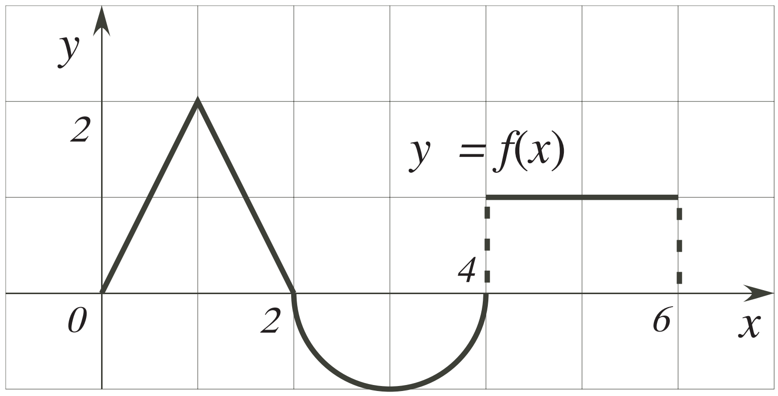
Math 10350 – Example Set 15B
(Section 5.2, 5.4, 5.5, 5.6 & 5.7)

1. Use geometry to compute the definite integral $\int_{-5}^{0} \sqrt{25 - x^2}\,dx$ ← left hemisphere

$\longrightarrow y = \sqrt{25 - x^2}$
$y^2 = 25 - x^2$
$x^2 + y^2 = 25$
i.e. circle of radius 5
positive semi-circle
$\int_{-5}^{0} \sqrt{25 - x^2}\,dx$ is the area bounded by the left-upper quarter
$= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(5)^2 = \frac{25}{4}\pi$



2. Consider the graph of $f(x)$ above. Using geometry, find the value of all the definite integrals below:

a. $\int_0^2 f(x)\,dx \overset{?}{=} \triangle = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$

c. $\int_2^4 f(x)\,dx \overset{?}{=} \frown = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1)^2 = \frac{1}{2}\pi$

b. $\int_1^4 f(x)\,dx \overset{?}{=} \triangle + \frown = 2 + \frac{1}{2}\pi$

d. $\int_0^6 f(x)\,dx \overset{?}{=} \triangle + \frown + \square = 2 + \frac{1}{2}\pi + (2)(1) = 4 + \frac{1}{2}\pi$

Properties of Definite Integral (5.2). Let $a < b < c$ and k be real numbers. Let $f(x)$ and $g(x)$ be continuous functions. Then we have the following:

i. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

ii. $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)dx$

iii. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$

We also define:

iv. $\int_a^a f(x)\,dx = 0$

v. $\int_b^a f(x)\,dx = -\int_a^b f(x)dx$

3. Given that $\int_0^2 f(x)dx = \int_2^3 f(x)dx = 5$, find

a. $\int_0^3 f(x)dx \overset{?}{=} \int_0^2 f(x)dx + \int_2^3 f(x)dx = 5 + 5 = 10$

c. $\int_0^2 f(x)dx + \int_3^2 f(x)dx \overset{?}{=} \int_0^2 f(x)dx - \int_2^3 f(x)dx$
$= 5 + 5 = 10$

b. $\int_0^2 [4f(x) + 2]dx \overset{?}{=} 4\int_0^2 f(x)dx + \int_0^2 2dx$
$= 4(5) + \int_0^2 2dx$ ← we don't know how to do this

Fundamental Theorem of Calculus (5.4). Let $F(x)$ be an anti-derivative of $f(x)$. Then

$$\int_a^b f(x)\,dx = \underline{F(b) - F(a)} = \underline{F(x)\Big|_a^b} \quad \text{(shorthand notation).}$$

In other words: Note that F(x) includes a plus c, this disappears as F(b) would have plus c and - F(a) would have -(plus c)

Total change in $F(x)$ over $[a, b]$ = $\int_a^b f(x)dx$ = ________

4. Evaluate the following definite integrals:

a. $\displaystyle\int_{-1}^{0} (1 + 3x - e^{-x})\,dx$

$F(x) = x + \frac{3}{2}x^2 + e^{-x} + C$

$F'(x) = 1 + 3x - e^{-x} = f(x)$

$\int_{-1}^{0} 1 + 3x - e^{-x}\,dx = F(0) - F(-1)$

$= x + \frac{3}{2}x^2 + e^{-x} \Big|_{-1}^{0}$

$= 0 + \frac{3}{2}(0)^2 + e^{-0} - \left((-1) + \frac{3}{2}(-1)^2 + e^{-(-1)}\right)$

$= 0 + 0 + 1 + 1 - \frac{3}{2} + e$

$= \frac{1}{2} + e$

b. $\displaystyle\int_{\pi/2}^{\pi} \cos\theta\,d\theta$

$F(\theta) = \sin(\theta) + C$

$F'(\theta) = \cos(\theta)$

$\int_{\pi/2}^{\pi} \cos(\theta)d\theta = \sin(\theta)\Big|_{\pi/2}^{\pi}$

$= \sin(\pi) - \sin(\pi/2)$

$= 0 - \frac{\sqrt{2}}{2}$

$= -\frac{\sqrt{2}}{2}$

c. $\displaystyle\int \sqrt{x-1}\,dx = \int (x-1)^{1/2}dx$

The format suggest power rule:

$\int ax^n = \frac{a}{n+1}x^{n+1} + C$

But we have $(x-1)^{1/2}$ and not $x^{1/2}$

Try: $F(x) = \frac{1}{3/2}(x-1)^{3/2} + C = \frac{2}{3}(x-1)^{3/2} + C$

$F'(x) = \frac{2}{3}\cdot\frac{3}{2}(x-1)^{1/2} = (x-1)^{1/2}$

$\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C$

d. $\displaystyle\int_1^5 \sqrt{x-1}\,dx$

$F(x) = \frac{2}{3}(x-1)^{3/2} + C$

$F'(x) = (x-1)^{1/2}$

$\int_1^5 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\Big|_1^5$

$= \frac{2}{3}(5-1)^{3/2} - \frac{2}{3}(1-1)^{3/2}$

$= \frac{2}{3}[(4)^{1/2}]^3 - \frac{2}{3}[(1)^{1/2}]^3$

$= \frac{2}{3}(2)^3 - \frac{2}{3}(1)^3$

$= \frac{16}{3} - \frac{2}{3}$

$= \frac{14}{3}$