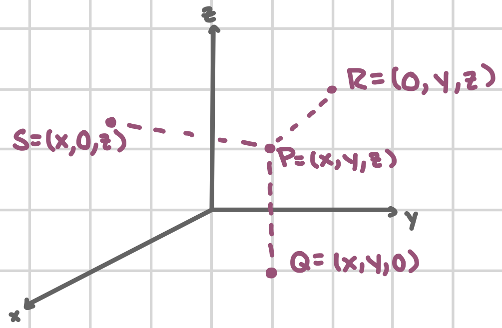
Standard 02: Lines

Introduction to the 3-D Coordinate System

The 3-D coordinate system is often denoted by $\mathbb{R}^3$, mimicking $\mathbb{R}^2$ for the 2-D coordinate system and $\mathbb{R}$ for the 1-D coordinate system. We can also bring this out to a n-dimensional coordinate system denoted by $\mathbb{R}^n$. Visually the 3-D coordinate system is shown below:

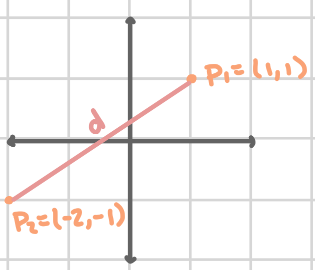


This is the standard placement of the axes with it assumed that only the positive directions are shown. We will add the negative axes only if needed and label them. The point $P=(x,y,z)$ is a general point sitting in 3-D space. We may use the word projection to describe going from the xyz-system to any of the 2-D planes, e.g. if you drop down to $z=0$ then we get the point $Q=(x,y,0)$ in the xy-plane. In addition, we can find points S and R in the xz and yz plane, respectively.

Properties of 3-D

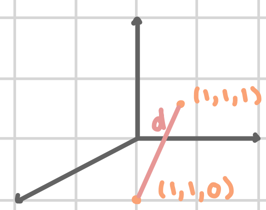
Many of the formulas you are familiar with in $\mathbb{R}^2$ have natural extensions into the $\mathbb{R}^3$ coordinate system. For example, the distance between two points:

2-D Space



$$
\begin{aligned}
d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
&= \sqrt{(1-(-2))^2+(1-(-1))^2} \\
&= \sqrt{(3)^2+(2)^2} \\
&= \sqrt{9+4} \\
&= \sqrt{15}
\end{aligned}
$$

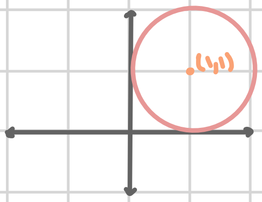
3-D Space



$$
\begin{aligned}
d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \\
&= \sqrt{(1-1)^2+(1-1)^2+(1-0)^2} \\
&= \sqrt{(0)^2+(0)^2+(1)^2} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

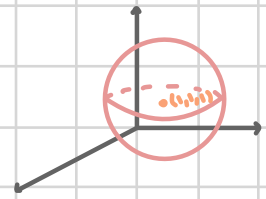
Likewise, the general equation of a circle with center (h,k) and radius r extends to a sphere with center (h,k,ℓ) and radius r:

2-D Space



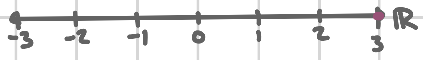
$$(x-h)^2+(y-k)^2=r^2$$
$$(x-1)^2+(y-1)^2=(1)^2$$

3-D Space

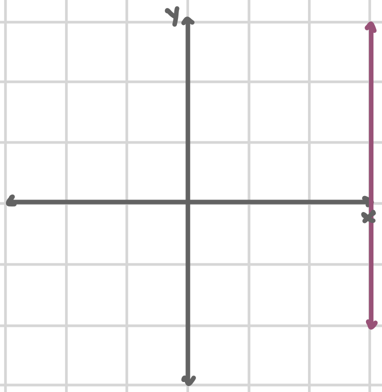


$$(x-h)^2+(y-k)^2+(z-\ell)^2=r^2$$
$$(x-1)^2+(y-1)^2+(z-1)^2=(1)^2$$

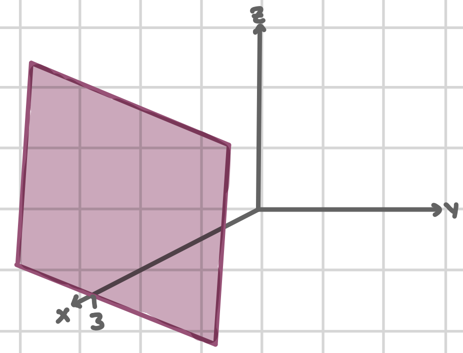
Not everything about $\mathbb{R}^2$ translates to $\mathbb{R}^3$ the way we expect. For example, let's graph $x=3$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.



$x=3$ is a point on $\mathbb{R}$



$x=3$ is a line in $\mathbb{R}^2$ of all points of the form $(3,y)$

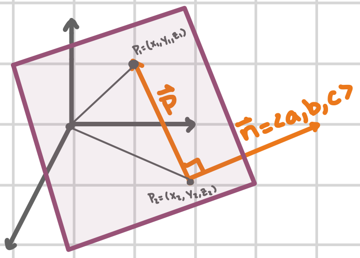


$x=3$ is a plane in $\mathbb{R}^3$ that contains all point $(3,y,z)$

Standard 03: Planes

Planes in 3D

Equation of a Plane in 3-D



A plane in 3D requires two things:
- a point in the plane $\vec{r}_0=(x_0,y_0,z_0)$
- the direction orthogonal to the plane $\vec{n}=\langle a,b,c\rangle$

vector equation = $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$.

standard equation = $ax+by+cz=d=\vec{n}\cdot\vec{r}_0$

example. Find the equation of the plane that contains the point $(7,2,-1)$ and is orthogonal to the line given by the parametrization $\vec{r}(t)=\langle 1-2t,3t,2-t\rangle$.

We are given the point $(7,2,-1)$ and a vector that is orthogonal. Since $\vec{r}(t)$ is orthogonal, its direction vector is orthogonal to the plane. So we have our two parts.

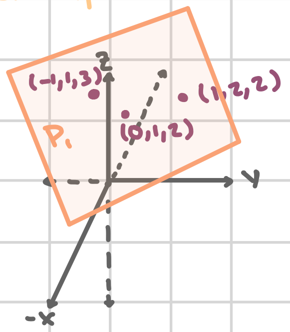
$\vec{r}_0=(7,2,-1)$ & $\vec{n}=\langle -2,3,-1\rangle$

$-2x+3y-1z=\langle -2,3,-1\rangle\cdot\langle 7,2,-1$

$-2x+3y-z=-14+6+1=-7$

$-2x+3y-z=-7$

example. Find an equation for P_1 the plane that goes through the points $(0,1,2)$, $(-1,1,3)$, and $(1,2,2)$.



To find the equation of a plane we need a point on the plane and normal vector $\vec{n}$. We are given three points to choose from and can find a normal vector by finding a cross product between two vectors in the plane, the vector between given points

work: $(0,1,2)$ to $(-1,1,2)$ is $\langle -1,0,1\rangle$ and $(0,1,2)$ to $(1,2,2)$ is $\langle 1,1,0\rangle$.

$$\vec{n}=\langle -1,0,1\rangle\times\langle 1,1,0\rangle=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-1&0&1\\1&1&0\end{vmatrix}=(0-1)\vec{i}-(0-1)\vec{j}+(-1-0)\vec{k}$$
$$=-1\vec{i}+1\vec{j}-1\vec{k}$$
$$=\langle -1,1,-1\rangle$$

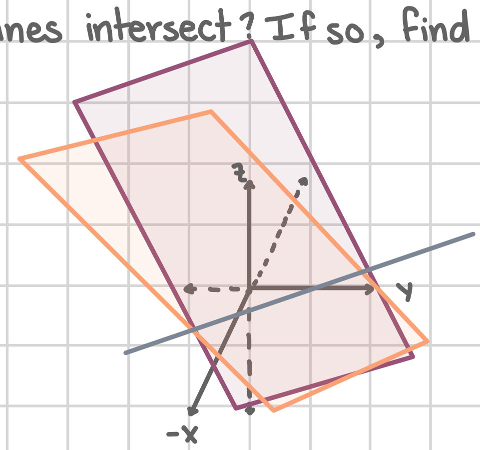
standard equation: $ax+by+cz=\vec{n}\cdot\vec{r}_0$

$-1x+1y-1z=\langle -1,1,-1\rangle\cdot\langle 0,1,2\rangle$

$-x+y-z=0+1-2$

$-x+y-z=-1$

example. Let P_1 be the plane found in the above example and P_2 be the plane described by $x-y+2z=1$. Do these planes intersect? If so, find the line of intersection and the cosine of the angle between the planes?



If the normal vectors of P_1 & P_2 are scalars of each other then they do not intersect, otherwise we can find L. Since L is contained in P_1 and P_2, L must be orthogonal to the normal vectors for each plane, $\vec{n}_1$ and $\vec{n}_2$. Thus the direction of L is $\vec{n}_1\times\vec{n}_2$. To find the initial point of the line, we must find a x,y,z s.t. P_1 and P_2 are satisfied.

Finding initial point:

$-x+y-z=-1$

$x-y+2z=1$

$0x+0y+1z=0$

$z=0$

if $z=0$ then $x+y=1$

$\vec{n}_1=\langle -1,1,-1\rangle$ & $\vec{n}_2=\langle 1,-1,2\rangle$

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-1&1&-1\\1&-1&2\end{vmatrix}$$

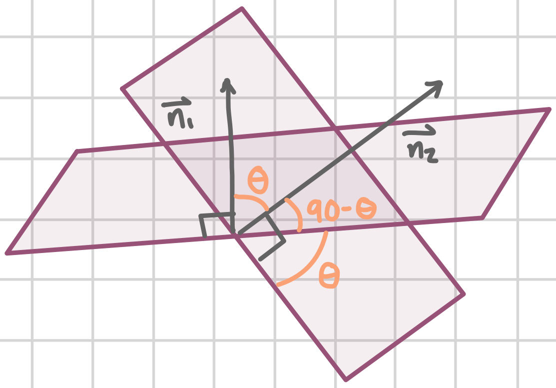
$=(2-1)\vec{i}-(-2-(-1))\vec{j}+(1-1)\vec{k}$

$=1\vec{i}-(-1)\vec{j}+0\vec{k}$

$=\langle 1,1,0\rangle$

using the point $(1,0,0)$ and $\vec{n}=\langle 1,1,0\rangle$

$\vec{r}(t)=\langle 1,0,0\rangle+t\langle 1,1,0\rangle$



The angle between 2 planes is found through finding the angle between their norms, this is equivalent due to complementary angles.

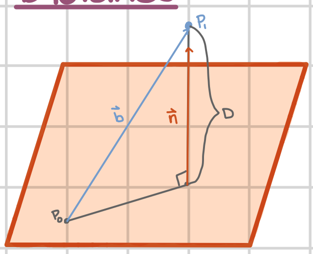
$\|\vec{n}_1\| \cdot \|\vec{n}_2\| \cos\theta = \vec{n}_1 \cdot \vec{n}_2$

$\sqrt{(-1)^2+(1)^2+(-1)^2}\,\sqrt{(1)^2+(-1)^2+(2)^2}\,\cos\theta = (-1)(1)+(1)(-1)+(-1)(2)$

$\sqrt{3}\,\sqrt{6}\,\cos\theta = -1-1-2$

$\cos\theta = {}^{-4}/\sqrt{18}$

Distance



Distance from a point P_1 to the plane.

Pick a point P_0 in the plane, name the distance between P_1 and P_0, $\vec{b}$. This may not be the shortest distance from the point to the plane, the shortest distance is on the normal vector of the plane, so we take $D = |\text{comp}_{\vec{n}}\,\vec{b}|$.

example. Find the distance from the point $(2,1,2)$ to the plane $x+y+z=1$.

To find a point on the plane set two variables to zero: $(0)+(0)+z=1 \Rightarrow P_0=(0,0,1)$

The vector between (x_1,y_1,z_1) & (x_2,y_2,z_2) is $\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$: $\langle 2-0, 1-0, 2-1\rangle = \langle 2,1,1\rangle$

The normal vector of $ax+by+cz=d$ is $\langle a,b,c\rangle$: $\vec{n} = \langle 1,1,1\rangle$

Recall the formula $\text{comp}_{\vec{n}}\,\vec{b} = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$: $\text{comp}_{\vec{n}}\,\vec{b} = \frac{\langle 1,1,1\rangle\cdot\langle 2,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \frac{(1)(2)+(1)(1)+(1)(1)}{\sqrt{1+1+1}} = \frac{4}{\sqrt{3}}$

Note that if two planes intersect then the distance between them is zero, so a question about distance only makes sense for parallel planes.