# ABELIAN PHASE CODING: A LINEAR PROJECTION FROM PREFRONTAL PHASE CODES TO THE ENTORHINAL GRID


ABSTRACT

To navigate complex environments, the mammalian brain integrates local sensory signals with a global model of the world. The prefrontal cortex (PFC) and the medial entorhinal cortex (MEC) are key components of this model. The PFC stores a high-dimensional phase code that tracks progress through abstract tasks [El-Gaby *et al.*, 2024]. The MEC stores a two-dimensional grid code that localises spatial features [Hafting *et al.*, 2005]. Together, they form a cognitive map of the agent's world [Behrens *et al.*, 2018]. Formally, we represent each cyclic buffer in PFC as a locally compact abelian group (LCAG), with a phase variable $z_j \in \mathbb{C}$ so that the joint state in PFC forms a complex vector $z \in \mathbb{C}^g$. By quotienting $\mathbb{C}^g$ by the lattice $\Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g$ (with $\Omega = \Omega^T$ and $\Im(\Omega) > 0$), we obtain a $g$-dimensional complex torus—an abelian variety—representing the agent's world model.

To project the PFC phase space onto the 2D hexagonal grid code in MEC, we aim to give these phase variables a structure based on a $\mathbb{Z}[\omega]$-module (with $\omega = e^{2\pi i/3}$), imposing an intrinsic hexagonal geometry on the phase space. A linear projection $\phi\colon \mathbb{C}^g \to \mathbb{C}$ then maps the PFC phase space onto a 2D lattice isomorphic to $\mathbb{Z}[\omega]$. The Riemann theta function $\Theta(z|\Omega)$—and its Fourier dual formulation developed by Weil (1964)—localises positions within the phase space, providing an interpretation of how the brain localises sensory features and integrates them with a global model structured as an abelian variety.

While the neuroscientific motivation is clear, the mathematical formalism—especially the resonance mechanisms enforcing the $\mathbb{Z}[\omega]$ symmetry—remains heuristic. We seek mathematical insights to rigorously formalise the abelian variety structure, elucidate the role of theta functions in neural integration, and produce biologically realistic simulations.


SUPPORTING FIGURES

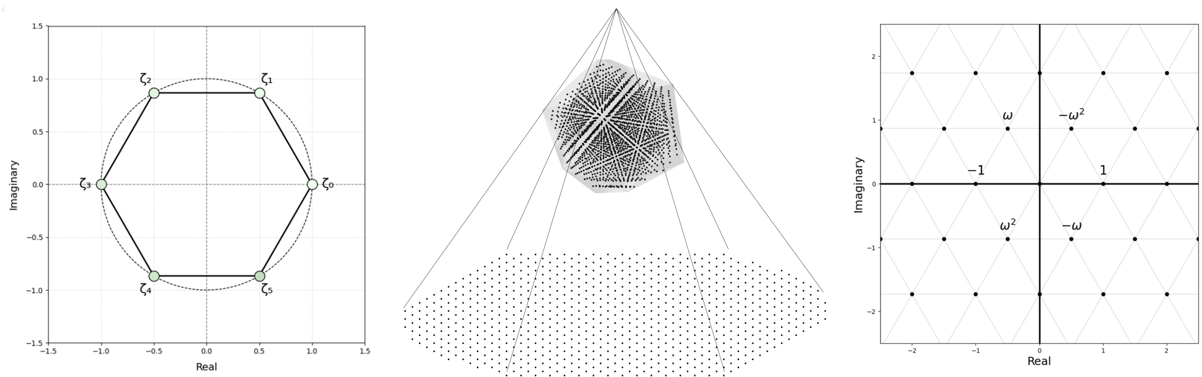


Figure 1. Sketch: If PFC cyclic buffers have a hexagonal structure in phase space (left) then a projection from a high-dimensional lattice defined by the abelian variety can linearly map onto a 2D hexagonal lattice (center). In ideal conditions, this 2D lattice is isomorphic to $\mathbb{Z}[\omega]$ (right). Note: This is just a high-level idea of what we want the model to be able to do. Defining the abelian variety, the projection, and the theta function is the end goal.

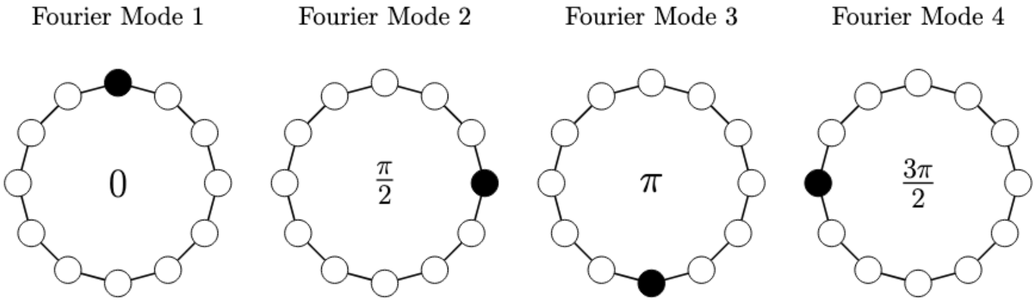


FIGURE 2. We model prefrontal cortex (PFC) as a set of $g$ cyclic buffers, with each buffer tracking the current phase of a Fourier mode in the agent's world model. We model each PFC buffer with a period $\omega$, and we use the period matrix $\Omega$ to construct an abelian variety: $\mathbb{C}^g/\mathbb{Z}^g+\Omega\,\mathbb{Z}^g$; with $\Omega \in \mathbb{H}_g$.

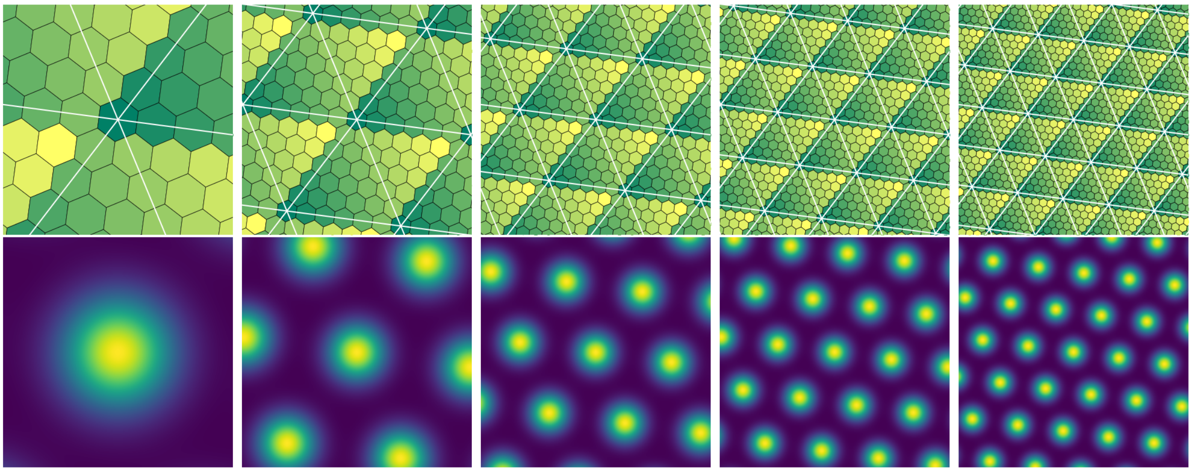

FIGURE 3. Ideal grid code in MEC based on a theta function $\Theta(z|\Omega)$: with lattice cosets and harmonically scaled modules. This is an idealisation, but it is where we start to see how neuroscience and algebraic geometry connect.

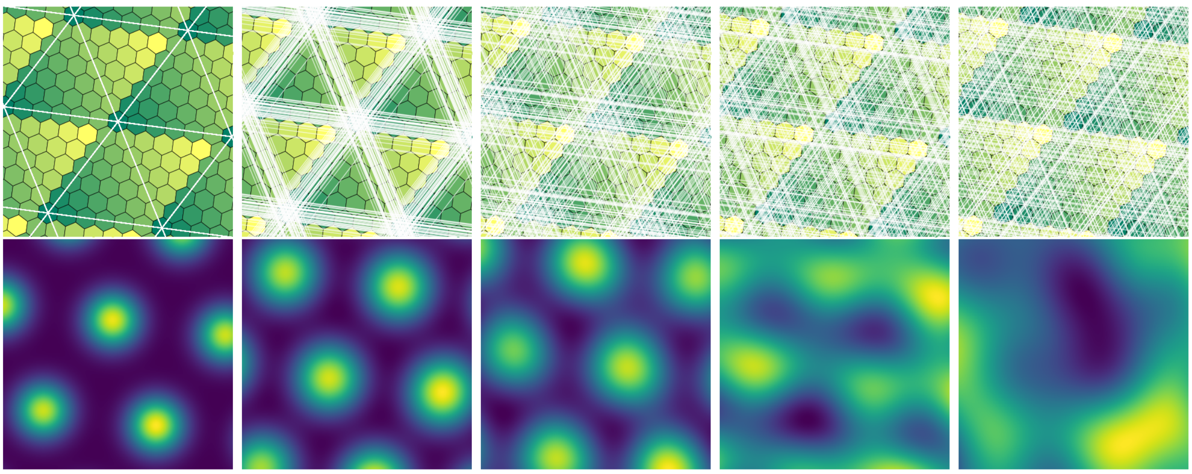

FIGURE 4. Perfectly linear projections $\phi\colon \mathbb{C}^g \to \mathbb{C}$ are biologically unrealistic. Here we model the progressive loss of grid coherence with an addtional noise variable: $\tilde{\phi}(z) = \phi(z)+\eta : \eta \sim \mathcal{N}(0,\, \sigma^2)$ in the $\tilde{\phi}\colon \mathbb{C}^g \to \mathbb{C}$ projection.

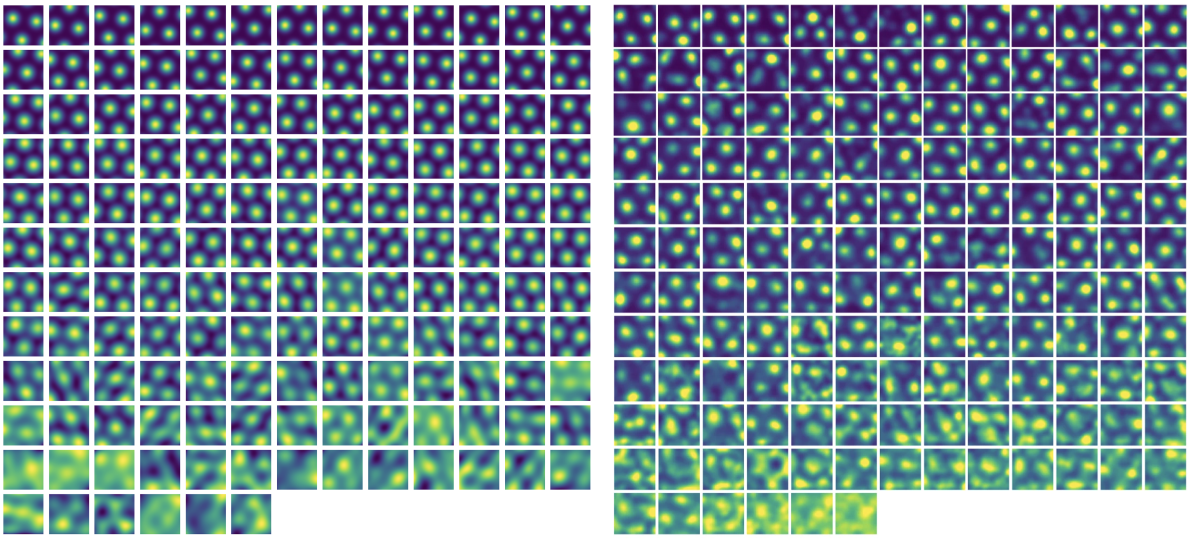

FIGURE 5. A comparison of decreasing grid coherence in our projection model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). Note: no one has ever modelled this coherence effect, it supports our model.

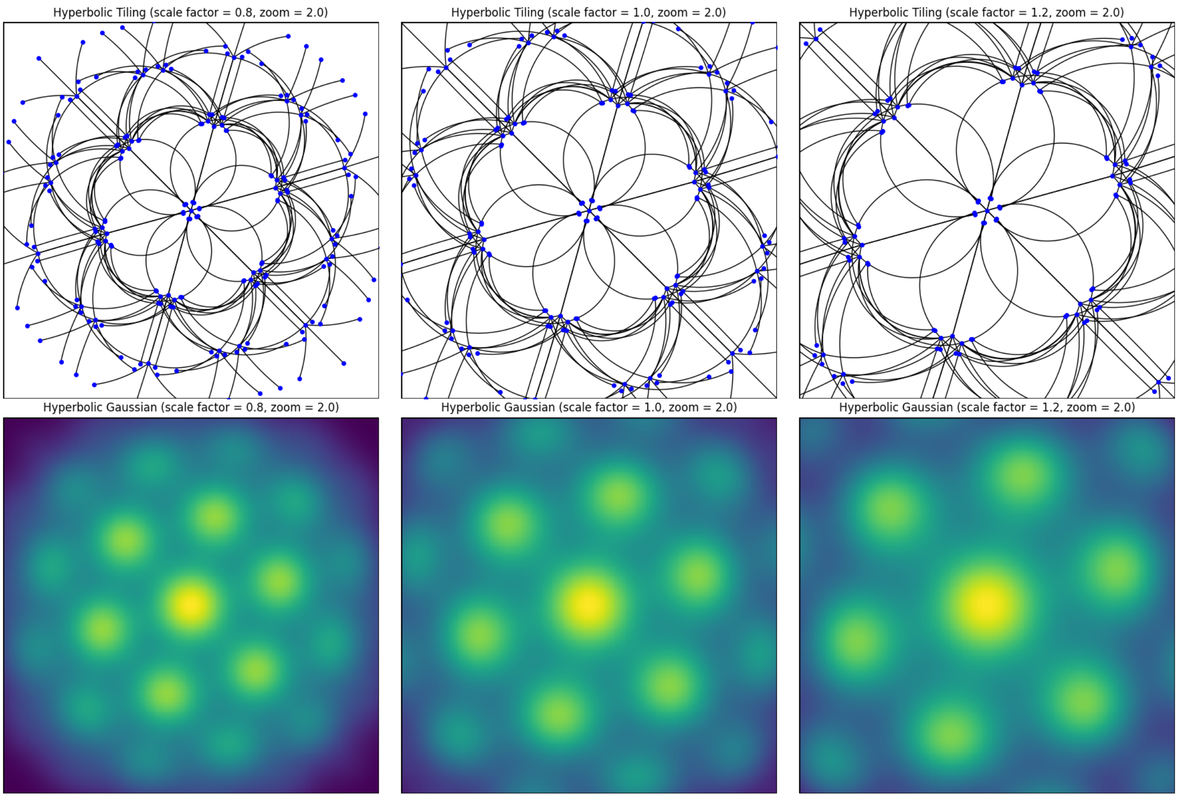


FIGURE 6. Preliminary explorations of resonance patterns in hyperbolic phase space. The central panel displays a hexagonally tiled structure that approximates empirical grid cell data suprisingly well, featuring reduced grid scales near boundaries and a 7.5° offset relative to environmental axes. This approach employs hyperbolic geometry to enhance information encoding and probe potential links between neural coding and automorphic forms. The goal is to relate analytic signals to geometric representations; this is where theoretical neuroscience and pure mathematics inter-connect.

## References


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