Project 5 FYS4150
Partial Differential equations

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1 Abstract

2 Introduction

In this project we want to study the numerical stability and errors of the forward Euler, backward Euler, and Crank-Nicolson discretization schemes. In order to do so we study the diffusion equation

3 Methods

3.1 Discretization

We wish to discretize the partial differential equation in order to create a system of linear equations which we can then solve using computationally.

We start off by discretizing the time interval such that after j time-steps the time is given by

$$t_j = j\,\Delta t \quad j \geq 0 \tag{1}$$

We do the same for the spacial interval and split it into step lengths of the same size given by

$$\Delta x = \frac{1}{n+1} \tag{2}$$

such that the position after i steps is given by

$$x_i = i\,\Delta x \quad 0 \leq i \leq n+1 \tag{3}$$

More details regarding discretization of domains can be found in chapter 10 of the lecture notes. [Hjort-Jensen, 2015] Below we apply three well-known discretization schemes to the diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0,\ x \in [0,L] \tag{4}$$

$$u_t = u_{xx} \tag{5}$$

3.1.1 Forward Euler

We discretize the time dependent part of our differential equation according to the forward Euler scheme centered at time t.

$$u_t \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t} \tag{6}$$

$$u_{xx} \approx \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2} \tag{7}$$

$$= \frac{u(x_i+\Delta x, t_j) - 2u(x_i,t_j) + u(x_i-\Delta x, t_j)}{\Delta x^2} \tag{8}$$

3.1.2 Backward Euler

$$u_t \approx \frac{u(x, t) - u(x,t-\Delta t)}{\Delta t} = \frac{u(x_i, t_j) - u(x_i, t_j - \Delta t)}{\Delta t} \tag{9}$$

$$u_{xx} \approx \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2} \tag{10}$$

$$= \frac{u(x_i+\Delta x, t_j) - 2u(x_i,t_j) + u(x_i-\Delta x, t_j)}{\Delta x^2} \tag{11}$$

3.1.3 Crank-Nicolson

The implicit Crank-Nicolson scheme with a time-centered scheme at $(x, t+\Delta t/2)$ as opposed to (x,t) for the Euler schemes.

$$u_t \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t}, \tag{12}$$

while the corresponding spatial second-order derivative is approximated as

$$u_{xx} \approx \frac{1}{2}\Bigg(\frac{u(x_i+\Delta x, t_j) - 2u(x_i,t_j) + u(x_i-\Delta x, t_j)}{\Delta x^2} + \tag{13}$$

$$\frac{u(x_i+\Delta x, t_j+\Delta t) - 2u(x_i,t_j+\Delta t) + u(x_i-\Delta x, t_j+\Delta t)}{\Delta x^2}\Bigg) \tag{14}$$

3.2

[Hjort-Jensen, 2015] [Tveito and Winther, 2009]

3.3 Implementation

All our code, calculations, and plots used can be found in my GitHub repository.

4 Results

4.1 Analytic expressions

4.2 Numerical comparison

4.3 Most likely state

4.4 Numerical studies of phase transitions

5 Conclusions

References

[Hjort-Jensen, 2015] Hjort-Jensen, M. (2015). *Computational physics*. Accessible at course github repository. 551 pages.

[Tveito and Winther, 2009] Tveito, A. and Winther, R. (2009). *Introduction to Partial Diffirential Equations*. Springer-Verlag.