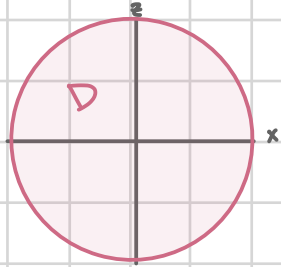
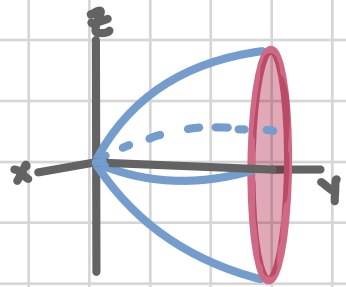


## Standard 14: Triple Integrals - Cylindrical & Spherical Coordinates

Recall the last example from rectangular coordinates:

**example.** Evaluate  $\iiint_E \sqrt{3x^2 + 3z^2} \, dV$  where  $E$  is the solid bounded by  $y = 2x^2 + 2z^2$  and the plane  $y = 8$ .



$$D = \{(x, z) \mid x^2 + z^2 \leq 4\}$$

$$\begin{aligned} \iiint_E \sqrt{3x^2 + 3z^2} \, dV &= \iint_D \left[ \int_{2x^2+2z^2}^8 \sqrt{3x^2 + 3z^2} \, dy \right] dA \\ &= \int_{-4}^4 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2x^2+2z^2}^8 \sqrt{3x^2 + 3z^2} \, dy \, dz \, dx \end{aligned}$$

The projection onto the  $xz$ -plane, the region  $D$ , is a disk. The equation of the disk comes from  $8 = 2x^2 + 2z^2$ . This region can be best described using something like polar coordinates for the  $xz$ -plane. Instead of the usual  $x = r \cos \theta$  and  $z = r \sin \theta$ , we use  $x = r \cos \theta$  and  $z = r \sin \theta$ . Thus the region  $D = \{(x, z) \mid x^2 + z^2 \leq 4\}$  becomes  $D = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2\}$ . We also have to reflect these changes in the integral:

$$\begin{aligned} \iiint_E \sqrt{3x^2 + 3z^2} \, dV &= \int_{-4}^4 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2x^2+2z^2}^8 \sqrt{3x^2 + 3z^2} \, dy \, dz \, dx \\ &= \int_0^{2\pi} \int_0^2 \int_{2r^2}^8 \sqrt{3r^2} \cdot r \, dy \, d\theta \, dr \end{aligned}$$

$$dA = r \, dr \, d\theta$$

### cylindrical coordinates

This set of conversions is called cylindrical coordinates and is an extension of polar coordinates into three dimensions. In the example above we used the conversions for  $E$ 's in which the  $D$  is in the  $xz$ -plane, there is a different set for each type of  $E$  we have seen:

- (i)  $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$     (ii)  $E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$     (iii)  $E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$x^2 + y^2 = r^2$$

$$dV = r \, dz \, dr \, d\theta$$

$$y = r \cos \theta$$

$$z = r \sin \theta$$

$$x = x$$

$$y^2 + z^2 = r^2$$

$$dV = r \, dx \, dr \, d\theta$$

$$x = r \cos \theta$$

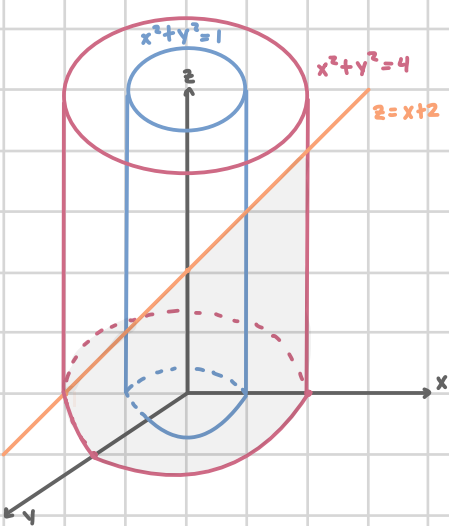
$$z = r \sin \theta$$

$$y = y$$

$$x^2 + z^2 = r^2$$

$$dV = r \, dy \, dr \, d\theta$$

**example.** Set up  $\iiint_E y \, dV$  where  $E$  is the region that lies below the plane  $z = x + 2$ , above the  $xy$ -plane and between the cylinders  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$0 \leq z \leq x + 2$$

$$\text{becomes}$$

$$0 \leq z \leq r \cos \theta + 2$$

$$D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$$

$$= \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$\iiint_E y \, dV = \int_0^{2\pi} \int_1^2 \int_0^{r \cos \theta + 2} r \sin \theta \cdot r \, dz \, dr \, d\theta$$

**example.** Convert the following integral into cylindrical coordinates:  $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-x}^{x^2+y^2} xyz \, dz \, dy \, dx$ .

Rectangular coordinate bounds:

$$0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}, -x \leq z \leq x^2 + y^2$$

Cylindrical conversion:

$$x = r \cos \theta, y = r \sin \theta, z = z, dV = r \, dz \, dr \, d\theta$$

Sketch  $xy$ -plane:



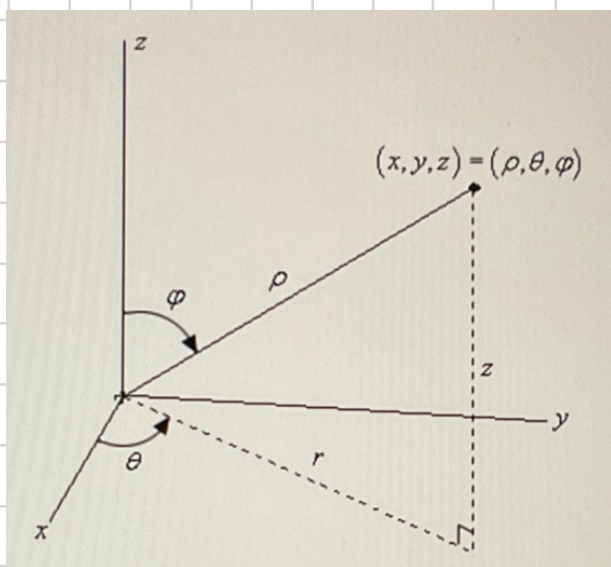
Rewrite bounds:

$$0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}, -r \cos \theta \leq z \leq r^2$$

$$\int_0^{\pi/2} \int_0^1 \int_{-r \cos \theta}^{r^2} (r \cos \theta)(r \sin \theta)(z) \cdot r \, dz \, dr \, d\theta$$

## spherical coordinates

There is another extension of polar coordinates into three dimensions; it is given by rotating polar coordinates. Spherical coordinates are  $(\rho, \theta, \varphi)$  where  $\rho$  is the distance from the origin,  $\theta$  is the angle made with the positive  $x$ -axis in the  $xy$ -plane, and  $\varphi$  is the angle made with the positive  $z$ -axis. Here is a visual of the conversion from rectangular to spherical:



$$x = \rho \sin(\varphi) \cos(\theta)$$

$$y = \rho \sin(\varphi) \sin(\theta)$$

$$z = \rho \cos(\varphi)$$

$$x^2 + y^2 + z^2 = \rho^2$$

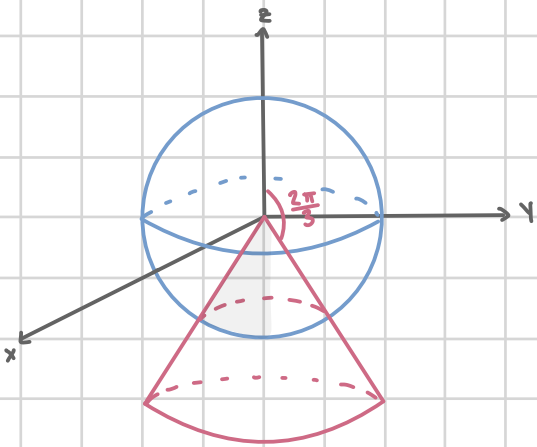
$$dV = \rho^2 \sin(\varphi) d\rho d\theta d\varphi$$

with the restriction:

$$\rho \geq 0$$

$$0 \leq \varphi \leq \pi$$

**example.** Set up  $\iiint_E z x dV$  where  $E$  is inside both  $x^2 + y^2 + z^2 = 2$  and the cone that makes an angle of  $\pi/3$  with negative  $z$ -axis and has  $x \leq 0$ .



$$0 \leq \rho \leq 2$$

$$\frac{2}{3}\pi \leq \varphi \leq \pi$$

$$\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$$

be careful when computing  $\varphi \neq \theta$

they care about the positive axis

$$\iiint_E z x dV = \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \int_{\frac{2}{3}\pi}^{\pi} \int_0^2 (\rho \cos(\varphi)) (\rho \sin(\varphi) \cos(\theta)) \rho^2 \sin(\varphi) d\rho d\theta d\varphi$$