PHYS 457 - Homework 1

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1. Consider a small grain of sand that has 1 gram of mass. Suppose one can measure its position to $1\,\text{nm} = 1.0 \times 10^{-9}\text{m}$ precision. What is its speed uncertainty in m/sec?
   You can use the fact that for $m = 1\,\text{gram}$, $mc^2 \approx 6 \times 10^{32}\,\text{eV}$. Do you think this Δv is measurable?

$m = 1g$
$\Delta x = 1nm = 10^{-9} m$
$\Delta v = ?$

uncertainty principle: $\Delta x \Delta p \geq \hbar/2$

$$\Delta x (m \Delta v) \geq \hbar/2$$

$$\Delta v \geq \frac{\hbar}{2} \cdot \frac{1}{\Delta x m} \cdot \frac{c^2}{c^2}$$

$$\geq \frac{\hbar c \, c}{2 \Delta x mc^2}$$

$$\gtrsim \frac{200 eV\,nm \cdot 3 \times 10^8 m/s}{2(1nm)(6 \times 10^{32} eV)}$$

$$\Delta v \gtrsim 50 \times 10^{-24} m/s$$

→ very very small so most probably not measurable

2. Following the method we used to show $\langle x|\hat{p}|\psi\rangle = (-i\hbar)d\psi(x)/dx$ in the lecture note (in 1-D), show that

$$\langle x|\hat{p}^2|\psi\rangle = -\hbar^2 \frac{d^2}{dx^2}\psi(x) \tag{1}$$

where $\psi(x) = \langle x|\psi\rangle$.

$$\langle x|\hat{p}^2|\psi\rangle = \langle x|\hat{I}_p \hat{p}^2|\psi\rangle$$

$\hat{I}_p = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p|$

$$= \int \frac{dp}{2\pi\hbar} \langle x|p\rangle\langle p|\hat{p}^2|\psi\rangle$$

$$= \int \frac{dp}{2\pi\hbar} p^2 \langle x|p\rangle\langle p|\psi\rangle$$

$\langle x|p\rangle = e^{ixp/\hbar}$

$$= \int \frac{dp}{2\pi\hbar} \underbrace{p^2 e^{ixp/\hbar}}_{-\hbar^2 \frac{d^2}{dx^2}(e^{ixp/\hbar})} \langle p|\psi\rangle$$

$$= -\hbar^2 \frac{d^2}{dx^2} \int \frac{dp}{2\pi\hbar} \underbrace{e^{ixp/\hbar}}_{\langle x|p\rangle} \langle p|\psi\rangle$$

$$= -\hbar^2 \frac{d^2}{dx^2} \psi(x)$$

3. Since $\hat{x}$ and $\hat{p}$ (1-D) are operators, their commutator $[\hat{x}, \hat{p}]$ is also an operator. Hence, we should be able to express it as

$$[\hat{x}, \hat{p}] = \int dx \int dx' \, |x\rangle\langle x|[\hat{x}, \hat{p}]|x'\rangle\langle x'| \tag{2}$$

Let's evaluate the matrix element of this operator.

(a) Show that

$$\langle x|[\hat{x}, \hat{p}]|x'\rangle = \int \frac{dp}{2\pi\hbar} \langle x|\hat{x}\hat{p}|p\rangle\langle p|x'\rangle - \int \frac{dp}{2\pi\hbar} \langle x|p\rangle\langle p|\hat{p}\hat{x}|x'\rangle \tag{3}$$

$$\langle x|[\hat{x},\hat{p}]|x'\rangle = \langle x|\hat{x}\hat{p} - \hat{p}\hat{x}|x'\rangle$$
$$= \langle x|\hat{x}\hat{p}\hat{I}_p|x'\rangle - \langle x|\hat{I}_p\hat{p}\hat{x}|x'\rangle \quad \hat{I}_p = \int \frac{dp}{2\pi\hbar}|p\rangle\langle p|$$
$$= \int \frac{dp}{2\pi\hbar} \langle x|\hat{x}\hat{p}|p\rangle\langle p|x'\rangle - \int \frac{dp}{2\pi\hbar} \langle x|p\rangle\langle p|\hat{p}\hat{x}|x'\rangle$$

(b) Show that this is equal to

$$\langle x|[\hat{x}, \hat{p}]|x'\rangle = \int \frac{dp}{2\pi\hbar} (x - x') p e^{i(x-x')p/\hbar} \tag{4}$$

from a), $$\langle x|[\hat{x},\hat{p}]|x'\rangle = \int \frac{dp}{2\pi\hbar} xp \underbrace{\langle x|p\rangle}_{e^{ixp/\hbar}} \underbrace{\langle p|x'\rangle}_{e^{-ix'p/\hbar}} - \int \frac{dp}{2\pi\hbar} \underbrace{\langle x|p\rangle}_{e^{ixp/\hbar}} px' \underbrace{\langle p|x'\rangle}_{e^{-ix'p/\hbar}}$$
$$= \int \frac{dp}{2\pi\hbar} xp e^{i(x-x')p/\hbar} - \int \frac{dp}{2\pi\hbar} px' e^{i(x-x')p/\hbar}$$
$$= \int \frac{dp}{2\pi\hbar} (x-x') p e^{i(x-x')p/\hbar}$$

(c) Show that the above is equal to

$$\langle x|[\hat{x}, \hat{p}]|x'\rangle = \frac{\hbar}{i} \int \frac{dp}{2\pi\hbar} p \frac{d}{dp} e^{i(x-x')p/\hbar} \tag{5}$$

$$\langle x|[\hat{x},\hat{p}]|x'\rangle = \int \frac{dp}{2\pi\hbar} p \underbrace{(x-x') e^{i(x-x')p/\hbar}}_{= \frac{\hbar}{i}\frac{d}{dp} e^{i(x-x')p/\hbar}}$$
$$= \frac{\hbar}{i} \int \frac{dp}{2\pi\hbar} p \frac{d}{dp} e^{i(x-x')p/\hbar}$$

(d) Assuming that (one can show this using the convergence factor $e^{-\epsilon|p|}$) integration-by-part is valid and the boundary terms vanish, show that

$$\langle x|[\hat{x},\hat{p}]|x'\rangle = i\hbar\delta(x-x') \tag{6}$$

$$\langle x|[\hat{x},\hat{p}]|x'\rangle = \int \frac{dp}{2\pi\hbar} p(x-x')e^{i(x-x')p/\hbar}$$

$$u = \frac{p(x-x')}{2\pi\hbar}, \quad dv = e^{i(x-x')p/\hbar}dp$$

$$du = \frac{(x-x')}{2\pi\hbar}dp, \quad v = \frac{\hbar}{i(x-x')}e^{i(x-x')p/\hbar}$$

$$= \underbrace{\frac{p(x-x')}{2\pi\hbar}\cdot\frac{\hbar}{i(x-x')}\Bigg|_{p_1}^{p_2}}_{\to 0 \text{ boundary terms}} - \int \frac{\cancel{\hbar}}{i\cancel{(x-x')}}e^{i(x-x')p/\hbar}\cdot\frac{\cancel{(x-x')}}{2\pi\cancel{\hbar}}dp$$

$$= \frac{-1}{2\pi i}\int e^{i(x-x')p/\hbar}dp$$

$$p = \hbar k, \quad dp = \hbar dk$$

$$= \frac{i}{2\pi}\int e^{i(x-x')k}\hbar dk$$

$$= i\hbar\int\frac{1}{2\pi}e^{i(x-x')k}dk$$

$$= i\hbar\delta(x-x')$$

(e) Show or argue that this means $[\hat{x},\hat{p}] = i\hbar$.

we know $\langle x|x'\rangle = \delta(x-x')$ so the $i\hbar$ factor must come from $[\hat{x},\hat{p}]$

4. Some operator identities:

(a) Show that $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$$[\hat{A},\hat{B}\hat{C}] = \hat{A}(\hat{B}\hat{C}) - (\hat{B}\hat{C})\hat{A}$$
$$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C}$$
$$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$$
$$= [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$$

(b) Show that $[\hat{B}\hat{C}, \hat{A}] = [\hat{B}, \hat{A}]\hat{C} + \hat{B}[\hat{C}, \hat{A}]$

$$[\hat{B}\hat{C},\hat{A}] = \hat{B}\hat{C}\hat{A} - \hat{A}\hat{B}\hat{C}$$
$$= \hat{B}\hat{C}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C}$$
$$= \hat{B}(\hat{C}\hat{A} - \hat{A}\hat{C}) + (\hat{B}\hat{A} - \hat{A}\hat{B})\hat{C}$$
$$= \hat{B}[\hat{C},\hat{A}] + [\hat{B},\hat{A}]\hat{C}$$

(c) Using only the commutator relationship $[\hat{x}, \hat{p}] = i\hbar$ (we are in 1-D) show that

$$[\hat{p}, \hat{x}^n] = -i\hbar n\hat{x}^{n-1} \quad (7)$$

You should use mathematical induction. That is, show that this formula works for $n = 1$. Assume that it also works for n and then show that the formula is valid for $n+1$ using $\hat{x}^{n+1} = \hat{x}\hat{x}^n$.

① base case: $n=1$, $[\hat{p},\hat{x}] = -i\hbar(1)\hat{x}^{1-1} = -i\hbar = -[\hat{x},\hat{p}]$ ✓

② Assuming $[\hat{p},\hat{x}^n] = -i\hbar n\hat{x}^{n-1}$ for n, for $n+1$ we have

$$[\hat{p},\hat{x}^{n+1}] = [\hat{p},\hat{x}\hat{x}^n]$$ using property from a)
$$= \underbrace{[\hat{p},\hat{x}^n]}_{=-i\hbar n\hat{x}^{n-1}\text{ (assumption)}}\hat{x} + \hat{x}^n\underbrace{[\hat{p},\hat{x}]}_{=-i\hbar\text{ (base case)}}$$
$$= -i\hbar n\hat{x}^{n-1}\hat{x} + \hat{x}^n(-i\hbar)$$
$$= -i\hbar n\hat{x}^n - i\hbar\hat{x}^n$$
$$= -i\hbar(n+1)\hat{x}^n$$
as expected

$\rightarrow$ by induction, $[\hat{p},\hat{x}^n] = -i\hbar n\hat{x}^{n-1}$ holds for all n

(d) Using only the commutator relationship $[\hat{x}, \hat{p}] = i\hbar$ (we are in 1-D), show or argue that

$$[\hat{x}, \hat{p}^n] = i\hbar n\hat{p}^{n-1} \quad (8)$$

The proof in c) only doesn't assume anything about $\hat{x}$ & $\hat{p}$ except that $[\hat{x},\hat{p}] = i\hbar$ and that $[\hat{p},\hat{x}] = -[\hat{x},\hat{p}]$. We had $[\hat{p},\hat{x}^n] = -i\hbar n\hat{x}^{n-1}$ so now flipping $\hat{p}$ & $\hat{x}$ should give us the same expression but with an additional $-$

$\Rightarrow [\hat{x},\hat{p}^n] = i\hbar n\hat{p}^{n-1}$

(we can also prove this by induction as in c))

5. Let's show that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots$$
$$= \hat{B} + \sum_{n=1}^{\infty} \frac{\mathcal{D}_A^n \hat{B}}{n!} \tag{9}$$

where I used $\mathcal{D}_A$ as a short hand notation for the "commutator with $\hat{A}$" such that

$$\mathcal{D}_A \hat{Q} = [\hat{A},\hat{Q}] \tag{10}$$
$$\mathcal{D}_A^2 \hat{Q} = \mathcal{D}_A(\mathcal{D}_A \hat{Q})$$
$$= [\hat{A},[\hat{A},\hat{Q}]] \tag{11}$$
$$\cdots \tag{12}$$

for any operator $\hat{Q}$.

(a) Show that

$$\frac{d}{dt}e^{t\hat{A}} = \hat{A}e^{t\hat{A}} = e^{t\hat{A}}\hat{A} \tag{13}$$

where t is a c-number parameter. You may use the definition

$$e^{t\hat{A}} = \sum_{n=0}^{\infty} \frac{t^n \hat{A}^n}{n!} \tag{14}$$

where $\hat{A}^0 = \hat{1}$ is understood to be the identity operator. You may also use the fact that $\hat{A}$ commutes with $\hat{A}$.

$$\frac{d}{dt}e^{t\hat{A}} = \frac{d}{dt}\sum_0^{\infty}\frac{t^n\hat{A}^n}{n!} = \sum_0^{\infty}\frac{d}{dt}\left(\frac{t^n\hat{A}^n}{n!}\right)$$
$$= \sum_0^{\infty}\frac{n\hat{A}^n}{n!}t^{n-1}$$

0th term = 0 so can start at 1

$$= \sum_1^{\infty}\frac{\hat{A}^n}{(n-1)!}t^{n-1}$$

shift back to 0

$$= \sum_0^{\infty}\frac{\hat{A}^{n+1}}{n!}t^n$$
$$= \hat{A}\sum_0^{\infty}\frac{\hat{A}^n}{n!}t^n$$

*since $\hat{A}$ commutes with itself, we can also take out the $\hat{A}$ at the back and write this as $\left(\sum\frac{\hat{A}^n}{n!}t^n\right)\hat{A} = e^{t\hat{A}}\hat{A}$

$$= \hat{A}e^{t\hat{A}}$$
$$= e^{t\hat{A}}\hat{A}$$

(b) Let

$$\hat{O}(t) = e^{t\hat{A}}\hat{B}e^{-t\hat{A}} \tag{15}$$

Show that

$$\frac{d^n}{dt^n}\hat{O}(t) = e^{t\hat{A}}\left(\mathcal{D}_A^n\hat{B}\right)e^{-t\hat{A}} \tag{16}$$

where $\mathcal{D}_A$ only applies to the $\hat{B}$ part, not the exponential behind. You may use the mathematical induction. That is, show that this formula works for $n = 1$. Then assuming that this is true for an arbitrary n, and then show that the formula works for $n + 1$.

① Base case: $n=1$, $\frac{d}{dt}\hat{O}(t) = \frac{d}{dt}\underbrace{e^{t\hat{A}}}\underbrace{\hat{B}e^{-t\hat{A}}}$

$$= \frac{d}{dt}(e^{t\hat{A}})\hat{B}e^{-t\hat{A}} + e^{t\hat{A}}\frac{d}{dt}(\hat{B}e^{-t\hat{A}})$$

$$= e^{t\hat{A}}\hat{A}\hat{B}e^{-t\hat{A}} + e^{t\hat{A}}\hat{B}(-\hat{A}e^{-t\hat{A}})$$

$$= e^{t\hat{A}}(\underbrace{\hat{A}\hat{B}-\hat{B}\hat{A}}_{D_A\hat{B}})e^{-t\hat{A}}$$

$$= e^{t\hat{A}}(D_A\hat{B})e^{-t\hat{A}}$$

② Assume $\frac{d^n}{dt^n}\hat{O}(t) = e^{t\hat{A}}(D_A^n\hat{B})e^{-t\hat{A}}$ holds for n, then for $n+1$ we have:

$$\frac{d^{n+1}}{dt^n}\hat{O}(t) = \frac{d^{n+1}}{dt^n}(e^{t\hat{A}}\hat{B}e^{-t\hat{A}}) = \frac{d}{dt}\Big[\underbrace{\frac{d^n}{dt^n}e^{t\hat{A}}\hat{B}e^{-t\hat{A}}}_{=e^{t\hat{A}}(D_A^n\hat{B})e^{-t\hat{A}} \text{ by assumption}}\Big]$$

$$= \frac{d}{dt}(e^{t\hat{A}}(D_A^n\hat{B})e^{-t\hat{A}})$$

Not a fxn of t!

$$= e^{t\hat{A}}\hat{A}(D_A^n\hat{B})e^{-t\hat{A}} + e^{t\hat{A}}(D_A^n\hat{B})(-\hat{A})e^{-t\hat{A}}$$

$$= e^{t\hat{A}}(\underbrace{\hat{A}D_A^n\hat{B} - D_A^n\hat{B}\hat{A}})e^{-t\hat{A}}$$

D_A^n isn't an operator so we can bring it out. We get $D_A^n\underbrace{(\hat{A}\hat{B}-\hat{B}\hat{A})}_{D_A^1\hat{B} \text{ from base case}}$

$$= e^{t\hat{A}}(D_A^n D_A^1\hat{B})e^{-t\hat{A}}$$

$$\frac{d^{n+1}}{dt^n}\hat{O}(t) = e^{t\hat{A}}(D_A^{n+1}\hat{B})e^{-t\hat{A}}$$

$\Rightarrow$ by induction, $\frac{d^n}{dt^n}\hat{O}(t) = e^{t\hat{A}}(D_A^n\hat{B})e^{-t\hat{A}}$ holds for all n.

(c) Taylor-expand $\hat{O}(t)$ and use the idea that the Taylor expansion is unique to prove Eq.(9).

$$\hookrightarrow e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \sum_{1}^{\infty} \frac{D_{\hat{A}}^{n}\hat{B}}{n!}$$

① Taylor expanding around 0:

$$\hat{O}(t) = \sum_{0}^{\infty} \frac{t^n}{n!}\,\hat{O}^{(n)}(0)$$

$$= \sum_{0}^{\infty} \frac{t^n}{n!}\left(e^{t\hat{A}}(D_{\hat{A}}^{n}\hat{B})e^{-t\hat{A}}\right)_{t=0}$$

$$= \sum_{0}^{\infty} \frac{t^n}{n!}(D_{\hat{A}}^{n}\hat{B})$$

② equation 9 is $\hat{O}(1)$ so we have $\hat{O}(1) = e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{0}^{\infty} \frac{1^n}{n!}(D_{\hat{A}}^{n}\hat{B})$

$$= \hat{B} + \sum_{1}^{\infty} \frac{D_{\hat{A}}^{n}\hat{B}}{n!}$$

(d) Use Eq.(9) to prove that

$$e^{ia\hat{p}_x/\hbar}\hat{x}e^{-ia\hat{p}_x/\hbar} = \hat{x} + a \tag{17}$$

$$e^{ia\hat{p}_x/\hbar}\hat{x}e^{-ia\hat{p}_x/\hbar} = \hat{x} + \sum_{1}^{\infty}\frac{D_{\hat{A}}^{n}\hat{x}}{n!} = \hat{x} + D_{\hat{A}}\hat{x} + \frac{1}{2}D_{\hat{A}}^{2}\hat{x} + \cdots$$

eq. 9 w/ $\hat{A} = ia\hat{p}_x/\hbar$, $\hat{B} = \hat{x}$

$$= \hat{x} + \underbrace{[\hat{A},\hat{x}]}_{} + \frac{1}{2}[\hat{A},\underbrace{[\hat{A},\hat{x}]}_{a}] + \cdots$$

↳ higher order terms all go to 0 bc of the commutator w/ a scalar

$$= [ia\hat{p}_x/\hbar, \hat{x}]$$

0 since a is a scalar

$$= \frac{ia}{\hbar}[\hat{p}_x, \hat{x}]$$

$$= \frac{ia}{\hbar}(-i\hbar)$$

$$= a$$

$$= \hat{x} + a$$

6. Consider a potential given by

$$V(x) = \frac{b}{m}|x| \tag{18}$$

where $E_0 > 0$ and $k > 0$. Following the method we used to estimate of the hydrogen atom during the class and also in the lecture note, estimate the ground-state energy of the 1-D Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{19}$$

You can use

$$\frac{d}{dx}|x| = \text{sign}(x) \tag{20}$$

without proof.

① $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = \frac{\hat{p}^2}{2m} + \frac{b}{m}|x|$

from uncertainty principle, $p \sim \frac{\hbar}{a}$ where a is the size of the ground state

$\rightarrow \hat{H} = \frac{\hbar^2}{2ma^2} + \frac{b}{m}|a|$

② To get ground state energy, we can minimize H by setting $\frac{\partial H}{\partial a} = 0$

$\frac{\partial H}{\partial a} = \frac{-2\hbar^2}{2m} a^{-3} + \frac{b}{m}\,\text{sign}(a)$

positive since a is a radius

$0 = \frac{-\hbar^2}{\cancel{m}} \frac{1}{a^3} + \frac{b}{\cancel{m}}$

$\frac{\hbar^2}{a^3} = b$

$a = \left(\frac{\hbar^2}{b}\right)^{1/3}$

③ $E_0 = \frac{\hbar^2}{2ma^2} + \frac{b}{m}|a|$

$= \frac{\hbar^2}{2m}\left(\frac{b}{\hbar^2}\right)^{2/3} + \frac{b}{m}\left(\frac{\hbar^2}{b}\right)^{1/3}$

$= \frac{b^{2/3}\hbar^{2/3}}{2m} + \frac{b^{2/3}\hbar^{2/3}}{m}$

$E_0 = \frac{3}{2}\frac{b^{2/3}\hbar^{2/3}}{m}$