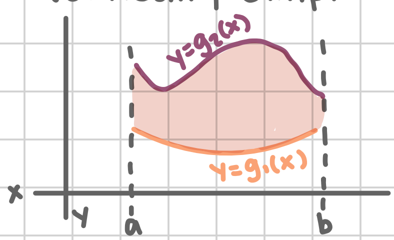
Double Integrals - General

Over General Regions

So far we have been working under the assumption the region we are working over is a rectangle, but this isn't always the case.

The integral over any region D can be described in two ways:

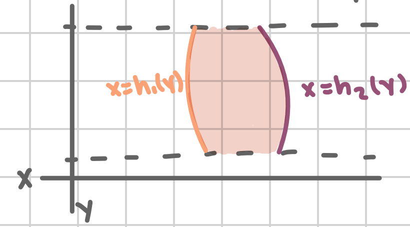
(i) Vertically Simple



$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

(ii) Horizontally Simple

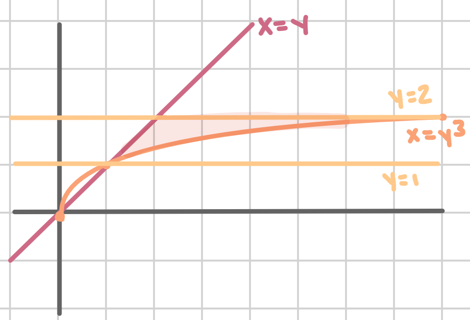


$D=\{(x,y)\mid h_1(y)\le x\le h_2(y),\ c\le y\le d\}$

$\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Examples:

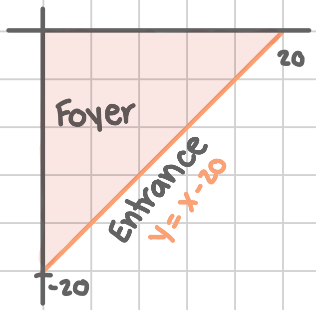
1. Compute $\iint_D e^{x/y}\,dA$ where $D=\{(x,y)\mid 1\le y\le 2,\ y\le x\le y^3\}$



This horizontally simple i.e. every horizontal line I draw horizontal line in the shaded region is bounded on top by one function and on the bottom by another function.

$$
\begin{aligned}
\iint_D e^{x/y}\,dA &= \int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy\\
&= \int_1^2 \left[y e^{x/y}\right]_y^{y^3} dy\\
&= \int_1^2 y e^{y^2} - y e^1\,dy\\
&= \left[\tfrac{1}{2}e^{y^2}-\tfrac{1}{2}y^2e^1\right]_1^2\\
&= \tfrac{1}{2}e^4-2e^1
\end{aligned}
$$

2. Compute the volume of the foyer in the house from last lecture.



The foyer is both vertically and horizontally simple

(i) vertically $D=\{(x,y)\mid 0\le x\le 20,\ x-20\le y\le 0\}$

$$\int_0^{20}\int_{x-20}^{0} 14-\frac{1}{100}(x^2+y^2)\,dy\,dx$$

(ii) horizontally $D=\{(x,y)\mid -20\le y\le 0,\ 0\le x\le y+20\}$

$$\int_{-20}^{0}\int_0^{y+20} 14-\frac{1}{100}(x^2+y^2)\,dx\,dy$$

(i) vertically $D=\{(x,y)\mid 0\le x\le 20,\ x-20\le y\le 0\}$

$$\int_0^{20}\int_{x-20}^{0} 14-\frac{1}{100}(x^2+y^2)\,dy\,dx$$

$$=\int_0^{20}\int_{x-20}^{0} 14-\frac{1}{100}x^2-\frac{1}{100}y^2\ dy\,dx$$

$$=\int_0^{20} 14y-\frac{1}{100}x^2y-\frac{1}{100}\cdot\frac{1}{3}y^3\Big]_{x-20}^{0}\ dx$$

$$=\int_0^{20} 14(x-20)-\frac{1}{100}x^2(x-20)-\frac{1}{300}(x-20)^3\,dx$$

$$=\int_0^{20} 14x-280-\frac{1}{100}x^3-\frac{1}{5}x^2-\frac{1}{300}x^3-\frac{1}{5}x^2-4x+\frac{80}{3}\ dx$$

$$=\int_0^{20}-\frac{1}{75}x^3-\frac{2}{5}x^2+10x+\frac{760}{3}\ dx$$

$$=-\frac{1}{300}x^4-\frac{2}{15}x^3+5x^2+\frac{760}{3}x\ \Big]_0^{20}$$

$$=-\frac{1}{300}(20)^4-\frac{2}{15}(20)^3+5(20)^2+\frac{760}{3}(20)$$

$$=\frac{7600}{3}$$

(ii) horizontally $D=\{(x,y)\mid -20\le y\le 0,\ 0\le x\le y+20\}$

$$\int_{-20}^{0}\int_0^{y+20} 14-\frac{1}{100}(x^2+y^2)\,dx\,dy$$

$$=\int_{-20}^{0}\int_0^{y+20} 14-\frac{1}{100}x^2-\frac{1}{100}y^2\ dx\,dy$$

$$=\int_{-20}^{0} 14x-\frac{1}{300}x^3-\frac{1}{300}y^2x\Big]_0^{y+20}\ dy$$

$$=\int_{-20}^{0} 14(y+20)-\frac{1}{300}(y+20)^3-\frac{1}{300}y^2(y+20)\,dy$$

$$=\int_{-20}^{0}-\frac{1}{75}y^3-\frac{2}{5}y^2+10y+\frac{760}{3}$$

$$=\frac{-1}{300}y^4-\frac{3}{15}y^3+5y^2+\frac{760}{3}y\ \Big]_{-20}^{0}$$

$$=0-\left(\frac{-1}{300}(-20)^4-\frac{3}{15}(-20)^3+5(-20)^2+\frac{760}{3}(-20)\right)$$

$$=\frac{7600}{3}$$

Exit Ticket Numerical Integration

Numerical Integration We can estimate the integral $\int_a^b f(x)dx$ using the following formulas,

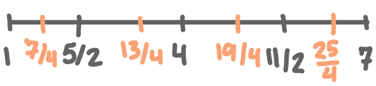
1. **midpoint:** $\int_a^b f(x)dx \approx \Delta x\,[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)]$
2. **trapezoid:** $\int_a^b f(x)dx \approx \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1} + f(x_n)]$
3. **simpson's:** $\int_a^b f(x)dx \approx \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2)... + 2f(x_{n-1}) + 4f(x_{n-1}) + f(x_n)]$

where n is the number is subintervals and $\Delta x = \frac{b-a}{n}$

Estimate the following integrals using each of the rules above: (with n = 4)

1. $\int_1^7 \frac{1}{x^3+1}dx$

$\Delta x = \frac{7-1}{4} = \frac{6}{4} = \frac{3}{2}$



midpoint:

$= \frac{3}{2}\left[\frac{1}{(7/4)^3+1} + \frac{1}{(13/4)^3+1} + \frac{1}{(19/4)^3+1} + \frac{1}{(25/4)^3+1}\right]$

trapezoid:

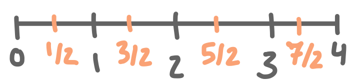
$= \frac{3}{4}\left[\frac{1}{(1)^3+1} + 2\cdot\frac{1}{(5/2)^3+1} + 2\cdot\frac{1}{(4)^3+1} + 2\cdot\frac{1}{(11/2)^3+1} + \frac{1}{(7)^3+1}\right]$

simpson's:

$= \frac{1}{2}\left[\frac{1}{(1)^3+1} + 4\cdot\frac{1}{(5/2)^3+1} + 2\cdot\frac{1}{(4)^3+1} + 4\cdot\frac{1}{(11/2)^3+1} + \frac{1}{(7)^3+1}\right]$

2. $\int_0^4 \cos(1+\sqrt{x})dx$

$\Delta x = \frac{4-0}{4} = 1$



midpoint:

$= 1\left[\cos(1+\sqrt{\tfrac{1}{2}}) + \cos(1+\sqrt{\tfrac{3}{2}}) + \cos(1+\sqrt{\tfrac{5}{2}}) + \cos(1+\sqrt{\tfrac{7}{2}})\right]$

trapezoid:

$= \frac{1}{2}\left[\cos(1) + 2\cos(2) + 2\cos(1+\sqrt{2}) + 2\cos(1+\sqrt{3}) + \cos(3)\right]$

simpson's:

$= \frac{1}{3}\left[\cos(1) + 4\cos(2) + 2\cos(1+\sqrt{2}) + 4\cos(1+\sqrt{3}) + \cos(3)\right]$