Math 251W: Foundations of Advanced Mathematics

Portfolio Assignment 4: §2.4 & 2.5

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Problem 2.4.3

For the following proposition, give a list of conditional statements you would prove to prove the entire stated equivlanece. Then prove **one** of those conditional statements.

proposition: Given a and b are positive integers, the following are equivalent:

i a and b are relatively prime.

ii $a + b$ and b are relatively prime.

iii a and $a + b$ are relatively prime.

scratchwork I will call the first predicate $A(a, b)$, the second $B(a, b)$, and the third $C(a, b)$. To prove that these are equivalent for all positive integers a and b, it must be shown that all six conditionals are true, but I need only prove three implications, and hypothetical syllogism will fill in the rest. The choice of which ones is arbitrary, so long as the conditionals are not the same. In this case, I will take the conditional statement $(\forall a, b \in \mathbb{Z}^+)(A(a, b) \to B(a, b))$.

$\boxed{\text{proof}}$ (Contradiction) The negation of this statement is that a and b are not relatively prime, and that $a + b$ and a are relatively prime.

By way of contradiction, assume that a and b are not relatively prime. We also assume that $a + b$ and a are relatively prime.

By definition, two numbers are are relatively prime if ± 1 is the only common factor. Assuming a and b are not relatively prime, there exists integers i, j, and k, such that $k \neq \pm 1$ and is a common factor, i is a factor of a, and j is a factor of b (that is to say that k divides both a and b, i divides a, and j divides b). By definition, this can be written $a = ik$ and $b = jk$.

By substitution and the distributive property, $a + b$ is equal to $k(i + j)$, and a is equal to ik. By definition of divides, k divides both a and $a + b$, meaning k is a factor of both numbers. By the definition of relatively prime numbers, we know that $a + b$ and a not relatively prime, which is a contradiction.

Thus, by way of contradiction, we have shown that if a and b are relatively prime, then $a + b$ and a are also relatively prime.

Problem 2.4.6 Fill in, and then prove, the following proposition.

proposition The only triple primes (i.e. primes of the form $p, p + 2$, and $p + 4$ for some integer p) are 3, 5, and 7.

$\boxed{\text{proof}}$ (Contradiction) Let x be an arbitrary integer, greater than two, such that $3|x$, and let k be an integer equal to 1, or 2. If k could equal three, by the distributive property and substitution, $x + k$ would be another number divisible by three, and therefore not a prime number.

Let $y = x + k$ be the first number of a set of triple primes, other than the one mentioned above.

There are two cases which this proof can be broken up into. These are (1) $k = 1$ and (2) $k = 2$.

(1) Let $k = 1$. Our initial prime number $y = x + k = x + 1$ may be prime, as it is not divisible by three. However, the second prime number will have the value of $y + 2$. This, by substitution, is $k + 2 + x = 1 + 2 + x = 3 + x$. By the commutative property, $3|x$, hence the second prime number is divisible by three. This is a contradiction of the definition of a prime number, which rules out the case in which $k = 1$.
(2) Now we have $y = x + 2$. We know $x + 2$ is not divisible by three. However, the third term will have the value $y + 4$. By substitution $y + 4 = x + 2 + 4 = x + 6$. By the commutative property $y + 4 = 3(x + 2)$. By definition of divides, $3|y + 4$, which contradicts the definition of a prime number.

These cases account for all integers, thus the only triple primes are 3, 5, and 7.

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Problem 2.4.8 Prove the following

<u>proposition:</u> If n is an odd integer, then there exists and integer k such that $n^2 = 8k + 1$.

$\boxed{\text{proof}}$ (Direct)

By the definition of odds, there exists some integer j such that $n = 2j + 1$. By definition, all integers are either even or odd. This proof is divided into two cases. The first case is that j is odd, and the second is that j is even

Case 1: by definition of an odd number, there exists an integer x such that $j = 2x + 1$. By substitution, $n = 2j + 1 = 2(2x + 1) + 1$. By the distributive, associative, and commutative properties, $n = 4x + 3$.

Now we turn to n^2. By substitution, $n^2 = (2j + 1)^2 = (2(2x + 1) + 1)^2$. By the distributive, commutative, and associative properties, $n^2 = 16x^2 + 24x + 9$. By the commutative and associative properties, $n = 8(2x^2 + 3x + 1) + 1$. By the closure property of integers over multiplication and addition, $(2x^2 + 3x + 1)$ is an integer. Thus there exists an integer, k, such that $n^2 = 8k + 1$.

Case 2: By definition of an even number, there exists an integer y such that $j = 2y$. By substitution, $n = 4y + 1$. By substitution, the distributive property, and the associative property, $n^2 = 16y^2 + 8y + 1 = 8(2y^2 + y) + 1$. By the closure property of integers, $(2y^2 + y)$ is an integer. Thus, there exists an integer k such that $n^2 = 8k + 1$.

Thus, if n is an odd integer, there exists and integer k such that $n^2 = 8k + 1$.

Problem 2.5.8 Prove or give a counterexample

<u>proposition:</u>For each real number p, there exist real numbers q and r such that $q \sin(r/5) = p$.

$\boxed{\text{proof}}$

Let q be a real number, and $r = 5\pi/2$. Thus, by substitution $q \sin(5\pi/(5 * 2)) = p$, which is equivalent to $q = p$. q is a real number, as is p, thus for each value of p, there exists a value of q such that $p = q$. These instances satisfy the requirements, thus the statement is true.

Problem 2.5.9 Prove or give a counterexample

proposition:For each integer x, and for each integer y, there exists an integer z such that

$$z^2 + 2xz - y^2 = 0.$$

$\boxed{\text{counterexample}}$ Using a counter example, this can be proven false. Let $x = 5$, and $y = 2$. By substitution, the equation can be written as $z^2 + 10z - 4 = 0$. This, by a whole load of axioms and definitions, is equivalent to $z = (-10 \pm (100 + 16))/2$, which is not an integer.