Topology

August bergquist

December 22, 2022

$\boxed{\text{Theorem 3.35}}$ Show that the product topology on $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection fucntions, that is, the subbasis is $\mathcal{S} = \{\pi_X^{-1}(U) : U \in \mathcal{T}_X\} \cup \{\pi_Y^{-1}(V) : V \in \mathcal{T}_Y\}$.

$\boxed{\text{proof}}$ We will need to show that the set of finite intersections of elements in $\mathcal{S}$ forms a basis for $\mathcal{T}_{X\times Y}$. Let $\mathcal{B}$ denote the set of finite intsersections of elements in $\mathcal{S}$. We will now proceed by showing that $\mathcal{B}$ meets the requirements of Theorem 3.1 in the case of $\mathcal{T}_{X\times Y}$.

- We will first need to show that $\mathcal{B} \subseteq \mathcal{T}_{X\times Y}$. By definition of a topology, it will suffice to show that each element in $\mathcal{S}$ is open in $\mathcal{T}_{X\times Y}$. This is because $\mathcal{B}$ consists of finite intersections of elements of $\mathcal{S}$, and by definition (Theorem...)of a topology the finite intersection of open sets must be open.

  Let $W \in \mathcal{S}$ be arbitrary. By definition of the union and construction of $\mathcal{S}$ we know that $W \in \{\pi_X^{-1}(U) : U \in \mathcal{T}_X\}$ or $W \in \{\pi_Y^{-1}(V) : V \in \mathcal{T}_Y\}$. Call these cases one and two respectively.

  In the first case, we have $W = \pi_X^{-1}(U)$ for some open set U in $\mathcal{T}_X$. Recall from the properties of the projection function in foundations that $W = \pi_X^{-1}(U) = U \times Y$. Since both U and Y are open in their respective spaces, and since open sets in the product topology are the unions of such sets, we conclude that $W \in \mathcal{T}_{X\times Y}$.

  Likewise, in the second case notice that $W = \pi_Y^{-1}(V) = X \times V$ for some open set V. By the same reasoning in the first case, we conclude that $W \in \mathcal{T}_{X\times Y}$.

  Since in all cases $W \in \mathcal{T}_{X\times Y}$, and since W was arbitrary in $\mathcal{S}$, it follows that all elements of $\mathcal{S}$ are in $\mathcal{T}_{X\times Y}$. And as we have shown, this implies that $\mathcal{B} \subseteq \mathcal{T}_{X\times Y}$.

- Now we must show that for each set $W \in \mathcal{T}_{X\times Y}$ and for each point $p \in W$ there is some set $V \in \mathcal{B}$ such that $p \in V \subseteq U$.

  Let W be arbitrary in $\mathcal{T}_{X\times Y}$, and let p be arbitrary in W. Since by definition of the product topology $\mathcal{T}_{X\times Y}$ is the topology generated by the basis $\mathcal{V} = \{U \times V : U \in \mathcal{T}_X V \in \mathcal{T}_Y\}$, it follows by theorem 3.1 that there exists some basic set B of $\mathcal{V}$ such that $p \in B \subseteq W$. By definition of $\mathcal{V}$ there must exist open sets U and V in spaces X and Y respectively such that $B = U \times V$.
  Now consider the subbasic sets $M = \pi_X^{-1}(U) = U \times Y$ and $N = \pi_Y^{-1}(V) = X \times V$. Consider the intersection $(U \times Y) \cap (X \times V)$. From foundations, recall that $(U \times Y) \cap (X \times V) = (U \cap X) \times (Y \times V) = U \times V = B$. Hence B is basic in the basis generated by the subbasis $\mathcal{S}$. Hence there exists some element B of $\mathcal{B}$ such that $p \in B \subseteq W$. Since W was arbitrary in $\mathcal{T}_{X\times Y}$, and since p was arbitrary in W, it follows that the second requirement of Theorem 3.1 in the case of $\mathcal{B}$ is met.

Having shown that $\mathcal{B}$ satisfies all of the requirements of Theorem 3.1, it follows that $\mathcal{B}$ is a basis for the product topology. Q.E.D.