THE ENTORHINAL GRID CODE ENCODES LOCATION IN PREFRONTAL TASK SPACE

Introduction & Motivation

These are a set of notes to cover how I envision each section in the poster, and to explain the broader narrative. It's still a little rough, but the sections are as follows:

(1) Introduction
(2) Prefrontal Phase Space in Fourier Theoeretic Terms
(3) The Riemann Theta Function
(4) The τ-function as a naturalistic model of the Grid Code
(5) Using Mapper to decode Task structure From Prefrontal Phase Space
(6) Next steps: Formally linking mPFC phase space with the MEC grid code - ($\sim$*1 year*)
(7) Grid Expansion under Uncertainty [Barry et al. 2012]

1. Introduction

While it has long been known that HPC/MEC encodes spatial structure, it has recently been shown that the prefrontal cortex, and mPFC more specifically, encodes more abstract task structures [El-Gaby *et al.*, 2024]. We think it's very likely that these two sets of represention are connected, and they work together to form a cognitive map [Behrens *et al.*, 2018] that allows mammals and other intelligent agents to egnage in flexible behaviour in a complex world. Recent computational models have started to show how the mPFC task code and hippocampal place code may relate ("A Tale of Two Algorithms" [Whittington *et al.*, 2024]). In this poster, we make the claim that the MEC has a central role in how the mPFC task phase code relates to the hippocampal place code. This allows the MEC to perform vector navigation in prefrontal task space; a structured navigation between two states: $\phi : \xi \to \xi'$.

2. Prefrontal Phase Space in Fourier Theoeretic Terms

We show how mPFC cyclic buffers [El-Gaby *et al.*, 2024] can be seen in Forier theoretic terms. We show that cyclces and tasks are interconnected; this leads to a representation of task space in terms of a complex set of interconnected loops. Formally, this system of interconnected task-loops can be modelled by a period matrix Ω. This forms a high-dimension torus that algebraciy represents mPFC task space. More formally, we might instead call this mPFC *phase* space, as we expect it is not just tasks being tracked, but the dynamics of the world. The full set of these dynamics forms a predictive model of the world; they tell you how you expect the world to change over time, and what actions can direct the future state of the world during intelligent behaviour.

Experimental work shows that the prefrontal cortex (PFC) maintains several cyclic (or ring) buffers [El-Gaby et al. 2024]. We can model each as tracking a phase variable $z_j \in \mathbb{C}$. Combining g such buffers yields a high-dimensional phase space $\mathbb{C}^g$, where interactions among these modes are captured by a lattice Λ generated from a period matrix Ω. Formally,

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1g} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{g1} & \omega_{g2} & \cdots & \omega_{gg} \end{bmatrix}, \quad \Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g. \tag{1}$$

The way to read this is to think of how tasks and goals (and the dynamics of the world more generally) are inter-related. We don't focus on tasks in the world one at a time: as I write a paper, I also get closer to conference goals, funding goals, closer to bedtime, etc. The period matrix Ω captures the algebraic structure of the phase space that you navigate through life.

Identifying points in $\mathbb{C}^g$ modulo this lattice Λ yields a g-dimensional *complex torus*, as 'illustrated' in Figure 1. In simple terms, you are always at some vector (z - see §3) in the phase space $\mathbb{C}^g/\Lambda$, as identified by the set of active phase cells across all prefrontal buffers.

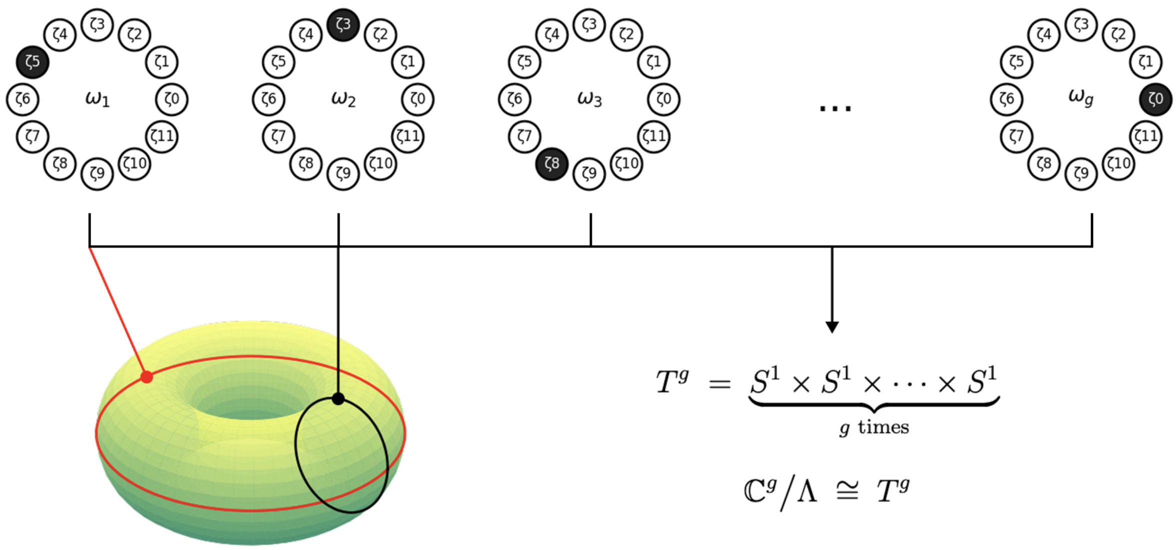

FIGURE 1. The Cartesian product of g circles (each representing a prefrontal cyclic buffer) forms a g-dimensional complex torus T^g, which is isomorphic to $\mathbb{C}^g/\Lambda$ for some g-dimensional lattice Λ. (e.g., you can see how two prefrontal buffers identify a point on a 2-torus.). A torus is probably an idealisation of the actual biological reality, but you get the idea of a big inter-connected space of loops defined by the period matrix Ω and how the different mPFC buffers relate to each other - or, progress over time.

3. The Riemann Theta Function

Location in this high-dimension mPFC phase space can naturally be localised by the Riemann theta function $\Theta(z|\Omega)$. We keep the maths here light, but we note the centrality of the Riemann theta function in modern mathematics. In essence, the Riemann theta function $\Theta(z|\Omega)$ allows us to localise position (z) in the prefrontal phase space structured by (Ω). The period matrix Ω defines a big space of loops (a high-dimensional torus), and it is the *topology* of this space of loops that we will ultimately use to connect MEC to PFC in terms that allow us to make experimental predictions and falsify this theory. The end goal is to show how the structured system (Ω) allows you to use the MEC to vector navigate in the mPFC phase space. MEC tracks where you are *locally* in the big space of loops, in a way that the respects the physical geometry of your local environment from HPC: see next: §4.

Formally linking the cognitive map to the Riemann theta function would be a *highly* significant step in linking modern neuroscience with modern pure mathematcs. If true: *it is hard to over-state how big this could be.* It opens up a wide range of possible collaborations between the two fields, and could be very exciting direction in modern mathematics/science.

4. The τ-function as a naturalistic model of the Grid Code

While the the Riemann theta function itself is highly-structured, we show how it relates to the τ-function, which includes additional non-linearities that we naturally expect in a representation of the world. The τ-function provides a direct solution to shallow water waves; using this fact, we can easily model wehere grid cells form in complex environments: just fill the room with water! We discuss coherence effects and provide 3d Renders of the grid code, see: (https://webgpu.aufbau.io/experience/gridcode/) - which provides a model of the grid code using the τ-function. Lots of nice 3D graphics...

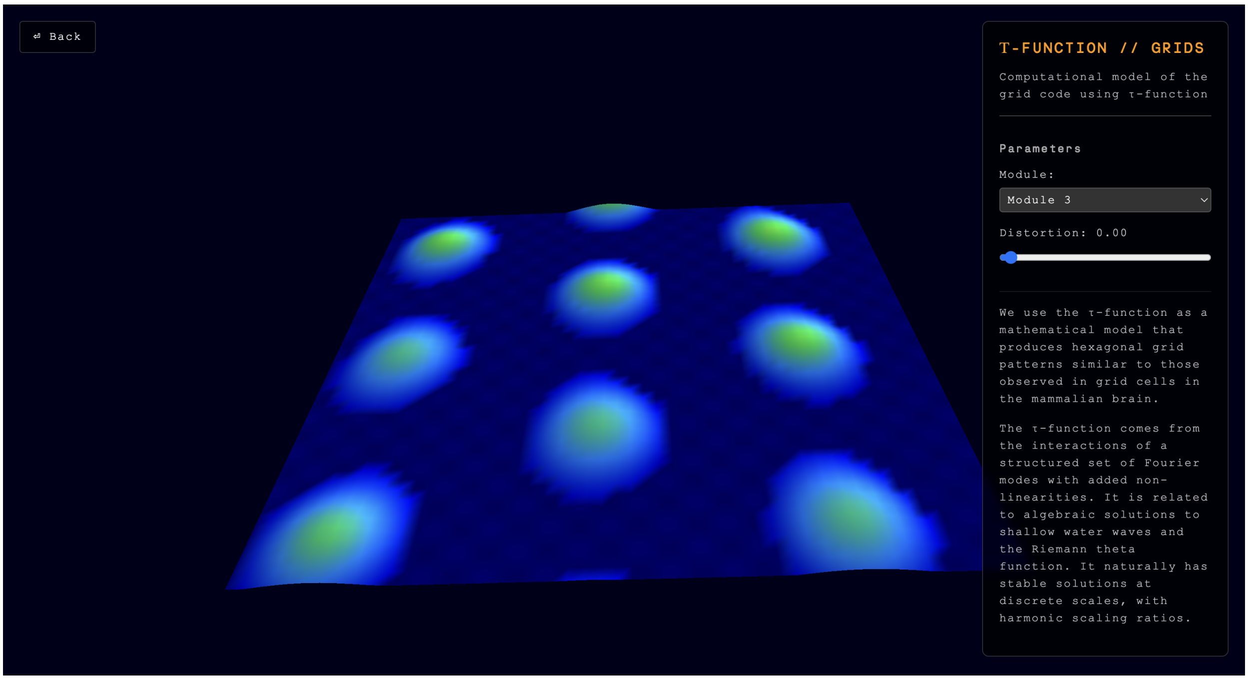

Figure 2. I only finished the tool late last night before writing this, check it out for yourself. I think it can generate a lot of neat 3D graphics. While it still needs some tweaking, it nicely captures wuite organic looking grid cells?

The τ-function forms the basis of linking tht topological phase-spaace information in mPFC with the local representations in MEC. Working out what exactly is happening here is non-trivial (see §6), but we expect individual grid cells to be encoding local aspects of the mPFC topology, such that the full grid code can be used to navigate mPFC phase space in a way that respects the geometry of your environment - the space within which your tasks live.

Below are a couple additional figures (we can maybe remove them) that explore other aspects. [Figure 3] captures an idea of how we can use shallow water waves (and thus the τ-function) to model how the grid code fills complex environments. [Figure 4] shows how adding an incoherence metric (adding noise in the wave equation) captures actual experimental data suprisignly well.

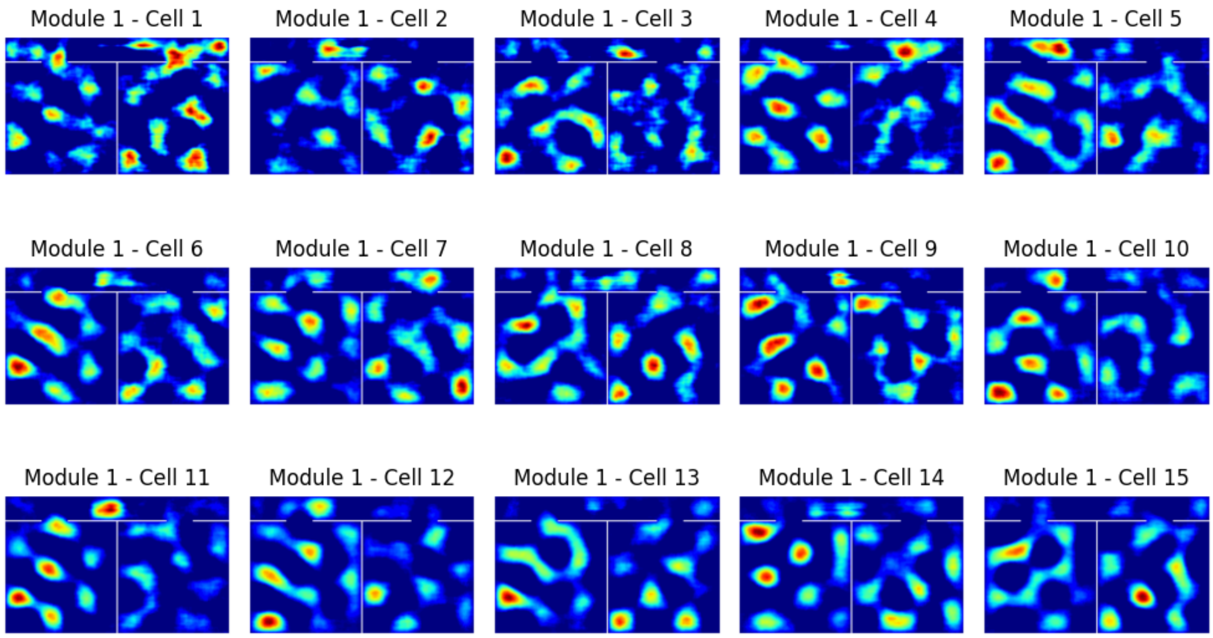

FIGURE 3. While I need to improve the rendering here (eg. this is not hexagonal or structured enough as the ones above), the τ-function produces a scalar wave field that naturally mirrors how grid cells tile complex environments. Crucially, it does so in a way that produces phase offset grids, which other current methods [Dordek *et al.*, 2016; Stachenfekd *et al.*, 2017] do not. It is still Lalcacians under the hood, but the Fourier angle add a dynamic edge that tiles the space more fluidly in a way that *seeems* to more realistic to how MEC the grid code looks and how it fills the local environment.

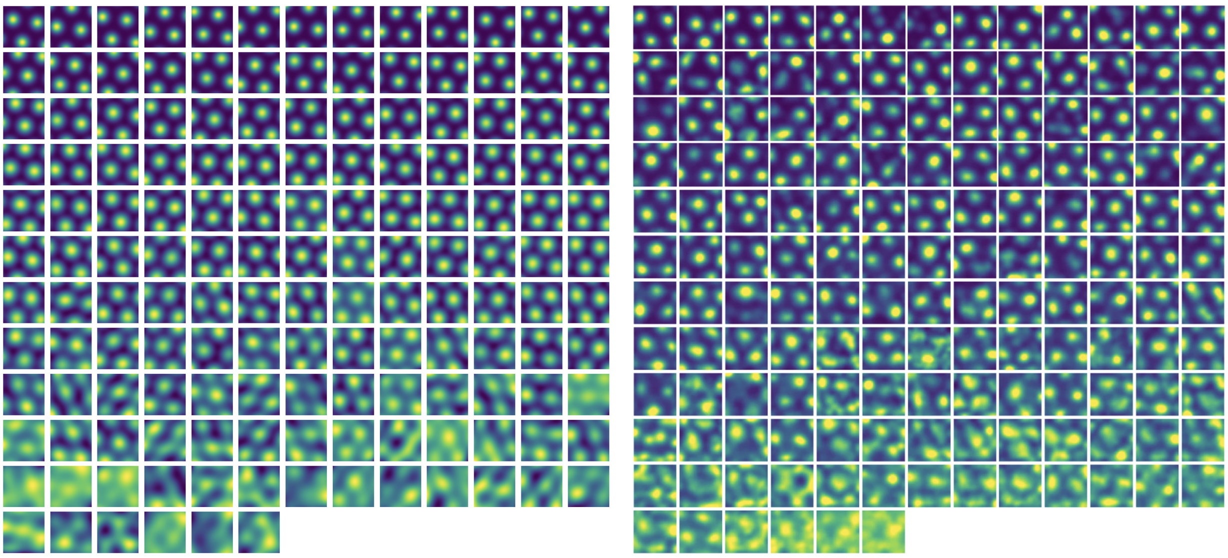

FIGURE 4. A comparison of decreasing grid coherence in our model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). It already looks good, but I still need to order my cells by spatial information as the Mosers did, and capture the non-linearities a little better. Maybe we can ditch this figure, it tells a story that confuses the 'Prefrontal Vector Navigation' one.

5. Using Mapper to decode Task structure From Prefrontal Phase Space

There is a growing branch of applied mathematics (starting somewhere in the early 2000s) called 'Topological Data Science' (TDA). TDA is behind recent papers such as the Toroidal grid cell paper [Gardner *et al.*, 2022]. We use an algorithm from TDA (Mapper) that allows us to decode a graph of mPFC phase space structure from neural data. Essentially, it looks for cycles, and builds a graph (a *Reeb graph*) that captures the full set of loops in the data. The Reeb graph essentially decodes the topology it finds in the dataset that it is given. We ran Mapper on the data from El-Gaby *et al.* (2024) and found some very simple cyclical topology. This was somewhat expected, as the animals were given very simple cyclic tasks.

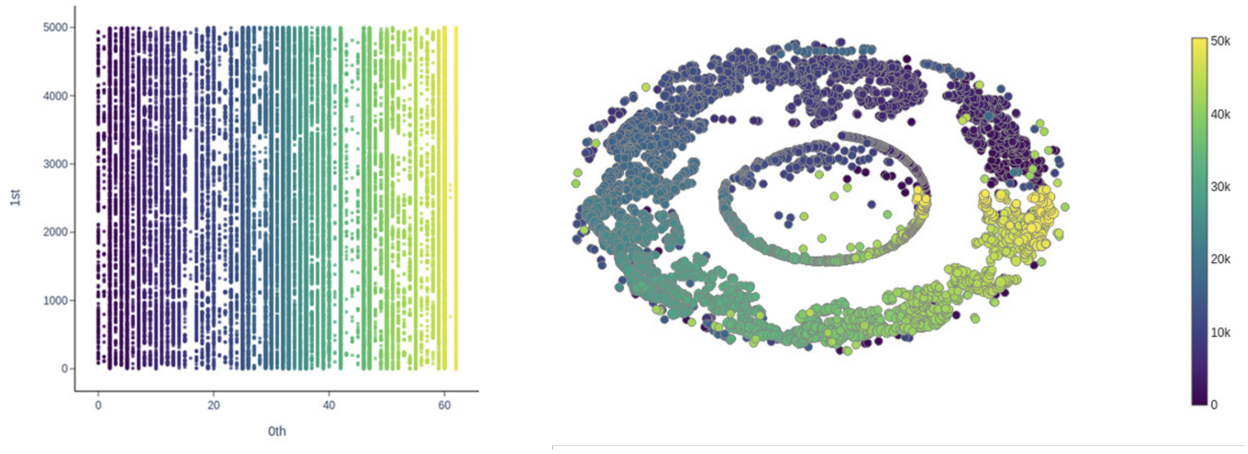

Figure 5. On the left we have spikes from the actual mPFC data in EL-Gaby *et al.* (2024) We ran Mapper on this data and found two cycles. This is natural, as the mice had two cyclical tasks to do. I left Miguel to do this by himself, so he only did some rough preliminary analysis. Once working out the experimental data from myself a bit better, I expect we can produce something cooler. Especially looking at the global structure of mPFC space.

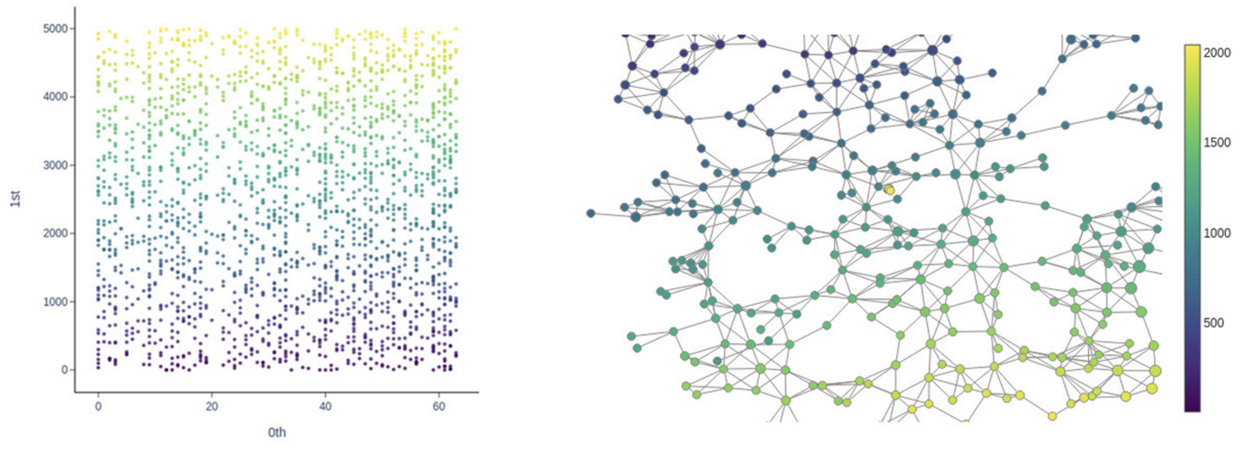

Figure 6. We then generated data from a (*made-up*) more abstract task space. Running mapper on this data allowed us to deconstruct the topology of this task space. Given a suitable experimental setup, this should allow us to decode task-space/phase-space strucre from future mPFC experiments with more complex task structures. This forms an experimental basis for decoding topological information from mPFC. These methods give experimentalists useful tools for getting data about the topology of the spaces stored in mPFC.

6. Next steps: Formally linking mPFC phase space with the MEC grid code

This is the crux of the paper and it ultimately comes down to 'do we publish this now to get feedback', or 'do we wait until we have a more complete model'. The local representations in MEC (from the τ-function) derive from the Rieman Theta Function $\Theta(z|\Omega)$ that localises positon in a big space of loops structured by the period matrix Ω. Up til now, we have described more of a theoretical bridge, based on the two being based on a shared topology - albeit a topology that is merged with local spatiak data to allow MEC vector navigation through mPFC task space that respects local environemntal structure. Our model complements the only other model of this ("A Tale of Two Algorithms" [Whittington *et al.*, 2024]) while aiming to extend the framework by including MEC and showing its centrality in the formal link. This is not a critque of that paper, but an additional view on the connection.

If representations in MEC/mPFC are actually related, we should be able to make experimental predictions, e.g. co-recording mPFC data and MEC data in less trivial spatial and environmental setups and comparing the topological information decoded from either set of representations. if the two systems are connected, there should be some decent form of overlap in the topological information decoded here. Acutually working out methods for doing this isn't that trivial, and is probably a years worth of work. Luckily, the maths we are working with *excites* mathematicians in Leipzig/Dresden at the MPIs, so getting help on the more complex parts - and then making them 'simple' - should be fairly straightforward.

I think that getting feedback at this midpoint is useful. I think the chance of getting scooped is pretty low. But ultimately this is your choice. In this doc I worrry that I have too many angles, so it may be on you to help me narrow it down.

7. Grid Expansion under Uncertainty [Barry *et al.* 2012]

Finally, I completely skipped out the 'grid expansion in uncertainty angle [Barry *et al.* 2012]. The Riemann theta function explains the funtional role of this expansion perfectly, but this angle should probably be ditched from the poster and just be kept for the poster; in breif:

There is a neat extension of the theta function [Weil 1964] that rephrases it in Fourier dual terms. In Weil's formulation, you implicitly have co-ordinates in a dual space: a local direct space, and the global Fourier dual space. For us, that means you can simultaneously index local information (i.e. HPC) and global phase information (i.e. PFC) at the same time. The MEC (or the theta function) then acts as the bridge that formally links the two domains and localises position within the phase space. The theta function integrates local information with respect to global symmetries structured via a lattice. The scale of the lattice dictactes which domain you are interating over; a finer lattice is integration over local information in the direct space, a coarser lattice is integration over global information in the dual space.

While we expect the grid expansion to relate to basic Hebbian mechanisms (i.e. when the MEC model fails to agree with HPC data, the grid coherence decreases and the grid code falls into a stable state with a coarser grid.). In formal terms though (e.g using the ideas of Marr's levels, that different levels explain different aspects of neural computation), then during uncertainty, a coarser lattice means you integrate more over the global information stored in mPFC than the local information in HPC. I anticipate this would correlate with an increase in low-theta band phase coupling between MEC-PFC during these periods of uncertainty. It would be interesting to look at the data, I have no time, maybe a student?

In effect, during uncertianty you resort to big picture narratives (mPFC global model) to work out "what the hell is going on", then once you've figured this out, the grid code naturally becomes finer, and you start focussing on local details again (HPC local information).

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