi}G_x\hat{V}$$

$$\hat{V}\hat{G}_x + \underbrace{\hat{G}_x^\dagger}_{\hat{G}_x^\dagger = -\hat{G}_x}\hat{V} = G_x\hat{V}$$

$$\hat{V}\hat{G}_x - \hat{G}_x\hat{V} = G_x\hat{V}$$

$$-[\hat{G}_x, \hat{V}] = G_x\hat{V}$$

we can repeat this for any $i = x, y, z$ since the choice of x was arbitrary

$\Rightarrow -[\hat{G}_i, \hat{V}] = G_i\hat{V}$

(c) Show that

$$\hat{R}^\dagger(\phi_y\mathbf{e}_y)\hat{R}^\dagger(\phi_x\mathbf{e}_x)\hat{\mathbf{V}}\hat{R}(\phi_x\mathbf{e}_x)\hat{R}(\phi_y\mathbf{e}_y) = \mathsf{R}(\phi_x\mathbf{e}_x)\mathsf{R}(\phi_y\mathbf{e}_y)\hat{\mathbf{V}} \tag{31}$$

Note the order of operations.

$$\hat{R}^\dagger(\varphi_y e_y)\underbrace{\hat{R}^\dagger(\varphi_x e_x)\hat{V}\hat{R}(\varphi_x e_x)}_{=R(\varphi_x e_x)\hat{V}}\hat{R}(\varphi_y e_y) = \hat{R}^\dagger(\varphi_y e_y)(R(\varphi_x e_x)\hat{V})\hat{R}(\varphi_y e_y)$$

$$= R(\varphi_x e_x)\underbrace{\hat{R}^\dagger(\varphi_y e_y)\hat{V}\hat{R}(\varphi_y e_y)}_{R(\varphi_y e_y)\hat{V}}$$

$$= R(\varphi_x e_x)R(\varphi_y e_y)\hat{V}$$

(d) Show that the following

$$\underbrace{\hat{R}^\dagger(\phi_y \mathbf{e}_y)\hat{R}^\dagger(\phi_x \mathbf{e}_x)\hat{\mathbf{V}}\hat{R}(\phi_x \mathbf{e}_x)\hat{R}(\phi_y \mathbf{e}_y)}_{①} - \overbrace{\hat{R}^\dagger(\phi_x \mathbf{e}_x)\hat{R}^\dagger(\phi_y \mathbf{e}_y)\hat{\mathbf{V}}\hat{R}(\phi_y \mathbf{e}_y)\hat{R}(\phi_x \mathbf{e}_x)}^{②} = \left(\mathsf{R}(\phi_x \mathbf{e}_x)\mathsf{R}(\phi_y \mathbf{e}_y) - \mathsf{R}(\phi_y \mathbf{e}_y)\mathsf{R}(\phi_x \mathbf{e}_x)\right)\hat{\mathbf{V}} \quad (32)$$

leads to

$$[\hat{G}_y, [\hat{G}_x, \hat{\mathbf{V}}]] - [\hat{G}_x, [\hat{G}_y, \hat{\mathbf{V}}]] = \mathsf{G}_z \hat{\mathbf{V}} \quad (33)$$

Assume that $\phi_x \ll 1$, $\phi_y \ll 1$.

LHS

$$
\begin{aligned}
① &= (\mathbb{1} - \varphi_y \cdot \hat{G}_y)(\mathbb{1} - \varphi_x \cdot \hat{G}_x)\hat{V}(\mathbb{1} + \varphi_x \cdot \hat{G}_x)(\mathbb{1} + \varphi_y \cdot \hat{G}_y) \\
&= (\mathbb{1} - \varphi_y \hat{G}_y)(\hat{V} - \varphi_x \hat{G}_x \hat{V})(\mathbb{1} + \varphi_x \cdot \hat{G}_x)(\mathbb{1} + \varphi_y \cdot \hat{G}_y) \\
&= (\hat{V} - \varphi_x \hat{G}_x \hat{V} - \varphi_y \hat{G}_y \hat{V} + \varphi_y \hat{G}_y \varphi_x \hat{G}_x \hat{V})(\mathbb{1} + \varphi_x \cdot \hat{G}_x)(\mathbb{1} + \varphi_y \cdot \hat{G}_y) \\
&= (\hat{V} - \varphi_x \hat{G}_x \hat{V} - \varphi_y \hat{G}_y \hat{V} + \varphi_y \hat{G}_y \varphi_x \hat{G}_x \hat{V})(\mathbb{1} + \varphi_y \hat{G}_y + \varphi_x \hat{G}_x + \varphi_x \varphi_y \hat{G}_x \hat{G}_y) \\
&= \hat{V} + \varphi_y \hat{V} \hat{G}_y + \varphi_x \hat{V} \hat{G}_x + \varphi_x \varphi_y \hat{V} \hat{G}_x \hat{G}_y \\
&\quad - \varphi_x \hat{G}_x \hat{V} - \varphi_x \varphi_y \hat{G}_x \hat{V} \hat{G}_y + \mathcal{O}(\varphi_x^2) \\
&\quad - \varphi_y \hat{G}_y \hat{V} - \varphi_x \varphi_y \hat{G}_y \hat{V} \hat{G}_x + \mathcal{O}(\varphi_y^2) \\
&\quad + \varphi_x \varphi_y \hat{G}_y \hat{G}_x \hat{V} + \mathcal{O}(\varphi_x^2 \varphi_y^2)
\end{aligned}
$$

ignore higher order terms of φ since $\varphi_{x,y} \ll 1$

$$
\begin{aligned}
&= \hat{V} + \varphi_x [\hat{V}, \hat{G}_x] + \varphi_y [\hat{V}, \hat{G}_y] + \varphi_x \varphi_y \left(\hat{G}_y (\hat{G}_x \hat{V} - \hat{V} \hat{G}_x) - (\hat{G}_x \hat{V} - \hat{V} \hat{G}_x)\hat{G}_y\right) \\
&= \hat{V} + \varphi_x [\hat{V}, \hat{G}_x] + \varphi_y [\hat{V}, \hat{G}_y] + \varphi_x \varphi_y [\hat{G}_y, [\hat{G}_x, \hat{V}]]
\end{aligned}
$$

Now for ②, we get the same thing except we switch the x & y indices (since ② = ① w/ switched indices)

$$
\begin{aligned}
② &= (\mathbb{1} - \varphi_x \cdot \hat{G}_x)(\mathbb{1} - \varphi_y \cdot \hat{G}_y)\hat{V}(\mathbb{1} + \varphi_y \cdot \hat{G}_y)(\mathbb{1} + \varphi_x \cdot \hat{G}_x) \\
&\;\vdots \\
&= \hat{V} + \varphi_y [\hat{V}, \hat{G}_y] + \varphi_y [\hat{V}, \hat{G}_y] + \varphi_x \varphi_y [\hat{G}_x, [\hat{G}_y, \hat{V}]]
\end{aligned}
$$

$$\Rightarrow ① - ② = \varphi_x \varphi_y \left([\hat{G}_y, [\hat{G}_x, \hat{V}]] - [\hat{G}_x, [\hat{G}_y, \hat{V}]]\right)$$

RHS

$$
\begin{aligned}
(R(\varphi_x e_x) R(\varphi_y e_y) - R(\varphi_y e_y) R(\varphi_x e_x))\hat{V} &= ((1 + \varphi_x G_x)(1 + \varphi_y G_y) - (1 + \varphi_y G_y)(1 + \varphi_x G_x))\hat{V} \\
&= (\cancel{1} + \cancel{\varphi_y G_y} + \cancel{\varphi_x G_x} + \varphi_x \varphi_y G_x G_y - \cancel{1} - \cancel{\varphi_x G_x} - \cancel{\varphi_y G_y} - \varphi_x \varphi_y G_y G_x)\hat{V} \\
&= \varphi_x \varphi_y \underbrace{[G_x, G_y]}_{= G_z} \hat{V} \\
&= \varphi_x \varphi_y G_z \hat{V}
\end{aligned}
$$

So we get LHS = RHS

$$\cancel{\varphi_x \varphi_y}\left([\hat{G}_y, [\hat{G}_x, \hat{V}]] - [\hat{G}_x, [\hat{G}_y, \hat{V}]]\right) = \cancel{\varphi_x \varphi_y} G_z \hat{V}$$

$$[\hat{G}_y, [\hat{G}_x, \hat{V}]] - [\hat{G}_x, [\hat{G}_y, \hat{V}]] = G_z \hat{V}$$

(e) By expanding the commutators explicitly, show that for any 3 operators $\hat{A}, \hat{B}, \hat{C}$,

$$\underbrace{[\hat{A},[\hat{B},\hat{C}]]}_{①} + \underbrace{[\hat{B},[\hat{C},\hat{A}]]}_{②} + \underbrace{[\hat{C},[\hat{A},\hat{B}]]}_{③} = 0 \tag{34}$$

Note: I will write the operators $\hat{A}, \hat{B}, \hat{C}$ without the ^ to make it easier to write

$$\left.\begin{aligned}
① &= [A,[B,C]] = [A, BC-CB] = A(BC-CB)-(BC-CB)A = \cancel{ABC}-\cancel{ACB}-\cancel{BCA}+\cancel{CBA} \\
② &= [B,[C,A]] = [B, CA-AC] = B(CA-AC)-(CA-AC)B = \cancel{BCA}-\cancel{BAC}-\cancel{CAB}+\cancel{ACB} \\
③ &= [C,[A,B]] = [C, AB-BA] = C(AB-BA)-(AB-BA)C = \cancel{CAB}-\cancel{CBA}-\cancel{ABC}+\cancel{BAC}
\end{aligned}\right\} \text{sum} = 0$$

(f) Show that

$$[\hat{G}_y,[\hat{G}_x,\hat{\mathbf{V}}]] - [\hat{G}_x,[\hat{G}_y,\hat{\mathbf{V}}]] = -[[\hat{G}_x,\hat{G}_y],\hat{\mathbf{V}}] \tag{35}$$

we can rewrite the equation in the form of the property in e)

$$\begin{aligned}
[\hat{G}_y[\hat{G}_x,\hat{V}]] + [\hat{G}_x,[\hat{V},\hat{G}_y]] &+ [[\hat{G}_x,\hat{G}_y],\hat{V}] = 0 \\
'' \qquad &- [\hat{V},[\hat{G}_x,\hat{G}_y]] = 0 \\
'' \qquad &+ [\hat{V},[\hat{G}_y,\hat{G}_x]] = 0
\end{aligned}$$

So now the 3 terms are written as $[\cdots,[\cdots,\cdots]]$ with the 3 permutations of where each operator can be, ie, the form of e)

(g) Show or argue that

$$[\hat{G}_x,\hat{G}_y] = \hat{G}_z \tag{36}$$

$$\begin{aligned}
[\hat{G}_y,[\hat{G}_x,\hat{V}]] - [\hat{G}_x,[\hat{G}_y,\hat{V}]] &= -[[\hat{G}_x,\hat{G}_y],\hat{V}] \quad \text{from f)} \\
&= G_z\hat{V} \quad \text{from d)}
\end{aligned}$$

$$\begin{aligned}
\rightarrow [[\hat{G}_x,\hat{G}_y],\hat{V}] &= -\underbrace{G_z\hat{V}} \\
&= -[\hat{G}_z,\hat{V}] \\
&= [\hat{G}_z,\hat{V}]
\end{aligned}$$

$\Rightarrow [\hat{G}_x,\hat{G}_y] = \hat{G}_z$

(h) Define

$$\hat{\mathbf{G}} = -i\hat{\mathbf{L}}/\hbar \tag{37}$$

Show that $\hat{\mathbf{L}}$ is Hermitian.

Hermitian $\rightarrow$ show $\hat{L} = \hat{L}^{\dagger}$

we have $\hat{L} = i\hbar\hat{G}$ so

$$\begin{aligned}
\hat{L}^{\dagger} = (\hat{L}^T)^* &= (i\hbar\hat{G}^T)^* \\
&= -i\hbar\hat{G}^{\dagger} \quad \hat{G}^{\dagger} = -\hat{G} \\
&= i\hbar\hat{G}
\end{aligned}$$

$\hat{L}^{\dagger} = \hat{L}$

(i) From $[\hat{G}_i, \hat{G}_j] = \epsilon_{ijk}\hat{G}_k$, show that

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \tag{38}$$

$$\begin{aligned}
[\hat{L}_i, \hat{L}_j] &= \hat{L}_i\hat{L}_j - \hat{L}_j\hat{L}_i \\
&= (i\hbar)^2 (\hat{G}_i\hat{G}_j - \hat{G}_j\hat{G}_i) \\
&= (i\hbar)^2 [\hat{G}_i, \hat{G}_j] \\
&= (i\hbar)^2 \varepsilon_{ijk}\hat{G}_k
\end{aligned}$$

$\downarrow \hat{L}_k = i\hbar\hat{G}_k$

$[\hat{L}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\hat{L}_k$