Variable Metric Forward Backward algorithm

Image reconstruction corrupted by a dependent Gaussian noise

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Outline

1. Introduction
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Context

Image Reconstruction

- We observe data $z \in \mathbb{R}^M$, related to the original image $\bar{x} \in \mathbb{R}^N$ through $z = H\bar{x} + w$, where
  - $H \in \mathbb{R}^{M \times N}$ observation matrix,
  - $w \in \mathbb{R}^M$ additive noise.
- Objective: Produce an estimate $\hat{x} \in \mathbb{R}^N$ of the target image $\bar{x}$ from the observed data z.

$\bar{x}$

z

$\hat{x}$

Minimization problem

Penalized optimization problem

$$\text{Find} \quad \hat{x} \in \operatorname{Argmin}\{G = F + R\}, \tag{1}$$

$F \rightsquigarrow$ Data fidelity term, related to noise

$R \rightsquigarrow$ Regularization term, related to some *a priori* informations

- F has an L-Lipschitz gradient on dom R, i.e.
$$\big(\forall (x,y) \in (\text{dom } R)^2\big) \quad \|\nabla F(x) - \nabla F(y)\| \leq L\|x-y\|,$$
- $R\colon \mathbb{R}^N \to (-\infty, +\infty]$ proper, lsc, convex and continuous on its domain,
- G is coercive, i.e. $\lim_{\|x\|\to+\infty} G(x) = +\infty$.

$$\Rightarrow \operatorname{Argmin} G \neq \varnothing$$

Proximity operator

Proximity operator

Let $\psi\colon \mathbb{R}^N \to (-\infty, +\infty]$ proper, lsc, convex. Let $x \in \mathbb{R}^N$.

$$\operatorname{prox}_\psi(x) = \underset{y\in\mathbb{R}^N}{\operatorname{argmin}}\ \psi(y) + \frac{1}{2}\|y - x\|^2.$$

$\rightsquigarrow$ Characterization: $p = \operatorname{prox}_\psi(x) \Leftrightarrow x - p \in \partial\psi(p)$.

Particular case:

Let $\mathcal{C} \subset \mathbb{R}^N$ be a convex set, $x \in \mathbb{R}^N$, $\iota_\mathcal{C}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise} \end{cases}$.

$$\psi(x) = \iota_\mathcal{C}(x) \Rightarrow \operatorname{prox}_\psi = \mathrm{P}_\mathcal{C}.$$

Proximity operator relative to a metric

Let $U \in \mathbb{R}^{N\times N}$ be a symmetric positive definite matrix. Let $x \in \mathbb{R}^N$.

- Weighted norm: $\|x\|_U = \left(x^\top U x\right)^{1/2}$.
- Loewner partial ordering on $\mathbb{R}^{N\times N}$ $(\forall (U_1, U_2) \in (\mathbb{R}^{N\times N})^2)$

$$U_1 \succeq U_2 \quad \Leftrightarrow \quad x^\top U_1 x \geq x^\top U_2 x.$$

Proximity operator relative to the metric induced by U

Let $\psi \colon \mathbb{R}^N \to (-\infty, +\infty]$ proper, lsc, convex.

$$\mathrm{prox}_{U,\psi}(x) = \underset{y\in\mathbb{R}^N}{\mathrm{argmin}}\ \psi(y) + \frac{1}{2}\|y - x\|_U^2.$$

$\rightsquigarrow$ $\mathrm{prox}_{\mathrm{I_N},\psi} = \mathrm{prox}_\psi$.

$\rightsquigarrow$ Characterization: $p = \mathrm{prox}_{U,\psi}(x) \Leftrightarrow U(x - p) \in \partial\psi(p)$.

Forward-Backward algorithm

FB Algorithm

$$
\begin{array}{l}
x_0 \in \mathbb{R}^N \\
\text{For } \; k = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\bar{y}_k = x_k - \gamma_k \nabla F(x_k), \\
y_k = \mathrm{prox}_{\gamma_k R}(\bar{y}_k), \\
x_{k+1} = x_k + \lambda_k (y_k - x_k),
\end{array}
\right.
\end{array}
$$

Convergence is established if:

- [Combettes and Pesquet - 2007]
  - $\rightsquigarrow$ F convex with L-Lipschitzian gradient, R convex lsc proper,
  - $\rightsquigarrow$ $(\gamma_k)_{k \in \mathbb{N}}$ and $(\lambda_k)_{k \in \mathbb{N}}$ bounded.
- [Attouch, Bolte and Svaiter - 2011]
  - $\rightsquigarrow$ F and R non convex, F with L-Lipschitzian gradient,
  - $\rightsquigarrow$ $\lambda_k \equiv 1$ and $(\gamma_k)_{k \in \mathbb{N}}$ bounded.

Variable Metric Forward-Backward algorithm

VMFB Algorithm

$$
\begin{array}{l}
x_0 \in \mathbb{R}^N \\
\text{For} \quad k = 0, 1, \ldots \\
\left\lfloor \begin{array}{l}
\bar{y}_k = x_k - \gamma_k A_k^{-1} \nabla F(x_k), \\
y_k = \operatorname{prox}_{\gamma_k^{-1} A_k, R}(\bar{y}_k), \\
x_{k+1} = x_k + \lambda_k (y_k - x_k),
\end{array} \right.
\end{array}
$$

Convergence is established if:

- [Combettes and Vũ - 2013]
  - ⇝ F convex with L-Lipschitzian gradient, R convex lsc proper,
  - ⇝ $(\gamma_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ bounded,
  - ⇝ $\exists (\eta_k)_{k\in\mathbb{N}} \in \ell_1^+(\mathbb{N})$, such that $(\forall k \in \mathbb{N})$ $(1+\eta_k)A_{k+1} \succeq A_k$,
  - ⇝ $\exists (\underline{\nu}, \overline{\nu}) \in (0, +\infty)^2$ such that $(\forall k \in \mathbb{N})$ $\underline{\nu}\, \mathrm{I}_N \preceq A_k \preceq \overline{\nu}\, \mathrm{I}_N$.
- Non convex case ?

Our contribution [Chouzenoux *et al.* - 2013]

- Convergence of the VMFB algorithm for F non convex ?
  $\rightsquigarrow$ Kurdyka-Łojasiewicz Inequality.
- Choice of variable metric $(A_k)_{k\in\mathbb{N}}$?
  $\rightsquigarrow$ Majorize-Minimize principle.
- Calculation of the proximity operator ?
  $\rightsquigarrow$ Inexact VMFB algorithm.

Majorize-Minimize assumption

MM Assumption

- For every $k \in \mathbb{N}$, there exists a symmetric positive definite matrix $A_k \in \mathbb{R}^{N \times N}$ such that for every $x \in \mathbb{R}^N$

$$Q(x, x_k) = F(x_k) + (x - x_k)^\top \nabla F(x_k) + \frac{1}{2}(x - x_k)^\top A_k (x - x_k),$$

  is a *majorant function* of F at x_k on $\operatorname{dom} R$, i.e.,

$$F(x_k) = Q(x_k, x_k) \quad \text{and} \quad (\forall x \in \operatorname{dom} R) \quad F(x) \leq Q(x, x_k).$$

- There exists $(\underline{\nu}, \overline{\nu}) \in (0, +\infty)^2$ such that $(\forall k \in \mathbb{N})$ $\underline{\nu}\, \mathrm{I}_N \preceq A_k \preceq \overline{\nu}\, \mathrm{I}_N$.

F is differentiable with an L-Lipschitzian gradient on $\operatorname{dom} R$	$\Rightarrow$	$A_k \equiv L\, \mathrm{I}_\mathrm{N}$ satisfies the above assumption [Bertsekas - 1999]

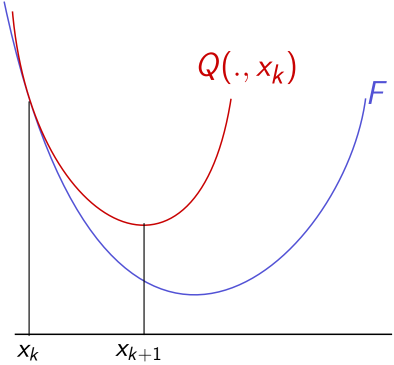
Majorize-Minimize algorithm [Jacobson and Fessler - 2007]

MM Algorithm

$$x_{k+1} \in \underset{x}{\mathrm{Argmin}}\, Q(x, x_k)$$

$\Leftrightarrow$ VMFB Algorithm with

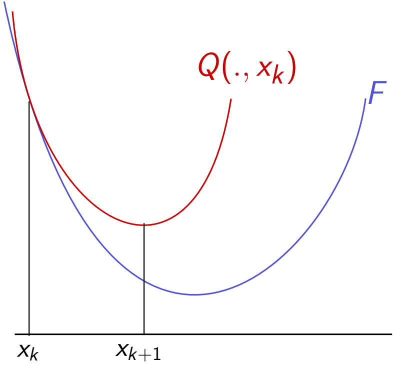
- $R \equiv 0$
- $\lambda_k \equiv 1$
- $\gamma_k \equiv 1$



Majorize-Minimize algorithm [Jacobson and Fessler - 2007]

MM Algorithm

$$x_{k+1} \in \underset{x}{\operatorname{Argmin}}\, Q(x, x_k) + R(x)$$



$\Leftrightarrow$ VMFB Algorithm with

- $\lambda_k \equiv 1$
- $\gamma_k \equiv 1$

Proposed algorithm

VMFB Algorithm

$x_0 \in \operatorname{dom} R$

For $k = 0, 1, \ldots$

$$
\left\lfloor
\begin{array}{l}
\bar{y}_k = x_k - \gamma_k A_k^{-1} \nabla F(x_k), \\
y_k = \operatorname{prox}_{\gamma_k^{-1} A_k, R}(\bar{y}_k), \\
x_{k+1} = (1 - \lambda_k) x_k + \lambda_k y_k,
\end{array}
\right.
$$

where

- $\exists (\underline{\eta}, \overline{\eta}) \in (0, +\infty)^2$ such that $(\forall k \in \mathbb{N})$ $\underline{\eta} \leq \gamma_k \lambda_k \leq 2 - \overline{\eta}$.
- $\exists \underline{\lambda} \in (0, +\infty)$ such that $(\forall k \in \mathbb{N})$ $\underline{\lambda} \leq \lambda_k \leq 1$.

Proposed algorithm

Inexact VMFB Algorithm

$x_0 \in \operatorname{dom} R, \tau \in (0, +\infty)$
For $k = 0, 1, \ldots$
Find $y_k \in \mathbb{R}^N$ and $r(y_k) \in \partial R(y_k)$ such that
$$R(y_k) + (y_k - x_k)^\top \nabla F(x_k) + \gamma_k^{-1} \|y_k - x_k\|_{A_k}^2 \le R(x_k),$$
$$\|\nabla F(x_k) + r(y_k)\| \le \tau \|y_k - x_k\|_{A_k},$$
$x_{k+1} = (1 - \lambda_k) x_k + \lambda_k y_k,$

where

- $\exists (\underline{\eta}, \overline{\eta}) \in (0, +\infty)^2$ such that $(\forall k \in \mathbb{N})$ $\underline{\eta} \le \gamma_k \lambda_k \le 2 - \overline{\eta}$.
- $\exists \underline{\lambda} \in (0, +\infty)$ such that $(\forall k \in \mathbb{N})$ $\underline{\lambda} \le \lambda_k \le 1$.

Inexact proximal step

$$\begin{cases} y_k = \operatorname{prox}_{\gamma_k^{-1} A_k, R}(x_k - \gamma_k A_k^{-1} \nabla F(x_k)) \\ \text{Convexity of } R \end{cases}$$

$$\Leftrightarrow (\exists r(y_k) \in \partial R(y_k)) \quad \begin{cases} r(y_k) = -\nabla F(x_k) + \gamma_k^{-1} A_k (x_k - y_k) \\ (y_k - x_k)^\top r(y_k) \geq R(y_k) - R(x_k). \end{cases}$$

$$\Rightarrow \begin{cases} R(y_k) + (y_k - x_k)^\top \nabla F(x_k) + \gamma_k^{-1} \|y_k - x_k\|_{A_k}^2 \leq R(x_k), \\ \|\nabla F(x_k) + r(y_k)\| = \gamma_k^{-1} \|A_k (y_k - x_k)\| \leq \gamma_k^{-1} \sqrt{\overline{\nu}} \|y_k - x_k\|_{A_k} \\ \qquad \leq \underline{\eta}^{-1} \sqrt{\overline{\nu}} \|y_k - x_k\|_{A_k} \\ \qquad \rightsquigarrow \tau = \underline{\eta}^{-1} \sqrt{\overline{\nu}} \end{cases}$$

Assumptions

- R proper lsc convex and continuous on dom R, F differentiable, ∇F L-Lipschitz on dom R, G is coercive.

- G satisfies the *Kurdyka-Łojasiewicz* inequality:
For every $\xi \in \mathbb{R}$, for every bounded $E \subset \mathbb{R}^N$, there exist $\kappa, \zeta > 0$ and $\theta \in [0, 1)$ such that, for every $x \in E$ such that $|G(x) - \xi| \leq \zeta$,
$$(\forall r(x) \in \partial R(x)) \qquad \|\nabla F(x) + r(x)\| \geq \kappa |G(x) - \xi|^{\theta}.$$

- $(A_k)_{k\in\mathbb{N}}$ satisfies the *majorization conditions*.

- $(\lambda_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ bounded.

- Decreasing assumption: There exists $\underline{\alpha} \in (0, 1]$ such that
$$(\forall k \in \mathbb{N}) \quad G(x_{k+1}) \leq (1 - \underline{\alpha}) G(x_k) + \underline{\alpha} G(y_k).$$

Convergence results

Descent Property

There exists $\mu \in (0, +\infty)$, such that

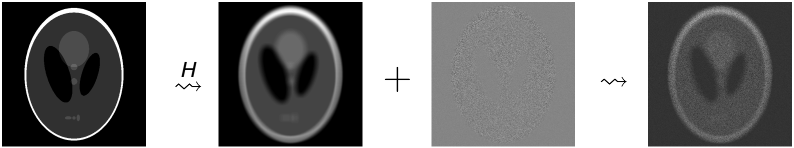
$$(\forall k \in \mathbb{N}) \quad G(x_{k+1}) \leq G(x_k) - \frac{\mu}{2}\|x_{k+1} - x_k\|^2.$$

Convergence theorem

Let $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ be sequences generated by the (inexact) VMFB algorithm.

- Global convergence:
  $\rightsquigarrow$ $\lim x_k = \lim y_k = \hat{x}$, where $\hat{x}$ is a critical point of G.
  $\rightsquigarrow$ $\lim G(x_k) = \lim G(y_k) = G(\hat{x})$.
- Local convergence:
  If $(\exists \upsilon > 0)$ such that $G(x_0) \leq \inf_{x\in\mathbb{R}^N} G(x) + \upsilon$,
  then $\lim x_k = \lim y_k = \hat{x}$, where $\hat{x}$ is a solution to Problem (1).

Image reconstruction under signal-dependent noise



$\overline{x} \in [0,+\infty)^N$ $\quad H\overline{x}$ $\quad w(H\overline{x})$ $\quad z \in [0,+\infty)^M$

- **Observation matrix:** $H \in [0,+\infty)^{M\times N}$.
- **Signal-dependent noise:** $w(H\bar{x}) = \left(w^{(m)}([H\bar{x}]^{(m)})\right)_{1\leq m\leq M}$, with $(\forall m \in \{1,\dots,M\})$
  - $a^{(m)} \in [0,+\infty)$, $b^{(m)} \in (0,+\infty)$,
  - $w^{(m)}([H\bar{x}]^{(m)})$ realization of $W^{(m)} \sim \mathcal{N}\left(0, a^{(m)}[H\bar{x}]^{(m)} + b^{(m)}\right)$.

OBJECTIVE: Produce an estimate $\hat{x} \in [0,+\infty)^N$ of the target image $\bar{x}$ from the observed data z.

Optimization problem

Solve Problem (1): Find $\hat{x} \in \text{Argmin}\{G = F + R\}$ where

Data fidelity term (neg-log-likelihood of the data)

$$F(x) = \begin{cases} F_1(x) + F_2(x) & \text{if } x \in [0, +\infty)^N \\ +\infty & \text{otherwise} \end{cases}, \text{ where}$$

- $F_1(x) = \frac{1}{2} \sum_{m=1}^{M} \frac{([Hx]^{(m)} - z^{(m)})^2}{a^{(m)}[Hx]^{(m)} + b^{(m)}} \rightsquigarrow$ Convex function.
- $F_2(x) = \frac{1}{2} \sum_{m=1}^{M} \log\left(a^{(m)}[Hx]^{(m)} + b^{(m)}\right) \rightsquigarrow$ Concave function.

Penalization term

$(\forall x \in \mathbb{R}^N)\ R(x) = R_1(x) + R_2(x)$, where

- $R_1 \rightsquigarrow \iota_{[x_{min}, x_{max}]^N}(x) = \begin{cases} 0 & \text{if } x \in [x_{min}, x_{max}]^N \\ +\infty & \text{otherwise} \end{cases}.$
- $R_2 \rightsquigarrow$ Sparsity prior in analysis frame or Non Local Total Variation.

Majorization strategy for F_1

$F_1 \rightsquigarrow$ Convex and additive separable function.

$(\forall x \in [0,+\infty)^N)\ F_1(x) = \sum_{m=1}^{M} \rho_1^{(m)}([Hx]^{(m)})$,

where $(\forall m \in \{1,\dots,M\})\ (\forall u \in [0,+\infty))\ \rho_1^{(m)}(u) = \dfrac{1}{2}\dfrac{(u-z^{(m)})^2}{a^{(m)}u + b^{(m)}}$.

Then $(\forall k \in \mathbb{N})$ a majorant function of F_1 on $[0,+\infty)^N$ at x_k is given by

$$\begin{cases} Q_1(\cdot, x_k) = F_1(x_k) + (\cdot - x_k)^\top \nabla F_1(x_k) + (\cdot - x_k)^\top A_k(\cdot - x_k) \\ A_k = \mathrm{Diag}(P^\top \omega(Hx_k)) + \varepsilon\, \mathrm{I}_N \qquad \text{for } \varepsilon \geq 0. \end{cases}$$

with
- $\omega\colon (v^{(m)})_{1\leq m\leq M} \in [0,+\infty)^M \mapsto \left(\omega^{(m)}(v^{(m)})\right)_{1\leq m\leq M} \in \mathbb{R}^M$, where

$$(\forall m \in \{1,\dots,M\}) \quad \omega^{(m)}(u) = \begin{cases} \ddot{\rho}_1^{(m)}(0) & \text{if } u = 0, \\ 2\frac{\rho_1^{(m)}(0) - \rho_1^{(m)}(u) + u\dot{\rho}_1^{(m)}(u)}{u^2} & \text{if } u > 0. \end{cases}$$

- $(\forall m \in \{1,\dots,M\})\ (\forall n \in \{1,\dots,N\})\ P^{(m,n)} = H^{(m,n)} \sum_{p=1}^{N} H^{(m,p)}$.

$\rightsquigarrow$ Proof based on the concavity of $\dot{\rho}_1^{(m)}$ and Jensen's inequality ([Erdogan and Fessler - 1999]).

Implementation

Construction of the majorant

$$F(x) = \begin{cases} F_1(x) + F_2(x) & \text{if } x \in [0, +\infty)^N \\ +\infty & \text{otherwise} \end{cases}, \text{ where}$$

- $F_1 \rightsquigarrow$ Convex function.
  $\rightsquigarrow$ Majorized at x_k by $Q_1(\cdot, x_k)$.
- $F_2 \rightsquigarrow$ Concave function.
  $\rightsquigarrow$ Majorized at x_k by $Q_2(\cdot, x_k) = F_2(x_k) + (\cdot - x_k)^\top \nabla F_2(x_k)$.

Implementation

Construction of the majorant

$$F(x) = \begin{cases} F_1(x) + F_2(x) & \text{if } x \in [0, +\infty)^N \\ +\infty & \text{otherwise} \end{cases}, \text{ where}$$

- $F_1 \rightsquigarrow$ Convex function.
  $\rightsquigarrow$ Majorized at x_k by $Q_1(\cdot, x_k)$.
- $F_2 \rightsquigarrow$ Concave function.
  $\rightsquigarrow$ Majorized at x_k by $Q_2(\cdot, x_k) = F_2(x_k) + (\cdot - x_k)^\top \nabla F_2(x_k)$.

Backward step

$$y_k = \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \left\{ R(x) + \frac{1}{2} \|x - \bar{y}_k\|^2_{\gamma_k^{-1} A_k} \right\} \text{ with } \bar{y}_k = x_k - \gamma_k A_k^{-1} \nabla F(x_k)$$

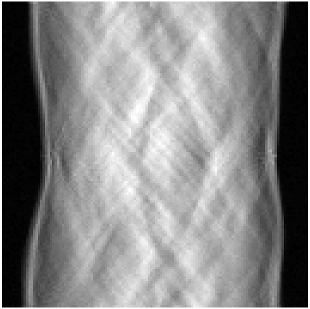
$$\Leftrightarrow y_k = \gamma_k^{1/2} A_k^{-1/2} \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \left\{ R(\gamma_k^{1/2} A_k^{-1/2} x) + \frac{1}{2} \|x - \gamma_k^{-1/2} A_k^{1/2} \bar{y}_k\|^2 \right\}$$

$\rightsquigarrow$ Dual Forward-Backward Algorithm [Combettes *et al.* - 2011]

Reconstruction with sparsity prior

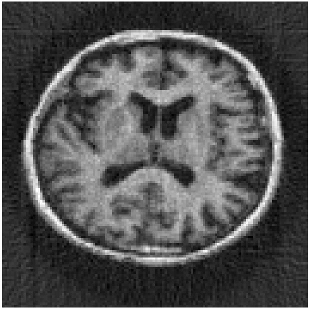
- H: Radon matrix modeling $M = 16384$ parallel projections from 128 acquisitions lines and 128 angles.
- $(\forall m \in \{1, \ldots, M\})\ a^{(m)} = 0.01$ and $b^{(m)} = 0.1$

Original image *Zubal*



Degraded sinogram

Results: Restored images



FBP: SNR=7 dB

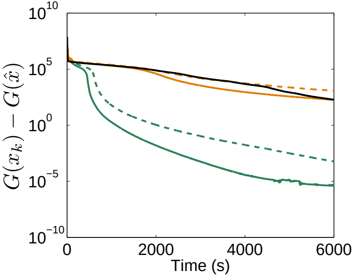
VMFB: SNR=18.9 dB

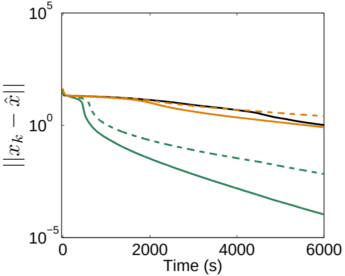
Results

VMFB Algorithm with • $\lambda_k \equiv 1$ and $\gamma_k \equiv 1.9$ (solid line)
• $\lambda_k \equiv 1$ and $\gamma_k \equiv 1$ (dashed line)

FB Algorithm with • $\lambda_k \equiv 1$ and $\gamma_k \equiv 1.9$ (solid line)
• $\lambda_k \equiv 1$ and $\gamma_k \equiv 1$ (dashed line)

FISTA

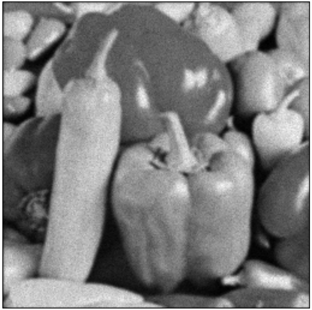




Deblurring with Non Local Total Variation

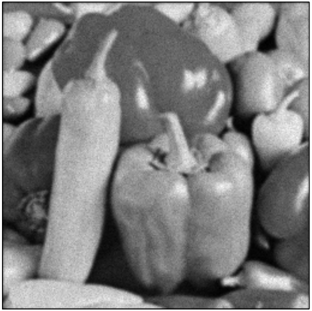
- H: Blur operator corresponding to a truncated Gaussian kernel of standard deviation 1 and size 7×7.
- $(\forall m \in \{1, \dots, M\})$, $a^{(m)} = 0.5$ and $b^{(m)} = 1$

Original image *Peppers*



Degraded image: SNR=21.85 dB

Results: Restored images



Degraded image: SNR=21.85 dB

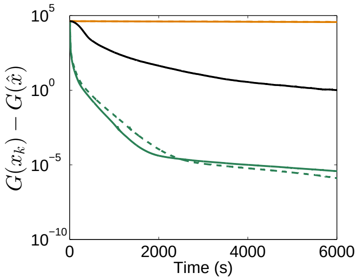
Restored image: SNR=27.11 dB

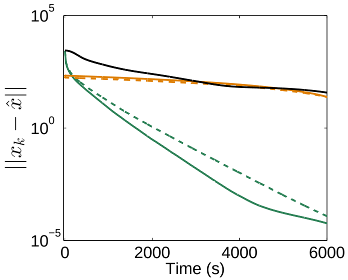
Results

VMFB Algorithm with • $\lambda_k \equiv 1$ and $\gamma_k \equiv 1.9$ (solid line)
• $\lambda_k \equiv 1$ and $\gamma_k \equiv 1$ (dashed line)

FB Algorithm with • $\lambda_k \equiv 1$ and $\gamma_k \equiv 1.9$ (solid line)
• $\lambda_k \equiv 1$ and $\gamma_k \equiv 1$ (dashed line)

FISTA





Conclusion

- Convergence of the VMFB algorithm for the sum of a non convex differentiable function F and a non smooth convex function R.
- Choice of variable metric $(A_k)_{k \in \mathbb{N}}$ based on MM principle.
- Inexact VMFB algorithm for the calculation of the proximity operator.

The variable metric strategy leads to a significant acceleration in terms of decay of both the objective function and the error on the iterates in each experiment.

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Thank you !

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