$\boxed{\text{Lemma: some useful facts from foundations}}$ Given a set X and sets U and A such that $U \subseteq X$ and $A \subseteq X$, the following are true:

1. $(X - A) \cup U = X - (A - U)$;
2. $(X - A) \cap U = U - A$.

$\boxed{\text{2.15: proposition}}$ In a topological space $(X, \mathcal{T})$, and given a closed subset $A \subseteq X$ and an open set $U \in \mathcal{T}$, $A - U$ is closed and $U - A$ is open.

$\boxed{\text{proof}}$ Let $(X, \mathcal{T})$ A and U be instantiated as in the proposition. From the first part of the lemma it follows that $X - (A - U) = (X - A) \cup U$. Furthermore, sense A is closed, it follows by theorem 2.14 that $X - A$ is closed. Since $X - A$ is open and so is U, it follows from the definition of a topology that the union $(X - A) \cap U = X - (A - U)$ is also open. Since $X - (A - U)$ is open, it follows from theorem 2.14 that $A - U$ is closed.

Furthermore, since $X - A$ is open and so is U, it follows from the definition of a topology that the intersection $(X - A) \cap U$ must also be open. From the second part of the lemma it follows that $(X - A) \cap U = U - A$, hence $U - A$ is open. Q.E.D.

$\boxed{\text{definition}}$ Let $\lim A$ denote the set of all limit points of A. Define the closure of A as $\overline{A} = A \cup \lim A$

$\boxed{\text{theorem 2.13}}$ The closure of the closure is closed. That is, given a subset $A \subseteq X$ in a topological space $(X, \mathcal{T})$, $\overline{A} = \overline{\overline{A}}$.

$\boxed{\text{proof}}$ Since what we're really trying to prove here is a set equality proof, this will be broken up into two subset proofs.

- $\subseteq$ We get this one for free. Since $\overline{\overline{A}} = \overline{A} \cup \lim \overline{A}$, it follows that $\overline{A} \subseteq \overline{\overline{A}}$.

- $\supseteq$ Suppose by way of contradiction that there exists some $x \in \overline{\overline{A}}$ such that $x \notin \overline{A}$. Then by definition of the closure it follows that x must be a limit point of $\overline{A}$ (since the closure is the set along with its limit points).

  Furthermore, x cannot be a limit point of A, for if it were, it would be in $\overline{A}$ by definition of the closure. Hence $x \notin \overline{A}$. Since x is not a limit point of A, it follows by negating the definition of a limit point that there exists some open set U containing x such that $(U - x) \cap A = \emptyset$. Furthermore, $x \notin A$ either, as if it were, it would be in $\overline{A}$. Hence $U \cap A = \emptyset$.

  Since U is an open set containing x, and since x is a limit point of $\overline{A}$, it follows by definition of a limit point that $(U - \{x\}) \cap \overline{A} \neq \emptyset$. Since $(U - \{x\}) \cap \overline{A}$ is not the open set, there must be something in it, call it z. By intersection it follows that $z \in U - \{x\}$ and $z \in \overline{A}$, and by the set compliment $z \in U$.

  Now suppose by way of contradiction (this only applies on this paragraph)that $z \in A$. Then since $z \in A$ and $z \in U$, it follows that $z \in U \cap A$, but we have supposed $U \cap A$ to be empty. Hence we arrive at a contradiction and conclude that $z \notin A$. Since $z \notin A$ and $z \in \overline{A}$ it follows that z is a limit point of A.

  Since z is a limit point of A, and since U is an open set containing z, it follows by definition of a limit point that $(U - \{z\}) \cap A \neq \emptyset$. Hence there must be some y such that $y \in (U - \{y\}) \cap A$. By intersection and the set compliment it follows that $y \in U$ and $y \in A$, hence $y \in U \cap A$, which we have supposed to be empty, arriving us at a contradiction. So there cannot be any element in $\overline{\overline{A}}$ that is not in $\overline{A}$, so $\overline{\overline{A}} \subseteq \overline{A}$.