Integral Review

textbook: Calculus Early Transcendentals 3^{rd} edition page 440

random numbers: 18, 17, 21, 25, 32, 58, 43, 7, 3, 4

18. $\int_4^9 \frac{1}{(x^2-1)^2}\,dx$ improper: $x^2-1=0 \Rightarrow x=\pm 1$

u-substition: $u=t^2-1$ then $du=2t\,dt$ but no $t\,dt$ exists
partial fractions!

factor denominator: $(t^2-1)^2 = ((t+1)(t-1))^2 = (t+1)^2(t-1)^2$

controls A or Ax+B; controls # of repeated fractions

factors	deg	multi
$(x+1)$	1	2
$(x-1)$	1	2

(x+1) is a degree 1 factor

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{A}{(x+1)^1} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)^1} + \frac{D}{(x-1)^2}$$

(x-1) has multiplicity 2

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{A}{(x+1)^1} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)^1} + \frac{D}{(x-1)^2}$$

this is time consuming to solve
a test would have you stop here

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{A}{(x+1)}\cdot\frac{(x+1)(x-1)^2}{(x+1)(x-1)^2} + \frac{B}{(x+1)^2}\cdot\frac{(x-1)^2}{(x-1)^2} + \frac{C}{(x-1)^1}\cdot\frac{(x-1)(x+1)^2}{(x-1)(x+1)^2} + \frac{D}{(x-1)^2}\cdot\frac{(x+1)^2}{(x+1)^2}$$

$1 = A(x+1)(x-1)^2 + B(x-1)^2 + C(x-1)(x+1)^2 + D(x+1)^2$

when $x=1$: $1 = A(2)(0)^2 + B(0)^2 + C(0)(2)^2 + D(2)^2$
$1 = 4D$
$D = 1/4$

when $x=-1$: $1 = A(0)(-2)^2 + B(-2)^2 + C(-2)(0)^2 + D(0)^2$
$1 = 4B$
$B = 1/4$

$1 = A(x^3-x^2-x+1) + B(x^2-2x+1) + C(x^3+x^2-x-1) + D(x^2+2x+1)$

x^3: $0x^3 = Ax^3 + Cx^3$
x^2: $0x^3 = -Ax^2 + Bx^2 + Cx^2 + Dx^2$
x: $0x = -Ax - 2Bx - Cx + 2Dx$
c: $1 = A + B - C + D$

plug in $B=D=\frac{1}{4}$

$0 = A + C$
$0 = -A + \frac{1}{4} + C + \frac{1}{4}$
$0 = -A - \frac{1}{2} - C + \frac{1}{2}$
$1 = A + \frac{1}{4} - C + \frac{1}{4}$

$0 = A + C$
$-\frac{1}{2} = -A + C$
$0 = -A - C$
$\frac{1}{2} = A - C$

add
$-\frac{1}{2} = 2C$
$-\frac{1}{4} = C$

plug in

$$\int_4^9 \frac{1}{(x^2-1)^2}\,dx = \int \frac{1/4}{(x+1)} + \frac{1/4}{(x+1)^2} + \frac{-1/4}{(x-1)} + \frac{1/4}{(x-1)^2}\,dx = \frac{1}{4}\ln|x+1| - \frac{1}{4}(x+1)^{-1} - \frac{1}{4}\ln|x-1| - \frac{1}{4}(x-1)^{-1}\Big|_4^9$$
$$= \frac{1}{4}\left[\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{(x+1)} - \frac{1}{(x-1)}\right]_4^9$$

17. $\int \frac{6x+4}{x^2-1}\,dx$

u-substitution: $u=x^2-1$ then $du = x\,dx$ but we have $(6x+1)\cdot dx$
separate it

$\int \frac{6x+4}{x^2-1}\,dx = \int \frac{6x}{x^2-1} + \frac{4}{x^2-1}\,dx = \int \frac{6x}{x^2-1}\,dx + \int \frac{4}{x^2-1}\,dx$

u-sub

not arctan $\left(\int \frac{1}{x^2+1}\,dx = \arctan(x)+c\right)$
partial fractions

you could do partial fractions from the beginning but I find partial fractions with only a constant on top easier

$\int \frac{4}{x^2-1}\,dx$

factor denominator: $x^2-1 = (x+1)(x-1)$

factors	deg	multi
$(x+1)$	1	1
$(x-1)$	1	1

$\frac{4}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$4 = A(x-1) + B(x+1)$

when $x=1$: $4 = A(0) + 2B$
$2 = B$
when $x=-1$: $4 = -2A + B(0)$
$-2 = A$

$\int \frac{6x+4}{x^2-1}\,dx = \int \frac{6x}{x^2-1} + \frac{-2}{(x+1)} + \frac{2}{(x-1)}\,dx$

$= 6\ln|x^2-1|\cdot\frac{1}{2} - 2\ln|x+1| + 2\ln|x-1| + c$

$= 3\ln|x^2-1| - 2\ln|x+1| + 2\ln|x-1| + c$

check: $3\frac{1}{x^2-1}\cdot 2x - 2\frac{1}{x+1} + 2\frac{1}{x-1}$

$= \frac{6x}{x^2-1} - \frac{2}{x+1} + \frac{2}{x-1}$ ✓

21. $\int_0^1 \ln(4-2x)\,dx$ improper: $4-2x=0$ when $x=2$

if you see ln in integral then integration by parts

integration by parts or ultra-violet voodoo ($uv-\int v\,du$)

pick u by: the dv is everything left

L og
I nv. trig.
A lgebra
T rig.
E xponential

we run into a log immediately:

$u=\ln(4-2x)$ $du=\frac{1}{4-2x}\cdot -2\,dx$

$dv=1\,dx$ $v=x$

$uv-\int v\,du$

$=(\ln(4-2x))\cdot(x)\Big|_0^1-\int_0^1(x)\cdot\left(\frac{-2}{4-2x}\right)dx$

$=x\cdot\ln(4-2x)\Big|_0^1-\int_0^1\frac{-x}{2-x}\,dx$

u-substitution in a commonly forgotten way

$u=2-x \Rightarrow x=2-u$

$du=-1dx$ a,b are u-bounds

u has different bounds than x, but I don't want to solve for them so I denote that they are different using a and b

$=x\cdot\ln(4-2x)\Big|_0^1-\int_a^b\frac{2-u}{u}\,du$

$=x\cdot\ln(4-2x)\Big|_0^1-\int_a^b\frac{2}{u}-\frac{u}{u}\,du$

$=x\cdot\ln(4-2x)\Big|_0^1-\int_a^b\frac{2}{u}-1\,du$

$=x\cdot\ln(4-2x)\Big|_0^1-2\ln|u|-u\Big|_a^b$

$=x\cdot\ln(4-2x)-2\ln|2-x|-(2-x)\Big|_0^1$

25. $\int_0^{\pi/4} \sin(3x)\cos(5x)\,dx$

trigonometric integration:
identities given on test

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$A = 3x \quad B = 5x$

$= \int_0^{\pi/4} \frac{1}{2}[\sin(3x+5x) + \sin(3x-5x)]\,dx$

$= \frac{1}{2}\int_0^{\pi/4} \sin(8x) + \sin(-2x)\,\,dx$

$= \frac{1}{2}\left[\frac{1}{8}\sin(8x) - \frac{1}{2}\sin(-2x)\right]_0^{\pi/4}$

32. $\int_{\pi/2}^{\pi} \cot^2\left(\frac{\theta}{2}\right) d\theta$

$\csc^2 x = \cot^2(x) + 1$
$\cot^2 x = \csc^2(x) - 1$

$= \int_{\pi/2}^{\pi} \csc^2\left(\frac{\theta}{2}\right) - 1\,\, d\theta$

$\left.\begin{array}{l} \frac{d}{dx}[\cot(x)] = -\csc^2(x) \\ \int \csc^2(x)\,dx = -\cot(x) \end{array}\right\}$ this is not a derivative or integral we think about often, it would appear in a list of integrals on a test

$= -2\cot\left(\frac{\theta}{2}\right) - \theta \Big|_{\pi/2}^{\pi}$

58. $\int \sin(x)\cdot\underbrace{\cosh(x)}\,dx$
hyperbolic trig; not part of course

43. $\int \frac{16}{(x-2)^2(x^2+4)}\,dx$ ← too complex for a test
set up but do not solve

factor denominator: $(x-2)^2(x^2+4)$

factors	deg	multi
$(x-2)$	1	2
(x^2+4)	2	1

x^2+4 is a deg 2 factor

$$\frac{16}{(x-2)^2(x^2+4)} = \frac{A}{(x-2)^1} + \frac{B}{(x-2)^2} + \frac{Cx+D}{(x^2+4)}$$

roots method fails as this never disappears

$\int \frac{1}{x(x^2-1)^{3/2}}\,dx$

partial fractions only works when exponents are whole numbers

this uses a method called trig. substitution which we do not learn in detail but you have seen in mobius

we would tell you take $x = \sec\theta$ and solve

$x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\sec\theta(\sec^2\theta-1)^{3/2}} \cdot \sec\theta \cdot \tan\theta\,d\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$= \int \frac{1}{\sec\theta(\tan^2\theta)^{3/2}}\, \sec\theta \cdot \tan\theta \cdot d\theta$

$= \int \frac{1}{\cancel{\sec\theta} \cdot \tan^{\cancel{3}\,2}\theta} \cdot \cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta$

$= \int \frac{1}{\tan^2\theta}\,d\theta$

$= \int \cot^2\theta\,d\theta$

$\cot^2\theta = \csc^2\theta - 1$

$= \int \csc^2\theta - 1\,d\theta$

$= -\cot(\theta) - \theta + c$

3. $\int \cos^3\theta \sin^8\theta\,d\theta$

trigonometric integration:
identities given on test
replace lowest odd power

$= \int \cos\theta \cdot \cos^2\theta \cdot \sin^8\theta\,d\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$= \int \cos\theta(1-\sin^2\theta)\cdot\sin^8\theta \, d\theta$

$= \int (\sin^8\theta - \sin^{10}\theta)\cdot\cos\theta \, d\theta$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$= \int u^8 - u^{10} \, du$

$= \frac{1}{9}u^9 - \frac{1}{11}u^{11} + c$

$= \frac{1}{9}\sin^9\theta - \frac{1}{11}\sin^{11}\theta + c$

4. $\int xe^{-12x} dx$

u-substitution: $u = -12x$ then $du = -12\,dx$ but we have $x \cdot dx$

two unrelated things being multiplied => integration by parts

LIATE => algebra is hit first

$u = x$ $\quad du = dx$
$dv = e^{-12x} dx$ $\quad v = -\frac{1}{12}e^{-12x}$

$uv - \int v \, du$

$= (x)(-\frac{1}{12}e^{-12x}) - \int -\frac{1}{12}e^{-12x} dx$

$= -\frac{1}{12}xe^{-12x} + \frac{1}{12}\int e^{-12x} dx$

$= -\frac{1}{12}xe^{-12x} + \frac{1}{12}(-\frac{1}{12}e^{-12x}) + c$

$= -\frac{1}{12}xe^{-12x} - \frac{1}{144}e^{-12x} + c$