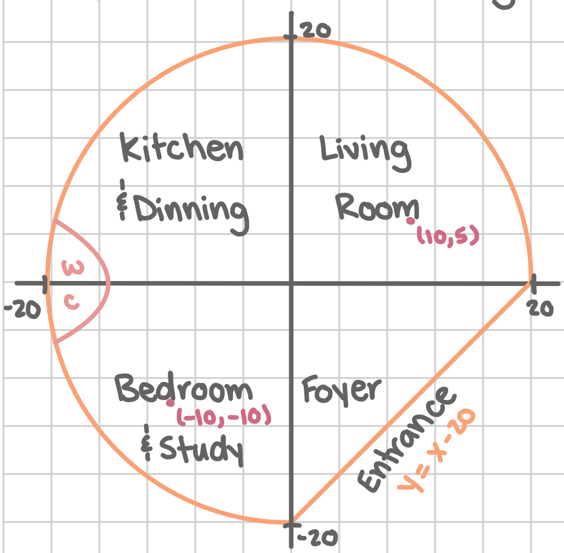
Two Variable Functions & Double Integrals

So far we have exclusively looked at functions of the form $y=f(x)$ and $x=h(y)$, but not all curves or equations follow this form. For example, a circle. The equation of a circle centered at $(0,0)$ with radius r is given by $x^2+y^2=r$. This equation can not be transformed into one of the forms mentioned above. We can solve for x or y but are left with $y=\pm\sqrt{r^2-x^2}$ or $x=\pm\sqrt{r^2-y^2}$ but these are technically two functions each, the top and bottom hemispheres or the left and right hemisphere.

Two Variable Functions

Let us get familiar with these two variable functions with a visual example. Let's say we have a house with a circular floor plan with a cut-off for the entrance. The height at any point in this house can be given by the two variable function $f(x,y)=14-\frac{1}{100}(x^2+y^2)$. Using the picture below answer the questions to the side.



(a) What region of the house are you in at $(10,5)$ and $(-10,-10)$?

$(10,5)$ is in the living room

$(-10,-10)$ is in the living room & study

(b) What is the height at $(10,5)$ and $(-10,-10)$?

$f(10,5)=14-\frac{1}{100}\left((10)^2+(5)^2\right)=12.75$

$f(-10,-10)=14-\frac{1}{100}\left((-10)^2+(-10)^2\right)=12$

(c) Where is the highest point?

Highest point happens when we subtract 0 from 14, i.e. $(0,0)$.

(d) What is the height around the circular perimeter?

Around circular edge $f(x,y)=14-\frac{1}{100}(x^2+y^2)=14-\frac{1}{100}(r^2)=14-\frac{1}{100}((20)^2)=14-4=10$

(e) Write down the height function along the entrance in terms of x with a range.

Using the equation of the line $y=x-20$, $f(x,y)=f(x,x-20)=14-\frac{1}{100}(x^2+(x-20)^2)$

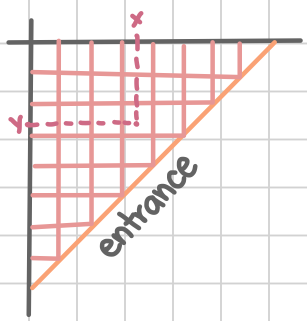
(f) Parameterize (or describe) the foyer region in terms of x and y.

x ranges from 0 to 20 and y ranges (in x terms) from $x-20$ to 0

(g) Use the ideas of Riemann sums to find the internal volume of the house enclosed by the foyer.

Volume enclosed by the surface $f(x,y)=f(x,x-20)=14-\frac{1}{100}(x^2-y^2)$ over the region

$D = \{(x,y) \mid 0 \le x \le 20,\ x-20 \le y \le 0\}$.



Divide the region D up into small rectangular patches with evenly spaced horizontal and vertical lines. We describe each square by its center (x,y) with width Δx and length Δy so it has area $A = \Delta x \cdot \Delta y$. Since this square is infinitely small, we can assume the height to be $f(x,y)$. Thus $\Delta V = \Delta x \cdot \Delta y \cdot f(x,y) = \left(14 - \frac{1}{100}(x^2+y^2)\right) \cdot \Delta x \cdot \Delta y$. Just like before we can sum over the variables to get

$$V = \lim_{n,m\to\infty} \sum^{n} \sum^{m} 14 - \frac{1}{100}(x^2+y^2)\, \Delta y\, \Delta x.$$

Double Integrals

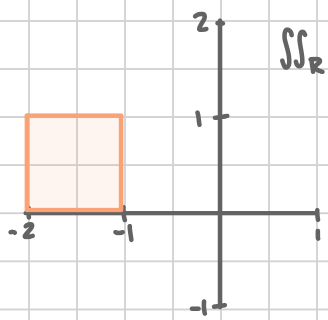
Rectangular Coordinates

The double integration of $f(x,y)$ over the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$ is

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy \quad \text{(if it exists).}$$

Example.

1. Compute $\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA$ over $R = \underset{\text{x-values}}{[-2,-1]} \times \underset{\text{y-values}}{[0,1]}$.



sin(πy) is a constant with respect to x i.e. $\int y\,dx = yx + C$

$$\begin{aligned}
\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA &= \int_0^1 \int_{-2}^{-1} x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dx\,dy \\
&= \int_0^1 \left[\tfrac{1}{3}x^3y^2 + \tfrac{1}{\pi}\sin(\pi x) + x\sin(\pi y)\right]_{-2}^{-1} dy \\
&= \int_0^1 \tfrac{1}{3}(-1)^3y^2 + \tfrac{1}{\pi}(0) - \sin(\pi y) - \tfrac{1}{3}(-2)^3y^2 - \tfrac{1}{\pi}(0) + 2\sin(\pi y)\,dy \\
&= \int_0^1 \tfrac{7}{3}y^2 + \sin(\pi y)\,dy \\
&= \left[\tfrac{7}{9}y^3 - \tfrac{1}{\pi}\cos(\pi y)\right]_0^1 \\
&= \tfrac{7}{9} + \tfrac{2}{\pi}
\end{aligned}$$

Since the bounds for both x and y are constants, we can swap dxdy to dydx.

$$\begin{aligned}
\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA &= \int_{-2}^{-1} \int_0^1 x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dy\,dx \\
&= \int_{-2}^{-1} \left[\tfrac{1}{3}x^2y^3 + y\cos(\pi x) - \tfrac{1}{\pi}\cos(\pi y)\right]_0^1 dx \\
&= \int_{-2}^{-1} \tfrac{1}{3}x^2 + \cos(\pi x) - \tfrac{1}{\pi}\cos(\pi) - 0 - 0 + \tfrac{1}{\pi}\cos(0)\,dx \\
&= \int_{-2}^{-1} \tfrac{1}{3}x^2 + \cos(\pi x) + \tfrac{1}{\pi} + \tfrac{1}{\pi}\,dx \\
&= \tfrac{1}{9}x^3 + \sin(\pi x) + \tfrac{2}{\pi}x \Big|_{-2}^{-1} \\
&= \tfrac{1}{9}(-1)^3 + 0 + \tfrac{2}{\pi}(-1) - \tfrac{1}{9}(-2)^3 + 0 - \tfrac{2}{\pi}(-2) \\
&= \tfrac{7}{9} + \tfrac{2}{\pi}
\end{aligned}$$

Exit Ticket Improper Integrals

> **Improper Integrals**
>
> 1. If $\int_a^c f(x)dx$ exists for every $t > a$, then $\int_a^\infty f(x)dx = \lim_{c \to \infty} \int_a^c f(x)dx$ provided that the limit exists and is finite.
> 2. If $\int_c^a f(x)dx$ exists for every $c < b$, then $\int_{-\infty}^b f(x)dx = \lim_{c \to -\infty} \int_c^b f(x)dx$ provided that the limit exists and is finite.
> 3. If $f(x)$ is continuous on the interval $[a, b)$ and not at $x = b$, then $\int_a^b f(x)dx = \lim_{c \to b} \int_a^c f(x)dx$ provided that the limit exists and is finite.
> 4. If $f(x)$ is continuous on the interval $(a, b]$ and not at $x = a$, then $\int_a^b f(x)dx = \lim_{c \to a} \int_c^b f(x)dx$ provided that the limit exists and is finite.
> 5. If $f(x)$ is not continuous $x = t$ where $a < t < b$, then $\int_a^b f(x)dx = \lim_{c \to t} \left[\int_a^c f(x)dx + \int_c^b f(x)dx\right]$ provided that the limit exists and is finite.
>
> The integral is considered **convergent** if the limit exists and is finite and **divergent** if the limit doesn't exist or is infinite.

Solve the following integrals using the concept above:

1. $\int_0^\infty \frac{1}{x} dx$

$= \lim_{c \to 0} \int_c^1 \frac{1}{x} dx + \lim_{d \to \infty} \int_1^\infty \frac{1}{x} dx$

$= \lim_{c \to 0^+} [\ln|x|]_c^1 + \lim_{d \to \infty} [\ln|x|]_1^d$

$= \ln|1| - \lim_{c \to 0^+} \ln|c| + \ln|1| - \lim_{d \to \infty} \ln|d|$

$= 0 - (-\infty) + 0 - \infty$ $\therefore$ divergent

2. $\int_{-5}^1 \frac{1}{10 + 2x} dx$

$= \lim_{c \to -5^+} \int_c^1 \frac{1}{10+2x} dx$

$= \lim_{c \to -5^+} \frac{1}{2} \ln|10+2x| \Big|_c^1$

$= \frac{1}{2} \ln|12| - \frac{1}{2} \lim_{c \to -5^+} \ln|10+2c|$

$= \frac{1}{2} \ln(12) - \frac{1}{2}(-\infty)$ $\therefore$ divergent

3. $\int_1^4 \frac{1}{x^2 + x - 6} dx = \int_1^4 \frac{1}{(x-2)(x+3)} dx$

$= \int_1^4 \frac{1/5}{(x-2)} - \frac{1/5}{(x+3)} dx$

$= \int_1^2 \frac{1/5}{(x-2)} - \frac{1/5}{(x+3)} dx + \int_2^4 \frac{1/5}{(x-2)} - \frac{1/5}{(x+3)} dx$

$= \lim_{c \to 2^-} \int_1^c \frac{1/5}{(x-2)} - \frac{1/5}{(x+3)} dx + \lim_{k \to 2^+} \int_k^4 \frac{1/5}{(x-2)} - \frac{1/5}{(x+3)} dx$

$= \lim_{c \to 2^-} \frac{1}{5} \ln|x-2| - \frac{1}{5} \ln|x+3| \Big|_1^c + \lim_{k \to 2^+} \frac{1}{5} \ln|x-2| - \frac{1}{5} \ln|x+3| \Big|_k^4$

4. $\int_{-\infty}^0 \frac{e^{\frac{1}{x}}}{x^2} dx$

$= \lim_{c \to -\infty} \int_c^{-1} \frac{1}{x^2} e^{\frac{1}{x}} dx + \lim_{d \to 0^-} \int_{-1}^d \frac{1}{x^2} e^{\frac{1}{x}} dx$

$= \lim_{c \to -\infty} -e^{\frac{1}{x}} \Big|_c^{-1} + \lim_{d \to 0^-} -e^{\frac{1}{x}} \Big|_{-1}^d$

$= -e^{-1} + \lim_{c \to \infty} e^{\frac{1}{c}} - \underbrace{\lim_{d \to 0^-} e^{\frac{1}{d}}}_{e^{-\infty} = \frac{1}{e^{\infty}} = 0} + e^{-1}$

$= 1 - 0$ $\therefore$ converges