Math 251W: Foundations of Advanced Mathematics

Portfolio problems from sections 3.4, 4.1, & 4.2

August: name

Problem 3.4.3(vi)

proposition: If I is a nonempty set, $\{A_i\}_{i\in I}$ is a family of sets indexed by I, and B is a set, then $B - (\bigcap_{i\in I} A_i) = \bigcup_{i\in I}(B - A_i)$.

$\boxed{\text{Proof}}$ (direct)

Let I be an arbitrary non-empty set, $\{A_i\}_{i\in I}$ be an arbitrary family of sets indexed by I, and B be an arbitrary set.

Let x be an arbitrary element of $B - (\bigcap_{i\in I} A_i)$. By definition of set difference, $x \in B$ and $x \notin \bigcap_{i\in I} A_i$. Because of this, by definition of the intersection of an indexed family of sets, there exists some $k \in I$ such that $x \notin A_k$. Since x is in B and not in A_k, by definition of set difference $x \in B - A_k$. We know by definition of the union of an indexed family of sets that $x \in \bigcup_{i\in I}(B - A_i)$. Hence, since x is an arbitrary element of $B - (\bigcap_{i\in I} A_i)$, by definition of subsets, $B - (\bigcap_{i\in I} A_i) \subseteq \bigcup_{i\in I}(B - A_i)$

Now let y be an arbitrary element of $\bigcup_{i\in I}(B - A_i)$. By definition of the union of an indexed family of sets, $y \in B - A_j$ for at least one j in I. Thus, by the definition of set difference, $y \in B$ and $y \notin A_j$. Hence, definition of the intersection of an indexed family of sets, we know that $y \notin \bigcap_{i\in I} A_i$. Thus, by definition of set difference, $y \in B - (\bigcap_{i\in I} A_i)$. Since y is an arbitrary element of $\bigcup_{i\in I}(B - A_i)$, by definition of subsets, $B - (\bigcap_{i\in I} A_i) \supseteq \bigcup_{i\in I}(B - A_i)$

Thus, by definition of set equality, for all families of sets $\{A_i\}_{i\in I}$ indexed by a nonempty set I, and for all sets B, $B - (\bigcap_{i\in I} A_i) = \bigcup_{i\in I}(B - A_i)$.

Problem 3.4.6(2)

<u>proposition:</u> If I is a nonempty set, $\{A_i\}_{i\in I}$ is a family of sets indexed by I, and B is a set, then $B \times (\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(B \times A_i)$.

$\boxed{\text{Proof}}$ (Direct)

Let I be an arbitrary nonempty set, let B be a set, and let $\{A_i\}_{i\in I}$ be an arbitrary family of sets indexed by I.
Let (a, b) be an arbitrary element of $B \times (\bigcap_{i\in I} A_i)$. By definition of the Cartesian product, $a \in B$ and $b \in \bigcap_{i\in I} A_i$. Let j be an arbitrary element of I. By definition of the intersection of an indexed family of sets, $a \in A_j$. By definition of the Cartesian product, $(a, b) \in B \times A_j$. Furthermore, by definition of the intersection of an indexed family of sets, and since j is arbitrary, $(a, b) \in \bigcap_{i\in I} A_i$. Because (a, b) is an arbitrary element of $B \times (\bigcap_{i\in I} A_i)$, $B \times (\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I}(B \times A_i)$.

Let (x, y) be an arbitrary element of $\bigcap_{i\in I}(B \times A_i)$. Let k be an arbitrary element of I. By definition of the intersection of an indexed family of sets, $(x, y) \in B \times A_k$. Furthermore, by definition of the Cartesian product, $x \in B$ and $y \in A_k$. Since k is an arbitrary element of I, $y \in A_i$ for all i in I. By definition of the intersection of an indexed family of sets, $y \in \bigcap_{i\in I} A_i$. By definition of the Cartesian product, $(x, y) \in B \times \bigcap_{i\in I} A_i$. Since (x, y) is an arbitrary element of $\bigcap_{i\in I}(B \times A_i) \subseteq B \times (\bigcap_{i\in I} A_i)$

Thus, by definition of set equality, $B \times (\bigcap_{i\in I} A_i) = \bigcap_{i\in I}(B \times A_i)$.

Problem 4.1.8

proposition: Given $A, B \subseteq X$, $\chi_A = \chi_B$ iff $A = B$

$\boxed{\text{proof}}$ (Direct)

Let A and B be arbitrary subsets of X. Since the proposition is biconditional, this proof will be divided into two sections.

1. For this section, suppose $\chi_A = \chi_B$.

   - Let a be an arbitrary element in A. By definition of the characteristic function, we know that the domain of χ_A and χ_B is X. By assumption, since A and B are subsets of the domain of χ_A and χ_B, a is in the domain of both of these functions. Thus, $\chi_A(a)$ and $\chi_B(a)$ must be defined. By definition of the characteristic function, since a is in A, $\chi_A(a) = 1$. Furthermore, by supposition, $\chi_B(a) = 1 = \chi_A(a)$. By definition of the characteristic function, a must be in B. Thus, since a is an arbitrary element of A, $A \subseteq B$.
   - Let b be an arbitrary element of B. Since B is in the domain of χ_A and χ_B, $\chi_A(b)$ and $\chi_B(b)$ are defined. Furthermore, by definition of the characteristic function, since $b \in B$, $\chi_B(b) = 1$. Furthermore, since the two functions are equal, their mapping must also be equal. Hence $\chi_A(b) = \chi_B(b) = 1$. By definition of the characteristic function, b must be in A. Since b is an arbitrary element of B, $A \supseteq B$.

   Since $A \supseteq B$ and $A \subseteq B$, by definition of set equality, $A = B$.

2. Now suppose that $A = B$. By definition of the characteristic function, both the domain and codomain are equal for χ_A and χ_B. This being the case, all that remains to be proven is that the functions map the same. To do this, let n be an arbitrary element of X.

   - Case 1: Consider the case where n is in A. By definition of set equality, n is in B as well. Furthermore, by definition of the characteristic function, $\chi_A(n) = 1 = \chi_B(n)$. Thus, whenever n is in A and B, $\chi_A(n) = \chi_B(n)$.
   - Case 2: Now consider the case where n is not in A. By definition of set equality, n is not in B. Furthermore, by definition of the characteristic function, $\chi_A(n) = 0 = \chi_B(n)$. Thus, whenever n is not in A and not in B, the $\chi_A(n) = \chi_B(n)$.

   Since $A = B$, n is either in both A and B, or in neither. Thus, this covers all cases, proving that for all n in the domain, $\chi_A(n) = \chi_B(n)$. Hence, whenever $A = B$, $\chi_A = \chi_B$.

By this we have shown that for arbitrary subsets of X, A and B, $A = B$ if and only if $\chi_A = \chi_B$.