Standard 13: Triple Integrals - Rectangular

Triple Integrals

The next step up from integrating over a two-dimensional region is integrating over a three-dimensional region. It should come as no surprise that we use a triple integral to integrate over a three-dimensional region. The general triple integral is $\iiint_E f(x,y,z)\,dV$.

rectangular-prism region

Over the box $B=[a,b]\times[c,d]\times[r,s]=\{(x,y,z)\mid a\le x\le b,\ c\le y\le d,\ r\le z\le s\}$, the triple integral $\iiint_B f(x,y,z)\,dV=\int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$.

Note: we could rewrite this integral six different ways, $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, $dz\,dy\,dx$.

example. Evaluate $\iiint_B 8xyz\,dV$ for $B=[2,3]\times[1,2]\times[0,1]$.

$$\begin{aligned}\iiint_E 8xyz\,dx\,dy\,dz &= \int_2^3\int_1^2\int_0^1 8xyz\,dz\,dy\,dx\\ &= \int_2^3\int_1^2 4xyz^2\Big|_0^1\,dy\,dx\\ &= \int_2^3\int_1^2 4xy\,dy\,dx\\ &= \int_2^3 2xy^2\Big|_1^2\,dx\\ &= \int_2^3 2x(2)^2-2x(1)^2\,dx\\ &= \int_2^3 6x\,dx\\ &= 3x^2\Big|_2^3\\ &= 3(3)^2-3(2)^2\\ &= 27-12\\ &= 15\end{aligned}$$

$$\begin{aligned}&= \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy\\ &= \int_1^2\int_2^3 4xyz^2\Big|_0^1\,dx\,dy\\ &= \int_1^2\int_2^3 4xy\,dx\,dy\\ &= \int_1^2 2x^2y\Big|_2^3\,dy\\ &= \int_1^2 2y(3)^2-2y(2)^2\,dy\\ &= \int_1^2 18y-8y\,dy\\ &= \int_1^2 10y\,dy\\ &= 5y^2\Big|_1^2\\ &= 5(2)^2-5(1)^2\\ &= 15\end{aligned}$$

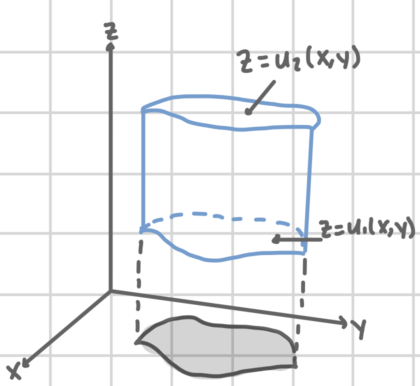
$$\begin{aligned}&= \int_0^1\int_1^2\int_2^3 8xyz\,dx\,dy\,dz\\ &= \int_0^1\int_1^2 4x^2yz\Big|_2^3\,dy\,dz\\ &= \int_0^1\int_1^2 4(3)^2yz-4(2)^2yz\,dy\,dz\\ &= \int_0^1\int_1^2 36yz-16yz\,dy\,dz\\ &= \int_0^1\int_1^2 20yz\,dy\,dz\\ &= \int_0^1 10y^2z\Big|_1^2\,dz\\ &= \int_0^1 10(2)^2z-10(1)^2z\,dz\\ &= \int_0^1 30z\,dz\\ &= 15z\Big|_0^1\\ &= 15\end{aligned}$$

Recall the calculus I explanation of the double integral: $A=\int_a^b g_2(x)-g_1(x)\,dx=\int_a^b\int_{g_1(x)}^{g_2(x)} 1\,dy\,dx$. We can make a similar assertion for triple integrals: $V=\iint_R u_2(x,y)-u_1(x,y)\,dA=\iint_R\int_{u_1(x,y)}^{u_2(x,y)} 1\,dz\,dA=\iiint_E 1\,dV$.

general region

We have three ways to describe a general region in three-dimension:

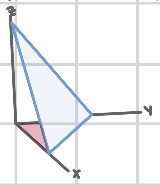
(i) $E=\{(x,y,z)\mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$

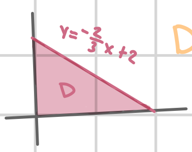


D is the shadow region in the xy-plane

$\iiint_E f(x,y,z)\,dV=\iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]dA$

example. Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.



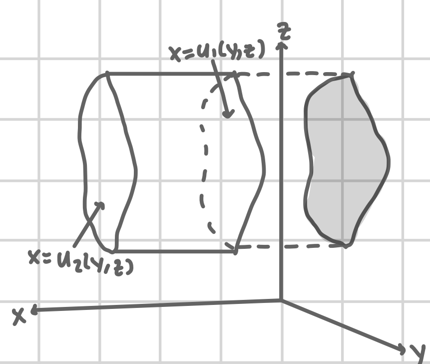


$D=\{(x,y)\mid 0\le x\le 3,\ 0\le y\le -\frac{2}{3}x+2\}$

$\iiint_E 2x\,dV=\iint_D\left[\int_0^{6-2x-3y} 2x\,dz\right]dA$

$=\int_0^3\int_0^{-2/3x+2}\int_0^{6-2x-3y} 2x\,dz\,dy\,dx$

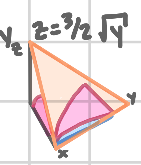
(ii) $E=\{(x,y,z)\mid (y,z)\in D,\ u_1(y,z)\le x\le u_2(y,z)\}$

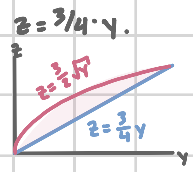


D is the shadow region in the yz-plane

$\iiint_E f(x,y,z)\,dV=\iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right]dA$

example. Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z=3/2\sqrt{y}$ and $z=3/4\cdot y$.



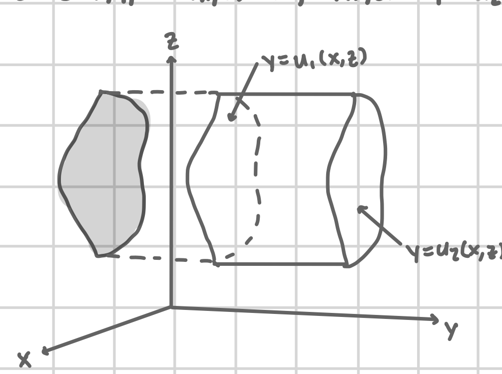


$D=\{(y,z)\mid 0\le y\le 4,\ \frac{3}{4}y\le z\le\frac{3}{2}\sqrt{y}\}$

$\iiint_E 1\,dV=\iint_D\left[\int_0^{8-y-z} 1\,dx\right]dA$

$=\int_0^4\int_{3y/4}^{3\sqrt{y}/4}\int_0^{8-y-z} 1\,dx\,dz\,dy$

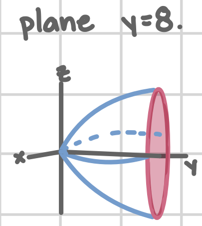
(iii) $E=\{(x,y,z)\mid (x,z)\in D,\ u_1(x,z)\le y\le u_2(x,z)\}$



D is the shadow region in the xz-plane

$\iiint_E f(x,y,z)\,dV=\iint_D\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,]\,dA$

example. Evaluate $\iiint_E\sqrt{3x^2+3z^2}\,dV$ where E is the solid bounded by $y=2x^2+2z^2$ and the plane $y=8$.



$D=\{(x,y)\mid x^2+y^2\le 4\}$

$=\{(r,\theta)\mid 0\le\theta\le 2\pi,\ 0\le r\le 2\}$

use $x=r\cos\theta$, $z=r\sin\theta$, $x^2+z^2=r^2$

$\iiint_E\sqrt{3x^2+3z^2}\,dV=\iint_A\left[\int_{2x^2+2z^2}^{8}\sqrt{3x^2+3z^2}\,dy\right]dA$

$=\int_0^{2\pi}\int_0^2\left[\int_{2r^2}^{8}\sqrt{3r^2}\,dy\right]r\,dr\,d\theta$