Standard 09: Local Extrema

Local Extrema

It is time to, once again, extend an idea from calculus I into multi-variable.

1. A function $f(x,y)$ has a local (or relative) minimum at the point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some region around (a,b).
2. A function $f(x,y)$ has a local (or relative) maximum at the point (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some region around (a,b).

The words local and relative highlight the fact that the local minimum may not be the smallest value the function will ever take. We just require it to be the smallest in a small area around it. The same holds for the local maximum and it being the highest value. We will cover how to find the absolute extremas over both a bounded region and the entire domain.

In calculus I we found the local extrema by using the first derivative to find critical points ($f'(x)=0$ or DNE) then testing the surrounding points to determine if the critical point is a maximum or minimum. We need to have an analogous test for multivariable calculus.

critical points

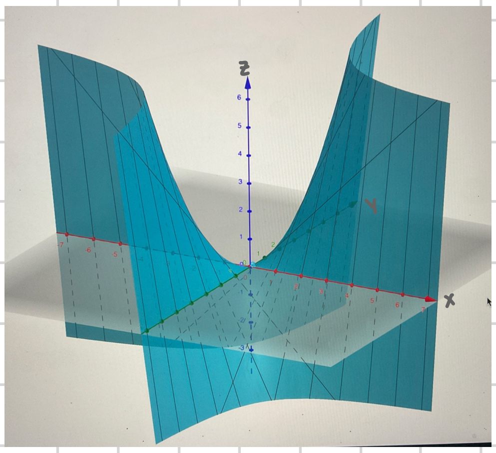
We extend the idea of a critical points to functions of two variables. The point (a,b) is a critical point of $f(x,y)$ if one of the following is true:

(i) $\nabla f(a,b) = \vec{0}$ i.e. $f_x = 0$ and $f_y = 0$

(ii) $f_x(a,b)$ and/or $f_y(a,b)$ doesn't exist

If the point (a,b) is a local extrema of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exists at (a,b) then (a,b) is a critical point of $f(x,y)$ and $\nabla f(a,b) = \vec{0}$. Note that this does not say every critical point is a local extrema, only that every local extrema is a critical point.

Consider the function $f(x,y) = xy$. The two first order partial derivatives are $f_x(x,y) = y$ and $f_y(x,y) = x$. The only critical point for the function is $(0,0)$. If we move in the positive x and positive y direction the function increases. The same thing happens in the negative x and negative y direction. If we move in any direction where the x and y differ in sign then the function decreases. No matter the region around the origin, there will be points larger and smaller than $f(0,0) = 0$. Therefore $(0,0)$ can not be a local minimum or maximum. We call this phenomenom a saddle point.



second derivative test

Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$. We have four cases:

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then there is a local minimum at (a,b).
2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then there is a local maximum at (a,b).
3. If $D(a,b) < 0$ then the point (a,b) is a saddle point.
4. If $D(a,b) = 0$ then the test is inconclusive. i.e. it could be a local extrema or a saddle point, we don't know

In calculus III the procedure is.

1. Find critical points.
   critical points appear when the gradient is 0, i.e. $\nabla f = \vec{0}$
2. Compute the determinant of the Hessian of f at critical points.
   $D(a,b) = \det(H(f(a,b))) = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2$
3. Compare $D(a,b)$ and $f_{xx}(a,b)$
   - if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum value.
   - if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum value.
   - if $D(a,b) < 0$ then (a,b) is a saddle point.
   - if $D(a,b) = 0$ then the test fails.

example. Find and classify the critical points of
$f(x,y) = x^2 + xy + y^2 - 4y = 0$

1. Find critical points (a,b) s.t. $\nabla f(a,b) = \vec{0}$
   $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x+y,\ x+2y-4 \rangle$
   $2x+y=0$ $\quad x = 4-2y$ $\quad y = 8/3$
   $x+2y-4=0$ $\quad 2(4-2y)+y=0$ $\quad x = -4/3$
2. Compute $D(x,y)$ at (a,b)
   $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$
   $D(a,b) = (2)(2) - (1)^2 = 3$
3. Compare $D(a,b)$ and $f_{xx}(a,b)$
   $D(a,b) = 3 > 0$ $\qquad f_{xx}(a,b) = 2 > 0$
   there is a minimum at $(-4/3, 8/3)$

Absolute Extrema

We want to optimize the function, or find max/min, on a given region. The maximum and minimum of the closed and bounded area must occur somewhere in the region or on its boundary. We know how to find local extrema using critical points, now we just need to test if this is in the region and if the boundary contains another extrema, e.g.  on $[0,1]$ 1 is the max

Process:

1. Critical Points in region
2. Critical Points on boundary
3. Plug into function to identify

example. Find the absolute extremas of $f(x,y) = 2x^2 - y^2 + 6y$ on $x^2 + y^2 \le 16$

1. Critical Points in region
   $f_x = 4x = 0 \Rightarrow x = 0$ $\qquad$ Check it is in region:
   $f_y = -2y + 6 = 0 \quad y = 3$ $\qquad (0)^2 + (3)^2 = 9 \le 16$ ✓
2. Critical Points on boundary
   $x^2 + y^2 = 16 \Rightarrow x^2 = 16 - y^2$ $\quad$ plug into $f(x,y)$ to make $g(y)$
   $g(y) = 2(16 - y^2) - y^2 + 6y = 32 - 3y^2 + 6y$ $\quad$ use calc 1 to find min/max
   $g'(x,y) = -6y + 6 = 0 \Rightarrow y = 1$ $\quad$ (we must consider endpoints $|y| = 4$)
   solve for x: if $y = 1$, $x^2 = 16 - 1 = 15$ $\quad x = \pm\sqrt{15}$
   if $y = -4$, $x^2 = 16 - 16 = 0$ $\quad x = 0$
   if $y = 4$, $x^2 = 16 - 16 = 0$ $\quad x = 0$
3. Test all points
   $f(0,3) = 9$
   $f(\sqrt{15},1) = 35$ maximum
   $f(\sqrt{15},1) = 35$ maximum
   $f(0,-4) = -40$ minimum
   $f(0,4) = 8$