Geometric Group Theory Notes

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Lecture 00: July XX, 2021

Lecture Title

1. Section Heading
definition. notes
example. notes

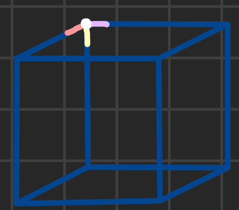
2. Introduction
prof. Rachel Skipper & Mark Pengitore
text. Office Hours with a Geometric Group Theorists
theme. connecting "algebra" and "geometry" through the study of symmetries

Lecture 01: July 12, 2021

Groups

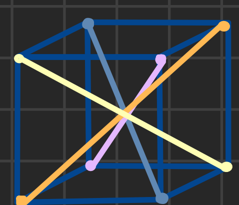
1. Symmetries of a Cube.

Fundamental question: What are the symmetries of a cube?



1) Choose a corner
   - Choose where it goes under line symmetry
     = 8 choices
2) Label the edges off this vertex
   - Choose where to send the edges of that vertex
     = 3 choices

total: $8 \cdot 3 = 24$ symmetries



Facts:
1) Any symmetry of the cube gives a permutation of the 4 diagonal lines.
2) Any permutation of the 4 diagonals gives a unique symmetry of the cube.

To see 2:
First observe that swapping any two diagonals corresponds to a symmetry where we skewer the cube through the middle of two edges that the diagonals connect.

def. rigid symmetries: All of the ways we can pick up the cube, rotate it around, and return it to the same spot.

The # of symmetries of the cubes = # perms of diagonals $= 4! = 24$

def. A group is a set together with a way of multiplying elements of the set such that:

(1) There is an element 1 in the set such that $1 \cdot g = g \cdot 1 = g$ for all g in the set G. This element 1 is called the identity

(2) For every g in G there is an element h in G called the inverse of g such that $g \cdot h = h \cdot g = 1$. Sometimes h is written as g^{-1}.

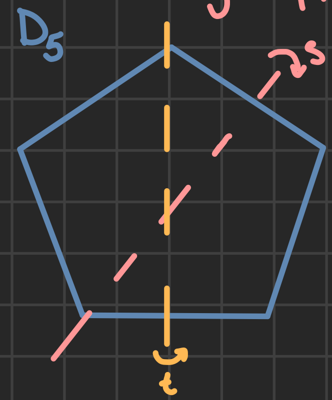
(3) Multiplication in G is associative, that is $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all f, g, h in G.

def. Multiplying in a group means we take two element f, g and $f \cdot g$ gives us something else in the group.

2. Rational numbers without 0 under standard multiplication

1) identity is 1
2) for any $\frac{p}{q}$ the inverse is $\frac{q}{p}$
3) associative comes by def. of multiple two rationals

3. Dihedral group, D_n, symmetries of the n-gon



Doing t and then s gives a rotation.
t and s "generate" the group

def. Formally, this means every element of the group can be obtained by multiplying combos of t, s, t^{-1}, s^{-1}

4. Symmetric Group, S_n

def. The symmetric group S_n is the set of all permutations of $\{1, \dots, n\}$ (i.e. bijective functions from $\{1,\dots,n\} \to \{1,\dots,n\}$) with multiplication being function composition.

def. A cycle in S_n is a permutation describle by $i_1 \mapsto i_2 \mapsto \dots \mapsto i_k \to i_1$, we will write as $(i_1\ i_2 \cdots i_k)$.

example. $1 \mapsto 3 \mapsto 5 \mapsto 5 \mapsto 7$, fixes 2,4,6,8 an element in S_8 $(1\,3\,5\,7)$

def. transpositions are cycles of length 2. **ex.** $1 \mapsto 4 \mapsto 1 = (1\,4)$

Every permutation can be written as a product of disjoint cycles
ex. $1 \mapsto 2 \mapsto 5 \mapsto 1$, $3 \mapsto 4 \mapsto 3$ $\quad (1\,2\,5)(3\,4)$

Symmetric group can be thought of as the symmetries of a set of points

$(1\,2\,5)(3\,4) \in S_5$
$= (34)(23)(34)(45)(34)(23)(12)$

Lecture 02: July 14, 2021

Groups

5. Cyclic Group of order n, $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$

$2+2=4$, $2+3=5=0$

- identity is 0

inverse of 0 is 0

- inverse of m in given by $n-m$; $m+n-m = n = n-n = 0$
- associativity $(a+b)+c$

↳ $(a+b) \geq n$, $(b+c) \geq n$, $(a+b)+c \geq n$, $a+(b+c) \geq n$

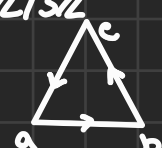
generators for $\mathbb{Z}/n\mathbb{Z}$ are given by m s.t. $\gcd(m,n)=1$

↳ $\{1\}$ is a generator and by the Euclidean algorithm there is integers t_1, t_2 where $1 = t_1 n + t_2 m = t_2 m = 1$ so $\{1\} = \{t_2 m\}$

Q: What object is $\mathbb{Z}/n\mathbb{Z}$ the symmetries of?

$\mathbb{Z}/8\mathbb{Z}$

$\mathbb{Z}/3\mathbb{Z}$



1 = send a to b, b to c, c to a

2 = send a to c, b to a, c to b

0 = do nothing

Note. $\mathbb{Z}/n\mathbb{Z}$ corresponds to rotations of an n-gon $\subseteq D_n$

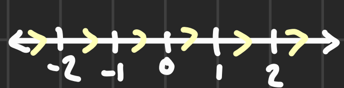
we missing the reflection so we do not have all of D_n

Put directions on sides and take all rigid symmetries that preserves arrows i.e. up arrow cannot go to down arrow

Infinite Groups

1. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ under addition

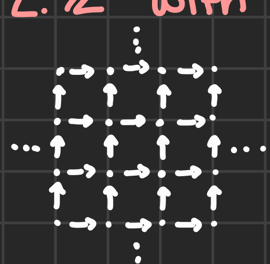
Q: What are the integers symmetries of?



if $a \in \mathbb{Z}$ then 2 sends a to $a+2$

we still need to remove reflections

2. $\mathbb{Z}^2$ with $(m_1, n_1) + (m_2, n_2) = (m_1 + n_1, m_2 + n_2)$



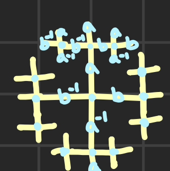
element of $\mathbb{Z}^2$ are symmetries that preserves arrow direction

3. $\mathbb{R}$ with addition
4. $\mathbb{Q}$ with addition
5. $\mathbb{R}^+$ with multiplication

6. Group product on the Alphabet

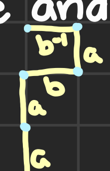
a word in letters a,b is an finite string of letters
the empty word is a finite string of no letters
multiply two words by concatenation
example. $(aba, bb) \to ababb$ but $(bb, aba) \mapsto bbaba$
Since b is in the set then b^{-1} should be an element s.t. $bb^{-1}=1$
def. a reduced word is a word in a, a^{-1}, b, b^{-1} with no instances of aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$.
ex. $aa^{-1}bb^{-1}ab \mapsto 1bb^{-1}ab \mapsto 11ab \mapsto 1ab \mapsto ab$
exercise. Every word can be written as a reduced word
def. $F_2 = \{$ reduced words with 1 in $a, b, a^{-1}, b^{-1} \}$ and product is given by the concatenation then reverse
- identity is 1
- inverse of a word is reverse word and replace with inverse

example. $(abbab)^{-1} = b^{-1}a^{-1}b^{-1}b^{-1}a^{-1}$
Q: What is this the symmetries of? a tree!



every up arrow is an "a"
—— down —— "a^{-1}"
—— right —— a "b"
—— left —— a "b^{-1}"

Take a point in the tree and follow the tree to get there
example. $(aabab^{-1})$



Higher free groups are defined similarly
F_3 is reduced words in a,b,c
F_{26} is reduced words in a,…,z

Lecture 03: July 16, 2021

Functions on Groups

Review

Groups: Start with a multiplication on a set G, $G\times G \to G$, $g\cdot h \mapsto gh$.

def. A group G is a set together with a way of multiplying elements such that

1) there is an identity element 1 with $1\cdot g = g\cdot 1 = g$ $\quad \overset{\text{for all}}{\forall} g \overset{\text{in}}{\in} G$
2) for every $g \in G$, there is an inverse g^{-1}: $gg^{-1} = g^{-1}g = 1$
3) multiplication is associative

Subgroups

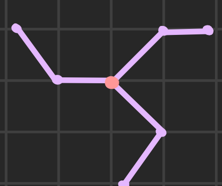
def. A <u>subgroup</u> of a group G is a subset of G that is a group on its own (same multiplication and identity)

example. $G = (\mathbb{Z}, +)$, $H = \{\text{even integers}\} = 2\mathbb{Z}$

Functions that Preserve Group Structure

def. A <u>homomorphism</u> from a group G to a group H is a function $f: G \to H$ so that for all a,b in G, $f(\underbrace{a\cdot b}_{\text{in } G}) = \underbrace{f(a)\cdot f(b)}_{\text{in } H}$. This is called preserving multiplication.

def. An <u>isomorphism</u> is a homomorphism that is also a bijection.



Boric Molecule

only has rotational symmetry
- do nothing
- rotate clockwise 120°
- rotate clockwise 240°

$\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$
$0 \mapsto$ do nothing
$1 \mapsto$ rotate 120°
$2 \mapsto$ rotate 240°

This is 1-1 and onto as well as preserves multiplication

lemma. If $f: G \to H$ is a homomorphism, then $f(1_G) = 1_H$.

Injective Homomorphisms

If we have an injective homomorphism, $f: G \to H$, it gives a way of thinking of G as a subgroup of H by image of G.

examples:

injective map symbol \hook right arrow

1. $\mathbb{Z}/n\mathbb{Z} \hookrightarrow D_n$, $1 \mapsto 1$ "click"/rotation
2. $\mathbb{Z}/2\mathbb{Z} \hookrightarrow D_n$, $1 \mapsto$ reflection
3. $\mathbb{Z}/2\mathbb{Z} \hookrightarrow S_n$, $1 \mapsto$ product of disjoint transpositions
4. $\mathbb{Z} \hookrightarrow \mathbb{Z}$, $x \mapsto mx$ for some m
5. $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$, $1 \mapsto (1,a)$ for any a
6. $\mathbb{Z} \hookrightarrow F_2$, fix $w \neq 1$, $n \mapsto w^n$

Non-injective Homomorphisms

examples:

1. $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, even $\mapsto 0$ & odd $\mapsto 1$
2. $\mathbb{R}\setminus\{0\} \to \mathbb{Z}/2\mathbb{Z}$, neg $\mapsto 1$ & pos $\mapsto 0$
3. $S_n \to \mathbb{Z}/2\mathbb{Z}$, odd crossing $\mapsto 1$ & even crossing $\mapsto 0$
4. $F_2 \to \mathbb{Z}^2$, word $\mapsto$ (# of exp. on a, # of exp. on b) $a^2ba^{-1}b^3 \mapsto (1,4)$

Kernels

def. The kernel of a homomorphism $f: G \to H$ is the set of elements g in G such that $f(g)=1$, the identity in H.

exercise. Show the kernel is a subgroup

def. A normal subgroup N of a group G is a subgroup where for all n in N and for all g in G we have gng^{-1} is in N.

lemma. The kernel of a homomorphism is a normal subgroup.

proof. The kernel being a subgroup is an exercise for the reader. Let $f: G \to H$ be a homomorphism and let N be the kernel of f. Let n be in N, let g be in G. Then $f(gng^{-1}) = f(g)f(n)f(g^{-1})$. n is in the kernel so $f(g)\cdot 1 \cdot f(g^{-1}) = f(g)f(g^{-1}) = f(gg^{-1}) = f(1) = 1$. #

Non-injective Homomorphisms & Kernels

examples:

1. $K = 2\mathbb{Z}$ = evens
2. K = positives
3. K = even crossings
4. K = adds to 0 exponents

Lecture 04: July 19, 2021

Normal Subgroups

Review

def. If G, H are groups, a homomorphism is a function $f: G \to H$ s.t. $f(ab) = f(a)f(b)$ for all $a, b \in G$.

def. The kernel of the homomorphism is the set $K := \{k \in G \mid f(k) = 1\}$.

def. The kernel is a normal subgroup. A subgroup $H \leq G$ is normal in G if for all n in N and for all g in G, gng^{-1} is in N.

Quotient Groups

def. Let G be a group and let N be a normal subgroup of G. Declare g_1, g_2 in G to be equivalent if $g_1 g_2^{-1}$ is in N. Denote the equivalence class of g by $[g]$. Elements of the quotient group G/N are these equivalence classes. Multiplication: $[g_1][g_2] = [g_1 g_2]$.

claim: N being normal is exactly the property required to make this multiplication well-defined.

Then there is a homomorphism from G to G/N defined by $g \mapsto [g]$. It's kernel is N. The kernel is $[1]$. An element h is in $[1]$ by def. if $h \cdot h^{-1}$ is in N, $h \cdot 1^{-1} \cdot h \cdot 1 = h$.

Theorem (1st Isomorphism Theorem). If $f: G \to H$ is a surjective homomorphism with kernel K, then H is isomorphic to G/K.

Group Presentations

1. Start with a list of generators

ex F_2 - two generators
$\mathbb{Z}^2$ - two generators
D_n - two generators

} not the same group

2. A list of equations that imply the rest of the list

ex If $gh = k$ and $h = pq$ then $gpq = k$.

Example 1: $\mathbb{Z}/n\mathbb{Z}$

Think of as $\{a^0, a^1, \dots, a^{n-1}\}$ so that $a^i a^j = a^{i+j \pmod n}$

1. generator is a^1
2. only necessary equation is $a^n = 1$

claim: $a^{n-2} a^3 = a$ is implied by $a^n = 1$

$a^{n+1} = a^n a^1 = 1 \cdot a^1 = a^1 = a$

Presentation: $\langle a \mid a^n = 1 \rangle$ a relation

Example 2: $\mathbb{Z}^2$

$\langle x, y \mid xy = yx \rangle$

Turn the relation into a relator: $xy - yx \Rightarrow xyx^{-1}y^{-1} = 1$

$\langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle$

note. $\mathbb{Z}^2$ in this way we are really defining an isomorphism that takes $(1,0) \mapsto x$ and $(0,1) \mapsto y$ and so $(1,0)+(0,1) \mapsto xy$

claim: $x^n y^m$ and $x^p y^q$ commute

proof. $x^n y^m x^p y^q$

$x^n y^{m-1} yx x^{p-1} y^q$

$x^n y^{m-1} xy x^{p-1} y^q$

$\vdots$

$x^p y^q x^n y^m$

$x^2 y^3 xy$

$x^2 y^2 yxy$

$x^2 y^2 xyy$

$x^2 y^2 xy^2$

$\vdots$

If $G = \langle S \mid R \rangle$ with $S = \{\text{gen.}\}$ and $R = \{\text{rel.}\}$. Then G is iso. to $F(S)/N$ where N is the smallest normal subgroup of $F(S)$ containing all elements of R.

Theorem. Every group is a quotient of a free group

Lecture 5: July 21, 2021

Group Actions

Review
- what does it mean for a group to be a symmetry
- groups can be realized as reflections and rotations of n-gons
- object (n-gon) on which our group does something to
  ↪ our group acts on something we like while preserving

def. an action of G on a set X is a function $G \times X \to X$ where $(g,x) \mapsto g \cdot x$ satisfies the following:
(i) $1 \cdot x = x$ for all x in X everything in X is fixed by identity
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ $g,h \in G$ action is compatible w/ multiplication in G

notation. If $A \subseteq X$, $g \cdot A = \{g \cdot a \mid a \text{ in } A\}$ when everything is clear from context we write gx gA.

Let S_X = symmetry. Not only does each element of G give a permutation of X. It does it in a way so that multiplication in G and composition of permutations on S_X are compatible, we actually get a homomorphism $f: G \to S_X$.
exercise. Show a group action of G on X is equiv. to the existence of a homomorphism $f: G \to S_X$. This homo. may not be injective.

If X is any set, G any group. G acts on X trivially, i.e. $gx = x$

examples.
1. S_n acts on $\{1, \ldots, n\}$
2. D_n acts on vertices of n-gons
3. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ via multi. by ± 1
4. $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(a,b) \cdot (x,y) = (a+x, b+y)$

Terminology
def. If x in X and G acts on X then the orbit of x under the action $G \cdot x = \{gx \mid g \text{ in } G\}$
def. If $G \cdot x = X$ then the action is transitive
def. If G acts on X trivially then $G \cdot x = \{x\}$
def. If no non identity of G fixes any x in X, the action of G is free

G acting on G

1. G acts on itself by left multiplication, $g \cdot h = gh$
2. G acts on itself via conjugation, $g \cdot h = ghg^{-1}$

exercise. action of G on itself by left mult. is injective

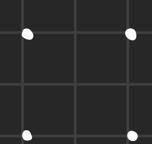
Theorem. Every group is isomorphic to a subgroup of a symmetric group via $G \to S_G$ by left multiplication.
note. groups are always symmetries of something

Introduce 1st collection of geometric objects that groups act on

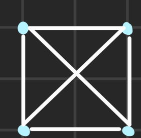
def. A graph, $\Gamma = \Gamma(V, E)$ with V being a set of vertices (non empty) and E a set of edges (can be empty), with an end point function $f: E \to V \times V$

examples.

1. disconnected

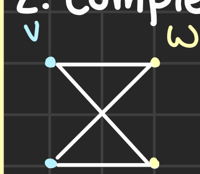


2. complete graph



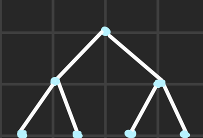
V is the set of n vertices
There is an edge $e \in E$ for every two distinct vertices

2. complete bipartied



$V = \{v_1, \ldots, v_n\}$
$W = \{w_1, \ldots, w_m\}$
edge for each $\{v_i, w_j\}$

4. trees connected graph with no loops



Definitions

loopless - no edge between vertice & itself
simple - no more than one edge between vert.
directed - edges have arrows

Group action on Graphs

A group G acts on a graph Γ if the following holds:
(i) for g in G, v in V, $g \cdot v$ is in V
(ii) G acts on edges
(iii) if v, w are endpoints of edge e, the $g \cdot v$ & $g \cdot w$ are endpoints on $g \cdot e$
action on a graph Γ is an action on vertices and edges that are compatible with endpoints

Cayley graphs

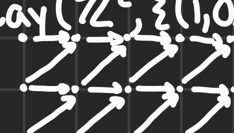
def. G a group, S a generating subset ($1 \notin S$). The Cayley graph for G with respect to S is the directed graph $\Gamma(G, S)$ given as follows: vertices = elements of G edges = directed edge from g to gs for each g in G and s in S

example.

1. $Cay(\mathbb{Z}, \{1\})$

2. $Cay(\mathbb{Z}, \{2, 3\}$

3. $Cay(\mathbb{Z}^2, \{(1,0)(1,1)$



Symmetries of graphs

$f_V: V \to V$ and $f_E: E \to E$ are automorphism and if v, w are endpoints of the edge e, the $f_V(v)$ and $f_V(w)$ are endpoints of $f_E(e)$

$\mathrm{Aut}(\Gamma)$ = group under composition.

- when G acts on a graph Γ, we get a homomorphism $\varphi: G \to \mathrm{Aut}(\Gamma)$

Theorem. Let G be a group, S a generating set. Then G is iso. to $\mathrm{Aut}(\mathrm{Cay}(G,S))$

Outline. G acts on itself by left mult., this gives us an action on vertices of Cayley graph $g \cdot h \Rightarrow gh$. $h \to hs$ with $s \in S$, $h \in G$ so $h \to hs$ goes to $gh \to ghs$. So for a fixed g, we call this automorphism φ_g. The map $g \mapsto \varphi_g$ is an iso. from G to $\mathrm{Aut}(\mathrm{Cay}(G,S))$

Lecture 6: July 23, 2021

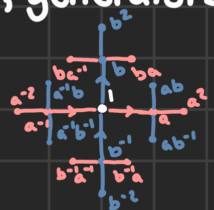
Free Groups

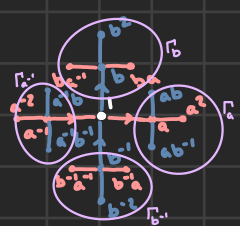
Review

Theorem. Let S be a generating set for a group G. Then the map $G \to \mathrm{Aut}(\Gamma(G,S))$ is an isomorphism

Free Groups

F_2, generators a,b





Let's take a and multiply on the left to get a symmetry of $\Gamma(\Gamma_a, \{a,b\})$.

- a takes Γ_a to Γ_a
- a takes Γ_b to Γ_a
- a take $\Gamma_{b^{-1}}$ to Γ_a
- a takes $\Gamma_{a^{-1}}$ to $\Gamma_b \cup \Gamma_{b^{-1}} \cup \Gamma_{a^{-1}} \cup \Gamma_{\{1\}}$

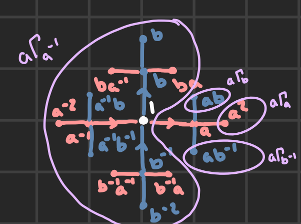
What about $\Gamma_{a^{-1}}$?

$a \cdot \Gamma_{a^{-1}} = a \cdot a^{-1} \cdot \omega$ for some ω

- ω can start with b, b^{-1}, a^{-1}

- if ω starts with a then $a^{-1}\omega$ is not reduced

$a\Gamma_{?}$



Metric Spaces

def. A <u>metric space</u> is a set X together with a function $d: X \times X \to \mathbb{R}$ with:

(i) positive definiteness
- $d(x,y) \geq 0$ for all x,y in X
- $d(x,y) = 0$ if and only if $x = y$

(ii) symmetry
- $d(x,y) = d(y,x)$ for all x in X

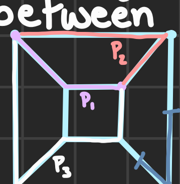
(iii) triangle inequality

$d(x,y) + d(y,z) \geq d(x,z)$ for all x,y,z in X

The function d is called a <u>metric</u>.

example. The path metric on a graph

Let Γ be a connected graph, then the distance between any two edges is the length of the shortest path between them.



P_2, P_1 are <u>geodesic</u>, the shortest path

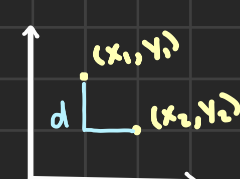
distance between midpoints is 1

For points on edges think of a single edge as a unit distance, i.e. mid + mid $= \frac{1}{2} + \frac{1}{2} = 1$

This is the <u>geometric realization</u> of Γ

example. Taxi cab metric on $\mathbb{R}^2$

$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$



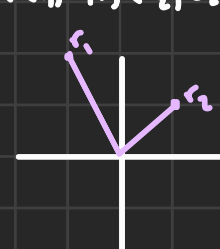
on $\mathbb{R}^n$: $d((a_1,\dots,a_n),(b_1,\dots,b_n)) = \sum_{i=1}^{n} |a_i - b_i|$

example. supremum metric, $\mathbb{R}^3$

$d((x_1,y_1,z_1),(x_2,y_2,z_2)) = \max\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\}$

example. The hub metric on $\mathbb{R}^2$ with polar coordinates (r,θ)

$$d((r_1,\theta_1),(r_2,\theta_2)) = \begin{cases} |r_1 - r_2| & \text{if } \theta_1 = \theta_2 \\ r_1 + r_2 & \text{if } \theta_1 \neq \theta_2 \end{cases}$$



Group as Metric Spaces

Let G be a group with generating set S. Then we can declare the distance between g and h to be the length of the word $g^{-1}h$ with respect to S.

note. This is the same as the path metric on $\Gamma(G,S)$.

example. $\mathbb{Z}^2$ with generating set $\{(1,0),(0,1)\}$ is the taxi cab metric

Isometry

def. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. An isometry is a bijective function from $f: X \to Y$ such that:
$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all x_1 and x_2 in X.

Lecture 07: July 26, 2021

Isometries

Isometry

recall. if X, Y are metric spaces with metrics d_X and d_Y respectively. Then a bijective function $f: X \to Y$ is an isometry if $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all x_1, x_2 in X.

our interest. isometries from $X \to X$

examples:

1) Rotation about a point in $\mathbb{R}^n$ and translations in $\mathbb{R}^n$ with Euclidean
   Translations are also isometries in taxicab
2) Reflections across a plane in $\mathbb{R}^3$ with Euclidean metric
3) In the hub metric, rotations about the origin

note. When we talk about symmetries of metric spaces, we mean isometries.

We are looking at homomorphism from a group G to the group of $\text{Isom}(X)$ (under function composition)

What this means

G is a group, X is a metric space. An action of G on X is a map $G \times X \to X$ s.t.

(i) $1 \cdot x = x$ for all $x \in X$
(ii) $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $g, h \in G$ and $x \in X$
(iii) $d(g \cdot x, g \cdot y) = d(x, y)$ for all g in G and x, y in X

examples:

1) D_n acts as isometries of the n-gon in the plane ← Euclidean
2) S_n acts on an n-simplex by permuting the corners
   in $\mathbb{R}^n$ under Euclidean
   - 0-simplex •
   - 1-simplex —
   - 2-simplex △
   - 3-simplex  (tetrahedron)
3) $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translation (w/ Euclidean or taxicab)
4) $\mathbb{Z}/n\mathbb{Z}$ by rotations about the origin (w/ Euclidean or hub)
5) Trivial action of any group on any metric space $(g \cdot x \mapsto x)$

torsion

def. An element $\overset{\neq 1}{g}$ in G is a <u>torsion element</u> if $g^n = 1$ for some n in $\mathbb{N}_{\{0\}}$

def. A group is <u>torsion free</u> if it has no torsion element

recall. An action of G on X is free if $g \cdot x = x$ means $g = 1$

Theorem. Suppose that G is a group that has a free action by isometries on $\mathbb{R}^n$ with the Euclidean metric. Then G is torsion free.

proof. Suppose that $g^m = 1$ for some g in G and $m \geq 1$. Let v be in $\mathbb{R}^n$. Let $\Theta = \{v, g\cdot v, g^2\cdot v, \ldots, g^{m-1} v\}$. Every finite set in $\mathbb{R}^n$ has a unique centroid (a point that minimizes the sum of distances to points in Θ) w. Notice that the element g fixes the set Θ as $g\cdot\Theta = \{g\cdot v, g^2\cdot v, \ldots g^m\cdot v\} = \Theta$ and g acts by isometries, so g fixes the centroid. Since g fixes a point, g must be the identity. $\#$

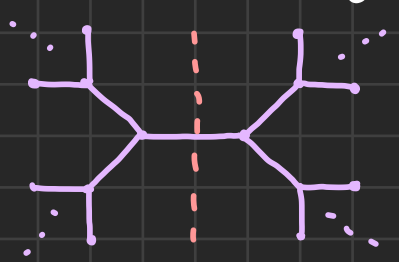
Groups Acting on Trees

Recall: last time, we saw if a group is free then it acts on a tree namely its Cayley graph

Theorem. If G is a free group, it acts freely on a tree.

Theorem. If a group G acts freely on a tree, then G is a free group.

Corollary (Nielsen Schreier Thm.). Every subgroup of a free group is a free group.



A group action on a tree is a group action on the vertices of the tree that respects edge endpoints

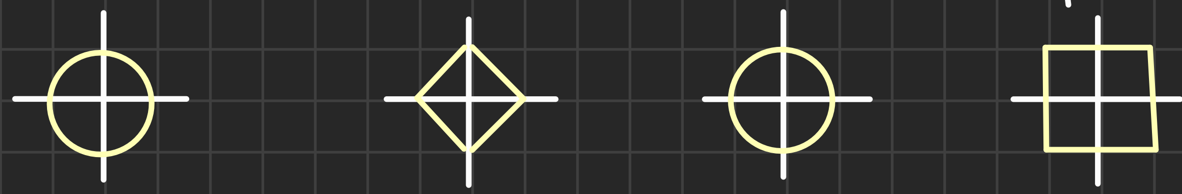
reflection through an edge is a not free action as it fixes the edge it goes through

Lecture XX : July 28, 2021

Problem Set 6, Problem 2

open sets in different metrics

Euclidean Taxicab Hub Sup



Problem Set 5, Problem 1

Show that G acts on itself via left multiplication and conjugation.

left mult: $G \times \overset{=G}{X} \to G$, $(g,h) \mapsto g \cdot h = gh$

(i) $1 \cdot h = h$ $\forall h \in X = G$ by properties of 1 in G

(ii) $g_1 \cdot (g_2 \cdot h) = (g_1 \cdot g_n) \cdot h$ by associativity of G

conjugation: $G \times \overset{=G}{X} \to G$, $(g,h) \mapsto ghg^{-1}$

(i) $1 \cdot h = 1h1^{-1} = 1h1 = h$ by properties of 1

(ii) $g_1 \cdot (g_2 \cdot h) = g_1 (g_2 h) g_1^{-1} = g_1 (g_2 h g_2^{-1}) g_1^{-1} = g_1 g_2 h (g_1 g_2)^{-1} = g_1 g_2 \cdot h$

Problem Set 5, Problem 2

Left multiplication induces an injective homomorphism $G \to S_G$.

Define $G \to S_G$, $g \mapsto \begin{pmatrix} G \to G \\ h \mapsto gh \end{pmatrix}$. an injective homomorphism induces a faithful action

conjugation?

Let H be a subgroup of G. Then $H^g = \{gHg^{-1} : g \in G\}$.

Furthermore, $G \times H^g \to H^g$, $g, (hHh^{-1}) \mapsto g(hHh^{-1})g^{-1}$!

Orbits?

orbit of H under conjugation $= \{ gHg^{-1} \in X : \exists g' \text{ s.t. } g'H = gHg^{-1} \}$

$= H^g$ transitive action = one orbit

stabilizers of H?

$\{g \in G : g \cdot H = H\} = \mathrm{stab}_G(H)$

↳ 1 will always be in the set

Lecture 8: July 30, 2021

Proving a Theorem

recall

Theorem 3.1. If a group G acts freely on a tree then G is a free group.
Theorem 3.2. Any subgroup of a free group is a free group.

motivation

geometry, combinatorics, graph structure, etc. of our tree gives us properties of G

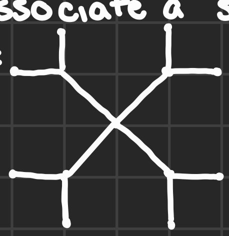
proof sketch of 3.1

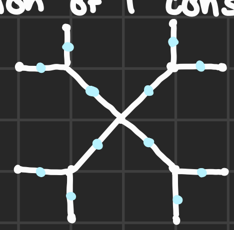
outline. Let G act freely on tree T.
1) break up T into smaller trees that respect the action of G (G-tiling)
2) use these subtrees and how they connect together to find gen. set
3) use uniqueness of paths in trees to show generating set is free

step 1: tiling the tree (hardest step)

def. A G-tiling of T is a collection of subtrees $\{T_i\}$ satisfying
(i) T_i and T_j don't share an edge when $i \neq j$,
in fact they touch at most at one vertex
(ii) union of $\{T_i\}$ is the whole tree, cover T with all tiles $\{T_i\}$
(iii) compatible with G-action,
we want a subtree T_0 in T so that $\{gT_0\}$ are all of the tiles

Let v be a vertex of T and denote Gv as its orbit. since G acts freely, there is a 1-1 correspondence between elements of G and vertices in the orbit of v. for each g in G, we will construct a tree T_g where $h \cdot T_g = T_{hg}$ and if w is a vertex then w in T_g for some g. moreover if e is an edge in T, then e is in T_g for some g. associate a subdivision of T constructed as follows:

T= 



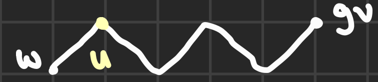
barycentric subdivision

for any g in G, vertices of T_g are given by

$$v(T_g) = \left\{ \begin{array}{l} \text{all vertices } w \text{ where} \\ d(w, gv) \leq d(w, g'v) \text{ for all } g' \end{array} \right\}$$

this means all vertices where g moves v the closest away. every other element of G moves v further away from w than g. capturing the collection of vertices that our element g moves v to the closets from edges of T_g, $E(T_g) = \{e$

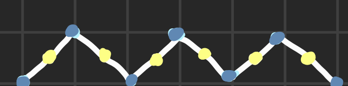
subclaim: T_g is connected for all g.
we will show a path in T between w and v is in T_g <gv is in T_g>.
Assume that this is not the case.



$\left.\begin{array}{l} d(w,gv)=n \\ d(u,gv)=n-1 \end{array}\right\}$ if u is not in T_g then $\exists g'$ where $d(v,g'v)=m<n-1$

this means there is a g' which move v a smaller distances from w than g. $d(w,g'v) \le d(w,v) + d(v,g'v)$. Since w and u are right next to each other $d(w,u)=1$. so $d(w,g'v) \le 1 + d(v,g'v) \le 1+m < 1+n-1 = n$ however $d(w,gv) \le d(w,g'v)$ but then $n \le d(w,g'v) < n$ a contradiction. u must be in T_g. we can repeat this process to show the path is in T_g and T_g is a subtree.

subclaim. union of T_g contains all of T
it is easy to see all vertices of T lie in some T_g. we want to show each edge of T lies in some T_g.



vertices in T
blue & yellow are vertices in T_0

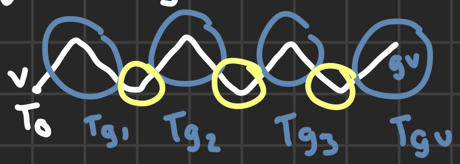
any path in T_0 alternates between blue & yellow vertices. if e is an edege, one endpoint is blue and the other is yellow. if our original vertex v is blue then the distance between another blue vertex is even and between a yellow is odd. so orbits of blue vertices are blue and same for yellow. assume v is blue, e an edge with endpoints v and w. for simplicity, let v be in T_g. since v is in T_g, distance between u and v, $d(v,gv) = d(v,Gv)$ and $d(w,Gv)$ is the distance between w and an orbit of v. suppose $d(v,gv) \ge d(w,Gv)$ w.l.o.g. then $d(v,gv) \le d(w,Gv) \le d(w,gv)$, $d(v,gv) \le d(w,u) + d(u,gv) \le 1 + d(v,gv)$. $d(v,gv) \le d(w,gv) \le 1 + d(v,gv)$ and since these are all integers $d(w,gv) = 1 + d(v,gv)$.
$\Rightarrow$ union of T_g is all T.

subclaim. $hT_g = T_{gh}$
if u is in T_h, $d(v,hv) \le d(v,kv)$ for any k. for gv to be in T_{gh} we need $d(gv,(gh)v) \le d(gv,kv)$ for any k. G acts on isometries so $d(gv,(gh)v) = d(g^{-1}gu, g^{-1}(gh)v) \le d(g^{-1}gu, g^{-1}ku) \Rightarrow d(u,hv) \le d(u,g^{-1}kv)$ so $d(gu,(gh)v) \le d(gu,kv)$ for any k. Thus $gT_h \subset T_{gh}$ and the other inclusion is similar.

we now have a G-tiling. write $T_0 = T_1$

Step 2: Find a generating set.

$S = \{g \text{ in } G \mid (gT_0) \cap T_0 \text{ is nonempty}\}$. we claim S is a symmetric generating set.



T_{g_i} and $T_{g_{i+1}}$ meet at one point

$T_{g_i} \cap T_{g_{i+1}} = \{s\}$

$g_i^{-1}(T_{g_i} \cap T_{g_{i+1}}) = \{g_i^{-1} \cdot s)$

$T_{g_i^{-1}g_i} \cap T_{g_i^{-1}g_{i+1}} = \{g_i^{-1}s)$

$T_1 \cap T_2 = \{g_i^{-1}s\}$ $T_0 \cap T_{g_i^{-1}g_{i+1}}$ is a single vertex. by def. $g_i^{-1}g_{i+1}$ is in S. since this holds for each i, we build g with letters from S using paths in T.

Step 3:

every spelling of g gives a path in T. $g = s_1 \cdots s_n$

this gives a 1-1 correspondence between paths in T and spellings of g

g is generated by S and S is a free graph.

Lecture 10: August 2, 2021

Free Group

recall. free group of rank n is written F_n and is the set of reduced words w/ concatenation as a product, $\{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_n, a_n^{-1}\}$ also, reduced means no subword/substring has an $a_i a_i^{-1}$ pair.

What do we know?

- every subgroup $H \leq F_n$ is free
- what if we want to know more?
- if you are given a list of k elements, do they generate a free subgroup of rank what?

example. $\{a_1, a_1^{-1}\}$ generates $\mathbb{Z}$ despite having two generators. $\{a_1, a_2, a_1a_2a_1a_1\}$ $a_1a_2a_1a_1$ is a word in $\langle a_1, a_2 \rangle$ so the group is the same as $\{a_1, a_2\}$ the free group of rank 2.

- what about $\{abab^{-1}, ab^2, bab, ba^3b^{-1}\}$ inside of $\{a, b\}$?
- Let H be the subgroup generated by the first set.
  - ↳ all reduced words in the elements and their inverses.

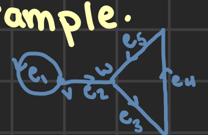
example. $(ab^2)(bab)^{-1} = (ab^2)(b^{-1}a^{-1}b^{-1}) = aba^{-1}b^{-1}$.

Systematic Way to Answer the Following:

1) What is the rank of the free group H?
2) Is $ab^2a^{-2}ba^3b^{-1}$ an element of H? Is b?
3) Does H have finite index in F_2
4) Is H a normal subgroup of F_2

Let Γ be a directed graph. Since each edge has a preferred direction, we have a starting and ending vertex. Additionally, if we go the opposite direction of the arrow on an edge, we swap the starting and ending vertices.

example.



an edge path in Γ is a string of edges $\alpha = e_0 \cdots e_k$ where ending vertex of e_i is starting of e_{i+1}

example. $e_1e_2e_3$ nonexample. e_2e_4

a path in Γ is closed if its starting and ending point are the same

example. $e_2e_3e_4e_5e_2^{-1}$

a path is tight if $e_{i-1} \neq e_i^{-1}$ at any point on a path

i.e. no back tracking example. $e_1e_2e_3$

we can tighten a path by removing all instances of ee^{-1} pairs

example. $e_1e_2e_2^{-1}e$ tightens to e_1e_1

observe that tight paths in Γ are like reduced words in a free group

Fundamental Group of a Graph

suppose Γ is a directed graph and v is a vertex in Γ. $\pi_1(\Gamma, v)$ is the set of all tight closed edge paths at v, call them loops based at v.

- the trivial edge path, not moving, the empty path ~ the empty word
- if α, β are in $\pi_1(\Gamma, v)$, the starting vertex of β is the ending vertex of α, so the concatenation $\alpha \cdot \beta$ makes senses and is a closed edge path (but not necessarily tight)

↳ $\alpha\beta$ is a loop that follows α then β, we can tighten $\alpha\beta$ to get a tightened closed path γ in $\pi_1(\Gamma, v)$.

example. $\alpha = e_1 e_1$, $\beta = e_1^{-1} e_2 e_3 e_4 e_5 e_2^{-1} \to \alpha\beta = e_1 e_1 e_1^{-1} e_3 e_4 e_5 e_2^{-1} \to \gamma = e_1 e_2 e_3 e_4 e_5 e_2^{-1}$

if we write 1 as the trivial path, then

(i) $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$ for any α

(ii) $(e_1 \cdots e_k)^{-1} \in \pi_1(\Gamma, v) = e_k^{-1} \cdots e_1^{-1}$

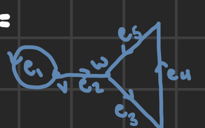
(iii) $(\alpha\beta)\gamma = \alpha(\beta\alpha)$ is shown the same way as the free group

Theorem. $\pi_1(\Gamma, v)$ is a group, called the fundamental group of Γ based at v.

Theorem. Suppose Γ is a connected graph w/ finitely many edges then for any vertex v in Γ, $\pi_1(\Gamma, v)$ is isomorphic to F_n where $n = 1 + \#\text{edges} - \#\text{vertices}$

example.

$\Gamma =$



$\pi_1(\Gamma, v) = F_{1+5-4} = F_2$

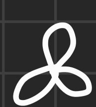
Lecture 11: August 4, 2021

The Fundamental Group

Theorem. Let Γ be a connected graph with finitely many edges. Then for every vertex, v in Γ, we have $\pi_1(\Gamma, v) = F_n$, where $n = 1 + E - V$

sketch of proof.

1) Basic example: rose of n-pedals

$R_3 =$  gives $\pi_1(R_n, v) = F_{1+n-1} = F_n$

WTS that each petal can be identified with a generator.
If F_n is generated by $a_1, \dots, a_n$ and R_n has petals $e_1, \dots, e_n$ then we send a_i to e_i.

example.

$F_3 = \langle a_1, a_2, a_3 \rangle \rightleftarrows R_3$ 

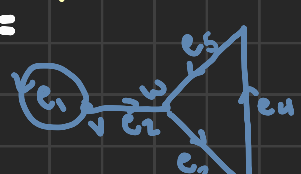
$g = a_1 a_2^{-1} a_2^{-1} a_1 a_3 a_1^{-1} \mapsto e_1 e_2^{-1} e_2^{-1} e_1 e_3 e_1^{-1}$

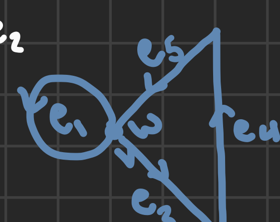
In general, a word in F_n can be written as $g = a_{i_1}^{\varepsilon_1} \cdots a_{i_k}^{\varepsilon_k}$ with a_{i_j} some generator and $\varepsilon_i \in \{1, -1\}$ maps to $\alpha = e_{i_1}^{\varepsilon_1} \cdots e_{i_k}^{\varepsilon_k}$ in $\pi_1(R_n, v)$. Reduced words without $a_t a_t^{-1}$ go to tight paths. The inverse map is given by replacing e_i's with a_i's; a path in R_n spells a word in F_n. These maps are bijections. Since reduction in F_n is the same as tightening in R_n, word structure is preserved. Which means this is an isomorphism.

2) Reduce General case to R_n

Suppose e is an edge in Γ where the starting and ending vertex are distinct, i.e. $e = \{v_1, v_2\}$ with $v_1 \neq v_2$. Define a collasped graph $\Gamma\downarrow_e$ as $V(\Gamma\downarrow_e) = V(\Gamma) - v_1$ or $V(\Gamma\downarrow_e) = V(\Gamma) - v_2$ and $E(\Gamma\downarrow_e) = E(\Gamma) - e$. WLOG take the first vertex collaspe. If e' is an edge in Γ whose initial vertex is v_1, then inside of $\Gamma\downarrow_e$ its new intial vertex is v_2.

example.

$\Gamma =$ 

$\Gamma\downarrow_{e_2}$ 

group homomorphism $\pi_1(\Gamma, v) \to \pi_1(\Gamma\downarrow_e, v)$ where we remove all instances of e and e^{-1} in closed tight paths

in $\Gamma\downarrow_e$ v_1 and v_2 are the same.

Check two things:

(i) this map is injective

any loop based at v, whenever it crosses the edge e, it must cross e^{-1} eventually. every non empty loop goes to a non empty loop so $\pi_1(\Gamma, v) \to \pi_1(\Gamma \downarrow e, v)$ is injective

(i) this map is surjective

put back in the collasped edge and follow loop

Take away, $\pi_1(\Gamma, v) \cong \pi_1(\Gamma \downarrow e, v)$ when the start and end point of e are distinct. We have removed 1-edge and 1-vertex, so $1 + E - V$ is equal for Γ and $\Gamma \downarrow e$. Repeat this process until you have a rose. $\pi_1(\Gamma, v) = \pi_1(\Gamma \downarrow e_1, v) = \pi_1(\Gamma \downarrow e_1 \downarrow e_2, v) = \ldots = \pi_1(R_n, v)$.

How to use Graphs to study subgroups

F_2 = free group of rank 2 on $\{a, b\}$

$H = \{abab^{-1}, ab^2, bab, ba^3b^{-1}\}$ = tightening loops concatenated from H

map Δ to Γ by sending each of Δ to appropriate edge of Γ. this is a homomorphism.

$p: \pi_1(\Delta, v) \to \pi_1(\Gamma, v)$

if p is injective then $p(\pi_1(\Delta, v)) = H$ and $\pi_1(\Delta, v) = F_4$ means $H = F_4$.

Q₁: How can we tell if p is injective?

Q₂: If p is not injective, can we modify Δ to make it injective.

A: Folding

idea due to Stallings

motivation: understand maps between graphs and the induced homo.

Folding

Γ-graph, e_1 & e_2 with same starting vertex.
build a new graph.

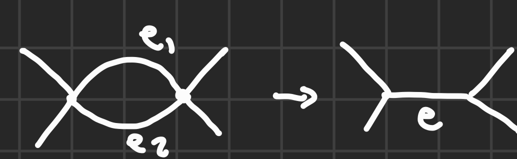
Step 1: remove e_1 and e_2 from Γ and place a new edge e that starts where e_1 and e_2 did.

step 2: edges that started at e_1 or e_2 now start at e.

two types:

1)

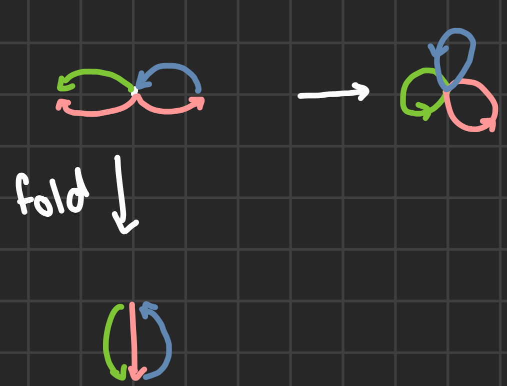
2)



Fact 1: If $\Gamma_1 \to \Gamma_2$ is a type 1 map fold, then $\pi_1(\Gamma_1, v) \to \pi_1(\Gamma_2, v)$ is an isomorphism

$\Gamma \to \Gamma_{e_1 = e_2}$ where $e_1, e_2 \mapsto e$ and everything else goes to obvious

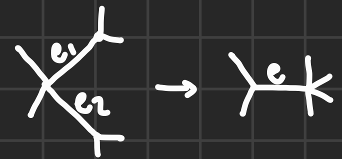
Fact 2: If $\Gamma_1 \to \Gamma_2$ is a type 2 map then $\pi_1(\Gamma_1, v) \to \pi_1(\Gamma_2, v)$ is surjective but not injective.



Lecture 12: August 6, 2021

Folding

type 1



type 2

$\Gamma \to \Gamma_{e_1=e_2}$
$e_1, e_2 \mapsto e$
everything else stays

fact. If $\Gamma \to \Delta$ is a type 1 fold then the induced homomorphism $\pi_1(\Gamma, v) \to \pi_1(\Delta, v)$ is an isomorphism. If $\Gamma \to \Delta$ is a type 2 fold then the induced homomorphism is surjective but not inj.

If $\Gamma \to \Delta$ is a map of graphs where e_1 and e_2 start at some vertex and go to the same edge in Δ, we have a factorization.

$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d] & \Delta \\ \Gamma_{e_1=e_2} \arrow[ur] & \end{tikzcd}$$

f is same as folding e_1 and e_2 and then send everything else to its original assignment.

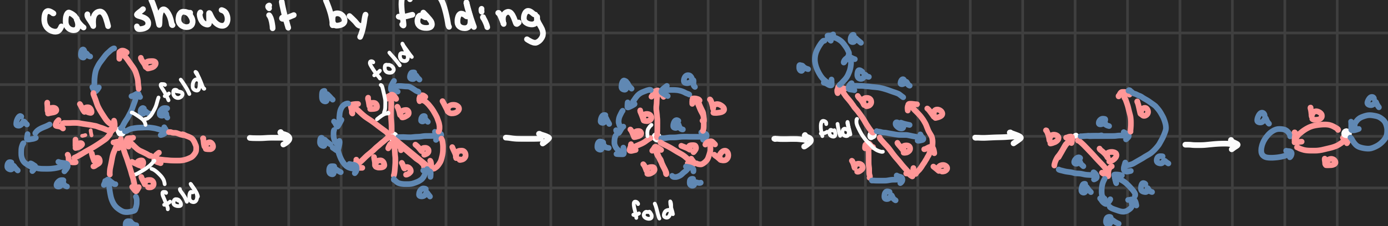
Q. What happens if we do not fold anything in $f: \Gamma \to \Delta$
↳ if e_1 and e_2 start at the same place and distinct, then so are images
↳ you can show that tight edge paths go to tight edge paths and tight loops go to tight loops

A: If we don't fold anything then the induced homomorphism is injective and we call these maps immersions

lemma. If $\Gamma \to \Delta$ is an immersion then the induced homomorphism $\pi_1(\Gamma, v) \to \pi_1(\Delta, v)$ is injective.

consequence. If $\Gamma \to \Delta$ is a map of graphs between finite graphs then there is a factorization $\Gamma = \Gamma_0 \to \Gamma_1 \to \dots \to \Gamma_k \to \Delta$ where $\Gamma_i \to \Gamma_{i+1}$ is a fold and $\Gamma_k \to \Delta$ is an immersion

example. $F_2 = \{a, b\}$, $H = \{abab^{-1}, ab^2, bab, ba^3b^{-1}\}$
we know already by previous work that it is free; however, we can show it by folding



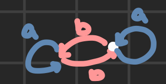
Every map was a type 1 fold, so $\pi_1(\Gamma, w) \cong \pi_1(\Delta_H, w)$ so $\pi_1(\Delta_H, w) \to \pi_1(R_2, w)$ is injective and $\mathrm{im}(\pi_1(\Delta_H, w)) \in \pi_1(R_2, w)$ is H. so $H \cong \pi_1(\Delta_H, w)$ which is the free group of rank $1+E+V$ therefore $\pi_1(\Delta_H, w) = F_3$. Hence H is a free group of rank 3 and we even have a generating set given by the last graph, $\{b^2, a, bab\}$

We can generalize this to get a new proof that each finitely generated subgroup of a free group is free.

Membership Problem

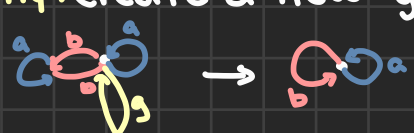
Q. Given F_2 and H. Can we tell if given an element g, whether g is in H or not?

↳ we can by looking at Δ_H.



g gives us a word with letters a, b. Start at w and label the paths given by g in Δ_H. If we get back to w then g is in H, if not then it is not.

why? Create a new graph $\Delta_{H,g}$. where we send each path to the appropriate color and we send g to tight loop labelled by spelling of g



the induced homomorphism $\pi_1(\Delta_{H,g}, w) \to \pi_1(R_2, w)$ has an image group generated by $\{b^2, a, bab, g\}$.

we now fold. If after folding we get back Δ_H. $\pi_1(\Delta_{H,g}, w)$ is equal to $\pi_1(\Delta_H, w)$ and $\mathrm{image}(\pi_1(\Delta_{H,g}, w))$ is $\{b^2, a, bab, g\}$. So the group generated by $\{b^2, a, bab, g\}$ is the same as the one generated by $\{b^2, a, bab\}$ so g was always in this group. If you end up with something else then g is not in H.