Abstract Algebra

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$\boxed{\text{proposition: 11}}$ Let $G = \mathbb{Z}_4 \bigoplus U(4) = \{(0,1),(0,3),(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}, H =< (2,3) >= \{(2,3),(0,1)\}$, and $K =< (2,1) >= \{(2,1),(0,1)\}$. Show that G/H is not isomorphic to G/K.

$\boxed{\text{proof}}$ Let G, H and K be defined as above. Consider the function $\phi : G/H \longrightarrow G/K$, such that $G/H = \{aH|a \in G\}$ and $G/K = \{bK|b \in G\}$.
Finding the factor groups, we have $G/H = \{[(0,1)H = (2,3)H = \{(2,3),(0,1)\}], [(0,3)H = (2,1)H = \{(2,1),(0,3)\}], [(1,1)H = (3,3)H\{(3,3),(1,1)\}], [(1,3)H = (3,1)H = \{(3,1),(1,3)\}]\}$.
Furthermore, $G/K = \{[(0,1)K = (2,1)K = \{(2,1),(0,1)\}], [(0,3)K = (2,3)K = \{(2,3),(0,3)\}], [(1,1)K = (3,1)K = \{(1,1),(3,1)\}], [(1,3)K = (3,3)K = \{(1,3),(3,3)\}]\}$.

Notice that $[(3,1)K][(3,1)K] = [(3,1)(3,1)]K = (2,1)K = K$. Hence the element order $(3,1)K$ is 2. Likewise, $((1,1)K)^2 = (2,1)K = K$, $[(1,3)K]^2 = (2,1)K = K$, K is the identity hence its order is one, hence the orders of these elements are $2, 2$, and 1 respectively. Note that there are no elements of order 4.

Luckily, we only need to type one calculation to finish the proof. Notice that $[(1,1)H]^4 = H$. Also, none of the elements in the cycle for $(1,1)$ are in H, besides $(0,1)$ which is the identity. Hence for all elements in this cycle other than the identity $aH \neq H$. Hence the order of H is also the order of $(1,1)$, which is four.

Since there is an element in G/H of order four, and since there isn't one in G/K, it follows since isomorphisms preserve element order that G/H is not isomorphic to G/K. Q.E.D.

$\boxed{\text{proposition 55}}$ In D_4, let $K = \{R_0, D\}$ and let $L = \{R_0, D, D', R_{180}\}$. Show that K is normal in L and L is normal in D_4, but K is not normal in D_4, hence normality is not transitive.

$\boxed{\text{proof}}$ Let us first show that K is normal in L. Let KL be defined as above. We have to show that $xKx^{-1} \subseteq K$, for all $x \in L$. We are thus going to look at all the elements separately. Obviously, for R_0, it follows that $xR_0x^{-1} = xx^{-1} = R_0 \in K$ for all elements in L. Let us therefore consider D. The cases $x = R_0$ and $x = D$ are clear and follow by properties of closure and inverses. Furthermore, by analyzing the caley table for D_4 we see that $D'DD'^{-1} = D'DD' = D \in K$ and $R_{180}DR_{180}^{-1} = R_{180}DR_{180} = D \in K$. Thus, K is normal in L.

Let us now show that L is normal in D_4. Again, $xR_0x^{-1} = xx^{-1} = R_0 \in L$ for all $x \in D_4$. We have already looked at the cases $x \in \{R_0, D, D', R_{180}\}$ in xDx^{-1}. It remains to show the cases such that $x \in \{R_{90}, R_{270}, V, H\}$. Therefore, consider $R_{90}DR_{90}^{-1} = R_{90}DR_{270} = D' \in L$, $R_{270}DR_{90} = D' \in L$, $VDV^{-1} = VDV = D' \in L$ and $HDH^{-1} = HDH = D' \in L$. Analogously, by replacing D by D', it follows that $xD'x^{-1} = y \in L$ for all $x \in D_4$. Lastly, we still need to consider $xR_{180}x^{-1}$ for all $x \in D_4$. Again, $x = R_0x = R_{180}$ follow by group properties. Consider: $R_{90}R_{180}R_{270} = R_{90}R_{180}R_{270} = R_{180} \in L$, and $xR_{180}x^{-1}xR_{180}x = R_{180} \in L$, for $x \in \{V, H, D, D'\}$. Hence, it follows that $xLx^{-1} \subseteq L$ for all $x \in D_4$ and L is normal in D_4.

In order to show that K is not normal in D_4, consider, for example, $R_{90}DR_{270} = D'$, which is not in K. Thus, K is not normal in D_4 and normality is not transitive.

$\boxed{\text{proposition 68}}$ If N is a characteristic subgroup of G, show that N is a normal subgroup of G.

$\boxed{\text{proof}}$ Let G and N be defined as above. Consider the inner automorphism of G induced by $a \in G$: $\phi_a(y) = aya^{-1}$. Since ϕ_a is an automorphism, it follows by properties of characteristic subgroups that $\phi_a(N) = aNa^{-1} = N$ for some $a \in G$. Since this is exactly the definition of a normal subgroup, it follows that N is normal in G. qed.