Week 13: April 13th, 2023

Section 14.3: Estimating Partial Derivatives

Estimating	One Varible	Two Variables (y)	Two Variable (x)
Forward Difference	$f'(x)= \frac{f(x+h)-f(x)}{h}$	$f_y= \frac{f(x,y+h)-f(x,y)}{h}$	$f_x= \frac{f(x+h,y)+f(x,y)}{h}$
Backward Difference	$f'(x)= \frac{f(x)-f(x-h)}{h}$	$f_y= \frac{f(x,y)-f(x,y-h)}{h}$	$f_x= \frac{f(x,y)-f(x-h,y)}{h}$
Central Difference	$f'(x)= \frac{f(x+h)-f(x-h)}{2h}$	$f_y= \frac{f(x,y+h)-f(x,y-h)}{2h}$	$f_x= \frac{f(x+h,y)-f(x-h,y)}{2h}$

$h=\Delta x$ or $h=\Delta y$

$h=\Delta x$ or $h=\Delta y$

$2h=\Delta x$ or $2h=\Delta y$

example. A two-variable function $f(x,y)$ has selected values given by

y\x	2.5	3.0	3.5
-1.0	6.0	6.5	8.0
-1.5	6.5	7.0	8.5
-2.0	5.8	6.9	7.8

(a) Write down three estimates for the value of $\frac{\partial f}{\partial y}(3.5,-1.5)$. State what estimates they are.

Central difference: $\frac{1}{\Delta y}[f(x,y+h)-f(x,y-h)]$

$$\frac{\partial f}{\partial y}(3.5,-1.5) = \frac{1}{\Delta y}[f(3.5,-1.0)-f(3.5,-2.0)] \quad ; \quad \Delta y = -1-(-2) = 1$$
$$= 1[8-7.8]$$
$$= 1[0.2]$$
$$= 0.2$$

Forward difference: $\frac{1}{\Delta y}[f(x,y+h)-f(x,y)]$

$$\frac{\partial f}{\partial y}(3.5,-1.5) = \frac{1}{\Delta y}[f(3.5,-1.0)-f(3.5,-1.5)] \quad ; \quad \Delta y = -1-(-1.5) = 0.5$$
$$= \frac{1}{0.5}[8-8.5]$$
$$= 2[-0.5]$$
$$= -1$$

Backward difference: $\frac{1}{\Delta y}[f(x,y)-f(x,y-h)]$

$$\frac{\partial f}{\partial y}(3.5,-1.5) = \frac{1}{\Delta y}[f(3.5,-1.5)-f(3.5,-2.0)] \quad ; \quad \Delta y = -1.5-(-2) = 0.5$$
$$= \frac{1}{0.5}[8.5-7.8]$$
$$= 2[0.7]$$
$$= 1.4$$

(b) Estimate $\frac{\partial f}{\partial x}(3.5,-1.5)$. Which did you use?

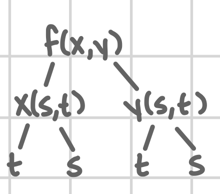
We can only look backwards:

$$\frac{\partial f}{\partial x}(3.5,-1.5) = \frac{1}{\Delta x}[f(3.5,-1.5)-f(3.0,-1.5)] \quad ; \quad \Delta x = 3.5-3 = 0.5$$
$$= \frac{1}{0.5}[8.5-7]$$
$$= 2[1.5]$$
$$= 3$$

Section 14.6: Chain Rule and Implicit Differentation

Chain Rule

The chain rule allows us to take a derivative down a tree of functions:



The derivative down each line is the top over the bottom

$\frac{df}{ds} = \frac{df}{dx}\cdot\frac{dx}{ds} + \frac{df}{dy}\cdot\frac{dy}{ds}$ $\qquad$ $\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}$

This replicates the process of plugging in the parameterizations $f(x(s,t),y(s,t))$ and taking the derivative example. Let $w = 3x\cos \pi y$. If $x = u^2 + v^2$, $y = \frac{v}{u}$, find the following partial derivatives at the given point. Use a tree diagram.

(a) $\frac{\partial w}{\partial u}$ at $u = 1$ and $v = -1$

$w(u,v) = 3x(u,v)\cdot\cos(\pi y(u,v))$

$\frac{\partial w}{\partial u} = 3\frac{\partial x}{\partial u}\cos(\pi y) + (-\sin(\pi y))\cdot\pi\frac{\partial y}{\partial u}\cdot 3x$

$= 3(2u)\cos(\pi y) - \sin(\pi y)\cdot\pi(-vu^{-2})\cdot 3x$

$= 6(1)\cos(\pi(1)) - \sin(\pi(1))\cdot\pi(-1\cdot(1)^{-2})\cdot 3(2)$

$= 6\cos(\pi) + 6\pi\sin(\pi)$

$= -6 + 0$

$= -6$

$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u^2+v^2) = 2u$

$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\frac{v}{u}\right) = -1\cdot v\ u^{-2}$

@ $u=1, v=1$

$x = u^2 + y^2 = 2$

$y = \frac{v}{u} = 1$

(b) $\frac{\partial w}{\partial v}$

$\frac{\partial w}{\partial v} = 3\frac{\partial x}{\partial v}\cos(\pi y) + (-\sin(\pi y))\cdot\frac{\partial y}{\partial v}\cdot\pi\cdot 3x$

$= 3(2v)\cos(\pi y) - \sin(\pi y)\cdot\left(\frac{1}{u}\right)\cdot\pi\cdot 3x$

$= 6v\cos(\pi y) - 3\pi\frac{1}{u}x\sin(\pi y)$

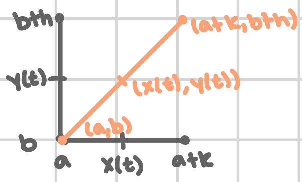
$= 6v\cos\left(\pi\frac{v}{u}\right) - 3\pi\frac{1}{u}(x^2+y^2)\sin\left(\pi\frac{v}{u}\right)$

$\frac{\partial x}{\partial v} = 2v$

$\frac{\partial y}{\partial v} = \frac{1}{u}$

Linear Approximation

Consider a particle moving from point (a,b) to point $(a+k, b+h)$. If the particle travels at a constant speed and the total duration of the motion is 1 second, find in terms of time t (in seconds), a formula for the position (x,y).



$(x(0), y(0)) = (a,b)$

$(x(1), y(1)) = (a+k, b+h)$

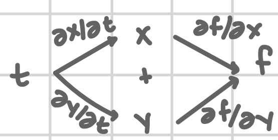
$(x(t), y(t)) = (a+tk, b+ht)$

w/ $\frac{dx}{dt} = k$, $\frac{dy}{dt} = h$

Consider a function $f(x,y)$ such that its first partial derivatives exist for all points near (a,b). If (x,y) is a point on the line segment found above, find a formula for the rate of change of f with respect to t.

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

$= k\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y}$



For a small change in time Δt, let the corresponding change in x be from a be Δx, the corresponding change in y from b be Δy and Δf be the corresponding change in f from $f(a,b)$. Then we have $\frac{\Delta f}{\Delta t} \approx \frac{df}{dt}\big|_{t=0}$. We want to show that $\Delta f \approx \frac{\partial f}{\partial x}(a,b)\cdot\Delta x + \frac{\partial f}{\partial y}(a,b)\cdot\Delta y$ where $\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$. This Δf is called the Linear Approximation of change in f when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.

$\Delta t \longrightarrow \Delta x \approx \frac{dx}{dt}$

$\Delta y \approx \frac{dy}{dt}$

$\frac{\Delta f}{\Delta t} = \frac{f(t+h) - f(t)}{h}$

$f(t+h) - f(t) \approx \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y$

example. A two-variable function $f(x,y)$ has selected values given by

y\x	2.5	3.0	3.5
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Using the central difference estimate for $\frac{\partial f}{\partial y}(3.5,-1.5)$ and the estimate of $\frac{\partial f}{\partial x}(3.5,-1.5)$ approximate the value of $f(3.2,-1.1)$.

formula: $f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

estimates: $\frac{\partial f}{\partial y}(3.5,-1.5) \approx 0.2$, $\frac{\partial f}{\partial x}(3,-1.5) \approx 3$

linear approximation: $f(3.5,-1.1) \approx f(3.5,-1.5) + \frac{\partial f}{\partial x}(3.5,-1.5)(3.2-3.5) + \frac{\partial f}{\partial y}(3.5,-1.5)(-1.1+1.5)$

$= 8.5 + 3(-0.3) + 0.2(0.4)$

$= 7.68$