A random block-coordinate primal-dual proximal algorithm with application to 3D mesh denoising

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PROBLEM FORMULATION

Minimization problem

$$\underset{x_1\in H_1,\dots,x_p\in H_p}{\text{minimize}} \sum_{j=1}^{p}\big(f_j(x_j)+h_j(x_j)\big)+\sum_{k=1}^{q}(g_k\,\square\, l_k)\Big(\sum_{j=1}^{p} L_{k,j}x_j\Big) \qquad (P)$$

$(\forall j\in\{1,\dots,p\})(\forall k\in\{1,\dots,q\})$

- H_j and G_k separable real Hilbert spaces,
- f_j and g_k proper lsc convex functions defined resp. on H_j and G_k,
- $h_j\colon H_j\to\,]-\infty,+\infty]$ convex Lipschitz differentiable function,
- l_k proper lsc strongly convex function defined on G_k,
- $L_{k,j}\colon H_j\to G_k$ linear bounded operator,
- Inf-convolution:
$$g_k\,\square\, l_k\colon G_k\to[-\infty,+\infty]\colon u\mapsto \inf_{z\in G_k}\big(g_k(z)+l_k(u-z)\big).$$

Dual formulation

$$\underset{v_1\in G_1,\dots,v_q\in G_q}{\text{minimize}} \sum_{j=1}^{p}(f_j^*\,\square\, h_j^*)\Big(-\sum_{k=1}^{q} L_{k,j}^* v_k\Big)+\sum_{k=1}^{q}\big(g_k^*(v_k)+l_k^*(v_k)\big) \qquad (D)$$

- Legendre-Fenchel conjugate function:
$$f_j^*\colon H_j\to[-\infty,+\infty]\colon v\mapsto \sup_{x\in H_j}\big(\langle x\mid v\rangle - f_j(v)\big).$$

Objective

★ Find $\widehat{x}=(\widehat{x}_j)_{1\leqslant j\leqslant p}$ solution to the primal problem (P).

★ Find $\widehat{v}=(\widehat{v}_k)_{1\leqslant k\leqslant q}$ solution to the dual problem (D).

Contributions

Combine usual primal-dual splitting methods [1] to a block-coordinate strategy [2], using recent advances in stochastic optimization [3].

★ Ability to handle high dimensional problems.

★ No inversion of linear operators needed.

★ Arbitrary random activation of the p blocks of variables.

PROPOSED ALGORITHM

Let $(\forall n\in\mathbb{N})$ $\lambda_n\in\,]0,1]$ such that $\inf_{n\in\mathbb{N}}\lambda_n>0$. Let $\boldsymbol{x}_0\in H_1\oplus\cdots\oplus H_p$ and $\boldsymbol{v}_0\in G_1\oplus\cdots\oplus G_q$ be random variables.

For $n=0,1,\dots$
- Select randomly a vector of binary variables $\boldsymbol{\varepsilon}_n=(\varepsilon_{j,n})_{1\leqslant j\leqslant p}$.
- for $k=1,\dots,q$
  - $\eta_{k,n}=\max_{1\leqslant j\leqslant p}\{\varepsilon_{j,n}\mid L_{k,j}\neq 0\}$
  - $u_{k,n}\simeq \eta_{k,n}\,\mathrm{prox}_{g_k^*}^{U_k^{-1}}\Big(v_{k,n}-U_k\big(\nabla l_k^*(v_{k,n})-\sum_{j=1}^{p}L_{k,j}x_{j,n}\big)\Big)$
  - $v_{k,n+1}=v_{k,n}+\lambda_n\eta_{k,n}(u_{k,n}-v_{k,n})$
- for $j=1,\dots,p$
  - $y_{j,n}\simeq \varepsilon_{j,n}\,\mathrm{prox}_{f_j}^{W_j^{-1}}\Big(x_{j,n}-W_j\big(\nabla h_j(x_{j,n})+\sum_{k=1}^{q}L_{k,j}^*(2u_{k,n}-v_{k,n})\big)\Big)$
  - $x_{j,n+1}=x_{j,n}+\lambda_n\varepsilon_{j,n}(y_{j,n}-x_{j,n})$

$(\forall j\in\{1,\dots,p\})(\forall k\in\{1,\dots,q\})$

★ W_j strongly positive self-adjoint linear operator from H_j to H_j

★ U_k strongly positive self-adjoint linear operator from G_k to G_k

★ $\mathrm{prox}_{f_j}^{W_j^{-1}}\colon H_j\to H_j\colon x\mapsto \underset{y\in H_j}{\mathrm{argmin}}\; f_j(y)+\frac{1}{2}\langle x-y\mid W_j^{-1}(x-y)\rangle.$

★ $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ independently distributed in $\{0,1\}^p\setminus\{\boldsymbol{0}\}$

★ $(\forall n\in\mathbb{N})$ $\boldsymbol{\varepsilon}_n$ and $(\boldsymbol{x}_{n'},\boldsymbol{v}_{n'})_{0\leqslant n'\leqslant n}$ are independent,

★ $(\forall n\in\mathbb{N})$ $P[\varepsilon_{j,0}=1]>0$

Convergence results

Under appropriate assumptions on $(W_j)_{1\leqslant j\leqslant p}$ and $(U_k)_{1\leqslant k\leqslant q}$:

★ $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly a.s. to a random variable solution to (P)

★ $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ converges weakly a.s. to a random variable solution to (D)

APPLICATION TO 3D MESH DENOISING

Mesh denoising problem

★ $\mathcal{V}=\mathcal{V}_1\cup\mathcal{V}_2$: set of vertices of the mesh, with $|\mathcal{V}|=p$

$$\mathbf{z}=\overline{\mathbf{x}}+\mathbf{b}$$

★ $\overline{\mathbf{x}}=(\overline{x}_j)_{1\leqslant j\leqslant p}\in\mathbb{R}^{3\times p}$
- ⇝ original spatial coordinates of a 3D mesh
- ⇝ $(\forall j\in\{1,\dots,p\})$ $\overline{x}_j\in\mathbb{R}^3$: 3D coordinates of the j-th vertex

★ $\mathbf{b}=(b_j)_{1\leqslant j\leqslant p}\in\mathbb{R}^{3\times p}$: additive independent noise
- $(\forall j\in\mathcal{V}_1)$ $b_j\sim\mathcal{N}(0,\sigma_1^2)$
- $(\forall j\in\mathcal{V}_2)$ $b_j\sim\pi\mathcal{N}(0,\sigma_2^2)+(1-\pi)\mathcal{N}(0,(\sigma_2')^2)$, with $\pi\in\,]0,1[$

Objective

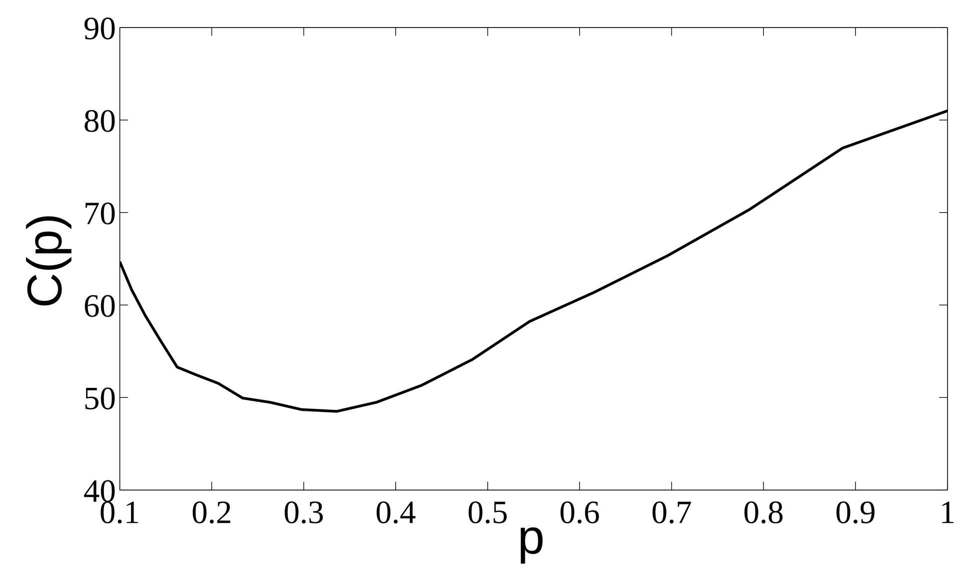
Estimate original spatial coordinates $\overline{\mathbf{x}}$ from the noisy observation $\mathbf{z}$.

Implementation details

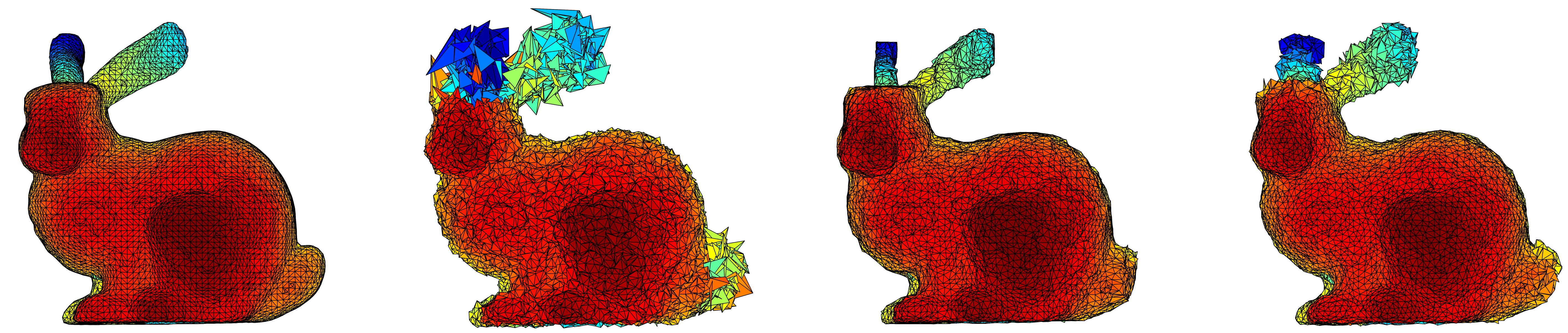
★ a block ≡ a vertex and $q=p$

★ $(\forall j\in\{1,\dots,p\})$ $(\forall k\in\{1,\dots,p\})$
- f_j ⇝ box constraint
- h_j ⇝ $\ell_2-\ell_1$ Huber data fidelity function
- $g_k(\sum_{j=1}^{p}L_{k,j}\,\cdot)$ ⇝ total variation penalization term
- $l_k=\iota_{\{0\}}$ ⇝ $g_k\,\square\,\iota_{\{0\}}=g_k$

★ $(\forall j\in\{1,\dots,p\})(\forall n\in\mathbb{N})$ $P[\varepsilon_{j,n}=1]=\begin{cases}p & \text{if } j\in\mathcal{V}_1\\ 1 & \text{otherwise}\end{cases}$



C(p) ≡ number of operations computed during the reconstruction process



From left to right: original mesh, noisy mesh (MSE $=5.45\times10^{-6}$), reconstructed mesh using the proposed method (MSE $=8.89\times10^{-7}$) and the smoothing Laplacian method (MSE $=1.29\times10^{-6}$)

[1] N. Komodakis and J.-C. Pesquet, “Playing with duality: An overview of recent primal-dual approaches for solving largescale optimization problems”, *Accepted for publication in IEEE Signal Process. Mag.*, 2014.

[2] E. Chouzenoux, J.-C. Pesquet, and A. Repetti, “A block coordinate variable metric forward-backward algorithm”, 2013, http://www.optimization-online.org/DB_HTML/2013/12/4178.html.

[3] P. L. Combettes and J.-C. Pesquet, “Stochastic quasi- Fejèr block-coordinate fixed point iterations with random sweeping,” 2014, http://www.optimization-online.org/DB_HTML/2014/04/4333.html.