Topology

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$\boxed{12.5}$ If α, α', β, β', are paths in a space X such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$, and $\alpha(1) = \beta(0)$, then $\cdot\beta \sim \alpha' \cdot \beta'$.

$\boxed{\text{proof}}$

For path equivalence to even make sense in this context, we will have to verify that $\alpha \cdot \beta(0) = \alpha' \cdot \beta'(0)$ and $\alpha \cdot \beta(1) = \alpha' \cdot \beta'(1)$. This is pretty much trivial, as by definition of the concatenation of paths and construction of α, α', β and β', $\alpha \cdot \beta(0) = \alpha(0) = \alpha'(0) = \alpha' \cdot \beta'(0)$ and $\alpha \cdot \beta(1) = \beta(2-1) = \beta(1) = \beta'(1) = \beta'(2-1) - \alpha' \cdot \beta'(1)$.

By definition of path equivalences, there exists homotopies H_α and H_β from α to α' and from β to β' respectively. We must construct a homotopy from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$. Consider the function $H : [0,1]^2 \to X$ defined

$$H(s,t) = \begin{cases} H_\alpha(2s,t) & 0 \le s \le 1/2 \\ H_\beta(2s-1,t) & 1/2 \le s \le 1 \end{cases}.$$

First, we will show that H is continuous. Since H_α and H_β are both continuous functions from $[0,1]^2$ to X, all that we need to show is that they agree at $s = 1/2$. For the first case of the definition of H, we have $H(1/2,t) = H_\alpha(2(1/2),t) = H_\alpha(1,t)$. Since H_α is a homotopy from α to α', it follows by definition of a homotopy that $H_\alpha(1,t) = \alpha(1)$. Furthermore, by the second definition, $H(1/2,t) = H_\beta(2(1/2)-1,t) = H_\beta(0,t)$ for all $t \in [0,1]$. Also, since H_β is a homotopy from β to β', it follows that $H_\beta(0,t) = \beta(0)$. Furthermore, by construction $\beta(0) = \alpha(1)$. So both of the piecewise definitions of H agree on their overlap.

We must now show that the remaining requirements of a homotopy from $\alpha\beta$ to $\alpha' \cdot \beta'$ are met.

- First, we must show that $H(s,0) = \alpha' \cdot \beta'(s)$ for all $s \in [0,1]$. By our definition of H, we have

  $$H(s,0) = \begin{cases} H_\alpha(2s,0) & 0 \le s \le 1/2 \\ H_\beta(2s-1,0) & 1/2 \le s \le 1 \end{cases}.$$

  Furthermore, by construction of H_α as a homotopy from α to α' (I'm getting tired of writing this sentence lol), $H_\alpha(2s,0) = \alpha(2s)$ for all whenever $2s \in [0,1]$ (which is true whenever $0 \le s \le 1/2$). Similarly, $H_\beta(2s-1,0) = \beta(2s-1)$ whenever $2s-1 \in [0,1]$ (which happens if and only if $1/2 \le s \le 1$). Hence, substituting back into $H(s,0)$, we have

  $$H(s,0) = \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

  This is just the definition of $\alpha \cdot \beta$, hence $H(s,0) = \alpha \cdot \beta(s)$ for all $s \in [0,1]$.

- Now we want to show that $H(s,1) = \alpha' \cdot \beta'(s)$ for all $s \in [0,1]$. Plugging in $t = 1$, we have

  $$H(s,1) = \begin{cases} H_\alpha(2s,1) & 0 \le s \le 1/2 \\ H_\beta(2s-1,1) & 1/2 \le s \le 1/2 \end{cases}.$$

Similar to as before, we recall that by definition of a homotopy from α to α', $H_\alpha(2s,1) = \alpha'(2s)$ whenever $2s \in [0,1]$ (which, as we have already pointed out, happens when $0 \leq s \leq 1/2$). Similarly, since H_β is a homotopy from β to β', $H_\beta(2s-1,1) = \beta'(2s-1)$ whenever $2s-1 \in [0,1]$ (which, as we have noted, happens when $1/2 \leq s \leq 1$). Hence

$$H(s,1) = \begin{cases} \alpha'(2s) & 0 \leq s \leq 1/2 \\ \beta'(2s-1) & 1/2 \leq s \leq 1 \end{cases}.$$

This is just the definition of $\alpha' \cdot \beta'$ over the domain $[0,1]$, hence $H(s,1) = \alpha' \cdot \beta'(s)$ for all $s \in [0,1]$.

- Now we must show that $H(0,t) = \alpha \cdot \beta(0)$ for all $t \in [0,1]$. By construction of H, and since the only first piecewise condition on the definition of H is satisfied by $s = 0$, we have $H(s,1) = H_\alpha(2(0),t) = H_\alpha(0,t)$. But since H_α is a homotopy from α to α', it follows by definition of a homotopy that $H_\alpha(0,t) = \alpha(0)$ for all for all $t \in [0,1]$. Furthermore, by definition of the concatenation of paths, $\alpha(0) = \alpha \cdot \beta(0)$ for all $t \in [0,1]$, hence $H(0,t) = \alpha \cdot \beta(0) = \alpha' \cdot \beta'(0)$.

- Finally, we must show that $H(1,t) = \alpha \cdot \beta(1) = \alpha' \cdot \beta'(1)$. Since $s = 1$ only satisfies the second requirements for the second case of the definition of H, we know that $H(1,t) = H_\beta(2(1)-1,t) = H_\beta(1,t)$. Moreover, since H_β is a homotopy from β to β', we know that $H_\beta(1,t) = \beta'(1) = \beta(1)$. Finally, by definition of the concatenation of paths, $H_\beta(1,t) = \beta(1) = \alpha \cdot \beta(1) = \alpha' \cdot \beta'(1)$, which is our desired result.

Having shown that H meets all of the requirements for being a homotopy from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$, it follows that

$\boxed{\text{Exercise 12.4}}$ Let α and β be paths in $\mathbb{R}$ such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Show that $\alpha \sim \beta$.

$\boxed{\text{proof}}$ Consider the function $H : [0,1]^2 \to \mathbb{R}$, defined $H(s,t) = (1-t)\alpha(s) + t\beta(s)$ for all $(s, t \in [0,1]^2)$. We will show that H is a homotopy from α to β.

First, we notice that since α and β are continuous functions of s, and since $(1-t)$ and t are continuous functions of t (in $\mathbb{R}_{std}$), and since the product and sum of continuous real functions is always continuous, $H(s,t) = (1-t)\alpha(s) + t\beta(s)$ really is continuous. It remains to be shown that H meets the other requirements for a homotopy.

- First, we must verify that $H(s,0) = \alpha(s)$ for every $s \in [0,1]$. By our definition of H, $H(s,0) = (1-0)\alpha(s) + (0)\beta(s) = \alpha(s)$ for all $s \in [0,1]$, which is the desired result.

- Now we must show that $H(s,1) = \beta(s)$ for all $s \in [0,1]$. By our definition of H we have $H(s,1) = (1-1)\alpha(s) + (1)\beta(s) = \beta(s)$ for all $s \in [0,1]$, which is our desired result.

- Now we must show that $H(0,t) = \alpha(0)$ for all $t \in [0,1]$. Recall that $\alpha(0) = \beta(0)$. Hence by our definition of H we have $H(0,t) = (1-t)\alpha(0) + t\beta(0) = \alpha(0) - t\alpha(0) + t\alpha(0) = \alpha(0) = \beta(0)$ for all $t \in [0,1]$, which is our desired result.

- Finally, we must show that $H(1,t) = \alpha(1) = \beta(1)$. Recall that $\alpha(1) = \beta(1)$, hence by definition of H, we have $H(1,t) = (1-t)\alpha(1) + t\beta(1) = \alpha(1) - t\alpha(1) + t\beta(1) = \alpha(1) - t\alpha(1) + t\alpha(1) = \alpha(1) = \beta(1)$.

Having shown that H meets all of the requirements for being a homotopy from α to β, it follows that $\alpha \sim \beta$. Q.E.D.