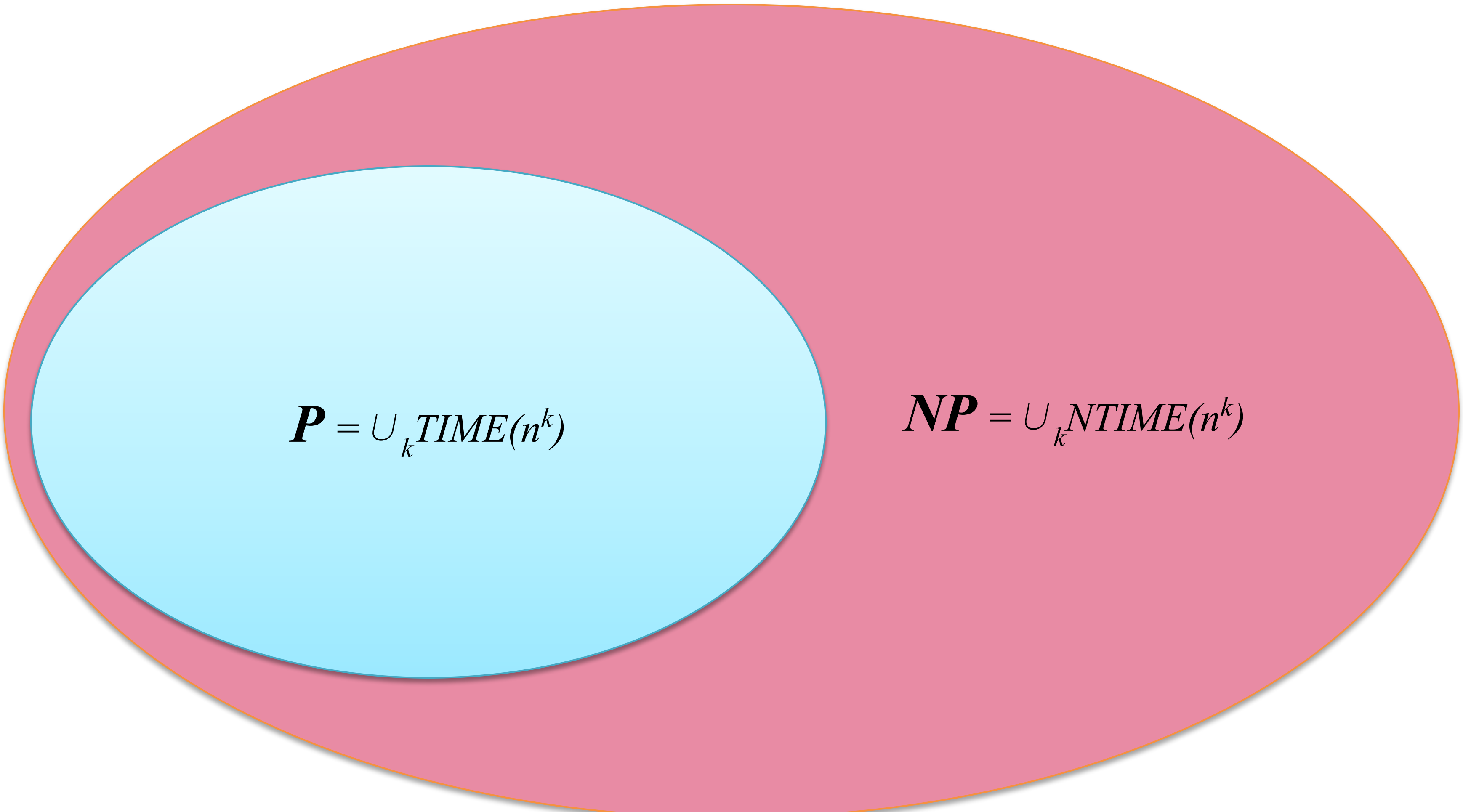
NP-completeness

Sipser 7.4 (pages 271 – 283)

The classes P and NP



A famous NP problem

- CNF satisfiability ($CNFSAT$):
  Given a boolean formula B in conjunctive normal for (CNF), is there a truth assignment that satisfies B?

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee \overline{x_2})$$

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Yes! x1 = T
x2 = F

$CNFSAT \in NP$

- Verifier:

  V = "On input $<B,c>$:
  1. Test whether c is a truth assignment for B's variables. If not, *reject*.
  2. Evaluate B using c.
     If B evaluates to true, *accept*; otherwise, *reject*."

- NTM:

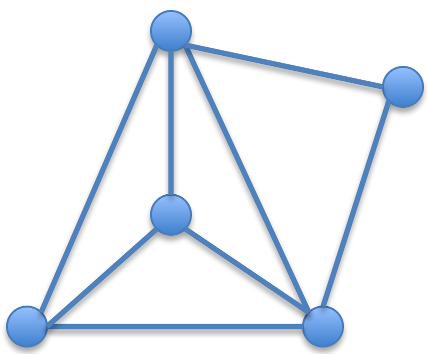
  N = "On input $<B>$:
  1. Nondeterministically assign true or false to each of B's variables.
  2. Evaluate B using c.
     If B evaluates to true, *accept*; otherwise, *reject*."

A graph theory NP problem

- *CLIQUE*:

  Given a graph $G = (V, E)$ and an integer t, does G contain K_t as a subgraph?

  – Is $<G,3> \in$ *CLIQUE?*



A graph theory NP problem

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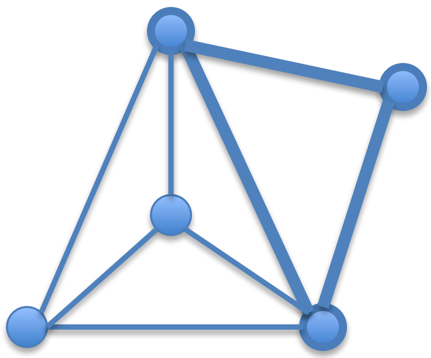

A graph theory NP problem

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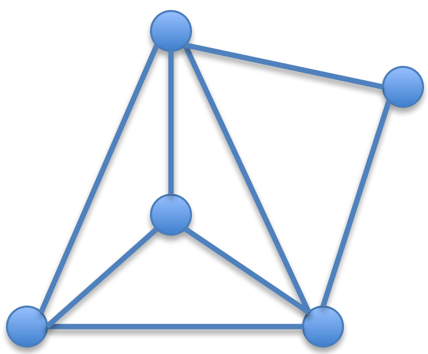

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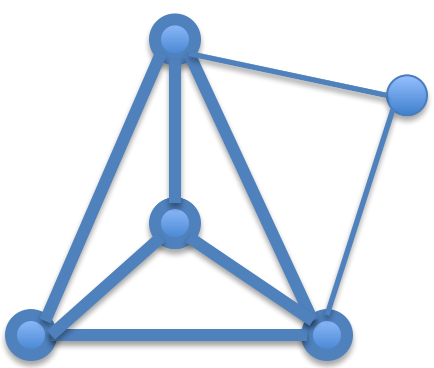
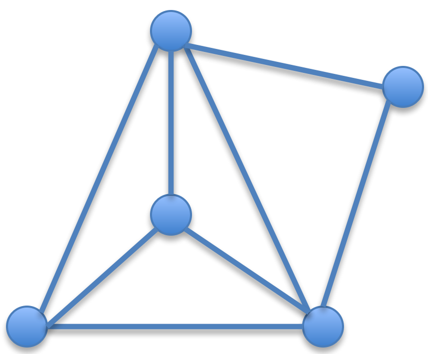


A graph theory NP problem

- *CLIQUE*:

  Given a graph $G = (V, E)$ and an integer t, does G contain K_t as a subgraph?

  - Is $<G,5> \in CLIQUE?$



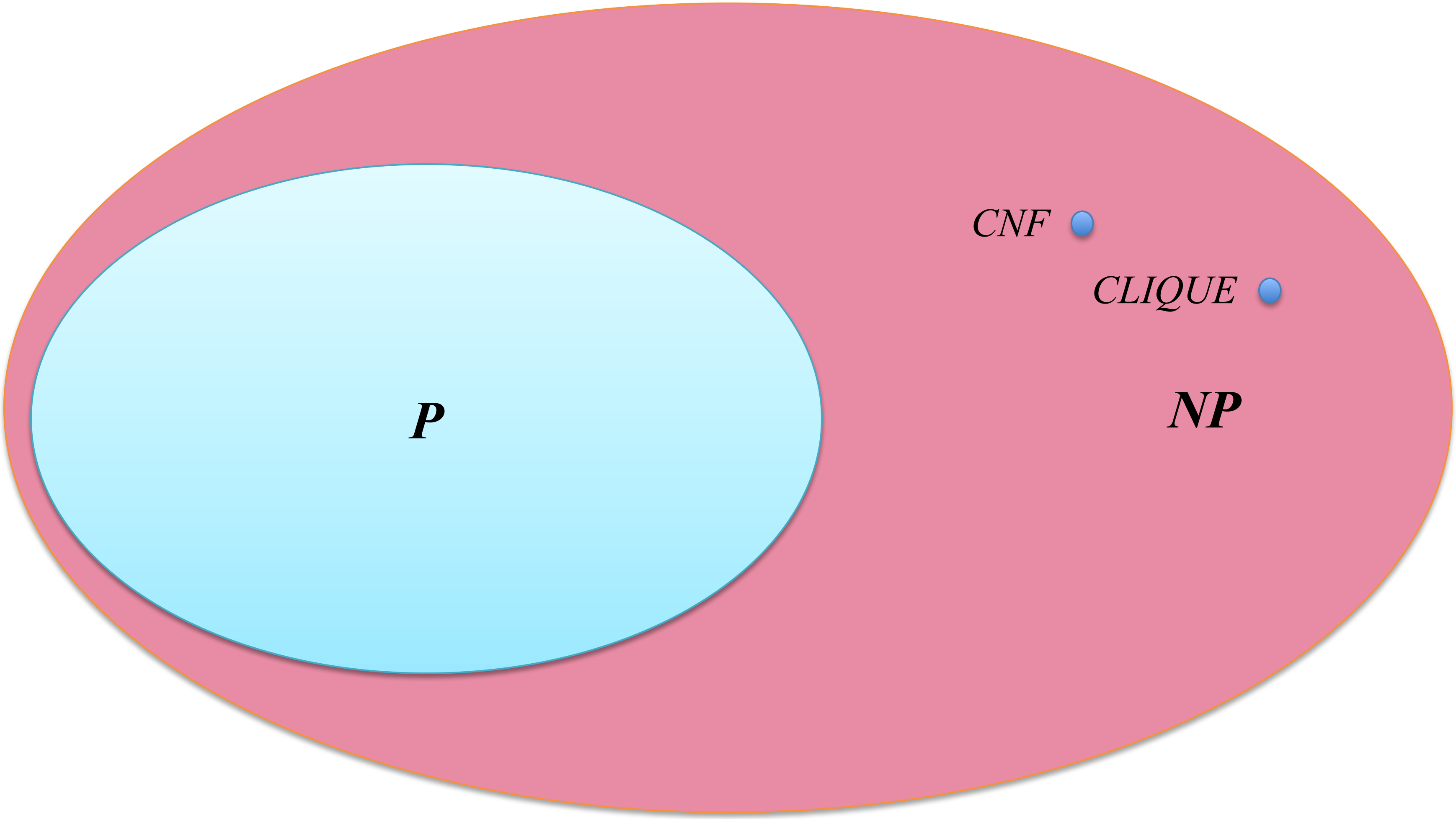
$CLIQUE \in NP$

- Verifier:

  V = "On input $<<G,t>,S>$:
  1. Test whether S is a set of t nodes of G
  2. Test whether G contains all edges connecting nodes in S
  3. If both pass, *accept*; otherwise, *reject*."
- NTM:

  N = "On input $<G,t>$:
  1. Nondeterministically select a subset S of t nodes of G
  2. Test whether G contains all edges connecting nodes in S
  3. If yes, *accept*; otherwise, *reject*."

Which problem is harder?



Recall…

- Definition 5.17: A function $f{:}\Sigma^*\rightarrow\Sigma^*$ is a **computable function** if ∃ some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Polynomial time computable functions

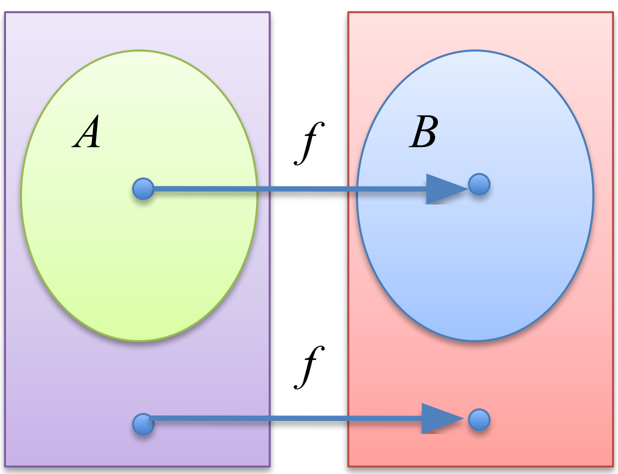
- Definition 7.28: A function $f{:}\Sigma^*\rightarrow\Sigma^*$ is a **polynomial time computable function** if ∃ some polynomial time Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Recall…

- Definition 5.20:

  Language A is *mapping reducible* to language B, written $A \leq_m B$, if there is a computable function $f: \Sigma^* \rightarrow \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

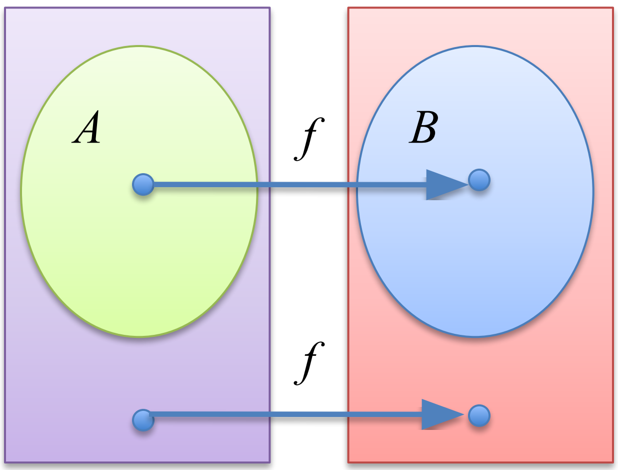


Polynomial time mapping reducibility

- Definition 7.29:

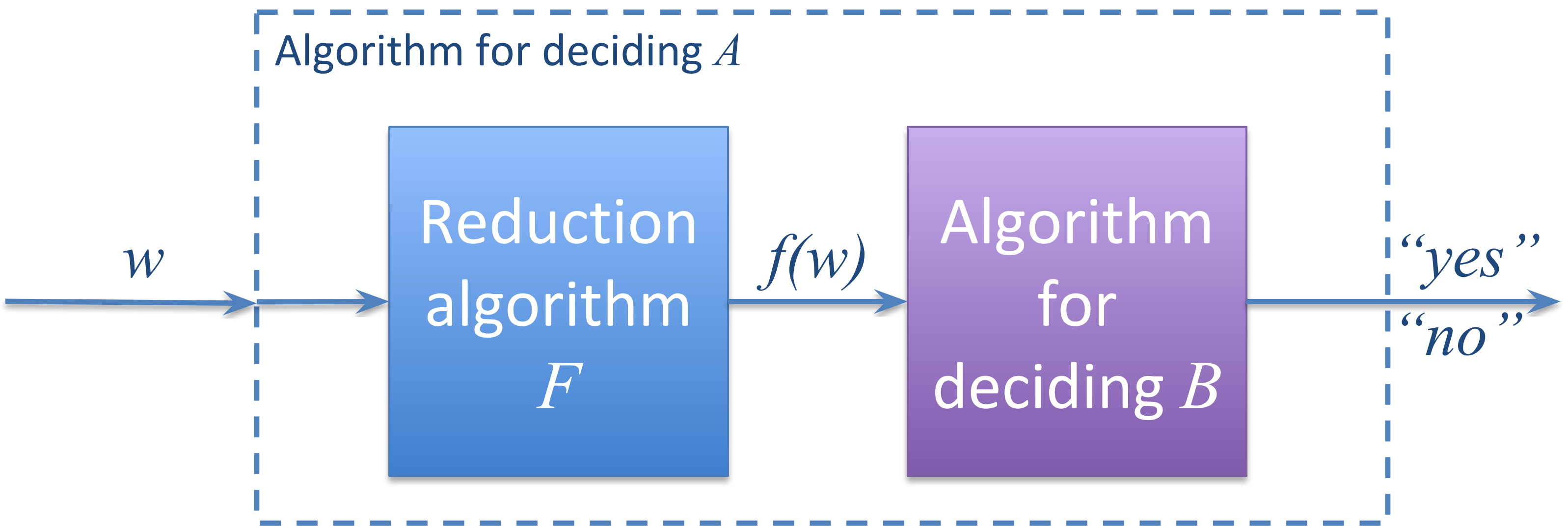
  Language A is *polynomial time mapping reducible* to language B, written $A \leq_p B$, if there is a polynomial time computable function $f: \Sigma^* \to \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$



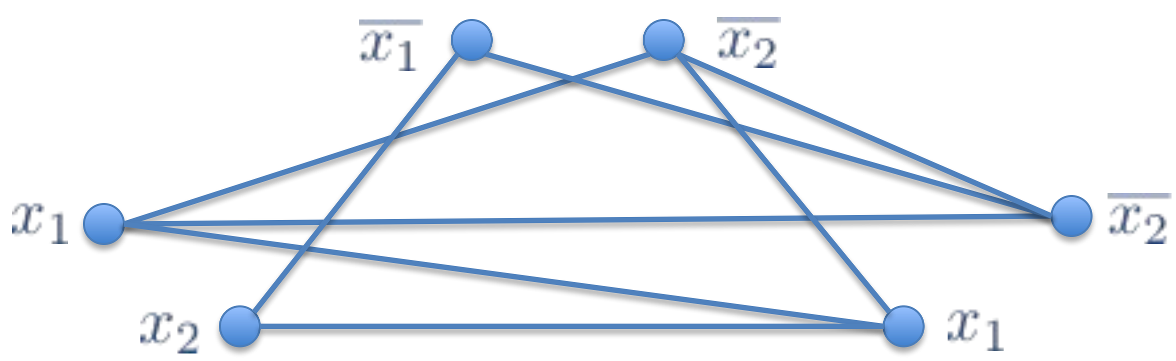
Intuitively, A is no harder than B

- Theorem 7.31:
  If $A \leq_p B$ and $B \in P$, then $A \in P$.
- Proof:



$CNF \leq_p CLIQUE$

- Given a boolean formula B in CNF, we show how to construct a graph G and an integer t such that G has a clique of size t $\Leftrightarrow B$ is satisfiable.
- Given $(x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee \overline{x_2})$ the construction would yield

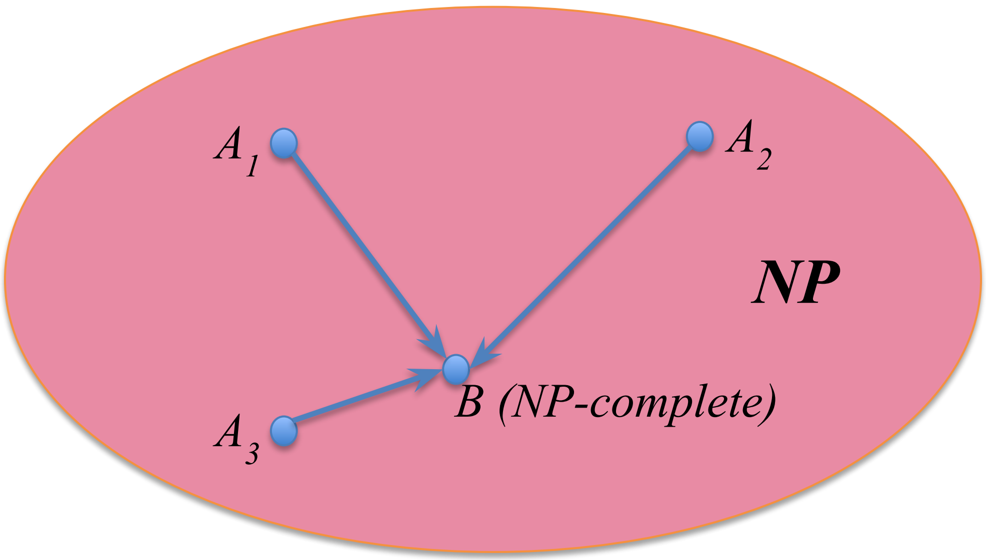


NP's hardest problems

- Definition 7.34:

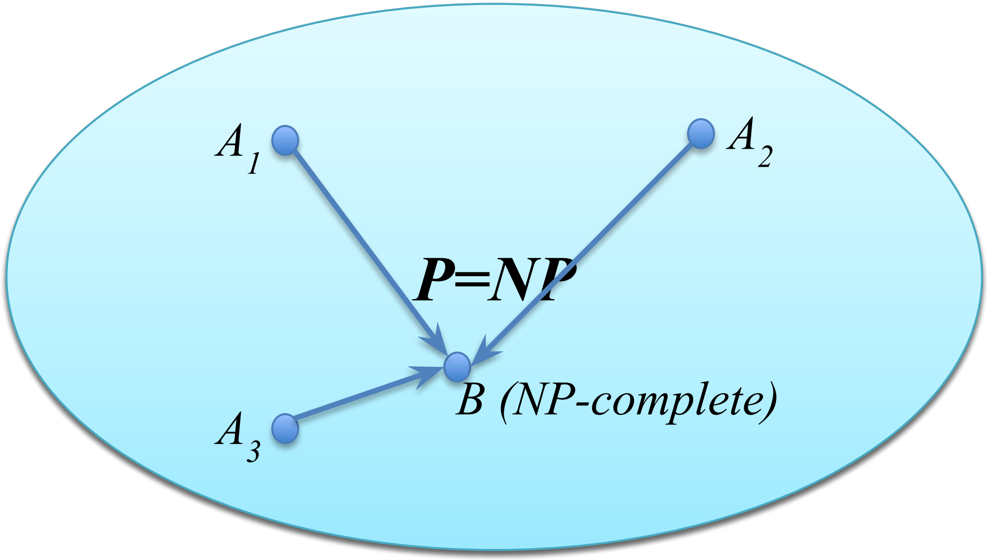
  A language B is **NP-complete** if

  1. $B \in NP$
  2. $A \leq_p B$, for all $A \in NP$



P=NP?

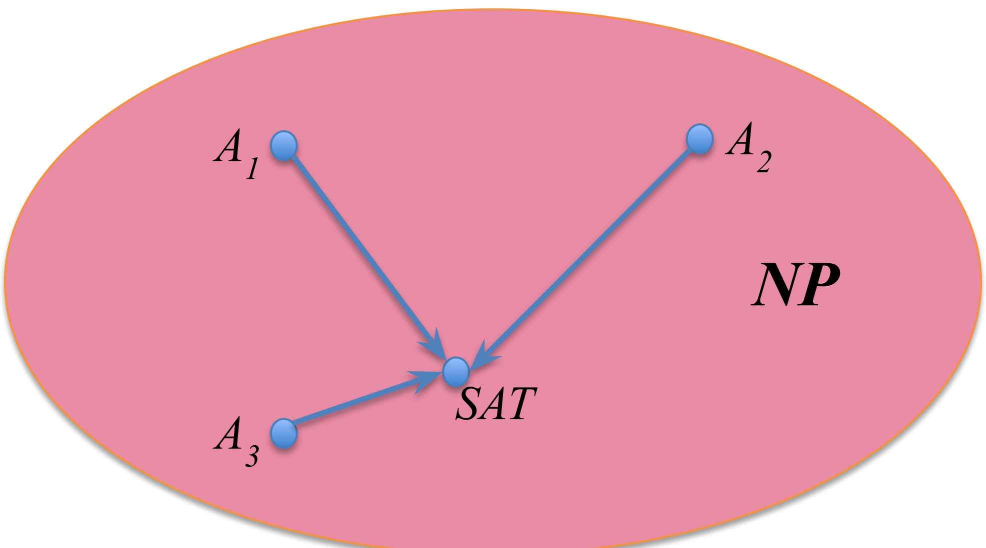
- Theorem 7.35: If B is NP-complete and $B \in P$, then $P = NP$.



Cook-Levin Theorem

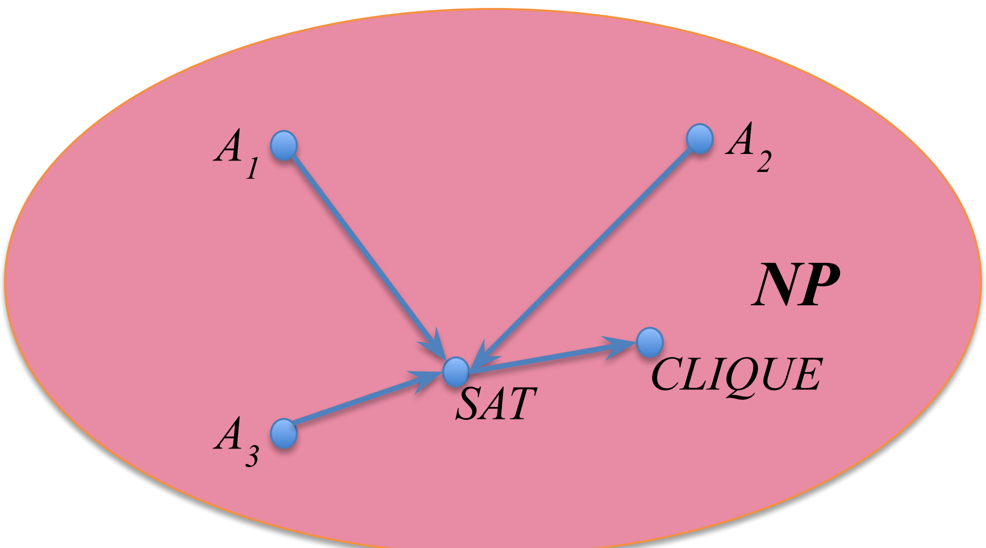
- SAT is NP-complete.

  (If $A \in NP$, then $A \leq_p SAT$.)

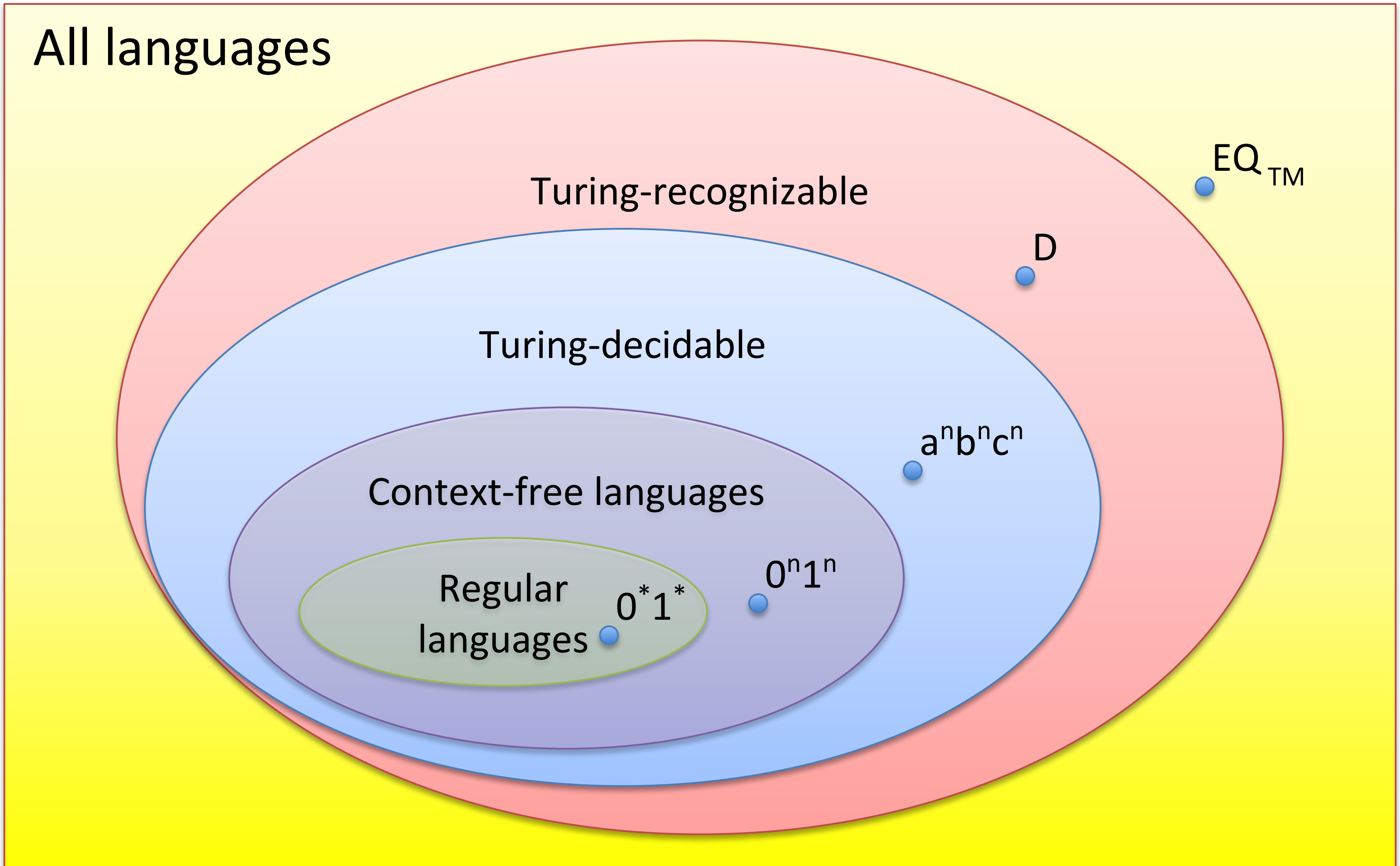


But that's not the only one!

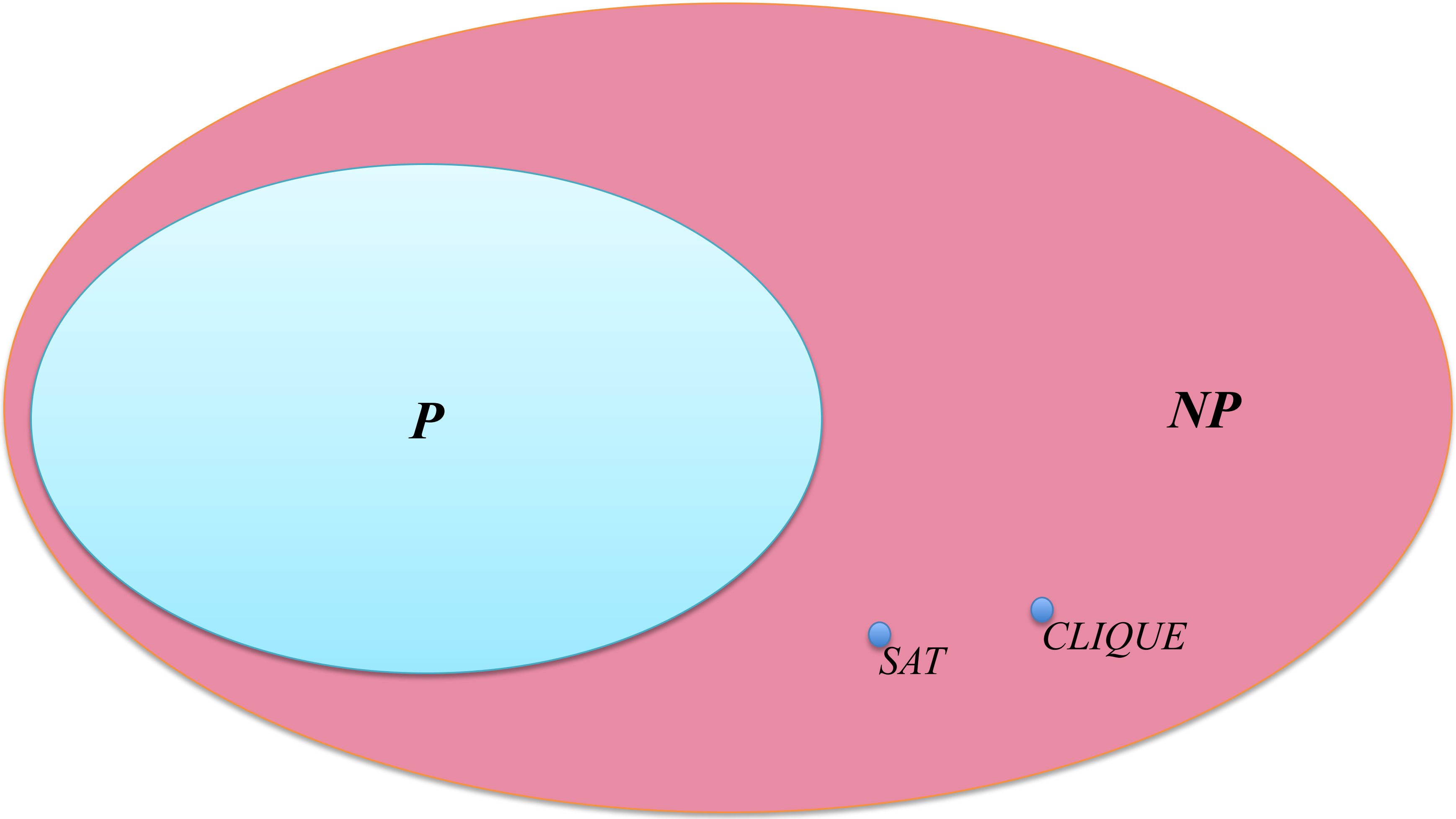
- *CLIQUE* is NP-complete (why?)



Hierarchy of languages



The classes P and NP



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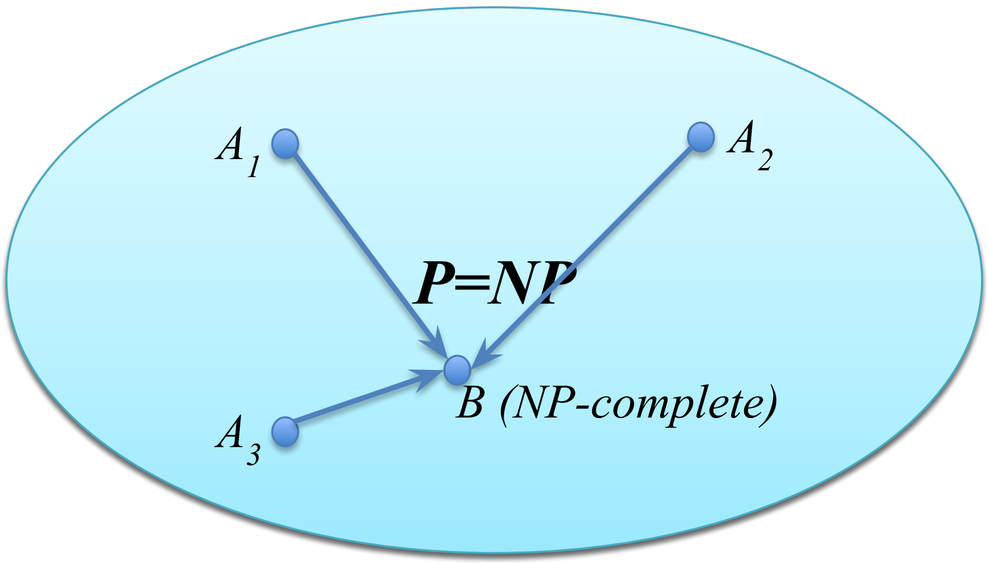
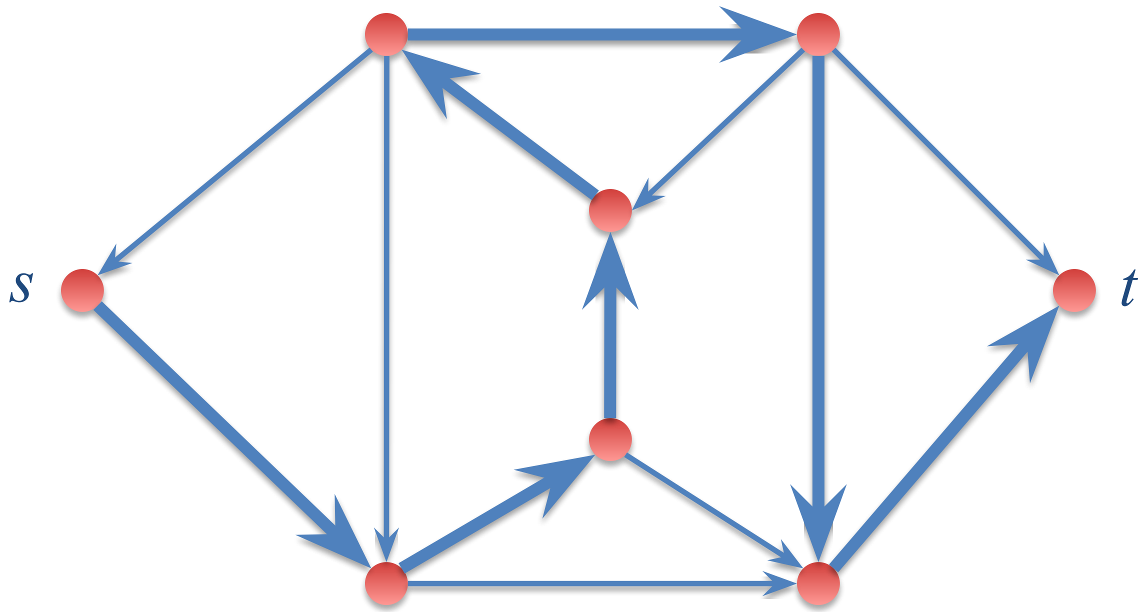


Hamiltonian paths

- *HAMPATH* = {<G,s,t> | ∃ Hamiltonian path from s to t}



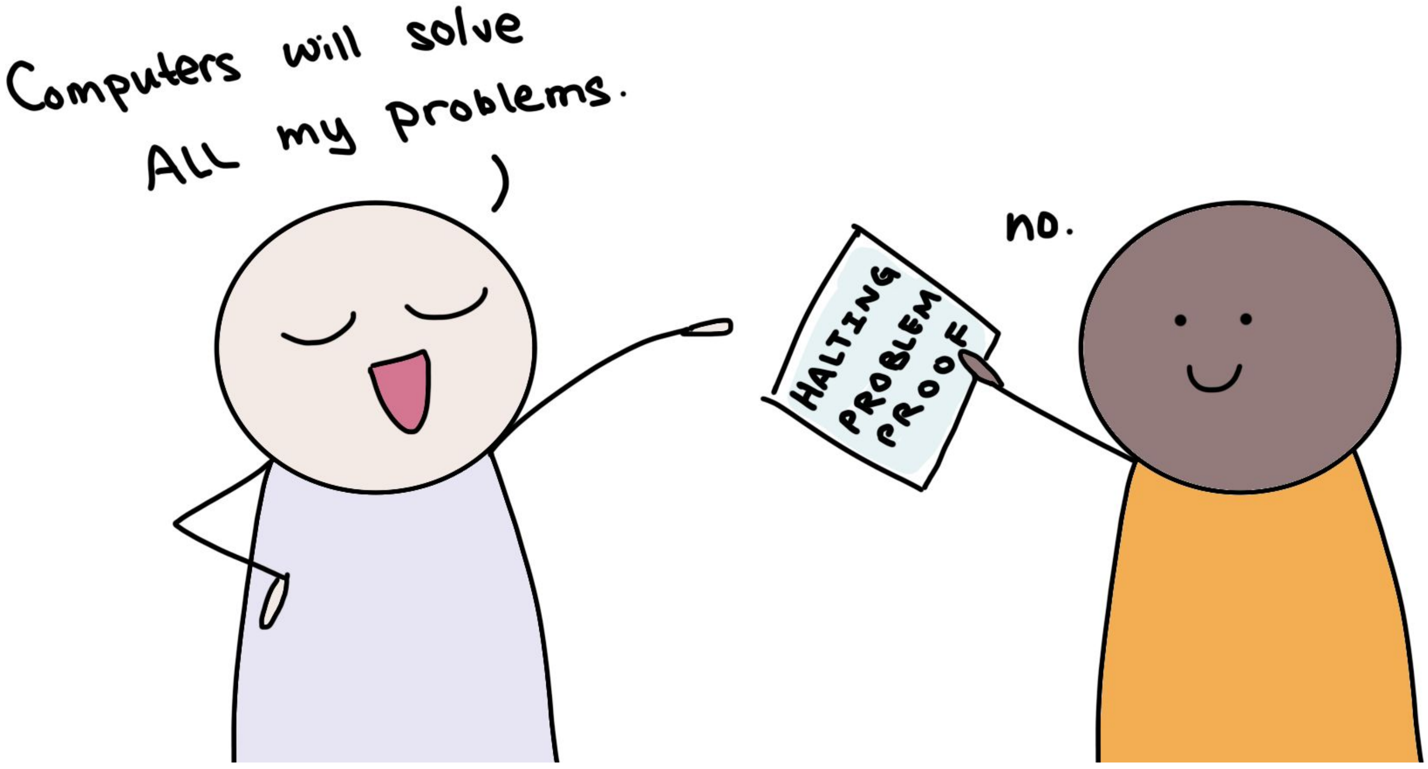
And…if that's not enough

- There are more than 3000 known NP-complete problems!

http://en.wikipedia.org/wiki/List_of_NP-complete_problems

Other types of complexity

- Space complexity
- Circuit complexity
- Descriptive complexity
- Randomized complexity
- Quantum complexity
- …



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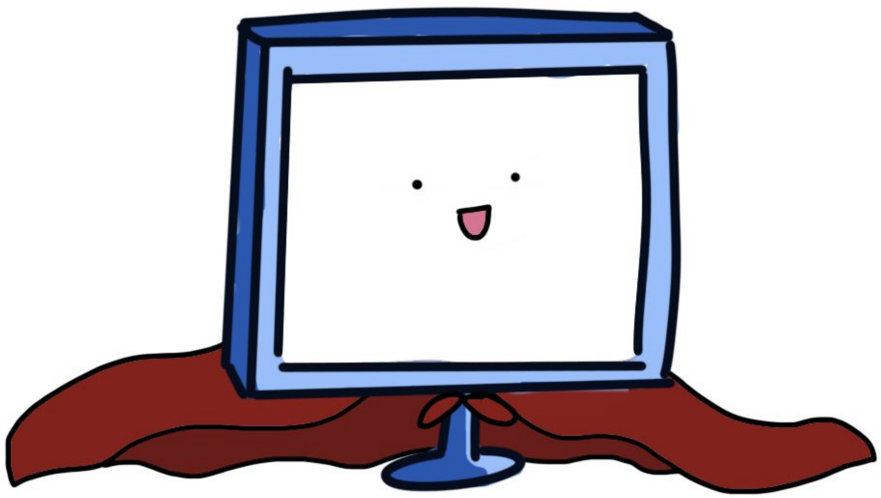
I can't solve the Halting Problem.



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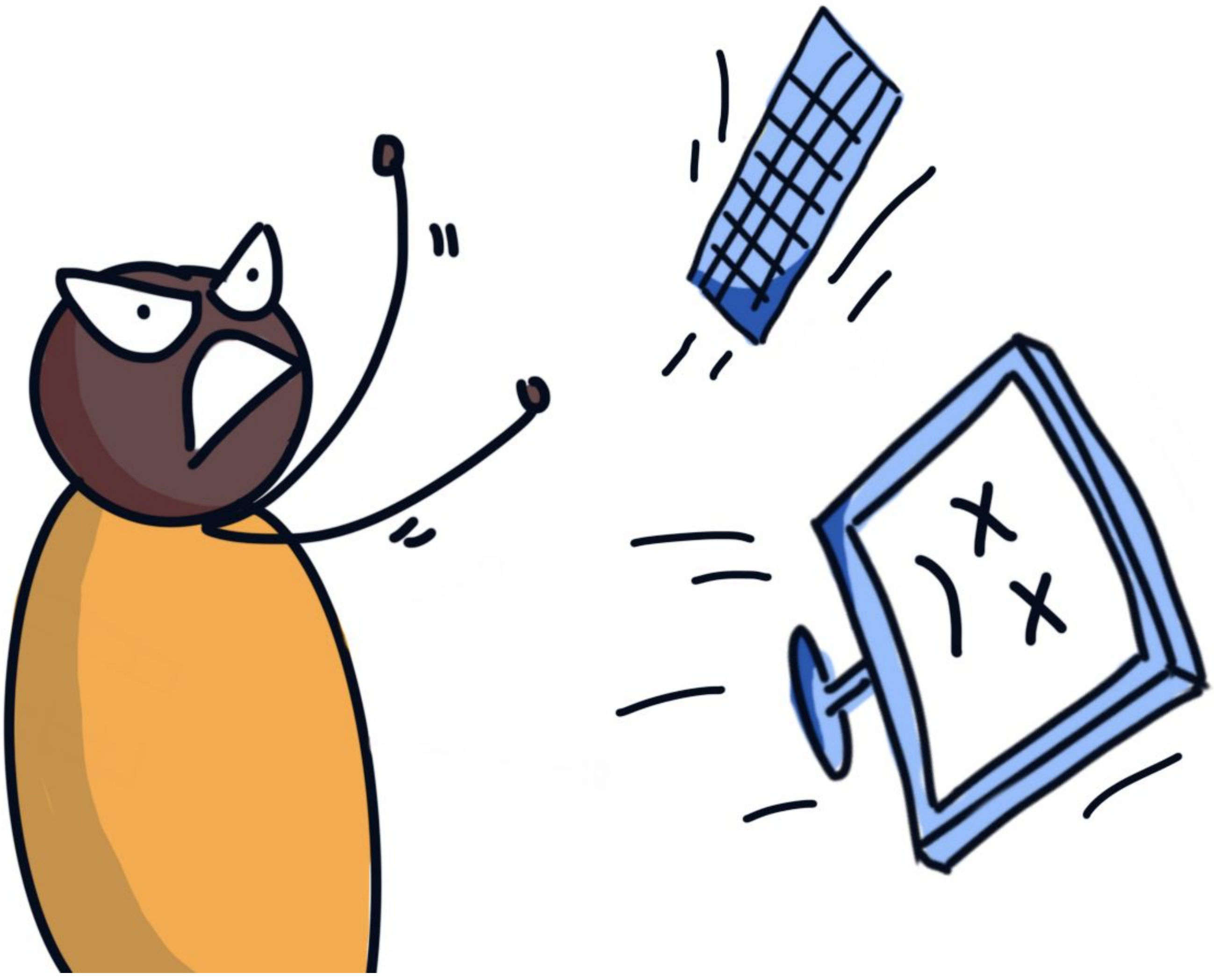

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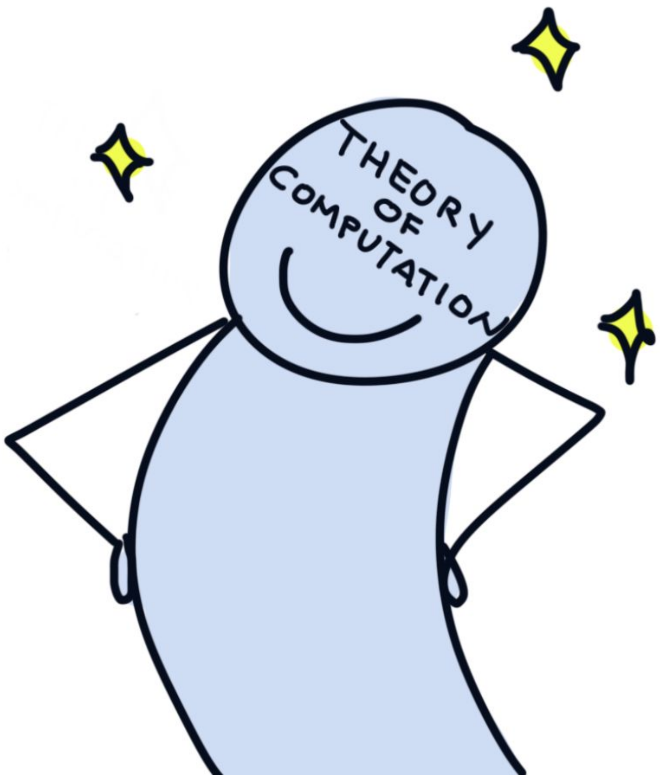

WHAT ARE THE FUNDAMENTAL CAPABILITIES & LIMITATIONS OF COMPUTING?

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