Standard 04: Calculus of Curves

Calculus of Curves

In single-variable calculus, you learnt how to take the limit, derivative, and integral of single variable functions, $f(x)$. This section aims to extend these ideas to vector-valued function, $\vec{r} = \langle x(t), y(t), z(t) \rangle$. These ideas can be extended to $\mathbb{R}^n$.

limits

The limit of a vector-valued function is intuitive: $\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle x(t), y(t), z(t) \rangle = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$. You simply take the limit of each component in the parameterization.

We can extend the definition of continuity of a single variable function to a definition of continuity for vector-valued functions: a vector-valued function $\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

example. Is the vector-valued function $\vec{r}(t) = \langle t\cos(t), t, t\sin(t) \rangle$ continuous at $t = \pi$?

$\lim_{t\to\pi} \vec{r}(t) = \langle \lim_{t\to\pi} t\cos(t), \lim_{t\to\pi} t, \lim_{t\to\pi} t\sin(t) \rangle$
$= \langle \pi \cdot -1, \pi, \pi \cdot 0 \rangle$
$= \langle -\pi, \pi, 0 \rangle$

$\vec{r}(\pi) = \langle \pi\cos(\pi), \pi, \pi\sin(\pi) \rangle$
$= \langle \pi \cdot -1, \pi, \pi \cdot 0 \rangle$
$= \langle -\pi, \pi, 0 \rangle$

$\vec{r}(t)$ is continuous at π.

derivatives

One of the main applications of limits from single-variable calculus is derivatives. We can extend the single-variable definition the same way we did for limits. The derivative for a vector-valued function, $\vec{v}(t)$, is defined as follows:

$$\frac{d}{dt}(\vec{r}(t)) = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h} \rangle = \langle x'(t), y'(t), z'(t) \rangle = \langle \frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t) \rangle.$$

example. Find the derivative of $\vec{r}(t) = \langle t\cos(t), t, t\sin(t) \rangle$.

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$x(t) = t\cos(t)$ $\quad x'(t) = 1\cdot\cos(t) + \sin(t)\cdot 1$

$y(t) = t$ $\quad y'(t) = 1$

$z(t) = t\sin(t)$ $\quad z'(t) = 1\cdot\sin(t) + (-\cos(t))\cdot 1$

$\vec{r}\,'(t) = \langle \cos(t)+\sin(t), 1, \sin(t)-\cos(t) \rangle$

properties:

$(\vec{u}+\vec{v})' = \vec{u}\,' + \vec{v}\,'$

$(c\vec{u})' = c\cdot\vec{u}\,'$

$(\vec{u}(\vec{v}(t)))' = \vec{u}\,'(\vec{v}(t))\cdot\vec{v}\,'(t)$

$(\vec{u}\cdot\vec{v})' = \vec{u}\,'\vec{v} + \vec{u}\,\vec{v}\,'$

$(\vec{u}\bullet\vec{v})' = \vec{u}\,'\bullet\vec{v} + \vec{u}\bullet\vec{v}\,'$

$(\vec{u}\times\vec{v}) = \vec{u}\,'\times\vec{v} + \vec{u}\times\vec{v}\,'$

integration

Lastly, we extend the definition of integrals to vector-valued functions.

- indefinite integral for vector-valued function: $\int \vec{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle$ each component will have a $+C$, use $\vec{C} = \langle C_1, C_2, C_3 \rangle$ to simplify
- definite integral for vector-valued function: $\int_a^b \vec{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$

example. Compute the following integrals using $\vec{r}(t) = \langle t\sin(t^2), t, \cos(t) \rangle$

(i) $\int \vec{r}(t)\,dt$

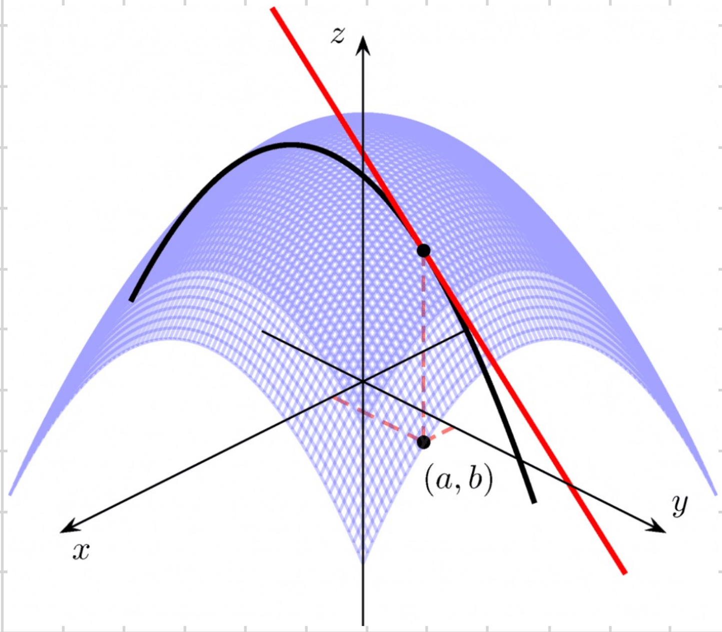
$\int \vec{r}(t)\,dt = \langle \int t\sin(t^2)\,dt, \int t\,dt, \int \cos(t)\,dt \rangle$
$= \langle -\frac{1}{2}\cos(t^2)+C_1, \frac{1}{2}t^2+C_2, \sin(t)+C_3 \rangle$
$= \langle -\frac{1}{2}\cos(t^2), \frac{1}{2}t^2, \sin(t) \rangle + \vec{C}$

(ii) $\int_0^\pi \vec{r}(t)\,dt$

$\int_0^\pi \vec{r}(t)\,dt = \langle \int_0^\pi t\sin(t^2), \int_0^\pi t\,dt, \int_0^\pi \cos(t)\,dt \rangle$
$= \langle -\frac{1}{2}\cos(t^2)\Big]_0^\pi, \frac{1}{2}t^2\Big]_0^\pi, \sin(t)\Big]_0^\pi \rangle$
$= \langle -\frac{1}{2}(\cos(\pi^2)-1), \frac{1}{2}(\pi^2), 0 \rangle$

applications

Recall from Calculus I that the derivative of a function is the slope of the tangent line. For vector-valued functions, the derivative gives a tangent vector that points in the direction of increasing t-values. This vector is used as a direction vector for the tangent.



Given the vector-valued function, $\vec{r}(t)$, we call $\vec{r}\,'(t)$ the <u>tangent vector</u> provided it exists and is not $\vec{0}$. The tangent line to $\vec{r}(t)$ at the point P is then the line that passes through the point P and is parallel to the tangent vector. If $\vec{r}\,'(t)=\vec{0}$ we would have a vector with no magnitude and no direction.

Given that $\vec{r}\,'(t)\neq\vec{0}$, the <u>unit tangent vector</u> to the curve is given by $\vec{T}(t)=\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

$\vec{r}(t)$ shown in black, tangent line shown in red i.e. $v(t)=\vec{P}+t\,\vec{r}\,'(t)$

example. Find the general formula for the unit tangent vector and the vector equation of the tangent line to the curve given by $\vec{r}(t)=\langle t\cos(t), t, t\sin(t)\rangle$ at $t=\pi$.

(i) $\vec{r}(t)=\langle t\cos(t), t, t\sin(t)\rangle$

$\vec{r}\,'(t)=\langle \cos(t)+\sin(t), 1, \sin(t)-\cos(t)\rangle$

$\|\vec{r}(t)\| = \sqrt{(\cos(t)+\sin(t))^2+(1)^2+(\sin(t)-\cos(t))^2}$

$= \sqrt{\cos^2(t)+2\cos(t)\sin(t)+\sin^2(t)+1+\sin^2(t)-2\cos(t)\sin(t)+\cos^2(t)}$

$= \sqrt{2\cos^2(t)+2\sin^2(t)+1}$

$T(t)=\frac{\vec{r}\,'(t)}{\|\vec{r}(t)\|}=\left\langle \frac{\cos(t)+\sin(t)}{\sqrt{2\cos^2(t)+2\sin^2(t)+1}}, \frac{1}{\sqrt{2\cos^2(t)+2\sin^2(t)+1}}, \frac{\sin(t)-\cos(t)}{\sqrt{2\cos^2(t)+2\sin^2(t)+1}}\right\rangle$

(ii) $T(\pi)=\left\langle \frac{\cos(\pi)+\sin(\pi)}{\sqrt{2\cos^2(\pi)+2\sin^2(\pi)+1}}, \frac{1}{\sqrt{2\cos^2(\pi)+2\sin^2(\pi)+1}}, \frac{\sin(\pi)-\cos(\pi)}{\sqrt{2\cos^2(\pi)+2\sin^2(\pi)+1}}\right\rangle$

$=\left\langle \frac{-1+0}{\sqrt{2(-1)^2+2(0)+1}}, \frac{1}{\sqrt{2(-1)+2(0)+1}}, \frac{0-(-1)}{\sqrt{2(-1)+2(0)+1}}\right\rangle$

$=\left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$

sometimes it is easier to find $\vec{r}\,'(a)$ and $\|\vec{r}\,'(a)\|$ separately and combine them for $T(a)$

$P=\vec{r}(\pi)=\langle \pi\cos(\pi), \pi, \pi\sin(\pi)\rangle = \langle -\pi, \pi, 0\rangle$

tangent line: $\vec{v}(t)=\langle -\pi,\pi,0\rangle + t\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$

Suppose two curves $\vec{r}_1$ and $\vec{r}_2$ intersect at a point P. Then the angle they intersect at can be determined by finding the angle of intersection of the tangent vector vectors at the point P.

example. Find the angle of intersection for the curves $\vec{r}_1(t)=\langle\cos(t), -\sin(t), t\rangle$ and $\vec{r}_2(s)=\langle -s, s^2-1, \ln(s)+\pi\rangle$ at the point $P=(-1,0,\pi)$

First solve for the values t and v: $\vec{r}_1(\pi)=\langle\cos(\pi), -\sin(\pi), \pi\rangle=\langle -1,0,\pi\rangle$ & $\vec{r}_2(1)=\langle -1, (1)^2-1, \ln(1)+\pi\rangle=\langle -1,0,\pi\rangle$.

Next find the tangent vectors: $\vec{r}_1\,'(t)=\langle -\sin(t), -\cos(t), 1\rangle$ & $\vec{r}_2\,'(s)=\langle -1, 2s, \frac{1}{s}\rangle$

Tangent vectors at our given t & s: $\vec{r}_1\,'(\pi)=\langle 0,1,1\rangle$ & $\vec{r}_2\,'(1)=\langle -1,2,1\rangle$

Angle between tangents at P: $\vec{r}_1\cdot\vec{r}_2=\|\vec{r}_1\|\cdot\|\vec{r}_2\|\cos(\theta)$

$\langle 0,-1,1\rangle\cdot\langle -1,2,1\rangle = \|\langle 0,-1,1\rangle\|\cdot\|\langle -1,2,1\rangle\|\cdot\cos(\theta)$

$(0)(-1)+(-1)(2)+(1)(1)=\sqrt{(0)^2+(-1)^2+(1)^2}\,\sqrt{(-1)^2+(2)^2+(1)^2}\,\cos(\theta)$

$0-2+1=\sqrt{2}\,\sqrt{6}\,\cos\theta$

$\frac{-1}{2\sqrt{3}}=\cos(\theta)$

$\theta=\arccos\left(-\frac{\sqrt{3}}{6}\right)$