Week 10: March 23rd, 2023

Section 9.1: Separable Differential Equations

Differential Equations

A differential equation is an equation that involves an unknown function and its first or higher derivatives.

examples. $\frac{dy}{dx} = 1 - 6e^{2x}$
$\frac{dy}{dt} + \frac{1}{t+30} \cdot y = 4$] first order differential equations
$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = 0$] second order (homogeneous) diff. eq.
$\frac{dy}{dx} = F(x,y)$] general first order differential equations

Separation of Variables

Separable differential equation: $\frac{dy}{dx} = p(x)q(x)$

examples. $\frac{dy}{dx} = 1-6e^{2x}$ $\quad p(x) = 1-6e^{2x}$; $q(y) = 1$
$y'(x) = 3x^2y$ $\quad p(x) = 3x^2$; $q(y) = y$

How to solve? Method of separation.

$$\frac{dy}{dx} = p(x)q(x) \iff y'(x) = p(x)q(y)$$
$$\iff \frac{1}{q(y)}y'(x) = p(x)$$
$$\iff \int \frac{1}{q(y)}y'(x)\,dx = \int p(x)\,dx$$
$$\iff \int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

Simplified explanation:

$$\frac{dy}{dx} = p(x)q(y) \iff \frac{1}{q(y)}\,dy = p(x)\,dx$$
$$\iff \int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

example. Solve for (i) all solutions and (ii) the solution satisfying $y(0)=5$ of the differential equation $\frac{dy}{dx} = 1-6e^{2x}$.

(i) $\frac{dy}{dx} = 1-6e^{2x}$
$\int \frac{dy}{dx}\,dx = \int 1-6e^{2x}\,dx$
$y = x - 3e^{2x} + c$

(ii) @ $y(0)=5$
$0 - 3e^0 + c = 5$
$-3 + c = 5$
$c = 8$
$y = x + 3e^{2x} + 8$

example. Solve for the general solution of $y'(x) = 3x^2y$. Find the particular solution such that $y(0) = -2$.

$y'(x) = 3x^2y(x)$
$\frac{1}{y(x)}y'(x) = 3x^2$
$\int \frac{1}{y(x)}y'(x)\,dx = \int 3x^2\,dx$
$u = y(x) \quad du = y'(x)\,dx$
$\int \frac{1}{u}\,du = \int 3x^2\,dx$
$\ln|u| + c_1 = x^3 + c_2$
$\ln|y(x)| = x^2 + (c_2 - c_1)$
$e^{\ln|y(x)|} = e^{x^2 + (c_2-c_1)}$
$|y(x)| = e^{c_2-c_1} \cdot e^{x^2}$
$y(x) = \pm Ae^{x^2}$

@ $y(0) = -2$
$Ae^0 = -2$
$A = -2$

$y(x) = -2e^{x^5}$

example. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. A roast turkey is taken from an oven when its temperature is 185°F and is placed on a table in a room where the temperature is 75°F. Temperature of the turkey falls to 150°F after half an hour. Apply Newton's Law of Cooling to find the temperature of the turkey after 45 minutes.

Newton's Law of Coding: $y' = -k(y - T_0)$; $y(t)$ = temp. of object at time t, k = cooling constant, T_0 = ambient temp.

What we know: $T_0 = 75°F$, $y(0) = 185°F$, $y(30) = 150°F$,

Need to find: k, $y(t)$, $y(45)$

$y' = -k(y-75)$

$\frac{1}{y-75} y' = -k$

$\int \frac{1}{y-75} y' dt = \int -k\, dt$

$\ln|y-75| = -kt + c$

$y - 75 = e^{-kt+c}$

$y = e^c e^{-kt} + 75$

$y = Ae^{-kt} + 75$

@ $y(0) = 185$

$185 = Ae^0 + 75$

$185 = A + 75$

$110 = A$

@ $y(30) = 150$

$110 e^{-30k} + 75 = 150$

$110 e^{-30k} = 75$

$e^{-30k} = 75/110$

$-30k = \ln(\frac{75}{110})$

$k = -\frac{1}{30} \ln(\frac{75}{110})$

$y(t) = 110 e^{-\frac{1}{30}\ln(\frac{75}{110})t} + 75$

$y(45) = 110 e^{-\frac{1}{30}\ln(\frac{75}{110})45} + 75$

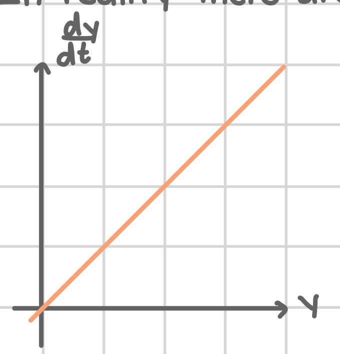
$= 110 e^{-\frac{3}{2}\ln(\frac{75}{110})} + 75$

Population Modeling

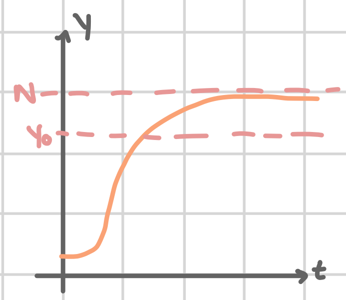
A population y of a single species with unrestricted growth is given by the differential equation $\frac{dy}{dt} = ky$ for $k > 0$. The issue is that this model allows growth with unlimited resources. There are no caps or restrictions.

The logistic Model

In reality there are restrictions like a max population.



unlimited population



max population N

$\frac{dy}{dt}$ with restrictions

$\frac{dy}{dt} = Cy(N-y) = \underbrace{CN}_{k} y(1 - \frac{y}{N})$; N = carrying capacity, k = intrinsic growth rate

example. A population of fish grows with growth constant (intrinsic growth rate) of 0.5 in a lake of carrying capacity of 10 thousand is modeled by the logistic differential equation $\frac{dp}{dt} = 0.5p(1 - \frac{p}{10})$ where p is its population measured in thousands. If the initial population is 5 thousand, find a formula for $p(t)$.

$\frac{dp}{dt} = 0.5p(1 - \frac{p}{10})$

$\frac{1}{p(1-\frac{p}{10})} \cdot dp = 0.5 dt$

$\underbrace{\int \frac{1}{p(1-\frac{p}{10})} dp}_{①} = \underbrace{\int 0.5 dt}_{②}$

① partial fraction

$\frac{10}{p(10-p)} = \frac{A}{p} + \frac{B}{10-p}$

$10 = A(10-p) + Bp$

$B - A = 0$

$10A = 10$

$A = 1$, $B = 1$

$\int \frac{1}{p} + \frac{1}{10-p} dp$

$= \ln|p| - \ln|10-p| + c_1$

② $\int 0.5 dt$

$= 0.5t + c_2$

$\ln|p| - \ln|10-p| = 0.5t + c$

@ $p(0) = 5$

$\ln|5| - \ln|10-5| = 0.5(0) + c$

$\ln|5| - \ln|5| = c$

$0 = c$

$\ln\left|\frac{p}{10-p}\right| = 0.5t$

$\frac{p}{10-p} = e^{0.5t}$

$p = e^{0.5t}(10-p)$

$p = 10e^{0.5t} - pe^{0.5t}$

$p + pe^{0.5t} = 10e^{0.5t}$

$p(1 + e^{0.5t}) = 10e^{0.5t}$

$p = \frac{10e^{0.5t}}{1 + e^{0.5t}}$

First Order Linear Differential Equations

example. Solve the initial value problem: $xy' + y = e^{2x}$ and $y(1) = 0$.

Note the left hand side $xy' + 1y$, this resembles a product rule $\frac{dy}{dx}(xy) = xy' + 1y$.

$\frac{dy}{dx}(xy) = e^{2x}$

$\int \frac{dy}{dx}(xy)\,dx = \int e^{2x}\,dx$

$xy = \frac{1}{2}e^{2x} + c$

$y = \frac{1}{2x}e^{2x} + \frac{c}{x}$

@ $y(1) = 0$

$0 = \frac{1}{2(1)}e^{2(1)} + \frac{c}{(1)}$

$0 = \frac{1}{2}e^{2} + c$

$c = -\frac{1}{2}e^{2}$

$y = \frac{1}{2x}e^{2x} + \frac{(-e^2/2)}{x}$

$y = \frac{1}{2x}e^{2x} - \frac{e^2}{2x}$

$y = \frac{1}{2x}(e^{2x} - e^2)$

Standard form: $y' + A(x)y = B(x)$

The key to solve it is to have something like $\frac{d}{dx}(\alpha(x)y)$ on the left hand side (LHS) and a function only in terms of x on the right hand side (RHS).

$\frac{d}{dx}(\alpha(x)y) = \alpha(x)y' + \alpha'(x)y$

In order to get $y' + A(x)y$ to resemble $\frac{d}{dx}(\alpha(x)y)$ we must multiply by $\alpha(x)$ on both sides:

$\alpha(x)y + \alpha(x)A(x)y = \alpha(x)B(x)$

For the two LHSs to be equivalent we need a function $\alpha(x)$ such that:

$\alpha'(x) = \alpha(x)A(x)$

We can use separation of variables to solve this:

$\frac{d\alpha}{dx} = \alpha A(x)$

$\frac{1}{\alpha}d\alpha = A(x)\,dx$

$\int \frac{1}{\alpha}d\alpha = \int A(x)\,dx$

$\ln|\alpha| = \int A(x)\,dx$

$|\alpha| = e^{\int A(x)dx}$

All this to say:
$\alpha = e^{\int A(x)dx}$

We can now integrate both sides:

$\alpha(x)y + \alpha(x)A(x)y = \alpha(x)B(x)$

$\frac{d}{dx}(\alpha(x)y(x)) = \alpha(x)B(x)$

$\int \frac{d}{d(x)}(\alpha(x)y(x))\,dx = \int \alpha(x)B(x)\,dx$

$\alpha(x)y(x) = \int \alpha(x)B(x)\,dx$

$y(x) = \frac{1}{\alpha(x)}\int \alpha(x)B(x)\,dx$

All this to say:
$y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x)\,dx + c\right]$
is the general solution.

Steps to solve:

1. Write the equation in standard form $y' + A(x)y = B(x)$
2. Compute integrating factor $\alpha(x) = e^{\int A(x)dx}$
3. Find the general solution $y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x) + c\right]$
4. If required, find a particular solution.

example. $y' - (\tan(x))y = 1$; $y(0) = 3$

1. $A(x) = -\tan(x)$; $B(x) = 1$
2. $\alpha(x) = e^{\int -\tan(x)dx} = e^{\ln|\cos(x)|} = \cos(x)$
3. $y(x) = \frac{1}{\cos(x)}\left[\int(\cos(x))\cdot(1)\,dx + c\right]$

$= \frac{1}{\cos(x)}\left[\sin(x) + c\right]$

$= \frac{\sin(x)}{\cos(x)} + \frac{c}{\cos(x)}$

$= \tan(x) + \frac{c}{\cos(x)}$

4. @ $y(0) = 3$

$3 = \tan(0) + \frac{c}{\cos(0)}$

$3 = \frac{c}{1}$

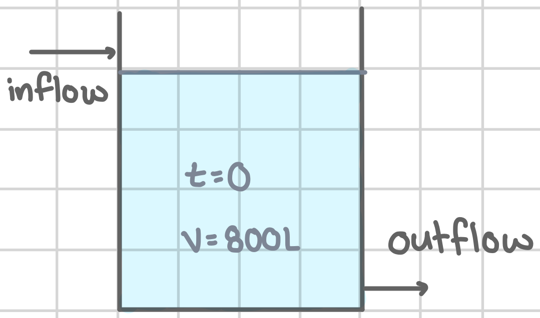
$3 = c$

$y(x) = \tan(x) + \frac{3}{\cos(x)}$

Mixing Tank Problem

example. (a) A tank contains 800L of fresh water. Brine that contains 0.05 kg of salt per liter enters the tank at a rate of 40 L/min. Brine is drained from the tank at the same rate of 40 L/min. Find an expression for the amount of salt in the tank at any time t.

(b) What if the brine is drained out at a slower rate of 10 L/min.? Find an expression for the amount of salt in the tank at any time t in this case.



Let $y(t)$ = amount of salt in the tank in kg at time t

$\frac{d}{dt}$ = salt inflow − salt outflow

= (concentration in)(inflow rate) − (concentration out)(outflow rate)

$= (0.05)\cdot(40) - \dfrac{\text{amount of salt in tank}, y(t)}{\text{volume of solution}, v(t)} \cdot (\text{outflow rate})$

(a) Outflow rate = 40 L/min.

$v(t) = 800L$ as flow in = flow out

outflow rate = 40 L/min.

$$\frac{dy}{dt} = 2 - \frac{y}{800}(40)$$

$$= 2 - \frac{y}{20}$$

$$\frac{dy}{dt} = \frac{40-y}{20}$$

$$\frac{1}{40-y}\,dy = \frac{1}{20}\,dt$$

$$\int \frac{1}{40-y}\,dy = \int \frac{1}{20}\,dt$$

$$-\ln|40-y| = \frac{1}{20}t + c$$

$$\ln|40-y| = -\frac{1}{20}t - c$$

$$|40-y| = e^{-\frac{1}{20}t - c}$$

$$40-y = ke^{-\frac{1}{20}t}$$

$$y = 40 - ke^{-\frac{1}{20}t}$$

@ $y(0) = 0$

$$0 = 40 - ke^{-\frac{1}{20}(0)}$$

$$0 = 40 - k(1)$$

$$k = 40$$

$$y = 40 - 40e^{-\frac{1}{20}t}$$

(b) Outflow rate = 10 L/min.

$v(t) = 800 + 30t$ as flow in = 40, flow out = 10

$$\frac{dy}{dt} = 2 - \frac{y}{800+30t}(10)$$

$$= 2 - \frac{1}{80+3t}y$$

$$\frac{dy}{dt} + \frac{1}{80+3t}y = 2$$ ← linear first order

1. $A(x) = \dfrac{1}{80+3t}$; $B(t) = 1$

2. $\alpha(t) = e^{\int \frac{1}{80+3t}dt}$

$$= e^{\frac{1}{3}\ln|80+3t|}$$

$$= e^{\ln(80+3t)^{1/3}}$$

$$= (80+3t)^{1/3}$$

3. $y(t) = \dfrac{1}{\alpha(t)}\left[\int \alpha(t)B(t)\,dt + c\right]$

$$= \frac{1}{(80+3t)^{1/3}}\left[\int \frac{1}{(80+3t)^{1/3}}(1)\,dt + c\right]$$

$$= \frac{1}{(80+3t)^{1/3}}\left[\frac{(80+3t)^{4/3}}{4} + c\right]$$

$$= \frac{1}{4}(80+3t) + (80+3t)^{-1/3}c$$

@ $y(0) = 0$

$$0 = \frac{1}{4}(80+3(0)) + (80+3(0))^{-1/3}c$$

$$0 = 20 + (80)^{-1/3}c$$

$$-20 = (80)^{-1/3}c$$

$$-20(80)^{1/3} = c$$

$$y = \frac{1}{4}(80+3t) - 20(80)^{1/3}(80+3t)^{-1/3}$$