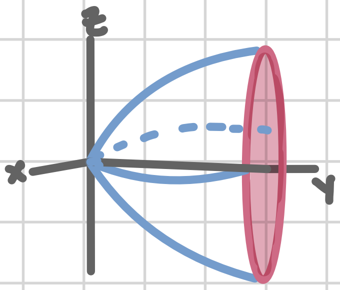
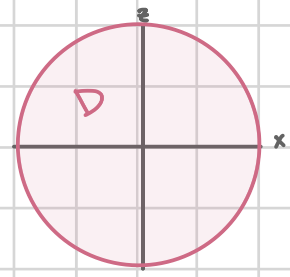
Standard 14: Triple Integrals - Cylindrical & Spherical Coo

Recall the last example from rectangular coordinates:

example. Evaluate $\iiint_E \sqrt{3x^2+3z^2}\,dV$ where E is the solid bounded by $y=2x^2+2z^2$ and the plane $y=8$.





$D=\{(x,y) \mid x^2+y^2 \leq 4\}$

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$$
$$= \int_{-4}^{4}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\,dz\,dx$$

The projection onto the xz-plane, the region D, is a disk. The equation of the disk comes from $8=2x^2+2z^2$. This region can be best described using something like polar coordinates for the xy-plane. Instead of the usual $x=r\cos\theta$ and $y=r\sin\theta$, we use $x=r\cos\theta$ and $z=r\sin\theta$. Thus The region $D=\{(x,y) \mid x^2+y^2 \leq 4\}$ becomes $D=\{(r,\theta) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2\}$. We also have to reflect these changes in the integral:

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \int_{-4}^{4}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2x^2+z^2}^{8} \sqrt{3x^2+3z^2}\,dy\,dz\,dx$$
$$= \int_0^2\int_0^{2\pi}\int_{2r}^{8} \sqrt{3r^2}\cdot r\,dy\,d\theta\,dr$$

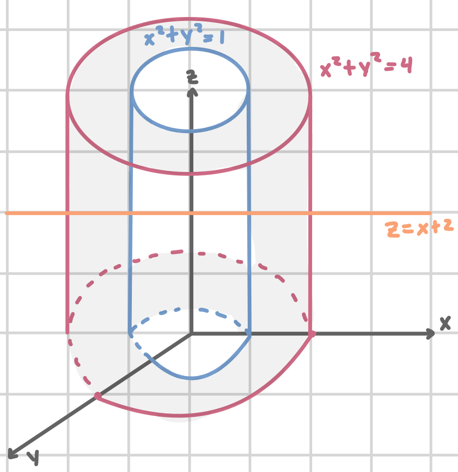
$dA = r\,dr\,d\theta$

cylindrical coordinates

This set of conversions is called cylindrical coordinates and is an extension of polar coordinates into three dimensions. In the example above we used the conversions for E's in which the D is in the xz-plane, there is a different set for each type of E we have seen:

(i) $E=\{(x,y,z) \mid (x,y)\in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$	(ii) $E=\{(x,y,z) \mid (y,z)\in D,\ u_1(y,z) \leq x \leq u_2(y,z)\}$	(iii) $E=\{(x,y,z) \mid (x,z)\in D,\ u_1(x,z) \leq y \leq u_2(x,z)\}$
$x=r\cos\theta$	$y=r\cos\theta$	$x=r\cos\theta$
$y=r\sin\theta$	$z=r\sin\theta$	$z=r\sin\theta$
$z=z$	$x=x$	$y=y$
$x^2+y^2=r^2$	$y^2+z^2=r^2$	$x^2+z^2=r^2$
$dV=r\,dz\,dr\,d\theta$	$dV=r\,dx\,dr\,d\theta$	$dV=r\,dy\,dr\,d\theta$

example. Set up $\iiint_E y\,dV$ where E is the region that lies below the plane $z=x+2$, above the xy-plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.



$x=r\cos\theta$
$y=r\sin\theta$
$z=z$

$0 \leq z \leq x+2$
becomes
$0 \leq z \leq r\cos\theta+2$

$D=\{(x,y) \mid 1 \leq x^2+y^2 \leq 4\}$
$=\{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\sin\theta\cdot r\,dz\,dr\,d\theta$$

example. Convert the following integral into cylindrical coordinates: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{-x}^{x^2+y^2} xyz\,dz\,dy\,dx$.

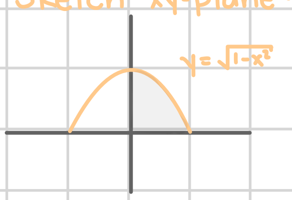
Rectangular coordinate bounds:

$0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2},\ -x \leq z \leq x^2+y^2$

Cylindrical conversion:

$x=r\cos\theta,\ y=r\sin\theta,\ z=z,\ dV=r\,dz\,dr\,d\theta$

Sketch xy-plane:



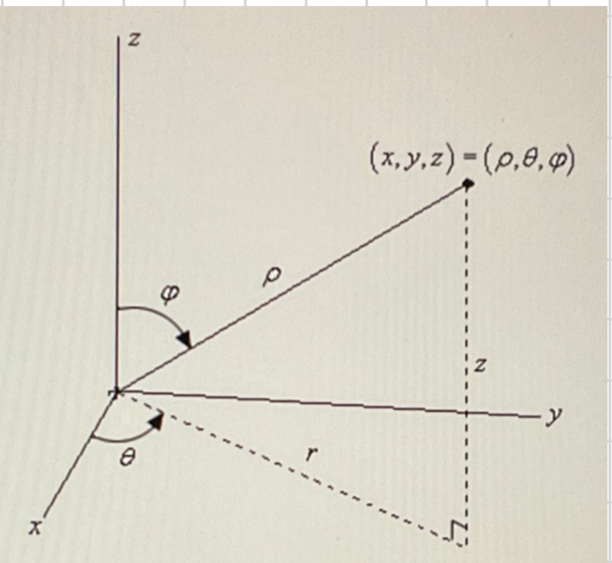
Rewrite bounds:

$0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2},\ -r\cos\theta \leq z \leq r^2$

$$\int_0^{\pi/2}\int_0^1\int_{-r\cos\theta}^{r^2} (r\cos\theta)(r\sin\theta)(z)\cdot r\,dz\,dr\,d\theta$$

spherical coordinates

There is another extension of polar coordinates into three dimensions; it is given by rotating polar coordinates. Spherical coordinates are (ρ, θ, φ) where ρ is the distance from the origin, θ is the angle made with the positive x-axis in the xy-plane, and φ is the angle made with the positive z-axis. Here is a visual of the conversion from rectangular to spherical:



$x = \rho \sin(\varphi)\cos(\theta)$

$y = \rho \sin(\varphi)\sin(\theta)$

$z = \rho \cos(\varphi)$

$x^2 + y^2 + z^2 = \rho^2$

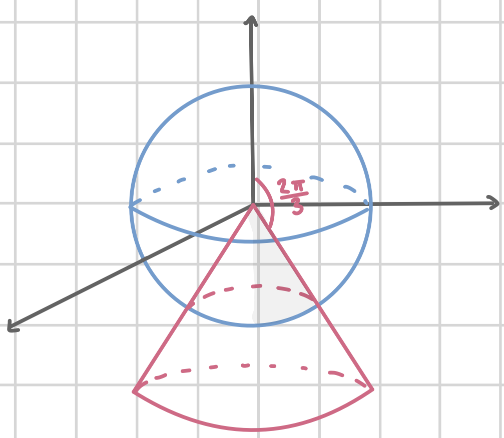
$dV = \rho^2 \sin(\varphi)\, d\rho\, d\theta\, d\varphi$

with the restriction:

$\rho \geq 0$

$0 \leq \varphi \leq \pi$

example. Set up $\iiint_E zx\, dV$ where E is inside both $x^2+y^2+z^2=2$ and the cone that makes an angle of $\pi/3$ with negative z-axis and has $x \leq 0$.



$0 \leq \rho \leq 2$

$\frac{2}{3}\pi \leq \varphi \leq \pi$

$\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$

be careful when computing φ & θ

they care about the positive axis

$$\iiint_E zx\, dV = \int_{\frac{2}{3}\pi}^{\pi} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \int_0^2 (\rho\cos(\varphi))(\rho\sin(\varphi)\cos(\theta))\,\rho^2 \sin(\varphi)\, d\rho\, d\theta\, d\varphi$$