Week 05: February 13th, 2023

Section 7.1: Integration by Parts

Let's start with some integrals we already know how to solve:
First, we see an integral that we know the formula for,

$\int e^x dx = e^x + c.$

Next, we consider something more complicated with a u and a u',

$\int x e^{x^2} dx$

$u = x^2$
$du = 2x dx \Rightarrow \frac{1}{2} du = x dx$

$= \frac{1}{2}\int e^u du$

$= \frac{1}{2} e^u + C$

$= \frac{1}{2} e^{x^2} + c$

These are problems we expect you to recognize and be able to solve with relative ease and speed.

Now we look at a new type of integral,

$\int x e^{6x} dx.$

If this contained only the x or only the e^{6x} then we could solve the integral. If the integrand was xe^{6x^2} then we also could solve it. However, neither is the case and we do not have the knowledge to solve this.

We introduced a new process called integration by parts. This method resembles the undoing of the product rule.

product rule: $(fg)' = f'g + g'f$
integrate both sides: $\int (fg)' dx = \int f'g\, dx + \int g'f\, dx$
solve for what we want: $\int g'f\, dx = \int (fg)' dx - \int f'g\, dx$
simplify: $\int f \cdot g' dx = f \cdot g - \int f'g\, dx$
substitution: $\int u dv = uv - \int v du$

I commonly write this formula
$\int u \cdot dv = uv - \int v du$
where $u = f(x)$, $v = g(x)$, $du = f'(x)\,dx$, $dv = g'(x)dx$ and use the phrase "ultra violet voodoo" to help

The challenging part of this method is deciding what should be u and what should be dv It is not always clear what should be u and what should be dv and sometime we will make the wrong choice and have to start over. We know we have made the right choice when we can fill out the formula and $\int v du$ is something we can integrate. I often use the acronym LIATE (log, inverse trig, algebra, trig, exponential) to pick u.

example. Evaluate the integral $\int x e^{6x} dx$.

$\int x e^{6x} dx$

$u = x \qquad dv = e^{6x} dx$
$du = dx \qquad v = \int e^{6x} dx = \frac{1}{6} e^{6x}$

$= x \cdot \frac{1}{6} e^{6x} - \int \frac{1}{6} e^{6x} \cdot dx$

$u = 6x$
$du = 6dx \Rightarrow \frac{1}{6} du = dx$

$= \frac{1}{6} x e^{6x} - \frac{1}{6}\int e^u \cdot \frac{1}{6} du$

$= \frac{1}{6} x e^{6x} - \frac{1}{36} e^u + c$

$= \boxed{\frac{1}{6} x e^{6x} - \frac{1}{36} e^{6x} + c}$

notice that picking u=x means the x drops out and the resulting integral is something we can do. picking $u = e^{6x}$ would just result in the integral getting worse when we aim for it to go away

example. Evaluate the follow integrals using integration by parts.

(a) $\int x^3 \ln(x)\, dx$

$u = \ln(x) \qquad dv = x^3 dx$
$du = \frac{1}{x} dx \qquad v = \frac{1}{4} x^4$

$= \ln(x) \cdot \frac{1}{4} x^4 - \int \frac{1}{4} x^4 \cdot \frac{1}{x} dx$

$= \frac{1}{4} x^4 \cdot \ln(x) - \frac{1}{4}\int x^3 dx$

$= \boxed{\frac{1}{4} x^4 \cdot \ln(x) - \frac{1}{4}\left(\frac{1}{4} x^4\right) + c}$

(b) $\int \arctan(x)\, dx$

$u = \arctan(x) \qquad dv = 1\, dx$
$du = \frac{1}{1+x^2} dx \qquad v = x$

$= x \arctan(x) - \int x \cdot \frac{1}{1+x^2} dx$

$u = 1+x^2$
$du = 2x\, dx$

$= \boxed{x \arctan(x) - \frac{1}{2} \cdot \ln(1+x^2) + c}$

(c) $\int x \cos(3x+2)\, dx$

$u = x \qquad dv = \cos(3x+2)$
$du = 1\, dx \qquad v = \frac{1}{3}\sin(3x+2)$

$= x \cdot \frac{1}{3}\sin(3x+2) - \int \frac{1}{3}\sin(3x+2) \cdot 1\, dx$

$u = 3x+2$
$du = 3dx$

$= \boxed{\frac{1}{3} x \sin(3x+2) - \frac{1}{3}\left(-\frac{1}{3}\cos(3x+2)\right) + c}$

Another integration technique that arises from integration by parts (rarely) can be seen below:

example. $\int x^2\cos(x)\,dx$

$u=x^2 \quad dv=\cos(x)\,dx$
$du=2x\,dx \quad v=\sin(x)$

$= x^2\cdot\sin(x) - \underbrace{\int \sin(x)\cdot 2x\,dx}_{\text{integration by parts (again)}}$

$u=2x \quad dv=\sin(x)\,dx$
$du=2dx \quad v=-\cos(x)$

$= x^2\cdot\sin(x) - \left[x\cdot -\cos(x) - \int -\cos(x)\cdot 2\,dx\right]$

$= x^2\cdot\sin(x) + x\cos(x) - 2\int\cos(x)\,dx$

$= \boxed{x^2\cdot\sin(x) + x\cos(x) - 2\sin(x) + c}$

example. $\int e^x\cos(x)\,dx$

$u=\cos(x) \quad dv=e^x dx$
$du=-\sin(x)dx \quad v=e^x dx$

$= \cos(x)\cdot e^x - \int e^x\cdot -\sin(x)\,dx$

$u=-\sin(x)dx \quad dv=e^x dx$
$du=-\cos(x)dx \quad v=e^x$

$= e^x\cos(x) - \left[-\sin(x)\cdot e^x - \int e^x\cdot -\cos(x)dx\right]$

$= e^x\cos(x) + e^x\sin(x) - \underbrace{\int e^x\cdot\cos(x)\,dx}_{\text{this is the original integral}}$

$2\int e^x\cos(x)\,dx = e^x\cdot\cos(x) + e^x\cdot\sin(x)$

$\Rightarrow \int e^x\cos(x)dx = \boxed{\frac{1}{2}\left(e^x\cdot\cos(x) + e^x\cdot\sin(x)\right)}$

Section 7.2: Trigonometric Integrals

Let us start with an integral that we know how to do,

$\int\cos(x)\cdot\sin^5(x)\,dx$

$u=\sin(x)$
$du=\cos(x)\,dx$

$= \int u^5\,du$

$= \frac{1}{6}u^6 + c$

$= \frac{1}{6}(\sin(x))^6 + c$

This integral is easy to do with a substitution because the presence of the cosine. Let us consider it without,

$\int\sin^5(x)\,dx$

Notice that we are unable to do the u-substitution without the cosine, so we may try to reintroduce it using identities,

$\int\sin^5(x)\,dx$

$= \int\sin^4(x)\cdot\sin(x)\,dx$

$= \int(\sin^2(x))^2\cdot\sin(x)dx$ utilize $\sin^2(x)+\cos^2(x)=1 \Rightarrow \sin^2(x)=1-\cos^2(x)$

$= \int(1-\cos^2(x))^2\cdot\sin(x)\,dx$

Now that we have both sine and cosine, we can reintroduce the u-substition $u=\cos(x)$

$\int(1-\cos^2(x))^2\cdot\sin(x)\,dx$

$u=\cos(x)$
$du=-\sin(x)\,dx$

$= -\int(1-u^2)^2\cdot du$

$= -\int 1-2u^2+u^4\,du$

$= -\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]+c$

$= -\cos(x)+\frac{2}{3}(\cos(x))^3-\frac{1}{5}(\cos(x))^5+c$

Notice that this rewriting and substitution worked because the exponent was odd, one sine stays and the rest get changed. It is often good practice to separate the odd function so that we have one sine (or cosine) and the rest cosine (or sine).

example. $\int\sin^6(x)\cos^3(x)\,dx$

$= \int\sin^6(x)\cdot\cos^2(x)\cdot\cos(x)\,dx$

$= \int\sin^6(x)\cdot(1-\sin^2(x))\cdot\cos(x)\,dx$

$u=\sin(x) \quad du=\cos(x)\,dx$

$= \int u^6(1-u^2)\cdot du$

$= \int u^6-u^8\,du$

$= \frac{1}{7}u^7-\frac{1}{9}u^9+c$

$= \boxed{\frac{1}{7}(\sin(x))^7-\frac{1}{9}(\sin(x))^9+c}$

Recap: $\int\sin^n(x)\cos^m(x)\,dx$

if n is odd: remove 1 sine, substitute the rest to cosine using $\sin^2(x)=1-\cos^2(x)$, use substitution $u=\cos(x)$

if m is odd: remove 1 cosine, substitute the rest to sine using $\cos^2(x)=1-\sin^2(x)$, use substitution $u=\sin(x)$

if n and m are odd: choose the one with the smallest exponent and follow that path

Now we ask ourselves, what if m and n are even?

example. $\int \sin^2(x)\cdot\cos^2(x)\,dx$

half-angle $= \int [\frac{1}{2}(1-\cos(2x))]\cdot[\frac{1}{2}\cdot(1+\cos(2x))]\,dx$

$= \frac{1}{4}\int 1-\cos^2(2x)\,dx$

half angle $= \frac{1}{4}\int 1-[\frac{1}{2}(1+\cos(4x))]\,dx$

$= \frac{1}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$

$= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$

$= \frac{1}{4}[\frac{1}{2}x-\frac{1}{8}\sin(4x)]+c$

$= \boxed{\frac{1}{8}x-\frac{1}{32}\sin(4x)+c}$

alternatively. $\int \sin^2(x)\cdot\cos^2(x)\,dx$

$= \int (\sin(x)\cdot\cos(x))^2\,dx$

double angle $= \int (\frac{1}{2}\sin(2x))^2\,dx$

$= \frac{1}{4}\int \sin^2(2x)\,dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\,dx$

$= \frac{1}{8}\int 1-\cos(4x)\,dx$

half angle $= \frac{1}{8}[x-\frac{1}{4}\sin(4x)]+c$

$= \boxed{\frac{1}{8}x-\frac{1}{32}\sin(4x)+c}$

In both of these examples we have sine and cosine of the same angle, but what if they are different?

example. $\int \cos(15x)\cos(4x)\,dx$

$= \int \frac{1}{2}[\cos(15x-4x)+\cos(15x+4x)]\,dx$

$= \frac{1}{2}\int \cos(11x)+\cos(19x)\,dx$

$= \boxed{\frac{1}{2}[\frac{1}{11}\sin(11x)+\frac{1}{19}\sin(19x)]+c}$

Now that we have covered all of the sine/cosine cases, we next consider secant/tangent cases.

example. Evaluate the following trigonometric integrals.

(a) $\int \sec^9(x)\tan^5(x)\,dx$

$= \int \sec^8(x)\tan^4(x)\cdot\tan(x)\sec(x)\,dx$

$= \int \sec^8(x)(\sec^2(x)-1)^2\cdot\tan(x)\sec(x)\,dx$

$u=\sec(x)$ $\quad du=\sec(x)\tan(x)$

$= \int u^8(u^2-1)^2\,du$

$= \int u^{12}-2u^{10}+u^8\,du$

$= \frac{1}{13}u^{13}-\frac{2}{11}u^{11}+\frac{1}{9}u^9+c$

$= \boxed{\frac{1}{13}(\sec(x))^{13}-\frac{2}{11}(\sec(x))^{11}+\frac{1}{9}(\sec(x))^9+c}$

(b) $\int \tan^3(x)\,dx$

$= \int \tan(x)\cdot\tan^2(x)\,dx$

$= \int \tan(x)\cdot(\sec^2(x)-1)\,dx$

$= \int \tan(x)\cdot\sec^2(x)\,dx-\int \tan(x)\,dx$

$= \int \tan(x)\cdot\sec^2(x)\,dx-\int \frac{\sin(x)}{\cos(x)}\,dx$

$u=\tan(x)$, $du=\sec^2(x)\,dx$ $\quad v=\cos(x)$, $dv=-\sin(x)\,dx$

$= \int u\,du-\int \frac{1}{v}\,dv$

$= \frac{1}{2}u^2-\ln|v|+c$

$= \boxed{\frac{1}{2}(\tan(x))^2-\ln|\sec(x)|+c}$

(c) $\int \sec(x)\,dx$

$= \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx$

$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$

$u=\sec(x)+\tan(x)$
$du=\sec(x)\tan(x)+\sec^2(x)\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u|+c$

$= \boxed{\ln|\sec(x)+\tan(x)|+c}$

List of Trigonometric Identities

$\cos^2\theta+\sin^2\theta=1$

$1+\tan^2\theta=\sec^2\theta$

$\cot^2\theta+1=\csc^2\theta$

$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$

$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$

$\sin\theta\cos\theta=\frac{1}{2}\sin(2\theta)$

$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$

$\sin A\sin B=-\frac{1}{2}[\cos(A+B)-\cos(A-B)]$

Section 7.5: Partial Fraction

Let's go back to algebra for a moment. How do you combine fractions with different denominators?

examples. $\frac{4}{x-3}-\frac{1}{x+2}$

$\frac{(x+2)}{(x+2)}\cdot\frac{4}{x-3}-\frac{1}{x+2}\cdot\frac{(x-3)}{(x-3)}$

$\frac{4x+8}{(x+2)(x-3)}-\frac{x-3}{(x+2)(x-3)}$

$\frac{4x+8-x+3}{(x+2)(x-3)}$

$\frac{3x+11}{(x+2)(x-3)}$

Like many in integral sections, we ask how can we go backwards?

i.e. $\frac{3x+11}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$ where A and B are expressions that may contain x.

Partial Fraction Decomposition

The easiest way to learn this method is to see it done.

example. $\frac{5x-4}{x^2-x-2}$ first check that the numerator has lower degree than denominator, if not do long division

$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$ factor the bottom

$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$ write a partial fraction for each factor (see chart for all expansions)

$5x-4 = A(x+1) + B(x-2)$ multiple by common denominator so it disappears

method 1: roots

Plug in roots to recieve 0 terms

$x=-1:\ 5(-1)-4 = A(-1+1) + B(-1-2)$

$-9 = A\cdot 0 + B\cdot -3$

$3 = B$

$x=2:\ 5(2)-4 = A(2+1) + B(2-2)$

$6 = 3A + 0B$

$2 = A$

$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$

method 2: linear algebra

multiple out

$5x-4 = Ax + A + Bx - 2B$

separate the powers

$5x = Ax + Bx$

$-4 = A - 2B$

solve system of linear equations

$5 = A + B$ => $A = 5-B$ => $A = 5-3$

$-4 = A - 2B$ $-4 = (5-B) - 2B$ $A = 2$

$-4 = 5 - 3B$

$-9 = -3B$

$3 = B$

$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$

Key Partial Fraction Rules

This method only works for proper fractions, i.e. the largest exponent is in the denominator.

To make something a proper fraction we can utilize polynomial long division.

The bottom can be made of linear factors (e.g. $x+2$) and irreducible quadratics (e.g. x^2+4)

When you have an irreducible quadratic factor, we use the partial fraction $\frac{Ax+B}{\text{irr. quadratic}}$ (e.g. $\frac{Ax+B}{x^2+4}$)

Sometimes the factors may have exponents (e.g. $\frac{1}{(x-2)^3}$), you need a partial fraction for each exponent over 1

example. $\frac{x^2+15}{(x+3)^2(x^2+3)}$

$(x+3)^2$ has an exponent of 2 so it splits into two partial fraction $\frac{A}{(x+3)}$ & $\frac{B}{(x+3)^2}$

(x^2+3) is a quadratic so it needs a partial fraction $\frac{Cx+D}{x^2+3}$

$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{Cx+D}{(x^2+3)}$

$x^2+15 = A(x+3)(x^2+3) + B(x^2+3) + (Cx+D)(x+3)^2$

method 1: roots

there is a root at $x=-3$

$(-3)^2+15 = A(-3+3)((-3)^2+3) + B((-3)^2+3) + (Cx+D)(-3+3)^2$

$9+15 = 0 + 12B + 0$

$24 = 12B$

$2 = B$

method 2: linear algebra

the root method will not work as we can't get down to just one of A,B,C

multiple out:

$x^2+15 = A(x^3+3x+3x^2+9) + B(x^2+3) + Cx(x^2+6x+9) + D(x^2+6x+9)$

$x^2+15 = Ax^3 + 3Ax + 3Ax^2 + 9A + Bx^2 + 3B + Cx^3 + 6Cx^2 + 9Cx + Dx^2 + 6Dx + 9D$

$x^2+15 = Ax^3 + Cx^3 + 3Ax^2 + Bx^2 + 6Cx^2 + Dx^2 + 3Ax + 9Cx + 6Dx + 9A + 3B + 9D$

plug in b=2 from method 1:

$x^2+15 = Ax^3 + Cx^3 + 3Ax^2 + 2x^2 + 6Cx^2 + Dx^2 + 3Ax + 9Cx + 6Dx + 9A + 6 + 9D$

separate the powers:

$x^3 : 0x^3 = Ax^3 + Cx^3$

$x^2 : 1x^2 = 3Ax^2 + 2x^2 + 6Cx^2 + Dx^2$

$x : 0x = 3Ax + 9Cx + 6Dx$

constants: $15 = 9A + 6 + 9D$

simplify:

$0 = A + C$

$-1 = 3A + 6C + D$

$0 = 3A + 9C + 6D$

$1 = A + D$

subtract eq. 4 from eq. 2

$0 = A + C$

$-2 = 2A + 6C$

$0 = 3A + 9C + 6D$

$1 = A + D$

row2 = row2 - 2 · row1

$R_2 = R_2 - 2R_1$

$0 = A + C$

$-2 = 4C$

$0 = 3A + 9C + 6D$

$1 = A + D$

Use row 3 to solve

$C = -1/2$

$A = 1/2$

$D = 1/2$

$$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{1/2}{(x+3)} + \frac{2}{(x+3)^2} + \frac{-1/2x + 1/2}{(x^2+3)}$$

Key Denominators and their Partial Fractions

Factor in denominator	Partial fraction in decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\frac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\frac{A_1}{ax^2+bx+c} + \frac{A_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_n}{(ax^2+bx+c)^n}$