Mapping Reducibility

Sipser 5.3 (pages 206-210)

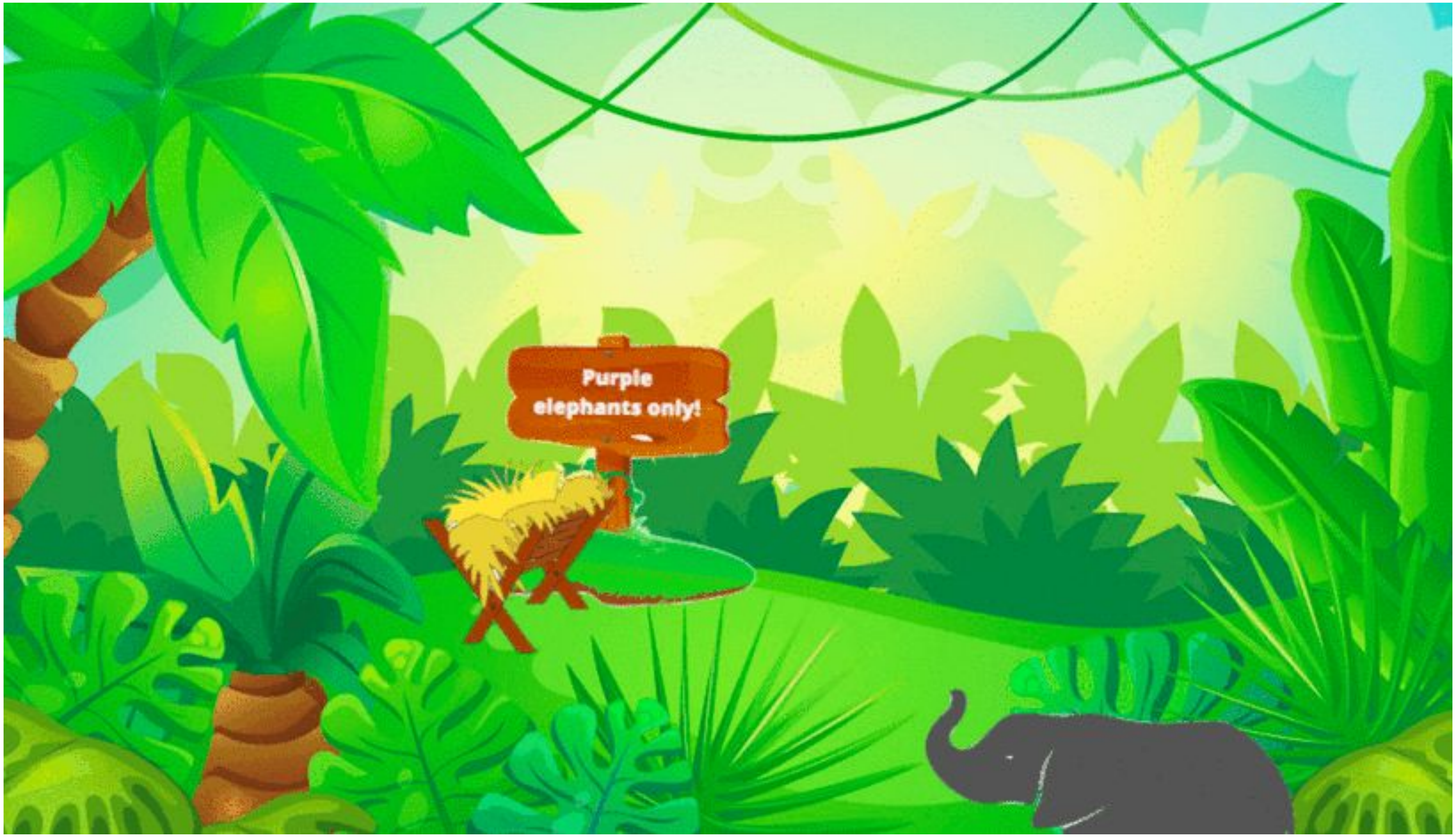


Image credit: Masha Lifshits (Fall 2019)

Computable functions

- Definition 5.17: A function $f{:}\Sigma^*\rightarrow\Sigma^*$ is a **computable function** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

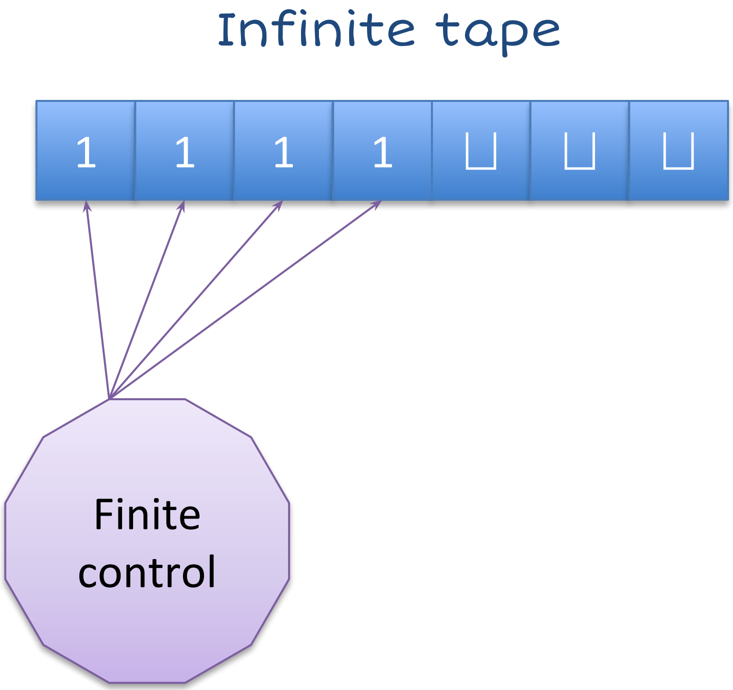
- Example: The increment function

$$inc{+}{+}{:}\{1\}^* \rightarrow \{1\}^*$$

is Turing-computable

Incrementing

- $inc{+}{+}: \{1\}^* \rightarrow \{1\}^*$



Transforming machines

F = "On input $<M>$:

1. Construct the machine

   M_∞ = "On input x:

   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, *loop*."

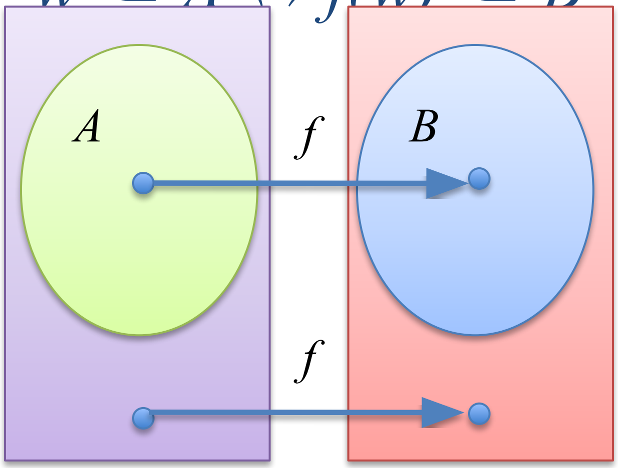
2. Output $<M_\infty>$."

Mapping reducibility

- Definition 5.20:

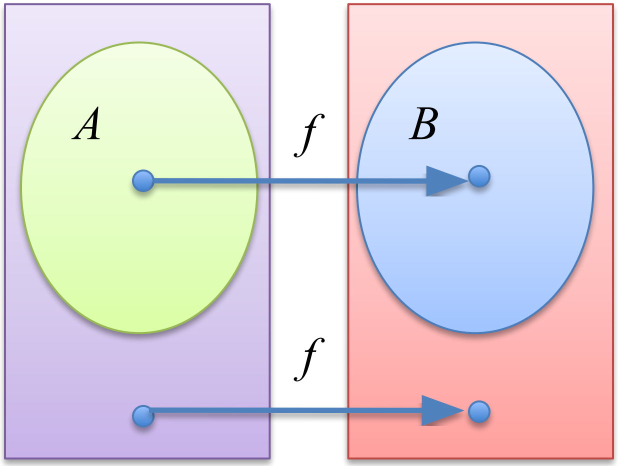
  Language A is *mapping reducible* to language B, written $A \leq_m B$, if there is a computable function $f{:}\Sigma^* \rightarrow \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$



Problem reduction

- Theorem 5.22:
  If $A \leq_m B$ and B is decidable,
  then A is decidable.



And… the contraposite

- Theorem 5.22:
  If $A \leq_m B$ and B is decidable,
  then A is decidable.

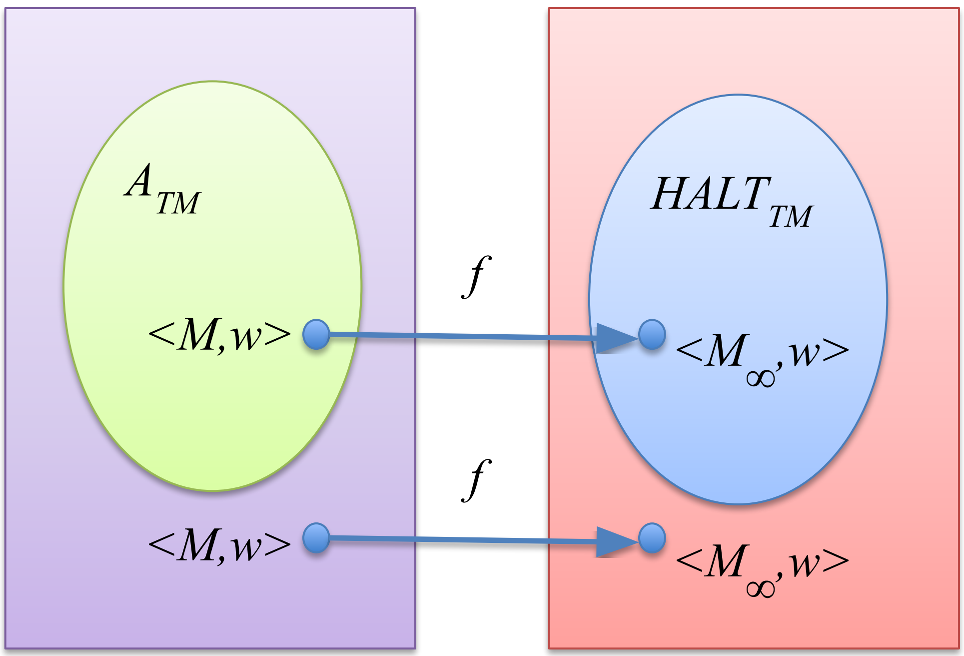
- Corollary 5.23:
  If $A \leq_m B$ and A is undecidable,
  then B is undecidable.

A familiar mapping reduction…

A_{TM} = {<M,w> | M is a TM and M accepts w}

$\leq_m$

$HALT_{TM}$ = {<M,w> | M is a TM & M halts on input w}



$A_{TM} \leq_m HALT_{TM}$

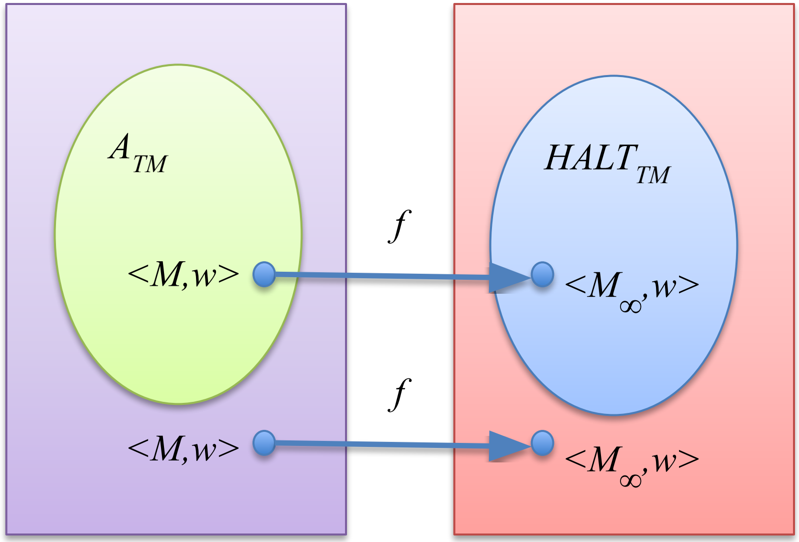
F = "On input $<M>$:

1. Construct the machine

   M_∞ = "On input x:

   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, *loop*."

2. Output $<M_\infty>$."

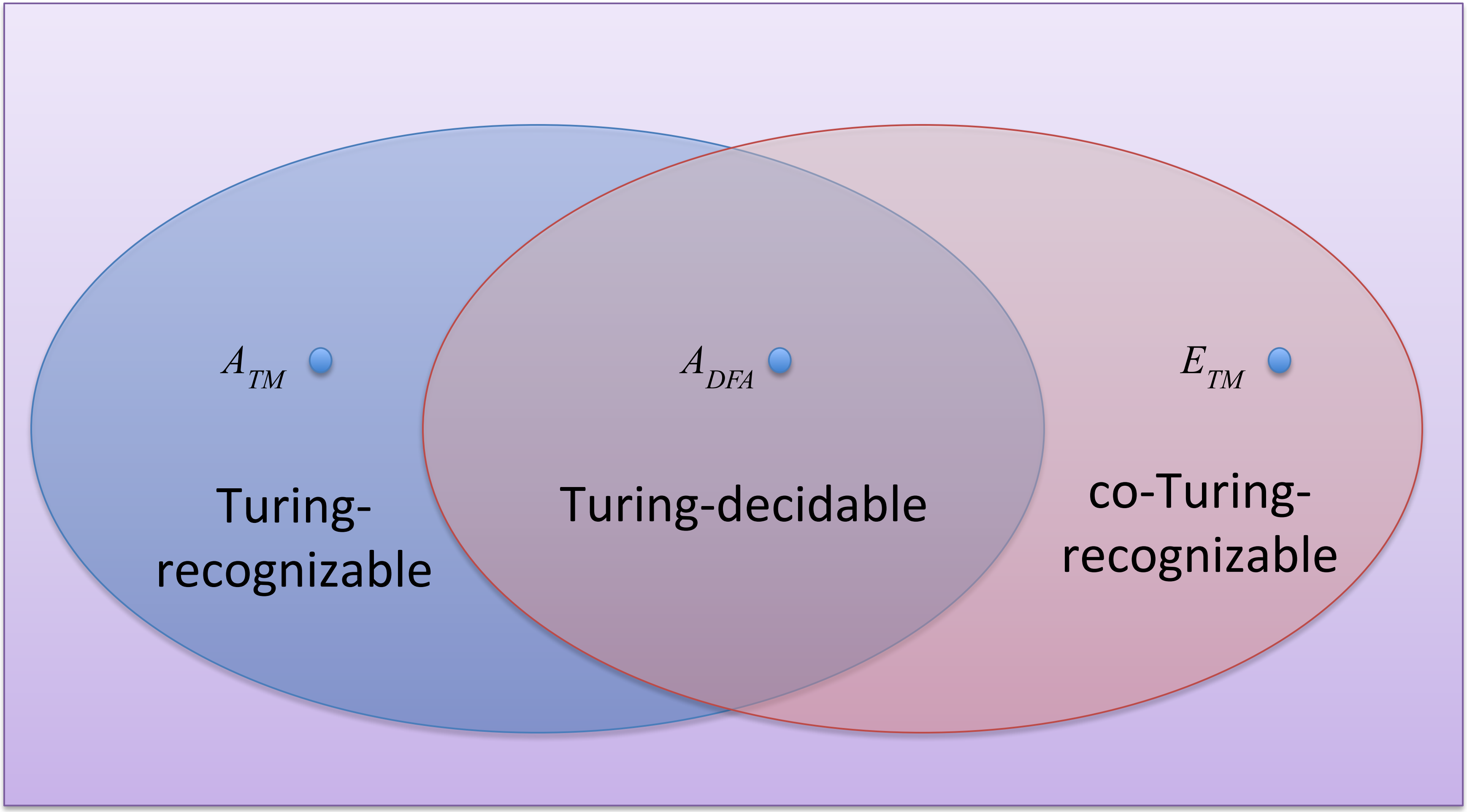


Similarly…

- Theorem 5.28:
  If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

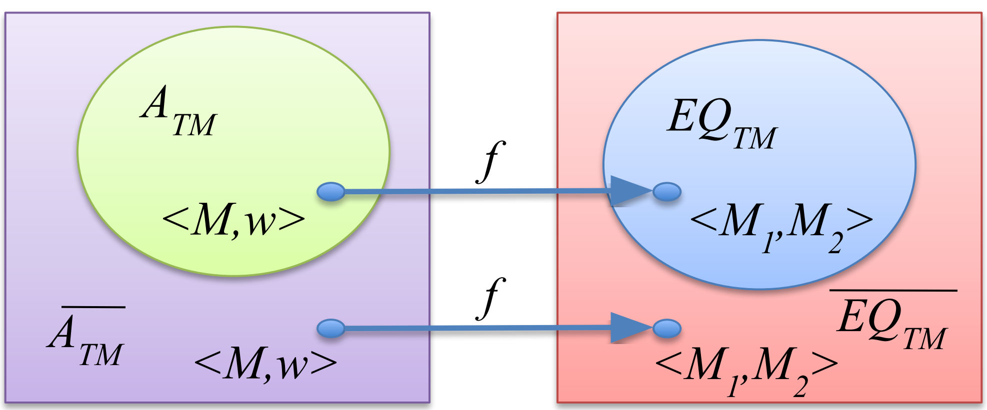
- Theorem 5.29:
  If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Solvable, half-solvable, hopeless



$EQ_{TM} = \{<M_1,M_2>|\ L(M_1)=L(M_2)\}$ is hopeless

- Theorem 5.30:
  EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable
- Proof:
  - What if we show $A_{TM} \leq_m EQ_{TM}$?



$A_{TM} \leq_m EQ_{TM}$

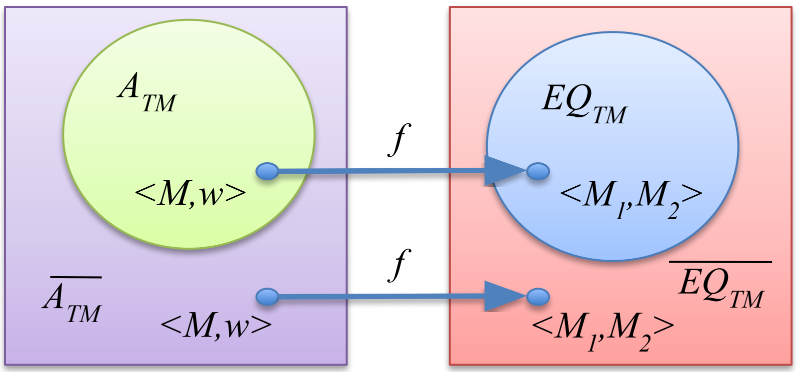
- G = "On input $<M,w>$:
  1. Construction the following two machines:

     M_1 = "On any input:

     1. *Accept.*"

     M_2 = "On any input:

     2. Run M on w.
     3. If it accepts, *accept*."
  2. Output $<M_1,M_2>$."



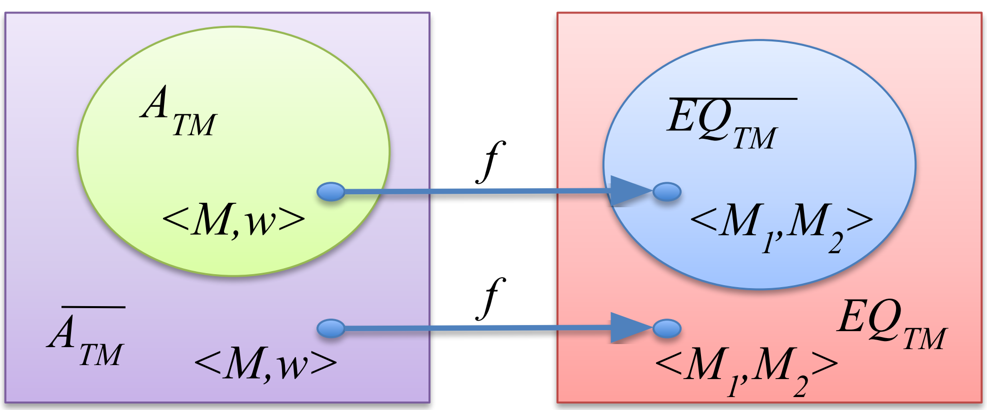
EQ_{TM} is not Turing-recognizable

- Theorem 5.30:

  EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable

- Proof:

  Show $A_{TM} \leq_m \overline{EQ_{TM}}$



$$A_{TM} \leq_m \overline{EQ_{TM}}$$

- G = "On input $<M,w>$:
  1. Construction the following two machines:

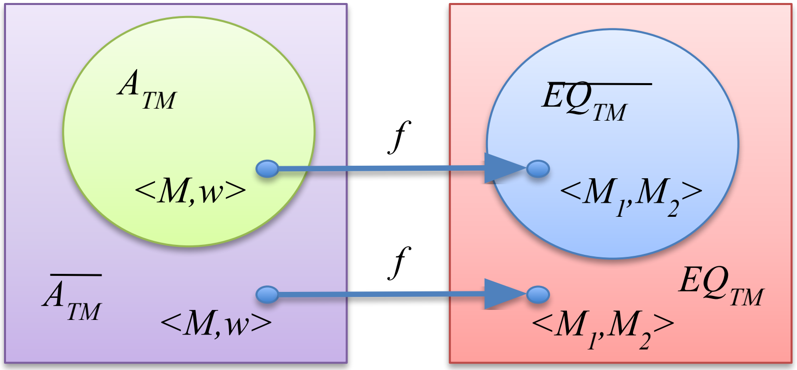
     M_1 = "On any input:

     1. *Reject*."

     M_2 = "On any input:

     2. Run M on w.
     3. If it accepts, *accept*."

  2. Output $<M_1,M_2>$."



Solvable, half-solvable, hopeless