Standard 06: TNB Frame, Normal Plane, and Osculating Plane

Tangent, Normal, and Binormal Vectors

In this section we want to look at an application of derivatives for vector-valued functions. We build on an application we saw last time: the unit tangent vector.

unit tangent vector

Provided $\vec{r}\,'(t) \neq \vec{0}$, the unit tangent vector to the curve is given by $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

example. Find the general formula for the unit tangent vector to the curve given by $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t)\rangle$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$\vec{r}\,'(t) = \langle 1, 3\cos(t), -3\sin(t)\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(1)^2 + (3\cos(t))^2 + (-3\sin(t))^2}$

$= \sqrt{1 + 9\cos^2(t) + 9\sin^2(t)}$

$= \sqrt{1 + 9(\cos^2(t) + \sin^2(t))}$

$= \sqrt{1+9} = \sqrt{10}$

$T'(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \right\rangle$

unit normal vector

Similarly, the unit normal vector to the curve is defined to be $\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$.

example. Find the general formula for the unit normal vector to the curve given by $\vec{r}\,'(t) = \langle t, 3\sin(t), 3\cos(t)\rangle$.

$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$, $\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \right\rangle$

$\vec{T}\,'(t) = \left\langle 0, \frac{-3}{\sqrt{10}}\sin(t), \frac{-3}{\sqrt{10}}\cos(t) \right\rangle$

$\|\vec{T}\,'(t)\| = \sqrt{0^2 + \left(\frac{-3}{\sqrt{10}}\sin(t)\right)^2 + \left(\frac{-3}{\sqrt{10}}\cos(t)\right)^2}$

$= \sqrt{\frac{9}{10}\sin^2(t) + \frac{9}{10}\cos^2(t)}$

$= \sqrt{\frac{9}{10}(\sin^2(t) + \cos^2(t))}$

$= \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$

$\vec{N}(t) = \left\langle \frac{0}{3/\sqrt{10}}, \frac{-3\sin(t)/\sqrt{10}}{3/\sqrt{10}}, \frac{-3\cos(t)/\sqrt{10}}{3/\sqrt{10}} \right\rangle$

$= \langle 0, -\sin(t), -\cos(t)\rangle$

Fun Facts about $\vec{N}(t)$

- $\vec{N}(t) \perp \vec{T}(t)$
- $\vec{N}(t) \perp \vec{r}(t)$
- $\vec{N}(t)$ should remind you of $\vec{n} = \langle a, b, c\rangle$
- If $\|\vec{r}(t)\| = c$ for all t then $\vec{r}\,'(t) \perp \vec{r}(t)$

ex. $\|\vec{T}(t)\| = 1$ for all t then $\vec{N}(t) \perp \vec{T}(t)$

unit binormal vector

Lastly, we define the unit binormal vector of a curve to be $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

example. Find the general formula for the unit binormal for the curve given by $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t)\rangle$.

$\vec{B} = \vec{T}(t) \times \vec{N}(t)$, $\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \right\rangle$, $\vec{N}(t) = \langle 0, -\sin(t), -\cos(t)\rangle$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3\cos(t)}{\sqrt{10}} & \frac{-3\sin(t)}{\sqrt{10}} \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix}$$

$= \left(\frac{-3}{\sqrt{10}}\cos^2(t) - \frac{3}{\sqrt{10}}\sin^2(t)\right)\vec{i} - \left(\frac{-1}{\sqrt{10}}\cos(t) - 0\right)\vec{j} + \left(\frac{-1}{\sqrt{10}}\sin(t) - 0\right)\vec{k}$

$= \left(\frac{-3}{\sqrt{10}}(\cos^2(t) + \sin^2(t))\right)\vec{i} + \left(\frac{1}{\sqrt{10}}\cos(t)\right)\vec{j} - \left(\frac{1}{\sqrt{10}}\sin(t)\right)\vec{k}$

$= \frac{-3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos(t)\vec{j} + \frac{1}{\sqrt{10}}\sin(t)\vec{k}$

$= \left\langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\cos(t), \frac{1}{\sqrt{10}}\sin(t) \right\rangle$

Curvature

We take a small break to talk about curvature. It will not be tested but is important to note.

The curvature of a smooth (i.e. everywhere differentiable, $\vec{r}\,'(t) \neq \vec{0}$) measures how fast a curve is changing direction at a given point. There are several formulas for determining curvature for a curve:

$$\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| \quad \text{magnitude of the derivative of } \vec{T}(t) \text{ reparameterized in terms of arclength}$$

$$= \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} \quad \text{my favorite to use}$$

$$= \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

example. Determine the curvature for $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$.

$\vec{r}\,'(t) = \langle 1, 3\cos(t), -3\sin(t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{10}$

$\vec{T}(t) = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos(t), \frac{-3}{\sqrt{10}}\sin(t) \rangle$

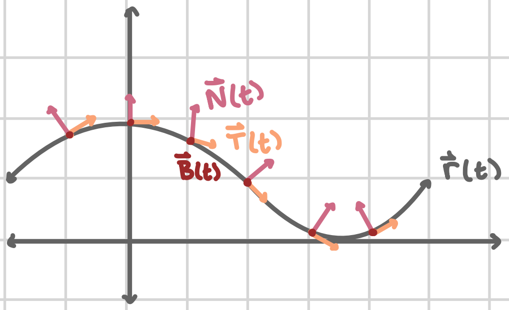
$\vec{T}\,'(t) = \langle 0, \frac{-3}{\sqrt{10}}\sin(t), \frac{-3}{\sqrt{10}}\cos(t) \rangle$

$\|\vec{T}\,'(t)\| = \frac{3}{\sqrt{10}}$

$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$

Ferret-Serre Frame "T-N-B Frame"

The T-N-B Frame consists of: the unit tangent vector $\vec{T}(t)$, the unit normal vector $\vec{N}(t)$, and the binormal vector $\vec{B}(t)$.



- unit tangent vector: "tangent", in the direction of, to the curve
- unit normal vector: orthogonal to the unit tangent vector and the curve
- unit binormal vector: orthogonal to both unit tangent vector and unit binormal vector

From these vectors we can define three planes:

(i) the normal plane: perpendicular to the curve $\vec{r}(t)$, contains the normal & binormal vectors, orthogonal to the tangent vector
$\vec{T}(t) \cdot \langle \vec{r}(t) - \vec{r}_0(t) \rangle = 0$

(ii) the osculating plane: captures the motion of $\vec{r}(t)$, contains the tangent & normal vectors, orthogonal to the binormal vector
$\vec{B}(t) \cdot \langle \vec{r}(t) - \vec{r}_0(t) \rangle = 0$

(iii) the rectifying plane: will not be tested, contains the tangent & binormal vectors, orthogonal to the normal vector
$\vec{N}(t) \cdot \langle \vec{r}(t) - \vec{r}_0(t) \rangle = 0$

example. Find the normal and osculating planes of the curve $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$ at the point $(0,0,3)$.

Step 1: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$\vec{r}\,'(t) = \langle 1, 3\cos(t), -3\sin(t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(1)^2 + (3\cos(t))^2 + (-3\sin(t))^2}$
$= \sqrt{1 + 9\cos^2(t) + 9\sin^2(t)}$
$= \sqrt{1 + 9(\cos^2(t) + \sin^2(t))}$
$= \sqrt{1+9} = \sqrt{10}$

$T'(t) = \langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \rangle$

$\vec{T}\,'(t) = \langle 0, \frac{-3}{\sqrt{10}}\sin(t), \frac{-3}{\sqrt{10}}\cos(t) \rangle$

$\|\vec{T}\,'(t)\| = \sqrt{0^2 + (\frac{-3}{\sqrt{10}}\sin(t))^2 + (\frac{-3}{\sqrt{10}}\cos(t))^2}$
$= \sqrt{\frac{9}{10}\sin^2(t) + \frac{9}{10}\cos^2(t)}$
$= \sqrt{\frac{9}{10}(\sin^2(t) + \cos^2(t))}$
$= \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$

$\vec{N}(t) = \langle \frac{0}{3/\sqrt{10}}, \frac{-3\sin(t)/\sqrt{10}}{3/\sqrt{10}}, \frac{-3\cos(t)/\sqrt{10}}{3/\sqrt{10}} \rangle$
$= \langle 0, -\sin(t), -\cos(t) \rangle$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3\cos(t)}{\sqrt{10}} & \frac{-3\sin(t)}{\sqrt{10}} \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix}$$

$= (\frac{-3}{\sqrt{10}}\cos^2(t) - \frac{3}{\sqrt{10}}\sin^2(t))\vec{i} - (\frac{-1}{\sqrt{10}}\cos(t) - 0)\vec{j} + (\frac{-1}{\sqrt{10}}\sin(t) - 0)\vec{k}$
$= (\frac{-3}{\sqrt{10}}(\cos^2(t) + \sin^2(t)))\vec{i} + (\frac{1}{\sqrt{10}}\cos(t))\vec{j} - (\frac{1}{\sqrt{10}}\sin(t))\vec{k}$
$= \frac{-3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos(t)\vec{j} + \frac{1}{\sqrt{10}}\sin(t)\vec{k}$
$= \langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\cos(t), \frac{1}{\sqrt{10}}\sin(t) \rangle$

Step 2: Find the time related to the point & plug in

$t = 0 \qquad 3\sin(t) = 0 \qquad 3\cos(t) = 1$

$\vec{T}(0) = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0 \rangle \qquad \vec{N}(0) = \langle 0, 0, 1 \rangle \qquad \vec{B}(0) = \langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \rangle$

Step 3: Fill out the planes

normal plane: $\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0 \rangle \cdot \langle x-0, y-0, z-3 \rangle = 0$

osculating plane: $\langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \rangle \cdot \langle x-0, y-0, z-3 \rangle = 0$