ABELIAN PHASE CODING: A LINEAR PROJECTION FROM PREFRONTAL PHASE CODES TO THE ENTORHINAL GRID

ABSTRACT

To navigate complex environments, the mammalian brain integrates local sensory signals with a global internal model. In our framework, both the prefrontal cortex (PFC) and the medial entorhinal cortex (MEC) are key components of this global model. The high-dimensional phase coding in the PFC [El-Gaby *et al.*, 2024] and the two-dimensional hexagonal grid pattern in the MEC [Hafting *et al.*, 2005] together establish a cognitive map [Behrens *et al.*, 2018]. Formally, we represent each PFC working memory buffer—a cyclic process modeled as a locally compact abelian group (LCAG)—with a phase variable $z_j \in \mathbb{C}$, so that their joint state forms a complex vector $z \in \mathbb{C}^g$. By quotienting $\mathbb{C}^g$ by the lattice $\Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g$ (with $\Omega = \Omega^T$ and $\Im(\Omega) > 0$), we obtain a g-dimensional complex torus—an abelian variety [Mumford 1980]—representing the brain's model of the world.

To capture the observed 60° symmetry in grid cell firing, we aim to endow a subset of these phase variables with a structure based on a $\mathbb{Z}[\omega]$-module (with $\omega = e^{2\pi i/3}$), imposing an intrinsic hexagonal geometry on the phase space. A linear projection $\phi\colon \mathbb{C}^g \to \mathbb{C}$ maps the high-dimensional torus onto a two-dimensional lattice isomorphic to $\mathbb{Z}[\omega]$. The Riemann theta function $\Theta(z|\Omega)$—and its Fourier dual formulation as developed by Weil (1964)—localises positions within the phase space, providing an interpretation of how the brain may integrate local sensory inputs with a global model structured as an abelian variety.

While the neuroscientific motivation is clear, the mathematical formalism—especially the resonance mechanisms enforcing the $\mathbb{Z}[\omega]$ symmetry—remains heuristic. We seek mathematical insights to rigorously formalise the abelian variety structure, elucidate the role of theta functions in neural integration, and produce biologically realistic simulations.

SUPPORTING FIGURES

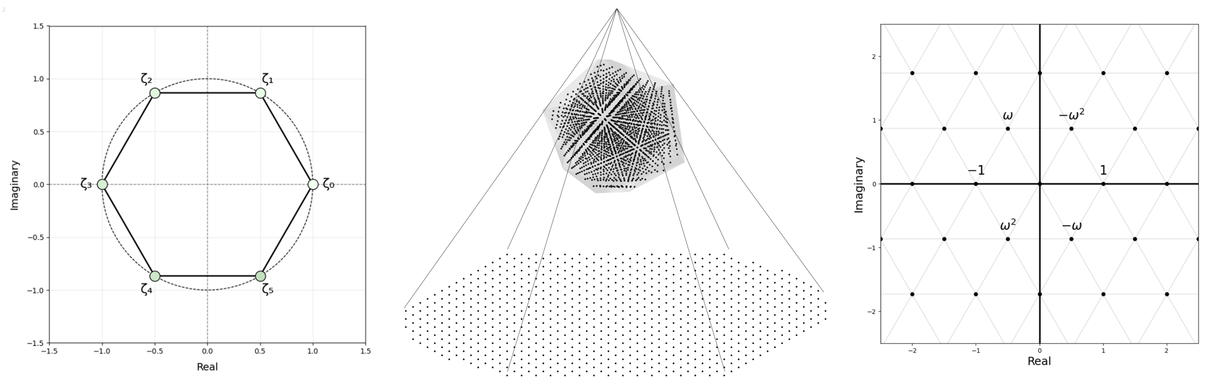

Figure 1. Sketch: If PFC cyclic buffers have a hexagonal structure in phase space (left) then a projection from a high-dimensional lattice defined by the abelian variety can linearly map onto a 2D hexagonal lattice (center). In ideal conditions, this 2D lattice is isomorphic to $\mathbb{Z}[\omega]$ (right). Note: This is just a high-level idea of what we want the model to be able to do. Defining the abelian variety, the projection, and the theta function is the end goal.

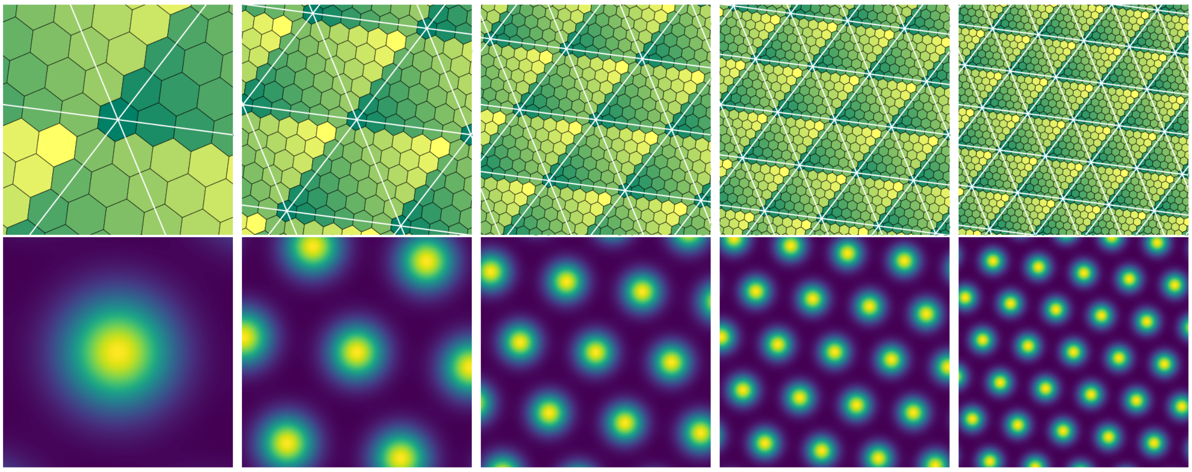

FIGURE 2. Ideal grid code based on a theta function: with coset shifts and scaling.

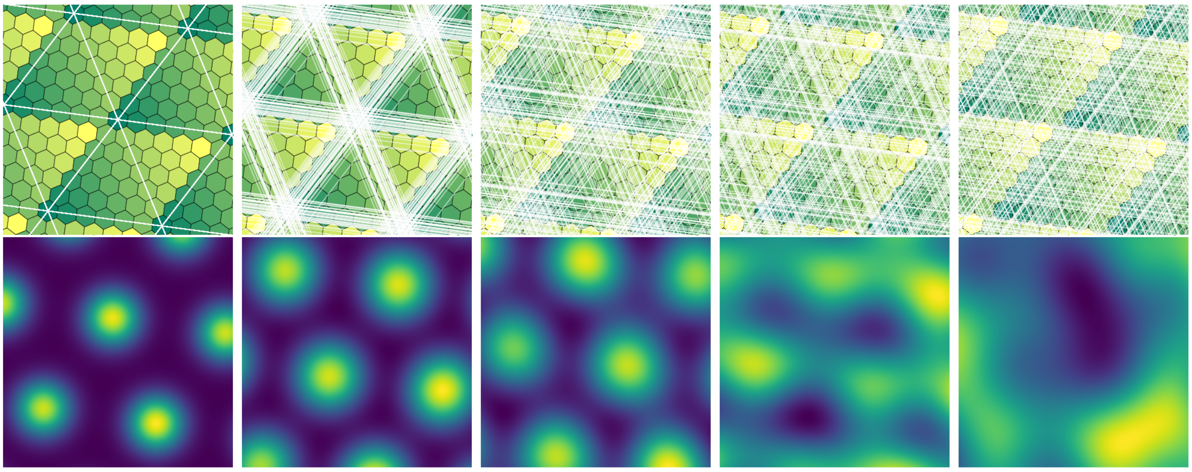

FIGURE 3. Progressive loss of grid coherence in an inexact linear projection model.

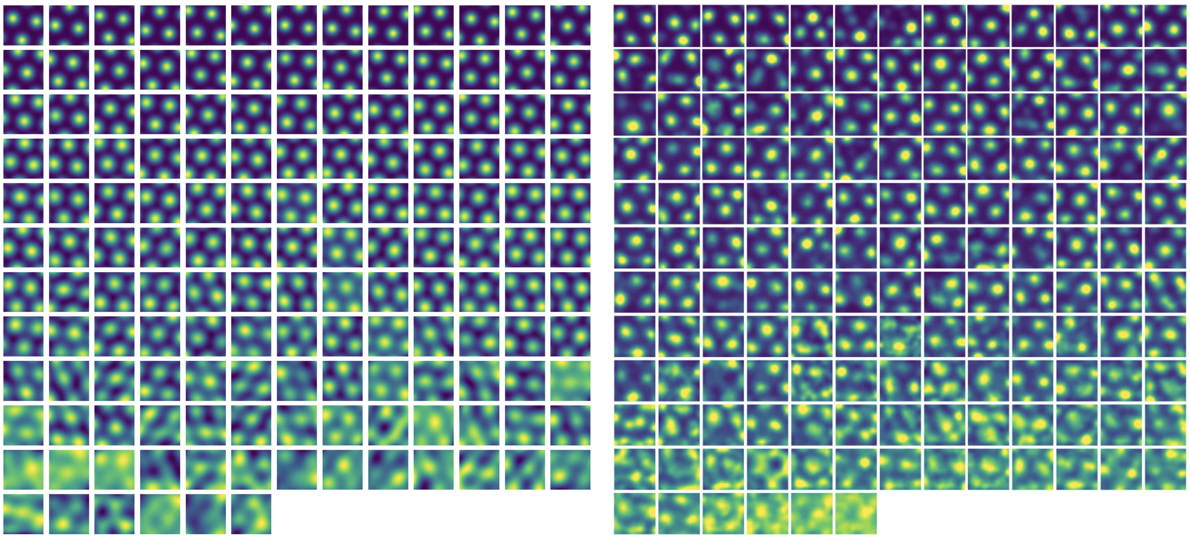

FIGURE 4. A comparison of decreasing grid coherence in our projection model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). Note: no one has modelled this coherence effect, it supports the approach.

References

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