Math 251W: Foundations of Advanced Mathematics

Solutions to Portfolio problems from sections 4.2, 4.3, & 4.4

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Problem 4.2.5

2 proposition: Given sets $A \subseteq X$ and $B \subseteq Y$, $A \times B = \pi_1^*(A) \cap \pi_2^*(B)$

$\boxed{\text{proof}}$ (direct) Let A, B, arbitrary sets, and X and Y be arbitrary supersets such that $A \subseteq X$ and $B \subseteq Y$. Since this proof involves set equality, it will be broken into two parts.

- Let (x, y) be an arbitrary element of $A \times B$. By definition of the Cartesian product, x is in A and y is in B. Furthermore, by definition of subsets, $x \in X$ and $y \in Y$.

  By definition of the projection map, for all elements $n \in Y$ and elements $m \in X$, $\pi_1((x, n)) = x$ and $\pi_2(m, y) = y$. By definition of the preimage, $(x, n) \in \pi_1^*(x)$ and $(m, y) \in \pi_2^*(y)$. Furthermore, since $A \subseteq X$ and $B \subseteq Y$, and by definition of intersection, $(x, y) \in \pi_1^*(A) \cap \pi_2^*(B)$. Thus, since (x, y) is arbitrary, $A \times B \subseteq \pi_1^*(A) \cap \pi_2^*(B)$.
- Let (i, j) be an arbitrary element of $\pi_1^*(A) \cap \pi_2^*(B)$. By definition of intersection, $(i, j) \in \pi_1^*(A)$ and $(i, j) \in \pi_2^*(B)$. By definition of the projection map $i \in A$ and $j \in B$. Thus, by definition of the Cartesian product, $(i, j) \in A \times B$. Since (i, j) is arbitrary, $A \times B \supseteq \pi_1^*(A) \cap \pi_2^*(B)$

By the definition of set equality, for any sets $A \subseteq X$ and $B \subseteq Y$, $A \times B = \pi_1^*(A) \cap \pi_2^*(B)$.

3 Given sets X and Y. Let $P \subseteq X \times Y$. Does $P = \pi_1(P) \times \pi_2(P)$? Give a proof or counterexample.

$\boxed{\text{counterproof}}$

Consider sets $X = \{1, 2\}$, $Y = \{a, b\}$. Let P be the set $\{(1, a), (2, b)\}$. By definition of the projection map for π_1 and π_2, $\pi_1(P) \times \pi_2(P) = \{(1, a), (1, b), (2, a), (2, b)\}$, which clearly does not equal P.

Thus this statement is false.

Problem 4.2.6

viii proposition: Given $f : A \to B$ is a function, and $\{V_j\}_{j \in J}$ is a family of sets such that for all $j \in J, V_j \subseteq B$, $f^*\left(\bigcup_{j \in J} V_j\right) = \bigcup_{j \in J} f^*(V_j)$.

$\boxed{\text{proof}}$ (Direct)

Let A and B be arbitrary sets, $f : A \to B$ be an arbitrary function, J be a nonempty set, and $\{V_j\}_{i \in J}$ be an arbitrary indexed family of sets such that for all $j \in J$, $V_j \subseteq B$.

Let x be an arbitrary element of $f^*(\bigcup_{j \in J} V_j)$. By definition of the preimage $f(x) \in \bigcup_{j \in J} V_j$. Furthermore, by the definition of the union of a family of sets, there exists some element k

in J such that $f(x) \in V_k$. By definition of the preimage, $x \in f^*(V_k)$. By definition of the union, $x \in \bigcup_{j\in J} f^*(V_j)$. Since x is arbitrary, $f^*\left(\bigcup_{j\in J} V_j\right) \subseteq \bigcup_{j\in J} f^*(V_j)$.

Now let y be an arbitrary element of $\bigcup_{j\in J} f^*(V_j)$. By definition of the union, there exists some i in J such that $x \in f^*(V_i)$. By definition of the preimage, $f(x) \in V_i$. Furthermore, by definition of the union, $f(x) \in \bigcup_{j\in J}$. Thus, by definition of the preimage, $x \in f^*\left(\bigcup_{j\in J} V_j\right)$. Since x is arbitrary, $f^*\left(\bigcup_{j\in J} V_j\right) \supseteq \bigcup_{j\in J} f^*(V_j)$.

By definition of set equality, given $f : A \to B$ is a function, and $\{V_j\}_{j\in J}$ is a family of sets such that for all $j \in J, V_j \subseteq B$, $f^*\left(\bigcup_{j\in J} V_j\right) = \bigcup_{j\in J} f^*(V_j)$.

Problem 4.2.8

1 Find an example of a function $f : A \to B$ together with sets $X \subseteq A$ and $Y \subseteq B$ such that $f_*(X) = Y$ and $X \neq f^*(Y)$.

Consider the case where $X = \{a, b\}$, $Y = \{1\}$, $A = \{a, b, c\}$, and $B = \{1, 2, 3, 4\}$. Let $f(a) = 1$, $f(c) = 1$, and $f(b) = 1$. All elements in X map to all elements in Y, which by definition of the image means that $f_*(X) = Y$.

Now, consider $f^*(Y)$. By definition of the preimage, $f^*(Y) = \{a, b, c\} \neq X$.

More generally, consider the case where $f_*(X) = Y$. By definition of the image, all elements in X map to each element in Y. Now consider the an element in the domain, $a \in A$, such that $f(a) \in Y$ and $a \notin X$. By definition of the preimage, $a \in f^*(X)$. Since we have defined a not to be an element of X, $f^*(Y) \neq X$.

Problem 4.2.11

proposition: Given $f : A \to B$ is a function, and $P, Q \subseteq A$, $f_*(P) - f_*(Q) \subseteq f_*(P - Q)$.

$\boxed{\text{proof}}$ (Direct) Let A, B, P, and Q be arbitrary sets such that $P, Q \subseteq A$, and $f : A \to B$ be an arbitrary function.

Let y be an arbitrary element of $f_*(P) - f_*(Q)$. By definition of set difference, $y \in f_*(P)$ and $y \notin f_*(Q)$. Because P and Q are subsets of A, by definition of the image, there exists some element x in P such that $f(x) \in P$, and $f(x) \notin Q$. By definition of set difference, $f(x) \in P - Q$. By definition of the image, $f(x) \in f_*(P - Q)$. It has already be stated that $y = f(x)$, hence $y \in f_*(P - Q)$. Since y is an arbitrary element of $f_*(P) - f_*(Q)$, $f_*(P) - f_*(Q) \subseteq f_*(P - Q)$.

Thus, given $f : A \to B$ is a function, and $P, Q \subseteq A$, $f_*(P) - f_*(Q) \subseteq f_*(P - Q)$.