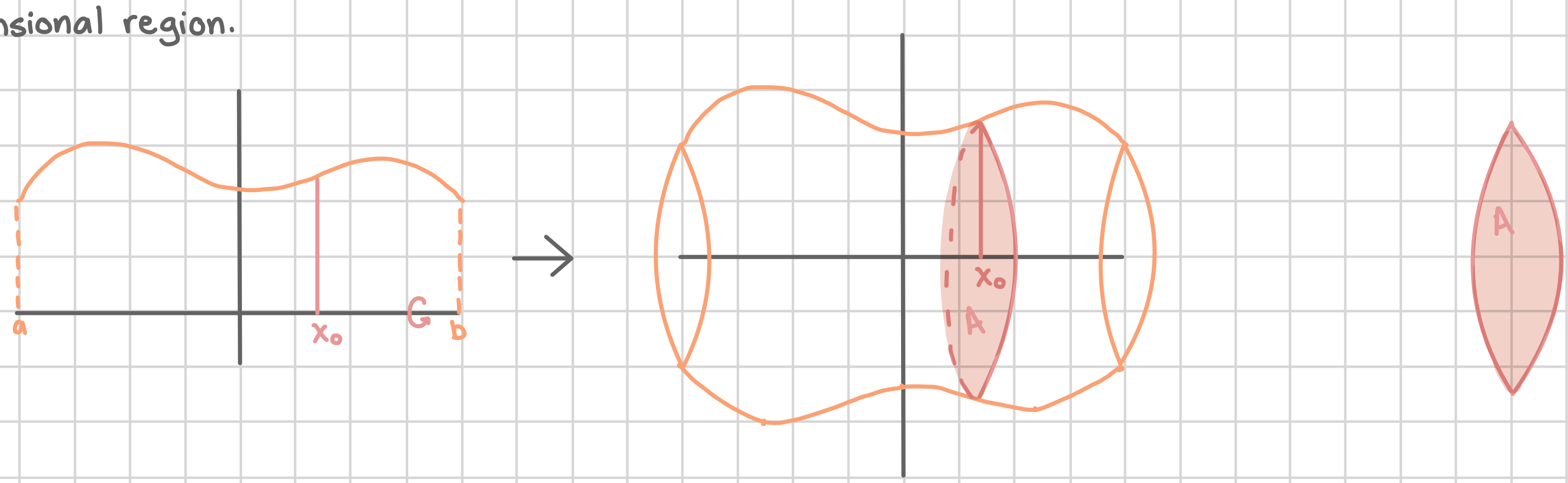
Week 04: February 9th, 2023

Section 6.3: Volumes of Revolution

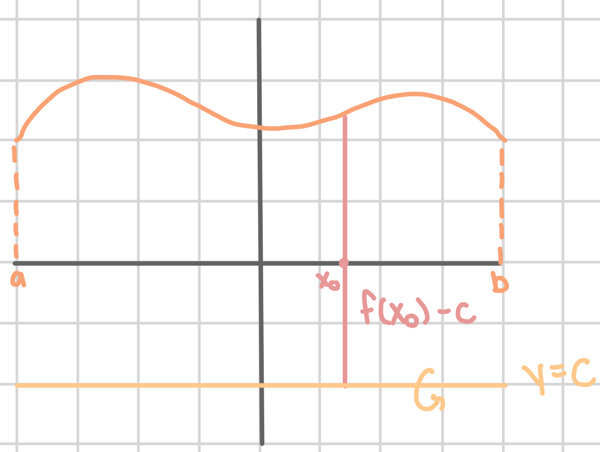
In this section we will start looking at the volume of a solid of revolution. To get a solid of revolution we start out with a function, $y=f(x)$, on an interval $[a,b]$. We then rotate this curve about a given axis to get the surface of the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.

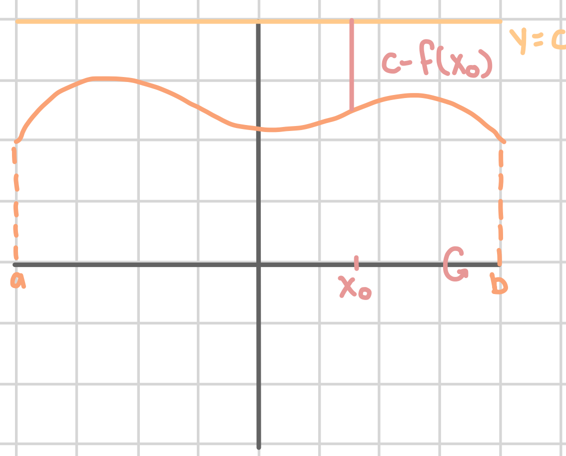


In the area and volume formulas are $V=\int_a^b A(x)\,dx$ and $V=\int_c^d A(y)\,dy$ where $A(x)$ and $A(y)$ are the cross-sectional area functions of the solid. There are many ways to get the cross-sectional area. One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross-section will be either a solid disk if the object is solid (as the example above) or a ring if we've hollowed out a portion of the solid.

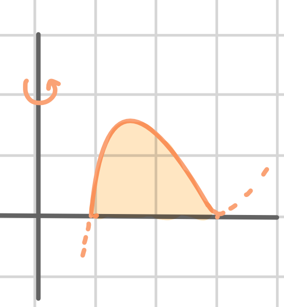
In the case that we get a solid disk the area is, $A=\pi(\text{radius})^2$ where the radius will depend upon the function and the axis of rotation. In the case that we get we get a ring the area is, $A=\pi((\text{outer radius})^2-(\text{inner radius})^2)$ where both of the radii will depend on the functions given and the axis of rotation. Note that these are the same concept as the solid disk just has inner radius 0. Whether the area is a function of x or y will depend upon the axis of rotation.

We can also find the volume of the solid of revolution from rotating the graph of the function $f(x)$ for $a\le x\le b$ about a line, like $y=c$. We utilize the formula $V=\int_a^b \pi[f(x)-c]^2\,dx$.

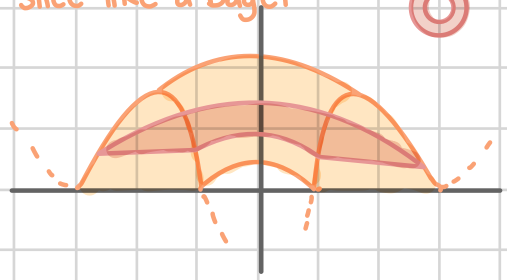




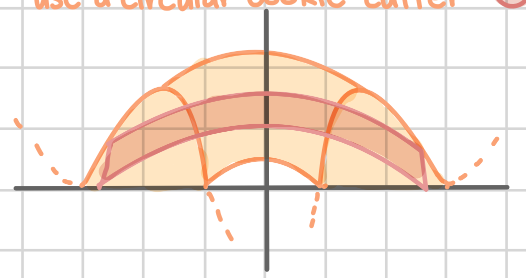
The second method is called the method of cylinders or the method of shells. In these cases we use the area formula $A=2\pi(\text{radius})(\text{height})$.



slice like a bagel



use a circular cookie cutter

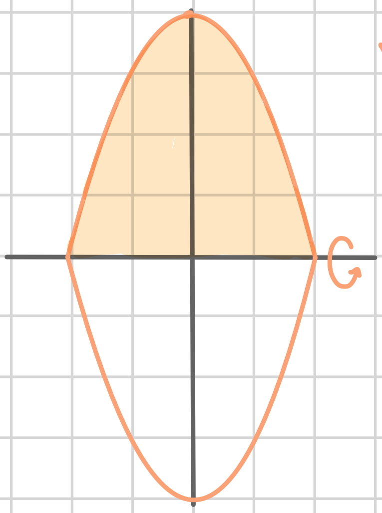


Issues with ring method:
- outer and inner radius are described by the same $f(x)$.
- hard to identify top y bound

Instead we do cylinders

example. Find the volume of the solid formed by rotating the region between the curve $y=4-x^2$ and x-axis for $-2 \le x \le 2$ about (a) the x-axis, (b) the line $y=-1$, and (c) the line $x=3$.

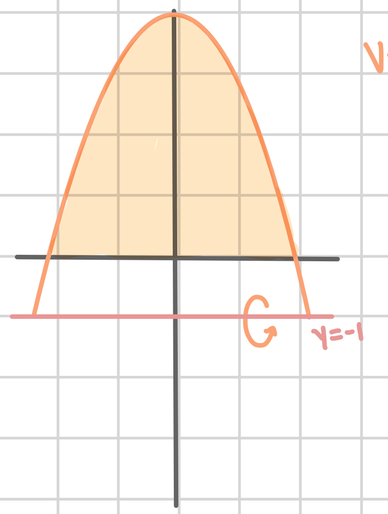
(a) the x-axis



$$\begin{aligned} V &= \int_{-2}^{2} \pi(16-8x^2+x^4)\,dx \\ &= \pi\int_{-2}^{2} 16-8x^2+x^4\,dx \\ &= \pi\left[16x-\tfrac{8}{3}x^3+\tfrac{1}{5}x^5\right]_{-2}^{2} \\ &= \pi\left[16(2)-\tfrac{8}{3}(2)^3+\tfrac{1}{5}(2)^5 - \left(16(-2)-\tfrac{8}{3}(-2)^3+\tfrac{1}{5}(-2)^5\right)\right] \\ &= \pi\left[64-\tfrac{128}{3}+\tfrac{64}{5}\right] \\ &= \pi\left[\tfrac{960}{5}-\tfrac{640}{3}+\tfrac{192}{3}\right] \\ &= \boxed{\pi\,\tfrac{512}{15}} \end{aligned}$$

$$\begin{aligned} A(x) &= \pi(\text{radius})^2 \\ &= \pi(f(x))^2 \\ &= \pi(4-x^2)^2 \\ &= \pi(16-8x^2+x^4) \end{aligned}$$

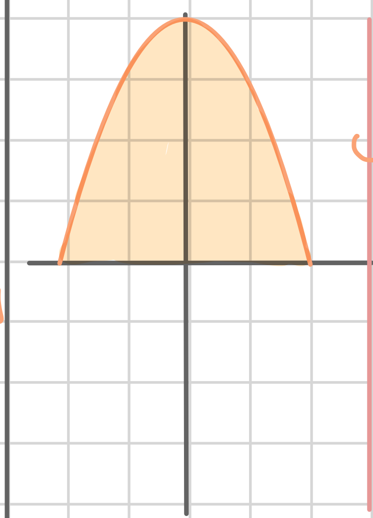
(b) the line $y=-1$



$$\begin{aligned} V &= \int_{-2}^{2} \pi(25-2x^2+x^4)\,dx \\ &= \pi\int_{-2}^{2} (25-2x^2+x^4)\,dx \\ &= \pi\left[25x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5\right]_{-2}^{2} \\ &= \pi\left[25(2)-\tfrac{2}{3}(2)^3+\tfrac{1}{5}(2)^5 - \left(25(-2)-\tfrac{2}{3}(-2)^3+\tfrac{1}{5}(-2)^5\right)\right] \\ &= \pi\left[100-\tfrac{32}{3}+\tfrac{64}{5}\right] \\ &= \pi\left[\tfrac{1500}{15}-\tfrac{160}{15}+\tfrac{192}{15}\right] \\ &= \boxed{\pi\,\tfrac{1532}{15}} \end{aligned}$$

$$\begin{aligned} A(x) &= \pi(4-x^2-(-1))^2 \\ &= \pi(5-x^2)^2 \\ &= \pi(25-2x^2+x^4) \end{aligned}$$

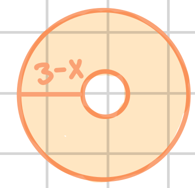
(c) the line $x=3$



height $= f(x_0)$

radius $= 3-x$

circumference $= 2\pi(3-x)$



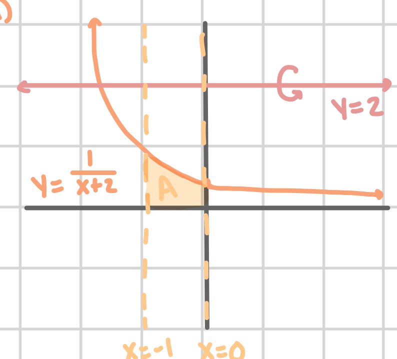
$$\begin{aligned} V &= \int_{-2}^{2} 2\pi(3-x)f(x)\,dx \\ &= 2\pi\int_{-2}^{2}(3-x)(4-x^2)\,dx \\ &= 2\pi\int_{-2}^{2}(12-3x^2-4x+x^3)\,dx \\ &= 2\pi\left[12x-x^3-2x^2+\tfrac{1}{4}x^4\right]_{-2}^{2} \\ &= 2\pi\left[12(2)-(2)^3-2(2)^2+\tfrac{1}{4}(2)^4 - \left(12(-2)-(-2)^3-2(-2)^2+\tfrac{1}{4}(-2)^4\right)\right] \\ &= 2\pi[48-16-16] \\ &= 2\pi[16] \\ &= \boxed{32\pi} \end{aligned}$$

example. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) $y=\frac{1}{x+2}$, $x=-1$, $x=0$, $y=0$; about the line $y=2$.

(b) $y=-x^2+2x$, $y=x$; about the y-axis

(a)



method 1

$V=$ volume of rotating $y=0$ − volume of rotating $y=\frac{1}{x+2}$

$$= \int_{-1}^{0} \pi(0-2)^2\,dx - \int_{-1}^{0} \pi\left(\tfrac{1}{x+2}-2\right)^2 dx$$

method 2

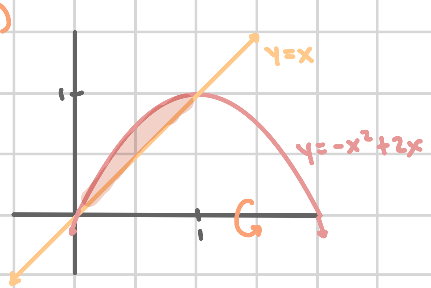
$$V = \int_a^b \pi(\text{outer}-\text{inner})^2\,dx$$

$$= \int_{-1}^{0} \pi\left(2^2-\left(\tfrac{1}{x+2}-2\right)^2\right)dx$$

outer radius is 2 becaus $y=0$ is distance 2 from $y=2$

similarly we have to subtract 2 to take into account the axis of rotation

(b)



$$\begin{aligned} V &= \int_0^1 \pi\left((-x^2+2x)^2-(x)^2\right)dx \\ &= \int_0^1 \pi\left(x^4-4x^3+4x^2-x^2\right)dx \\ &= \pi\int_0^1 x^4-4x^3+3x^2\,dx \\ &= \pi\left[\tfrac{1}{5}x^5-x^4+x^3\right]_0^1 \\ &= \pi\left(\tfrac{1}{5}-1+1\right) \\ &= \tfrac{1}{5}\pi \end{aligned}$$

Section 6.5: Work and Energy

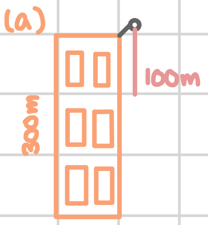
In this section we will be looking at the amount of work that is done by a force in moving an object. In physics you learn that work's formula is $W=F\cdot d$ where F is a constant force and d is the distance traveled over. However, force is not always constant.

Suppose that the force at any x is given by $F(x)$, then the work done by the force in moving the object from $x=a$ to $x=b$ is given by $W=\int_a^b F(x)\,dx$.

Notice that if $F(x)$ is a constant then $\int_a^b F\,dx = F\cdot x\Big]_a^b = F(b-a) = F\cdot d$ just as in physics.

example. A 300 kg chain 100 m in length is attached at one end to a crank on the top of a 300 m building, and the rest of the chain is allowed to hang freely on the side of the building from the crank.
(a) How much work is done when the whole chain is cranked up to the top of the tower?
(b) If a 20 kg weight is attached to the bottom end of the chain, how would the amount of work change?
(c) If a 20 kg weight is attached to the bottom end of the chain, how much work is done to crank 3/4 of the chain with the weight attached to the top of the tower?
Assume that the chain has uniform linear density. You may take the acceleration due to gravity as $g = 10\ m/s^2$.

(a)



density $= 300\text{ kg}/100\text{ m} = 3\text{ kg/m}$

$W_{x_0} =$ weight · displacement

$= (3\text{kg} \cdot 10\ m/s^2\ \Delta x) \cdot x$

$W_T = \sum_x 30x \cdot \Delta x$

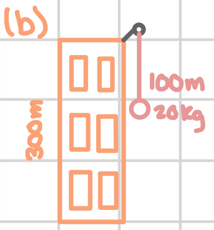
$W = \lim_{\Delta x \to 0^+} \sum 30x \cdot \Delta x$

$= \int_0^{100} 30\, x\, dx$

$= \frac{30}{2} x^2 \Big]_0^{100}$

$= 150{,}000$

(b)



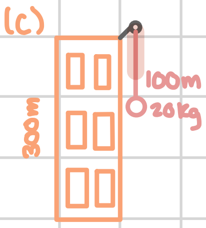
$W = W_{chain} + W_{ball}$

$= 150{,}000 + (20\text{ kg})(10\ m/s^2)(100\text{ m})$

$= 150{,}000 + 20{,}000$

$= 170{,}000$

(c)



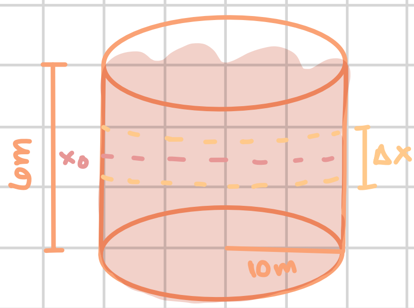
W = W(cranking chain)
+ W(displacing chain)
+ W(displacing weight)

$= \int_0^{75} 30x\, dx$

mass of 25m of chain $+ (25\text{m})(3\text{kg/m})(75\text{m})$

weight · gravity $+ (20\text{ kg})(10\ m/s)(75\text{m})$

example. A cisten is shaped like a cylinder of height 6m and radius 10 m. The circular opening (10 m in radius) of the cistern is at ground level and the rest of the cistern is buried below ground. Compute the amount of work done in pumping all the water out of the cistern from ground level if it is filled completely with water. Mass density is 100 kg/m³. You may take the acceleration due to gravity as $g = 10\ m/s^2$.



Amount of work of displacing Δx portion of water = total weight · displacement

$= (\text{volume} \cdot \text{density} \cdot \text{gravity}) \cdot (6 - x_0)$

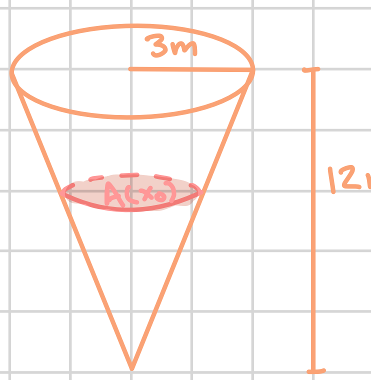
$= (100\,\pi\,\Delta x)(1000)(10) \cdot (6 - x_0)$

Total work for all water $\approx \sum_x 100\pi\,(10{,}000)(6-x)\,\Delta x$

Final answer $= \lim_{\Delta x \to 0} \sum_x 1{,}000{,}000\,\pi\,(6-x)\,\Delta x$

$= \int_0^6 1000000\,\pi\,(6-x)\,dx$

example. A tank is shaped like an inverted right circular cone of height 12m with circular opening of radius 3m. Assuming that the tank is filled halfway up with a certain kind of oil, compute the amount of work done in pumping all the oil to a level 2m above the opening of the tank. Density of the oil is 500 kg/m³. You may take the acceleration due to gravity as $g = 10\ m/s^2$.



$\int_a^b$ (Area of cross-section) · (density) · (gravity) · (displacement) · dx

$= \int_0^6 (\pi \frac{1}{16} x^2)(500)(10)(14-x)\, dx$