Math 251W: Foundations of Advanced Mathematics

Solutions to Portfolio problems from sections 4.3, & 4.4

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Problem 4.3.10

proposition: Let $f : A \to B$ be a function.

i If f has two distinct left inverses, it has no right inverse. inverse.

$\boxed{\text{proof}}$ (Contradiction) Let $f : S \to B$ be an arbitrary function. Suppose by way of contradiction that there exists some $f : A \to B$ such that there exists two distinct left inverses and there exists a right inverse. Let the left inverses of f be $g, h : B \to A$, and let the right inverse be $m : B \to A$.

By definition of the left inverse, $h \circ f = g \circ f = 1_A$. By definition of the identity function by substitution, $(g \circ f)(x) = g(f(x)) = x$ for all x in A. Furthermore, $(h \circ f)(x) = h(f(x)) = x$. By supposition, h and g are distinct, meaning that there exists some element in the domain, $y \in B$, such that $h(y) \neq g(y)$.

Consider $(g \circ f) \circ m$. This composition is well defined, as $g \circ f$'s domain is by definition m's codomain. Furthermore, by substitution, $(g \circ f) \circ m = (h \circ f) \circ m$. By the associative property of compositions, $g \circ (f \circ m) = h \circ (f \circ m)$. By the definition of m, and by substitution, $g \circ 1_B = h \circ 1_B$. Furthermore, by the identity property of compositions, $g = h$, which by assumption, cannot be the case. We have thus reached a contradiction.

Thus, whenever any function $f : A \to B$ has two left inverses, it cannot have a right inverse.

Problem 4.3.11

proposition: Let $f : A \to A_1 \times A_2 \times \cdots \times A_k$ be a function, and let $U_i \subseteq A_i$ for each $i \in \{1, \ldots, k\}$, then

$$f^*(U_1 \times U_2 \times \cdots \times U_k) = \bigcap_{i=1}^{k} f_i^*(U_i)$$

$\boxed{\text{proof}}$ (Intimidation) Let $f : A \to A_1 \times A_2 \times \cdots \times A_k$ be an arbitrary function, and let U_i be an arbitrary subset of A_i for each $i \in \{1, \ldots, k\}$.

Let a be an arbitrary element of $f^*(U_1 \times U_2 \times \cdots \times U_k)$. By definition of the preimage, there exists some $f(a) \in A_1 \times A_2 \times \cdots \times A_k$ such that $f(a) \in U_1 \times U_2 \times \cdots \times U_k$. By definition of the coordinate function, $f(a) = (f_1(a), \ldots, f_k(a))$, where $f_i = \pi_i \circ f$ for each $i \in \{1, \ldots, k\}$. By definition of the preimage, $a \in f_i^*(U_i)$ for each $i \in \{1, \ldots, k\}$. Thus, by definition of the intersection of an indeed family of sets, $a \in \bigcap_{i=1}^{k} f_i^*(U_i)$. Hence, since a is an arbitrary element in $f^*(U_1 \times U_2 \times \cdots \times U_k)$,

$$f^*(U_1 \times U_2 \times \cdots \times U_k) \subseteq \bigcap_{i=1}^{k} f_i^*(U_i).$$

let b be an arbitrary element of $\bigcap_{i=1}^{k} f_i^*(U_i)$. By definition of the intersection of an indexed family of sets, $b \in f_i^*(U_i)$ for each $i \in \{1, \ldots, k\}$. Futhermore, by defintion of the preimage, there exists some $f_i(b)$ such that $f_i(b) \in U_i$. By definition of the coordinate function, $f(b) = (f_1(b), \ldots, f_k)$. By definition of the cross product, $f(b) \in U_1 \times U_2 \times \cdots \times U_k$. By definition of the preimage, $b \in f^*(U_1 \times U_2 \times \cdots \times U_k)$. Hence, since b is an arbitrary element of $\bigcap_{i=1}^{k} f_i^*(U_i)$,

$$f^*(U_1 \times U_2 \times \cdots \times U_k) \supseteq \bigcap_{i=1}^{k} f_i^*(U_i).$$

By definition of set equality,

$$f^*(U_1 \times U_2 \times \cdots \times U_k) = \bigcap_{i=1}^{k} f_i^*(U_i).$$

Problem 4.4.7

proposition: The function $\phi : \mathcal{P}(A) \to \mathcal{P}(A)$, defined by $\phi(X) = A - X$ for all $X \in \mathcal{P}(A)$, is bijective.

$\boxed{\text{proof}}$ (Direct) Let A be an arbitrary set, and let $\phi : \mathcal{P}(A) \to \mathcal{P}(A)$ be an arbitrary function defined by $\phi(X) = A - X$ for all $X \subseteq A$.
For ϕ to be bijective, by definition it must be both injective and surjective. As a result, this proof will be divided into two subsections. The first will show that ϕ is injective, and the second will show that ϕ is surjective.

1. Let S and T be arbitrary elements of the codomain, $\mathcal{P}(A)$, such that $\phi(S) = \phi(T)$. By definition of ϕ, and by substitution, $A - S = A - T$. Using some theorem having to do with sets, since $S, T \subseteq A$, $S = T$. Thus, we have shown that for all S and T in the domain of ϕ, $\phi(S) = \phi(T)$ implies that $S = T$. By definition of injectivity, ϕ is injective.

2. Let Y be an arbitrary element of the codomain $\mathcal{P}(A)$. By definition of the powerset, $Y \subseteq A$. Furthermore, by some set theorem, there exists some subset $X \subseteq A$ such that $Y = A - X$. Thus, by definiton of ϕ, $\phi(X) = Y$. Thus, since Y is an arbitrary element of the codomain, for all Y in the codomain, there exists some X domain such that $\phi(X) = Y$. Thus, by definition of surjectivity, ϕ is surjective.

Having shown that ϕ is both surjective and injective, by definition of bijectivity, ϕ is bijective.

Problem 4.4.14

3 proposition: Given functions $f : A \to B$, and $g : B \to C$, if $g \circ f$ is bijective, then f must be injective and g must be surjective.

$\boxed{\text{proof}}$ (direct) Let $f : A \to B$ and $g : B \to C$ be arbitrary functions. Suppose $g \circ f$ is bijective. The first section of this proof will show that f is injective, and the second that g is surjective.

1. Let a, b be arbitrary elements of A such that $f(a) = f(b)$. By definition of composition, $(g \circ f)(x) = g(f(x))$ for all elements x in the domain, A. By definition of functions, because each function must map each element of the domain to exactly one element of the codomain, and since g is a function, we know that $g(f(a)) = g(f(b))$. Since this mapping is the same as the composition $g \circ f$, and because this is bijective, we know that $g \circ f$ is injective, hence $g(f(a)) = g(f(a))$ implies that $a = b$. Thus, for all elements a and b in the domain A, $f(a) = f(b)$ implies that $a = b$. By definition of injectivity, f is injective.
2. Let c be an arbitrary element of the codomain of c, C. By definition of composition, $g \circ f$ is bijective, and therefore surjective as well. Thus, by definition of surjectivity, and since c is in the codomain of $g \circ f$, there exists some a in the codomain of $g \circ f$, A such that $g \circ f(a) = c$. Furthermore, by definition of composition, $g \circ f(a) = g(f(a))$. By definition of f, $f(a)$ is in the codomain of f, which is B. Thus, there exists some element $b = f(a) \in B$ such that $g(b) = c$. By definition of g, B is the codomain of g, thus for each element c in the codomain of g there exists some b such that $c = g(b)$. Hence, by definition of surjectivity, g is surjective.

Thus, for any functions $f : A \to B$ and $g : B \to C$, if the composition $g \circ f$ is bijective, then f is surjective and f is injective.

Problem 4.4.17

proposition: Let $f : A \to B$ be a map. f is surjective iff $B - f_*(X) \subseteq f_*(A - X)$ for all $X \subseteq A$.

$\boxed{\text{proof}}$(Direct) Let X, A and B be arbitrary sets, and let $f : A \to B$ be an arbitrary map. Since this statement is biconditional, the proof will be divided into two subsections.

1. Suppose $f : A \to B$ is surjective. Let y be an arbitrary element of $B - f_*(X)$, and let $X \subseteq A$. By definition of set difference, $y \in B$ and $y \notin f_*(X)$. By definition of surjetivity, there exists some $x \in A$ such that $f(x) = y$. Hence, we know that $x \in A$ and $x \notin X$. Furthermore, by set difference, $x \in (A - X)$. By definition of the image, $f(x) = y \in f_*(A - X)$. By definition of subsets, $B - f_*(X) \subseteq f_*(A - X)$.
2. Now suppose $B - f_*(X) \subseteq f_*(A - X)$ for all $X \subseteq A$.
   - Let x be an arbitrary element of B.

     Somehow, $x \in f_*(A)$
   - Let y be an arbitrary element of $f_*(A)$.

     Somehow, $y \in B$

   By definition of set equality, $f_*(A) = B$. Since A and B are the domain and codomain of f, by definition of surjectivity f is surjective.