Taylor Polynomial & Taylor Series

In this section we want to represent functions with power series. Let us start with the geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r|<1$. If we take $a=1$ and $r=x$, then we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$. This means we can represent the function $f(x)=\frac{1}{1-x}$ with the power series $\sum_{n=0}^{\infty} x^n$ for $|x|<1$, we say that the function $f(x)=\frac{1}{1-x}$ has a power series representation $1+x+x^2+x^3+\dots$ centered at $x=0$ for the interval of convergence $-1<x<1$.

Taylor Series

A function $f(x)$ is said to be <u>analytic</u> if it has a convergent power series representation for each c,

$$f(x)=a_0+a_1(x-c)^1+a_2(x-c)^2+\dots+a_n(x-c)^n+\dots \quad \text{for } -r<(x-c)<r$$

where the coefficients a_i and radius of convergent r are to be determined. We call this series the <u>Taylor Series</u> of the function $f(x)$ centered at $x=c$.

Maclaurin Series.

For the special case of $c=0$, we get:

$$f(x)=a_0+a_1x^1+a_2x^2+a_3x^3+\dots \quad \text{for } -r<x<r.$$

We call this series the <u>Maclaurin Series</u> for $f(x)$ or the Taylor Series for $f(x)$ centered at $x=0$.

Derivation Taylor Series

Given a function with a power series representation about c,

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\dots$$

that has derivatives of every order that we can find. We can find the coefficients a_n.

First, evaluate every thing at $x=c$. Then $f(c)=a_0$.
So all terms except the first are zero and we now know $a_0=f(a)$.
This doesn't tell us much about the other a_i $(i>0)$. However, if we take the derivative of the function and its power series then plug in $x=c$, we get

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+\dots$$
$$f'(c)=a_1$$

We can continue this process with the second derivative

$$f''(x)=2a_2+3\cdot 2a_3(x-c)+4\cdot 3a_4(x-c)^2+\dots$$
$$f''(c)=2a_2$$

$a_3 = \frac{f'''(c)}{3\cdot 2}$

We can repeatively do this process to receive the general formula
$a_n = \frac{f^n(c)}{n!}$.

All together we have the following Taylor series for $f(x)$ about $x=c$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^4(c)}{4!}(x-c)^4 + \ldots$$

Similar to defining partial sums, the N^{th} degree Taylor polynomial of $f(x)$ is,

$$T_N(x) = \sum_{n=0}^{N} \frac{f^n(c)}{n!}(x-c)^n.$$

Note that there are Maclaurin equivalent for these when $c=0$.

We can also talk about the error between the function $f(x)$ and the N^{th} degree Taylor polynomial for a given N: $R(x) = f(x) - T_N(x)$

Examples:

1. Find the third Taylor polynomial for the function $\ln(x+2)$ centered at $x=-1$ and estimate $\ln(0.8)$.

$$T_4(x) = \sum_{n=0}^{N} \frac{f^n(c)}{n!}(x-c)^n$$

$$= \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{IV}(c)}{4!}(x-c)^4$$ T_4 = up to deg 4.

$f(x) = \ln(x+2)$	$f(-1) = \ln(1) = 0$	$(x-c) = (x-(-1))$
$f'(x) = \frac{1}{(x+2)}$	$f'(-1) = \frac{1}{1} = 1$	$= x+1$
$f''(x) = -\frac{1}{(x+2)^2}$	$f''(-1) = -\frac{1}{1^2} = -1$	
$f'''(x) = \frac{2}{(x+2)^3}$	$f'''(-1) = \frac{2}{1^3} = 2$	
$f^{IV}(x) = \frac{-6}{(x+2)^4}$	$f^{IV}(-1) = \frac{-6}{1^4} = -6$	

$$T_4(x) = 0 + \frac{1}{1!}(x+1)^1 + \frac{-1}{2!}(x+1)^2 + \frac{2}{3!}(x+1)^3 + \frac{-6}{4!}(x+1)^4$$

$$= (x+1) - \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^4$$

$$\ln(0.8) = \ln(\underset{x}{-1.2}+2) = (-1.2+1) - \frac{1}{2}(-1.2+1)^2 + \frac{1}{3}(-1.2+1)^3 + \frac{1}{2}(-1.2+1)^4$$

$T_4(x)$ is for $\ln(x+2)$

$$= (0.8) - \frac{1}{2}(0.8)^2 + \frac{1}{3}(0.8)^4$$

2. (a) Find the Taylor Series for $f(x)=e^x$ about $x=0$.

First we take note that $f^n=e^x$ and $f^n(0)=e^0=1$ for $n=0,1,2,3,\ldots$
Therefore, the Taylor series $f(x)=e^x$ about $x=0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b. Use the Maclaurin polynomial $T_4(x)$ for e^x to estimate $e^{0.2}$

$$e^x \approx T_4 = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$ ← go until this says 4 not 4 terms

$$e^{0.2} \approx T_4(0.2) = 1 + (0.2) + \frac{1}{2!}(0.2)^2 + \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 = 1.2214$$

c. Estimate the error of the estimate for $e^{0.2}$.

$$R(x) = f(x) - T(x)$$
$$= e^{0.2} - 1.2214$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}(0.2)^n - \sum_{n=0}^{4} \frac{1}{n!}(0.2)^n$$

$\sum_{n=a}^{c} x_n = \sum_{n=a}^{b} x_n + \sum_{n=b+1}^{c} x_n$ all must be x_n

$$= \sum_{n=5}^{\infty} \frac{1}{n!}(0.2)^n$$

d. Estimate the value of $\int_0^{0.2} e^{-x^2}dx$ using $T_4(x)$ for e^x.

$$\int_0^2 e^{-x^2}dx \approx \int_0^{0.} T_4(-x^2)\,dx$$

$$= \int_0^{0.2} 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 \, dx$$

$$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 \right]_0^{0.2}$$