10350 Final Extra Worksheet - Set 01 Name ______________________________

1. A vessel in the shape of a right regular cone with height 10 m and opening diameter 8 m. Water is slowly poured into the vessel and it is observed that the depth of the water is increasing at a rate of 0.1 m/s when the depth of the water is 2 m. (a) How fast is water filling the vessel at that same instant? (b) How fast is the diameter of the water surface change at the same moment?

Given:

$\frac{r}{h} = \frac{4}{10}$, $\frac{dh}{dt} = 0.1$ when $h = 2m$.

$10r = 4h$

(a) Find $\frac{dV}{dt}$.

$h = \frac{5}{2}r$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$

$= \frac{1}{3}\pi\left(\frac{4}{25}\right)h^3$

$\frac{dV}{dt} = \frac{4}{25}\pi h^2 \cdot \frac{dh}{dt}$

$= \frac{4}{25}\pi(2)\cdot(0.1)$

(b) Find $\frac{dr}{dt}$.

$r = \frac{2}{5}h$

$V = \frac{1}{3}\pi r^2\left(\frac{5}{2}r\right)$

$= \frac{1}{3}\left(\frac{5}{2}\right)\pi r^3$

$\frac{dV}{dt} = \frac{5}{2}\pi r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} \cdot \frac{2}{5\pi r^2} = \frac{dr}{dt}$

when $h = 2$

$r = \frac{2}{5}(2) = \frac{4}{5}$

$\frac{dr}{dt} = \left((0.1)\left(\frac{8}{25}\right)\pi\right)\cdot\left(\frac{2}{5\pi(4/5)^2}\right)$

2. Consider a function $f(x)$ defined for all x except $x = 0$ such that its **second derivative** is

$$f''(x) = \frac{(x-2)^2}{e^x - 1}.$$

Find all values of x for which $f(x)$ is **concave down**. What are the inflection points of $f(x)$?

$f''(x) = \frac{(x-2)^2}{e^x - 1} = 0 \longrightarrow$ DNE when $x = 0$ (given)

$(x-2)^2 = 0$

$x - 2 = 0$

$x = 2$

$-$	$+$	$+$
$\frac{pos}{neg}$	$\frac{pos}{pos}$	$\frac{pos}{pos}$

$0 \quad 2$

$\boxed{x = 0}$

3. Find the linear approximation of $f(x) = x^{2/3} - 3$ at $x = 8$.

$L(x) = f'(a)(x-a) + f(a)$

$f'(x) = \frac{2}{3}x^{-1/3}$

$f'(8) = \frac{2}{3} \cdot \frac{1}{\sqrt[3]{8}}$

$= \frac{2}{3} \cdot \frac{1}{2}$

$= \frac{1}{3}$

$f(8) = \sqrt[3]{(8)^2} - 3$

$= 4 - 3$

$= 1$

$L(x) = \frac{1}{3}(x-8) + 1$

4. An object moving on a straight line has position function

$$s(t) = e^{2t} + 4\csc(2t).$$

Find the acceleration $a(t)$ of the object at time t.

$v(t) = s'(t) = 2e^{2t} - 8\csc(2t)\cot(2t)$

$a(t) = s''(t) = 4e^{2t} - 8[2\csc(2t)\cot(2t)\cdot\cot(2t) - 2\csc^2(2t)\csc(2t)]$

$= 4e^{2t} + 16\left[\frac{1}{\sin(2t)} \cdot \frac{\cos^2(2t)}{\sin^2(2t)} + \frac{1}{\sin^2(2t)} \cdot \frac{1}{\sin(2t)}\right]$

$= 4e^{2t} + 16\left[\frac{\cos^2(2t)}{\sin^3(2t)} + \frac{1}{\sin^3(2t)}\right]$

$= 4e^{2t} + \frac{16(\cos^2(2t)+1)}{\sin^3(2t)}$

$= 4e^{2t} + 16[\csc(2t)\cot^2(2t) + \csc^3(2t)]$

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5. Find the equation of the tangent line to the curve at $t = 1$ given by the parametric equation

$$x = e^{t^2-1}; \qquad y = t\cos(\pi t)$$

$y - y_1 = m(x - x_1)$

$y + 1 = -\frac{1}{2}(x - 1)$

$y = -\frac{1}{2}x + \frac{1}{2} - 1$

$= -\frac{1}{2}x - \frac{1}{2}$

$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{\cos(\pi t) - \pi t \sin(\pi t)}{2t\, e^{t^2-1}}$

at $t=1$: $\frac{\cos(\pi) - \pi\sin(\pi)}{2e^0} = \frac{-1 - \pi \cdot 0}{2\ 1}$

$= -\frac{1}{2}$

$x_1 = e^{1-1} = e^0 = 1$

$y_1 = 1 \cdot \cos(\pi) = -1$

6. Find the absolute minimum value and absolute maximum value of the function on the given interval:

$$Q(x) = 3(x-1)^{1/3} - x + 5; \qquad -7 \le x \le 1.$$

$Q'(x) = (x-1)^{-2/3} - 1 = 0 \longrightarrow$ DNE $x = 1$

$\frac{1}{(x-1)^{2/3}} = 1$

$1 = (x-1)^{2/3}$

$1 = \sqrt[3]{(x-1)^2}$

$1 = (x-1)^2$

$\pm 1 = x - 1$

$1 = x - 1 \qquad -1 = x - 1$

$2 = x \qquad 0 = x$

out of domain

$f(-7) = 3(-7-1)^{1/3} - (-7) + 5 = 3(-2) + 7 + 5 = 6$

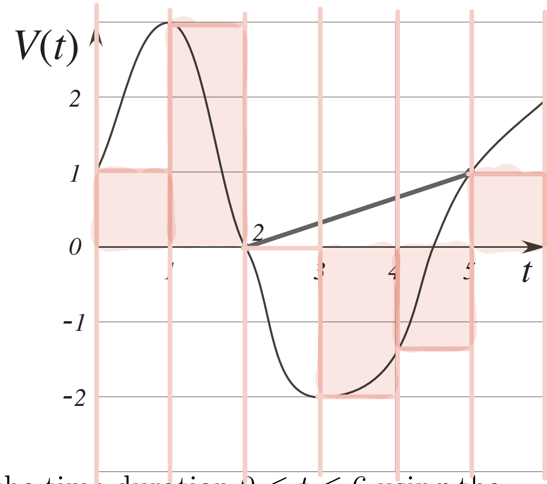
$f(0) = 0 - 0 + 5 = 5$

$f(1) = 3(0)^{1/3} - (1) + 5 = 4$

absolute min $= 6$

absolute max $= 4$

7. The velocity $V(t)$ ft/sec of a particle moving along a straight line at time t (in seconds) is shown in the figure.



7a. What is the average rate of change of the velocity of the particle over the time interval $2 \le t \le 5$?

$$\frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{1}{3}$$

7b. Estimate the total change in the position of the particle over the time duration $0 \le t \le 6$ using the Riemann sum for six equal segments and the left hand end points.

$$L_6 = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$= 1[f(0) + f(1) + f(2) + f(3) + f(4) + f(5)]$$

$$= 1[\cancel{1} + 3 + 0 \cancel{-2} - 1.3 \cancel{+1}]$$

$$= 1[3 - 1.3]$$

$$= 1.7$$

8. Evaluate the integral $\displaystyle\int_0^1 \frac{x+3}{(x^2+6x+5)^3}\,dx.$ $= \displaystyle\int_5^{12} \frac{1}{u^3} \cdot \frac{\cancel{(x+3)}}{1} \cdot \frac{1}{\cancel{(x+3)}} \cdot \frac{1}{2} \cdot du$

$x=0 \Rightarrow u=5$

$x=1 \Rightarrow u=1+6+5$
$=12$

$u = x^2+6x+5$

$du = 2x+6\,dx$

$du = 2(x+3)\,dx$

$\frac{1}{2}du = (x+3)dx$

$\frac{1}{2} \cdot \frac{1}{(x+3)} du = dx$

$$= \frac{1}{2}\int_5^{12} \frac{1}{u^3}\,du$$

$$= \frac{1}{2} \cdot -\frac{1}{2} u^{-2} \Big|_5^{12}$$

$$= -\frac{1}{4} u^{-2} \Big|_5^{12}$$

$$= -\frac{1}{4}\left(\frac{1}{(12)^2} - \frac{1}{(5)^2}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{144} - \frac{1}{25}\right)$$

9. The length (in mm) at time t (in seconds) of a straight metal rod being heated slowly is given by the function

$$L(t) = \sqrt{2t+1}$$

Using **calculus**, **estimate** the percentage change in length of the rod over the time duration $4 \leq t \leq 4.5$

$\Delta L = L'(a)(x-a)$

$L'(x) = \frac{1}{2}(2t-1)^{-1/2} \cdot 2$

$= \frac{1}{\sqrt{2t+1}}$

$L'(4) = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$\Delta L = \frac{1}{3}(x-4)$

10. Find $\frac{dy}{dx}$ if $4x^2 + xy = 3\ln(y)$.

$8x + [1 \cdot y + y' \cdot x] = 3 \frac{1}{y} \cdot y'$

$8x + y + xy' = 3 \cdot \frac{1}{y} \cdot y'$

$8x + y = 3 \cdot \frac{1}{y} \cdot y' - xy'$

$8x + y = y' [3 \cdot \frac{1}{y} - x]$

$\frac{8x+y}{3 \cdot \frac{1}{y} - x} = y'$

$\frac{8xy + y^2}{3 - xy} = y'$

11. Solve the initial value problem:

$$\frac{dy}{dx} = (e^x + 1)(e^x + 2); \qquad y(0) = 2$$

$y = \int (e^x+1)(e^x+2)\,dx$

$= \int e^{2x} + 3e^x + 2\,dx$

$= \frac{1}{2}e^{2x} + 3e^x + 2x + C$

$y(0) = \frac{1}{2}e^0 + 3e^0 + 0 + C$

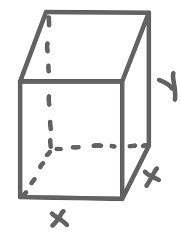
$2 = \frac{1}{2} + 3 + C$

$-1.5 = C$

$y = \frac{1}{2}e^{2x} + 3e^x + 2x - 1.5$

12. A box with a square base and a top is to be built with a volume of 20 cubic meters. The material for the base has density 3 gram per square meter, the material for the top has density 2 gram per square meter, and the material for the side has density 1 gram per square meter. What should the dimension of the box be so that its weight is minimum. Ans: 2m by 2m by 5m

$V = x \cdot x \cdot y = 20 \longrightarrow y = \frac{20}{x^2} \qquad 0 < x < \infty$



$W = \underbrace{\text{density} \cdot \text{area}}_{\text{base}} + \underbrace{\text{density} \cdot \text{area}}_{\text{top}} + \underbrace{\text{density} \cdot \text{area}}_{\text{side}} \times 4$

$= 3 \cdot (x^2) + 2 \cdot (x^2) + 1(xy) \cdot 4$

$= 3x^2 + 2x^2 + 4xy$

$= 5x^2 + 4xy$

$W(x) = 5x^2 + 4x\left(\frac{20}{x^2}\right)$

$= 5x^2 + \frac{80}{x}$

$W'(x) = 10x - \frac{80}{x^2} = 0$ DNE when $x = 0$

$10x = \frac{80}{x^2}$

$10x^3 = 80$

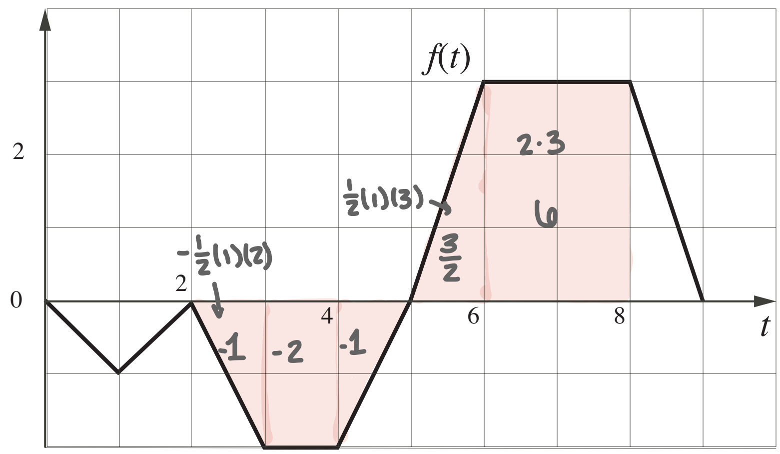
$x^3 = 8$

$x = 2$

$y = \frac{20}{(2)^2} = \frac{20}{4} = 5$

$\boxed{2 \times 2 \times 5}$

13.



The graph of $y = f(t)$ is given above.

13a. Find $\int_2^8 f(t)dt$.

$= -1 - 2 - 1 + \frac{3}{2} + 6$

$= \frac{3}{2} + 2$

$= 7/2$

13b. Evaluate the integral $\int_0^6 (|f(t)| - 3t^2 + 1)dt$.

$= \int_0^6 |f(t)|\,dt - \int_0^6 3t^2 + 1\,dt$

$= |-1| + |-2| + |-1| + |\frac{3}{2}| + |6| - t^3 + t \Big|_0^6$

$= 1 + 2 + 1 + \frac{3}{2} + 6 - (6)^3 + (6) - 0^3 - 0$

$= 15 + \frac{3}{2} - 6^3$