MAT4120

Exercises for Mathematical Optimization

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Contents

Abstract

This document contains my solutions to the exercises for the course MAT4120–Mathematical Optimization, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: `https://github.com/augustfe/MAT4120`.

1 The basic concepts

Exercise 1.1. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and assume that $x_1 \leq x_2$ and $y_1 \leq y_2$. Verify that the inequality $x_1 + y_1 \leq x_2 + y_2$ holds. Let now λ be a non-negative real number. Explain why $\lambda x_1 \leq \lambda x_2$ holds. What happens if λ is negative?

Solution 1.1. With $x_1 \leq x_2$, we have that

$$(x_1)_i \leq (x_2)_i \qquad \forall i = 1, \dots, n. \tag{1.1}$$

Component-wise, we then have

$$(x_1)_i + (y_1)_i \leq (x_2)_i + (y_2)_i \qquad \forall i = 1, \dots, n, \tag{1.2}$$

and thus $x_1 + y_1 \leq x_2 + y_2$. Similarly, if $\lambda \geq 0$, we have

$$\lambda(x_1)_i \leq \lambda(x_2)_i \qquad \forall i = 1, \dots, n, \tag{1.3}$$

and therefore $\lambda x_1 \leq \lambda x_2$. Finally, for $\lambda < 0$, the inequality reverses:

$$\lambda(x_1)_i \geq \lambda(x_2)_i \qquad \forall i = 1, \dots, n, \tag{1.4}$$

giving $\lambda x_1 \geq \lambda x_2$.

Example 1.2.1 *(The non-negative real vectors)* The sum of two non-negative numbers is again a non-negative number. Similarly, we see that the sum of two non-negative vectors is a non-negative vector. Moreover, if we multiply a non-negative vector by a non-negative number, we get another non-negative vector. These two properties may be summarized by saying that $\mathbb{R}^n_+$ is closed under addition and multiplication by non-negative scalars. We shall see that this means that $\mathbb{R}^n_+$ is a convex cone, a special type of convex set.

Exercise 1.2. Think about the question in Exercise 1.1 again, now in light of the properties explained in Example 1.2.1.

Solution 1.2. We can now rewrite $x_1 \leq x_2$ as $x_2 - x_1 \in \mathbb{R}^n_+$. We can now easily consider the first question as

$$(x_2 + y_2) - (x_1 + y_1) = (x_2 - x_1) + (y_2 - y_1) \in \mathbb{R}^n_+, \tag{1.5}$$

as $\mathbb{R}^n_+$ is closed under addition. Similarly, we can use the fact that $\mathbb{R}^n_+$ is closed under multiplication by non-negative scalars to see that

$$(\lambda x_2 - \lambda x_1) = \lambda(x_2 - x_1) \in \mathbb{R}^n_+, \tag{1.6}$$

for $\lambda \geq 0$. As $\mathbb{R}^n_+$ is not closed under multiplication by negative scalars, we cannot conclude that $\lambda x_1 \leq \lambda x_2$ for $\lambda < 0$.

Exercise 1.3. Let $a \in \mathbb{R}^n_+$ and assume that $x \leq y$. Show that $a^T x \leq a^T y$. What happens if we do not require a to be non-negative here?

Solution 1.3. With $a \in \mathbb{R}^n_+$ and $x \leq y$, we have that

$$x_i \leq y_i \qquad \forall i = 1, \dots, n, \tag{1.7}$$

and consequently

$$a_i x_i \leq a_i y_i \qquad \forall i = 1, \dots, n, \tag{1.8}$$

as shown previously. Written in vector notation, we therefore have

$$a^T x \leq a^T y. \tag{1.9}$$

With a not necessarily non-negative, we may have neither $a^T x \geq a^T y$ nor $a^T x \leq a^T y$, as we could have $a_i x_i > a_i y_i$ for some i.

Exercise 1.4. Show that every ball $B(a, r) := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$ is convex (where $a \in \mathbb{R}^n$ and $r \geq 0$).

Solution 1.4. Let $x, y \in B(a, r)$ for some $a \in \mathbb{R}^n$ and $r \geq 0$. Then, let $0 \leq \lambda \leq 1$ and consider $z = \lambda x + (1 - \lambda) y$. We then have

$$\begin{aligned} \|z - a\| &= \|\lambda(x - a) + (1 - \lambda)(y - a)\| \\ &\leq \lambda\|x - a\| + (1 - \lambda)\|y - a\| \\ &\leq \lambda r + (1 - \lambda) r = r, \end{aligned} \tag{1.10}$$

showing that $z \in B(a, r)$. $B(a, r)$ is therefore convex.

Exercise 1.5. Explain how you can write the LP problem $\max\{c^T x : Ax \leq b\}$ in the form $\max\{c^T x : Ax = b, x \geq O\}$

Solution 1.5. We introduce new slack variables $w \in \mathbb{R}^m_+$, where m is the number of rows/inequalities in A, defined by

$$w_j = b_j - (Ax)_j \quad \forall j = 1, \dots, m. \tag{1.11}$$

We can then rewrite our system of equations by setting $\tilde{A} = \begin{bmatrix} A & I \end{bmatrix}$, $\tilde{x} = \begin{bmatrix} x \\ w \end{bmatrix}$, and $\tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}$. We then have

$$\tilde{A}\tilde{x} = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = Ax + w = b, \tag{1.12}$$

and

$$\tilde{c}^T \tilde{x} = \begin{bmatrix} c \\ 0 \end{bmatrix}^T \begin{bmatrix} x \\ w \end{bmatrix} = c^T x. \tag{1.13}$$

Again, as we require $w \geq 0$, we then have $Ax \leq b$.

Exercise 1.6. Make a drawing of the standard simplices S_1, S_2, and S_3. Verify that each unit vector e_j lies in S_n (e_j has a one in position j, all other components are zero). Each $x \in S_n$ may be written as a linear combination $x = \sum_{j=1}^n \lambda_j e_j$ where each λ_j is non-negative and $\sum_{j=1}^n \lambda_j = 1$. How? Can this be done in several ways?

Solution 1.6. Fig. 1 shows the standard simplices S_1, S_2, and S_3. Clearly each unit vector e_j lies in S_n. Each $x \in S_n$ may be written as $\sum_{j=1}^n \lambda_j e_j$ where $\lambda_j = x_j$, i.e. the coordinate components of x.

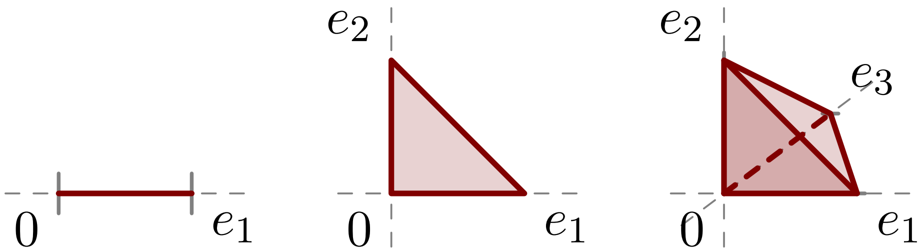

Figure 1: The simplices S_1, S_2, and S_3. For S_1, the standard simplex is the point e_1, for S_2, the standard simplex is the line segment between e_1 and e_2, and for S_3, the standard simplex is the triangle with vertices e_1, e_2, and e_3.

Exercise 1.7. Show that each convex cone is indeed a convex set.

Solution 1.7. To see that a convex cone is a convex set, let first $x_1, x_2 \in C$. Then let $0 \leq \lambda_1 \leq 1$ and $\lambda_2 = 1 - \lambda_1 \geq 0$. We then have by definition of the convex cone that

$$\lambda_1 x_1 + (1 - \lambda_1) x_1 = \lambda_1 x_1 + \lambda_2 x_2 \in C, \tag{1.14}$$

showing that the set is convex.

Exercise 1.8. Let $A \in \mathbb{R}^{m \times n}$ and consider the set $C = \{x \in \mathbb{R}^n : Ax \leq O\}$. Prove that C is a convex cone.

Solution 1.8. Let $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We then have

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A x_1 + \lambda_2 A x_2 \leq \lambda_1 O + \lambda_2 O = O, \tag{1.15}$$

showing that the set is a convex cone.

Polyhedral cone A convex cone of the form $\{x \in \mathbb{R}^n : Ax \leq O\}$ where $A \in \mathbb{R}^{m \times n}$ is called a *polyhedral cone*. Let $x_1, \ldots, x_t \in \mathbb{R}^n$ and let $C(x_1, \ldots, x_t)$ be the set of vectors of the form

$$u = \sum_{j=1}^{t} \lambda_j x_j, \tag{1.16}$$

where $\lambda_j \geq 0$ for each $j = 1, \ldots, t$.

Exercise 1.9. Prove that $C(x_1, \ldots, x_t)$ is a convex cone.

Solution 1.9. Let $C = C(x_1, \ldots, x_t)$ here for convenience. Let $u, v \in C$ with respective coefficients $\lambda_j, \mu_j \geq 0$ for $j = 1, \ldots, t$. Then, for arbitrary coefficients $\alpha, \beta \geq 0$, we have

$$\begin{aligned} A(\alpha u + \beta v) &= A\left(\alpha \sum_{j=1}^{t} \lambda_j x_j + \beta \sum_{j=1}^{t} \mu_j x_j\right) \\ &= \alpha A \sum_{j=1}^{t} \lambda_j x_j + \beta A \sum_{j=1}^{t} \mu_j x_j \\ &\leq \alpha O + \beta O = O, \end{aligned} \tag{1.17}$$

showing that $\alpha u + \beta v \in C$, and that C is a convex cone.

Exercise 1.10. Let $S = \{(x, y, z) : z \geq x^2 + y^2\} \subset \mathbb{R}^3$. Sketch the set and verify that it is a convex set. Is S a finitely generated cone?

Solution 1.10. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be two points in S, and $0 \leq \lambda \leq 1$. We then seek to show that $\lambda u + (1 - \lambda)v \in S$. Note that $f(x) = x^2$ is convex, i.e. that $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.

Considering the right hand side of the inequality first, we have component-wise that

$$\begin{aligned} (\lambda x_1 + (1 - \lambda)x_2)^2 &\leq \lambda x_1^2 + (1 - \lambda)x_2^2 \\ (\lambda y_1 + (1 - \lambda)y_2)^2 &\leq \lambda y_1^2 + (1 - \lambda)y_2^2, \end{aligned} \tag{1.18}$$

and therefore

$$\begin{aligned} (\lambda x_1 + (1 - \lambda)x_2)^2 + (\lambda y_1 + (1 - \lambda)y_2)^2 &\leq \lambda(x_1^2 + y_1^2) + (1 - \lambda)(x_2^2 + y_2^2) \\ &\leq \lambda z_1 + (1 - \lambda)z_2, \end{aligned} \tag{1.19}$$

and so $\lambda u + (1 - \lambda)v \in S$, showing that S is convex.

S is not a finitely generated cone, as it should then be closed under scaling. Consider e.g. $u = (1, 0, 1)$, which satisfies $1 \geq 1^2 + 0^2$ and so $u \in S$. Letting however $\lambda_1 = 2$ (and $\lambda_2 = 0$), we have $2u = (2, 0, 2) \notin S$ as $2 \ngeq 2^2 + 0^2$.

Exercise 1.11. Consider the linear system $0 \le x_i \le 1$ for $i = 1, \dots, n$, and let P denote the solution set. Explain how to solve a linear programming problem

$$\max\{c^T x : x \in P\}. \tag{1.20}$$

What if the linear system was $a_i \le x_i \le b_i$ for $i = 1, \dots, n$? Here we assume $a_i \le b_i$ for each i.

Solution 1.11. We choose the solution component-wise, by maximizing each component $c_i x_i$. If $c_i > 0$, simply let $x_i = 1$. If $c_i < 0$, increasing x_i decreases the objective function, so we let $x_i = 0$. The same argument holds in the alternate case, just choose $x_i = b_i$ or $x_i = a_i$ respectively based on the sign of c_i.

Exercise 1.12. Is the union of two convex sets again convex?

Solution 1.12. No. Let $A = [-2, -1]$ and $B = [1, 2]$. Then $A \cup B = [-2, -1] \cup [1, 2]$ is not convex, as e.g. $\frac{1}{2}1 + (1 - \frac{1}{2})(-1) = 0 \notin A \cup B$.

Exercise 1.13. Determine the sum $A + B$ in each of the following cases:

$$\begin{array}{rll} (i) & A = \{(x, y) : x^2 + y^2 \le 1\}, & B = \{(3, 4)\}; \\ (ii) & A = \{(x, y) : x^2 + y^2 \le 1\}, & B = [0, 1] \times \{0\}; \\ (iii) & A = \{(x, y) : x + 2y = 5\}, & B = \{(x, y) : x = y, 0 \le x \le 1\}; \\ (iv) & A = [0, 1] \times [1, 2], & B = [0, 2] \times [0, 2]. \end{array}$$

Solution 1.13. Taking the cases in turn:

(i) The sum $A + B$ is given by the set of points $\{u + (3, 4) : u \in A\}$, that is, the unit disk centred around $(3, 4)$.

(ii) The sum $A + B$ is given by those points that are either in the rectangle with corners at $(0, -1)$, $(1, -1)$, $(1, 1)$ and $(0, 1)$, or in the unit disks centred about $(0, 0)$ or $(1, 0)$.

(iii) Let $B = \{(\lambda, \lambda) : 0 \le \lambda \le 1\}$. Then we can write a point $(x, y) \in A + B$ as

$$(x, y) = (x_0 + \lambda, y_0 + \lambda). \tag{1.21}$$

Then

$$x + 2y = (x_0 + 2y_0) + 3\lambda = 5 + 3\lambda, \tag{1.22}$$

which gives us that

$$A + B = \{(x, y) : 5 \le x + 2y \le 8\}. \tag{1.23}$$

(iv) The sum $A + B$ is simply given by $[0, 3] \times [1, 4]$.

Exercise 1.14. More enumerated exercises...

(*i*) Prove that, for every $\lambda \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}^n$, it holds that $\lambda(A+B) = \lambda A + \lambda B$.

(*ii*) Is it true that $(\lambda + \mu)A = \lambda A + \mu A$ for every $\lambda, \mu \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$? If not, find a counterexample.

(*iii*) Show that, if $\lambda, \mu \geq 0$ and $A \subseteq \mathbb{R}^n$ is convex, then $(\lambda + \mu)A = \lambda A + \mu A$.

Solution 1.14. Taking the exercises in turn again...

(*i*) We have that

$$\begin{aligned}\lambda(A+B) &= \{\lambda(a+b) : a \in A, b \in B\} \\ &= \{\lambda a + \lambda b : a \in A, b \in B\} \\ &= \{a + b : a \in \lambda A, b \in \lambda B\} \\ &= \lambda A + \lambda B.\end{aligned}$$

(*ii*) No, it is not true. Consider $A = [1, 2]$, $\lambda = 1$ and $\mu = -1$. Then we have

$$(\lambda + \mu)A = 0A = \{0\} \quad \text{and} \quad \lambda A + \mu A = [1, 2] + [-2, -1] = [-1, 1]. \tag{1.24}$$

(*iii*) Let $\lambda, \mu \geq 0$. For any $a \in A$, we have that

$$(\lambda + \mu)a = \lambda a + \mu a \in \lambda A + \mu A, \tag{1.25}$$

so $(\lambda + \mu)A \subseteq \lambda A + \mu A$. For the reverse inclusion, let $u \in \lambda A + \mu A$. Then $u = \lambda a + \mu b$ for some $a, b \in A$. Scaling the factors, we have that

$$u = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b\right). \tag{1.26}$$

As both the inner coefficients are non-negative and sum to one, and $a, b \in A$, we have that

$$\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b \in A, \tag{1.27}$$

and $u \in (\lambda + \mu)A$. Therefore, $(\lambda + \mu)A = \lambda A + \mu A$ with the given assumptions.

Exercise 1.15. Show that if $C_1, \ldots, C_t \subseteq \mathbb{R}^n$ are all convex sets, then $C_1 \cap \cdots \cap C_t$ is convex. Do the same when all sets are affine (or linear subspaces, or convex cones). In fact, a similar result for the intersection of any family of convex sets. Explain this.

Solution 1.15. Let $x, y \in C_1 \cap \cdots \cap C_t$ and $0 \leq \lambda \leq 1$. Then $x, y \in C_i$ for all $i = 1, \ldots, t$. Since each C_i is convex, we have that

$$\lambda x + (1-\lambda)y \in C_i \tag{1.28}$$

for all $i = 1, \ldots, t$. Thus,

$$\lambda x + (1-\lambda)y \in C_1 \cap \cdots \cap C_t, \tag{1.29}$$

proving that the intersection is convex.

Suppose there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$. Then the set

$$C = \{x \in \mathbb{R}^n : Ax = b\} \tag{1.30}$$

is an affine set. Then let $x \in \bigcap_{i=1}^t C_i$, meaning that $A_i x = b_i$ for all $i = 1, \ldots, t$. Thus, x satisfies

$$(A_1 + A_2 + \cdots + A_t)x = b_1 + b_2 + \cdots + b_t, \tag{1.31}$$

and $\bigcap_{i=1}^t C_i$ is itself affine.

A similar argument shows that the closure property of convex sets is preserved under finite intersections.

Exercise 1.16. Consider a family (possibly infinite) of linear inequalities $a_i^T x \leq b, i \in I$, and C be its solution set, i.e., C is the set of points satisfying all the inequalities. Prove that C is a convex set.

Solution 1.16. Let $x, y \in C$, and $0 \leq \lambda \leq 1$. Then, for each $i \in I$, we have

$$a_i^T(\lambda x + (1-\lambda)y) = \lambda a_i^T x + (1-\lambda)a_i^T y \leq \lambda b_i + (1-\lambda)b_i = b_i. \tag{1.32}$$

Therefore, $\lambda x + (1-\lambda)y \in C$, and C is convex.

Exercise 1.17. Consider the unit disc $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ in $\mathbb{R}^2$. Find a family of linear inequalities as in the previous problem with solution set S.

Solution 1.17. Let $U = \{(\cos\theta, \sin\theta) : 0 \leq \theta < 2\pi\}$ be the unit circle in $\mathbb{R}^2$. Then, for each $(u, v) \in U$, we have the linear inequalities

$$ux_1 + vx_2 \leq 1. \tag{1.33}$$

This feels like a circular argument, but alright.

Exercise 1.18. Is the unit ball $B = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ a polyhedron?

Solution 1.18. I don't believe the unit ball is a polyhedron. A polyhedron requires a *finite* number of linear inequalities, i.e. constraints which can be written as $Ax \leq b$. The unit ball however is inherently smooth, without edges, and therefore requires an infinite number of linear constraints, as those applied in the previous exercise.

Exercise 1.19. Show that the unit ball $B_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ is convex. Here $\|x\|_\infty = \max_j |x_j|$ is the max norm of x. Show that B_∞ is a polyhedron. Illustrate when $n = 2$.

Solution 1.19. Let $x, y \in B_\infty$ and $0 \leq \lambda \leq 1$. We then have

$$\|\lambda x + (1-\lambda)y\|_\infty \leq \lambda \|x\|_\infty + (1-\lambda)\|y\|_\infty \leq \lambda + (1-\lambda) = 1, \tag{1.34}$$

so $\lambda x + (1-\lambda)y \in B_\infty$, and B_∞ is convex. B_∞ is a polyhedron, as it is described by the constraints

$$x_j \leq 1 \quad \text{and} \quad -x_j \leq 1 \qquad j = 1, \ldots, n. \tag{1.35}$$

When $n = 2$, B_∞ is the unit square, with corners $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, as seen in Fig. 2.

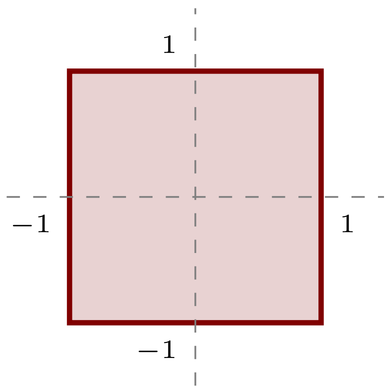

Figure 2: B_∞ when $n = 2$.

Exercise 1.20. Show that the unit ball $B_1 = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ is convex. Here $\|x\|_1 = \sum_{j=1}^n |x_j|$ is the absolute norm of x. Show that B_1 is a polyhedron. Illustrate when $n = 2$.

Solution 1.20. Let $x, y \in B_1$ and $0 \le \lambda \le 1$. Then we have

$$\begin{aligned}
\|\lambda x + (1-\lambda) y\|_1 &= \sum_{j=1}^{n} |\lambda x_j + (1-\lambda) y_j| \\
&\le \sum_{j=1}^{n} \lambda |x_j| + (1-\lambda)|y_j| = \lambda \sum_{j=1}^{n} |x_j| + (1-\lambda) \sum_{j=1}^{n} |y_j| \\
&\le \lambda + (1-\lambda) = 1.
\end{aligned} \tag{1.36}$$

Therefore, $\lambda x + (1-\lambda) y \in B_1$, and B_1 is convex.

B_1 is a polyhedron, as it is described by the constraints

$$\sum_{j=1}^{n} \sigma_j x_j \le 1, \quad \forall (\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n. \tag{1.37}$$

When $n = 2$, B_1 is the unit diamond, with corners $(\pm 1, 0)$ and $(0, \pm 1)$, as seen in Fig. 3.

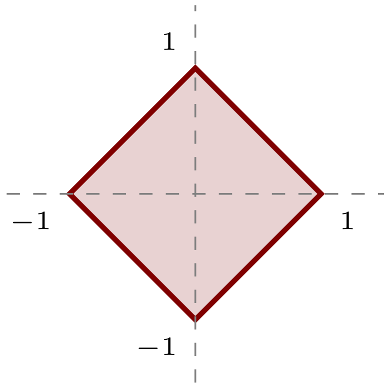

Figure 3: B_1 when $n = 2$.

Proposition 1.5.1 (*Affine sets*). Let C be a non-empty subset of $\mathbb{R}^n$. Then C is an affine set if an only if there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ for some m such that

$$C = \{x \in \mathbb{R}^n : Ax = b\}. \tag{1.38}$$

Moreover, C may be written as $C = L + x_0 = \{x + x_0 : x \in L\}$ for some linear subspace L of $\mathbb{R}^n$. The subspace L is unique.

Exercise 1.21. Prove Proposition 1.5.1.

Solution 1.21. Let $x_0 \in C$, where C is affine, and $L = \{x - x_0 : x \in C\}$. For an arbitrary $x \in C$, we have that

$$\lambda(x - x_0) = \lambda x + (1 - \lambda)x_0 - x_0 \in L. \tag{1.39}$$

Finish later, this is getting long.

Exercise 1.22. Let C be a non-empty affine set in $\mathbb{R}^n$. Define $L = C - C$. Show that L is a subspace and that $C = L + x_0$ for some vector x_0.

Solution 1.22. We have that

$$L = C - C = \{x - y : x, y \in C\}. \tag{1.40}$$

We have from the previous exercise that $C = L + x_0$ for a subspace L. Letting $x, y \in C$, we can express these as

$$x = \alpha + x_0 \quad \text{and} \quad y = \beta + x_0 \tag{1.41}$$

for some $\alpha, \beta \in L$. Then $x - y = \alpha - \beta \in L$, showing that $C - C \subseteq L$. For the reverse inclusion, let $\alpha \in L$. Then $\alpha = x - x_0$ for some $x \in C$, showing that $L \subseteq C - C$, proving that $L = C - C$.