$\boxed{\text{Theorem 4.47}}$ The quotient topology defines a topology.

$\boxed{\text{Proof}}$ Let X and Y be topological spaces, and let f be a surjective function $f : X \to Y$. Let $\mathcal{T} = \{U \in \mathcal{P}(Y) : f^{-1}(U) \in \mathcal{T}_X\}$, where $\mathcal{T}_X$ denotes topology on X. We will need to show that $\mathcal{T}$ meets all of the requirements as provided by the definition of a topology.

1. To show that $Y \in \mathcal{T}$, note by definition of a function, $f^{-1}(Y) = X$. Otherwise not every point in X would map to some point in Y under f, which violates the definition of a function. Furthermore, by definition of a topology, $f^{-1}(Y) = X$ must be open in $\mathcal{T}_X$, hence $Y \in \mathcal{T}$.

2. To show that $\emptyset \in \mathcal{T}$, consider $f^{-1}(\emptyset)$. Suppose by way of contradiction that some $x \in f^{-1}(\emptyset)$. Then x gets mapped to nothing, contradiction the supposition that f is a function. Hence there is nothing in $f^{-1}(\emptyset)$. In other words, $f^{-1}(\emptyset) = \emptyset$, which is open in X by definition of a topology. Hence $\emptyset \in \mathcal{T}$.

3. Now suppose that U and V are elements of $\mathcal{T}$. Then by definition of $\mathcal{T}$, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Furthermore, since by definition of a topology the union of open sets is open, $f^{-1}(U) \cap f^{-1}(V)$ must also be open in X. Recall from Foundations that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. Hence $f^{-1}(U \cap V)$ must be open in X. Hence by definition of the quotient topology, $U \cap V \in \mathcal{T}$.

4. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary collection of sets in Y such that $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$. Then by construction of $\mathcal{T}$, $f^{-1}(U_\alpha)$ is open in X for all $\alpha \in A$. Recall from Foundations that the inverse image of an arbitrary union is the union of the inverse images, hence $f^{-1}\big(\bigcup_{\in A} U_\alpha\big) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$. Since each $f^{-1}(U_\alpha)$ is open in X, and since by definition of a topology the arbitrary union of open sets must also be open, we conclude that $f^{-1}\big(\bigcup_{\in A} U_\alpha\big)$ is open in X as well. Hence by definition of the quotient topology, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Having shown that all four requirements of a topology are met in the case of the quotient topology on Y induced by f, we conclude that the quotient topology really is a topology. Q.E.D.