$\boxed{\text{Theorem 7.29}}$ Suppose $f : X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff. Then f is a homeomorphism.

$\boxed{\text{proof}}$ We know already that f is bijective, and that it is continuous. In showing that f is a homeomorphism, it remains to be shown that $f^{-1} : Y \to X$ is continuous.

To show that f^{-1} is continuous, let U be open in X. Noting that $(f^{-1})^{-1} = f$, consider $f(U)$. We need to show that $f(U)$ is open in Y.

Since f is continuous and surjective, and since X is compact, it follows by Theorem 7.15 that Y is compact.

Since U is open and X is open in X, it follows by Theorem 2.15 that $X - U$ is a closed map.

Since X is compact and Y is Hausdorff, and since f is continuous from X to Y, we know by Theorem 7.24 that f is closed.

Because f is closed, and since $X - U$ is closed, it follows by definition of a closed map that $f(X - U)$ is closed. By properties of functions, $f(X - U) = f(x) - f(U)$, hence $f(X) - f(U)$ is closed. Furthermore, since f is bijective, it is surjective, and by definition of surjectivity, $f(X) = Y$. Substituting, we have $f(X) - f(U) = Y - f(U)$, hence $Y - f(U)$ is closed.

Since $Y - f(U)$ is closed, and since Y is open in Y, it follows by Theorem 2.15 that $f(U) = (f^{-1})^{-1}(U)$ is open.

Since U was an arbitrary open set in the codomain of f^{-1}, and since its inverse image is open in the domain of f^{-1}, it follows that all open sets in X are mapped to by open sets in Y under f^{-1}. So by definition of a continuous map, f^{-1} is continuous.

Since $f : X \to Y$ is a continuous bijection whose inverse is also continuous, it follows by definition of a homeomorphism that f is a homeomorphism.

$\boxed{\text{exersize 6.7}}$ If A and B are compact subsets f X, then $A \cup B$ is compact. Suggest and prove a generalization.

$\boxed{\text{generalization}}$ Given a finite collection of compact subsets $\{S_i\}_{i<n}$ of X for some $n \in \mathbb{N}$, the union $\bigcup_{i=1}^{n} S_i$ is also compact.

$\boxed{\text{proof}}$

Let $\mathcal{C}$ be an open cover for $\bigcup_{i=1}^{n} S_i$. Let S_j be an arbitrary member of $\{S_i\}_{i<n}$. Notice that since each $S_j \subseteq \bigcup_{i=1}^{n} S_i$, and since by definition of a subcover, $\bigcup_{i=1}^{n} S_i \subseteq_{U \in \mathcal{C}} U$, it follows by the transitivity of subsets that $S_j \subseteq_{U \in \mathcal{C}} U$. Hence $\mathcal{C}$ forms an open cover for S_j as well. Since $\mathcal{C}$ forms a cover for S_j, and since $\mathcal{C}$ is open, we know by definition of compactness and the construction of S_j as a member of a collection of compact sets that $\mathcal{C}$ has a finite subcover for S_j, call it $\mathcal{C}_j$. Since S_j was arbitrary, it follows that for each element $S_k$of $\{S_i\}_{i<n}$ there is some finite subcover $\mathcal{C}_k \subseteq \mathcal{C}$ that covers S_k. For convenience, let the union of the elements of all such $\mathcal{C}_k$ be denoted as U_k. Let $\mathcal{S}$ be the union of all such $\mathcal{C}_k$. Then since each S_k is a subset of its corresponding U_k, it follows from a theorem in foundations that $\bigcup_{i=1}^{n} S_i \subseteq \bigcup_{i \in A} U_i = \bigcup_{C \in \mathcal{S}} C$. Hence by definition of a cover, $\mathcal{S}$ forms a cover for $\bigcup_{i=1}^{n} S_i$. Furthermore, since the union of a finite collection of finite sets is finite, $\mathcal{S}$ is finite. So we have a finite subcover for $\mathcal{C}$ of $\bigcup_{i=1}^{n} S_i$. Since $\mathcal{C}$ was an arbitrary open cover for $\bigcup_{i=1}^{n} S_i$, it follows that each open cover for $\bigcup_{i=1}^{n} S_i$ has a finite subcover. Hence by definition of compactness, $\bigcup_{i=1}^{n} S_i$ is compact.