ABELIAN PHASE CODING: A LINEAR PROJECTION FROM PREFRONTAL PHASE CODES TO THE ENTORHINAL GRID

ABSTRACT

The mammalian brain integrates local sensory signals with a global model of the world. We propose a mathematical framework that bridges high-dimensional phase coding in the prefrontal cortex (PFC) [El-Gaby *et al.*, 2024] with the two-dimensional hexagonal grid structure observed in the medial entorhinal cortex (MEC) [Hafting *et al.*, 2005]. In our model, each PFC working memory buffer–a cyclic process, modelled as a locally compact abelian group (LCAG)—is represented by a phase variable $z_j \in \mathbb{C}$, so that the joint state forms a complex vector $z \in \mathbb{C}^g$. By quotienting $\mathbb{C}^g$ with respect to a lattice

$$\Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g,$$

where the period matrix Ω satisfies Riemann's bilinear relations ($\Omega = \Omega^T;\ \Im(\Omega) > 0$), we obtain a g-dimensional complex torus that is an abelian variety [Mumford 1983].

To capture the observed 60° symmetry in grid cell firing, we aim to endow a subset of these phase variables with a structure based on a $\mathbb{Z}[\omega]$-module (with $\omega = e^{2\pi i/3}$), imposing an intrinsic hexagonal geometry on the phase space. A linear projection $\phi\colon \mathbb{C}^g \to \mathbb{C}$ maps the high-dimensional torus onto a two-dimensional lattice isomorphic to $\mathbb{Z}[\omega]$. The Riemann theta function $\Theta(z|\Omega)$—and its Fourier dual formulation as developed by Weil (1964)—localises positions within the phase space, providing an interpretation of how the brain may integrate local sensory inputs with a global model structured as an abelian variety.

While the neuroscientific motivation is clear, the mathematical formalism—especially the resonance mechanisms enforcing the $\mathbb{Z}[\omega]$ symmetry—remains heuristic. We seek mathematical insights to rigorously formalise the abelian variety structure, elucidate the role of theta functions in neural integration, and produce biologically realistic simulations.

SUPPORTING FIGURES

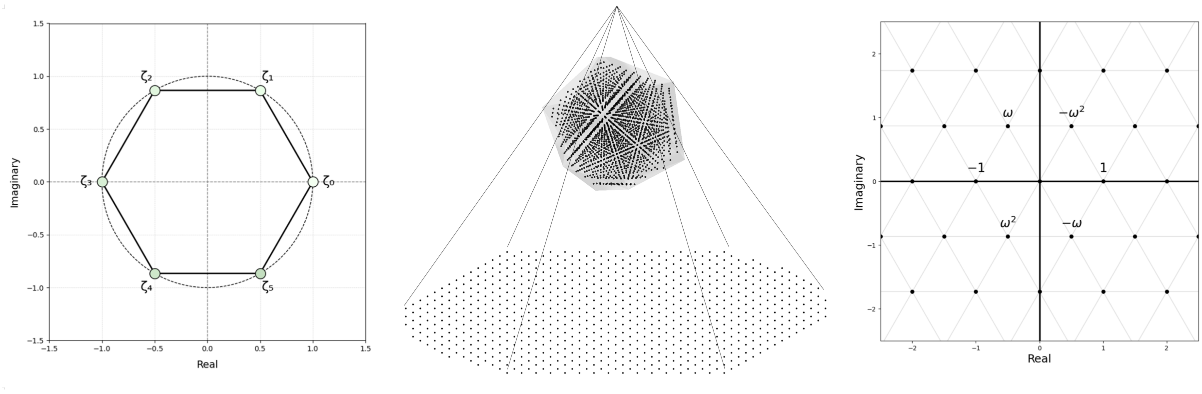

FIGURE 1. Sketch: If PFC cyclic buffers have a hexagonal structure in phase space (left) then a projection from a high-dimensional lattice defined by the abelian variety can linearly map onto a 2D hexagonal lattice (center). In ideal conditions, this 2D lattice is isomorphic to $\mathbb{Z}[\omega]$ (right). Note: This is just a high-level idea of what we want the model to be able to do. Defining the abelian variety, the projection, and the theta function is the end goal.

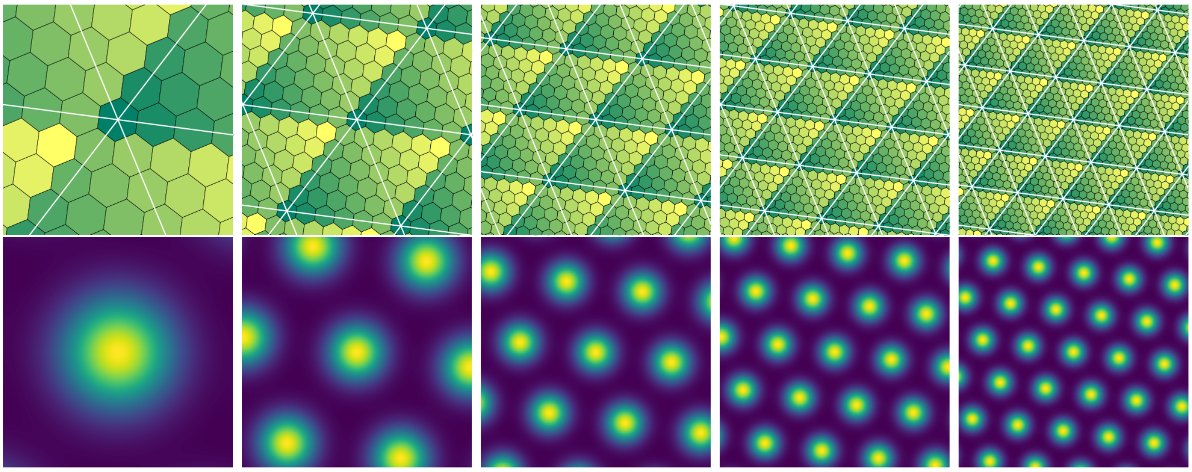

FIGURE 2. Ideal grid code based on a theta function: with coset shifts and scaling.

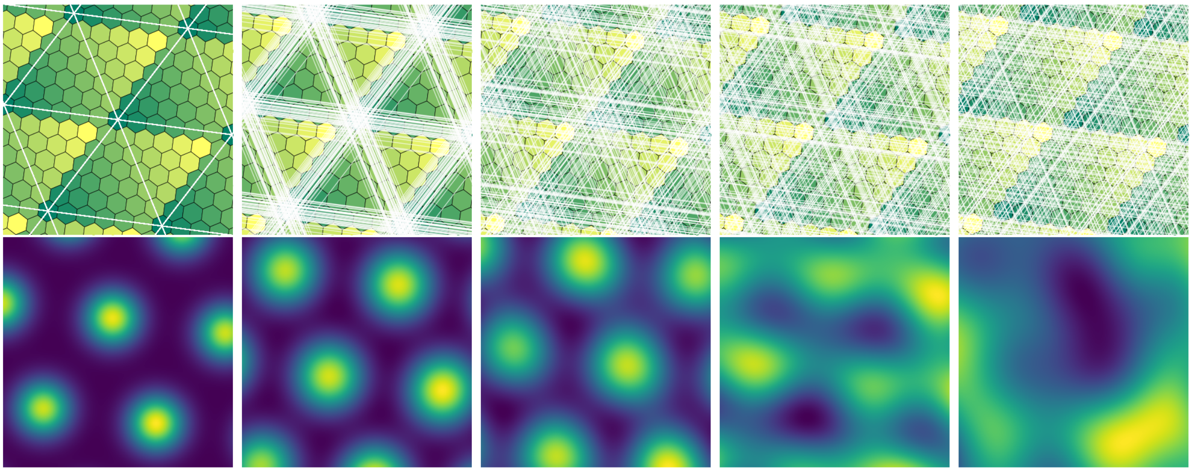

FIGURE 3. Progressive loss of grid coherence in an inexact linear projection model.

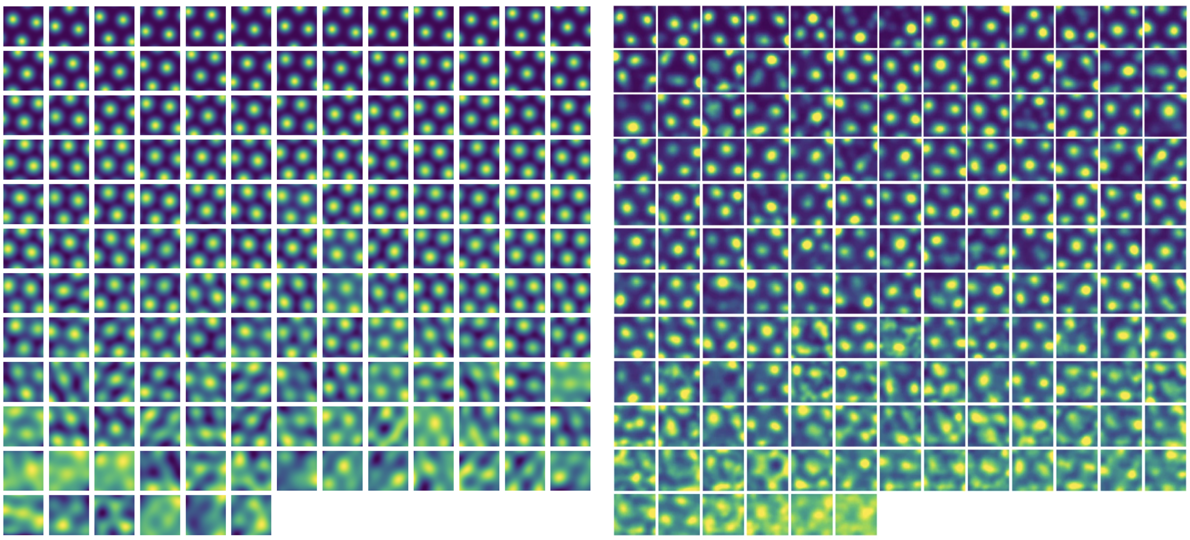

FIGURE 4. A comparison of decreasing grid coherence in our projection model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). Note: no one has modelled this coherence effect, it supports the approach.

References

[1] **[El-Gaby *et al.*, 2024]**
El-Gaby, M., Harris, A. L., Whittington, J. C. R., Dorrell, W., Bhomick, A., Walton, M. E., Akam, T., & Behrens, T. E. J. (2024). *A cellular basis for mapping behavioural structure. Nature*, 636, 671–680.

[2] **[Hafting *et al.*, 2005]**
Hafting, T., Fyhn, M., Molden, S., Moser, M. B., & Moser, E. I. (2005). *Microstructure of a spatial map in the entorhinal cortex. Nature*, 436(7052), 801–806. https://doi.org/10.1038/nature03721

[3] **[Gardner *et al.*, 2022]**
Gardner, R. J., Hermansen, E., Pachitariu, M., Burak, Y., Baas, N. A., Dunn, B. A., Moser, M.-B., & Moser, E. I. (2022). *Toroidal topology of population activity in grid cells. Nature*, 601(7893), 347–353. https://doi.org/10.1038/s41586-021-04268-7

[4] **[Mumford, 1983]**
Mumford, D. (1983). *Tata Lectures on Theta I.* Progress in Mathematics, Vol. 28. Birkhäuser.

[5] **[Weil, 1964]**
Weil, A. (1964). *Sur certains groupes d'opérateurs unitaires* (E. Peterson, Trans.). *Acta Mathematica*, 111, 143–211.