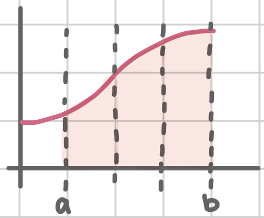
Area Between Curves

Area under a curve

Recall the Riemann sum definition of the integral:



$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

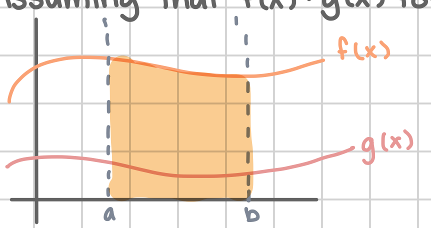
height of rectangle

width of rectangle

The integral adds up the area between the curve and the x-axis by summing up little rectangles. How might one add up the area between two curves?

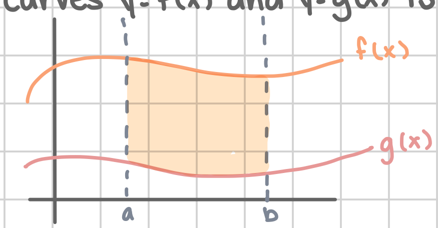
Area between curves

Assuming that $f(x) > g(x)$ for $a < x < b$, the area between the curves $y = f(x)$ and $y = g(x)$ is:



Area between the graph of f(x) and the x-axis is given by $\int_a^b f(x)\,dx$

Area between the graph of g(x) and the x-axis is given by $\int_a^b g(x)\,dx$



Area between the graph of f(x) and the graph of g(x) is given by $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Note that is important that f(x) is the "top" function and g(x) is <u>always</u> below it. For each rectangle the height has a distance of $f(x) - g(x)$.

Examples:

1. Find the area enclosed by the graphs of $y = x^3 - x$ and $y = 3x$.

Find intercepts:

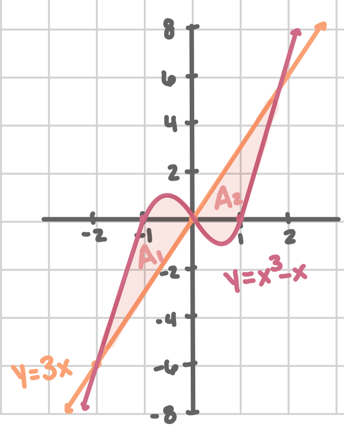
$x^3 - x = 3x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$

$x = -2, 0, 2$



Our first instict is to do:

$$\int_{-2}^{2} (\text{top} - \text{bottom})\,dx$$

But which function is the "top" function?

We run into an issue when trying to define the height of our rectangles. For some x's $y=x^3-x$ is larger than $y=3x$ and for other x values they switch. Our area formula only works when one function is always the "top" function.

The fix is to split the graph into 2 regions and integrate over each region. The first region $-2 \le x \le 0$ has $y=x^3-x$ as the "top" function and $y=3x$ as the "bottom". We can now apply the area formula to $-2 \le x \le 0$. Similarly, the second region has a "top" and "bottom" function. We can combined them to get:

parenthesis are important here

$$\begin{aligned}
\text{Area} = A_1 + A_2 &= \int_{-2}^{0} (x^3-x)-(3x)\,dx + \int_0^2 (3x)-(x^3-x)\,dx \\
&= \int_{-2}^{0} x^3-x-3x\,dx + \int_0^2 3x-x^3+x\,dx \\
&= \int_{-2}^{0} x^3-4x\,dx + \int_0^2 -x^3+4x\,dx \\
&= \left[\tfrac{1}{4}x^4-2x^2\right]_{-2}^{0} + \left[-\tfrac{1}{4}x^4+2x^2\right]_0^2 \\
&= \left[\tfrac{1}{4}(0)^4-2(0)^2\right]-\left[\tfrac{1}{4}(-2)^4-2(-2)^2\right]+\left[-\tfrac{1}{4}(2)^4+2(2)^2\right]-\left[-\tfrac{1}{4}(0)^4+2(0)^2\right] \\
&= [0-0]-\left[\tfrac{1}{4}(16)-2(4)\right]+\left[-\tfrac{1}{4}(16)+2(4)\right]-[0+0] \\
&= -[4-8]+[-4+8] \\
&= -[-4]+[4] \\
&= 4+4 \\
&= 8
\end{aligned}$$

A note on top and bottom functions:
If you ever switch your top and bottom functions you will get a negative area:

$$\begin{aligned}
A &= \int_a^b f(x)-g(x)\,dx \\
&= \int_a^b -1(-f(x)+g(x))\,dx \\
&= \int_a^b -1(g(x)-f(x))\,dx \\
&= -\int_a^b g(x)-f(x)\,dx
\end{aligned}$$

If I ever get a negative area but know that one is always the top function, then I just absolute value my answer.

Volume of a Solid with Uniform Cross-section

We can use similar reasoning to set up volume formula for shapes with uniform cross-sections (or slices). If the cross-sections are perpendicular to the x-axis, then the area of the cross-section will be functions of x determined by their 2D shape and denoted $A(x)$. The volume of such a solid will add up all of the slices along some range $a \le x \le b$ and be calculated using $V=\int_a^b A(x)\,dx$. As a Riemann sum this is $V=\sum A(x)\Delta x$ where Δx is some tiny width and $A(x)$ is the cross-section. Similarly, if the cross-sections are perpendicular to the y-axis, we get area function $A(y)$, a range of integration $a \le y \le b$, and a volume formula $V=\int_a^b A(y)\,dy$.

Example:

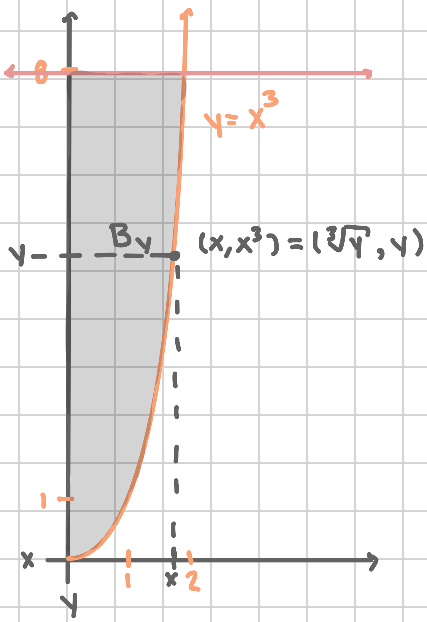
2. Consider a solid whose base is the region bounded by the lines $y = x^3$, $y = 8$, and the y-axis. Find the volume of the solid in each of the following cases:

a. The cross-sections perpendicular to the y-axis are squares.

b. The cross-sections perpendicular to the y-axis are rectangles of height $\sqrt{y}$.

c. The cross sections perpendicular to the y-axis are semicircles.

a. The cross sections perpendicular to the y-axis are squares.



$A(y)$ = Area of cross-section at y

= Area of square with side B_y

$= (B_y)^2$

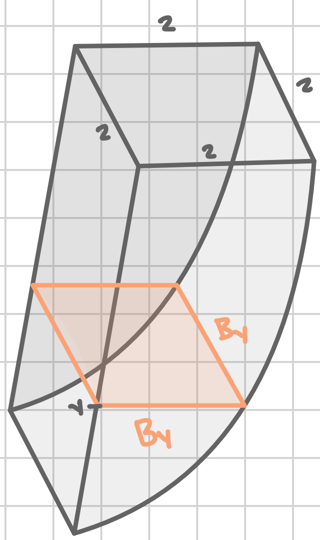
B_y = Base of cross-section when cutting at y

$= \sqrt[3]{y}$

interval of integration

"lowest y" $\leq y \leq$ "highest y"

$0 \leq y \leq 8$



$V = \int_0^8 y^{2/3}\,dy$

$= \left[\frac{3}{5} y^{5/3}\right]_0^8$

$= \frac{3}{5}\left((8)^{5/3} - (0)^{5/3}\right)$

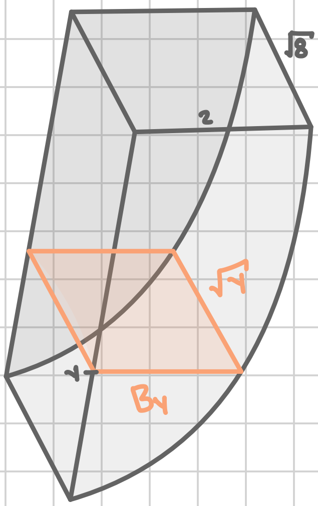
$= \frac{3}{5}(8^{1/3})^5$

$= \frac{3}{5}(2)^5$

$= \frac{3}{5}(32)$

$= \frac{96}{5}$

b. The cross sections perpendicular to the y-axis are rectangles of height $\sqrt{y}$.



$A(y) =$ Area of cross-section at y

$=$ Area of rectangle with base $\sqrt[3]{y}$ and height $\sqrt{y}$

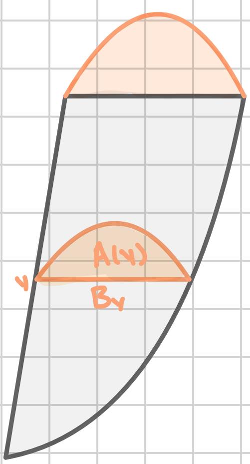
$= \sqrt[3]{y}\,\sqrt{y}$

$= y^{1/3}\,y^{1/2}$

$= y^{5/6}$

$$\int_0^8 y^{5/6}\,dy$$
$$= \frac{6}{11} y^{11/6} \Big]_0^8$$
$$= \frac{6}{11}\left(8^{11/6} - 0^{11/6}\right)$$
$$= \frac{6}{11}(8)^{11/6}$$

c. The cross sections perpendicular to the y-axis are semicircles.



$\text{Area}(y) =$ Area of cross-section at y

$=$ Area of semicircle of diameter B_x

$=$ Area of semicircle of diameter $\sqrt[3]{y}$

$= \frac{1}{2}\pi r^2$ where $r = \frac{1}{2}(\sqrt[3]{y})$

$= \frac{1}{2}\left(\pi\left(\frac{1}{2}\sqrt[3]{y}\right)^2\right)$

$= \frac{1}{2}\pi\left(\frac{1}{4}y^{2/3}\right)$

$= \frac{1}{8}\pi y^{2/3}$

$$V = \int_0^8 \frac{1}{8}\pi y^{2/3}\,dy$$
$$= \frac{1}{8}\pi\left(\frac{3}{5}y^{5/3}\right)\Big]_0^8$$
$$= \frac{3}{40}\pi\left(8^{5/3} - 0^{5/3}\right)$$
$$= \frac{3}{40}\pi(8)^{5/3}$$

Exit Ticket Inverse Trigonometric Functions

Fill in the derivatives and integrals:

1. $\frac{d}{dx}[\arcsin(x)] =$

2. $\int \frac{1}{\sqrt{1-x^2}}dx =$

3. $\frac{d}{dx}[\arctan(x)] =$

4. $\int \frac{1}{1+x^2}dx =$

Use the rules above to find the integrals below:

1. $\int \frac{1}{1+9x^2}dx$

$= \int \frac{1}{1+(3x)^2}dx$

$= \arctan(3x)\cdot\frac{1}{3} + C$

2. $\frac{d}{dx}\left[\arcsin\left(\frac{3}{4}x\right)\right]$

$= \frac{1}{\sqrt{1-(\frac{3}{4}x)^2}}\cdot\frac{3}{4}$

$= \frac{3}{4\sqrt{1-\frac{9}{16}\cdot x^2}}$

$= \frac{3}{\sqrt{4-9x^2}}$

3. $\int \frac{3}{\sqrt{9-4x^2}}dx$

$= \int \frac{3}{\sqrt{9(1-\frac{4}{9}x^2)}}dx$

$= \int \frac{3}{3\sqrt{1-(\frac{2}{3}x)^2}}dx$

$= \arcsin(\frac{2}{3}x)\cdot\frac{3}{2} + C$

4. $\frac{d}{dx}[\arctan(x^2)]$

$= \frac{1}{1+(x^2)^2}\cdot 2x$

$= \frac{2x}{1+x^4}$

5. $\int \frac{5x+1}{4+9x^2}dx$

$= \int \frac{5x}{4+9x^2} + \frac{1}{4+9x^2}dx$

$u = 4+9x^2$

$du = 18x\,dx$

$= \int \frac{5}{u}\cdot\frac{1}{18}du + \int \frac{1}{4}\cdot\frac{1}{1+\frac{9}{4}x^2}dx$

$= \frac{5}{18}\int \frac{1}{u}du + \frac{1}{4}\int \frac{1}{1+(\frac{3}{2}x)^2}dx$

$= -\frac{5}{18}\ln|u| + \frac{1}{4}\arctan(\frac{3}{2}x)\cdot\frac{2}{3} + C$

6. $\frac{d}{dx}[\arcsin(x+1)]$

$= \frac{1}{\sqrt{1-(x+1)^2}}\cdot 1$

$= \frac{1}{\sqrt{1-(x^2+2x+1)}}$

$= \frac{1}{\sqrt{-x^2-2x}}$