Topology

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$\boxed{\text{Theorem 3.16}}$ Suppose that X is a set and $\mathcal{S}$ is a collection of subsets X. Then $\mathcal{S}$ is a subbasis for some topology on X if and only if every point of X is in some element of $\mathcal{S}$.

$\boxed{\text{proof}}$ Since this statement is biconditional, this proof will be broken into two parts. Let $\mathcal{B}$ denote the set of all finite intersections of sets in $\mathcal{S}$.

$\Rightarrow$ Suppose that $\mathcal{S}$ is a subbasis for some topology on X. Then by definition, the set of all intersections of members of $\mathcal{S}$, $\mathcal{B}$, forms a basis for a topology. Call this topology $\mathcal{T}$. Let p be arbitrary in X. Since by definition of a topology X must be open in $\mathcal{T}$, we have an open containing p. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, we know that there must exist some basic set V of $\mathcal{B}$ such that $p \in V \subseteq X$. By construction of $\mathcal{B}$, $V = \bigcap_{i=0}^{n} S_i$ for elements $S_0, \ldots, S_n \in \mathcal{S}$ and for some $n \in \mathbb{N}^0$. Since p is in this intersection, it follows that for each S_j being intersected over, $p \in S_j$. So there must be some $S \in \mathcal{S}$ such that $p \in S$. Since p is arbitrary in X, it follows that all points of X are in some element of $\mathcal{S}$.

$\Leftarrow$ Suppose that each point in X is in some element of $\mathcal{S}$. We will now use theorem 3.3 to show that $\mathcal{B}$ is the basis for some topology on X, which will show that $\mathcal{S}$ forms the subbasis for some topology on X.

1. First we will need to show that every element of X is in some member of $\mathcal{B}$. Let p be an arbitrary element of X. By our supposition, there exists some $S \in \mathcal{S}$ such that $p \in S$. Consider the intersection of only S, which will just be S. Hence S is an element of $\mathcal{B}$ which contains p. Since p was arbitrary in X, it follows that all points of X are in some member of $\mathcal{B}$.
2. Now let U and V be arbitrary elements of $\mathcal{B}$, and let p be arbitrary in $U \cap V$. Since U and V are elements of $\mathcal{B}$, by construction of $\mathcal{B}$ we know that U and V are both finite intsersections of elements in $\mathcal{S}$, hence their union is as well. This means that $U \cap V$ is itself in $\mathcal{B}$. So there is a basic element in $\mathcal{B}$, namely $U \cap V$, such that $p \in U \cap V \subseteq U \cap V$.. Since U and V are arbitrary in $\mathcal{B}$, and p is arbitrary in $U \cap V$, it follows that the second criteria of Theorem 3.3 is met.

Since both the first and second criteria of Theorem 3.3 are satisfied, it follow that $\mathcal{B}$ is the basis for some topology on X. Since $\mathcal{B}$ is the set of all finite intersections of members of $\mathcal{S}$, it follows that $\mathcal{S}$ forms the subbasis for some topology on X.

$\boxed{\text{Theorem 3.30}}$ Let $(X, \mathcal{T})$ be a topological space, and let Y be a subset of X. Given a basis $\mathcal{B}$ of X, the set $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace $\mathcal{T}_Y$.

$\boxed{\text{proof}}$ We will use Theorem 3.1.

1. First, let x be arbitrary in Y. Since $Y \subseteq X$, and since $\mathcal{B}$ is a basis for X, it follows by Theorem 3.1 that there exists some B such that $x \in B$. Since $x \in B$ and $x \in Y$, it follows by definition of the intersection that $x \in B \cap Y = B_Y$. Since $B \in \mathcal{B}$, it follows by construction of $\mathcal{B}_Y$ that $B_Y \in \mathcal{B}_Y$. Hence there exists some basic set of $\mathcal{B}_Y$ which contains x, namely B_Y. Since x is arbitrary in Y, the first requirement of Theorem 3.1 is satisfied in the case of $\mathcal{B}_Y$.

2. Now let U_Y be an arbitrary open set in the subspace induced by Y. Let p be an arbitrary point in U_Y. By definition of the subspace induced by Y we know that $U_Y = U \cap Y$ for some open U set in the space X. By definition of the intersection we know that $p \in U$ and $p \in Y$. Since $p \in U$ which is open in X, and $\mathcal{B}$ forms a basis for the space X, it follows by Theorem 3.1 that there must exist some basic set $V \in \mathcal{B}$ such that $p \in V \subseteq U$.

   Consider the set $V_Y = V \cap Y \in \mathcal{B}_Y$. Since, as we have shown, $p \in Y$ and $p \in V$, it follows by definition of the intersection that $p \in V \cap Y = V_Y$. We now need to show that $V_Y \subseteq U_Y$. To do so, let a be arbitrary in V_Y. Then by definition of V_Y and by definition of the intersection it follows that $a \in V$ and $a \in Y$. Since by construction $V \subseteq U$, it follows by definition of a subset that $a \in U$ as well. Since $a \in U$ and $a \in Y$, it follows by definition of the intersection that $a \in U \cap Y = U_Y$. Since a was arbitrary in V, it follows that $V_Y \subseteq U_Y$.

   Having shown that there exists some basic set V_Y of $\mathcal{B}_Y$ such that $p \in V_Y \subseteq U_Y$, and since U_Y is arbitrary in Y and p is arbitrary in U_Y, it follows that the second requirement of Theorem 3.1 is met.

Having shown that the requirements of Theorem 3.1 are met in the case of $\mathcal{B}_Y$ and Y, it follows that $\mathcal{B}_Y$ forms a basis for the subspace induced by Y.