ABELIAN PHASE CODING: A LINEAR PROJECTION FROM PREFRONTAL PHASE CODES TO THE ENTORHINAL GRID

ABSTRACT

We propose a mathematical framework that bridges high-dimensional phase coding in the prefrontal cortex (PFC) with the two-dimensional hexagonal grid structure observed in the medial entorhinal cortex (MEC). In our model, each PFC working memory buffer (cyclic buffers that track phases of dynamical processes, and can be modelled as LCAGs) is represented by a phase variable $z_j \in \mathbb{C}$, and the joint state of the full phase space forms a complex vector $z \in \mathbb{C}^g$. By quotienting $\mathbb{C}^g$ with respect to a lattice

$$\Lambda = \mathbb{Z}^g + \Omega\,\mathbb{Z}^g,$$

where the period matrix Ω satisfies Riemann's bilinear relations ($\Omega = \Omega^T; \quad \Im(\Omega) > 0$), we obtain a g-dimensional complex torus that is an abelian variety.

Crucially, to capture the observed 60° symmetry in grid cell firing, we want to endow a subset of these phase variables with something akin to $\mathbb{Z}[\omega]$-module structure (with $\omega = e^{2\pi i/3}$), thus imposing an intrinsic hexagonal geometry on the phase space. A linear projection $\phi : \mathbb{C}^g \to \mathbb{C}$ then maps the high-dimensional torus onto a two-dimensional lattice isomorphic to $\mathbb{Z}[\omega]$. The Riemann theta function is employed to localize positions within the phase space, linking global phase dynamics to local information via a lattice.

While the neuroscientific motivation is clear—relating prefrontal phase codes to entorhinal grid representations—the mathematical formalism, particularly regarding the resonance mechanisms enforcing the $\mathbb{Z}[\omega]$ symmetry, remains only heuristic. We seek mathematical insights and rigor in formalizing the structure of the abelian variety, clarifying the role of theta functions, and extending the framework to model observed distortions under noise.

SUPPORTING FIGURES

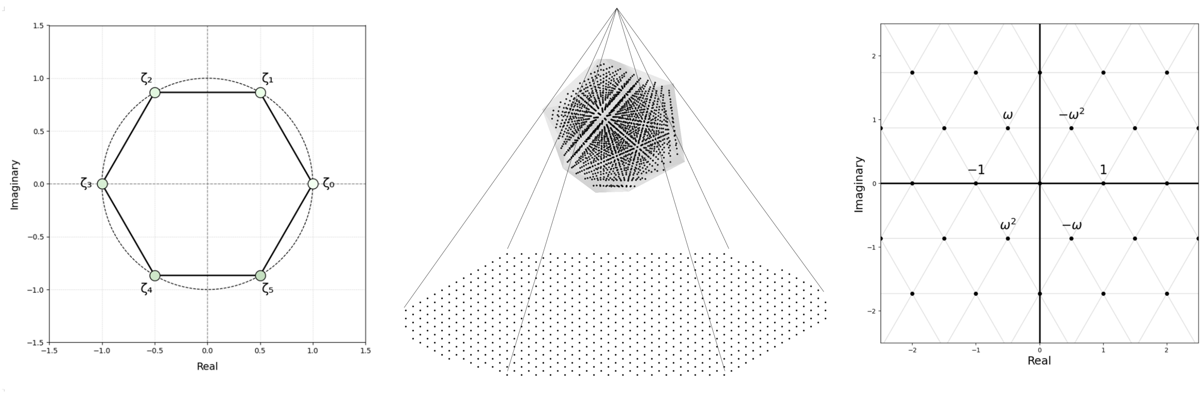

Figure 1. Sketch: If PFC cyclic buffers have a hexagonal structure in phase space (left) then a projection from a high-dimensional lattice defined by the abelian variety can linearly map onto a 2D hexagonal lattice (center). In ideal conditions, this 2D lattice is isomorphic to $\mathbb{Z}[\omega]$ (right). Note: This is just a high-level idea of what we want the model to be able to do. Defining the abelian variety, the projection, and the theta function correctly are the key formal problems. Then ideally, making some good falsifiable predctions.

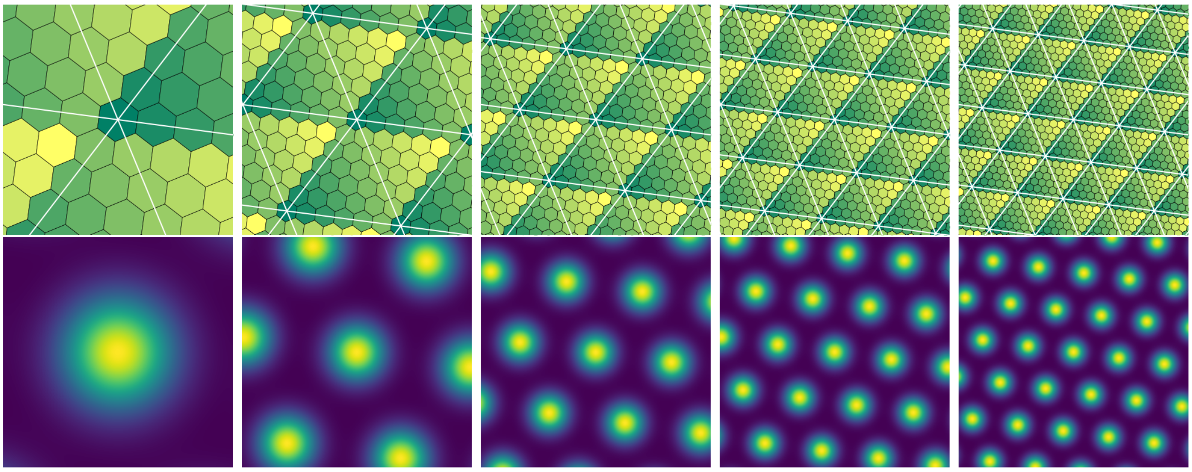

FIGURE 2. Ideal grid code from a theta function: with coset shifts and scaling.

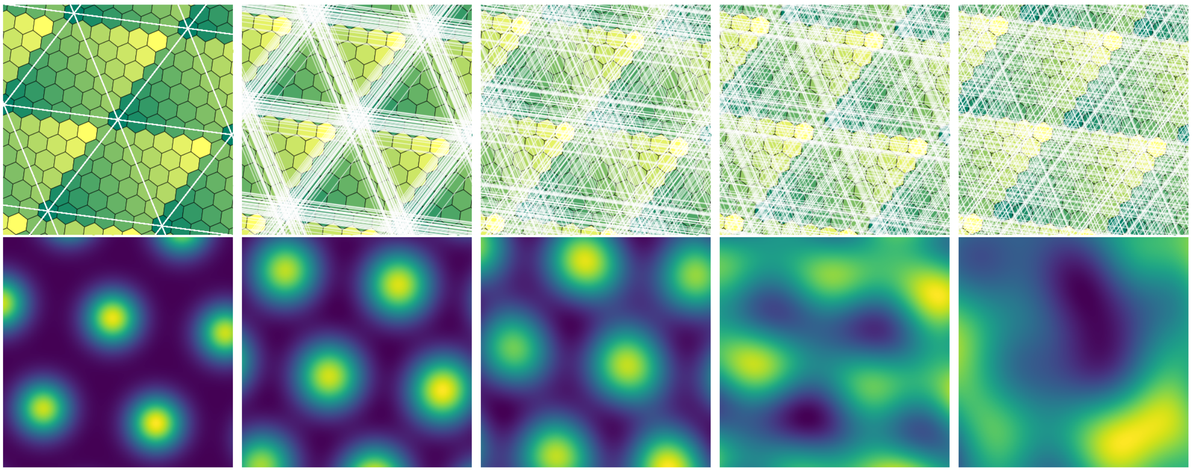

FIGURE 3. Decreasing projection coherence in the linear projection model.

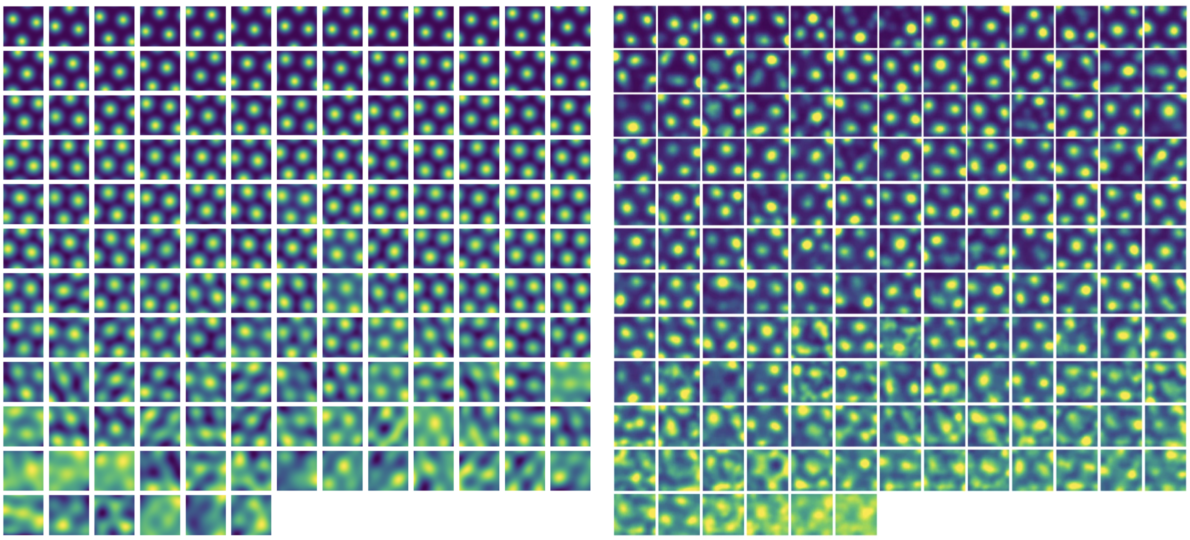

FIGURE 4. A comparison of decreasing grid coherence in our projection model (left) vs. experimental recordings (right) from Gardner *et al.* (2022). Note: no one has modelled this coherence effect, it supports the approach.