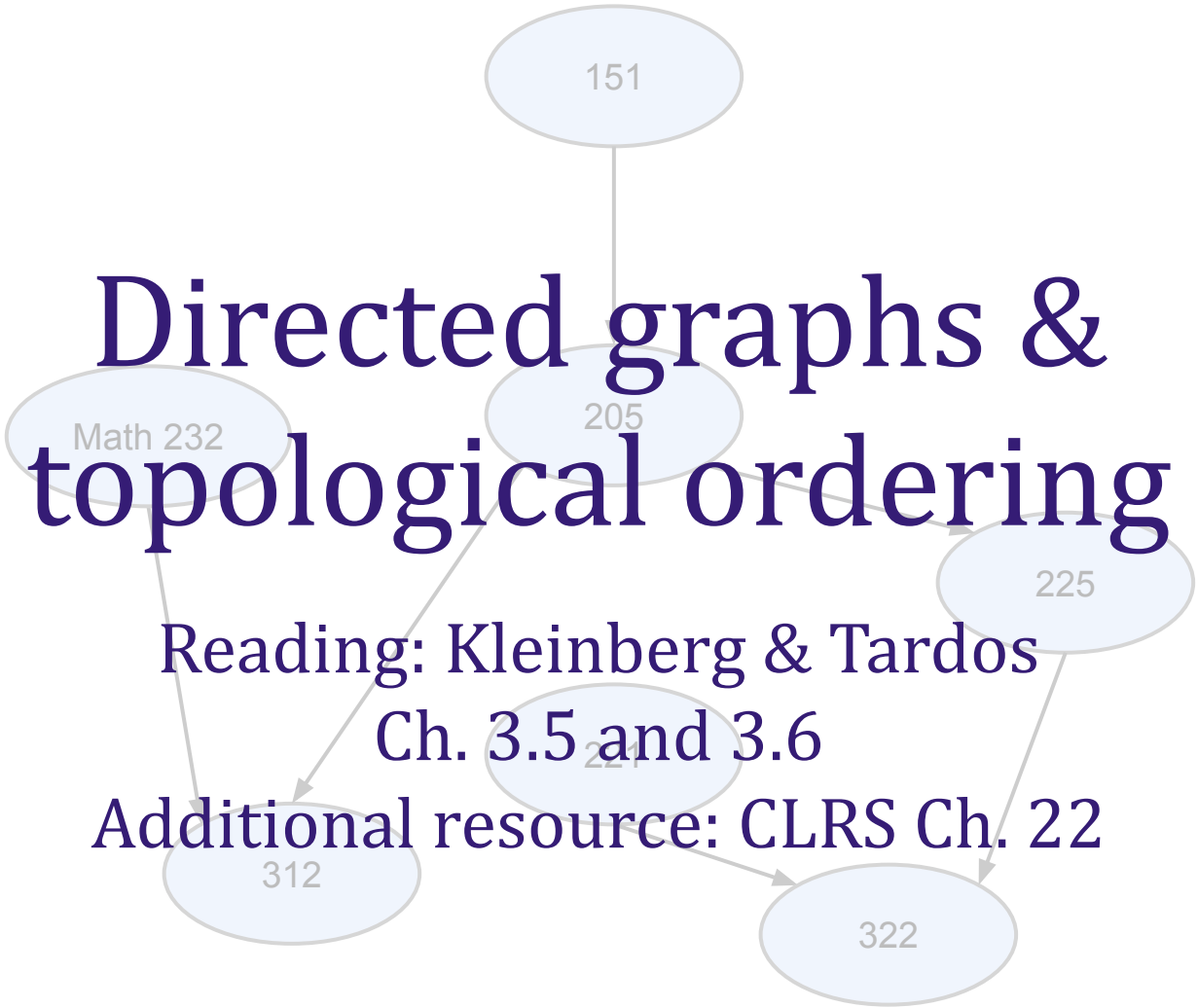
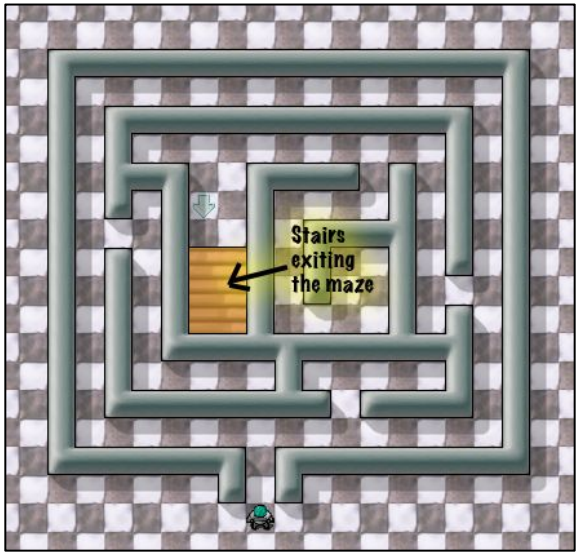


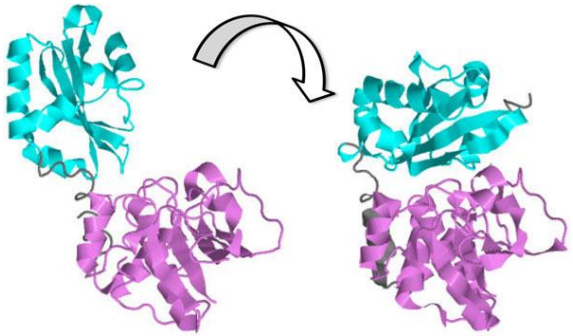
Directed graphs & topological ordering

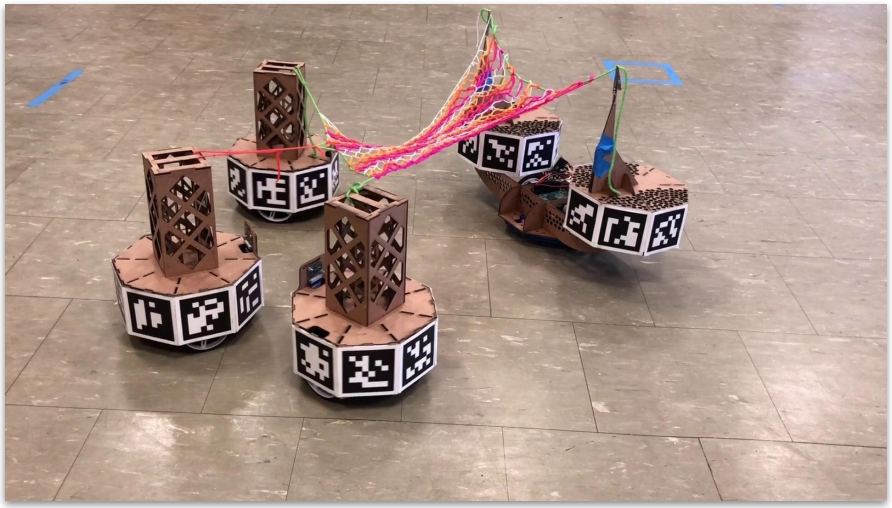
Reading: Kleinberg & Tardos
Ch. 3.5 and 3.6
Additional resource: CLRS Ch. 22

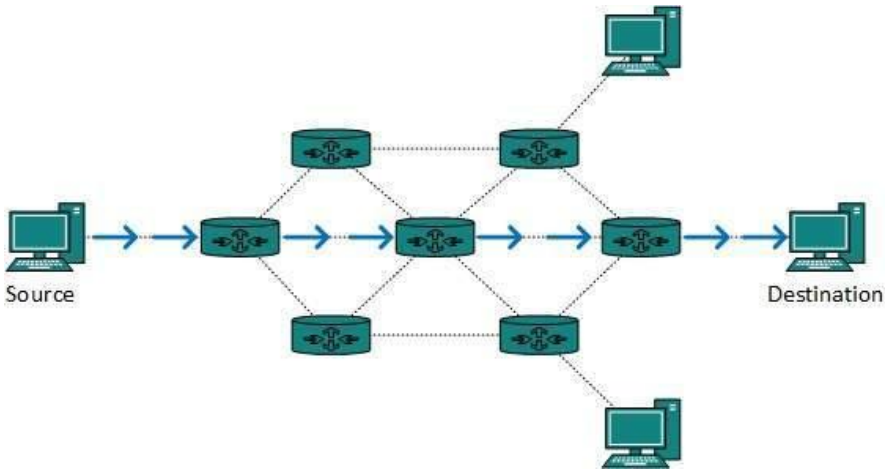
Recall…

What do these have in common?





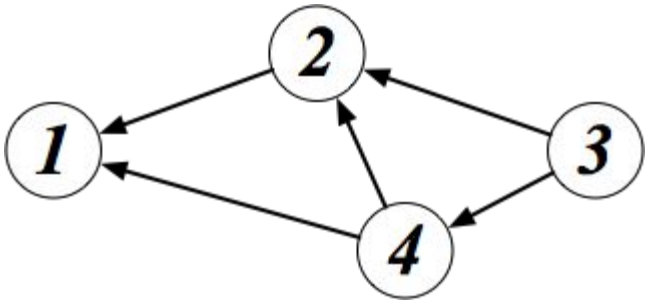




Directed graphs

Directed graph: definitions and terminology

- A ***graph*** $G = (V, E)$
  - V is a set of ***vertices***
  - $E \subseteq V \times V$ is the set of ***edges***

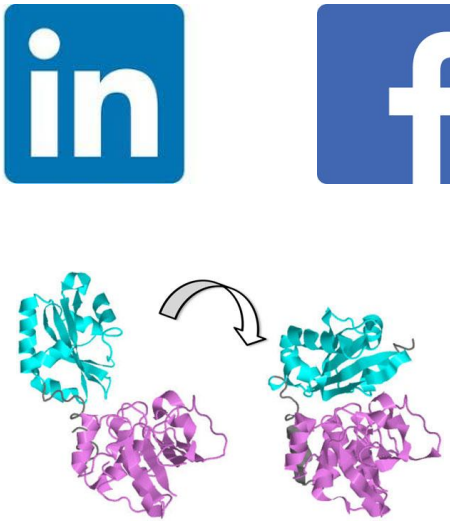


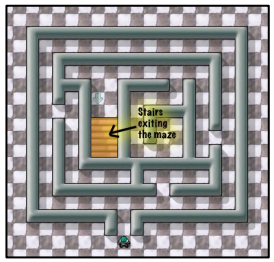
- ***directed edge*** $e = (u,v)$
  - u: ***source***
  - v: ***target***
  - **outgoing edge** from u
  - **Incoming edge** to v

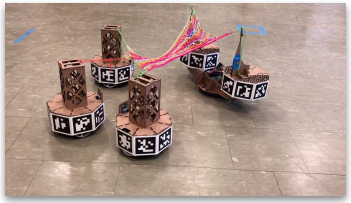
Directed graphs: examples

undirected

directed



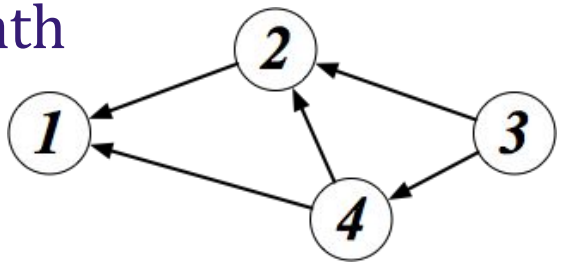




Directed graph: paths

- (Directed) path, cycle: same as for undirected

  $v_1, v_2, \ldots, v_{k-1}, v_k$
- What about connectivity?
  - Hmm… $s \rightarrow t$ path doesn't mean $t \rightarrow s$ path



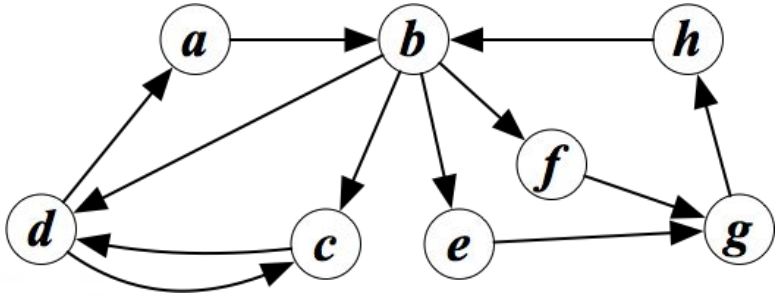
- t is ***reachable*** from s if there is a $s \rightarrow t$ path
- How to find all nodes reachable from u?
  - Example: all words that start with letter "a"
- What is the length of the shortest directed path from s to t?

Directed graph traversals

- DFS/BFS: same as for undirected
  - "neighbors": outgoing neighbors
- Adjacency lists
  - Outgoing neighbors
  - Incoming neighbors

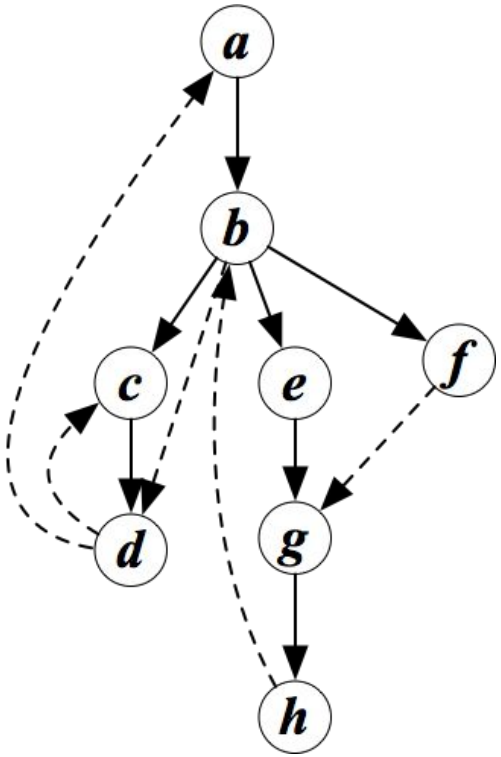
- Problem: Find all nodes s with $s \rightarrow t$ path?
- Algorithm: BFS following edges in reverse direction.

DFS on a directed graph



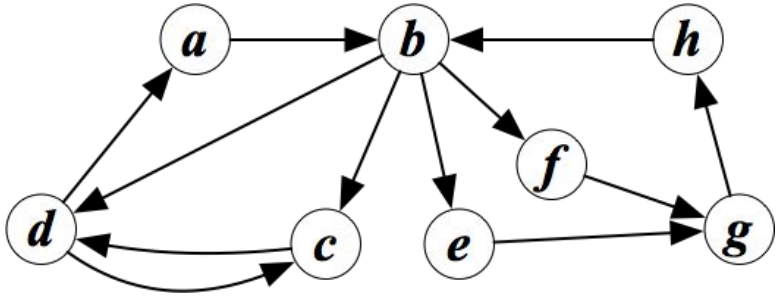

```
DFS(G)
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
 1  time = time + 1          // white vertex u has just been discovered
 2  u.d = time
 3  u.color = GRAY
 4  for each v ∈ G.Adj[u]    // explore edge (u, v)
 5      if v.color == WHITE
 6          v.π = u
 7          DFS-VISIT(G, v)
 8  u.color = BLACK          // blacken u; it is finished
 9  time = time + 1
10  u.f = time
```



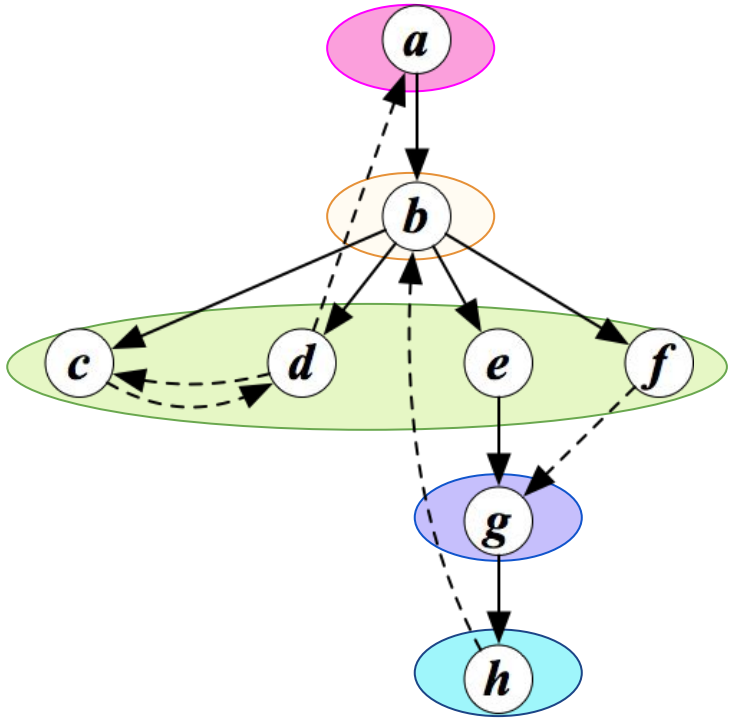
From CLRS

BFS on a directed graph



BFS(*G*,*v*):

```
for each vertex u in G.V
        u.visited = false
v.layer = 0
v.parent = null
Q = empty queue
Q.enqueue(v)
while !Q.empty()
        x = Q.dequeue()
        for each neighbor y of x
                if !(y.visited)
                        y.visited = true
                        y.parent = x
                        y.layer = x.layer + 1
                        Q.enqueue(y)
```

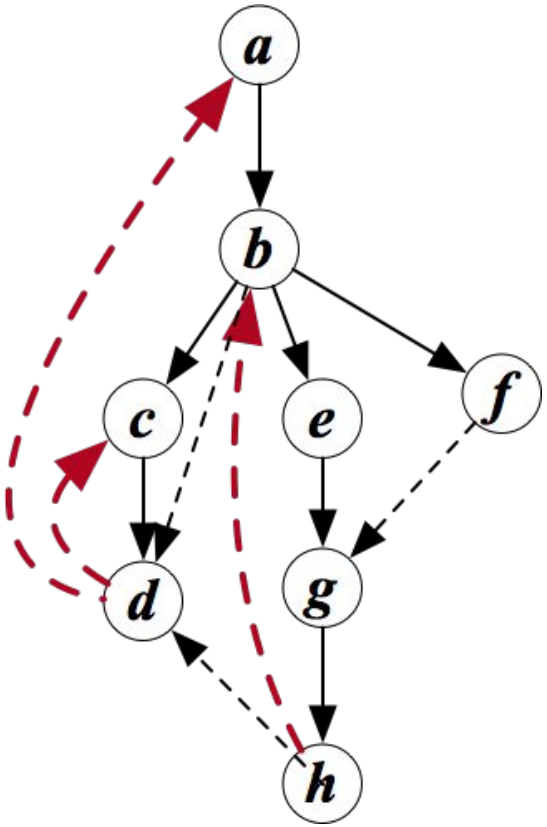


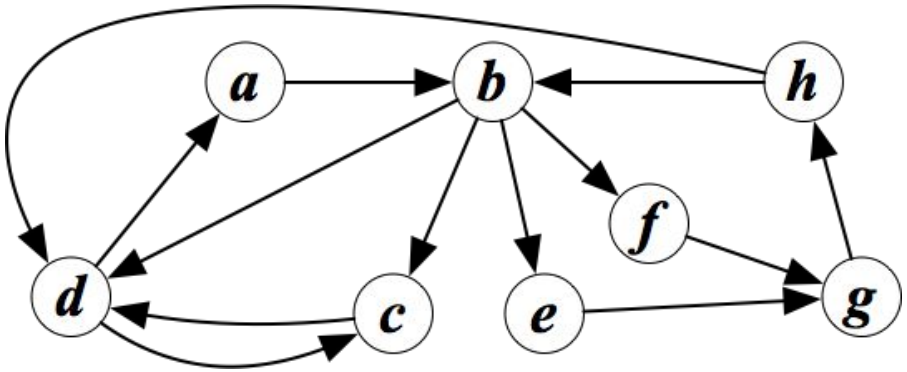
[CLRS]

Directed DFS: non-tree edges

3 types:

- **Back**: to an ancestor

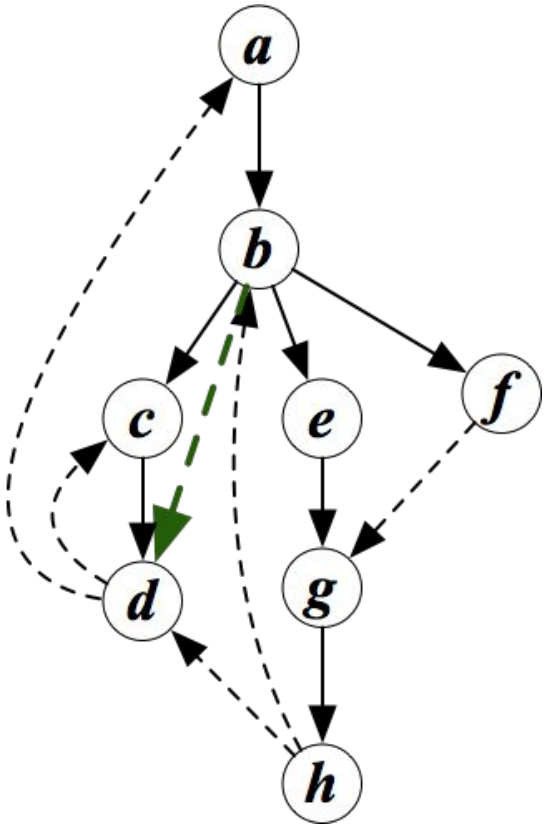


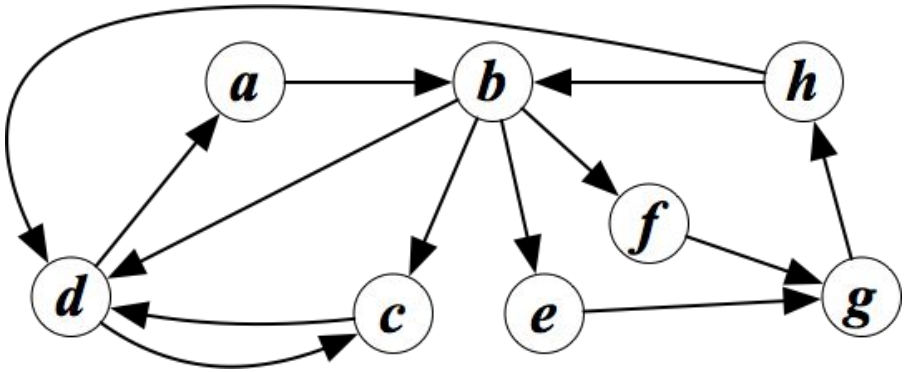


Directed DFS: non-tree edges

3 types:

- **Back**: to an ancestor
- **Forward**: to a descendant

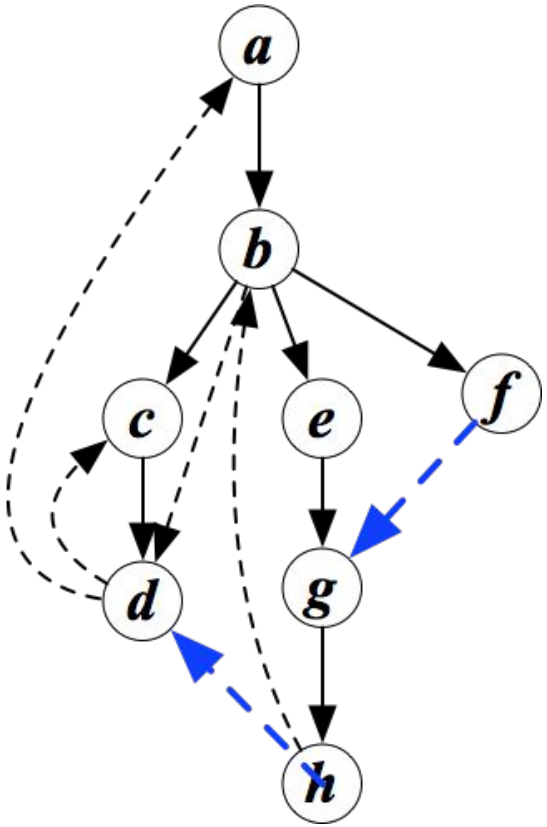


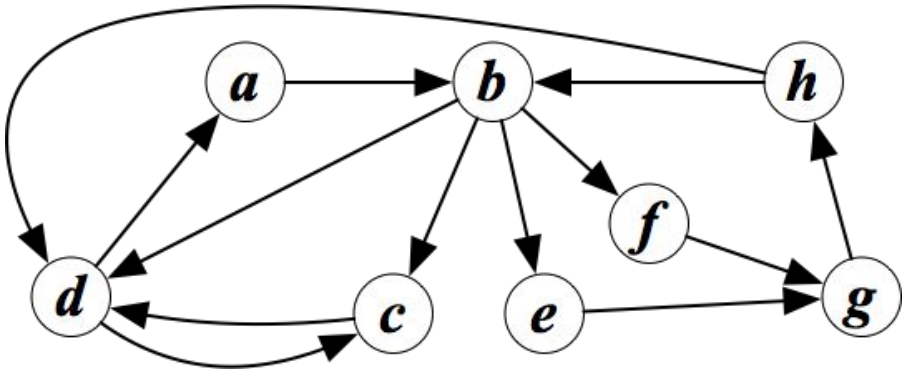


Directed DFS: non-tree edges

3 types:

- **Back**: to an ancestor
- **Forward**: to a descendant
- **Cross**: to neither an ancestor nor descendant

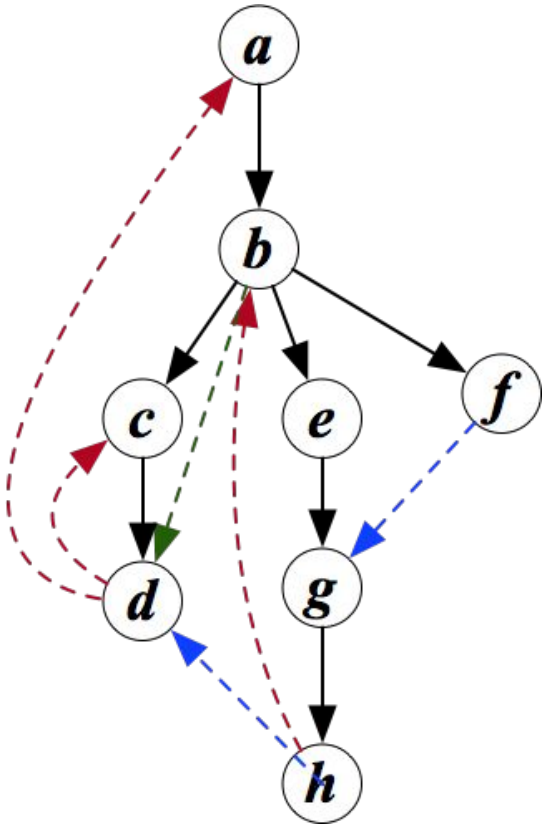


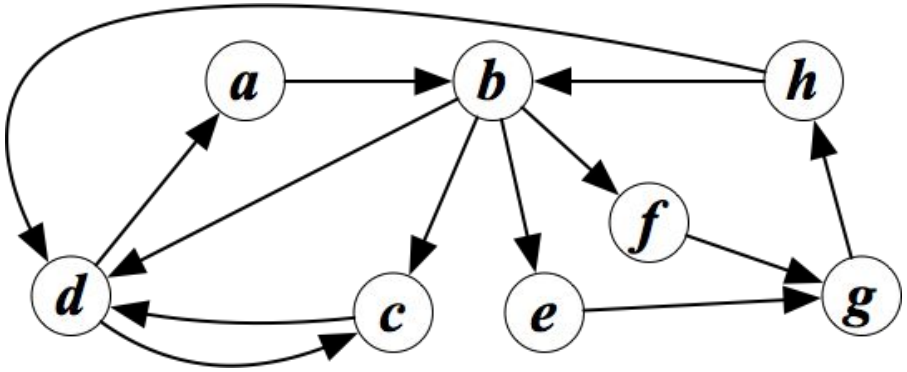


Directed DFS: non-tree edges

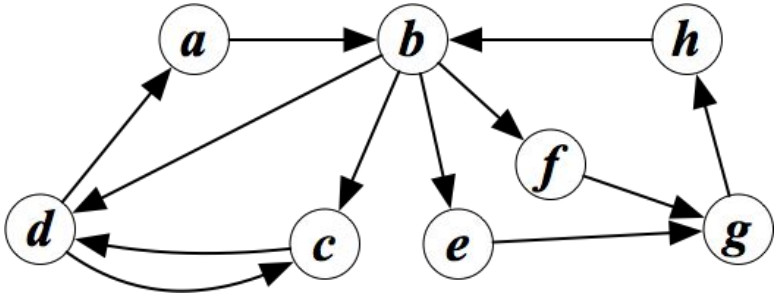
3 types:

- **Back**: to ancestor
- **Forward**: to descendant
- **Cross**: to neither ancestor nor descendant

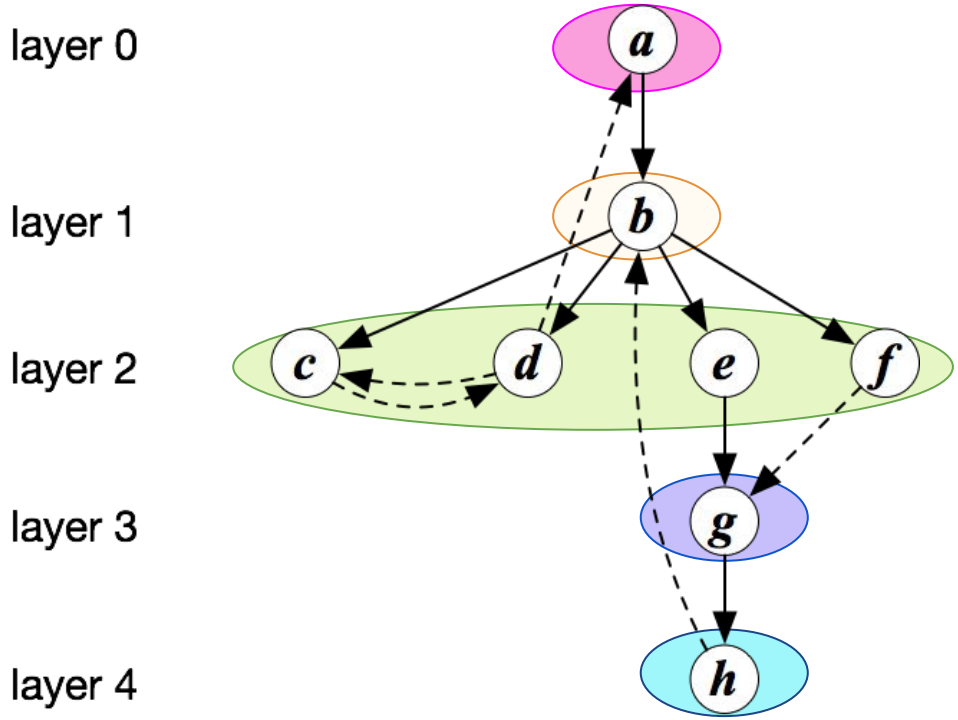




Directed BFS: non-tree edges

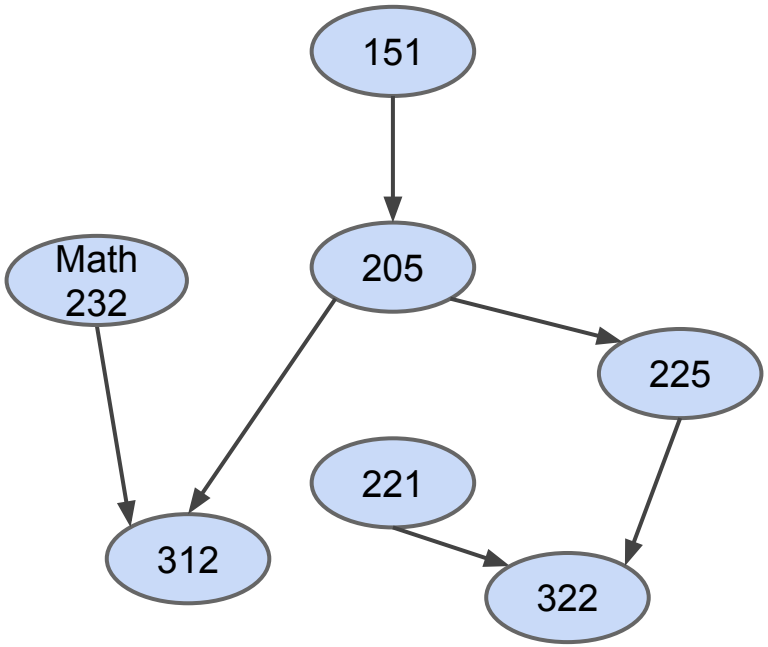


- One level down
  (tree or non-tree)
  - Why not > 1?
    same reason as undirected
    (would add to next level)
- Same level (non-tree)
- Any levels up (non-tree)



Directed acyclic graphs

- DAG: directed acyclic graph
  - No (directed) cycles
- Models dependencies

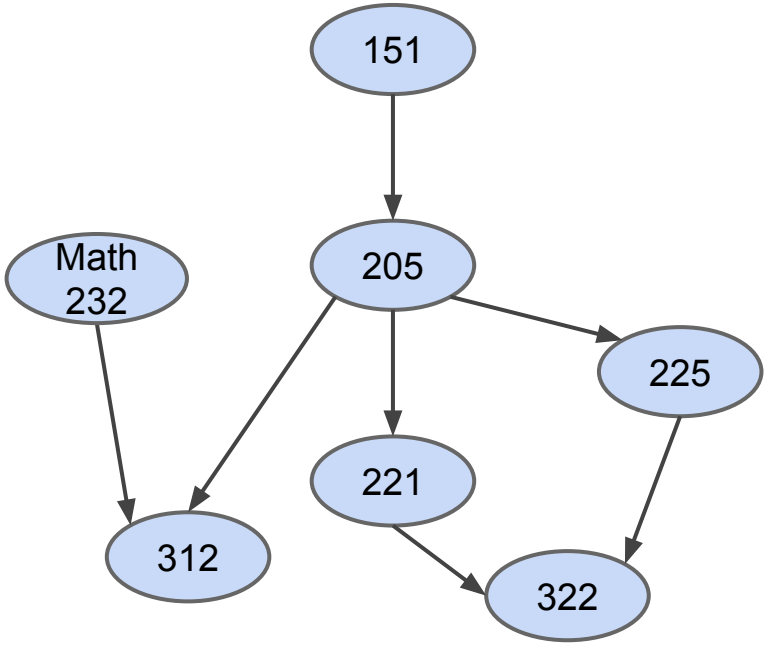


DAGs & topological sorting

- **topological ordering**: ordering of vertices such that all edges go "forward"

  $v_1, v_2, \ldots, v_n$

  - If $(v_i, v_j) \in E$, then $i < j$
- Example: ordering of classes so prerequisites satisfied
- Question: topological ordering of DAG always possible?



DAGs & topological sorting

- Claim: topological sorting => DAG
- Proof (sketch):

Assume G has topological sort $v_1, v_2, \ldots, v_n$ such that

if $(v_i, v_j) \in E$, then $i < j$

Suppose, for contradiction, G has cycle $u_1, u_2, \ldots, \ldots, u_k$.

By definition of cycle/path, there exist edges for each pair $u_a u_{a+1}$.

By definition of topological sort, only forward edges

=> u_i before u_j if $i < j$ in the sort

But, by definition of cycle, $u_1 = u_k$, giving the contradiction:

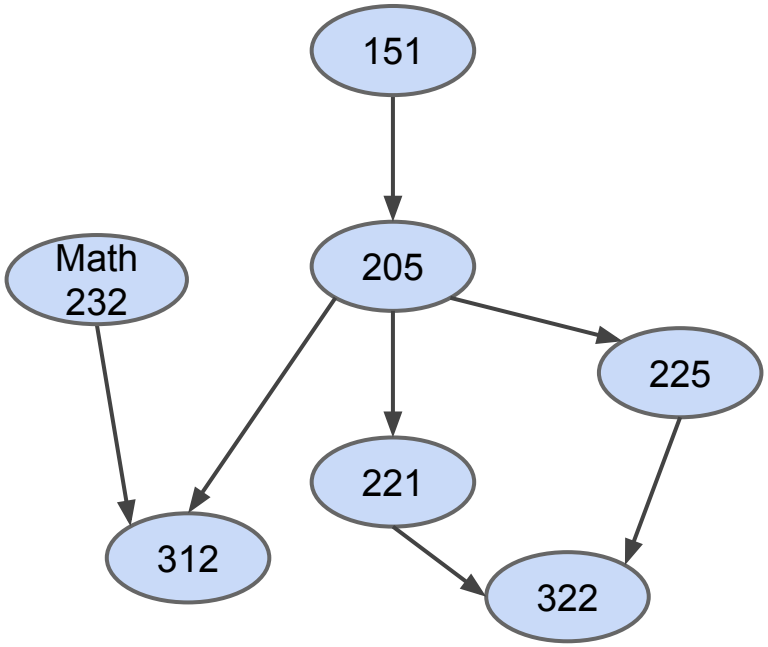
$u_{k-1} u_k$ forward edge, so u_{k-1} before u_k in the sort

$k = 1$, so u_{k-1} before u_1 in the sort

=> contradiction!

DAGs & topological sorting

- What about the other way?
- Does DAG => topological sorting?



DAGs & topological sorting

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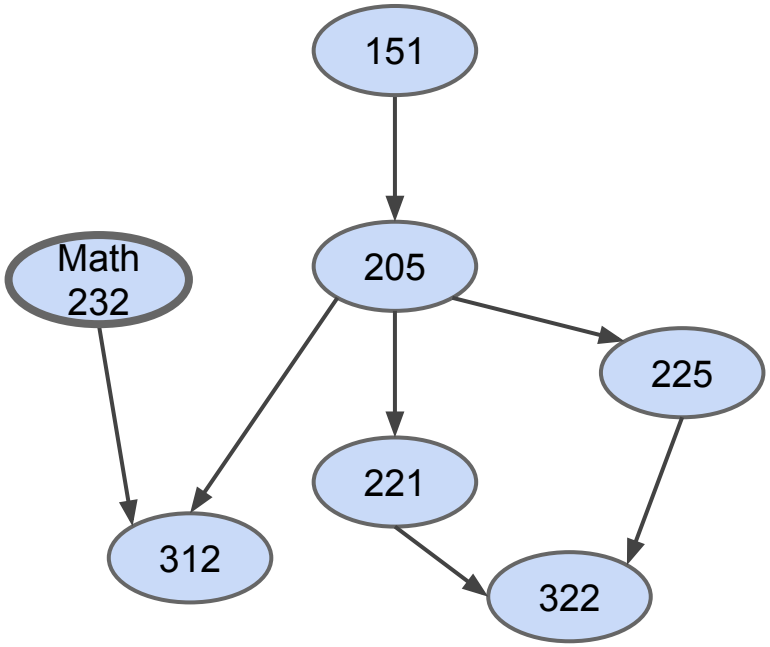

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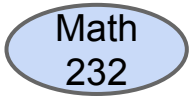

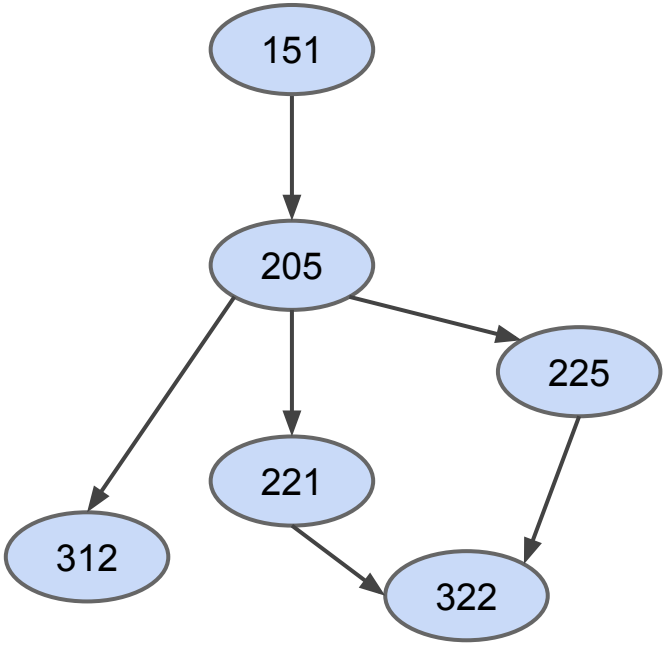

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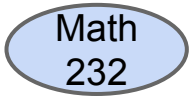

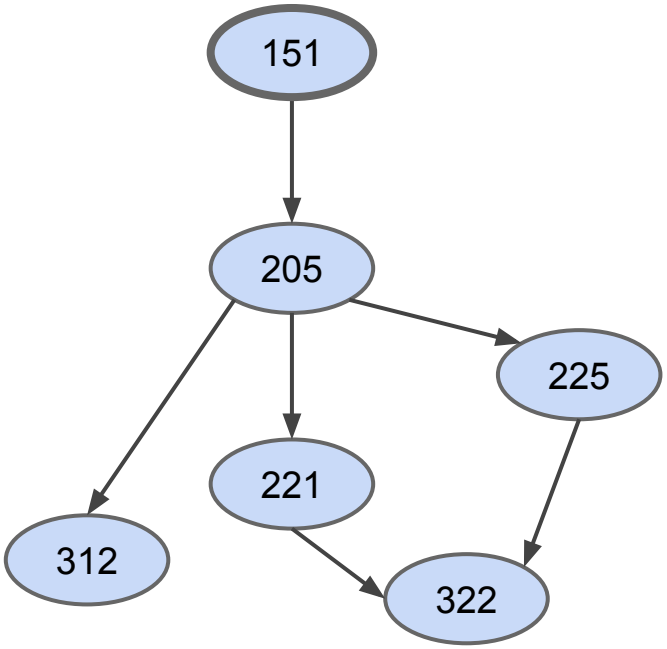

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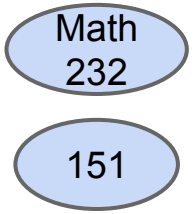

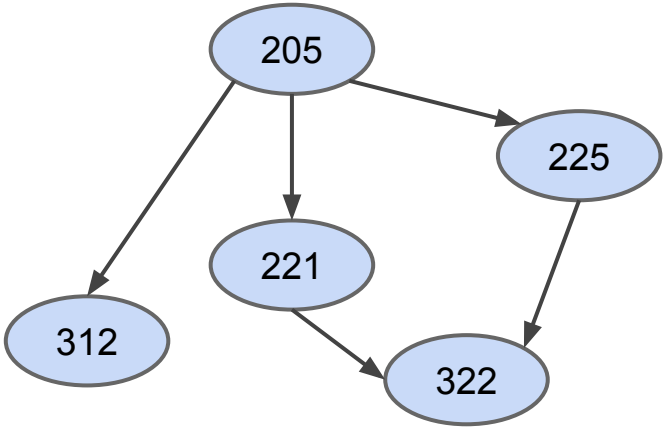

DAGs & topological sorting

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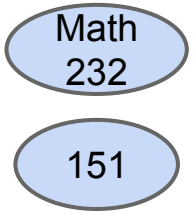

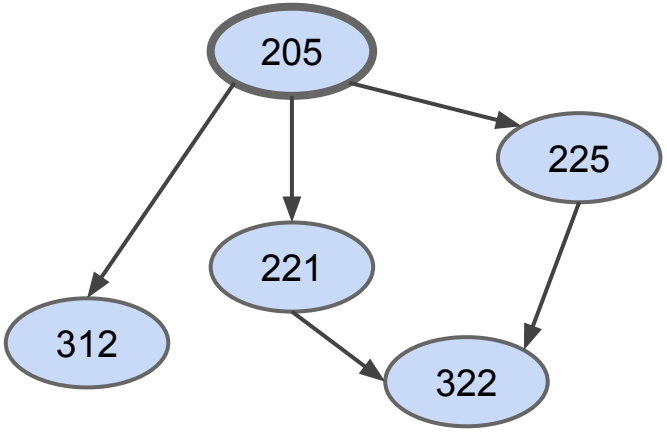

DAGs & topological sorting

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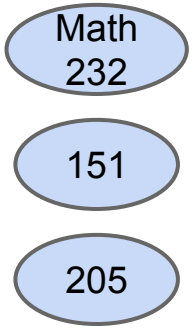

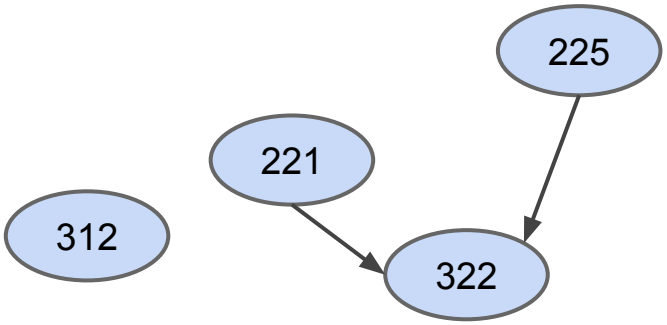

DAGs & topological sorting

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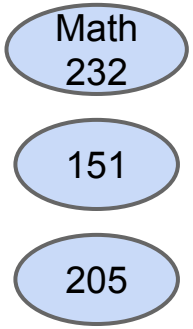

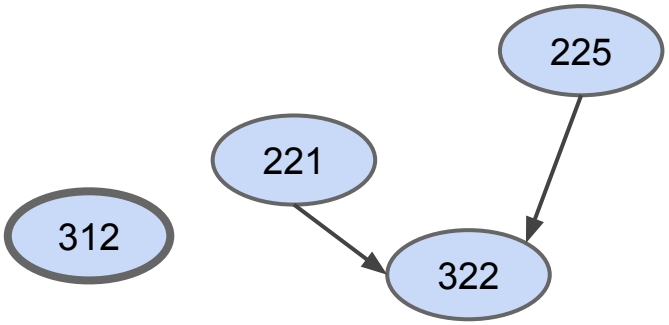

DAGs & topological sorting

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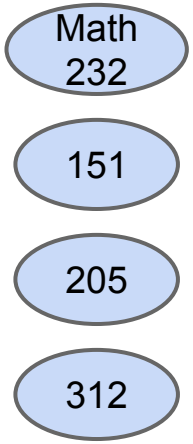

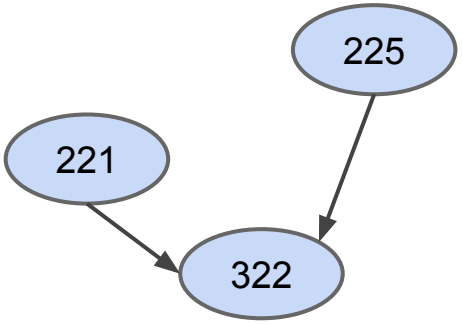

DAGs & topological sorting

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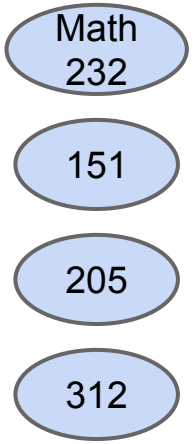
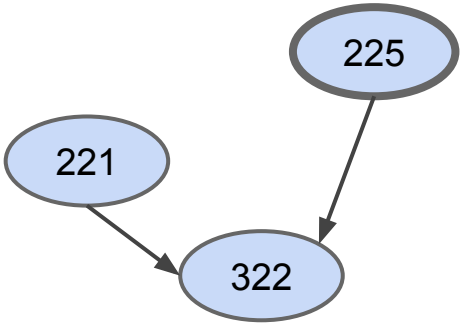
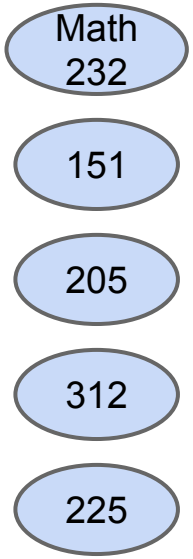
- Does DAG => topological sorting?

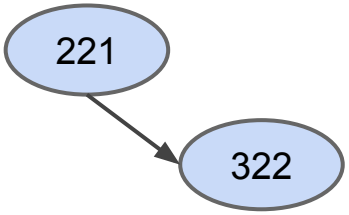




DAGs & topological sorting

- What about the other way?
- Does DAG => topological sorting?





DAGs & topological sorting

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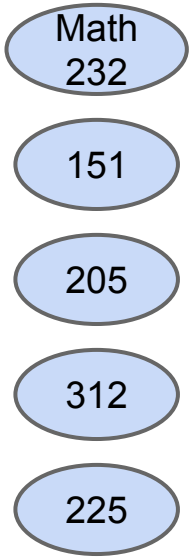

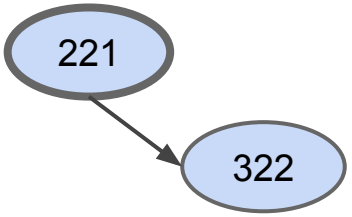

DAGs & topological sorting

- What about the other way?
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DAGs & topological sorting

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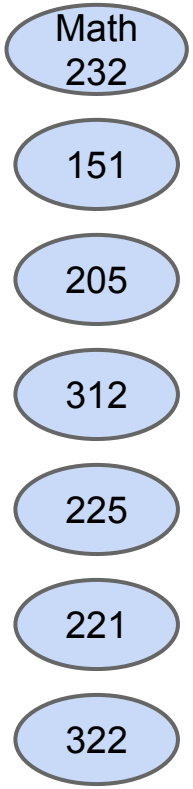


DAG => topological ordering?

Claim: Every DAG has a vertex with no incoming edges.

Proof: *(feels familiar…)*
Suppose, for a contradiction, all v in DAG have incoming edges.
Construct a sequence of $n+1$ vertices $v_1, v_2, \ldots, v_n, v_{n+1}$ such that v_{i+1} is an incoming neighbor to v_i.
There are $n+1$ variables ("nicknames") for n vertices.
By the Pigeonhole Principle, there must be some vertex assigned to >1 variable (some vertex has multiple "nicknames", some hole has multiple pigeons).
Thus, there is a cycle, giving the contradiction.

DAG <=> topological ordering

1. Every DAG has a vertex with no incoming edges.
2. Removing any vertex from DAG => still DAG.
3. Vertex with no incoming edges can be first in topological ordering.

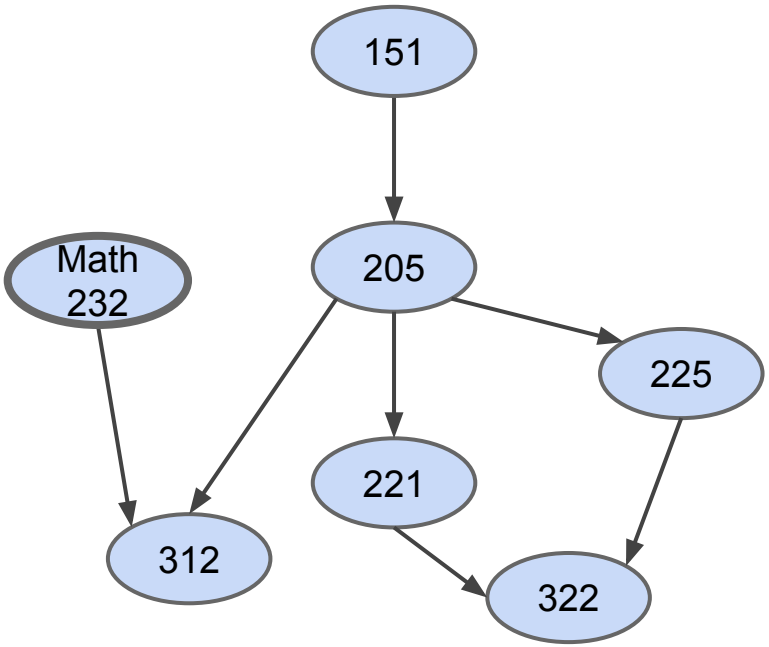
… by induction …

Theorem

G is a DAG

if and only if

G has a topological ordering.



Algorithm

topo-sort(DAG $G = (V, E)$):

    init sequence $S = ( )$
    while $V \neq \varnothing$
        find $v \in V$ with no incoming edges
        place v at end of S
        remove v and its (outgoing) edges from V
    return S

Correct?

Running time? $O(nm)$

Optimization for topological sorting

topo-sort(DAG $G = (V, E)$):

init sequence $S = ( )$

while $V \neq \varnothing$

find $v \in V$ with no incoming edges

place v at end of S

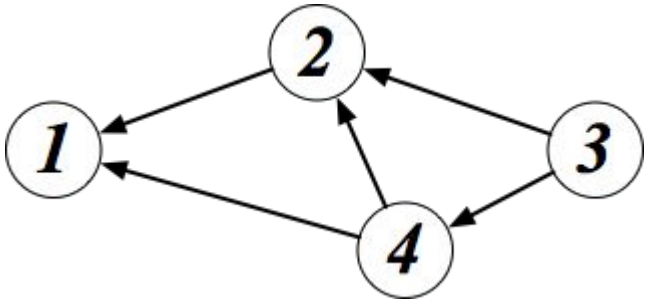
remove v and its (outgoing) edges from V

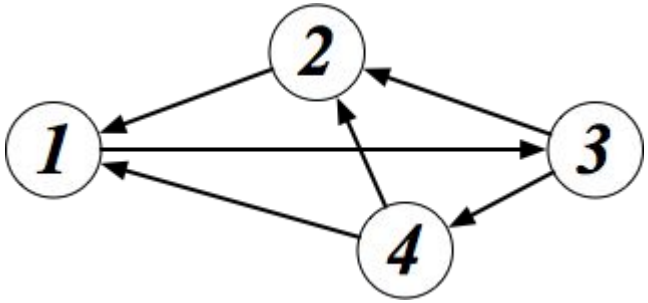
return S

- don't search every time for nodes w/o incoming edges
- keep and update incoming edge count for each node:
  - Initialization: O(m+n)
  - Update: constant-time
- keep set of nodes with no incoming edges
  - add node when its count becomes zero
- Running time: O(m+n)

Directed graph: connectivity

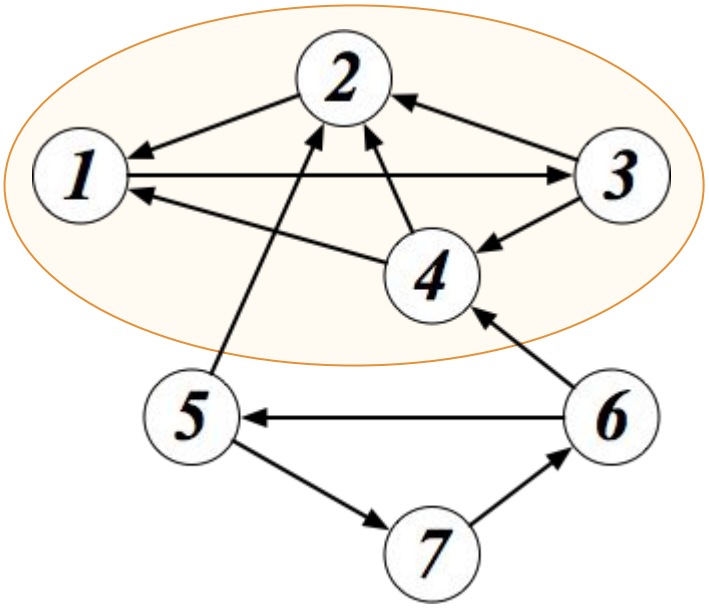
- graph is ***strongly connected***:
  exists path for every *ordered* pair of vertices



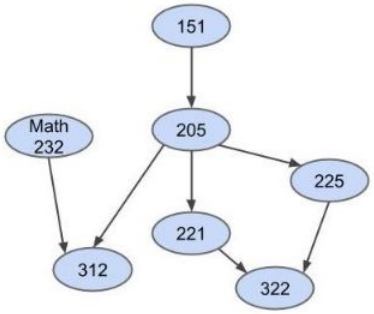


Directed graph: connectivity

- graph: ***strongly connected***:
  exists path for all *ordered* vertex pairs
- ***strongly connected component (SCC)***:
  maximal set of vertices with path for all *ordered* vertex pairs



Which is not a topological ordering of the given DAG?



MATH 232, 151, 205, 221, 225, 312, 322

151, 205, 221, MATH 232, 225, 312, 322

151, 205, 225, 221, MATH 232, 312, 322

They are all topological orderings.

I don't know.

Consider the graph H whose nodes are SCCs and there is an edge from C to D if any node in C has an edge to D. Which of the following is always true?

H is strongly connected

H has a cycle

H has at least $\frac{n}{2}$ nodes

H is a DAG

I don't know.