NON-CONVEX BLIND DECONVOLUTION APPROACH FOR SPARSE RADIO INTERFEROMETRIC IMAGING

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STATE OF THE ART: STEFCAL ALGORITHM

✓ Direction-independent effects (DIE) calibration method
✓ Alternated direction implicit approach minimizing a least-squares criterion
✗ No convergence guarantees when combined with an imaging method
✗ Not adapted for direction-dependent effects (DDE) calibration

CONTRIBUTIONS: DDE SELF-CALIBRATION METHOD

★ Joint DDE calibration and imaging approach
★ Flexible choices for the regularization function for the image
★ DDEs modelled as smooth functions of the sky
★ Efficient non-convex optimization algorithm with convergence guarantees

BLIND DECONVOLUTION PROBLEM

- Interferometer with n_a antennas: $M = n_a(n_a-1)/2$ visibilities
- Measurement acquired by antenna pair (α,β), at the discrete spatial frequency $k_{\alpha,\beta} = k_\alpha - k_\beta$: $\boxed{y_{\alpha,\beta} = \sum_n \overline{d}_\alpha(n)\overline{d}^*_\beta(n)\overline{x}(n)\mathrm{e}^{-2i\pi k_{\alpha,\beta}\frac{n}{N}} + w_{\alpha,\beta}}$

⋆ $\overline{\boldsymbol{x}} = \big(\overline{x}(n)\big)_n \in \mathbb{R}^N \rightsquigarrow$ unknown original image
⋆ $\overline{\boldsymbol{d}}_\alpha = \big(\overline{d}_\alpha(n)\big)_n \in \mathbb{C}^N \rightsquigarrow$ unknown DDE associated with antenna α
⋆ $\boldsymbol{w} = \big(w_{\alpha,\beta}\big)_{\alpha,\beta} \in \mathbb{C}^M \rightsquigarrow$ realization of an additive i.i.d. Gaussian noise

IMAGING PROBLEM:

- $\overline{\boldsymbol{x}} = \boldsymbol{b} + \overline{\boldsymbol{\epsilon}}$
  - ⋆ $\boldsymbol{b} \rightsquigarrow$ known approximation of the brightest sources of $\overline{\boldsymbol{x}}$ with support $\mathcal{S}$
  - ⋆ $\overline{\boldsymbol{\epsilon}} \rightsquigarrow$ unknown faint sources and errors on the bright sources of $\overline{\boldsymbol{x}}$
- Find an estimate of $\overline{\boldsymbol{\epsilon}}$ from $\boxed{\boldsymbol{y} = \overline{\boldsymbol{G}}\boldsymbol{F}(\boldsymbol{b}+\overline{\boldsymbol{\epsilon}}) + \boldsymbol{w}}$
  - ⋆ $\boldsymbol{F} \in \mathbb{C}^{N\times N} \rightsquigarrow$ Fourier matrix
  - ⋆ $\overline{\boldsymbol{G}} \in \mathbb{C}^{TM\times N} \rightsquigarrow$ contains on each row the antenna-based gain for the pair (α,β) centred in $k_{\alpha,\beta}$

CALIBRATION PROBLEM:

- $\overline{\boldsymbol{d}}_\alpha$ characterized by $S \ll N$ non-zero Fourier coefficients stored in $\overline{\boldsymbol{u}}_\alpha \in \mathbb{C}^S$
- Find an estimate of $(\overline{\boldsymbol{U}}_1, \overline{\boldsymbol{U}}_2)$ from $\boxed{\boldsymbol{Y} = \mathcal{D}_1(\overline{\boldsymbol{U}}_1)\overline{\boldsymbol{X}}\mathcal{D}_2(\overline{\boldsymbol{U}}_2)^\top + \boldsymbol{W}}$
  - ⋆ $(\forall i \in \{1,2\}) \quad \overline{\boldsymbol{U}}_i = \big(\overline{\boldsymbol{u}}_\alpha\big)_{1\le\alpha\le n_a} \in \mathbb{C}^{n_a\times S}$
  - ⋆ $\mathcal{D}_1(\overline{\boldsymbol{U}}_1) \in \mathbb{C}^{n_a\times N} \rightsquigarrow$ sparse matrix containing on each row α the Fourier kernel $\overline{\boldsymbol{u}}_\alpha$ flipped and centred in k_α
  - ⋆ $\mathcal{D}_2(\overline{\boldsymbol{U}}_2) \in \mathbb{C}^{n_a\times N} \rightsquigarrow$ sparse matrix containing on each row α the Fourier kernel $\overline{\boldsymbol{u}}^*_\alpha$ centred in $-k_\alpha$
  - ⋆ $\overline{\boldsymbol{X}} \in \mathbb{C}^{N\times N} \rightsquigarrow$ contains on each line/column a shifted version of the Fourier transform of $\overline{\boldsymbol{x}}$

GLOBAL MINIMIZATION PROBLEM:

$$\boxed{\underset{\boldsymbol{\epsilon}, \boldsymbol{U}_1, \boldsymbol{U}_2}{\text{minimize}}\; F(\boldsymbol{\epsilon}, \boldsymbol{U}_1, \boldsymbol{U}_2) + R(\boldsymbol{\epsilon}) + P(\boldsymbol{U}_1, \boldsymbol{U}_2)}$$

- $F \rightsquigarrow$ differentiable least-squares data fidelity term
- $R \rightsquigarrow$ regularization term promoting sparsity of the image:
  $R(\boldsymbol{\epsilon}) = \sum_n r\big(\epsilon(n)\big)$ with $r\big(\epsilon(n)\big) = \begin{cases} \lambda|\epsilon(n)| + \iota_{[0,+\infty[}\big(\epsilon(n)\big), & \text{if } n \notin \mathcal{S} \\ \iota_{[-\eta,+\eta]}\big(\epsilon(n)\big), & \text{if } n \in \mathcal{S} \end{cases}$
  where $\lambda > 0$ and $\eta \ge 0$
- $P \rightsquigarrow$ regularization term for the DDEs:
  $P(\boldsymbol{U}_1, \boldsymbol{U}_2) = \vartheta\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|^2 + \iota_{\mathbb{D}}(\boldsymbol{U}_1) + \iota_{\mathbb{D}}(\boldsymbol{U}_2)$
  where $\vartheta > 0$ and $\mathbb{D} \subset \mathbb{C}^{n_a\times S}$ is a constraint set for the amplitude of the Fourier coefficients of the DDEs

PROPOSED OPTIMIZATION METHOD

⋆ Block-coordinate forward-backward algorithm alternating between the estimation of $\overline{\boldsymbol{\epsilon}}$, $\overline{\boldsymbol{U}}_1$ and $\overline{\boldsymbol{U}}_2$
⋆ Convergence to a critical point of the objective function

ALGORITHM DESCRIPTION

⋆ At each iteration $k = 0, 1, \ldots$ choose to update either the DDEs $(\boldsymbol{U}_1^{(k)}, \boldsymbol{U}_2^{(k)})$ or the image $\boldsymbol{\epsilon}^{(k)}$

⋆ If the DDEs are updated:

$\boldsymbol{U}_1^{(k,0)} = \boldsymbol{U}_1^{(k)},\; \boldsymbol{U}_2^{(k,0)} = \boldsymbol{U}_2^{(k)}$.
For $\ell = 0, \ldots, L-1$
$\Big\lfloor\; \boldsymbol{U}_1^{(k,\ell+1)} = \mathrm{Proj}_{\mathbb{D}}\Big(\boldsymbol{U}_1^{(k,\ell)} - \gamma_1^k \nabla_{\boldsymbol{U}_1} F\big(\boldsymbol{\epsilon}^{(k)}, \boldsymbol{U}_1^{(k,\ell)}, \boldsymbol{U}_2^{(k)}\big) - \gamma_1^k\big(\boldsymbol{U}_1^{(k,\ell)} - \boldsymbol{U}_2^{(k)}\big)\Big)$,
$\boldsymbol{U}_1^{(k+1)} = \boldsymbol{U}_1^{(k,L)}$.
For $\ell = 0, \ldots, L-1$
$\Big\lfloor\; \boldsymbol{U}_2^{(k,\ell+1)} = \mathrm{Proj}_{\mathbb{D}}\Big(\boldsymbol{U}_2^{(k,\ell)} - \gamma_2^k \nabla_{\boldsymbol{U}_2} F\big(\boldsymbol{\epsilon}^{(k)}, \boldsymbol{U}_1^{(k+1)}, \boldsymbol{U}_2^{(k,\ell)}\big) - \gamma_2^k\big(\boldsymbol{U}_2^{(k,\ell)} - \boldsymbol{U}_1^{(k+1)}\big)\Big)$,
$\boldsymbol{U}_2^{(k+1)} = \boldsymbol{U}_2^{(k,L)}$.
$\boldsymbol{\epsilon}^{(k+1)} = \boldsymbol{\epsilon}^{(k)}$.

⋆ If the image is updated:

$\boldsymbol{\epsilon}^{(k,0)} = \boldsymbol{\epsilon}^{(k)}$.
For $j = 0, \ldots, J-1$
$\Big\lfloor\; \boldsymbol{\epsilon}^{(k,j+1)} = \mathrm{prox}_{\tau^k R}\Big(\boldsymbol{\epsilon}^{(k,j)} - \tau^k \nabla_{\boldsymbol{\epsilon}} F\big(\boldsymbol{\epsilon}^{(k,j)}, \boldsymbol{U}_1^{(k)}, \boldsymbol{U}_2^{(k)}\big)\Big)$,
$\boldsymbol{\epsilon}^{(k+1)} = \boldsymbol{\epsilon}^{(k,J)}$.
$\boldsymbol{U}_1^{(k+1)} = \boldsymbol{U}_1^{(k)},\; \boldsymbol{U}_2^{(k+1)} = \boldsymbol{U}_2^{(k)}$.

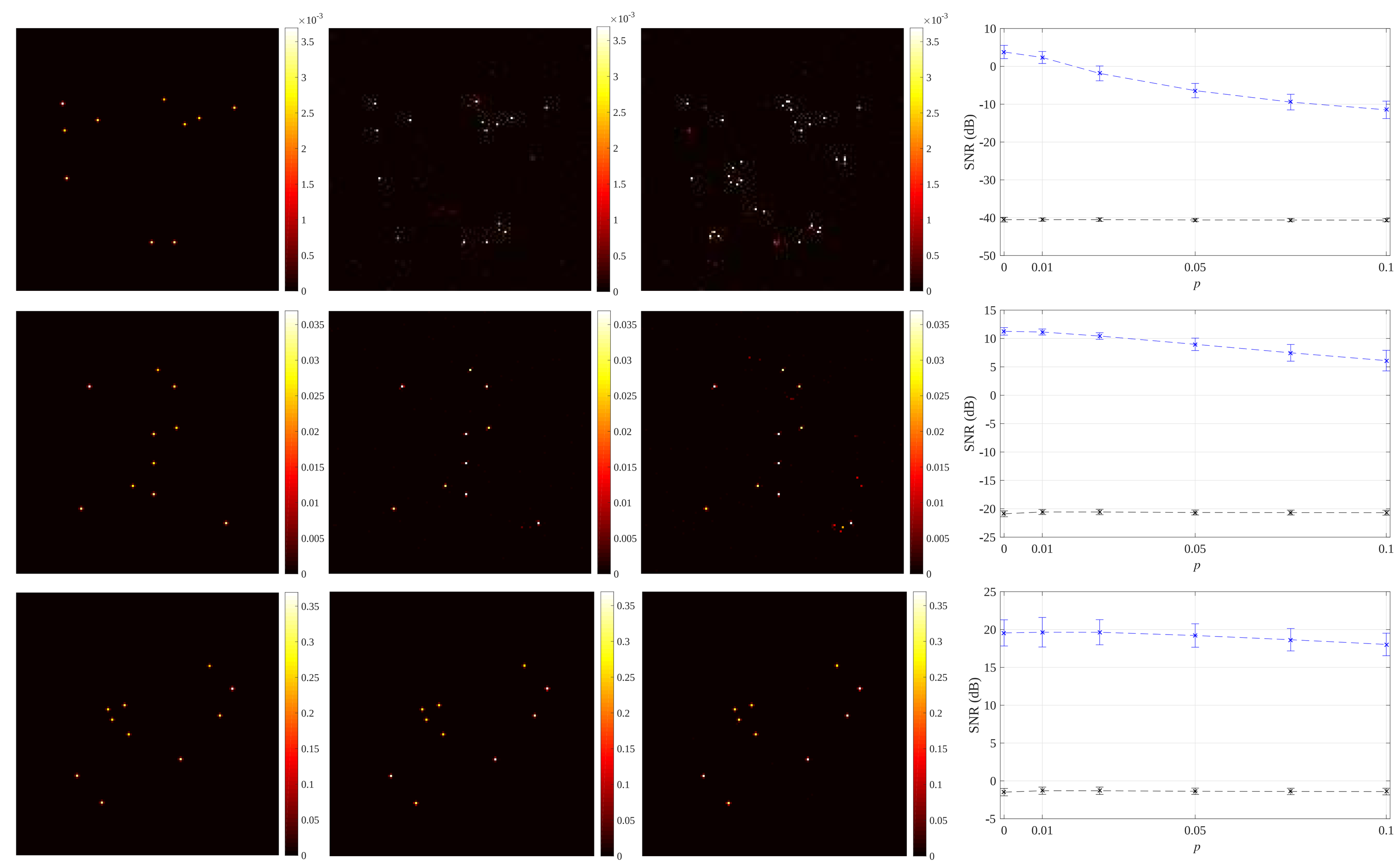
SIMULATION RESULTS

⋆ Dimensions: $n_a = 200$, $N = 128 \times 128$, and $S = 7 \times 7$

⋆ $\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}_1 + \overline{\boldsymbol{x}}_2$
- Bright sources $\overline{\boldsymbol{x}}_1$ approximately known, with $\mathsf{E}(\overline{\boldsymbol{x}}_1) = 10$
- Faint sources $\overline{\boldsymbol{x}}_2$ unknown with $\mathsf{E}(\overline{\boldsymbol{x}}_2) \in \{0.01, 0.1, 1\}$

⋆ Known approximation $\boldsymbol{b}$ of $\overline{\boldsymbol{x}}_1$ such that $b(n) = \big(1 + p\, z(n)\big)\overline{x}_1(n)$, with $z(n) \sim \mathcal{N}(0,1)$ and $p \in \{0, 0.01, 0.025, 0.05, 0.075, 0.1\}$

⋆ Objective: Estimate the faint sources belonging to $\overline{\boldsymbol{x}}_2$



Results obtained considering $\mathsf{E}(\overline{\boldsymbol{x}}_2) = 0.01$ (top row), $\mathsf{E}(\overline{\boldsymbol{x}}_2) = 0.1$ (middle row) and $\mathsf{E}(\overline{\boldsymbol{x}}_2) = 1$ (bottom row). From left to right:
- True $\overline{\boldsymbol{x}}_2$;
- Reconstructions obtained using the proposed method considering (col. 2) $p = 0.01$ and (col. 3) $p = 0.1$;
- The graphs depict the SNR values of the reconstructed second level $\overline{\boldsymbol{x}}_2$ obtained varying $p \in \{0, 0.01, 0.025, 0.05, 0.075, 0.1\}$. Black curves are obtained using the StEFCAL method combined with an imaging algorithm, and blue curves correspond to the reconstructions obtained using the proposed method. Results are given for an average over 10 realizations varying the antenna distribution, the random image, the approximation $\boldsymbol{b}$, and the DDEs.

[1] A. Repetti, J. Birdi, A. Dabbech, and Y. Wiaux, *Non-convex optimization for self-calibration of direction-dependent effects in radio interferometric imaging*, Accepted for publication in **MNRAS**, May 2017.
[2] E. Chouzenoux, J.-C. Pesquet, and A. Repetti, *A block coordinate variable metric forward-backward algorithm*, **J. Global Optim.**, vol. 66, no. 3, pp. 457–485, Nov. 2016.
[3] S. Salvini and S. J. Wijnholds, *Fast gain calibration in radio astronomy using alternating direction implicit methods: Analysis and applications*, **A & A**, vol. 571, p. A97, 2014.