Math 10350 – Example Set 07C
Section 4.1 Linear Approximation and Applications

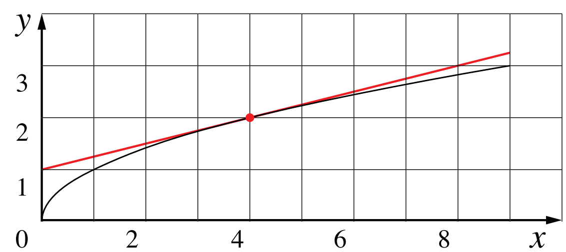
1. The population of wolves $w(t)$ and wild boars $p(t)$ in the thousands are given by the equations:

$$w(t) = 3\sin t + 5; \qquad p(t) = 2\cos t + 5.$$

(a) What is the rate of change of w with respect to p at $t = \dfrac{\pi}{4}$? (b) Find a relation between w and p by eliminating t. (c) Draw the graph of the p and w relationship in a p-w coordinate plane, (d) Describe what is happening between the two populations as time t progresses. Hint: Input different values of t and trace the curve you drew.

1. Find the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.

(b) Write down the linearization (linear approximation) of $f(x) = \sqrt{x}$ at $x = 4$.



linearization: $f(x) \approx f'(x_1)(x - x_1) + f(x_1)$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f(4) = \sqrt{4} = 2$

$f(x) = \frac{1}{4}(x-4)+2$

$= \frac{1}{4}x - 1 + 2$

$= \frac{1}{4}x + 1$

(c) Using your answer in (b), estimate the following values and comment on their accuracy with a calculator:

(i) $f(4.05) \overset{?}{\approx}$ 2.0125 (ii) $f(3.9) \overset{?}{\approx}$ 1.975 (iii) $f(5) \overset{?}{\approx}$ 2.25

2. Find the linearization (tangent line approximation) of e^x at $x = 0$. Estimate $e^{0.04}$. Draw a graph to illustrate your estimation. Is your estimate an overestimate or underestimate?

Linear Approximation of change in a function. The linearization of $f(x)$ at $x = a$ is often used in estimating the change Δf of a function $f(x)$ as x changes from a to $a + \Delta x$ is often difficult to compute exactly. Draw in the graph below to show where Δf is.

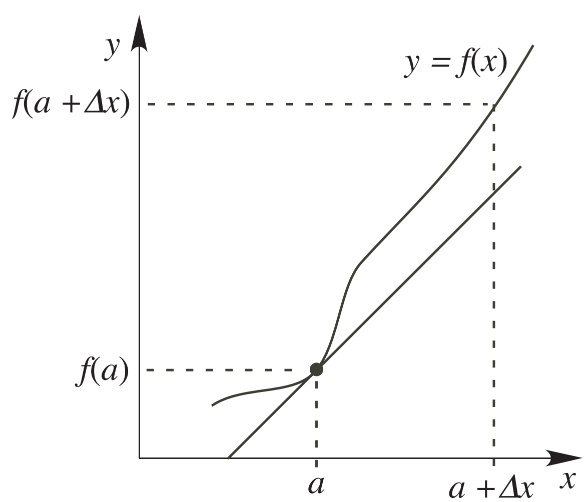
(a) Exact value of $\Delta f =$ $f'(a)(x-a)$

(b) For small Δx, the linear approximation of $f(x)$ at $x = a$ gives:

$$\boxed{\Delta f \approx f'(a)\Delta x}$$

(c) Such estimates for Δf are often used to approximate change and percentage change.

percent change : $100 \cdot \frac{\Delta f}{f}$



3. (Concept Test) If $g(3) = 4$ and $g'(3) = -1$. Estimate Δg and the percentage change of g as x changes from 3 to 3.01. Estimate $g(3.01)$.

1. The population of wolves $w(t)$ and wild boars $p(t)$ in the thousands are given by the equations:

$$w(t) = 3\sin t + 5; \qquad p(t) = 2\cos t + 5.$$

(a) What is the rate of change of w with respect to p at $t = \frac{\pi}{4}$? (b) Find a relation between w and p by eliminating t. (c) Draw the graph of the p and w relationship in a p-w coordinate plane, (d) Describe what is happening between the two populations as time t progresses. Hint: Input different values of t and trace the curve you drew.

(a) find $\frac{dw}{dp}$

$\frac{dw}{dp} \cdot \frac{dp}{dt} = \frac{dw}{dt}$

$\frac{dw}{dp} = \left(\frac{dw}{dt}\right) / \left(\frac{dp}{dt}\right)$

$\frac{dw}{dt} = 3\cos(t) + 0$

$\frac{dp}{dt} = -2\sin(t) + 0$

$\frac{dw}{dp} = \frac{3\cos(t)}{-2\sin(t)}$

$\frac{dw}{dp} = -\frac{3}{2}\cot(t)$

$\frac{dw}{dp}\left(\frac{\pi}{4}\right) = -\frac{3}{2}\cot\left(\frac{\pi}{4}\right) = -\frac{3}{2}(1) = -\frac{3}{2}$

(b) $w = 3\sin t + 5 \rightarrow t = \arcsin\left(\frac{w-5}{3}\right)$

$p = 2\cos t + 5$

$p = 2\cos\left(\arcsin\left(\frac{w-5}{3}\right)\right) + 5$

$p = 2\left(\frac{\sqrt{9-(w-5)^2}}{3}\right) + 5$

$p = 2\left(\frac{\sqrt{9-(w^2-10w+25)}}{3}\right) + 5$

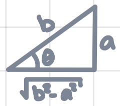
$= \frac{2}{3}\sqrt{9-w^2+10w-25} + 5$

$= \frac{2}{3}\sqrt{-w^2+10w-16} + 5$

$= \frac{2}{3}\sqrt{-(w^2-10w+16)} + 5$

$= \frac{2}{3}\sqrt{-(w-8)(w-2)} + 5$

simplify $\cos(\arcsin(\frac{a}{b}))$



$\arcsin(\frac{a}{b}) = \theta$

$\frac{a}{b} = \sin\theta = \frac{opp}{hyp}$

$\cos(\theta) = \frac{\sqrt{b^2-a^2}}{b}$

$a = w-5,\ b = 3$

(b) alternative: utilize $\sin^2(t) + \cos^2(t) = 1$

$w = 3\sin t + 5 \Rightarrow \sin t = \frac{w-5}{3}$

$p = 2\cos t + 5 \Rightarrow \cos t = \frac{p-5}{2}$

$1 = (\sin t)^2 + (\cos t)^2 = \left(\frac{w-5}{3}\right)^2 + \left(\frac{p-5}{2}\right)^2$

$1 = \frac{1}{9}(w-5)^2 + \frac{1}{4}(p-5)^2$

(c) graph $(x,y) \rightarrow (p,w)$

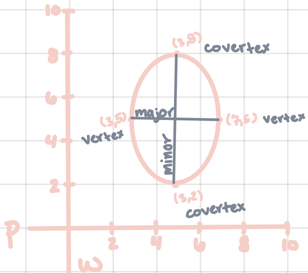
elipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

↳ vertices: $(h \pm a, k)$

↳ length of major axis: $2a$

↳ covertices: $(h, k \pm b)$

↳ length of minor axis: $2b$



2. Find the linearization (tangent line approximation) of e^x at $x = 0$. Estimate $e^{0.04}$. Draw a graph to illustrate your estimation. Is your estimate an overestimate or underestimate?

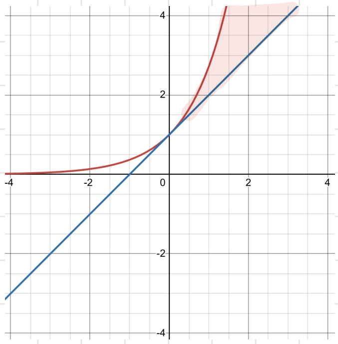
linerization: $f(x) \approx f'(x_1)(x-x_1) + f(a)$

$f'(x) = e^x$

$f'(0) = e^0 = 1$

$f(0) = e^0 = 1$

$f(x) \approx 1(x-0) + 1 = x + 1$



underestimate

$f(0.04) \approx 0.04 + 1 = 1.04$

$f(0.04) = e^{0.04} = 1.048$

$f(1) = e = 2.71$

$f(1) \approx 1 + 1 = 2$

3. (Concept Test) If $g(3) = 4$ and $g'(3) = -1$. Estimate Δg and the percentage change of g as x changes from 3 to 3.01. Estimate $g(3.01)$.

linearization: $f(x) \approx f'(x_1)(x-x_1) + f(x_1)$

$\Delta g = g(x) - g(x_1) \approx g'(x_1)(x-x_1)$

$\Delta g \approx g'(3)(x-3)$

$= -1(x-3)$

$= -x+3$

$g(x) \approx -1(x-3)+4$

$= -x+3+4$

$= -x+10$

$g(3.01) = -3.01+10$

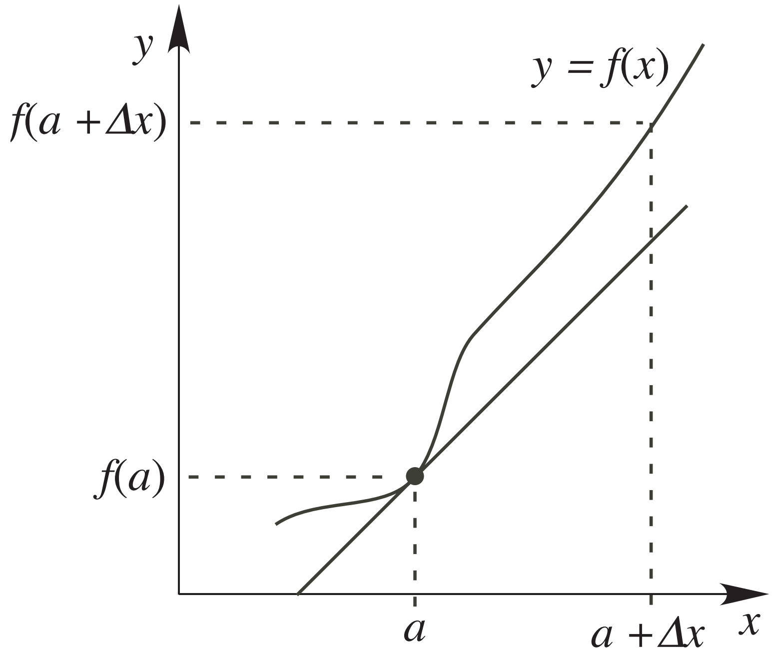
$= 6.99$

percent: $100 \cdot \frac{\Delta g}{g}$

$100 \cdot \frac{-3.01+3}{4}$

$= 100 \cdot \frac{-0.01}{4} = -\frac{1}{4} = -25\%$

Summary: Linearization of a Differentiable Function at $x = a$



The linear approximation (or linerization or tangent line approx.) of a differentiable function $f(x)$ at $x = a$ is given by the function of the ______ to the graph of $f(x)$ at $x = a$.

$$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$

this is just the equation of the tangent line at the point $(a, f(a))$ in a simplified format

(a) Exact value of $\Delta f = f(a+\Delta x) - f(a)$

(b) For small Δx, the change in $f(x)$ as x changes from a to $a + \Delta x$ is given by:

$$\boxed{\Delta f \approx f'(a)\,\Delta x}$$

(~~c~~) Such estimates for Δf are often used to approximate change and percentage change.