Week 07: March 3rd, 2023

Section 7.7 Improper Integrals

Infinite Integral

In an infinite integral one, or both, of the limits of integration are infinite. We have solved integrals over definite intervals, but these are integrals over infinite intervals. Consider the integral $\int_1^\infty \frac{1}{x^2}\,dx$.

This is an integral we have seen before, but because of the infinity we can not just integrate and "plug in." Let us recall what an integral is. We are asking for the area under the curve $f(x)=\frac{1}{x^2}$ over the interval $[1,\infty]$. This is still hard to compute, so instead we consider the area under the curve $f(x)=\frac{1}{x^2}$ over the interval $[1,t]$ where $t>1$ and t is finite. This is something we can do:

$A_t = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1-\frac{1}{t}$.

Now we can consider the area under $f(x)$ on $[1,\infty)$ simply by taking the limit of A_t as t goes to infinity.

$A = \lim_{t\to\infty} A_t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1.$

This is how we tackle the integral itself,

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t\\ &= \lim_{t\to\infty}\left[1-\frac{1}{t}\right]\\ &= 1.\end{aligned}$$

$\lim_{t\to\infty}\left(\frac{1}{t}\right) \approx \frac{1}{\text{infinitely big}} \approx \text{infinitely small} \approx 0$

This is how we deal with these kinds of integrals in general. We will replace the infinity with a variable (I like c), do the integral and take the limit of the result as the variable goes to infinity.

In this example the area under the curve over an infinite interval was not infinity as one might expect. Instead we got a small number. This isn't always the case. We call these integrals convergent if the limit exists and is a finite number and divergent if the limit either doesn't exist or is (plus or minus) infinity.

1. If $\int_a^t f(x)\,dx$ exists for every $t>a$ then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ provided the limit exists and is finite.
2. If $\int_t^b f(x)\,dx$ exists for every $t<b$ then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$ provided the limit exists and is finite.
3. If $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ are both convergent then, $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ where c is any number.

Note that both integrals must converge for this integral to be convergent.

example. Determine if the following integral is convergent or divergent and if it's convergent find its value,

(a) $\int_1^\infty \frac{1}{x}\,dx$

$= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty}\left[\ln|x|\right]_1^t$

$= \lim_{t\to\infty}\left[\ln(t)-\ln(1)\right]$

$= \infty$

$\therefore$ divergent

(b) $\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{t\to-\infty}\int_t^0 \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{t\to-\infty}\left[-2\sqrt{3-x}\right]_t^0$

$= \lim_{t\to-\infty}\left(-2\sqrt{3}+2\sqrt{3-t}\right)$

$= -2\sqrt{3}+\infty$

$= \infty$

$\therefore$ divergent

(c) $\int_{-\infty}^\infty x e^{-x^2}\,dx$

$= \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$

$= \lim_{t\to-\infty}\int_t^0 xe^{-x^2}\,dx + \lim_{s\to\infty}\int_0^s xe^{-x^2}\,dx$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0 + \lim_{s\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^s$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}+\frac{1}{2}e^{-t^2}\right] + \lim_{s\to\infty}\left[-\frac{1}{2}e^{-s^2}+\frac{1}{2}\right]$

$= -\frac{1}{2}+\frac{1}{2}$

$= 0$

$\therefore$ convergent

(c) $\int_{-2}^\infty \sin(x)\,dx$

$= \lim_{t\to\infty}\int_{-2}^t \sin(x)\,dx$

$= \lim_{t\to\infty}\left[-\cos(x)\right]_{-2}^t$

$= \lim_{t\to\infty}\left[\cos(2)-\cos(t)\right]$

limit does not exists!

$\therefore$ divergent

Discontinuous Integrand

The second type of improper integral is a discontinuous integral.

1. If $f(x)$ is continuous on the interval $[a,b)$ and not continuous at $x=b$ then $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$ provided the limit exists and is finite. Note as well that we do need to use a left-hand limit here since the interval of integration is entirely on the left side of the upper limit.
2. If $f(x)$ is continuous on the interval $(a,b]$ and not continuous at $x=a$ then $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$ provided the limit exists and is finite. In this case we need to use a right-hand limit here since the interval of integration is entirely on the right side of the lower limit.
3. If $f(x)$ is not continuous at $x=c$ where $a<c<b$ and $\int_a^c f(x)dx$ and $\int_a^b f(x)dx$ are both convergent then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. Note that both integrals must be convergent for the total to be convergent.
4. If $f(x)$ is not continuous at $x=a$ and $x=b$ and if $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are both convergent then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where c is any number. Again, both must be convergent for the total to be.

example. Determine if the following integral is convergent or divergent. If it is convergent find its value.

(a) $\int_0^3 \frac{1}{\sqrt{3-x}}dx$

$= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}dx$

$= \lim_{t\to 3^-} \left[-2\sqrt{3-x}\right]_0^t$

$= \lim_{t\to 3^-} \left(2\sqrt{3} - 2\sqrt{3-t}\right)$

$= 2\sqrt{3}$

$\therefore$ converge

(b) $\int_{-2}^3 \frac{1}{x^3}dx$

$= \int_{-2}^0 \frac{1}{x^3}dx + \int_0^3 \frac{1}{x^3}dx$

$= \lim_{t\to 0^-} \int_{-2}^t \frac{1}{x^3}dx + \lim_{s\to 3^+} \int_s^3 \frac{1}{x^3}dx$

$= \lim_{t\to 0^-} \left[-\frac{1}{2x^2}\right]_{-2}^t + \lim_{s\to 3^+} \left[-\frac{1}{2x^2}\right]_s^3$

$= \lim_{t\to 0^-} \left(\frac{1}{8} - \frac{1}{2t^2}\right) + \lim_{s\to 3^+} \left(\frac{1}{18} - \frac{1}{2s^2}\right)$

$= -\infty + -\infty$

$= -\infty$

$\therefore$ diverge

(c) $\int_0^\infty \frac{1}{x^2}dx$

$= \int_0^1 \frac{1}{x^2}dx + \int_1^\infty \frac{1}{x^2}dx$

$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}dx + \lim_{s\to\infty} \int_1^s \frac{1}{x^2}dx$

$= \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1 + \lim_{s\to\infty} \left[-\frac{1}{x}\right]_1^s$

$= \lim_{t\to 0^+} \left[-1 + \frac{1}{t}\right] + \lim_{s\to\infty} \left[-1 + \frac{1}{s}\right]$

$= \infty + 0$

$= \infty$

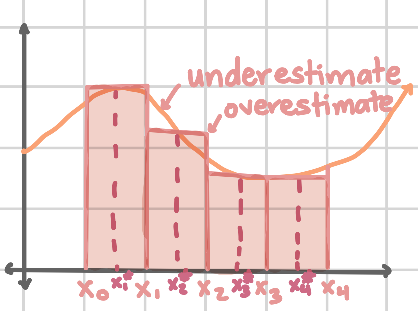
$\therefore$ diverge

Section 7.8: Numerical Integration

So far we have seen integrals that we can compute, but sometimes we run into integrals that we can not compute. The most popular example of this is $\int_0^2 e^{x^2}dx$. Instead of computing the integral, we aim to estimate the values of such definite integrals. Commonly this is done by estimating the area under the curve using shapes we know the area of.

Midpoint Rule

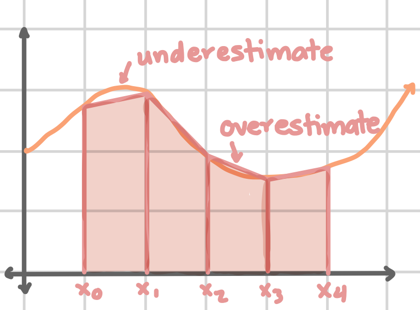
This rule should be familar to you from calculus I (or calculus A). We divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$, we denote these subintervals by $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ where $x_0=a$ and $x_n=b$.



For each interval let x_i^* be the midpoint of the interval. We then sketch rectangles for each subinterval with a height of $f(x_i^*)$. The image shows an example with $n=4$. We can easily find the area for each of these rectangles, the general formula is $\int_a^b f(x)dx \approx \Delta x\cdot f(x_1^*) + \Delta x\cdot f(x_2^*) + \ldots + \Delta x\cdot f(x_n^*) = \Delta x\left[f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*)\right]$.

Trapezoid Rule

We once again split our interval $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

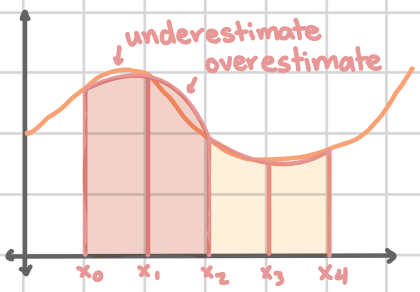


For each subinterval we draw a straight line that is equal to the function values at either endpoint of the interval. The image shows an example of $n=4$. You will notice that the resulting shapes are trapezoids, hence the name of the rule. We can now use the area formula $A_i = \frac{\Delta x}{2}\left(f(x_{i-1}) + f(x_i)\right)$. We add up each trapezoid to find the general formula $\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left(f(x_0)+f(x_1)\right) + \frac{\Delta x}{2}\left(f(x_1)+f(x_2)\right) + \ldots + \frac{\Delta x}{2}\left(f(x_{n-1}) + f(x_n)\right)$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2\cdot f(x_1) + 2\cdot f(x_2) + \ldots + 2\cdot f(x_{n-1}) + f(x_n)\right].$$

Simpson's Rule

The final method requires the number of subintervals, n, to be even. The width of each subinterval will still be $\Delta x = \frac{b-a}{n}$. The necessity of n being even will be obvious in a bit.



Unlike the trapezoid rule, which used a straight line approximation, Simpson's rule approximates the function using a quadratic that agrees with 3 points of the function from our subintervals. The image shows an example with $n=4$. The approximations are colored differently to help see the three points and why we need an even number of subintervals.

We can now utilize the area of these approximations, $A_i = \frac{\Delta x}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))$, to approximate the integral,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2)) + \frac{\Delta x}{3}(f(x_2) + 4f(x_3) + f(x_4)) + \ldots + \frac{\Delta x}{3}(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

$$= \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Notice that all of the odd subscripts are multiplied by 4 and all of the even (except first and last) are multiplied by 2.

example. Use $n=4$ and all three rules to approximate the value of the following integral: $\int_0^2 e^{x^2}\,dx$.

In each case the width of the subintervals will be $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, i.e. $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$.

Midpoint: $\int_0^2 e^{x^2}dx \approx \frac{1}{2}(e^{(0.25)^2} + e^{(0.75)^2} + e^{(1.25)^2} + e^{(1.75)^2}) = 14.4856$

Trapezoid: $\int_0^2 e^{x^2}dx \approx \frac{1/2}{2}(e^{(0)^2} + 2e^{(0.5)^2} + 2e^{(1)^2} + 2e^{(1.5)^2} + e^{(2)^2}) = 20.6446$

Simpson's: $\int_0^2 e^{x^2}dx \approx \frac{1/2}{3}(e^{(0)^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2}) = 17.3536$

example. Find $f(5)$ using trapezoid rule with $n=4$ and $f(1)=1$.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f'(x)	3.0	3.5	2.0	1.5	2.5	2.0	3.0	3.0	2.5

We are going to have to use previous knowledge and some algebra to find what we are looking for.

Recall the fundamental theorem of calculus, $\int_a^b f'(x)\,dx = f(b) - f(a)$ so $f(b) = f(a) + \int_a^b f'(x)\,dx$.

$$f(5) = f(1) + \int_0^5 f'(x)\,dx$$
$$\approx f(1) + \frac{\Delta x}{2}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)]$$
$$= f(1) + \frac{1}{2}[3.0 + 2(2.0) + 2(2.5) + 2(3.0) + 2.5]$$
$$= 1 + \frac{1}{2}[3 + 4 + 5 + 6 + 2.5]$$
$$= 1 + \frac{1}{2}[20.5]$$
$$= 1 + 10.25$$
$$= 11.25$$