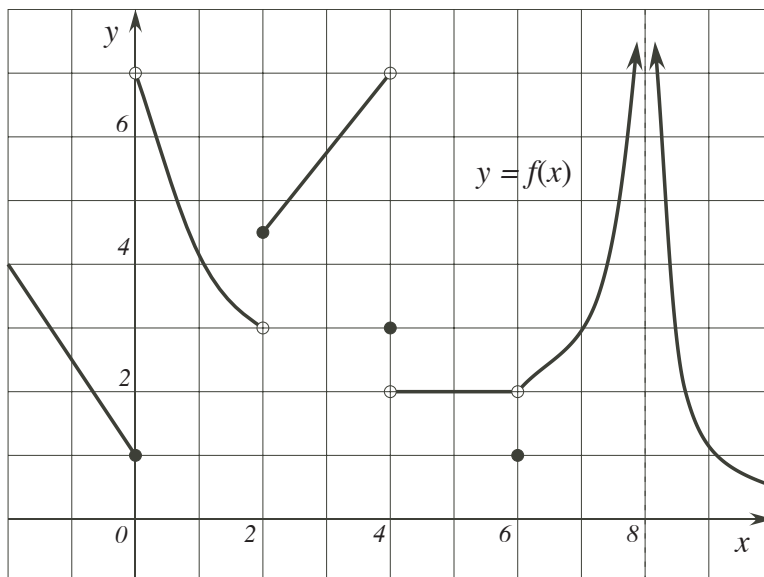


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1. The graph of $y = f(x)$ for $-2 \leq x \leq 10$ is shown above. Find the values of the following limits if it exists:

(a) $\lim_{x \rightarrow 6} [2f(x)] \stackrel{?}{=} 2[\lim_{x \rightarrow 6} f(x)] = 2[2] = 4$

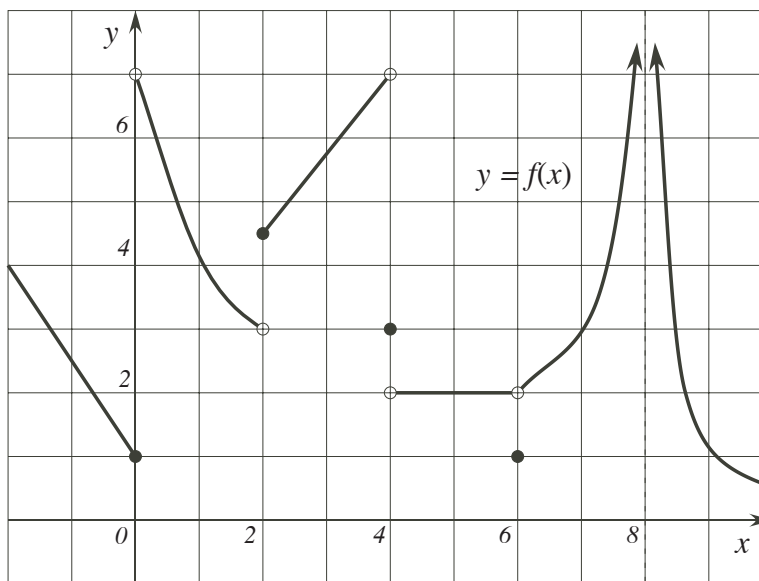
(b) $\lim_{x \rightarrow 2^-} [x \cdot f(x)] \stackrel{?}{=} \lim_{x \rightarrow 2^-} x \cdot \lim_{x \rightarrow 2^-} f(x) = 2 \cdot 3 = 6$

(c) $\lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} \stackrel{?}{=} \frac{\text{rise}}{\text{run}} = \frac{2.5}{2} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$

(d) $\lim_{x \rightarrow 0^+} \frac{\sqrt{f(x)}}{f(x) + 1} \stackrel{?}{=} \frac{\lim_{x \rightarrow 0^+} \sqrt{f(x)}}{\lim_{x \rightarrow 0^+} f(x) + 1} = \frac{\sqrt{7}}{7+1} = \frac{\sqrt{7}}{8}$

$\lim_{x \rightarrow 0^+} f(x) = 7$

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2. The graph of $y = f(x)$ for $-2 < x < 10$ is shown above.

a. Find all values of x in $(-2, 10)$ for which the function $y = f(x)$ is discontinuous.

$$x = 0, 2, 4, 6, 8$$

b. Amongst the values of x in (a), which of them are places where there is a removable discontinuity?

$$x = 6$$

$$\lim_{x \rightarrow c} f(x) \neq f(c)$$

c. Amongst the values of x in (a), which of them are places where there is a jump discontinuity?

$$x = 0, 2, 4$$

$$\lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x)$$

$x = 8$ is a special case:

$\lim_{x \rightarrow 8^-} f(x) = \infty$ and $\lim_{x \rightarrow 8^+} f(x) = -\infty$ so $\lim_{x \rightarrow 8} f(x) = \infty$, but we can not "plug" this hole with $(8, \infty)$. In fact it is deemed as neither a jump or a removable discontinuity.

d. Amongst the values of x in (a), which of them are places where $f(x)$ is left or right continuous? Explain your choices.

Left continuous:

$$x = 0$$

Right continuous:

$$x = 2$$

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3. The position of a particle at time t seconds moving on a straight line is given by $s(t) = \frac{1}{2t+1}$ meters. Answer the following questions below.

a. Find the average velocity of the particle between $t = 2$ and $t = 2 + h$. Simplify your answers as far as possible assuming $h \neq 0$.

average: $\frac{f(b) - f(a)}{b - a}$

$$\begin{aligned} & \frac{f(2+h) - f(2)}{2+h - 2} \\ &= \frac{\frac{1}{2(2+h)+1} - \frac{1}{2(2)+1}}{h} \\ &= \frac{\frac{1}{4+2h+1} - \frac{1}{5}}{h} \\ &= \frac{\frac{1}{2h+5} - \frac{1}{5}}{h} \\ &= \frac{\frac{5 - (2h+5)}{5(2h+5)}}{\frac{h}{1}} \\ &= \frac{\frac{5-2h-5}{5(2h+5)}}{\frac{h}{1}} \\ &= \frac{-2h}{5(2h+5)} \cdot \frac{1}{h} \\ &= \frac{-2}{10h+25} \end{aligned}$$

b. Find the difference quotient of $s(t)$ at $t = 2$.

difference quotient: $\frac{f(x+h) - f(x)}{h}$

$$\begin{aligned} &= \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} \\ &= \frac{\frac{1}{2t+2h+1} - \frac{1}{2t+1}}{h} \end{aligned}$$

$$\begin{aligned} &= \frac{\frac{2t+1 - (2t+2h+1)}{(2t+2h+1)(2t+1)}}{h} \\ &= \frac{\cancel{2t} + 1 - \cancel{2t} - 2h - \cancel{1}}{(2t+2h+1)(2t+1)} \\ &= \frac{-2h}{(2t+2h+1)(2t+1)} \cdot \frac{1}{h} \end{aligned}$$

$$\begin{aligned} &= \frac{-2}{(2t+2h+1)(2t+1)} \\ & @ t = 2: \\ &= \frac{-2}{(2(2)+2h+1)(2(2)+1)} \\ &= \frac{-2}{(2h+5)(5)} \\ &= \frac{-2}{10h+25} \end{aligned}$$

- c. Explain using limits how one can find the slope of the tangent line to $s(t) = \frac{1}{2t+1}$ at $t = 2$.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2}{10h+25} \\ &= \frac{-2}{25} \end{aligned}$$

- d. Find the instantaneous velocity of the particle at $t = 2$. Draw a graph to illustrate your answer.

$$\begin{aligned} \text{IROC: } \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \\ @t=2: \lim_{h \rightarrow 0} \frac{-2}{10h+25} \\ &= \frac{-2}{25} \end{aligned}$$

- e. Find the equation of the tangent line to $s(t) = \frac{1}{2t+1}$ at $t = 2$.

$$\begin{aligned} \text{tangent line: } y - f(x_1) &= f'(x_1)(x - x_1) \\ m &= -\frac{2}{25} \\ x_1 &= 2 \\ y_1 &= \frac{1}{2(2)+1} = \frac{1}{5} \\ y - \frac{1}{5} &= -\frac{2}{25}(x - 2) \\ y - \frac{1}{5} &= -\frac{2}{25}x - \frac{4}{25} \\ +\frac{1}{5} & \quad +\frac{4}{25} \\ y &= -\frac{2}{25}x + \frac{1}{25} \end{aligned}$$

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4. Solve for x at which the two curves intersect:
$$\begin{cases} y = 3 \ln(x) - 2 \\ y = \ln(3 - x^3) \end{cases}$$

$$3 \ln(x) - 2 = \ln(3 - x^3)$$

$$\ln(x^3) - \ln(3 - x^3) = 2$$

$$\ln\left(\frac{x^3}{3 - x^3}\right) = 2$$

$$\cancel{e^{\ln}}\left(\frac{x^3}{3 - x^3}\right) = e^2$$

$$\frac{x^3}{3 - x^3} = e^2$$

$$x^3 = e^2(3 - x^3)$$

$$x^3 = 3e^2 - e^2 x^3$$

$$x^3 + e^2 x^3 = 3e^2$$

$$x^3(1 + e^2) = 3e^2$$

$$x^3 = \frac{3e^2}{1 + e^2}$$

$$x = \sqrt[3]{\frac{3e^2}{1 + e^2}}$$

5. Recall the compound interest formulae: $B = P \left(1 + \frac{r}{n}\right)^{nt}$; $B = Pe^{rt}$.

Find the annual interest rate of an account that quadruples its principal in 10 years if interest is compounded (a) monthly and (b) continuously.

(a) quadruples at $t=10$: $4P = P \left(1 + \frac{r}{12}\right)^{12(10)}$

$$4 = \left(1 + \frac{r}{12}\right)^{120}$$

$$4^{1/120} = \left(1 + \frac{r}{12}\right)$$

$$4^{1/120} - 1 = \frac{r}{12}$$

$$12(4^{1/120} - 1) = r$$

(b) quadruples at $t=10$: $4P = Pe^{r \cdot 10}$

$$4 = e^{10r}$$

$$\ln(4) = \ln(e^{10r})$$

$$\ln(4) = 10r$$

$$\frac{\ln(4)}{10} = r$$

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6. Consider the function

$$f(x) = \begin{cases} a(2^x) + 6 & -\infty < x \leq 1 \\ \frac{2x^2 - x - 1}{x - 1} & 1 < x < +\infty \end{cases}$$

Find the value of a such that $f(x)$ is continuous at $x = 1$.

continuous: $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$

given: left continuity i.e. $\lim_{x \rightarrow 1^-} f(x) = f(1)$

find a such that: $\lim_{x \rightarrow 1^-} f(x) = f(1)$

$$\lim_{x \rightarrow 1^-} \frac{2x^2 - x - 1}{x - 1} = a(2^1) + 6$$

$$\lim_{x \rightarrow 1^-} \frac{(2x+1)(x-1)}{x-1} = 2a + 6$$

$$\lim_{x \rightarrow 1^-} (2x+1) = 2a + 6$$

$$2(1)+1 = 2a + 6$$

$$3 = 2a + 6$$

$$-3 = 2a$$

$$-\frac{3}{2} = a$$

7. Find the equation of the tangent line to the graph of $y = 8e^x - 2x + 1$ which is parallel to the line whose equation is $-6x + y = 4$. Give your answer in the form $y = mx + b$.

parallel = same slope, tangent slope = derivative

find x where $f'(x) = \text{slope of given line}$

Step 1: Find slope of given line

$$-6x + y = 4$$

$$y = 6x + 4$$

$$m = 6$$

Step 2: Find x s.t. $f'(x) = m$

$$f'(x) = 8e^x - 2 = 6$$

$$8e^x = 8$$

$$e^x = 1$$

$$\ln(e^x) = \ln(1)$$

$$x = 0$$

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8. The profit, in thousands of dollars, from the sales of a certain ipod accessory is given by the formula

$$P(x) = -2x^2 + 12x - 13$$

where x is the number of dozens accessory sold.

(a) By completing the square, write $P(x)$ in the form $P(x) = a(x + b)^2 + c$. Show clearly all your steps.

$$\hookrightarrow \left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$$

$$P(x) = -2x^2 + 12x - 13$$

factor out a

$$= -2(x^2 - 6x) - 13$$

complete the square

$$b = -6, \left(\frac{1}{2}b\right)^2 = (-3)^2 = 9$$

$$= -2(x^2 - 6x + 9 - 9) - 13$$

$$= -2((x-3)^2 - 9) - 13$$

$$= -2(x-3)^2 + 18 - 13$$

$$= -2(x-3)^2 + 5$$

(b) Find the number of dozens of accessory that must be sold so profit is at its maximum. What is the maximum profit?

$$\text{maximum} = \text{vertex} = (-b, c)$$

$$\text{maximum} = (3, 5)$$

the maximum profit is 5 thousand dollars

when 3 dozen accessories are sold