Week 14: April $20^{th}$, 2023

## Section 10.1: Sequences

A sequence is a list of numbers written in a specific order. This list may or may not be infinite, but our class focuses on infinite sequences. A general sequence has terms $a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$, make sure to keep the n+1 in the subscript. Let's take a look at a couple of sequences:

example. Write down the first few terms of each of the following sequences:

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$ : $a_1 = \frac{1+1}{(1)^2} = \frac{2}{1} = 2$, $a_2 = \frac{2+1}{(2)^2} = \frac{3}{4}$, $a_3 = \frac{3+1}{(3)^2} = \frac{4}{9}$, $a_4 = \frac{4+1}{(4)^2} = \frac{5}{16}$, $a_5 = \frac{5+1}{(5)^2} = \frac{6}{25}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$ : $a_0 = \frac{(-1)^{0+1}}{2^0} = \frac{-1}{1} = -1$, $a_1 = \frac{(-1)^{1+1}}{2^1} = \frac{1}{2}$, $a_2 = \frac{(-1)^{2+1}}{2^2} = \frac{-1}{4}$, $a_3 = \frac{(-1)^{3+1}}{2^3} = \frac{1}{8}$, $a_4 = \frac{(-1)^{4+1}}{2^4} = \frac{-1}{16}$

(c) $\{b_n\}_{n=1}^{\infty}$ where $b_n$ is the $n^{th}$ digit of $\pi$: $a_1 = 3$, $a_2 = 1$, $a_3 = 4$, $a_4 = 1$, $a_5 = 5$

We can also use pattern recognition to solve for the general term of the sequence and the limit of the sequence as $n \to \infty$.

example. For each sequence below, write down (a) the formula for the general term of the sequence and (b) the limit of the sequence:

1. $\frac{2}{4\cdot 5}, \frac{2}{5\cdot 6}, \frac{2}{6\cdot 7}, \frac{2}{7\cdot 8}, \dots$

(a). $a_n = \frac{2}{(n)(n+1)}$

(b) $\lim_{n\to\infty} a_n = 0$

2. $2, -2, 2, -2, \dots$

(a) $a_n = 2^{n+1}$

(b) $\lim_{n\to\infty} a_n = \text{D.N.E.}$

## Section 10.2: Summing an Infinite Series

Start with a sequence $\{a_n\}_{n=1}^{\infty}$, we define a new sequence of "partial sums", $\{s_n\}_{n=1}^{\infty}$, by the following: $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, $s_4 = a_1 + a_2 + a_3 + a_4, \dots, s_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i$. Recall that $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_n$. We say that the infinite series converges if the sequence $\{s_n\}_{i=1}^{\infty}$ is convergent and its limit is finite. Similarly, if the sequence of partial sums is divergent then so is the series.

example. Determine if the following series are convergent or divergent. If it converges, determine its value.

(a) $\sum_{n=1}^{\infty} n$

$\{a_n\} = \{1, 2, 3, 4, 5, \dots\}$

$\{s_n\} = \{1, 3, 6, 10, 15, \dots\} = \left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$

divergent

(b) $\sum_{n=3}^{\infty} (\sqrt{n+1} - \sqrt{n})$

$\{a_n\}_{n=3}^{\infty} = \{\sqrt{4} - \sqrt{2}, \sqrt{5} - \sqrt{4}, \sqrt{6} - \sqrt{5}, \sqrt{7} - \sqrt{6}, \dots\}$

$\{s_n\}_{n=3}^{\infty} = \{\sqrt{4} - \sqrt{2}, -\sqrt{2} + \sqrt{5}, -\sqrt{2} + \sqrt{6}, \dots\}$

convergent to $-\sqrt{2}$

## Special Series

### Geometric Series

A geometric series is any series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1}$, or with an index shift $\sum_{n=0}^{\infty} ar^n$.

The partial sums are $s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$. The series will converge provided the partial sums form a convergent sequence, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} r^n$. The limit will exist and be finite provided $-1 < r < 1$, it will also be 0. Therefore, a geometric series will converge when $|r| < 1$ to $\frac{a}{1-r}$.

example. Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$

$= \sum_{n=1}^{\infty} 9^{-(n-2)} 4^{n+1}$

$= \sum_{n=1}^{\infty} \frac{4^{n-1}4^2}{9^{n-2}}$

$= \sum_{n=1}^{\infty} \frac{4^{n-1}4^2}{9^{n-1}9^{-1}}$

$= \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$

$a = 144$, $r = \frac{4}{9} < 1$

$= \frac{144}{1 - 4/9}$

$= \frac{9}{5}(144)$

$= \frac{1296}{5}$

(b) $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

$= \sum_{n=0}^{\infty} \frac{((-4)^3)^n}{5^n 5^{-1}}$

$= \sum_{n=0}^{\infty} \frac{5(-64)^n}{5^n}$

$= \sum_{n=0}^{\infty} 5\left(\frac{-64}{5}\right)^n$

$a = 5$, $r = -\frac{64}{5} > 1$

series diverges

## Telescoping Series

A telescoping series is best shown through an example.

example. Determine if the following series converges or diverges. If it converges find its value.

(a) $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

$$s_n = \sum_{i=0}^{n} \frac{1}{i^2+3i+2}$$
$$= \sum_{i=0}^{n} \frac{1}{(i+2)(i+1)}$$
$$= \sum_{i=0}^{n} \left(\frac{1}{i+1} - \frac{1}{i+2}\right)$$
$$= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right) + \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$= 1 - \frac{1}{n+2}$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)$$
$$= 1$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

$$s_n = \sum_{i=1}^{n} \left(\frac{1/2}{i+1} - \frac{1/2}{i+3}\right)$$
$$= \frac{1}{2}\sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right)$$
$$= \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \ldots + \overset{n-2}{\left(\frac{1}{n-1}-\frac{1}{n+1}\right)} + \overset{n-1}{\left(\frac{1}{n}-\frac{1}{n+2}\right)} + \overset{n}{\left(\frac{1}{n+1}-\frac{1}{n+3}\right)}\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right]$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$= \frac{5}{12}$$

It is not always obvious if a series is telescoping, you see it when you look at the partial sum. Not all partial fractions are telescoping, take $\sum_{n=1}^{\infty} \frac{3+2n}{n^2+3n+2} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$ for example.

## Ratio Test

Suppose we have the series $\sum a_n$. Define $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Then

(i) if $L<1$ the series is convergent

(ii) if $L>1$ the series is divergent

(iii) if $L=1$ the test fails

example. Determine if the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$
$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)}$$
$$= \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{(-10)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{-10(n+1)}{4^2(n+2)}\right|$$
$$= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2}$$
$$= \frac{10}{16}$$

converges as $L<1$

(b) $\sum_{n=0}^{\infty} \frac{n!}{5^n}$

$$a_n = \frac{n!}{5^n}$$
$$a_{n+1} = \frac{(n+1)!}{5^{n+1}}$$
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{1}{5} \cdot \frac{(n+1)n!}{n!}\right|$$
$$= \lim_{n\to\infty}\frac{n+1}{5}$$
$$= \infty$$

diverges

## Power Series

A power series about a is any series that can be written in the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ where $c$ and $a_n$ are numbers and the $a_n$'s are called the coefficients of the series. The same principles for converges apply, but now we have a variable that will affect when the series converges. There exists a number R such that the power series converges for $|x-a|<R$ and will diverge for $|x-a|>R$. This number is called the radius of convergence for the series.

example. Determine the radius of convergence for the following power series.

(a) $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$

ratio test

$$\lim_{k\to\infty}\left|\frac{(x-2)^{k+1}}{(k+1)^3} \cdot \frac{k^3}{(x-2)^k}\right|$$
$$= \lim_{k\to\infty}\left|(x-2)\cdot\frac{k^3}{(k+1)^3}\right|$$
$$= |x-2| \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^3$$
$$= |x-2| < 1$$

$x \in (1,3)$

radius $= 1$

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

ratio test

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$
$$= x\lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$
$$= x\cdot(0)$$
$$= 0 < 1$$

$x \in (-\infty, \infty)$

radius $= \infty$

(c) $\sum_{k=1}^{\infty} \frac{x^{2k}}{2k+1}$

ratio test

$$\lim_{k\to\infty}\left|\frac{x^{2(k+1)}}{2(k+1)+1}\cdot\frac{2k+1}{x^{2k}}\right|$$
$$= \lim_{k\to\infty}\left|\frac{x^{2k+2}}{2k+3}\cdot\frac{2k+1}{x^{2k}}\right|$$
$$= \lim_{k\to\infty}\left|(x^2)\cdot\frac{2k+1}{2k+3}\right|$$
$$= |x^2| \cdot \lim_{k\to\infty}\left(\frac{2k+1}{2k+3}\right)$$
$$= |x^2| < 1$$

$x \in (-1, 1)$

radius $= 1$

## Section 10.7 & 10.8: Taylor Polynomial & Taylor Series

In this section we want to represent functions with power series. Let us start with the geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r|<1$. If we take $a=1$ and $r=x$, then we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$. This means we can represent the function $f(x)=\frac{1}{1-x}$ with the power series $\sum_{n=0}^{\infty} x^n$ for $|x|<1$, we say that the function $f(x)=\frac{1}{1-x}$ has a power series representation $1+x+x^2+x^3+\dots$ centered at $x=0$ for the interval of convergence $-1<x<1$.

### Taylor Series

A function $f(x)$ is said to be analytic if it has a convergent power series representation for each $c$,

$$f(x)=a_0+a_1(x-c)^1+a_2(x-c)^2+\dots+a_n(x-c)^n+\dots \quad \text{for } -r<(x-c)<r$$

where the coefficients $a_i$ and radius of convergent $r$ are to be determined. We call this series the Taylor Series of the function $f(x)$ centered at $x=c$.

### Maclaurin Series.

For the special case of $c=0$, we get:

$$f(x)=a_0+a_1x^1+a_2x^2+a_3x^3+\dots \qquad \text{for } -r<x<r.$$

We call this series the Maclaurin Series for $f(x)$ or the Taylor Series for $f(x)$ centered at $x=0$.

### Derivation Taylor Series

Given a function with a power series representation about $c$,

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\dots$$

that has derivatives of every order that we can find. We can find the coefficients $a_n$.

First, evaluate every thing at $x=c$. Then

$$f(c)=a_0.$$

So all terms except the first are zero and we now know $a_0=f(a)$. This doesn't tell us much about the other $a_i$ $(i>0)$. However, if we take the derivative of the function and its power series then plug in $x=c$, we get

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+\dots$$

$$f'(c)=a_1$$

We can continue this process with the second derivative

$$f''(x)=2a_2+3\cdot2a_3(x-c)+4\cdot3a_4(x-c)^2+\dots$$

$$f''(c)=2a_2$$

$$a_2=\frac{f''(c)}{2}$$

And again for the third

$$f'''(x)=3\cdot2a_3+4\cdot3\cdot2a_4(x-c)+\dots$$

$$f'''(c)=3\cdot2a_3$$

$$a_3=\frac{f'''(c)}{3\cdot2}$$

We can repeatively do this process to receive the general formula

$$a_n=\frac{f^n(c)}{n!}.$$

All together we have the following Taylor series for $f(x)$ about $x=c$,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$$

$$=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\frac{f^4(c)}{4!}(x-c)^4+\dots$$

Similar to defining partial sums, we have the $n^{th}$ degree Taylor polynomial of $f(x)$ as,

$$T_n(x)=\sum_{i=0}^{\infty}\frac{f^i(c)}{i!}(x-c)^i.$$

Note that there are Maclaurin equivalent for these when $c=0$.

We can also talk about the error between the function $f(x)$ and the $n^{th}$ degree Taylor polynomial for a given $n$.

$R(x) = f(x) - T_n(x)$

example. Find the Taylor Series for $f(x)=e^x$ about $x=0$.

First we take note that $f^n = e^x$ for $n=0,1,2,3,\dots$ and $f^n(0)=e^0=1$ for $n=0,1,2,3,\dots$

Therefore, the Taylor series $f(x)=e^x$ about $x=0$ is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

example. Use the Maclaurin polynomial $T_4(x)$ for $e^x$ to estimate $e^{0.2}$

$e^x \approx T_4 = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$

$e^{0.2} \approx T_4(0.2) = 1 + (0.2) + \frac{1}{2!}(0.2)^2 + \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 = 1.2214$

example. Estimate the error of the estimate for $e^{0.2}$.

$$\begin{aligned} R(x) &= f(x) - T(x) \\ &= e^{0.2} - 1.2214 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(0.2)^n - \sum_{n=0}^{4} \frac{1}{n!}(0.2)^n \\ &= \sum_{n=5}^{\infty} \frac{1}{n!}(0.2)^n \end{aligned}$$

example. Estimate the value of $\int_0^{0.2} e^{-x^2}\,dx$ using $T_4(x)$ for $e^x$.

$$\begin{aligned} \int_0^{2} e^{-x^2}\,dx &\approx \int_0^{0.2} T_4(-x^2)\,dx \\ &= \int_0^{0.2} 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 \,dx \\ &= \left[ x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 \right]_0^{0.2} \end{aligned}$$