Double Integrals - Polar

To convert from rectangular to polar:

$r^2 = x^2 + y^2$

$\alpha = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ (sometimes α is not enough)

To convert polar to rectangular:

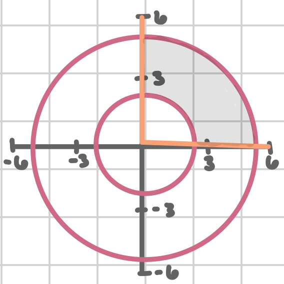
$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r \cdot dr\,d\theta$

Examples:

1. Compute $\iint_D 2xy\,dA$ where D is the region between the circle of radius 2 centered at the origin and the circle of radius 5 centered at the origin that lies in the first quadrant.



D: $2 \le r \le 5$ and $0 \le \theta \le \pi/2$

$$\iint_D 2xy\,dA = \int_0^{\pi/2}\int_2^5 2(r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_2^5 r^3\sin(2\theta)\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left[\tfrac{1}{4}r^4\sin(2\theta)\right]_2^5 d\theta$$
$$= \int_0^{\pi/2}\frac{609}{4}\sin(2\theta)\,d\theta$$
$$= \left[-\frac{609}{8}\cos(2\theta)\right]_0^{\pi/2}$$
$$= \frac{609}{4}$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

2. A variety of deep sea worm is distributed about a hydrothermal vent according to the population density $\rho(r,\theta) = \frac{800\sin^2(\theta/2)}{9+r^2}$ thousand per sq. miles where $1 \le r \le 3$ is the distance from the vent. Find the total population of the sea worms.

$\Delta P = \rho(r,\theta) \cdot \Delta A$

$P = \iint_D \rho(r,\theta) \cdot dA$

$$P = \int_1^3\int_0^{2\pi}\frac{800\sin^2(\theta/2)}{9+r^2}\cdot r\,d\theta\,dr$$
$$= 800\int_1^3\int_0^{2\pi}\underbrace{\frac{r}{9+r^2}}_{\text{constant for } d\theta}\cdot\left(\frac{1-\cos\theta}{2}\right)d\theta\,dr$$
$$= 800\int_1^3\frac{r}{9+r^2}\left[\frac{\theta-\sin\theta}{2}\right]_0^{2\pi}dr$$
$$= 800\int_1^3\frac{r}{9+r^2}\left[\pi\right]dr$$
$$= 800\pi\int_1^3\frac{r}{9+r^2}\,dr$$

$u = 9+r^2 \qquad du = 2r$

$$= 800\pi\int_{10}^{18}\frac{1}{u}\cdot\frac{1}{2}\,du$$
$$= 400\pi\,\ln|u|\Big|_{10}^{18}$$
$$= 400\pi(\ln|18| - \ln|10|)$$
$$= 400\pi\ln\left(\frac{9}{5}\right)$$

$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$