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Math 251W: Foundations of Advanced Mathematics

Portfolio problems from sections 6.1, 6.2, & 6.3

Problem 6.1.Bonus*: Prove the following

<u>proposition:</u> Given sets A and B. If A and B have the same cardinality, the power set $\mathcal{P}(A)$ has the same cardinality as the power set $\mathcal{P}(B)$ (i.e. $A \sim B \Rightarrow \mathcal{P}(A) \sim \mathcal{P}(B)$.)

$\boxed{\text{proof:}}$ Let A and B be arbitrary sets such that $A \sim B$. There are three cases, either both are empty, both are finite, or both are infinite.

Case 1: Suppose A and B are both empty. By definition, they have a cardinality of zero, and their powersets both have a cardinality of one.

Case 2: Suppose A and B are both finite. Then $|A| = |B| = n$ for some $n \in \mathbb{N}$. By definition of the powerset, $|\mathcal{P}(A)| = 2^n = |\mathcal{P}(B)|$.

Case 3: Suppose A and B are both infinite. Since $A \sim B$, there exists some bijective function $f : A \to B$. Consider a function $h : \mathcal{P}(A) \to \mathcal{P}(B)$ defined $h(X) = f_*(X)$ for all $X \subseteq A$, that is $X \in \mathcal{P}(A)$. This function is well defined, as X is a subset of the domain of f, A, and $f_*(X)$ is in the codomain of h. Now we need to show that this function is both surjective and injective, and therefore bijective.

- Let X be an arbitrary element in $\mathcal{P}(B)$. By definition of the powerset, $X \subseteq B$. Since f is surjective, $X \subseteq f_*(A)$. Thus, by definition of h, $X \in h_*(\mathcal{P}(A))$.

  Let Y be an arbitrary element in $h_*(\mathcal{P}(A))$. By definition of h

Somehow, this function is bijective. By definition, since there exists a bijective function $h : \mathcal{P}(A) \to \mathcal{P}(B)$, $\mathcal{P}(A) \sim \mathcal{P}B$.

Problem 6.1.6*: Prove the following

<u>proposition:</u> Let A and B be finite sets such that $|A| = |B|$. Given a function $f : A \to B$, the following are equivalent:

i. f is bijective

ii. f is injective

iii. f is surjective

$\boxed{\text{proof:}}$

Let A and B be sets such that $|A| = |B|$. Let $f : A \to B$ be an arbitrary function. By definition of a function, each element in the codomain, A, is mapped to exactly one element in the codomain, B. Since there are the same amount of elements in A as in B, and since each element in the domain A is mapped to an element in B, there are no elements in B which are not in the image of A. Thus, $f_*(A) = B$. By definition of surjectivity, f is surjective. Let x and y be arbitrary elements in A such that $x \neq y$. Since the domain and the codomain have the same cardinality, and since a function maps each element in the domain to exactly one element in the codomain,

$f(x) \neq f(y)$. By definition of bijectivity, f is bijective. Thus, for all functions $f : A \to B$, f is bijective, injective and surjective. Since this is true for all functions, f's bijectivity, injectivity, and surjectivity are equivalent, as all of them must be true for any function $f : A \to B$.

Problem 6.1.13: Prove the following

proposition: Given countable sets A and B, $A \times B$ is countable.

$\boxed{\text{proof}}$ Let A and B be sets such that both sets are countable. Consider the sets defined $B_a = \{(a,x)|x \in B\}$ for all $a \in A$. Consider the projection function $\pi_2 : B_a \to B$, for some arbitrary $a \in A$. By definition of the projection function, $\pi_2(a,x) = x$ for all $x \in B$. Consider another function $g : B \to B_a$, defined $g(x) = (a,x)$ for all $x \in B$. Consider the composition $g \circ \pi_2 : B_a \to B_a$, which is well defined since the domain of g is the codomain of π_2. By definition of both functions and of composition, $g \circ \pi_2((a,x)) = g(\pi_2(a,x)) = g(x) = (a,x)$ for all $(a,x) \in B_a$. Thus, the composition is by definition the identity map on B_a, hence g is a left inverse of π_2.

Furthermore, consider the composition $\pi_2 \circ g : B \to B$. Once again, since the codomain of g is the domain of π_2, the composition is well defined. By definition of both functions, $\pi_2 \circ g(x) = \pi_2(g(x)) = \pi_2((a,x)) = x$ for all $x \in B$. Therefore, by definition of the identity map, $\pi_2 \circ g = 1_B$. This shows that g is also a right inverse of π_2, meaning that it is the inverse of π_2.

Since π_2 has an inverse, it is by theorem bijective, thus showing that $B \sim B_a$. By this we conclude that B_a is countable.

Now consider the set defined $\bigcup_{a \in A} B_a$. By definition of B_a, and by definition of the union, $\bigcup_{a \in A} B_a = \{(a,b)|a \in A, b \in B\}$. By definition of the Cartesian product, $\bigcup_{a \in A} B_a = A \times B$. Since this is the union of an indexed family of sets, indexed by A, which is defined to be countable, we know by Theorem 6.1.10 that $\bigcup_{a \in A} B_a$ is also countable. Thus, by substitution, $A \times B$ is countable.

Problem 6.3.11: Prove the following

<u>proposition:</u>

$$P(n): \qquad \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n} \qquad \forall n\geq 2$$

$\boxed{\text{proof}}$(PMI-I) Let n be an arbitrary natural number such that $n \geq 2$. Consider the base-case where $n=2$. By substituting,

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{2^2}\right)=\frac{3}{4}=\frac{2+1}{2(2)}=\frac{n+1}{2n}.$$

Thus, the base case is true.
Suppose $k \geq 2$ is an arbitrary natural number such that

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2k}.$$

Consider $k+1$. By the associative, communitative, and distributive properties of real numbers, and substitution,

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}-\frac{1}{2k(k+1)}=\frac{(k+1)^2-1}{2k(k+1)}=\frac{(k+1)+1}{2(k+1)}.$$

Hence $P(k+1)$ holds. Since k is an arbitrary natural number such that $k \geq 2$, $P(k)$ implies $P(k+1)$ for all natural numbers greater than two. Thus, by way of induction,

$$P(n): \qquad \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n} \qquad \forall n\geq 2.$$

Problem 6.3.12: Prove the following

<u>proposition:</u> For all $(n \geq 2) \in \mathbb{N}$,

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}}>\sqrt{n}.$$

$\boxed{\text{proof:}}$ For proof by induction, consider the case where $n=2$. Hence we have

$$\left(\sum_{i=1}^{2}\frac{1}{\sqrt{i}}=1+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}}\right)>\left(\frac{2}{\sqrt{2}}=\sqrt{2}\right).$$

Thus, the base case holds.

Now suppose the statement holds for some particular $(n \geq 2) \in \mathbb{N}$, such that

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}}>\sqrt{n}.$$

By the associative property of addition,

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{1+n}}.$$

By induction hypothesis,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{1+n}} > \sqrt{n+1}$$

. By way of induction, it follows that for all $(n \geq 2) \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}.$$

Problem Irreducible Polynomials: Prove the following

proposition: Every reducible polynomial can be written as a product of irreducible polynomials. (AKA) for all $n \in \mathbb{N}$, reducible polynomials of degree n can be written as the product of irreducible polynomials.

$\boxed{\text{proof}}$ (Strong Induction) The lowest possible degree for a polynomial to be reducible is degree $n = 2$.

For proof by induction, consider the base case $n = 2$. Let p be an arbitrary reducible polynomial of degree 2. By definition of reducible polynomials, $p_2 = q_a r_b$, where q_a and r_b are polynomials of degrees a and b, such that $a, b < 2$. Furthermore, since $a, b < 2$, q_a and r_b are either of degree 1 or 0, both of which are degrees at which all polynomials are irreducible. Thus, p_2 is a product of irreducible polynomials for $n = 2$, and the base case holds.

For the induction hypothesis, suppose all reducible polynomials of degree $i >= n$ can be written as the product of reducible polynomials for some particular $n \in \mathbb{N}$.
Consider an arbitrary polynomial of degree $i + 1$, p_{i+1} such that p_{i+1} is reducible. By definition of reducible polynomials, $p_{i+1} = q_a r b$, where q_a and r_b are polynomials of degree a and b respectively, and a and b are both less than $i + 1$. By the induction hypothesis, these polynomials can be written as a product of irreducible polynomials. Furthermore, by substitution and the associative property, p_{i+1} can be written as the product of irreducible polynomials.

Thus, by way of strong induction, all reducible polynomials can be written as a product of irreducible polynomials.

Problem 6.4.6: Prove the following

<u>proposition:</u> For all $n \in \mathbb{N}$ such that $n > 5$, $F_n = 5F_{n-4} + 3F_{n-5}$.

$\boxed{\text{proof}}$ For proof by induction, consider the base case where $n = 6$. By definition of the Fibonacci sequence, $F_6 = 8$, $F_2 = 1$, and $F_1 = 1$. By substitution $F_6 = 5(1) + 3(1) = 8$. Thus the base case hold.

Suppose by way of strong induction that $F_i = 5F_{i-4} + 3F_{i-5}$ for all $i \leq n$ for some particular $(n > 6) \in \mathbb{N}$. Consider F_{i+1}. By definition of the Fibonacci sequence, $F_{i+1} = F_i + F_{i-2}$. By the induction hypothesis,

$$F_{i+1} = 5F_{i-4} + 3F_{i-5} + F_{i-2} = 5F_{i-4} + 3F_{i-5} + F_{i-2} + F_{i-3}$$

$$= 5F_{i-4} + 3F_{i-5} + F_{i-3} + F_{i-4} + F_{i-3}$$

$$= (3F_{i-4} + 3F_{i-5}) + 2F_{i-3} + 3F_{i-4}$$

$$F_{i+1} = 5F_{(i+1)-4} + 3F_{(i+1)-5}.$$

Thus, by way of strong induction, for all $n \in \mathbb{N}$ such that $n > 6$, $F_n = 5F_{n-4} + 3F_{n-5}$.

Problem 6.4.14(1i): Prove the following

Given $P_1 = 1$, $P_{n+1} = P_n + (3n+1)$, $T_1 = 1$, $T_{n+1} = T_n + (n+1)$, $L_1 = 1$, and $L_{n+1} = L_n + 1$ for all $n \in \mathbb{N}$,

<u>proposition:</u> $P_n = 3T_n - 2L_n$ for all $n \in \mathbb{N}$

$\boxed{\text{proof}}$ For proof by induction consider the base case where $n = 1$. It is given that $P_1 = 1$, and that $T_1 = 1$, and that $L_1 = 1$. Thus, $P_1 = 1 = 3(1) - 2(1) = 3T_1 - 2T_1$.

For our induction hypothesis, suppose that $P_n = 3T_n - 2L_n$ for for some particular $n \in \mathbb{N}$. Consider P_{n+1}. By definition of P, $P_{n+1} = P_n + (3n+1)$. By the induction hypothesis and by definition of T and L,

$$P_{n+1} = 3T_n - 2L_n + (3n+1) = 3(T_{n+1} - (n+1)) - 2(L_{n+1} - 1) + (3n+1)$$

$$= 3T_{n+1} - 3n - 3 - 2L_{n+1} + 2 + 3n + 1$$

$$P_{n+1} = 3T_{n+1} - 2L_{n+1}.$$

Thus, by way of induction, $P_n = 3T_n - 2L_n$ for all $n \in \mathbb{N}$.