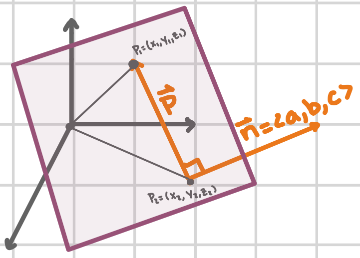
Standard 03: Planes

Planes in 3D

Equation of a Plane in 3-D



A plane in 3D requires two things:
- a point in the plane $\vec{r}_0 = (x_0, y_0, z_0)$
- the direction orthogonal to the plane $\vec{n} = \langle a,b,c \rangle$

vector equation = $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$.

standard equation = $ax + by + cz = d = \vec{n} \cdot \vec{r}_0$

example. Find the equation of the plane that contains the point $(7,2,-1)$ and is orthogonal to the line given by the parametrization $\vec{r}(t) = \langle 1-2t, 3t, 2-t \rangle$.

We are given the point $(7,2,-1)$ and a vector that is orthogonal. Since $\vec{r}(t)$ is orthogonal, its direction vector is orthogonal to the plane. So we have our two parts.

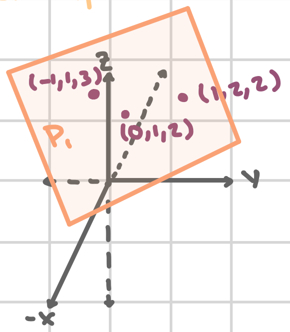
$\vec{r}_0 = (7,2,-1)$ & $\vec{n} = \langle -2,3,-1 \rangle$

$-2x + 3y - 1z = \langle -2,3,-1 \rangle \cdot \langle 7,2,-1$

$-2x + 3y - z = -14 + 6 + 1 = -7$

$-2x + 3y - z = -7$

example. Find an equation for P_1 the plane that goes through the points $(0,1,2)$, $(-1,1,3)$, and $(1,2,2)$.



To find the equation of a plane we need a point on the plane and normal vector $\vec{n}$. We are given three points to choose from and can find a normal vector by finding a cross product between two vectors in the plane, the vector between given points

work: $(0,1,2)$ to $(-1,1,2)$ is $\langle -1,0,1 \rangle$ and $(0,1,2)$ to $(1,2,2)$ is $\langle 1,1,0 \rangle$.

$$\vec{n} = \langle -1,0,1 \rangle \times \langle 1,1,0 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (0-1)\vec{i} - (0-1)\vec{j} + (-1-0)\vec{k}$$

$$= -1\vec{i} + 1\vec{j} - 1\vec{k}$$

$$= \langle -1, 1, -1 \rangle$$

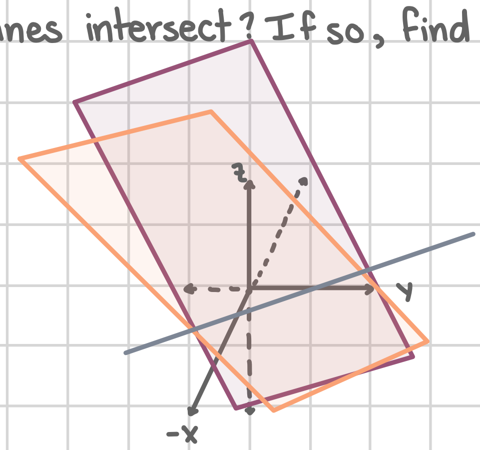
standard equation: $ax + by + cz = \vec{n} \cdot \vec{r}_0$

$-1x + 1y - 1z = \langle -1,1,-1 \rangle \cdot \langle 0,1,2 \rangle$

$-x + y - z = 0 + 1 - 2$

$-x + y - z = -1$

example. Let P_1 be the plane found in the above example and P_2 be the plane described by $x - y + 2z = 1$. Do these planes intersect? If so, find the line of intersection and the cosine of the angle between the planes?



If the normal vectors of P_1 & P_2 are scalars of each other then they do not intersect, otherwise we can find L. Since L is contained in P_1 and P_2, L must be orthogonal to the normal vectors for each plane, $\vec{n}_1$ and $\vec{n}_2$. Thus the direction of L is $\vec{n}_1 \times \vec{n}_2$. To find the initial point of the line, we must find a x,y,z s.t. P_1 and P_2 are satisfied.

Finding initial point:

$-x + y - z = -1$

$\underline{x - y + 2z = 1}$

$0x + 0y + 1z = 0$

$z = 0$

if $z = 0$ then $x + y = 1$

$\vec{n}_1 = \langle -1,1,-1 \rangle$ & $\vec{n}_2 = \langle 1,-1,2 \rangle$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

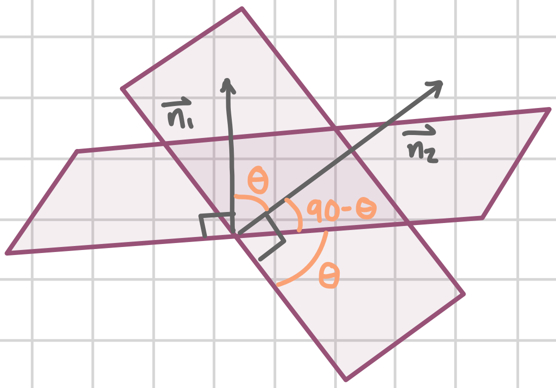
$$= (2-1)\vec{i} - (-2-(-1))\vec{j} + (1-1)\vec{k}$$

$$= 1\vec{i} - (-1)\vec{j} + 0\vec{k}$$

$$= \langle 1,1,0 \rangle$$

using the point $(1,0,0)$ and $\vec{n} = \langle 1,1,0 \rangle$

$\vec{r}(t) = \langle 1,0,0 \rangle + t\langle 1,1,0 \rangle$



The angle between 2 planes is found through finding the angle between their norms, this is equivalent due to complementary angles.

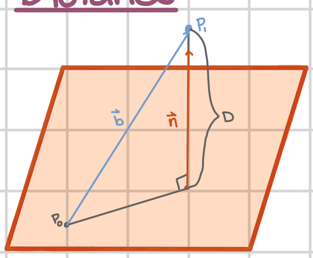
$\|\vec{n}_1\| \cdot \|\vec{n}_2\| \cos\theta = \vec{n}_1 \cdot \vec{n}_2$

$\sqrt{(-1)^2+(1)^2+(-1)^2}\,\sqrt{(1)^2+(-1)^2+(2)^2}\,\cos\theta = (-1)(1)+(1)(-1)+(-1)(2)$

$\sqrt{3}\,\sqrt{6}\,\cos\theta = -1-1-2$

$\cos\theta = {}^{-4}/\sqrt{18}$

Distance



Distance from a point P_1 to the plane.
Pick a point P_0 in the plane, name the distance between P_1 and P_0, $\vec{b}$. This may not be the shortest distance from the point to the plane, the shortest distance is on the normal vector of the plane, so we take $D = |\mathrm{comp}_{\vec{n}}\vec{b}|$.

example. Find the distance from the point $(2,1,2)$ to the plane $x+y+z=1$.
To find a point on the plane set two variables to zero: $(0)+(0)+z=1 \Rightarrow P_0=(0,0,1)$
The vector between (x_1,y_1,z_1) & (x_2,y_2,z_2) is $\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$: $\langle 2-0, 1-0, 2-1\rangle = \langle 2,1,1\rangle$
The normal vector of $ax+by+cz=d$ is $\langle a,b,c\rangle$: $\vec{n}=\langle 1,1,1\rangle$
Recall the formula $\mathrm{comp}_{\vec{n}}\vec{b} = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$: $\mathrm{comp}_{\vec{n}}\vec{b} = \frac{\langle 1,1,1\rangle \cdot \langle 2,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \frac{(1)(2)+(1)(1)+(1)(1)}{\sqrt{1+1+1}} = \frac{4}{\sqrt{3}}$

Note that if two planes intersect then the distance between them is zero, so a question about distance only makes sense for parallel planes.