ABELIAN PHASE CODING: A LINEAR PROJECTION FROM PREFRONTAL PHASE CODES TO THE ENTORHINAL GRID

ABSTRACT

To navigate complex environments, the mammalian brain integrates local sensory signals with a global model of the world. The prefrontal cortex (PFC) and the medial entorhinal cortex (MEC) are key components of this model. The PFC stores a high-dimensional phase code that tracks progress through abstract tasks [El-Gaby *et al.*, 2024]. The MEC stores a two-dimensional grid code that localises spatial features [Hafting *et al.*, 2005]. Together, they form a cognitive map of the agent's world [Behrens *et al.*, 2018]. Formally, we represent each cyclic buffer in PFC as a locally compact abelian group (LCAG), with a phase variable $z_j \in \mathbb{C}$ so that the joint state in PFC forms a complex vector $z \in \mathbb{C}^g$. By quotienting $\mathbb{C}^g$ by the lattice $\Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g$ (with $\Omega = \Omega^T$ and $\Im(\Omega) > 0$), we obtain a g-dimensional complex torus—an abelian variety—representing the agent's world model.

To project the PFC phase space onto the 2D hexagonal grid code in MEC, we aim to give these phase variables a structure based on a $\mathbb{Z}[\omega]$-module (with $\omega = e^{2\pi i/3}$), imposing an intrinsic hexagonal geometry on the phase space. A linear projection $\phi\colon \mathbb{C}^g \to \mathbb{C}$ then maps the PFC phase space onto a 2D lattice isomorphic to $\mathbb{Z}[\omega]$. The Riemann theta function $\Theta(z|\Omega)$—and its Fourier dual formulation developed by Weil (1964)—localises positions within the phase space, providing an interpretation of how the brain localises sensory features, and integrates them with a global model structured as an abelian variety.

While the neuroscientific motivation is clear, the mathematical formalism—especially the resonance mechanisms enforcing the $\mathbb{Z}[\omega]$ symmetry—remains heuristic. We seek mathematical insights to rigorously formalise the abelian variety structure, elucidate the role of theta functions in neural integration, and produce biologically realistic simulations.

SUPPORTING FIGURES

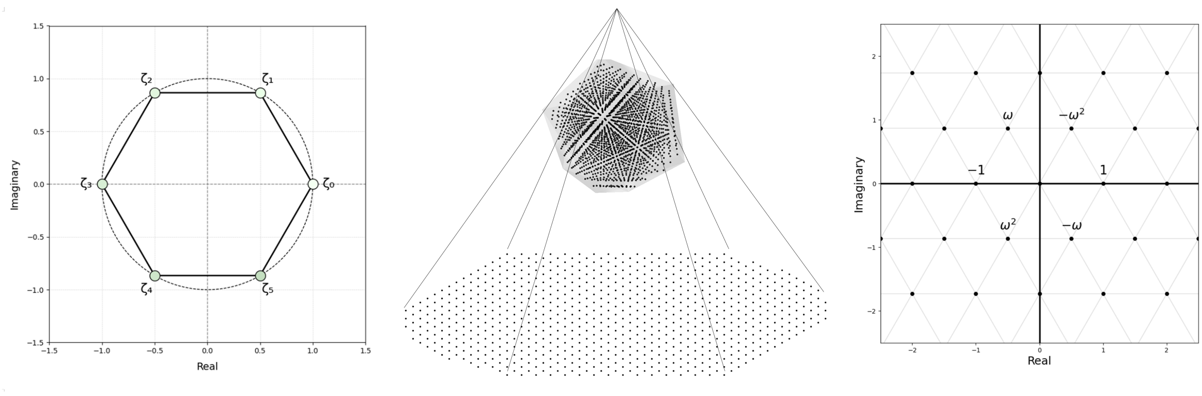

FIGURE 1. Sketch: If PFC cyclic buffers have a hexagonal structure in phase space (left) then a projection from a high-dimensional lattice defined by the abelian variety can linearly map onto a 2D hexagonal lattice (center). In ideal conditions, this 2D lattice is isomorphic to $\mathbb{Z}[\omega]$ (right). Note: This is just a high-level idea of what we want the model to be able to do. Defining the abelian variety, the projection, and the theta function is the end goal.

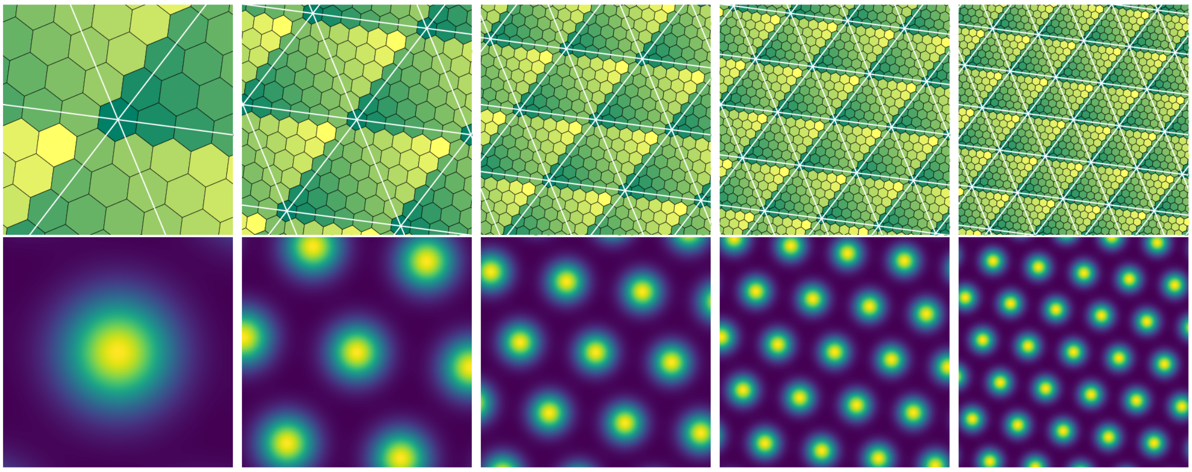

FIGURE 2. Ideal grid code based on a theta function: with coset shifts and scaling.

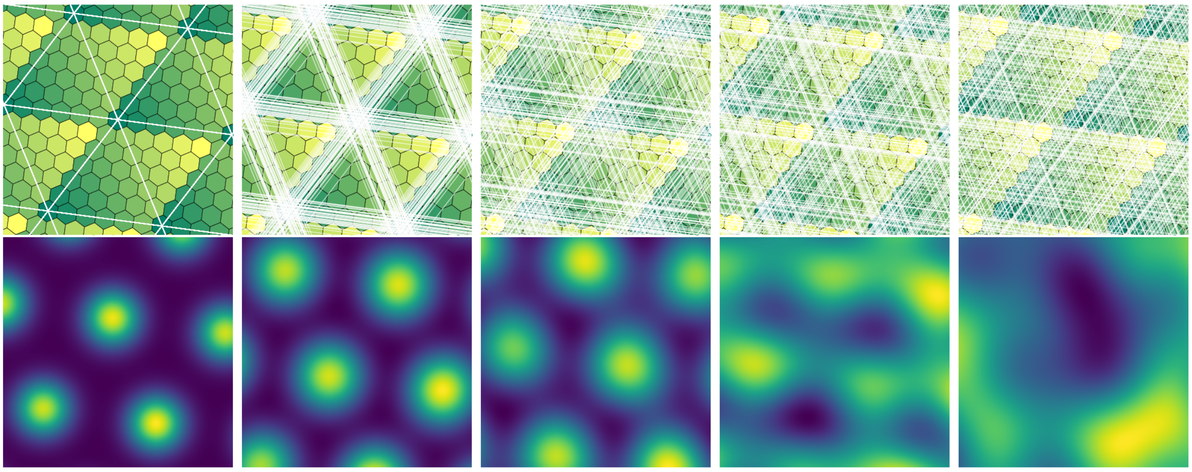

FIGURE 3. Progressive loss of grid coherence in an inexact linear projection model.

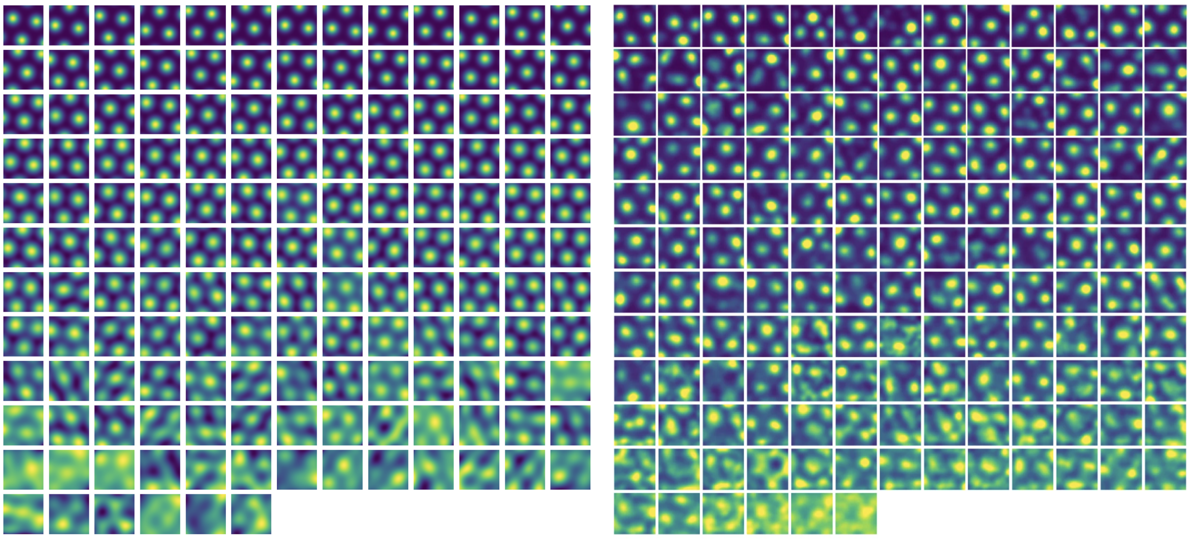

FIGURE 4. A comparison of decreasing grid coherence in our projection model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). Note: no one has modelled this coherence effect, it supports the approach.

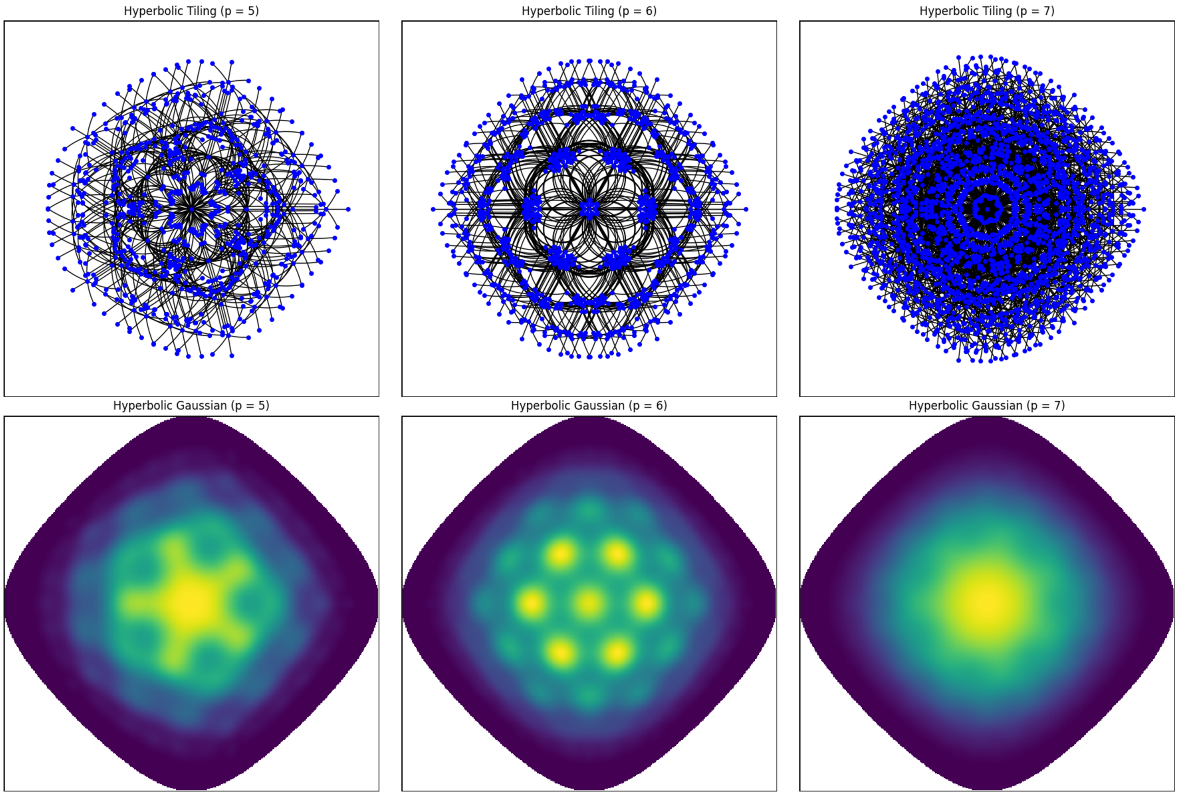

FIGURE 5. Preliminary sketches of resonance patterns in hyperbolic space. The hexagonal tiling (centre) matches empirical grid cell data suprisingly well. Notably, with smaller grids near boundaries, and a 7.5° offset from the walls. Hyperbolic space was considered as it has favourable properties for data compression. The intuition behind this approach (hyperbolic tilings in phase space) is motivated by *Esquisse d'un Programme* [Grothendieck 1984]. Ultimately, the hope is to unveil deeper links to algebraic geometry.

References

[1] **[Behrens *et al.*, 2018]**
Behrens, T. E. J., Muller, T. H., Whittington, J. C. R., Mark, S., Baram, A. B., Stachenfeld, K. L., & Kurth-Nelson, Z. (2018). *What is a cognitive map? Organizing knowledge for flexible behavior. Neuron*, 100(2), 490–509.

[2] **[El-Gaby *et al.*, 2024]**
El-Gaby, M., Harris, A. L., Whittington, J. C. R., Dorrell, W., Bhomick, A., Walton, M. E., Akam, T., & Behrens, T. E. J. (2024). *A cellular basis for mapping behavioural structure. Nature*, 636, 671–680.

[3] **[Gardner *et al.*, 2022]**
Gardner, R. J., Hermansen, E., Pachitariu, M., Burak, Y., Baas, N. A., Dunn, B. A., Moser, M.-B., & Moser, E. I. (2022). *Toroidal topology of population activity in grid cells. Nature*, 601(7893), 347–353. https://doi.org/10.1038/s41586-021-04268-7

[4] **[Grothendieck, 1984]**
Grothendieck, A. (1984). *Esquisse d'un programme* (translated and edited by L. Schneps & P. Lochak, in *Geometric Galois Actions, 1*, London Mathematical Society Lecture Note Series, Vol. 242, pp. 5–48). Cambridge University Press.

[5] **[Hafting *et al.*, 2005]**
Hafting, T., Fyhn, M., Molden, S., Moser, M. B., & Moser, E. I. (2005). *Microstructure of a spatial map in the entorhinal cortex. Nature*, 436(7052), 801–806. https://doi.org/10.1038/nature03721

[6] **[Mumford, 1983]**
Mumford, D. (1983). *Tata Lectures on Theta I.* Progress in Mathematics, Vol. 28. Birkhäuser.

[7] **[Weil, 1964]**
Weil, A. (1964). *Sur certains groupes d'opérateurs unitaires* (E. Peterson, Trans.). *Acta Mathematica*, 111, 143–211.