Regular languages: complement closure

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This is a proof for completely showing the *closure* of the class of regular languages under the complement operator. Note the **correctness** portion where we must show an "if and only if."

Theorem 1. *The class of regular languages is closed under the complement operator.*

Proof. Let A be a regular language. Then there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes A, i.e., $L(M) = A$.

We must show that $\overline{A}$ is a regular language. That is, we must show that there exists a DFA that recognizes $\overline{A}$. The proof of existence is by construction. We build the DFA $M' = (Q, \Sigma, \delta, q_0, F')$, where $F' = \overline{F}$, so that $L(M') = \overline{A}$.

We now show correctness of our construction, allowing us to conclude that $\overline{A}$ is a regular language; i.e., regular languages are closed under the complement operator.

Claim 1. *A string $w \in L(M')$ if and only if $w \in \overline{A}$.*

Proof. ($\Rightarrow$) Assume $w = w_1 w_2 \ldots w_n \in L(M')$, where $w_i \in \Sigma$ for all $i = 1, \ldots, n$. By definition of the language of a machine, M' accepts w. Then, by definition of a DFA accepting an input, there exists a sequence of states $s'_0, s'_1, \ldots, s'_n$ such that

1. $s'_0 = q_0$
2. $\delta(s'_i, w_{i+1}) = s'_{i+1}$ for all $i = 0, \ldots, n-1$
3. $s'_n \in F'$

We show that $w \in \overline{A}$, i.e., $w \notin A$. Since $L(M) = A$, this is equivalent to showing that M does not accept w; i.e., we must show there does not exist a sequence of states satisfying the three conditions of the definition of a DFA accepting an input string.

Suppose, for a contradiction, there did exist a sequence of states $s_0, s_1, \ldots, s_n$ such that

1. $s_0 = q_0$

2. $\delta(s_i, w_{i+1}) = s_{i+1}$ for all $i = 0, \ldots, n-1$

3. $s_n \in F$

Since $s'_0 = q_0$ from above, $s_0 = s'_0$. Also from above, $\delta(s'_0 = s_0, w_1) = s'_1$, so $s_1 = s'_1$. By applying this argument inductively on the length n of w, we can conclude that $s_i = s'_i$ for all $i = 0, \ldots, n$. In particular, $s_n = s'_n$, so $s_n \in F'$. However, $F' = \overline{F}$ and s_n cannot be in both F and its complement, giving the contradiction.

Thus, it must be the case that M does not accept w and $w \in \overline{A}$.

($\Leftarrow$) Assume $w = w_1 w_2 \ldots w_n \in \overline{A}$, where $w_i \in \Sigma$ for all $i = 1, \ldots, n$. Since $L(M) = A$, M does not accept w.

We show that $w \in L(M')$, i.e., we show that M' accepts w. Consider the sequence of states $s'_0, s'_1, \ldots, s'_n$ constructed as follows:

1. $s'_0 = q_0$

2. $s'_i = \delta(s'_{i-1}, w_i)$ for $i = 1, \ldots, n$

Notice that the sequence $s'_0, \ldots, s'_n$ satisfies the first two conditions of the definition of a DFA accepting an input for the machine M. Since M does not accept w, it must be the case that the third condition fails, i.e., $s'_n \notin F$. Therefore, $s'_n \in \overline{F}$; by construction $F' = \overline{F}$, so the sequence $s'_0, \ldots, s'_n$ shows that M' accepts w. Thus, $w \in L(M')$, completing the proof. $\square$

$\square$