Topology

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$\boxed{\text{theorem 2.22}}$ Let A and B be subsets of a topological space X. Then:

1. $A \subseteq B$ implies that $\overline{A} \subseteq \overline{B}$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$\boxed{\text{proof}}$

1. Suppose by way of contradiction that $A \subseteq B$ and $\overline{A} \not\subseteq \overline{B}$. Then there exists some $x \in \overline{A}$ that is not in $\overline{B}$. Since x being in A would imply that $x \in B$ by supposition, we know that $x \notin A$, hence $x \in \lim A$. Furthermore, x cannot be a limit point of B, otherwise it would be in $\overline{B}$. Hence, negating the definition of a limit point, there must exist some open set U containing x such that $(U - \{x\}) \cap B$ is empty. Furthermore, since $x \notin B$, it follows that $U \cap B$ is also empty.

   Since x is a limit point of A, and since U is an open set containing x, we know by definition of a limit point that $(U - \{x\}) \cap A \neq \emptyset$. Hence there must be some $y \in (U - \{x\}) \cap A$, and by intersection it follows that $y \in A$ and $y \in U$. Since $A \subseteq B$, we know that $y \in B$. Since $y \in U$ and $y \in B$, it follows by intersection that $y \in U \cap B$, contradicting that $U \cap B$ is empty.

2. Now to show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   $\subseteq$ Let $x \in \overline{A \cup B}$ be arbitrary. By definition of the closure it follows that (1) x is a limit point of $A \cup B$ but not in $A \cup B$ or (2) x is in $A \cup B$.

   (a) Suppose that $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in \overline{A}$ as well. Likewise, if $x \in B$, then $x \in \overline{B}$ as well. Hence $x \in \overline{A}$ and $x \in \overline{B}$. Hence by union, $x \in \overline{A} \cup \overline{B}$.

   (b) Suppose that x is a limit point of $A \cup B$ and $x \notin A \cup B$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$ as follows from the negation of union.

   Let U be an arbitrary open set that contains x. Since x is a limit point of $A \cup B$ it follows by definition of a limit point that $(U - \{x\}) \cap (A \cup B) \neq \emptyset$, so there must be something in it: call it y. Since $y \in (U - \{x\}) \cap (A \cup B)$ it follows by intersection that $x \in U - \{x\}$ and $x \in A \cup B$. Hence by union and set compliment it follows that $x \in U$ and $x \in A$ or $x \in B$. Since $x \notin A$ (for if it were, $x \in U \cap A$ since $x \in U$ as well), it follows that $x \in B$. Since $x \in B$, it follows that $x \in \overline{B}$ by definition of the closure.

   Since in either case, $x \in \overline{A} \cup \overline{B}$, it follows that $x \in \overline{A} \cup \overline{B}$. Since x was arbitrary in $\overline{A \cup B}$, it follows that all elements in $\overline{A \cup B}$ are also in $\overline{A} \cup \overline{B}$, hence $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

   $\supseteq$ Now let x be arbitrary in $\overline{A} \cup \overline{B}$. Then by union it follows that $x \in \overline{A}$ or $x \in \overline{B}$. Suppose without loss of generality (since there's nothing special about A that distinguishes it from B), that $x \in \overline{A}$. Then there are two cases as follow from the definition of the closure: (1) $x \in A$ or (2) $x \in \lim A$. If $x \in \lim A$ and $x \in A$, the nit is covered by the first case. As a result, we can for case (2) assume that $x \notin A$.

(a) Suppose that $x \in A$. Then $x \in A \cup B$ as well, hence by definition of the closure $x \in \overline{A \cup B}$.

(b) Suppose that $x \in \lim A$ but $x \notin A$.
Let U be an arbitrary open set containing x. Since x is a limit point of A, it follows by definition of a limit point that $(U - \{x\}) \cap A \neq \emptyset$. Hence there's something in it, why not call it y. By intersection $y \in A$, hence by union $y \in A \cup B$. Hence $y \in (U - \{x\}) \cap (A \cup B)$, and $(U - \{x\}) \cap (A \cup B) \neq \emptyset$. Since U was an arbitrary open set containing x, it follows that for all open sets V containing x, $(V - \{x\}) \cap (A \cup B) \neq \emptyset$, hence x is a limit point of $A \cup B$. Since x is a limit point of $A \cup B$, it follows by definition of the closure that $x \in \overline{A \cup B}$.

Having shown that in either case, $x \in \overline{A \cup B}$, and since x was arbitrary in $\overline{A} \cup \overline{B}$, it follows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Since $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, it follows that $\overline{A \cup B} - \overline{A} \cap \overline{B}$.

Q.E.D