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Orbit sizes and the central product group of order 16

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Abstract

This paper continues the study of linear group actions with no regular orbits where the largest orbit size equals the order of the abelian quotient of the group. In previous work of the first author with Yong Yang it was shown that if G is a finite solvable group and G a finite group and V a finite faithful completely reducible G-module, possibly of mixed characteristic, and M is the largest orbit size in the action of G on V then $|G/G'| \leq M$. In a continuation of this work the first author and his student Nathan Jones analyzed the first open case of when equality occurs and proved the following. If G is a finite nonabelian group and V a finite faithful irreducible G-module and $M = |G/G'|$ is the largest orbit of G on V and that there are exactly two orbits if size M on V, then $G = D_8$ and $V = V(2, 3)$. This paper is concerned with the next case, the one where, under otherwise the same hypotheses as before, we have three orbits of size $M = |G/G'|$. It turns out that again there is exactly one such action, the one where G is the central product of D_8 and C_4 is acting on the vector space of order 25.

Keywords Finite group · Orbit structure · Commutator subgroup · Dihedral group · Central product · Abelian quotient

Mathematics Subject Classification 20D10

1 Introduction

The study of the orbit structure in finite linear group actions has been intensely studied topic for many years. This paper is a continuation of some work that focuses on the study of the abelian quotient of a finite linear group and its relationship to the largest orbit size in

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the action. The first paper out of this group [6] by the first author and Y. Yang established the following result.

Theorem 1 [6] *Let G be a finite solvable group and V a finite faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Then*

$$|G/G'| \leq M.$$

More precisely, we have one of the following:

(a) $|G/G'| < M$;
(b) $|G/G'| = M$ *and G is abelian; or*
(c) $|G/G'| = M$, *G is nilpotent, and G has at least two different orbits of size M on V.*

The main conclusion of the above result, namely that $|G/G'| \leq M$, has since been generalized to arbitrary finite groups, see [7]. The full conclusion of Theorem 1 is true for arbitrary finite groups by very recent work of Qian and Yang [12, Theorem 2.9].

The next paper in the line up focuses on part (c) of the above theorem and considers the "smallest" case when there are exactly two orbits of size M and when V is assumed to be irreducible. The first author and N. Jones were are able to completely determine the group action in this case. The main result is as follows.

Theorem 2 [5] *Let G be a finite nonabelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly two orbits of size M on V. Then G is dihedral of order 8, and $V = V(2, 3)$.*

This result confirmed Conjecture 2.5 in [6], which was also included as Problem 18.44 in the 18th issue of the Kourovka Notebook [8].

In this paper we will continue the path of finding the next "smallest" case when there are exactly three orbits of size M with the otherwise same hypothesis as above in Theorem 2. It turns out that as for two orbits, there is again a unique action that satisfies these conditions, namely the following.

The action of the central product of the dihedral group of order eight, D_8, and the cyclic group of order four, C_4, denoted by $D_8 \circ C_4$, can be studied by the way its embedding in $GL(2, 5)$ acts naturally on $V = V(2, 5)$, the two-dimensional vector space over the field with five elements. The action is well understood, a breakdown of the action and its resulting orbits can be found in [14]. It has four orbits on V, namely three orbits of size 8 and one orbit of size 1. Thus it satisfies the hypotheses of the theorem. Note that the central product $D_8 \circ C_4$ is also known as the Pauli group, particularly in physics.

The main result of the paper hence is as follows.

Theorem 3 *Let G be a finite nonabelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly three orbits of size M on V. Then $G = D_8 \circ C_4$ and $V = V(2, 5)$.*

This confirms Conjecture 4.1 in [5] (note that $D_8 \circ C_4 \cong Q_8 \circ C_4$). We also remark that the action showing up in Theorem 3 prominently features in other contexts; for example, it is the second exception in [11, Theorem 2(c)], showing that in this action the acting group is very large compared to the module it is acting on.

The proof, as the proofs of the earlier results on this topic, is long and technical. It follows the same sequence of steps outlined in [5], but of course the current problem is "bigger", produces more subcases and requires new arguments and insights in many places. Along the way we will also take the opportunity to correct some errors in [5] that we noticed when working on the proof of Theorem 3G. We will point these places out in the proof. However, due to some oversight, Case 2.4 is very poorly presented in [5], and so we decided to redo the proof of this important case completely and write it down here in Section III, and the argument presented there will guide the handling of the corresponding case in the proof of the main result.

We conclude the paper by stating some open problems on the subject.

The notation used in this paper is fairly standard. $V(n, q)$ denotes the n-dimensional vector space over the field with q elements, and if H and G are groups, then $H \lessapprox G$ means that H is isomorphic to a subgroup of G.

2 Auxiliary results

First we restate two important lemmas from [6] for the convenience of the reader.

Lemma 1 [6, Lemma 2.1] *Let G be a finite group and $N \trianglelefteq G$. Then*

$$|G/G'| = |G/G'N| \cdot |N : N \cap G'|;$$

and

$$|G : G'| \quad \text{divides} \quad |G/N : (G/N)'| \cdot |N : N'|.$$

Lemma 2 [6, Lemma 2.2] *Let A be an abelian finite group and let V be a finite faithful completely reducible A-module. It is well known that A has a regular orbit on V. Write $V = V_1 \oplus \cdots \oplus V_n$ for irreducible A-modules V_i. Suppose that A has exactly one regular orbit on V. Then $A/C_A(V_i)$ is cyclic of order $|V_i| - 1$ for all i and $A \cong \times_{i=1}^{n} A/C_A(V_i)$ is of order $\prod_{i=1}^{n}(|V_i| - 1)$.*

3 Correction to a previous proof

This section serves as a correction to Case 2.4 in [5]. Due to an oversight an erroneous version of that case ended up in the paper. It seems to be best to completely replace it by the following argument. This new argument will be fairly easy to adapt to also be used in the proof of Theorem 3.

Case 2.4 There is "<" in the first and "=" in the second inequality.

Suppose we have equality in (4) and strict inequality in (3). That is

$$M \geq M_1M_2 \geq p|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|D/D'| = |G : G'|.$$

Because $|G : G'| = M$, we have equality everywhere, and $M = M_1M_2, M_1 = p|D : D'C_D(V_1)| > |D : D'C_D(V_1)|, M_2 = |C_D(V_1) : C_D(V_1)'|$. Again let M_D denote the largest orbit size of D on V, then $M_D \geq M_1M_2$ so $M_D = M$. By Theorem 1.1 $C_D(V_1)$ is abelian or has at least two orbits of size M_2 on W_1. We consider again some subcases.

Case 2.4.1 $C_D(V_1)$ has at least two orbits of size M_2 on W_1.

Let $w_1, w_2 \in W_1$ be representatives of such orbits.

Assume that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Because $M = M_1M_2$ we have that $(v_1 + w_1)^D, (v_1 + w_2)^D$, $(v_2 + w_2)^D$ and $(v_2 + w_1)^D$ are all distinct orbits of size $M_D = M$, contradicting there being only two orbits of size M. Therefore $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1. Let $v_1 \in V_1$ be a representative of this orbit.

Now let w_1, w_2 be representatives of two distinct orbits of size M_2 of $C_D(V_1)$ on W_1, then $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are two distinct D–orbits of size M, and if $C_D(V_1)$ had a third orbit of size M_2 on W_1, similarly we would get a third orbit of G of size M, a contradiction. Thus $C_D(V_1)$ exactly two orbits of size M_2 on W_1.

Now write $W_1 = \bigoplus_{i=1}^k X_i$ for a suitable $k \in \{1, \dots, n\}$ and irreducible $C_D(V_1)$-modules X_i $(i = 1, \dots, k)$. We may assume that $X_1 \leq V_2$. Then the intersection of all the $C_{C_D(V_1)}(X_i)$ is trivial, and hence

$$C_D(V_1) \lessapprox C_D(V_1)/C_{C_D(V_1)}(X_1) \times \cdots \times C_D(V_1)/C_{C_D(V_1)}(X_k) \quad (+)$$

Moreover, if we put $N_0 = C_D(V_1)$, $Z_0 = W_1$ and recursively for $i \geq 1$ let $Y_i \leq Z_{i-1}$ be an irreducible N_{i-1}-module, $N_i = C_{N_{i-1}}(Y_i)$, and Z_i be a $C_D(V_1)$-invariant complement of Y_i in Z_{i-1}, and put $t = i - 1$ and stop the process as soon as $Z_i = 0$ and $N_i = 1$, then we have that $\bigcap_{i=0}^t N_i = 1$ and $W_1 = \bigoplus_{i=0}^t Y_i$. Also, $R_{i-1} := N_{i-1}/N_i$ acts faithfully and irreducibly on Y_{i-1} for $i = 1, \dots, t$. Write M_{i-1}^* for the largest orbit size of N_{i-1}/N_i on Y_{i-1} for $i = 1, \dots, t$. Then by repeated use of Lemma 2.1 we see that

$$M_2 = |C_D(V_1) : C_D(V_1)'| \leq \prod_{i=1}^{t} |R_i : R_i'| \leq \prod_{i=1}^{t} M_i^* \leq M_2, \quad (++)$$

the last inequality easily following by considering the sum of representatives of orbits of size M_{i-1}^* of N_{i-1}/N_i on Y_{i-1}. Thus we have equality everywhere, and it follows that $|R_i : R_i'| = M_i^*$ for $i = 1, \dots, t$. It also follows that the elements of every orbit of $C_D(V_1)$ on W_1 of size M_2 have the form $y_1 + \dots, y_t$ for some $y_i \in Y_i$ $(i = 1, \dots, t)$ which lies in an orbit of size M_i^* of N_i/N_{i+1} on Y_i $\quad (+++)$.

Case 2.4.1.1 $C_D(V_1)$ is not abelian.

Put $C = C_D(V_1) \cap C_D(X_1) = C_{C_D(V_1)}(X_1)$. Then by (+) we may assume that $C_D(V_1)/C$ is nonabelian, and it also acts faithfully and irreducibly on X_1. We also clearly may assume that $Y_1 = X_1$ and hence with (++) and (+++) conclude that $C_D(V_1)/C$ has exactly two orbits of size of its abelian quotient on X_1. Hence we may apply induction and, in particular, get $p = 2, |X_1| = 9$ and $C_D(V_1)/C \cong D_8$. Moreover, since $C_D(V_1)$ has exactly two orbits of size M_2 on W_1, then from (+++) it follows that R_{i-1} has exactly one orbit of size M_{i-1}^* on Y_{i-1} for $i = 2, \dots, t$. This forces, for $i = 2, \dots, t$, that R_{i-1} is cyclic of order $2, |Y_{i-1}| = 3$, and hence $C_{C_D(V_1)}(X_1)$ is elementary abelian of order $p^{\dim W_1 - 2}$. Note that $W_1 = V_2$ since $p = 2$.

Assume that $k \geq 2$, so $t \geq 3$ (since the X_i all have dimension 2). Then we may assume that $X_2 = Y_1 \oplus Y_2$, and from the above we know that $C/C_C(X_2)$ is elementary abelian of order 4.

Now consider the action of $C_D(V_1)$ on X_1. We know that $C_D(V_1)$ is isomorphic to a subgroup of a direct product of k copies of D_8, and $C_D(V_1)/C$ is isomorphic to D_8 and has four noncentral involutions. If all of them have inverse images in $C_D(V_1)$ which act trivially on $X_2 \oplus \cdots \oplus X_k$, then $C_D(V_1)$ has a D_8 as a subgroup which acts trivially on $X_2 \oplus \cdots \oplus X_k$, and since the X_i are transitively permuted by D, it follows that $C_D(V_1)$ is isomorphic to a direct product of k copies of D_8; in particular, then $C/C_C(X_2) \cong C_D(V_1)/C_{C_D(V_1)}(X_2) \cong D_8$, contradicting the above observation that $C/C_C(X_2)$ is elementary abelian of order 4. Hence there exists an element $c \in C_D(V_1)$ such that $c \notin C$, $c^2 \in C$, and c acts nontrivially on at least one X_i for some $i \in \{2, \dots, k\}$, so without loss we may assume that c acts nontrivially on X_2. Now there is a $0 \neq x \in V_1$ such that c centralizes x. Since $c \notin C$ and $C/C_C(X_2)$ is elementary abelian of order 4, this shows that $C_D(x)/C_{C_D(x)}(X_2)$ has order divisible by 8, and thus $C_D(x)/C_{C_D(x)}(X_2)$ is isomorphic to D_8 and therefore has two orbits of size 4 on X_2. This allows us in an obvious way to construct two different orbits of size $M_2 = 4^k$ of $C_D(V_1)$ on $V_2 = W_1$ having representatives with x in their X_1-component; in addition to another orbit of size M_2 having a representative in the X_1-component from the second orbit of size 4 of $C_D(V_1)/C$ on X_1, giving us in total three distinct orbits of $C_D(V_1)$ on V_2, contradicting the current fact that $C_D(V_1)$ has exactly two orbits of size M_2 on V_2.

Hence our assumption that $k \geq 2$ was wrong, and we now have $k = 1$. So $W_1 = V_2 = X_1$ is of order 9, and $C_D(V_1) \cong D_8$ acts irreducibly on it and has two orbits of size $M_2 = 4$ on it. Hence $D_8 \times D_8 \cong C_D(V_2) \times C_D(V_1)$ is a normal subgroup of G. Now since $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 (as we saw above), it follows that $M_1 = 8$ and $D/C_D(V_1)$ must be at least of order 16, and thus $D/C_D(V_1)$ is a full Sylow 2-subgroup of GL $(2, 3)$, i.e., a semidihedral group of order 16. Moreover, $|G : G'| = M = M_1M_2 = 8 \cdot 4 = 2^5$ and $|G| = |G/D|\,|D/C_D(V_1)|\,|C_D(V_1)| = 2 \cdot 16 \cdot 8 = 2^8$. Therefore $|G'| = 2^3$. Now let $Z = C_D(V_1)' \times C_D(V_2)'$. Then $Z \leq D'$ is a Klein 4-group and $G'/Z = (G/Z)'$. Working in G/Z, we notice that $(C_D(V_1) \times C_D(V_2)')/Z$ is elementary abelian of order 2^4, and if $g \in G - D$, then gZ interchanges the two subgroups $C_D(V_i)Z/Z \cong C_D(V_i)/C_D(V_i)'$ $(i = 1, 2)$. Looking at the elements $[gZ, xZ] \in (G/Z)'$ for $x \in C_D(V_1)$ shows us that $|(G/Z)'| \geq |C_D(V_1)Z/Z| = 4$ so that altogether $2^3 = |G'| = |G'/Z||Z| \geq 4 \cdot 4 = 2^4$, which is a contradiction. This completes Case 2.4.1.1.

Case 2.4.1.2 $C_D(V_1)$ is abelian.

Then $C_D(V_1)$ has regular orbits on W_1, and thus $M_2 = |C_D(V_1)|$, so $C_D(V_1)$ has exactly two regular orbits on W_1.

Note that $M_2 = |C_D(V_1)|$ and so

$$
\begin{aligned}
M = M_1M_2 = M_1|C_D(V_1)| &= |G/G'| = p|D/D'| \\
&= p|D : D'C_D(V_1)||D'C_D(V_1) : D'| \\
&= M_1|D'C_D(V_1) : D'| \\
&= M_1|C_D(V_1) : (D' \cap C_D(V_1))|
\end{aligned}
$$

This forces $D' \cap C_D(V_1) = 1$. So if $x \in D$ and $c \in C_D(V_1)$, then $[x, c] \in D' \cap C_D(V_1) = 1$. This shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \dots p$.

Now we consider the k in (+).

First suppose that $k = 1$, then $W_1 = X_1$, but since $W_1 = V_2 \oplus \cdots \oplus V_p$, we see that $X_1 = V_2$ and $p = 2$. In particular, V_2 is an irreducible faithful $C_D(V_1)$-module, so $C_D(V_1)$ is cyclic and has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits, which shows that $(|V_2| - 1)/2 = |C_D(V_1)|$. Since $|C_D(V_1)|$ is a power of 2, by an elementary result from number theory (see [10, Proposition 3.1]) it follows that—if we write q for the characteristic of V—either V_2 is of dimension 1 and $|V_2|$ is a Fermat prime, or $|V_2| = 9$ and $|C_D(V_1)| = 4$. In the former case we get that $D/C_D(V_1)$ is abelian and hence D is abelian, and so G is abelian (since $G' = D'$), a contradiction. In the latter case we get that $D/C_D(V_1)$ must be at least of order 8 (since it has exactly one maximal orbit (of size M_1) on V_1, and it must be isomorphic to a subgroup the semidehedral group SD_{16}, as Sylow 2-subgroup of

GL $(2,3)$. However, all such subgroups have center of order 2, contradicting the fact that $|C_D(V_1)| = |C_D(V_2)| = 4$ and $C_D(V_1) \leq Z(D)$. This concludes the case that $k = 1$.

So let $k > 1$. Then define $X_0 = 0$ and $L_i = C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_i)/C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_{i+1})$ for $i = 0, \ldots, k-1$. As $k > 1$, we see that L_0 has exactly one regular orbit on X_1, because otherwise also L_1 would have at least two regular orbits on X_2 which ultimately would lead to $C_D(V_1)$ having at least four regular orbits on W_1, a contradiction. Since all orbits of L_0 on X_1 must be regular, we thus conclude that $|L_0| = |X_1| - 1$. Since $C_D(V_1)$ has exactly two regular orbits on W_1, it follows that there is exactly one $l \in \{1, \ldots, k\}$ such that L_{l-1} has exactly two regular orbits on X_l, whereas all the other L_i's have exactly one regular orbit on X_{i+1}. However, since L_{l-1} only has regular orbits on $X_l - \{0\}$, it is clear that the single regular orbit of size $|X_l| - 1$ of $C_D(V_1)/C_{C_D(V_1)}(X_l)$ on X_l splits into at least p regular orbits for L_{l-1} on X_L. This shows that $p = 2$. Hence $C_D(V_1)C_D(V_2) = C_D(V_1) \times C_D(V_2) \leq Z(G)$, and since $D/C_D(V_1)$ acts faithfully and irreducibly on V_1, we see that $C_D(V_1) \cong C_D(V_1)C_D(V_2)/C_D(V_2)$ is cyclic and thus has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits and we now can arrive at a contradiction just as in the case that $k = 1$

This concludes Case 2.4.1.2, and thus Case 2.4.1 is completed.

Case 2.4.2 $C_D(V_1)$ has exactly one orbit of size M_2 on W_1.

Then by Theorem 1.1 $C_D(V_1)$ is abelian and hence has regular orbits on W_1, so $M_2 = |C_D(V_1)|$ and the same argument as at the beginning of Case 2.4.1.2 shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \ldots p$.

Assume that $X_1 < V_2$ (where X_1 is as in (+)). Since $C_D(V_1)$ has exactly one regular orbit on W_1, it also has exactly one regular orbit on V_2, and since V_2 is not irreducible as $C_D(V_1)$-module, by Lemma 2.2 it is clear that $C_D(V_1)/C_{C_D(V_1)}(V_2)$ is not cyclic. But since $C_D(V_1) \leq Z(D)$, we see that

$$\begin{aligned} C_D(V_1)/C_{C_D(V_1)}(V_2) &= C_D(V_1)/C_D(V_1 \oplus V_2) = C_D(V_1)/(C_D(V_1) \cap C_D(V_2)) \\ &\cong C_D(V_1)C_D(V_2)/C_D(V_2) \end{aligned}$$

is a noncyclic central subgroup of $D/C_D(V_2)$. But on the other hand, $D/C_D(V_2)$ acts faithfully and irreducibly on V_2 and hence has a cyclic center, and we have a contradiction. This shows that $X_1 = V_2$, so V_2 is an irreducible $C_D(V_1)$-module and $C_D(V_1)$ has exactly two orbits (one of them being the trivial orbit) on V_2. Therefore again by [10, Proposition 3.1]) it follows that—if we write q for the characteristic of V—either

- $p = 2$, V_2 is of dimension 1 and $|V_2|$ is a Fermat prime; or
- $q = 2$ and $|C_D(V_1)/C_D(V_1 \oplus V_2)| = p$ is a Mersenne prime; or
- $p = 2$, $q = 3$, $|V_2| = 9$ and $|C_D(V_1)| = 8$.

In the first case we get (as earlier) that $D/C_D(V_1)$ is abelian and thus D is abelian, a contradiction. In the second case, since D is a p-group, with [10, Theorem 2.1] we see that $D/C_D(V_1)$ cyclic of order p and thus abelian, making D abelian, a contradiction. So we are left with the third case. Here we have that $D/C_D(V_1)$ is a subgroup of the semidihedral group of order 16, so $|G| \leq 2^9$, and $|G| \leq 2^8$ unless $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$. Moreover, since any $g \in G - D$ interchanges $C_D(V_1)$ and $C_D(V_2)$, by taking commutators of elements in $C_D(V_1)$ with g we easily see that $|D'| \geq 8$ and so $|G'| \geq 8$. Now D has an orbit of size $\geq 2^6$ on V (from the regular orbit of $C_D(V_1) \times C_D(V_2)$. So if $|G| \leq 2^8$, we get $2^5 < 2^6 \leq M = |G/G'| \leq 2^8/2^3 = 2^5$, a contradiction. This leaves us with $|G| = 2^9$, and

$D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$, but in this case for similar reasons as above we see that $|D'| \geq 2^4$ and thus get the contradiction $2^5 < 2^6 \leq M = |G/G'| \leq 2^9/2^4 = 2^5$.

This final contradiction concludes the proof of the theorem. $\Diamond$

4 The proof

The goal of this section is to prove the main result of this paper. Note that in [5] we gave an outline of the whole proof and its many cases; since here we follow the same outline, we do not repeat it here and refer the reader to [5] instead.

First, we restate the result,

Theorem 4 *Let G be a finite nonabelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly three orbits of size M on V. Then $G = D_8 \circ C_4$, the central product of the dihedral group of order 8 and the cyclic group of order 4, and $V = V(2, 5)$.*

Proof Assume that the result is not true and let G, V be a counterexample such that $|GV|$ is minimal. By [12, Theorem 2.9], which combines results of the first author, Qian, and Yang, we know that G must be nilpotent. So next we will make the reduction to p-groups. $\square$

Step 1 A Reduction to p-groups

Assume that G is not a p–group, then $|G|$ is divisible by at least two distinct primes, call them p and q. Let $P \in \mathrm{Syl}_p(G)$ and $H \in \mathrm{Hall}_q(G)$. Since G is nilpotent, we know that $P \lhd G$. Furthermore, we know that $G = P \times H$. By the hypothesis we know V is a finite G-module over a field, call it K. By [13, Lemma 10], there exists a field extension L of K such that if U is an irreducible summand of V viewed as an $LG-module$, then the permutation actions of G on V and U are permutation isomorphic. This allows us to consider the action of G on U instead of V. Through relabeling we can assume V is absolutely irreducible. Using [1, (3.6)] we may assume that $V = X_1 \otimes X_2$ where X_1 is a faithfully irreducible P-module and X_2 is a faithfully irreducible H-module. We can pick an $x_1 \in X_1$ and a $x_2 \in X_2$ such that $|x_1^P|$ is the largest orbit size of P on X_1, $|x_2^H|$ is the largest orbit size of H on X_2, and we have

$$|P/P'| \leq |x_1^P|,$$
$$|H/H'| \leq |x_2^H|.$$

If $g \in P$ and $g \in H$ such that $gh \in C_G(x_1 \otimes x_2)$ then $x1g = \alpha x_1$ and $x_2 h = \beta x_2$ where α, β are scalars in the field with $\alpha\beta = 1$ [9, Lemma 3.3]. Now g and h have coprime orders so we have that $\alpha = \beta = 1$ which gives

$$C_G(x_1 \otimes x_2) = C_P(x_1) \times C_H(x_2).$$

We now have the following

$$\begin{aligned} |G/G'| = M \geq |(x_1 \otimes x_2)^G| &= |G : C_p(x_1) \times C_H(x_2)| \\ &= |P : C_P(x_1)||H : C_H(x_2)| \\ &= |x_1^P||x_2^H| \geq |P/P'||H/H'| = |G/G'|. \end{aligned} \tag{1}$$

Since we have inequality everywhere in (1), we have $|x_1^P| = |P/P'|$ and $|x_2^H| = |H/H'|$. Therefore $|P/P'|$ is the largest orbit size of P on X_1 and similarly $|H/H'|$ is the largest orbit size of H on X_2. From now on let $M_1 = |P/P'|$ and $M_2 = |H/H'|$. We can now break this into four cases.

Case 1 P and H have exactly one maximal orbit of size M_1 and M_2, respectively. Then by Theorem 1, both P and H are abelian. But if P and H are both abelian then $G = P \times H$ is abelian. A contradiction to G being nonabelian.

Case 2 P and H each have at least two maximal orbits of size M_1 and M_2 respectively. Let $y_1 \in X_1$ be a representative of a second orbit of size M_1 on X_1 and $y_2 \in X_2$ be a representative of second orbit of size M_2 on X_2. Then using (1),

$$|G/G'| = |(x_1 \otimes x_2)^G| = |(x_1 \otimes y_2)^G| = |(y_1 \otimes x_2)^G| = |(y_1 \otimes y_2)^G| = M$$

we have four orbits of size M in the action of G on V, contradicting G having exactly three orbits of size M.

Case 3 P has exactly two orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2, or the opposite, H has exactly two orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. Consider the first option. Suppose P has exactly two orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2. Let $y_1 \in X_1$ represent the second orbit of P on X_1. Then $(x_1 \otimes x_2)^G$ and $(y_1 \otimes x_2)^G$ are two distinct orbits of size M of G on V. Since H has exactly one maximal orbit, by Theorem 1 H must be abelian. Therefore $|H/H'| = |H| = |x_2^H|$ and by Lemma 2$H \cong H/C_H(X_2)$ and H is a cyclic group of order $|X_2| - 1$. We know P cannot be abelian or else $G = P \times H$ would be abelian. Thus we can use [5] to get $P = D_8$ and $X_1 = V(2,3)$. This makes $\mathrm{char}(X_1) = 3$ and $\mathrm{char}(X_2) = 3$, leaving us with $|H| = 3^n - 1$ for some $n \in \mathbb{N}$. This makes $|H|$ even and contradicts that $P \in \mathrm{Syl}_2(G)$.

Now consider the second option, i.e., when H has exactly two orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. By Theorem 1 P is an abelian group and by Lemma 2 P is a cyclic group of order $|X_1| - 1$. We know H cannot be abelian or else G is, so we may use Theorem 2 to see that $H = D_8$ and $X_2 = V(2,3)$. This means that $|H| = 8$ and $\mathrm{char}(X_1) = \mathrm{char}(X_2) = 3$ and $|P|$ is even, contradicting that $\gcd(|H|, |G|) = 1$.

Case 4 P has exactly three orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2, or the opposite, H has exactly three orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. Consider the first option. Suppose P has exactly three orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2. Let $y_1, y_2 \in X_1$ represent the other two orbits of P on X_1. Then $(x_1 \otimes x_2)^G$, $(y_1 \otimes x_2)^G$ and $(y_2 \otimes x_2)^G$ are three distinct orbits of size M of G on V. Since H has exactly one maximal orbit, by Theorem 1 H must be abelian. Therefore $|H/H'| = |H| = |x_2^H|$ and by Lemma 2 $H \cong H/C_H(X_2)$ and H is a cyclic group of order $|X_2| - 1$. We know P cannot be abelian or else $G = P \times H$ would be abelian. Thus we can use induction to get $P = D_8 \circ C_4$ and $X_1 = V(2,5)$. This makes char $(X_1) = 5$ and char $(X_2) = 5$, leaving us with $|H| = 5^n - 1$ for some $n \in \mathbb{N}$. This makes $|H|$ even and contradicts that $P \in \mathrm{Syl}_2(G)$. Now consider when H has exactly three orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. By Theorem 1 P is an abelian group and by Lemma 2 P is a cyclic group of order $|X_1| - 1$. We know

H cannot be abelian or else G is, so we may use induction to see that $H = D_8 \circ C_4$ and $X_2 = V(2,5)$. This means that $|H| = 16$ and char(X_1) =char$(X_2) = 5$ and $|P|$ is even, contradicting that $\gcd(|H|, |G|) = 1$.

Step 2 A Reduction to the Case that V is Imprimitive

The next step is to show that V is imprimitive. We will recreate the proof from [5] for the sake of completeness. Assume that V is quasiprimitive. Using the proof of Theorem 3.3 in [10] we can write $G = S \times T$ where S is a 2-group and T is a cyclic group of odd order and by Corollary 1.3 in [10] G is cyclic, quaternion, dihedral, or semi-dihedral and $G \ncong D_8$. It is well known that the derived subgroup of the quaternion, dihedral, and semi-dihedral 2-groups have index 4. We also know there exists a $U \lhd G$ where U is cyclic, $|G : U| \leq 2$ and U has a regular orbit on V. This gives the inequality $M \geq |U| \geq |G|/2$. Since G is a nonabelian p−group, we have $|T| = 1$ and $G = S$, and $p = 2$. This makes G a nonabelian 2−group so $|G| > 4$ and $8 \mid |G|$. All together we have

$$M = |G/G'| = 4 \leq |G|/2 \leq M$$

We have equality here so $|G'| = 2$ and $|G| = 8$ making G the quaternion group. It is known that the quaternion group has a regular orbit on V [10, Lemma 4.2(a)] contradicting $M = 4$. Therefore V cannot be quasiprimitive. In particular, V is imprimitive.

Step 3 The Case Where V is Imprimitive

Since we know V is imprimitive now, then there exists a $D \trianglelefteq G$ with $|G : D| = p$ where p is prime and $V_D = V_1 \oplus \cdots \oplus V_p$ for irreducible D-modules V_i of V_D. There are two cases to consider, $D' < G'$ and $D' = G'$.

Step 3.1 The Case Where $D' < G'$

Suppose $D' < G'$, that is, $p|D'| \leq |G'|$. Then by using Theorem 1 we have the following inequality:

$$M \geq |D : D'| = \frac{|D|}{|D'|} = \frac{p|D|}{p|D'|} \geq \frac{|G|}{|G'|} = M.$$

Therefore $|D : D'| = M$. Since $\cap_{i=1}^{p} C_G(V_i) = 1$, we have that D is isomorphic to a subgroup of $D/C_D(V_1) \times \ldots \times D/C_D(V_p)$ [7, equation (4)]. From now on we will use the symbol $H \lessapprox G$ to denote the fact that H is isomorphic to a subgroup of G. Therefore

$$D \lessapprox D/C_D(V_1) \times \cdots \times D/C_D(V_p) = \bigtimes_{i=1}^{p} D/C_D(V_i) =: T. \tag{2}$$

The above equation tells us that if $D/C_D(V_1)$ is abelian for any $i = 1, \ldots, p$ then $D/C_D(V_i)$ is abelian for all $i = 1, \ldots, p$ and D will follow. This is an important fact that we will reference in the following arguments. We know from Theorem 1 that since $|D : D'| = M$ then one of the following is true: D is abelian, D has 2 orbits of size M on V_D or D has 3 orbits of size M on V_D. We know that D cannot have more than 3 orbits of size M on V_D or else G would have more than three orbits of size M on V or an orbit larger than size M on V. Recall that $V_D = V_1 \oplus V_2 \oplus \cdots \oplus V_p$, with each V_i being irreducible faithful D-modules. Let $W_i = \oplus_{j=1, j \neq i}^{p} V_j$. Write M_1 for the largest orbit size of D on V_1 and M_2 for the largest orbit size of $C_D(V_1)$ on W_1. Also let M_D be the largest orbit size of D on V. Let $x \in V_D$ be in a largest orbit of D on V_D. Write $x = x_1 + x_2$ for some $x_1 \in V_1$ and $x_2 \in W_1$. Observe that

$$\begin{aligned} M_D &= |D : C_D(x)| = |D : C_D(x_1) \cap C_D(x_2)| \\ &= |D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(x_2)| = |x_1^D||x_2^{C_D(x_1)}|. \end{aligned}$$

If $|x_1^D| < M_1$, then the same calculation would show that if $y_1 \in V_1$ with $|y_1^D| = M_1$, then $|D : C_D(y_1 + x_2)| > M_D$, contradicting the definition of M_D. Thus we have $|x_1^D| = M_1$. Moreover, $|x_2^{C_D(x_1)}| \geq |x_2^{C_D(V_1)}|$. We also can conclude that $|x_2^{C_D(x_1)}| \geq M_2$, because if $|x_2^{C_D(x_1)}| < M_2$, then let $y_2 \in W_1$ such that $|y_2^{C_D(V_1)}| = M_2$, and then

$$\begin{aligned} M_D = |(x_1 + y_2)^D| = |D : C_D(x_1) \cap C_D(y_2)| = |D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(y_2)| \\ M_1|y_2^{C_D(x_1)}| = M_1 M_2 > M_1|x_2^{C_D(x_1)}| = |D : C_D(x)| = M_D, \end{aligned}$$

a contradiction. Thus altogether we get $M \geq M_D \geq M_1 M_2$.Then

$$\begin{aligned} M &\geq |x^D| = |D : C_D(x)| = |D : C_D(x_1) \cap C_D(x_2)| \\ &= |D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(x_2)| = M_1|x_2^{C_D(x_1)}| \geq M_1|x_2^{C_D(W_1)}| = M_1 M_2. \end{aligned}$$

By applying Theorem 1 $|D : D'|$ divides $|D/C_D(V_1) : D/C_D(V_1)'||C_D(V_1)/C_D(V_1)'|$ and we have

$$M \geq M_1 M_2 \geq |D/C_D(V_1) : (D/C_D(V_1)')||C_D(V_1) : C_D(V_1)'| \geq |D : D'| = M.$$

This shows $M_1 M_2 = M_D = M$ and it follows that $M_1 = |D/C_D(V_1) : (D/C_D(V_1))'|$ and $M_2 = |C_D(V_1) : C_D(V_1)'|$. Suppose that there exists $y_1, z_1, a_1 \in V_1$ where y_1, z_1, a_1 are representatives of 3 new orbits of size M_1 of $D/C_D(V_1)$ acting on V_1. Then as we have seen previously, $x_1 + x_2, y_1 + x_2, z_1 + x_2$, and $a_1 + x_2$ are four representatives of orbits of size M of D on V_D. This contradicts our hypothesis. So we know $D/C_D(V_1)$ has at most 3 orbits of size M_1 on V_1. We also know that if $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 then $D/C_D(V_1)$ is abelian by Theorem 1 and by (2) D would also be abelian. Therefore we can split the case into D being abelian and D being nonabelian. We will look at the case that D is nonabelien first.

Step 3.1.1 The Case Where D is Nonabelian

When D is nonabelian then we know $D/C_D(V_1)$ must have either two or three orbits of size M_1 on V_1. We know by (2) that $D/C_D(V_1)$ is not abelian or else D would be abelian as a result. We also know that $C_D(V_1)$ must have exactly one orbit of size M_2 on W_1 or else G would have too many orbits of size M as a result, this is a similar argument as made in the reduction to p−groups section. Thus $D/C_D(V_1)$ has either exactly two orbits or exactly three orbits and $C_D(V_1)$ has exactly 1.

Step 3.1.1.1 The Case Where $D/C_D(V_1)$ has Exactly Two Orbits of Size M_1

When $D/C_D(V_1)$ has exactly two orbits of size M_1, by [5], $D/C_D(V_1) = D_8, V_1 = V(2, 3)$, and $p = 2$ so $|W_1| = |V_1| = |V_2|$. We have $D \lessapprox D/C_D(V_1) \times D/C_D(V_2) \cong D_8 \times D_8$ making $C_D(V_1) \lessapprox D_8$. If $C_D(V_1) \cong D_8$ then $C_D(V_1)$ has two orbits of size M_2 on V_2, contradicting $C_D(V_1)$ having exactly one orbit of size M_2 on $W_1 = V_2$. Therefore $C_D(V_1)$ is of order one, two or four. This makes $C_D(V_1)$ abelian which gives it at least two regular orbits (i.e, two orbits of size $M_2 = |C_D(V_1)|$) unless $C_D(V_1)$ is the Klein four-group. Since $C_D(V_1)$ has only one orbit of size M_2 on V_2, we conclude that $C_D(V_1)$ is the Klein four-group. Since $C_D(V_1)$ and $C_D(V_2)$ are conjugate under the action of G, then $C_D(V_2)$ is also a Klein four-group. Thus $C = C_D(V_1)C_D(V_2)$ is elementary abelian of order 16. Let $d \in D - C$, then $[d, C_D(V_1)] \leq C_D(V_1)$ is cyclic of order 2, $[d, C_D(V_2)] \leq C_D(V_2)$ is cyclic of order 2 and $C_D(V_1) \cap C_D(V_1) = 1$. It follows that $|D'| \geq 4$, so $16 = M = |D/D'| \leq 32/4 = 8$, a contradiction.

Step 3.1.1.2 The Case Where $D/C_D(V_1)$ has Exactly Three Orbits of Size M_1

When $D/C_D(V_1)$ has exactly three orbits of size M_1, by induction, we have $D/C_D(V_1) = D_8{\circ}C_4$, $V_1 = V(2,5)$, and $p = 2$ thus $|W_1| = |V_1| = |V_2|$. We have $D \lessapprox D/C_D(V_1) \times D/C_D(V_2) \cong (D_8{\circ}C_4) \times (D_8{\circ}C_4)$ making $C_D(V_1) \lessapprox D_8{\circ}C_4$. But, if $C_D(V_1) \cong D_8{\circ}C_4$ then $C_D(V_1)$ has three orbits of size M_2 on V_2; a contradiction to $C_D(V_1)$ having exactly one orbit of size M_2 on $W_1 = V_2$. Therefore $C_D(V_1)$ must be isomorphic to a proper subgroup of $D_8{\circ}C_4$ (i.e. $C_D(V_1)$ is one of the following: C_2, C_4, V_4, $C_2 \times C_4$, Q_8, D_8). All proper subgroups of $D_8{\circ}C_4$ have at least two regular orbits, except for the dihedral group of order 8. Therefore we can conclude $C_D(V_1)$ is D_8. Since $C_D(V_1)$ and $C_D(V_2)$ are conjugate under the action of G, then $C_D(V_2)$ is also a dihedral group of order 8. Let $v_1 \in V_1$ be in an orbit of size $M_1 = 8$ of $D/C_D(V_1)$ on V_1, such that v_1 is in a regular orbit of $C_D(V_2)$ on V_1. We claim that $|C_D(v_1)| = 16$. Note that $D_8 \cong C_D(V_1) \leq C_D(v_1)$ so $|C_D(v_1)| \geq 8$. Assume $|C_D(v_1)| = |C_D(V_1)| = 8$.Then $|v_1^D| = |D : C_D(v_1)| = 16$ and $v_1^D \leq V_1$ so D has an orbit of size 16 on V_1, a contradiction to $M_1 = 8$. Thus $|C(v_1)|$ must be greater than 8 and as a p-group, it must be of order at least 16. Since $|C_D(v_1) \cap C_D(V_2)| = 1$, then $|C_D(v_1)C_D(V_2)| \geq 16 \cdot 8 = |D|$. Combined with $C_(v_1)C_D(V_2) \leq D$ and $|D_D(v_1)C_D(V_2)| = |C_D(v_1)||C_D(V_2)|$, gives us $|C_D(v_1)| = 16$. Now $|C_D(v_1)| = 16$ and $C_D(v_1) \cap C_D(V_2) = 1$, therefore

$$C_D(v_1) \cong C_D(v_1)/(C_D(v_1) \cap C_D(V_2)) \cong C_D(v_1)C_D(V_2)/C_D(V_2) \cong D/C_D(V_2) \cong D_8{\circ}C_4$$

where the third isomorphism comes from the isomorphism theorems. Hence we have $C_D(v_1) \cong D_8{\circ}C_4$. Therefore $C_D(v_1)$ acts faithfully on V_2 and if $z_1, z_2, z_3 \in V_2$ are representatives of the three orbits of size eight in the action of $C_D(v_1)$ on V_2, then $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$ are representatives of three orbits of size $64 = M_D = M$ of D on V_D. Now let $w_1 \in V_1$ be in an orbit of D of size $M_1 = 8$ such that w_1 is not in a regular orbit of $C_D(V_2)$ on V_1. Let $z_4 \in V_2$ be in the (unique) regular orbit of $C_D(V_1)$ on V_2 (so it is of size $M_2 = 8$). Then clearly $w_1 + z_4$ is a representative of an orbit of size $64 = M$ of D on V that is different from the orbits containing $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$. Thus we have found four orbits of size $64 = M$ of D on V which contradicts our hypothesis. This concludes the case where D is nonabelian.

Step 3.1.2 The Case Where D is Abelian

When D is abelian, there are three possibilities; D has exactly one orbit of size M on V, D has exactly two orbits of size M on V, and D has exactly three orbits of size M on V. Clearly D cannot have four orbits of size M or else G has four orbits of size M on V or an orbit larger than M on V. Recall that $|G/G'| = M = M_D = |D| = |G|/p$, thus $|G'| = p$. We may use Theorem 3.2 from [4] to state that $p = 2$ and there exists a $v \in V$ such that $|C_G(v)| \leq 2$, in particular $|C_G(v)| = 2$ in our case or else G would have a regular orbit making $|G| = M = |D|$ which contradicts $p|D| = |G|$. This allows us to improve on the proof given by [5].

Step 3.1.2.1 The Case Where D has Exactly One Orbit of Size M

Since D is abelian, D has regular orbits, so $M = |D|$. Therefore D must have exactly one regular orbit. Then by Lemma 2, we have $D = \times_{i=1}^{p} C_D(W_i)$. We can note that G/D cycles these direct factors around, therefore $|G : G'| = p|C_D(W_1)|$ (see [5]). Additionally, $|G/G'| = |D| = |C_D(W_1)|^p$ which implies that $|C_D(W_1)|^{p-1} = p$. Since $p = 2$, then $|C_D(W_1)| = 2$. It follows that $|D| = |C_D(W_1)|^p = 4$, D is elementary abelian, $|G| = 8$, and G is nonabelian. Since $M = |D| = 4$ and $|G| = 8$, G does not have a regular orbit on V making G the dihedral group of order 8. Since D is elementary abelian and has exactly one regular orbit on V, we can conclude that $|V| = 9$. We know by [5] that $G = D_8$ has exactly 2 orbits of size M on $V = V(2,3)$. This contradicts that G has exactly three orbits of size M on V in the hypothesis. This concludes the case where D has exactly one orbit of size M on V.

Step 3.1.2.2 The Case Where D has Exactly Two Orbits of Size M

Assume that D has exactly two orbits of size M on V, we know that these are regular orbits as D is abelian. To keep consistent, we will denote the largest orbit size of the action of $D/C_D(V_1)$ on V_1 as M_1, and M_2 will denote the largest orbit size of $C_D(V_1)$ acting on $W_1 = V_2$. Recall that $D/C_D(V_1)$ has at most two orbits of size M_1 on V_1 if D has only 2 orbits. So for $i = 1, 2$, $D/C_D(V_i)$ are isomorphic and $D/C_D(V_i)$ has either one or two orbits of size M_1 acting on V_i. Suppose we have two orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then as argued previously $C_D(V_1)$ must have exactly one orbit of size M_2 on V_2 or else G would have too many orbits of size M on V. Since $D/C_D(V_1)$ has two regular orbits on V_1, then $D/C_D(V_2)$ will have at least two regular orbits on V_2, which immediately implies that $C_D(V_1)$ has two regular orbits on V_2, contradiction. Therefore we know that $D/C_D(V_1)$ has only one orbit of size M_1 in the action on V_1 and $C_D(V_1)$ has exactly two orbits of size M_2 on V_2. Thus we have $|D/C_D(V_1)| = |V_1| - 1$, and $D/C_D(V_1)$ is cyclic. Now we have $D \lnapprox D/C_D(V_i) =: T$ and T has exactly one regular orbit on V. Every regular orbit of T on V splits into $\frac{|T|}{|D|}$ regular orbits of D. Since D has no more than two orbits of size $M = M_D = |D|$, we see that $\frac{|T|}{|D|} \leq 2$. If $\frac{|T|}{|D|} = 1$, then $T = D$ and so $C_D(V_1) \cong \times_{i=2}^{p} D/C_D(V_i)$ has only one regular orbit on W_1, a contradiction. Therefore we know that $\frac{|T|}{|D|} = 2$. Then from Lemma 2 we have that $|T| = (|V_i| - 1)^p$, thus

$$(|V_1| - 1)^p = 2|D| = 2p^k \tag{3}$$

for appropriate k. We wish to show that $|C_D(V_1)| = 2$. Observe that $C_D(V_1) \cap C_D(V_2) = 1$ and so $C_D(V_1) \times C_D(V_2) = C_D(V_1)C_D(V_2) \leq D$. Let $g \in G - D$ and $(1, a) \in C_D(V_1) \times C_D(V_2)$, then G' contains the element

$$[(1, a), g] = (1, a)^{-1} g^{-1} (1, a) g = (1, a)^{-1} (1, a)^g = (1, a^{-1})(a^*, 1) = (a^*, a^{-1})$$

for suitable $a^* \in C_D(V_1)$. If there are more than two choices for $a \in C_D(V_2)$ then $|G'| \geq 3$. On the other hand, $\frac{|G|}{|G'|} = |D|$ and $\frac{|G|}{|D|} = 2$ so $|G'| = |G|/|D| = 2$. We conclude that $|C_D(V_1)| \leq 2$. If $|C_D(V_1)| = 1$ then $D = D/C_D(V_1)$, but we know that D has exactly two orbits of size M_1 on V_1 while $D/C_D(V_1)$ has only one orbit of size M_1 on V_1. Therefore we can say $|C_D(V_1)| = 2$. Therefore we can say $|C_D(V_1)| = 2$. We can now determine $|D|$ using $\frac{|T|}{|D|} = \frac{|D/C_D(V_1)||D/C_D(V_2)|}{|D|} = 2$. So $2|D| = \frac{|D|}{2}\frac{|D|}{2} = \frac{|D|^2}{4}$, or $|D| = 8$ and $|G| = 16$. From Lemma 2 $|D/C_D(V_1)| = 4 = |V_1| - 1$ giving us $|V_1| = 5, i = 1, 2$, and thus $V = \mathrm{GF}(5)^2$. Then $\{(1, 0), (2, 0), (3, 0), (4, 0)\}$ and $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$ are both orbits of D on V (as $D/C_D(V_1)$ has an orbit of size 4 on V_i), and their union is an orbit of size 8 of G on V. Moreover, if $a, b \in \mathrm{GF}(5) - \{0\}$, then $(a, b)^D$ will contain $(a, -b)$ as $C_D(V_1)$ acts as $x \to -x$ on V_2. Since $D/C_D(V_1)$ has an orbit of size 4 on V_1, we also see that $(a, b)^D$ will contain elements of the form $(1, *), (2, *), (3, *)$, and $(4, *)$. Altogether we see that $|(a, b)^D| = 8 = M$. Putting this together shows that, since $8 = M$, G has three orbits of size 8 on V. We know that G must be a subset of the Sylow 2-group of $GL(2, 5)$. There are three subgroups the the Sylow 2-group of $GL(2, 5)$ that G can possibly be: $C_4 \times C_4$, $M_4(2)$ (the maximum modular cyclic group $M_n(2)$,a semidirect product $C_2^{n-1} \rtimes C_2$ where C_2 acts on C_2^{n-1} by $x \mapsto x^{2^{n-2}+1}$), and our group $D_8 \circ C_4$. We know that $G \neq C_4 \times C_4$ as $C_4 \times C_4$ is abelian and G is not. By calculating the orbits of $M_4(2)$ on $V(2, 5)$ by matrix multiplication, we receive one orbit of size 8 and one orbit of size 16. But we know that our group has 3 orbits of size

$M = 8$. This leaves only $G = D_8 \circ C_4$, a second verification that $D_8 \circ C_4$ satisfies our hypothesis. This concludes the case that D has exactly two orbits of size M.

Step 3.1.2.3 The Case Where D has Exactly Three Orbits of Size M

Assume that D has exactly three orbits of size M on V, we know that these are regular orbits as D is abelian. We will denote the largest orbit size of the action of $D/C_D(V_1)$ on V_1 as M_1, and M_2 will denote the largest orbit size of $C_D(V_1)$ acting on W_1. Recall that $D/C_D(V_1)$ has at most three orbits of size M_1 on V_1. So the $D/C_D(V_i)$ for $i = 1, \dots, p$ are all isomorphic and $D/C_D(V_i)$ has either one, two, or three orbits of size M_1 acting on V_i. Suppose we have three orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then as argued previously $C_D(V_1)$ must have exactly one orbit of size M_2 on W_1 or else we have too many orbits as they do not combine. Suppose we have two orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then if $C_D(V_1)$ has two or three orbits of maximal size M_2 then we have too many orbits and if $C_D(V_1)$ has one orbit of maximal size then we do not have enough. Therefore we know that $D/C_D(V_1)$ has only one orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1 and $C_D(V_1)$ has exactly three orbits of size M_2 on W_1. Thus we have $|D/C_D(V_1)| = |V_1| - 1$, and $D/C_D(V_1)$ is cyclic. Now we have $D \lessapprox \times_{i=1}^{p} D/C_D(V_i) =: T$ and T has exactly one regular orbit on V. Every regular orbit of T on V splits into $\frac{|T|}{|D|}$ regular orbits of D. Since D has no more than three orbits of size $M = M_D = |D|$, we see that $\frac{|T|}{|D|} \leq 3$. If $\frac{|T|}{|D|} = 1$, then $T = D$ and so $C_D(V_1) \cong \times_{i=1}^{p} D/C_D(V_i)$ has only one regular orbit on W_1, a contradiction. Suppose $\frac{|T|}{|D|} = 2$. Then from Lemma 2 we have that $|T| = (|V_i| - 1)^p$, thus

$$(|V_1| - 1)^p = 2|D| = 2p^k \tag{4}$$

for appropriate k. Since $p = 2$, we have $V = V_1 \oplus V_2$ and $W_2 = V_2 \oplus .. \oplus V_p = V_2$. We know that $|C_D(V_1)| < |D/C_D(V_2)|$ and that $C_D(V_1) < D/C_D(V_2)$. So the regular orbit of $D/C_D(V_2)$ would split into $\frac{|D/C_D(V_2)|}{|C_D(V_1)|}$ regular orbits. This is impossible as $C_D(V_1)$ has 3 regular orbits. Therefore we are left with $\frac{|T|}{|D|} = 3$, then $p = 3$. But as stated previously in step 3.1.2, by Theorem 3.2 in [4], $p = 2$, a contradiction. This concludes the case where D has exactly three orbits and in turn concludes the broader cases where D is abelian and $D' < G'$

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Step 3.2 The Case Where $D' = G'$

We will consider the action of $D/C_D(V_1)$ on V_1 and $C_D(V_1)$ acting on W_1. Using Theorem 1, we have the following inequalities

$$|D : D'C_D(V_1)| \leq M_1 \tag{5}$$

$$|C_D(V_1) : C_D(V_1)'| \leq M_2 \tag{6}$$

where M_1 is the largest orbit size of $D/C_D(V_1)$ on V_1 and M_2 is the largest orbit size of $C_D(V_1)$ on W_1. There are now four cases to consider; strict inequality in (5) and (6), equality in (5) and strict inequality in (6), equality in both (5) and (6), and equality in (6) and strict inequality in (5).

Step 3.2.1 The Case Where We have Strict Inequality in (5) and (6)

First we consider the case where we have strict inequality in (5) and (6). Because G is a p-group we know that $p|D : D'C_D(V_1)| \leq M_1$ and $p|C_D(V_1) : C_D(V_1)'| \leq M_2$. Therefore

$$M \geq M_1 M_2 \geq p^2 |D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'|.$$

Recall $|G : D| = p$ and notice that $|D : D'| \leq M_D \leq pM_1M_2$ so

$$p^2|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|G : D||D : D'| = p|G : D'| = p|G : G'| > |G : G'|.$$

Putting the above equations together we have $|G : G'| < M$. This contradicts our hypothesis that $|G : G'| = M$, and therefore either (5) or (6) must be an equality.

Step 3.2.2 The Case Where We have Equality in (5) and Strict Inequality in (6)

Consider the case of equality in (5), that is $|D : D'C_D(V_1)| = M_1$. If $D/C_D(V_1)$ is abelian then by (2) we have D is abelian. If D is abelian then $1 = D' = G'$ and G is abelian, a contradiction. Therefore, we note that $D/C_D(V_1)$ cannot be abelian for the rest of the paper. By Theorem 1 we have that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Let $v_1, v_2 \in V_1$ be representatives of two different orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1. Let $w \in W_1$ be in an orbit of size M_2 in the action fo $C_D(V_1)$ on W_1. Because (6) is strict, we have $p|C_D(V_1) : C_D(V_1)'| \leq M_2$. This gives us the following,

$$\begin{aligned} M &\geq |(v_i + w)^G| \geq |v_i^{D/C_D(V_1)}||w^{C_D(V_1)}| = M_1M_2 \\ &\geq p|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq |G : G'| = M \end{aligned}$$

for $i = 1, 2$. This gives equality everywhere. Since G has exactly three orbits of size M on V, $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1. Suppose $D/C_D(V_1)$ has two orbits of size M_1 on V_1, then by [5] $D/C_D(V_1) \cong D_8, |V_1| = 9$ and $p = 2$. Thus $M_1 = 4$ and $M_2 \leq 4$. This gives us $C_D(V_1) \leq D_8$. If $C_D(V_1) \cong D_8$ then $|C_D(V_1) : C_D(V_1)'| = 4$ contradicting (6) is strict. Therefore $C_D(V_1)$ is size one, two, or four. This makes $C_D(V_1)$ abelian. That means $M_2 = |C_D(V_1)| = |C_D(V_1) : C_D(V_1)'|$ which contradicts (6) is strict. Thus $D/C_D(V_1)$ must have three orbits of size M_1 on V_1 and by induction $D/C_D(V_1) \cong D_8 \circ C_4$, $|V_1| = 25$ and $p = 2$. Thus $M_1 = 8$ and $M_2 \leq 8$. This gives us $C_D(V_1) \leq D_8 \circ C_4$. If $C_D(V_1) \cong D_8 \circ C_4$ then $|C_D(V_1) : C_D(V_1)'| = 8$ contradicting (6) is strict. Therefore $C_D(V_1)$ is size one, two, four, or eight. If $C_D(V_1)$ is of size one, two, or four then $C_D(V_1)$ is abelain and that means $M_2 = |C_D(V_1)| = |C_D(V_1) : C_D(V_1)'|$ which contradicts (6) is strict. So, $C_D(V_1)$ must be of size eight and thus be D_8, $C_2 \times C_4$, or Q_8 as it is a subgroup of $D_8 \circ C_4$. If $C_D(V_1)$ is $C_2 \times C_4$ or Q_8, then (6) cannot be strict because each have more than one maximal orbit. Which leaves $C_D(V_1)$ to be D_8. Then the action $D/C_D(V_1)$ on V_1 has three orbits of order eight and the action $C_D(V_1)$ on W_1 has one orbit of order 8 and four orbits of order four. Following the argument from Step 3.1.1.2. Let $v_1 \in V_1$ be in an orbit of size $M_1 = 8$ of $D/C_D(V_1)$ on V_1, such that v_1 is in a regular orbit of $C_D(V_2)$ on V_1. We claim that $|C_D(v_1)| = 16$. Note that $D_8 \cong C_D(V_1) \leq C_D(v_1)$ so $|C_D(v_1)| \geq 8$. Assume $|C_D(v_1)| = |C_D(V_1)| = 8$.Then $|v_1^D| = |D : C_D(v_1)| = 16$ and $v_1^D \leq V_1$ so D has an orbit of size 16 on V_1, a contradiction to $M_1 = 8$. Thus $|C(v_1)|$ must be greater than 8 and as a p-group, it must be of order at least 16. Since $|C_D(v_1) \cap C_D(V_2)| = 1$, then $|C_D(v_1)C_D(V_2)| \geq 16 \cdot 8 = |D|$. Combined with $C_(v_1)C_D(V_2) \leq D$ and $|D_D(v_1)C_D(V_2)| = |C_D(v_1)||C_D(V_2)|$, gives us $|C_D(v_1)| = 16$. Now $|C_D(v_1)| = 16$ and $C_D(v_1) \cap C_D(V_2) = 1$, therefore

$$C_D(v_1) \cong C_D(v_1)/(C_D(v_1) \cap C_D(V_2)) \cong C_D(v_1)C_D(V_2)/C_D(V_2) \cong D/C_D(V_2) \cong D_8 \circ C_4$$

where the third isomorphism comes from the isomorphism theorems. Hence we have $C_D(v_1) \cong D_8 \circ C_4$. Therefore $C_D(v_1)$ acts faithfully on V_2 and if $z_1, z_2, z_3 \in V_2$ are representatives of the three orbits of size eight in the action of $C_D(v_1)$ on V_2, then $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$ are representatives of three orbits of size $64 = M_D = M$ of D on V_D. Now let $w_1 \in V_1$ be in an orbit of D of size $M_1 = 8$ such that w_1 is not in a regular orbit of $C_D(V_2)$ on V_1. Let $z_4 \in V_2$ be in the (unique) regular orbit of $C_D(V_1)$ on V_2 (so it is of size $M_2 = 8$).

Then clearly $w_1 + z_4$ is a representative of an orbit of size $64 = M$ of D on V that is different from the orbits containing $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$. Thus we have found four orbits of size $64 = M$ of D on V which contradicts our hypothesis.

Step 3.2.3 The Case Where We have Equality in (5) and (6)

We now consider the case that (5) and (6) are equalities. That is
$|D : D'C_D(V_1)| = M_1$ and $|C_D(V_1) : C_D(V_1)'| = M_2$. Then

$$M = |G : G'| = p|D|/|D'| = p|C_D(V_1) : D'C_D(V_1)|C_D(V_1) : C_D(V_1) \cap D'| \leq pM_1M_2$$

also, $M \geq M_D \geq M_1M_2$ so

$$(\dagger) \quad M_1M_2 \leq M_D \leq M \leq pM_1M_2.$$

We know that exactly one of these inequalities is strict because $|D|/|D'| < p|D|/|D'| = |G|/|G'| = M$. We now have three cases to consider.

In all of these cases we know that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1, otherwise $D/C_D(V_1)$ would be abelian, making $D' \leq (\times_{i=1}^{p} D/C_D(V_i))' = 1$. Then we would have $D' = G' = 1$, contradicting that G is nonabelian. Throughout the following arguments we will let $v_1, v_2 \in V_1$ be representatives of two orbits of size M_1 on V_1.

Step 3.2.3.1 The Case Where the Last Inequality in † is Strict

That is, $M_1M_2 = M_D = M < pM_1M_2$. Assume that $v_3, v_4 \in V_1$ are a third and fourth orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1. Then let $w_1 \in W_1$ be a representative of an orbit of size M_2 in the action of $C_D(V_1)$ on W_1. This gives us $(v_1 + w_1)^D, (v_2 + w_1)^D$, $(v_3 + w_1)^D$, and $(v_4 + w_1)^D$, four orbits of size $M_D = M$ on V. Thus we have four orbits of size M in the action of G on V, a contradiction. Therefore $D/C_D(V_1)$ can only have two or three orbits of size M_1 on V_1. Let $w_1, w_2 \in W_1$ be representatives of two distinct orbits of size M_2 in the action of $C_D(V_1)$ on W_1. We see that $(v_1 + w_1)^D$, $(v_1 + w_2)^D$, $(v_2 + w_1)^D$, and $(v_2 + w_2)^D$ are four orbits of size $M_D = M$ on V, a contradiction. Therefore we know that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1 and by Theorem 1 we see that $C_D(V_1)$ is abelian.

Suppose $D/C_D(V_1)$ has two orbits of size M_1 on V_1. By [5] we have that $D/C_D(V_1) \cong D_8, p = 2$, and $V_1 = V_2 = V(2,3)$. Therefore we have $C_D(V_1) \times C_D(V_2) \lnapprox D \lnapprox D/C_D(V_1) \times D/C_D(V_2)$ and $C_D(V_1) \leq D/C_D(V_1) \cong D_8$. If $|C_D(V_1)| = 8$, then $C_D(V_1) \cong D_8$, contradicting $C_D(V_1)$ is abelian. If $|C_D(V_1)| = 4$, then $C_D(V_1)$ must be the Klein-4, since it has only one orbit of size 4 on V_2, which is shown to be a contradiction in [5] corresponding case. If $|C_D(V_1)| = 2$, then $C_D(V_1)$ is Z_2, the cyclic group of order two. If we calculate the orbits, we see that all subgroups of D_8 of order two have at least three orbits of size two in the action on V_2, a contradiction. If $|C_D(V_1)| = 1$, then $D/C_D(V_1)$ has two orbits of size four in the action of $D/C_D(V_1)$ on V_1 and $D/C_D(V_2)$ has two orbits of size four in the action of $D/C_D(V_2)$ on V_2. Therefore D has either four orbits of size four or an orbit of size eight. This contradicts that G has exactly three orbits of size $M = M_1M_2 = 4$ in the action of G on V.

Suppose $D/C_D(V_1)$ has three orbits of size M_1 on V_1. By induction we have that $D/C_D(V_1) \cong D_8 \circ C_4, p = 2$, and $V_1 = V_2 = V(2,5)$. Therefore we have $C_D(V_1) \times C_D(V_2) \lnapprox D \lnapprox D/C_D(V_1) \times D/C_D(V_2)$ and $C_D(V_1) \leq D/C_D(V_1) \cong D_8 \circ C_4$. If $|C_D(V_1)| = 16$, then $C_D(V_1) \cong D_8 \circ C_4$, contradicting $C_D(V_1)$ is abelian. If $|C_D(V_1)| = 8$, then $C_D(V_1)$ is either D_8, Q_8, or $C_2 \times C_4$. If $C_D(V_1) \cong D_8$ or Q_8 then we contradict $C_D(V_1)$ is abelian. If $C_D(V_1) = C_2 \times C_4$ then $C_D(V_1)$ has two orbits of size M_2 on W_1 contradicting the statement above that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. If $|C_D(V_1)| = 4$, by calculating the orbits of all subgroup of $D_8 \circ C_4$ that are of order 4, we see that they all have at least four orbits of size four, a contradiction. If $|C_D(V_1)| = 2$, then $C_D(V_1)$ is Z_2, the cyclic group of order two. We know that all subgroups of $D_8 \circ C_4$ of order two have at least ten orbits of size two in the action on V_2, a contradiction. If $|C_D(V_1)| = 1$, then $D/C_D(V_1)$ has three orbits of size eight in the action of $D/C_D(V_1)$ on V_1 and $D/C_D(V_2)$ has three orbits of size eight in the action of $D/C_D(V_2)$ on V_2. Therefore D has either six orbits of size eight or at least one orbit of size sixteen. This contradicts that G has exactly three orbits of size $M = M_1M_2 = 8$ in the action of G on V. This concludes the case where $M_1M_2 = M_D = M$.

Step 3.2.3.2 The Case Where the First Inequality in † is Strict

That is, $M_1M_2 < M_D = M = pM_1M_2$. We know $|D|/|D'| < M_D$ and $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. We claim that $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1 and that $C_D(V_1)$ has at least two orbits of size M_2 on W_1. This proof can be found in the corresponding argument in [5]; however, the argument has a few gaps and assumptions that have been corrected here. Assume $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. Since we have inequality in (6) , by [7, Theorem 2.3] we know that $C_D(V_1)$ must be abelian and thus the largest orbit of $C_D(V_1)$ on W_1 is a regular orbit of size $M_2 = |C_D(V_1)|$. It follows by [7, Lemma 2.2] that $M_2 = \prod_{i=2}^{p}(|V_i| - 1)$. But then $M_1 = |V_1| - 1$ and thus the largest orbit of size M_D on V is M_1M_2 as the corresponding orbit is $\{x_1 + \cdots + x_p | x_i \in V_i - \{0\}$ for $i = 1, \dots, p\}$ and there cannot be a larger orbit. So $M_D = M_1M_2$, a contradiction to the fact that $M_D = pM_1M_2$ in this case. Therefore $C_D(V_1)$ has at least two orbits of size M_2 on W_1. Now, let w_1, w_2 be representatives of two different orbits of size M_1 of $C_D(V_1)$ on W_1. We now aim to show $D/C_D(V_1)$ cannot have more than three orbits. Let $v_i \in V_1$ where $i = 1, 2, 3, 4$ be representative of four different orbits of size M_1 on V_1. Fix $i \in \{1, 2, 3, 4\}$. Consider $z_j = v_i + w_j$ for $j = 1, 2$. We can see that $|(z_j)^D| \geq M_1M_2$ for $j = 1, 2$. Since $M = pM_1M_2$, we have $|(Z_j)^D| \in \{M_1M_2, M\}$ for $j = 1, 2$. If $|(z_1)^D| = M_1M_2$ then $|(z_2)^M| > M_1M_2$ since any $g \in G - D$ cannot fix both orbits. So, $|(z_2)^D| = M$. Since this is true for all $i \in \{1, 2, 3, 4\}$, G has four orbits of size M on V a contradiction to our hypothesis. Thus $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1.

If there are exactly two orbits of size M_1 on V_1 then by [5] and we have

$$D/C_D(V_1) \cong D_8, p = 2, V_1 = V_2 = V(2, 3).$$

We also know $C_D(V_1) \times C_D(V_2) \lessapprox D \lessapprox D/C_D(V_1) \times D/C_D(V_1) \cong D_8 \times D_8$, and $C_D(V_1) \leq D_8$. This tells us $|C_D(V_1)| \in \{1, 2, 4, 8\}$. In order to have the complete proof, the following cases have been taken from [5].

If $|C_D(V_1)| = 8$, then $C_D(V_1) \cong D_8$. This means $D \cong D_8 \times D_8$, $|D| = 64$, and $|G| = 128$. By Lemma 2.8 [10, Lemma 2.8] we have $G \leq D_8 \wr Z_2$. Because $|D_8 \wr Z_2| = 128$ we have that $G = D_8 \wr Z_2$, which is known to not be metabelian by Satz 3.15.3 [2, Satz 3.15.3 (d)], that is $G'' \neq 1$. However, we have that $G' = D' = (D_8 \times D_8)'$ which is size four. This makes G' abelian and $G'' = 1$, a contradiction.

If $|C_D(V_1)| = 4$ we have that $|D| < |D/C_D(V_1) \times D/C_D(V_1)| = |D|^2/16 = 64$. That is $|D| = 32$. We also have $D' \leq (D_8 \times D_8)'$ so $|D'| \leq 4$. Suppose $|D'| = 4$ then $\frac{|D|}{|D'|} = \frac{32}{4} = 8$, so $M_D = 16 = M_1M_2$ a contradiction. Suppose that $|D'| = 2$. Notice $C_D(V_1) \cap C_D(V_2) = 1$ so $C_D(V_1) \times C_D(V_2) = 1$ and $C_D(V_1) \times C_D(V_2) = C_D(V_1)C_D(V_2)$. Let $g \in G - D$ and $(1, a) \in C_D(V_1) \times C_D(V_2)$. Then

$$[(1, a), g] = (1, a)^{-1}g^{-1}(1, a)g = (1, a)^{-1}(1, a)^g = (1, a^{-1})(a^*, 1) = (a^*, a^{-1})$$

for some $a^* \in C_D(V_1)$, and we have four choices for $a \in C_D(V_1)$. Thus $2 = |D'| = |G'| \geq 4$, a contradiction.

If $|C_D(V_1)| = 2$, then $\frac{|D|^2}{4} = 64$ or $|D| = 16$. As before, we know that $|D'| \in \{2, 4\}$. Suppose $|D'| = 4$, then $\frac{|D|}{|D'|}^4 = \frac{16}{4} = 4$, and $M_D = 8$. We know $(v_1, 0)$ and $(v_2, 0)$ are both in D-orbits of size four on V. Let $w \in V_2$ be a regular orbit. Then $(v_1 + w)$ would be in an orbit of size eight, contradicting $M_1 M_2 < M_D$. Suppose that $|D'| = 2$, then $\frac{|D|}{|D'|} = 8$ and $M_D = 16$. Thus $C_D(V_1)$ must have four orbits of size two on V_2. Let $w_i \in V_2$ for $i = 1, 2, 3, 4$ be representatives of these four orbits. Then we know that $C_D(V_1) = \{1, r\}$ as all other subgroups of D_8 of size two have only three orbits of size two on V_2. We also know there must exist a $d_i \in D$, $i = 1, 2, 3, 4$ where $w_1^{d_1} = w_2, w_2^{d_2} = w_3, w_3^{d_3} = w_4, w_4^{d_4} = w_1$. Without loss let $w_1 = (1, 0)$ and $w_2 = (1, -1)$. Then there exists a $d \in D$ with $(1, 0)^d = (1, -1)$ in the action of D on V_2, a contradiction.

If $|C_D(V_1)| = 1$ then $D \cong D_8$, but $D_8/D_8' = 4 = M_D$, a contradiction. This concludes the case from [5].

Therefore there are exactly three orbits of size M_1 on V_1 thus by induction, we have $D/C_D(V_1) \cong D_8 \circ C_4, p = 2, V_1 = V_2 = V(2, 5)$. We also know $C_D(V_1) \times C_D(V_2) \lessapprox D \lessapprox D/C_D(V_1) \times D/C_D(V_1) \cong (D_8 \circ C_4) \times (D_8 \circ C_4)$, and $C_D(V_1) \leq D_8 \circ C_4$. This tells us $|C_D(V_1)| \in \{1, 2, 4, 8, 16\}$.

If $|C_D(V_1)| = 16$, then $C_D(V_1) \cong D_8 \circ C_4$. This means $D \cong (D_8 \circ C_4) \times (D_8 \circ C_4)$, $|D| = 256$, and $|G| = 512$. By Lemma 2.8 [10, Lemma 2.8] we have $G \leq (D_8 \circ C_4) \wr Z_2$. Because $|(D_8 \circ C_4) \wr Z_2| = 512$ we have that $G = (D_8 \circ C_4) \wr Z_2$, which is not metabelian. However, we have that $G' = D' = ((D_8 \circ C_4) \times (D_8 \circ C_4))'$ which is of order four. This makes G' abelian and $G'' = 1$, a contradiction.

If $|C_D(V_1)| = 8$, then $C_D(V_1)$ is isomorphic to one of the following: D_8, Q_8 or $C_4 \times C_2$. The case where $C_D(V_1) \cong D_8$ can be seen in the case that $D/C_D(V_1)$ has exactly two orbits. Suppose $C_D(V_1) \cong Q_8$. Since $C_D(V_1)$ acts faithfully on V_2, there exists a regular orbit of size 8. Therefore $M_2 = 8$, but $M_2 = |C_D(V_1) : C_D(V_1)'| = 8/2 = 4$, a contradiction. Suppose $C_D(V_1) \cong C_4 \times C_2$. We know that $C_D(V_2)$ is also isomorphic to $C_4 \times C_2$ and therefore has 2 regular orbits. Let $v_1, v_2 \in V_1$ be representatives of regular orbits of $C_D(V_2)$ acting on V_1 and in orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1. Consider $C_D(v_1)$, $|C_{D/C_D(V_1)}(v_1)| = 2$ thus $|C_D(v_1)| = 16$. Let $w_1, w_2, w_3 \in V_2$ be representatives of the three maximal orbits of $D/C_D(V_1)$ on V_2. Now consider the action of G on $V = V_1 \oplus V_2$ of the form $(v_i + w_j)^G$ where $i \in 1, 2$ and $j \in 1, 2, 3$. Since elements in $G - D$ swap the orbits from V_1 to V_2, there exists a $g \in G - D$ such that in the action of $C_D(V_2)$ on V_1 the orbit is $(v_1)^D$ and in the action of $C_D(V_1)$ on V_2 the orbit is $(v_1^g)^D$. We have chosen g such that v_1^D has to be exactly one of the following, w_1, w_2, w_3. Without loss of generality we may assume $v_1^g = w_1$. The orbit $(v_1 + w_1)^G = (v_1 + v_1^g)^G$ is fixed and will be of order $M_2 = 64$, but we have two orbits $(v_1 + w_2)^G$ and $(v_1 + w_3)^G$ of maximal size $M = 128$. We can repeat this method with v_2 in place of v_1 to receive two more orbits of size $M = 128$, a contradiction to our hypothesis.

The cases $|C_D(V_1)| = 4$ and $|C_D(v_1)| = 2$ are the same as when $D/C_D(V_1)$ has exactly two orbits. If $|C_D(V_1)| = 1$, then $D \cong D_8 \circ C_4$, but $(D_8 \circ C_4)/(D_8 \circ C_4)' = 8 = M_D$, a contradiction.

Step 3.2.3.3 The Case Where the Second Inequality in † is Strict

That is, $M_1 M_2 = M_D < M = p M_1 M_2$. Since by Theorem 1$|D/D'| \leq M_D$, and since $M_1 M_2 = M_D < M = p M_1 M_2$ and $p|D/D'| = |G/G'| = M$ we see that

$|D/D'| = M_1M_2 = M_D < M = pM_1M_2$. We know that $D/C_D(V_1)$ has at least two orbits of size M_1 in the action of V_1. We now must find how many orbits $C_D(V_1)$ has of size M_2 on W_1.

Step 3.2.3.3.1 The Case Where $C_D(V_1)$ has Exactly One Orbit of Size M_2 on W_1

We note that this case can be handled by following word for word the corresponding argument in the proof of Theorem 1 in [7]; however, there was a small error in the argument presented there. A corrected version of this case, that replaces the corresponding argument in [7], can be found in [5] and has been recreated here for completeness.

Let us first assume that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. By Theorem 1, $C_D(V_1)$ is abelian. Therefore $C_D(V_i)$ is abelian for all $i = 1, \dots, p$, and it follows that for each i that $C_D(W_i)$ is abelian. We further claim that for each i, $C_D(W_i)$ has exactly one regular orbit of V_i. To see this, observe that $C_D(V_1)$ has exactly one regular orbit on W_1. Hence $C_D(V_1)/(C_D(V_1) \cap C_D(V_2))$ has exactly one regular orbit on V_2, and $(C_D(V_1) \cap C_D(V_2))/(C_D(V_1) \cap C_D(V_2) \cap C_D(V_3))$ has exactly one regular orbit on V_3. We can repeat this until we finally get

$$\left(\bigcap_{j=1}^{p-1} C_D(V_j)\right)/\left(\bigcap_{j=1}^{p} C_D(V_i)\right) \cong C_D(W_p)$$

has exactly one regular orbit on V_p. Since the actions of $C_D(W_i)$ on V_i are equivalent for all i, the claim is true. Let $A := \prod_{i=1}^{p} C_D(W_i)$. We see that $A \trianglelefteq G$, and $A = \times_{i=1}^{p} C_D(W_i)$ is an internal direct product because $C_D(W_i) \cap \prod_{j\in\{1,\dots,p\}-\{i\}} C_D(W_j)) = 1$, for $i = 1, \dots, p$ (all elements in $\prod_{j\in\{1,\dots,p\}-\{i\}} C_D(W_j)$) act trivially on V_i). Applying Lemma 2.2 to the action of $C_D(W_i)$ on V_i for all i. Putting this together thus shows that if we write $V_A = X_1 \oplus \cdots \oplus X_m$ for some $M \in \mathbb{N}$ and irreducible A-modules such that $V_1 = X_1 \oplus \cdots \oplus X_k$ for some $k \in \mathbb{N}$, then $m = kp$ and

$$|A| = \prod_{i=1}^{m}(|X_i| - 1) = (|X_1| - 1)^m \le M$$

and

$$M_1 \ge \prod_{i=1}^{k}(|X_i| - 1) = (|X_1| - 1)^k.$$

Now recall that $v_1, v_2 \in V_1$ are representatives of two orbits of size M_1 of $D/C_D(V_1)$ on V_1. Write $v_i = x_{i1} + \cdots + x_{ik}$ with $x_{ij} \in X_j$ for $j = 1, \dots, k$, $i = 1, 2$. We may assume that v_1 is in a regular orbit of $A/C_A(V_1)$, and thus $x_{1j} \neq 0$ for $j = 1, \dots, k$. But then $v_1^D = v_1^A = \{y_1 + \cdots + y_k | 0 \neq y_i \in X_i \text{ for } i = 1, \dots, k\}$, and this forces that $x_{2j} = 0$ for at least one $j \in \{1, \dots, k\}$. If we let $g \in G - D$ and put $z_i = v_i + \sum_{j=1}^{p-1} v_1^{g^j}$ for $i = 1, 2$, then it is clear that both z_1 and z_2 are in different orbits of size greater than or equal to $|A|$ of G. Hence $|G/G'| \ge |A|$.

Now let q be the characteristic of V and write $|X_1| = q^s$. Write $|A/C_A(X_1)| = p^t$. Then $p^t = q^s - 1$ and hence with [10, Proposition 3.1] we know that either $s = 1$, $p = 2$, and q is a Fermat prime; or $t = 1$, $q = 2$, and p is a Mersenne prime; or $s = 2$, $t = 3$, $p = 2$, and $q = 3$. Moreover, by [10, Theorem 2.1] we know that $N_G(X_1)/C_G(X_1)\Gamma(X_1)$ and since G is a p-group, altogether we conclude that

$$N_G(X_1)/C_G(X_1) \cong A/C_A(X_1)$$

unless possibly in the third case, when $|V_1| = 9$ and $N_G(X_1)/C_G(X_1) \cong \Gamma(3^2)$ is possible (in the first case this is clear, in the second it follows by Fermat's Little Theorem). For the moment suppose that $N_G(X_1)/C_G(X_1) \cong A/C_A(X_1)$. Because of the size of A with [10, Lemma 2.8] we conclude that $G \cong A/C_A(X_1) \wr G/A$ where G/A transitively and faithfully permutes the X_i ($i = 1, \dots, m$). Now with arguments similar to the one in the proof of [3, Lemma 2] we see that

$$|[A, G]| \geq |A/C_A(X_1)|^{m-1} = (|X_1| - 1)^{m-1}.$$

Moreover, by [1, Theorem 2.3] we have $|G : G'A| = |G/A : (G/A)'| \leq p^{m/p}$. Hence altogether we have

$$\begin{aligned}(|X_1| - 1)^m = |A| &\leq |G/G'| = |G : G'A||G'A : G'| \\ &\leq p^{m/p}|A : A \cap G'| \\ &\leq p^{m/p}|A : [A, G]| \\ &= p^{m/p}(|X_1| - 1)\end{aligned}$$

Now clearly $|X_1| - 1 \geq p$, and so it follows that

$$p^{m-1} \leq (|X_1| - 1)^{m-1} \leq p^{\frac{m}{p}}.$$

So $m - 1 \leq m/p$, and since $m \geq p$, we get that $p = m = 2$, $|X_1| = 3$, $k = 1$, $V_1 = X_1$, $|V| = 9$, $M = |G/G'| = 4$ and thus G is dihedral of order 8 acting on the nine elements of V. But then D is abelian, and since $G' = D'$, G is also abelian. This is a contradiction.

In the exceptional case $s = 2$, $t = 3$, $p = 2$, $q = 3$ above we have that the kernel K/A of the permutation action of G/A on the X_i is of order at most 2^m. So we see that $G/\Omega_2(A)$ has A as an abelian normal subgroup, and so similarly as above

$$\begin{aligned}|G/G'| &\leq |G : G'K||G'K : G'| \\ &\leq 2^{\frac{m}{2}}|K : K \cap G'| \\ &\leq 2^{\frac{m}{2}}|K/\Omega_1(A) : (K \cap G')\Omega_1(A)/\Omega_1(A)| \cdot |\Omega_1(A)|\end{aligned}$$

Now $|[K/\Omega_1(A), G/\Omega_1(A)]| \geq 4^{m-1}$, and thus altogether

$$\begin{aligned}2^{3m} = 8^m = (|X_1| - 1)^m &= |A| \\ &\leq |G/G'| \\ &\leq 2^{\frac{m}{2}}|K/\Omega_1(A) : [K/\Omega_1(A), G/\Omega_1(A)]| \cdot 2^m \\ &\leq 2^{\frac{m}{2}} \cdot \frac{8^m}{4^{m-1}} \cdot 2^m \\ &= 2^{\frac{3}{2}m+m-1} \cdot 8 = 2^{\frac{5}{2}m+2}\end{aligned}$$

Hence $3m \leq \frac{5}{2}m + 2$ or, equivalently, $m \leq 4$. Since $m = kp = 2k$, we have that $m = 2$ or $m = 4$.

If $m = 2$, then $k = 1$ and thus $X_i = V_i$ for $i = 1, 2$. But then $M_1 = 8 = |V_1| - 1$, and $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1, contradicting our observation above that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1.

If $m = 4$, then $k = 2$ and $|V_1| = 3^4$. Hence G is isomorphic to a subgroup of GL $(8, 3)$ and thus $|G| \leq 2^{19}$. As above, we know that $|G'| \geq |[A, G]| \geq (|X_1| - 1)^{m-1} = 8^3 = 2^9$, and hence $|G/G'| \leq 2^{10} < 2^{12} = |A| \leq M$, contradicting $|G/G'| = M$.

Therefore we now know that $C_D(V_1)$ has at least two orbits of size M_2 on W_1.

Step 3.2.3.3.2 The Case Where $C_D(V_1)$ has at Least Two Orbits of Size M_2 on W_1

The set up for this argument is the same as in [5] and deviates in the subcases that follow. Let w_1 and w_2 be representatives of such orbits. If there exists a $d \in C_D(v_1)$ such that $w_1^d = w_2$ we see that

$$M \geq M_D \geq M_1 p M_2 \geq p|D : D'| = |G/G']$$

contradicting that $M_D < M$. Therefore, no such d exists. This tells us that $(v_1 + w_1)$ and $(v_1 + w_2)$ lie in different D-orbits on V. Similarly $v_2 + w_1$ and $v_2 + w_1$ are in different D-orbits on V.

Now identify D with a subgroup of $\times_{i=1}^{p} D/C_D(V_i)$. Also let $g \in G - D$ and put

$$L_i = \sum_{j=0}^{p-1} (v_i^{g^j})^D := \left\{ \sum_{j=0}^{p-1} x_j | x_j \in (v_i^{g^j})^D \text{ for } j = 0, \dots, p-1 \right\} \subset V$$

for $i = 1, 2$. The L_i are clearly G-invariant subsets, and $L_1 \cap L_2 = \emptyset$. For any $x \in V_2 \oplus \cdots \oplus V_p$ it follows that if the orbit $(v_i + x)^D \subset L_i$ $(i \in \{1, 2\})$.

We now have four cases to consider. The D-orbits $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are both G–invariant. The D-orbits $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are both G-invariant. Lastly, one of the D-orbits $(v_1 + w_1)^D$ or $(v_1 + w_2)^D$ is not G-invariant, and at least one of the orbits D-orbits $(v_2 + w_1)^D$ or $(v_2 + w_2)^D$ is not G-invariant.

Step 3.2.3.3.2.1 The Case Where the D–orbits $(v_1 + w_1)^D$ and $(v_1 + w_1)^D$ are Both G–invariant

Then $v_1 + w_1 \in L_1$ and $v_1 + w_2 \in L_1$, and thus $v_2 + w_1 \notin L_2$ and $v_2 + w_2 \notin L_2$, that is $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are not G-invariant, so that

$$M \geq |(v_2 + w_i)^G| \geq p|(v_2 + w_i)^D| \geq pM_1M_2 \geq p|D/D'| = |G/G'| = M$$

for $i = 1, 2$. This tells us that $C_D(V_1)$ has either two or three orbits of size M_2 in the action on W_1, because otherwise the above argument shows that we would get more than three orbits of size M.

The following argument is a corrected version of the corresponding part in [5] and should serve as a replacement proof. Suppose that $D/C_D(V_1)$ has three orbits of size M_1 on V_1, let v_3 be a representative of a third such orbit. Arguing just as for v_2, we know that $(v_3 + w_1)^D$ and $(v_3 + w_2)^D$ are not G–invariant, and (also as above for v_2) we get $|(v_3 + w_i)^G| = M$ for $i = 1, 2$. But this gives us four orbits of size M_D and we can only have two or three, therefore at least two must be the same. There are six possible combinations as order does not matter.

Suppose $(v_2 + w_1)^D$ and $(v_3 + w_1)^D$ are G–conjugate, then there exists a $x \in G$ such that $(v_3 + w_1) = (v_2 + w_1)^x = v_2^x + w_1^x$. If $x \in D$ then $v_2^x \in V_1$ and $v_2^x = v_3$, but then $v_2^D = v_3^D$ a contradiction to them being representatives of different orbits. Thus $x \in G - D$. Since $w_1 \in \sum_{i=1}^{p-1} (v_1^{g^i})^D$, w_1^x has a component in V_1 and that component is in v_1^D so $v_3 \in v_1^D$ and thus $v_3^D = v_1^D$ a contradiction to them being representatives of different orbits. Therefore $(v_2 + w_1)^D$ and $(v_3 + w_1)^D$ are not G–conjugate. The same contradiction can be found for $(v_2 + w_2)^D$ and $(v_3 + w_2)^D$ being G–conjugate by changing the index as they also have the same element in W.

Suppose $(v_2+w_1)^D$ and $(v_3+w_2)^D$ are G–conjugate, then there exists a $x \in G$ such that $(v_3+w_2)=(v_1+w_1)^x=v_2^x+w_1^x$. If $x \in D$ then $v_2^x=v_3$ and $v_2^D=v_3^D$ a contradiction to them being representatives of different orbits. Thus $x \in G-D$. Since $w_2 \in \sum_{i=1}^{p-1}(v_1^{g^i})^D$, w_2^x has a component in V_1 and that component is in v_1^D. Therefore $v_3 \in v_1^D$ and thus $v_3^D=v_1^D$ a contradiction to them being representatives of different orbits. Therefore $(v_2+w_1)^D$ and $(v_3+w_2)^D$ cannot be G–conjugate. The same contradiction can be found for $(v_2+w_2)^D$ and $(v_2+w_1)^D$ being G–conjugate.

If $(v_2+w_1)^D$ and $(v_2+w_2)^D$ are G–conjugate then they are D–conjugate, otherwise if $x \in G-D$ then $v_2+w_2=(v_2+w_1)^x=v_2^x+w_1^x$ then since $w_2 \in \sum_{i=1}^{p-1}(v_1^{g^i})^D$, w_1^x has a component in V_1 and that component is in v_1^D so $v_2 \in v_1^D$ thus $v_2^D=v_1^D$ a contradiction to them representing different orbits. Thus if $(v_2+w_1)^D$ and $(v_2+w_2)^D$ are G–conjugate then they are D–conjugate. Since they are D–conjugate, there exists a $d \in D$ such that $v_2^d=v_2$ and $w_1^d=w_2$. We know $d \in C_D(v_2)$ but $d \notin C_D(V_1)$ and thus w_1, w_2 are in some $C_D(v_2)$–orbits. So the largest orbit size of $C_D(v_2)$ on W_1 is greater than or equal to pM_1. Then $|(v_2+w_1)^D|=M_1pM_2=M_D$ a contradiction to M_D being strictly less than pM_1M_2.

Thus there cannot be a third orbit of $D/C_D(V_1)$ on W_1 of size M_2 and by [5] $D/C_D(V_1) \cong D_8$, $p=2$, and $V_1=V(2,3)$. We know that $C_D(V_1) \times C_D(V_2) \leq D \leq D/C_D(V_1) \times D/C_D(V_2) \cong D_8 \times D_8$ and $C_D(V_1) \leq D/C_D(V_1) \cong D_8$. Then $|C_D(V_1)| \in \{2,4,8\}$.

If $|C_D(V_1)|=8$ then $D=D_8 \times D_8$, so $|D|=64$, $|D'|=4=|G'|$ and $|G|=128$. By [10, Lemma 2.8] we have $G \leq D_8 \wr Z_2$, so since $|G|=128$, we have $G=D_8 \wr Z_2$ contradicting $|G'|=4$.

If $|C_D(V_1)|=4$, then $C_D(V_1) \cong Z_4$ (because $C_D(V_1)$ has two orbits of size M_2 on V_2, therefore it is not the Klein-4). Thus $Z_4 \times Z_4 \leq D \leq D_8 \times D_8$, $|D|=|D/C_D(V_1)|\,|C_D(V_1)|=8 \cdot 4=32$.Because $D' \leq (D_8 \times D_8)'$ we have that $|D'| \in \{2,4\}$. If $|D'|=4$ then $M_D=\frac{|D|}{|D'|}=\frac{32}{4}=8$, contradicting that $Z_4 \times Z_4$ has a regular orbit (size 16) on V. Therefore $|D'|=|G'|=2$; but $C_D(V_1)=4$ which makes $|G'| \geq 4$, a contradiction.

If $|C_D(V_1)|=2$, then $C_D(V_1) \cong Z_2$. This means that $C_D(V_1)$ has at least three orbits of size $M_2=2$ on V_2, a contradiction.

Step 3.2.3.3.2.2 The Case Where the D–orbits $(v_2+w_1)^D$ and $(v_2+w_2)^D$ are Both G–invariant

This case will follow the same proof as in Step 3.2.3.3.2.1 if we replace v_1 with v_2.

Step 3.2.3.3.2.3 The Case Where the at Least One of the D-orbits $(v_1+w_1)^D$ or $(v_1+w_2)^D$ is Not G-invariant, and at Least One of the Orbits D-orbits $(v_2+w_1)^D$ or $(v_2+w_2)^D$ is Not G-invariant.

Without loss of generality, we may assume that $(v_1+w_1)^D$ is not G–invariant. If $(v_2+w_1)^D$ is also not G–invariant, we see that $(v_1+w_1)^G$ and $(v_1+w_1)^G$ are two distinct orbits of size M, because

$$M \geq (v_i+w_1)^G \geq pM_1M_2=|G/G'|=M$$

for $i=1,2$ and if we write $w_1=(x_2,\ldots,x_p) \in V_2 \oplus \cdots \oplus V_p$ then $v_1+w_1=(v_1,x_2,\ldots,x_p$ and $v_2+w_2=(v_2,x_2,\ldots,x_p)$ have a different number of components in the corresponding component of L_1 and cannot be conjugate in G.

Now we show that $D/C_D(V_1)$ cannot have a fourth orbit of size M_1 on V_1. Let $v_3, v_4 \in V_1$ be a third and fourth such orbit. If $(v_3+w_1)^D$ and $(v_3+w_2)^D$ are both G–invariant then we can follow the proof in Step 3.2.3.2.1 by replacing v_1 with v_3 to find a contradiction.

Similarly, we can replace v_1 with v_4 in the same argument to show $(v_4 + w_1)^D$ and $(v_4 + w_2)^D$ cannot both be G–invariant. If $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ are not G–invariant then $(v_3 + w_1)^G$ and $(v_4 + w_1)^G$ are a third and fourth orbit of size M on V, a contradiction. If both $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ are G–invariant then $(v_3 + w_2)^D$ and $(v_4 + w_2)^D$ must not be G–invariant. Thus $|(v_2 + w_2)^G| = |(v_4 + w_2)^G| = M$ and since there are only three orbits of size M, at least two orbits must be equal. Without loss, assume $(v_4 + w_2)^G = (v_j + w_1)^G$ for some $j \in \{1, 2, 3\}$. Then there exists a $g \in G - D$ with $(v_4 + w_2)^g = v_j + w_1$. Then $(v_4 + w_2)^G \subseteq L_4$ and $(v_j + w_1)^G \subseteq L_j$, a contradiction to $L_4 \cap L_j = \emptyset$. If one of $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ is not G–invariant then we have three orbits and can use induction to tell us $D/C_D(V_1) \cong D_8 \circ C_4$, $V_1 \cong V(2, 5)$, $p = 2$ and $V = V_1 \oplus V_2$, which is known to be a contradiction as seen in Step 3.2.3.2.1. Therefore there cannot be a fourth orbit of size M_1 when $D/C_D(V_1)$ acts on V_1.

If there is exactly three orbits of size M_1 of $D/C_D(V_1)$ then we can use induction to find $D/C_D(V_1) \cong D_8 \circ C_4$, $V_1 = V(2, 5)$, $p = 2$ and $V = V_1 \oplus V_2$, a contradiction. If there is exactly two orbits of size M_1 of $D/C_D(V_1)$ then we can use [5] to find $D/C_D(V_1) \cong D_8$, $V_1 = V(2, 3)$, $p = 2$ and $V = V_1 \oplus V_2$, a contradiction by step 3.2.3.2.1.

Therefore $(v_2 + w_1)^D$ is G-invariant and thus $(v_2 + w_2)^D$ is not G-invariant. A similar argument to the one above follows and we can again use induction and [5] to find $D/C_D(V_1) \cong D_8 \circ C_4$ and $D/C_D(V_1) \cong D_8$, respectively, a contradiction.

Thus $(v_1 + w_1)^D$ and $(v_2 + w_2)^D$ are both G-invariant. If $(v_1 + w_2)^D$ and $(v_2 + w_2)^D$ are G-conjugate, then $v_2 + w_1 \in L_2$ and $v_1 + w_2 \in L_1$. This means that $v_1 + w_1$ and $v_2 + w_2$ can only be conjugate in G if $p = 2$. We can now let $p = 2$.

Suppose there are four orbits of size M_2 in the action $C_D(V_1)$ on V_2. Let $w_3, w_4 \in W_1 = V_2$ be representative of such orbits. As above it follows that $v_1 + w_1$, $v_1 + w_2$, $v_1 + w_3$ and $v_1 + w_4$ all lie in different D–orbits on V and so do $v_2 + w_1$, $v_2 + w_2$, $v_2 + w_3$ and $v_2 + w_4$, and as above, with w_3 and w_4 in place of w_2 we see the following must be true.

At least one of the orbits $(v_1 + w_1)^D$ or $(v_1 + w_3)^D$ is not G–invariant, and at least one of the orbits $(v_2 + w_1)^D$ or $(v_2 + w_3)^D$ is not G–invariant. We already know that $(v_2 + w_1)^D$ is G–invariant, it follows that $(v_2 + w_3)^D$ is not G–invariant. The argument from earlier shows that $(v_2 + w_2)^G$ and $(v_2 + w_3)^G$ are different G–orbits. Assume there is a third orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1, let v_3 be a representative of this orbit. Then, at least two of the three orbits $(v_3 + w_1)^D$, $(v_3 + w_2)^D$ and $(v_3 + w_3)^D$ are not G–invariant. If $(v_3 + w_1)^D$ is G–invariant, then $(v_3 + w_2)^G$, $(v_1 + w_1)^G$, $(v_3 + w_3)^G$ and $(v_2 + w_1)^G$ are four G–orbits of size M, a contradiction. If $(v_3 + w_2)^D$ is G–invariant, then $(v_3 + w_1)^G$, $(v_1 + w_2)^G$, $(v_3 + w_3)^G$ and $(v_2 + w_2)^G$ are four G–orbits of size M, a contradiction. Lastly, if $(v_3 + w_3)^D$ is G–invariant, then $(v_3 + w_1)^G$, $(v_1 + w_3)^G$, $(v_3 + w_2)^G$ and $(v_2 + w_3)^G$ are four G–orbits of size M, a contradiction. Thus we see that $D/C_D(V_1)$ has exactly two orbits of size M_1 on V_1. By [5], we have that $D/C_D(V_1) \cong D_8$, a contradiction as proven above. This concludes section 2.3 entirely.

Step 3.2.4 The Case Where We Have Strict Inequality in (5) and Equality in (6)

Suppose we have equality in (4) and strict inequality in (3). That is

$$M \geq M_1 M_2 \geq p|D : D' C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|D/D'| = |G : G'|.$$

Because $|G : G'| = M$ we have equality everywhere, and $M = M_1 M_2, M_1 = p|D : D' C_D(V_1)| > |D : D' C_D(V_1)|, M_2 = |C_D(V_1) : C_D(V_1)'|$. Again let M_D denote the largest orbit size of D on V, then $M_D \geq M_1 M_2$ so $M_D = M$. By Theorem 1.1 $C_D(V_1)$ is abelian or has at least two orbits of size M_2 on W_1. We consider again some subcases.

Step 3.2.4.1 The Case Where $C_D(V_1)$ has at Least Two Orbits of Size M_2 on W_1

Let $w_1, w_2 \in W_1$ be representatives of such orbits.

Assume that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Because $M = M_1M_2$ we have that $(v_1 + w_1)^D, (v_1 + w_2)^D$, $(v_2 + w_2)^D$ and $(v_2 + w_1)^D$ are all distinct orbits of size $M_D = M$, contradicting there being only three orbits of size M. Therefore $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1. Let $v_1 \in V_1$ be a representative of this orbit.

Now let w_1, w_2 be representatives of two distinct orbits of size M_2 of $C_D(V_1)$ on W_1, then $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are two distinct D–orbits of size M, and if $C_D(V_1)$ had a third and fourth orbit of size M_2 on W_1, similarly we would get a third and fourth orbit on G of size M, a contradiction. Thus $C_D(V_1)$ has two or three orbits of size M_2 on W_1.

Now write $W_1 = \bigoplus_{i=1}^k X_i$ for a suitable $k \in \{1, \dots, n\}$ and irreducible $C_D(V_1)$-modules X_i $(i = 1, \dots, k)$. We may assume that $X_1 \le V_2$. Then the intersection of all the $C_{C_D(V_1)}(X_i)$ is trivial, and hence

$$C_D(V_1) \lessapprox C_D(V_1)/C_{C_D(V_1)}(X_1) \times \cdots \times C_D(V_1)/C_{C_D(V_1)}(X_k) \quad (+)$$

Moreover, if we put $N_0 = C_D(V_1)$, $Z_0 = W_1$ and recursively for $i \ge 1$ let $Y_i \le Z_{i-1}$ be an irreducible N_{i-1}-module, $N_i = C_{N_{i-1}}(Y_i)$, and Z_i be a $C_D(V_1)$-invariant complement of Y_i in Z_{i-1}, and put $t = i - 1$ and stop the process as soon as $Z_i = 0$ and $N_i = 1$, then we have that $\bigcap_{i=0}^t N_i = 1$ and $W_1 = \bigoplus_{i=0}^t Y_i$. Also, $R_{i-1} := N_{i-1}/N_i$ acts faithfully and irreducibly on Y_{i-1} for $i = 1, \dots, t$. Write M_{i-1}^* for the largest orbit size of N_{i-1}/N_i on Y_{i-1} for $i = 1, \dots, t$. Then by repeated use of Lemma 2.1 we see that

$$M_2 = |C_D(V_1) : C_D(V_1)'| \le \prod_{i=1}^t |R_i : R_i'| \le \prod_{i=1}^t M_i^* \le M_2, \quad (++)$$

the last inequality easily follows by considering the sum of representatives of orbits of size M_{i-1}^* of N_{i-1}/N_i on Y_{i-1}. Thus we have equality everywhere, and it follows that $|R_i : R_i'| = M_i^*$ for $i = 1, \dots, t$. It also follows that the elements of every orbit of $C_D(V_1)$ on W_1 of size M_2 have the form $y_1 + \dots, y_t$ for some $y_i \in Y_i$ $(i = 1, \dots, t)$ which lies in an orbit of size M_i^* of N_i/N_{i+1} on Y_i $(+++)$. We now split into two cases $C_D(V_1)$ is not abelian and $C_D(V_1)$ is abelian.

Step 3.2.4.1.1 The Case Where $C_D(V_1)$ is not Abelian

Put $C = C_D(V_1) \cap C_D(X_1) = C_{C_D(V_1)}(X_1)$. Then by (+) we may assume that $C_D(V_1)/C$ is nonabelian, and it also acts faithfully and irreducibly on X_1. We also clearly may assume that $Y_1 = X_1$ and hence with (++) and (+++) conclude that $C_D(V_1)/C$ has either two or three orbits of size of its abelian quotient on X_1. Hence we may apply induction and, in particular, get $p = 2, |X_1| = 25$ and $C_D(V_1)/C \cong D_8 \circ C_4$. Moreover, since $C_D(V_1)$ has either two or three orbits of size M_2 on W_1, then from (+++) it follows that R_{i-1} has exactly one orbit of size M_{i-1}^* on Y_{i-1} for $i = 2, \dots, t$. This forces, for $i = 2, \dots, t$, that R_{i-1} is cyclic of order $2, |Y_{i-1}| = 3$, and hence $C_{C_D(V_1)}(X_1)$ is elementary abelian of order $p^{\dim W_1 - 2}$. Note that $W_1 = V_2$ since $p = 2$.

Assume that $k \ge 2$, so $t \ge 3$ (since the X_i all have dimension 2). Then we may assume that $X_2 = Y_1 \oplus Y_2$, and from the above we know that $C/C_C(X_2)$ is the dihedral group of order eight, since it is a subgroup of $D_8 \circ C_4$ that has only one regular orbit.

Now consider the action of $C_D(V_1)$ on X_1. We know that $C_D(V_1)$ is isomorphic to a subgroup of a direct product of k copies of $D_8 \circ C_4$, and $C_D(V_1)/C$ is isomorphic to $D_8 \circ C_4$ and has six noncentral involutions. If all of them have inverse images in $C_D(V_1)$ which act trivially on $X_2 \oplus \cdots \oplus X_k$, then $C_D(V_1)$ has a $D_8 \circ C_4$ as a subgroup which acts trivially on $X_2 \oplus \cdots \oplus X_k$, and since the X_i are transitively permuted by D, it follows that $C_D(V_1)$ is isomorphic to a direct product of k copies of $D_8 \circ C_4$; in particular, then $C/C_C(X_2) \cong C_D(V_1)/C_{C_D(V_1)}(X_2) \cong D_8 \circ C_4$, contradicting the above observation that $C/C_C(X_2)$ is the dihedral group of order eight. Hence there exists an element $c \in C_D(V_1)$ such that $c \notin C$, $c^2 \in C$, and c acts nontrivially on at least one X_i for some $i \in \{2, \dots, k\}$, so without loss we may assume that c acts nontrivially on X_2. Now there is a $0 \neq x \in V_1$ such that c centralizes x. Since $c \notin C$ and $C/C_C(X_2)$ is dihedral of order eight, this shows that $C_D(x)/C_{C_D(x)}(X_2)$ has order divisible by 16,and thus $C_D(x)/C_{C_D(x)}(X_2)$ is isomorphic to $D_8 \circ C_4$ and therefore has three orbits of size 8 on X_2. This allows us in an obvious way to construct three different orbits of size $M_2 = 8^k$ of $C_D(V_1)$ on $V_2 = W_1$ having representatives with x in their X_1-component; in addition to another orbit of size M_2 having a representative in the X_1-component from the second orbit of size 8 of $C_D(V_1)/C$ on X_1, giving us in total four distinct orbits of $C_D(V_1)$ on V_2, contradicting the current fact that $C_D(V_1)$ has exactly three orbits of size M_2 on V_2.

Hence our assumption that $k \geq 2$ was wrong, and we now have $k = 1$. So $W_1 = V_2 = X_1$ is of order 25, and $C_D(V_1) \cong D_8 \circ C_4$ acts irreducibly on it and has three orbits of size $M_2 = 8$ on it. Hence $(D_8 \circ C_4) \times (D_8 \circ C_4) \cong C_D(V_2) \times C_D(V_1)$ is a normal subgroup of G. Now since $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 (as we saw above), it follows that $M_1 = 16$ and $D/C_D(V_1)$ must be at least of order 32, and thus $D/C_D(V_1)$ is a full Sylow 2-subgroup of $\mathrm{GL}(2,5)$, i.e., a semidihedral group of order 32. Moreover, $|G : G'| = M = M_1 M_2 = 16 \cdot 8 = 2^7$ and $|G| = |G/D|\,|D/C_D(V_1)|\,|C_D(V_1)| = 2 \cdot 32 \cdot 16 = 2^{10}$. Therefore $|G'| = 2^3$. Now let $Z = C_D(V_1)' \times C_D(V_2)'$. Then $Z \leq D'$ is a Klein 4-group and $G'/Z = (G/Z)'$. Working in G/Z, we notice that $(C_D(V_1) \times C_D(V_2)')/Z$ is elementary abelian of order 2^4, and if $g \in G - D$, then gZ interchanges the two subgroups $C_D(V_i)Z/Z \cong C_D(V_i)/C_D(V_i)'$ $(i = 1, 2)$. Looking at the elements $[gZ, xZ] \in (G/Z)'$ for $x \in C_D(V_1)$ shows us that $|(G/Z)'| \geq |C_D(V_1)Z/Z| = 4$ so that altogether $2^3 = |G'| = |G'/Z||Z| \geq 4 \cdot 4 = 2^4$, which is a contradiction. This completes Case 2.4.1.1 where $C_D(V_1)$ is not abelian.

Step 3.2.4.1.2 The Case Where $C_D(V_1)$ is Abelian

Then $C_D(V_1)$ has regular orbits on W_1, and thus $M_2 = |C_D(V_1)|$, so $C_D(V_1)$ has either two or three regular orbits on W_1.

Note that $M_2 = |C_D(V_1)|$ and so

$$\begin{aligned} M = M_1 M_2 = M_1 |C_D(V_1)| &= |G/G'| = p|D/D'| \\ &= p|D : D' C_D(V_1)||D' C_D(V_1) : D'| \\ &= M_1 |D' C_D(V_1) : D'| \\ &= M_1 |C_D(V_1) : (D' \cap C_D(V_1))| \end{aligned}$$

This forces $D' \cap C_D(V_1) = 1$. So if $x \in D$ and $c \in C_D(V_1)$, then $[x, c] \in D' \cap C_D(V_1) = 1$. This shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \dots p$.

Now we consider the k in (+).

First suppose that $k = 1$, then $W_1 = X_1$, but since $W_1 = V_2 \oplus \cdots \oplus V_p$, we see that $X_1 = V_2$ and $p = 2$. In particular, V_2 is an irreducible faithful $C_D(V_1)$-module, so $C_D(V_1)$ is cyclic and has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits, which shows that $(|V_2| - 1)/2 = |C_D(V_1)|$. Since $|C_D(V_1)|$ is a power of 2, by an elementary result from number theory (see [10, Proposition 3.1]) it follows that—if we write q for the characteristic of V—either V_2 is of dimension 1 and $|V_2|$ is a Fermat prime, or $|V_2| = 9$ and $|C_D(V_1)| = 4$.

In the former case we get that $D/C_D(V_1)$ is abelian and hence D is abelian, and so G is abelian (since $G' = D'$), a contradiction. In the latter case we get that $D/C_D(V_1)$ must be at least of order 8 (since it has exactly one maximal orbit (of size M_1) on V_1, and it must be isomorphic to a subgroup of the semidihedral group SD_{16}, as Sylow 2-subgroup of GL $(2,3)$. However, all such subgroups have center of order 2, contradicting the fact that $|C_D(V_1)| = |C_D(V_2)| = 4$ and $C_D(V_1) \leq Z(D)$. This concludes the case that $k = 1$.

So let $k > 1$. Then define $X_0 = 0$ and $L_i = C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_i)/C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_{i+1})$ for $i = 0, \ldots, k-1$. As $k > 1$, we see that L_0 has exactly one regular orbit on X_1, because otherwise also L_1 would have at least two regular orbits on X_2 which ultimately would lead to $C_D(V_1)$ having at least four regular orbits on W_1, a contradiction. Since all orbits of L_0 on X_1 must be regular, we thus conclude that $|L_0| = |X_1| - 1$. Since $C_D(V_1)$ has exactly two regular orbits on W_1, it follows that there is exactly one $l \in \{1, \ldots, k\}$ such that L_{l-1} has exactly two regular orbits on X_l, whereas all the other L_i's have exactly one regular orbit on X_{i+1}. However, since L_{l-1} only has regular orbits on $X_l - \{0\}$, it is clear that the single regular orbit of size $|X_l| - 1$ of $C_D(V_1)/C_{C_D(V_1)}(X_l)$ on X_l splits into at least p regular orbits for L_{l-1} on X_L. This shows that $p = 2$. Hence $C_D(V_1)C_D(V_2) = C_D(V_1) \times C_D(V_2) \leq Z(G)$, and since $D/C_D(V_1)$ acts faithfully and irreducibly on V_1, we see that $C_D(V_1) \cong C_D(V_1)C_D(V_2)/C_D(V_2)$ is cyclic and thus has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits and we now can arrive at a contradiction just as in the case that $k = 1$

This concludes Case 2.4.1.2 and thus Case 2.4.1 is completed and it is left to show that $C_D(V_1)$ cannot have exactly one orbit of size M_2 on W_1.

Step 3.2.4.2 The Case Where $C_D(V_1)$ has exactly one orbit of size M_2 on W_1

Then by Theorem 1.1 $C_D(V_1)$ is abelian and hence has regular orbits on W_1, so $M_2 = |C_D(V_1)|$ and the same argument as at the beginning of Case 2.4.1.2 shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \ldots p$.

Assume that $X_1 < V_2$ (where X_1 is as in (+)). Since $C_D(V_1)$ has exactly one regular orbit on W_1, it also has exactly one regular orbit on V_2, and since V_2 is not irreducible as $C_D(V_1)$-module, by Lemma 2.2 it is clear that $C_D(V_1)/C_{C_D(V_1)}(V_2)$ is not cyclic. But since $C_D(V_1) \leq Z(D)$, we see that

$$\begin{aligned} C_D(V_1)/C_{C_D(V_1)}(V_2) &= C_D(V_1)/C_D(V_1 \oplus V_2) = C_D(V_1)/(C_D(V_1) \cap C_D(V_2)) \\ &\cong C_D(V_1)C_D(V_2)/C_D(V_2) \end{aligned}$$

is a non-cyclic central subgroup of $D/C_D(V_2)$. But on the other hand, $D/C_D(V_2)$ acts faithfully and irreducibly on V_2 and hence has a cyclic center, and we have a contradiction. This shows that $X_1 = V_2$, so V_2 is an irreducible $C_D(V_1)$-module and $C_D(V_1)$ has either two or three (one of them being the trivial orbit) on V_2. Therefore again by [10, Proposition 3.1] it follows that—if we write q for the characteristic of V—either

- $p = 2$, V_2 is of dimension 1 and $|V_2|$ is a Fermat prime; or
- $q = 2$ and $|C_D(V_1)/C_D(V_1 \oplus V_2)| = p$ is a Mersenne prime; or
- $p = 2, q = 3, |V_2| = 9$ and $|C_D(V_1)| = 8$.

In the first case we get (as earlier) that $D/C_D(V_1)$ is abelian and thus D is abelian, a contradiction. In the second case, since D is a p-group, with [10, Theorem 2.1] we see that

$D/C_D(V_1)$ cyclic of order p and thus abelian, making D abelian, a contradiction. So we are left with the third case. Here we have that $D/C_D(V_1)$ is a subgroup of the semidihedral group of order 16, so $|G| \leq 2^9$, and $|G| \leq 2^8$ unless $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$. Moreover, since any $g \in G - D$ interchanges $C_D(V_1)$ and $C_D(V_2)$, by taking commutators of elements in $C_D(V_1)$ with g we easily see that $|D'| \geq 8$ and so $|G'| \geq 8$. Now D has an orbit of size $\geq 2^6$ on V (from the regular orbit of $C_D(V_1) \times C_D(V_2)$. So if $|G| \leq 2^8$, we get $2^5 < 2^6 \leq M = |G/G'| \leq 2^8/2^3 = 2^5$, a contradiction. This leaves us with $|G| = 2^9$, and $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$, but in this case for similar reasons as above we see that $|D'| \geq 2^4$ and thus get the contradiction $2^5 < 2^6 \leq M = |G/G'| \leq 2^9/2^4 = 2^5$.

This final contradiction concludes the proof of the theorem.

5 Outlook

What's next? It is natural to ask what happens if we replace the "three" in the hypothesis of our result by "four" and see what happens. Does such an action exist? We are not aware of any example and believe there is none, but of course proving this might be quite challenging, and we are not certain whether the techniques developed so far would be sufficient to prove it. The next step, going to five, might be the last one for which one could hope to classify all the groups with reasonable effort. We are aware of just one such action, and so we formulate the following

Conjecture 1 *Let G be a finite nonabelian group V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly five orbits of size M on V. Then $G = D_8 \circ Q_8$ and $V = V(4, 3)$.*

Here G is a central product of D_8 and Q_8 embedded in GL(4,3) acting naturally on $V = V(4, 3)$ by matrix multiplication. So $|G| = 32$ and G has five orbits of size $16 = |G/G'|$ on V.

The above conjecture perhaps is a little bold at this time since our evidence for its truth is weaker than for, say our earlier conjecture in [5] on three orbits which was proved here. Five seems to be the limit of what can be done in terms of classifications.

Another approach to further work on this topic would be to relax the irreducibility hypothesis on the earlier results to complete reducibility, or just assuming faithful action and $\mathrm{O}_p(G) = 1$, where p is the characteristic of the V (this would add some group actions to the conclusions).

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