MAT4120

Exercises for Mathematical Optimization

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Contents

Abstract

This document contains my solutions to the exercises for the course MAT4120–Mathematical Optimization, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MAT4120.

1 The basic concepts

Exercise 1.1. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and assume that $x_1 \leq x_2$ and $y_1 \leq y_2$. Verify that the inequality $x_1 + y_1 \leq x_2 + y_2$ holds. Let now λ be a non-negative real number. Explain why $\lambda x_1 \leq \lambda x_2$ holds. What happens if λ is negative?

Solution 1.1. With $x_1 \leq x_2$, we have that

$$(x_1)_i \leq (x_2)_i \qquad \forall i = 1, \ldots, n. \tag{1.1}$$

Component-wise, we then have

$$(x_1)_i + (y_1)_i \leq (x_2)_i + (y_2)_i \qquad \forall i = 1, \ldots, n, \tag{1.2}$$

and thus $x_1 + y_1 \leq x_2 + y_2$. Similarly, if $\lambda \geq 0$, we have

$$\lambda(x_1)_i \leq \lambda(x_2)_i \qquad \forall i = 1, \ldots, n, \tag{1.3}$$

and therefore $\lambda x_1 \leq \lambda x_2$. Finally, for $\lambda < 0$, the inequality reverses:

$$\lambda(x_1)_i \geq \lambda(x_2)_i \qquad \forall i = 1, \ldots, n, \tag{1.4}$$

giving $\lambda x_1 \geq \lambda x_2$.

Example 1.2.1 (*The non-negative real vectors*) The sum of two non-negative numbers is again a non-negative number. Similarly, we see that the sum of two non-negative vectors is a non-negative vector. Moreover, if we multiply a non-negative vector by a non-negative number, we get another non-negative vector. These two properties may be summarized by saying that $\mathbb{R}^n_+$ is closed under addition and multiplication by non-negative scalars. We shall see that this means that $\mathbb{R}^n_+$ is a convex cone, a special type of convex set.

Exercise 1.2. Think about the question in Exercise 1.1 again, now in light of the properties explained in Example 1.2.1.

Solution 1.2. We can now rewrite $x_1 \leq x_2$ as $x_2 - x_1 \in \mathbb{R}^n_+$. We can now easily consider the first question as

$$(x_2 + y_2) - (x_1 + y_1) = (x_2 - x_1) + (y_2 - y_1) \in \mathbb{R}^n_+, \tag{1.5}$$

as $\mathbb{R}^n_+$ is closed under addition. Similarly, we can use the fact that $\mathbb{R}^n_+$ is closed under multiplication by non-negative scalars to see that

$$(\lambda x_2 - \lambda x_1) = \lambda(x_2 - x_1) \in \mathbb{R}^n_+, \tag{1.6}$$

for $\lambda \geq 0$. As $\mathbb{R}^n_+$ is not closed under multiplication by negative scalars, we cannot conclude that $\lambda x_1 \leq \lambda x_2$ for $\lambda < 0$.

Exercise 1.3. Let $a \in \mathbb{R}^n_+$ and assume that $x \leq y$. Show that $a^T x \leq a^T y$. What happens if we do not require a to be non-negative here?

Solution 1.3. With $a \in \mathbb{R}^n_+$ and $x \leq y$, we have that

$$x_i \leq y_i \qquad \forall i = 1, \dots, n, \tag{1.7}$$

and consequently

$$a_i x_i \leq a_i y_i \qquad \forall i = 1, \dots, n, \tag{1.8}$$

as shown previously. Written in vector notation, we therefore have

$$a^T x \leq a^T y. \tag{1.9}$$

With a not necessarily non-negative, we may have neither $a^T x \geq a^T y$ nor $a^T x \leq a^T y$, as we could have $a_i x_i > a_i y_i$ for some i.

Exercise 1.4. Show that every ball $B(a, r) := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$ is convex (where $a \in \mathbb{R}^n$ and $r \geq 0$).

Solution 1.4. Let $x, y \in B(a, r)$ for some $a \in \mathbb{R}^n$ and $r \geq 0$. Then, let $0 \leq \lambda \leq 1$ and consider $z = \lambda x + (1 - \lambda) y$. We then have

$$\begin{aligned} \|z - a\| &= \|\lambda(x - a) + (1 - \lambda)(y - a)\| \\ &\leq \lambda \|x - a\| + (1 - \lambda)\|y - a\| \\ &\leq \lambda r + (1 - \lambda) r = r, \end{aligned} \tag{1.10}$$

showing that $z \in B(a, r)$. $B(a, r)$ is therefore convex.

Exercise 1.5. Explain how you can write the LP problem $\max\{c^T x : Ax \leq b\}$ in the form $\max\{c^T x : Ax = b, x \geq O\}$

Solution 1.5. We introduce new slack variables $w \in \mathbb{R}^m_+$, where m is the number of rows/inequalities in A, defined by

$$w_j = b_j - (Ax)_j \quad \forall j = 1, \dots, m. \tag{1.11}$$

We can then rewrite our system of equations by setting $\tilde{A} = \begin{bmatrix} A & I \end{bmatrix}$, $\tilde{x} = \begin{bmatrix} x \\ w \end{bmatrix}$, and $\tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}$. We then have

$$\tilde{A}\tilde{x} = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = Ax + w = b, \tag{1.12}$$

and

$$\tilde{c}^T \tilde{x} = \begin{bmatrix} c \\ 0 \end{bmatrix}^T \begin{bmatrix} x \\ w \end{bmatrix} = c^T x. \tag{1.13}$$

Again, as we require $w \geq 0$, we then have $Ax \leq b$.

Exercise 1.6. Make a drawing of the standard simplices S_1, S_2, and S_3. Verify that each unit vector e_j lies in S_n (e_j has a one in position j, all other components are zero). Each $x \in S_n$ may be written as a linear combination $x = \sum_{j=1}^{n} \lambda_j e_j$ where each λ_j is non-negative and $\sum_{j=1}^{n} \lambda_j = 1$. How? Can this be done in several ways?

Solution 1.6. Fig. 1 shows the standard simplices S_1, S_2, and S_3. Clearly each unit vector e_j lies in S_n. Each $x \in S_n$ may be written as $\sum_{j=1}^{n} \lambda_j e_j$ where $\lambda_j = x_j$, i.e. the coordinate components of x.

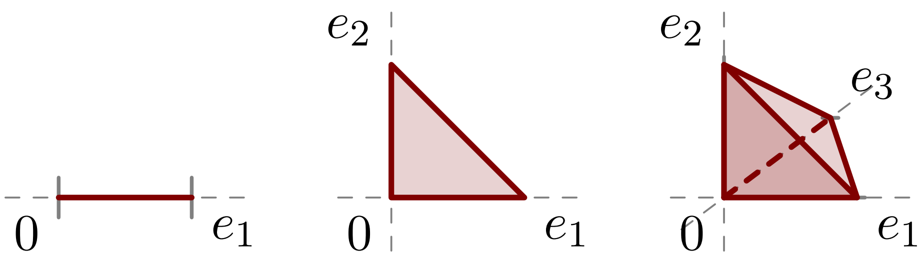

Figure 1: The simplices S_1, S_2, and S_3. For S_1, the standard simplex is the point e_1, for S_2, the standard simplex is the line segment between e_1 and e_2, and for S_3, the standard simplex is the triangle with vertices e_1, e_2, and e_3.

Exercise 1.7. Show that each convex cone is indeed a convex set.

Solution 1.7. To see that a convex cone is a convex set, let first $x_1, x_2 \in C$. Then let $0 \leq \lambda_1 \leq 1$ and $\lambda_2 = 1 - \lambda_1 \geq 0$. We then have by definition of the convex cone that

$$\lambda_1 x_1 + (1 - \lambda_1) x_1 = \lambda_1 x_1 + \lambda_2 x_2 \in C, \tag{1.14}$$

showing that the set is convex.

Exercise 1.8. Let $A \in \mathbb{R}^{m \times n}$ and consider the set $C = \{x \in \mathbb{R}^n : Ax \leq O\}$. Prove that C is a convex cone.

Solution 1.8. Let $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We then have

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A x_1 + \lambda_2 A x_2 \leq \lambda_1 O + \lambda_2 O = O, \tag{1.15}$$

showing that the set is a convex cone.

Polyhedral cone A convex cone of the form $\{x \in \mathbb{R}^n : Ax \leq O\}$ where $A \in \mathbb{R}^{m \times n}$ is called a *polyhedral cone*. Let $x_1, \dots, x_t \in \mathbb{R}^n$ and let $C(x_1, \dots, x_t)$ be the set of vectors of the form

$$u = \sum_{j=1}^{t} \lambda_j x_j, \tag{1.16}$$

where $\lambda_j \geq 0$ for each $j = 1, \dots, t$.

Exercise 1.9. Prove that $C(x_1, \dots, x_t)$ is a convex cone.

Solution 1.9. Let $C = C(x_1, \dots, x_t)$ here for convenience. Let $u, v \in C$ with respective coefficients $\lambda_j, \mu_j \geq 0$ for $j = 1, \dots, t$. Then, for arbitrary coefficients $\alpha, \beta \geq 0$, we have

$$\begin{aligned} A(\alpha u + \beta v) &= A\left(\alpha \sum_{j=1}^{t} \lambda_j x_j + \beta \sum_{j=1}^{t} \mu_j x_j\right) \\ &= \alpha A \sum_{j=1}^{t} \lambda_j x_j + \beta A \sum_{j=1}^{t} \mu_j x_j \\ &\leq \alpha O + \beta O = O, \end{aligned} \tag{1.17}$$

showing that $\alpha u + \beta v \in C$, and that C is a convex cone.

Exercise 1.10. Let $S = \{(x, y, z) : z \geq x^2 + y^2\} \subset \mathbb{R}^3$. Sketch the set and verify that it is a convex set. Is S a finitely generated cone?

Solution 1.10. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be two points in S, and $0 \leq \lambda \leq 1$. We then seek to show that $\lambda u + (1 - \lambda)v \in S$. Note that $f(x) = x^2$ is convex, i.e. that $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.

Considering the right hand side of the inequality first, we have component-wise that

$$\begin{aligned} (\lambda x_1 + (1 - \lambda)x_2)^2 &\leq \lambda x_1^2 + (1 - \lambda)x_2^2 \\ (\lambda y_1 + (1 - \lambda)y_2)^2 &\leq \lambda y_1^2 + (1 - \lambda)y_2^2, \end{aligned} \tag{1.18}$$

and therefore

$$\begin{aligned} (\lambda x_1 + (1 - \lambda)x_2)^2 + (\lambda y_1 + (1 - \lambda)y_2)^2 &\leq \lambda(x_1^2 + y_1^2) + (1 - \lambda)(x_2^2 + y_2^2) \\ &\leq \lambda z_1 + (1 - \lambda)z_2, \end{aligned} \tag{1.19}$$

and so $\lambda u + (1 - \lambda)v \in S$, showing that S is convex.

S is not a finitely generated cone, as it should then be closed under scaling. Consider e.g. $u = (1, 0, 1)$, which satisfies $1 \geq 1^2 + 0^2$ and so $u \in S$. Letting however $\lambda_1 = 2$ (and $\lambda_2 = 0$), we have $2u = (2, 0, 2) \notin S$ as $2 \not\geq 2^2 + 0^2$.

Exercise 1.11. Consider the linear system $0 \leq x_i \leq 1$ for $i = 1, \ldots, n$, and let P denote the solution set. Explain how to solve a linear programming problem

$$\max\{c^T x : x \in P\}. \tag{1.20}$$

What if the linear system was $a_i \leq x_i \leq b_i$ for $i = 1, \ldots, n$? Here we assume $a_i \leq b_i$ for each i.

Solution 1.11. We choose the solution component-wise, by maximizing each component $c_i x_i$. If $c_i > 0$, simply let $x_i = 1$. If $c_i < 0$, increasing x_i decreases the objective function, so we let $x_i = 0$. The same argument holds in the alternate case, just choose $x_i = b_i$ or $x_i = a_i$ respectively based on the sign of c_i.

Exercise 1.12. Is the union of two convex sets again convex?

Solution 1.12. No. Let $A = [-2, -1]$ and $B = [1, 2]$. Then $A \cup B = [-2, -1] \cup [1, 2]$ is not convex, as e.g. $\frac{1}{2}1 + (1 - \frac{1}{2})(-1) = 0 \notin A \cup B$.

Exercise 1.13. Determine the sum $A + B$ in each of the following cases:

$$\begin{array}{rll}
(i) & A = \{(x,y) : x^2 + y^2 \leq 1\}, & B = \{(3,4)\}; \\
(ii) & A = \{(x,y) : x^2 + y^2 \leq 1\}, & B = [0,1] \times \{0\}; \\
(iii) & A = \{(x,y) : x + 2y = 5\}, & B = \{(x,y) : x = y, 0 \leq x \leq 1\}; \\
(iv) & A = [0,1] \times [1,2], & B = [0,2] \times [0,2].
\end{array}$$

Solution 1.13. Taking the cases in turn:

(i) The sum $A + B$ is given by the set of points $\{u + (3,4) : u \in A\}$, that is, the unit disk centred around $(3,4)$.

(ii) The sum $A + B$ is given by those points that are either in the rectangle with corners at $(0,-1)$, $(1,-1)$, $(1,1)$ and $(0,1)$, or in the unit disks centred about $(0,0)$ or $(1,0)$.

(iii) Let $B = \{(\lambda, \lambda) : 0 \leq \lambda \leq 1\}$. Then we can write a point $(x,y) \in A + B$ as

$$(x,y) = (x_0 + \lambda, y_0 + \lambda). \tag{1.21}$$

Then

$$x + 2y = (x_0 + 2y_0) + 3\lambda = 5 + 3\lambda, \tag{1.22}$$

which gives us that

$$A + B = \{(x,y) : 5 \leq x + 2y \leq 8\}. \tag{1.23}$$

(iv) The sum $A + B$ is simply given by $[0,3] \times [1,4]$.

Exercise 1.14. More enumerated exercises...

(*i*) Prove that, for every $\lambda \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}^n$, it holds that $\lambda(A+B) = \lambda A+\lambda B$.

(*ii*) Is it true that $(\lambda + \mu)A = \lambda A + \mu A$ for every $\lambda, \mu \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$? If not, find a counterexample.

(*iii*) Show that, if $\lambda, \mu \geq 0$ and $A \subseteq \mathbb{R}^n$ is convex, then $(\lambda + \mu)A = \lambda A + \mu A$.

Solution 1.14. Taking the exercises in turn again...

(*i*) We have that

$$\begin{aligned}\lambda(A + B) &= \{\lambda(a + b) : a \in A, b \in B\}\\ &= \{\lambda a + \lambda b : a \in A, b \in B\}\\ &= \{a + b : a \in \lambda A, b \in \lambda B\}\\ &= \lambda A + \lambda B.\end{aligned}$$

(*ii*) No, it is not true. Consider $A = [1, 2]$, $\lambda = 1$ and $\mu = -1$. Then we have

$$(\lambda + \mu)A = 0A = \{0\} \quad \text{and} \quad \lambda A + \mu A = [1, 2] + [-2, -1] = [-1, 1]. \tag{1.24}$$

(*iii*) Let $\lambda, \mu \geq 0$. For any $a \in A$, we have that

$$(\lambda + \mu)a = \lambda a + \mu a \in \lambda A + \mu A, \tag{1.25}$$

so $(\lambda + \mu)A \subseteq \lambda A + \mu A$. For the reverse inclusion, let $u \in \lambda A + \mu A$. Then $u = \lambda a + \mu b$ for some $a, b \in A$. Scaling the factors, we have that

$$u = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b\right). \tag{1.26}$$

As both the inner coefficients are non-negative and sum to one, and $a, b \in A$, we have that

$$\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b \in A, \tag{1.27}$$

and $u \in (\lambda + \mu)A$. Therefore, $(\lambda + \mu)A = \lambda A + \mu A$ with the given assumptions.

Exercise 1.15. Show that if $C_1, \dots, C_t \subseteq \mathbb{R}^n$ are all convex sets, then $C_1 \cap \cdots \cap C_t$ is convex. Do the same when all sets are affine (or linear subspaces, or convex cones). In fact, a similar result for the intersection of any family of convex sets. Explain this.

Solution 1.15. Let $x, y \in C_1 \cap \cdots \cap C_t$ and $0 \leq \lambda \leq 1$. Then $x, y \in C_i$ for all $i = 1, \ldots, t$. Since each C_i is convex, we have that

$$\lambda x + (1 - \lambda)y \in C_i \tag{1.28}$$

for all $i = 1, \ldots, t$. Thus,

$$\lambda x + (1 - \lambda)y \in C_1 \cap \cdots \cap C_t, \tag{1.29}$$

proving that the intersection is convex.

Suppose there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$. Then the set

$$C = \{x \in \mathbb{R}^n : Ax = b\} \tag{1.30}$$

is an affine set. Then let $x \in \bigcap_{i=1}^t C_i$, meaning that $A_i x = b_i$ for all $i = 1, \ldots, t$. Thus, x satisfies

$$(A_1 + A_2 + \cdots + A_t)x = b_1 + b_2 + \cdots + b_t, \tag{1.31}$$

and $\bigcap_{i=1}^t C_i$ is itself affine.

A similar argument shows that the closure property of convex sets is preserved under finite intersections.

Exercise 1.16. Consider a family (possibly infinite) of linear inequalities $a_i^T x \leq b, i \in I$, and C be its solution set, i.e., C is the set of points satisfying all the inequalities. Prove that C is a convex set.

Solution 1.16. Let $x, y \in C$, and $0 \leq \lambda \leq 1$. Then, for each $i \in I$, we have

$$a_i^T(\lambda x + (1 - \lambda)y) = \lambda a_i^T x + (1 - \lambda)a_i^T y \leq \lambda b_i + (1 - \lambda)b_i = b_i. \tag{1.32}$$

Therefore, $\lambda x + (1 - \lambda)y \in C$, and C is convex.

Exercise 1.17. Consider the unit disc $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ in $\mathbb{R}^2$. Find a family of linear inequalities as in the previous problem with solution set S.

Solution 1.17. Let $U = \{(\cos\theta, \sin\theta) : 0 \leq \theta < 2\pi\}$ be the unit circle in $\mathbb{R}^2$. Then, for each $(u, v) \in U$, we have the linear inequalities

$$ux_1 + vx_2 \leq 1. \tag{1.33}$$

This feels like a circular argument, but alright.

Exercise 1.18. Is the unit ball $B = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ a polyhedron?

Solution 1.18. I don't believe the unit ball is a polyhedron. A polyhedron requires a *finite* number of linear inequalities, i.e. constraints which can be written as $Ax \leq b$. The unit ball however is inherently smooth, without edges, and therefore requires an infinite number of linear constraints, as those applied in the previous exercise.

Exercise 1.19. Show that the unit ball $B_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ is convex. Here $\|x\|_\infty = \max_j |x_j|$ is the max norm of x. Show that B_∞ is a polyhedron. Illustrate when $n = 2$.

Solution 1.19. Let $x, y \in B_\infty$ and $0 \leq \lambda \leq 1$. We then have

$$\|\lambda x + (1-\lambda)y\|_\infty \leq \lambda \|x\|_\infty + (1-\lambda)\|y\|_\infty \leq \lambda + (1-\lambda) = 1, \tag{1.34}$$

so $\lambda x + (1-\lambda)y \in B_\infty$, and B_∞ is convex. B_∞ is a polyhedron, as it is described by the constraints

$$x_j \leq 1 \quad \text{and} \quad -x_j \leq 1 \qquad j = 1, \dots, n. \tag{1.35}$$

When $n = 2$, B_∞ is the unit square, with corners $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, as seen in Fig. 2.

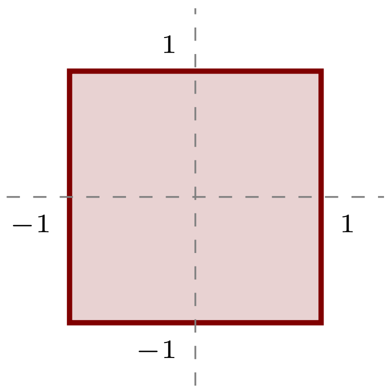

Figure 2: B_∞ when $n = 2$.

Exercise 1.20. Show that the unit ball $B_1 = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ is convex. Here $\|x\|_1 = \sum_{j=1}^n |x_j|$ is the absolute norm of x. Show that B_1 is a polyhedron. Illustrate when $n = 2$.

Solution 1.20. Let $x, y \in B_1$ and $0 \le \lambda \le 1$. Then we have

$$\begin{aligned}\|\lambda x + (1-\lambda)y\|_1 &= \sum_{j=1}^{n} |\lambda x_j + (1-\lambda)y_j| \\ &\le \sum_{j=1}^{n} \lambda|x_j| + (1-\lambda)|y_j| = \lambda\sum_{j=1}^{n}|x_j| + (1-\lambda)\sum_{j=1}^{n}|y_j| \\ &\le \lambda + (1-\lambda) = 1.\end{aligned} \tag{1.36}$$

Therefore, $\lambda x + (1-\lambda)y \in B_1$, and B_1 is convex.

B_1 is a polyhedron, as it is described by the constraints

$$\sum_{j=1}^{n} \sigma_j x_j \le 1, \quad \forall(\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n. \tag{1.37}$$

When $n = 2$, B_1 is the unit diamond, with corners $(\pm 1, 0)$ and $(0, \pm 1)$, as seen in Fig. 3.

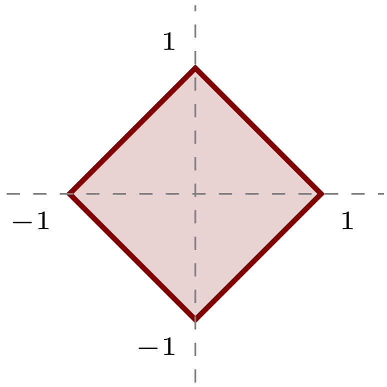

Figure 3: B_1 when $n = 2$.

Proposition 1.5.1 (*Affine sets*). Let C be a non-empty subset of $\mathbb{R}^n$. Then C is an affine set if an only if there is a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$ for some m such that

$$C = \{x \in \mathbb{R}^n : Ax = b\}. \tag{1.38}$$

Moreover, C may be written as $C = L + x_0 = \{x + x_0 : x \in L\}$ for some linear subspace L of $\mathbb{R}^n$. The subspace L is unique.

Exercise 1.21. Prove Proposition 1.5.1.

Solution 1.21. Let $x_0 \in C$, where C is affine, and $L = \{x - x_0 : x \in C\}$. For an arbitrary $x \in C$, we have that

$$\lambda(x - x_0) = \lambda x + (1 - \lambda)x_0 - x_0 \in L. \tag{1.39}$$

Finish later, this is getting long.

Exercise 1.22. Let C be a non-empty affine set in $\mathbb{R}^n$. Define $L = C - C$. Show that L is a subspace and that $C = L + x_0$ for some vector x_0.

Solution 1.22. We have that

$$L = C - C = \{x - y : x, y \in C\}. \tag{1.40}$$

We have from the previous exercise that $C = L + x_0$ for a subspace L. Letting $x, y \in C$, we can express these as

$$x = \alpha + x_0 \quad \text{and} \quad y = \beta + x_0 \tag{1.41}$$

for some $\alpha, \beta \in L$. Then $x - y = \alpha - \beta \in L$, showing that $C - C \subseteq L$. For the reverse inclusion, let $\alpha \in L$. Then $\alpha = x - x_0$ for some $x \in C$, showing that $L \subseteq C - C$, proving that $L = C - C$.

2 Convex hulls and Carathéodory's theorem

Exercise 2.1. Illustrate some combinations (linear, convex, non-negative) of two vectors in $\mathbb{R}^2$.

Solution 2.1. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- Linear combinations: $\alpha\mathbf{u} + \beta\mathbf{v}$ for $\alpha, \beta \in \mathbb{R}$.
- Convex combinations: $\alpha\mathbf{u} + \beta\mathbf{v}$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.
- Non-negative combinations: $\alpha\mathbf{u} + \beta\mathbf{v}$ for $\alpha, \beta \geq 0$.

The linear combinations fill the entire $\mathbb{R}^2$ plane, the convex combinations fill the line segment between $\mathbf{u}$ and $\mathbf{v}$, and the non-negative combinations fill the first quadrant of the plane.

Exercise 2.2. Choose your favourite three points x_1, x_2, x_3 in $\mathbb{R}^2$, but make sure that they do not all lie on the same line. Thus, the three points form the corners of a triangle C. Describe those points that are convex combinations of two of the three points. What about the interior of the triangle C, i.e., those points that lie in C but not on the boundary (the three sides): can these points be written as convex combinations of x_1, x_2 and x_3? If so, how?

Solution 2.2. Let's choose the points $x_1 = (0, 0)$, $x_2 = (1, 0)$, and $x_3 = (0, 1)$, i.e., the corners of a right triangle. The points that are convex combinations of two of the three points are those that lie on the edges of the triangle. The interior points, which do not lie on the boundary, can be expressed as

$$c = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \tag{2.1}$$

with $0 < \lambda_i < 1$ for $i = 1, 2, 3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

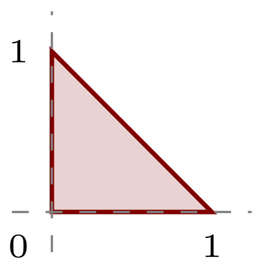

Figure 4: The unit triangle with points x_1, x_2, and x_3.

Exercise 2.3. Show that conv(S) is convex for all $S \subseteq \mathbb{R}^n$. (Hint: look at two convex combinations $\sum_j \lambda_j x_j$ and $\sum_j \mu_j y_j$, and note that both these points may be written as a convex combination of the same set of vectors.)

Solution 2.3. Let $u, v \in \text{conv}(S)$. Then there exists points $x_1, x_2, \ldots, x_n \in S$ and $y_1, y_2, \ldots, y_m \in S$ and coefficients $\lambda_i, \mu_j \geq 0$ with $\sum_i \lambda_i = 1$ and $\sum_j \mu_j = 1$ such that

$$u = \sum_{i=1}^{n} \lambda_i x_i, \quad v = \sum_{j=1}^{m} \mu_j y_j. \tag{2.2}$$

For any $0 \leq \theta \leq 1$, we have that

$$\begin{aligned} \theta u + (1-\theta)v &= \theta \sum_{i=1}^{n} \lambda_i x_i + (1-\theta) \sum_{j=1}^{m} \mu_j y_j \\ &= \sum_{i=1}^{n} (\theta\lambda_i) x_i + \sum_{j=1}^{m} ((1-\theta)\mu_j) y_j. \end{aligned} \tag{2.3}$$

The new coefficients are non-negative, and furthermore we have

$$\sum_{i=1}^{n} (\theta\lambda_i) + \sum_{j=1}^{m} ((1-\theta)\mu_j) = \theta \cdot 1 + (1-\theta) \cdot 1 = 1, \tag{2.4}$$

showing that $\theta u + (1-\theta)v$ is a convex combination of points in S, and therefore lies in conv(S). conv(S) is thus convex.

Exercise 2.4. Give an example of two distinct sets S and T having the same convex hull. It makes sense to look for a smallest possible subset S_0 of a set S such that $S = \text{conv}(S_0)$. We study this question later.

Solution 2.4. Let $S = \{(0,0), (1,0), (0,1)\}$ and $T = \{(0,0), (1,0), (0,1), (0.5, 0.5)\}$. Both sets have the same convex hull, which is the triangle formed by the points $(0,0)$, $(1,0)$, and $(0,1)$. The point $(0.5, 0.5)$ in set T lies within this triangle and does not change the convex hull.

Exercise 2.5. Prove that if $S \subseteq T$, then conv$(S) \subseteq$ conv(T).

Solution 2.5. Let $u \in \text{conv}(S)$. Then there exist points $x_1, x_2, \ldots, x_n \in S$ and coefficients $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ such that

$$u = \sum_{i=1}^{n} \lambda_i x_i. \tag{2.5}$$

Since $S \subseteq T$, we have that $x_i \in T$ for all i. Therefore, u is also a convex combination of points in T, and thus lies in conv(T). Hence, conv$(S) \subseteq$ conv(T).

Exercise 2.6. If S is convex, then $\text{conv}(S) = S$. Show this!

Solution 2.6. Since S is convex, for any $u, v \in S$ and any $0 \leq \lambda \leq 1$, we have $\lambda u + (1-\lambda)v \in S$. By the definition of convex hull, $\text{conv}(S)$ is the smallest convex set containing S. Since S is already convex and contains itself, it follows that $\text{conv}(S) = S$.

Exercise 2.7. Let $S = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$, this is the unit circle in $\mathbb{R}^2$. Determine $\text{conv}(S)$ and $\text{cone}(S)$.

Solution 2.7. The convex hull $\text{conv}(S)$ of the unit circle is the unit disk, i.e., the set $\{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$. This is because any point inside the unit circle can be expressed as a convex combination of points on the unit circle.

The conical hull $\text{cone}(S)$ of the unit circle is the entire $\mathbb{R}^2$ plane. This is because any point in $\mathbb{R}^2$ can be expressed as a non-negative combination of points on the unit circle, scaled appropriately. This can be seen simply by looking at the problem in polar coordinates.

Exercise 2.8. Does affine independence imply linear independence? Does linear independence imply affine independence? Prove or disprove!

Solution 2.8. Affine independence does not imply linear independence. Consider a set of linearly independent vectors $x_2, \ldots, x_t \in \mathbb{R}^n$ and let $x_1 = 0$. Then clearly the set $\{x_1, x_2, \ldots, x_t\}$ is affinely independent, but not linearly independent since x_1 is the zero vector.

Linear independence does imply affine independence. Suppose $\{x_1, x_2, \ldots, x_t\}$ is linearly independent. To show affine independence, we need to show that the only solution to

$$\sum_{i=1}^{t} \lambda_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{t} \lambda_i = 0 \tag{2.6}$$

is $\lambda_i = 0$ for all i. Since the vectors are linearly independent, the first equation implies that all λ_i must be zero. Thus, the set is affinely independent.

Exercise 2.9. Let $x_1, \ldots, x_t \in \mathbb{R}^n$ be affinely independent and let $w \in \mathbb{R}^n$. Show that $x_1 + w, \ldots, x_t + w$ are also affinely independent.

Solution 2.9. As $x_1, \ldots, x_t$ are affinely independent, we have that

$$\{x_2 - x_1, x_3 - x_1, \ldots, x_t - x_1\} \tag{2.7}$$

is linearly independent. We then have that

$$\{(x_2+w)-(x_1+w), \ldots, (x_t+w)-(x_1+w)\} = \{x_2-x_1, x_3-x_1, \ldots, x_t-x_1\}, \tag{2.8}$$

which is linearly independent. Thus, $x_1 + w, \ldots, x_t + w$ are affinely independent.

Dimension of a set. The *dimension* of a set $S \subseteq \mathbb{R}^n$, denoted by $\dim(S)$, is the maximal number of affinely independent points in S minus 1. So, for example in $\mathbb{R}^3$, the dimension of a point and a lines is 0 and 1 respectively, and the dimension of the plane $x_3 = 0$ is 2.

Exercise 2.10. Let L be a linear subspace of dimension (in the usual linear algebra sense) t. Check this this coincides with our new definition of dimension above. (Hint: add O to a "suitable" set of vectors).

Solution 2.10. Let $\{v_1, v_2, \ldots, v_t\}$ be a basis for the linear subspace L. Then the set $\{0, v_1, v_2, \ldots, v_t\}$ contains $t+1$ affinely independent points in L. To see this, note that the vectors $v_1, v_2, \ldots, v_t$ are linearly independent by definition of a basis. Therefore, the maximal number of affinely independent points in L is $t+1$, and thus $\dim(L) = t+1-1 = t$, which coincides with the usual definition of dimension in linear algebra.

Exercise 2.11. Consider a convex set C of dimension d. Then there are (and no more than) $d+1$ affinely independent points in C. Let $S = \{x_1, \ldots, x_{d+1}\}$ denote a set of such points. Then the set of all convex combinations of these vectors, i.e., $\mathrm{conv}(S)$ is a polytope contained in C and $\dim(S) = \dim(C)$. Moreover, let A be the set of all vectors of the form $\sum_{j=1}^t \lambda_j x_j$ where $\sum_{j=1}^t \lambda_j = 1$ (no sign restrictions of the λ's). Then A is an affine set containing C, and it is the smallest affine set with this property. A is called the *affine hull* of C.

Prove the last statements in the previous paragraph.

Solution 2.11. Consider two points $u, v \in A$ where

$$u = \sum_{j=1}^{t} \lambda_j x_j \quad \text{and} \quad v = \sum_{j=1}^{t} \mu_j x_j, \tag{2.9}$$

where $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \mu_j = 1$. Then, we have that

$$(1-\theta)u + \theta v = \sum_{j=1}^{t} ((1-\theta)\lambda_j + \theta\mu_j) x_j \in A \tag{2.10}$$

for any $\theta \in \mathbb{R}$ as the coefficients still sum to 1. A is therefore affine. Choosing $\lambda_j = \delta_{ij}$ (the Kronecker delta) shows that $x_i \in A$ for all i, and thus $\mathrm{conv}(C) = C \subseteq A$.

Exercise 2.12. Construct a set which is neither open nor closed.

Solution 2.12. Consider the interval $S = (0, 1]$ in $\mathbb{R}$. This set is not open because it contains the point 1, which is a limit point of the set. It is not closed because it does not contain the point 0, which is also a limit point of the set. Therefore, S is neither open nor closed.

Exercise 2.13. Show that $x^k \to x$ if and only if $x_j^k \to x_j$ for $j = 1, \ldots, n$. Thus convergence of a point sequence simply means that all the component sequences are convergent.

Solution 2.13. ($\Rightarrow$) Suppose $x^k \to x$. Then, as

$$|x_j^k - x_j| \leq \|x^k - x\|, \tag{2.11}$$

we have that $x_j^k \to x_j$ for each component j.

($\Leftarrow$) Conversely, suppose $x_j^k \to x_j$ for each component j. Then,

$$\|x^k - x\|^2 = \sum_{j=1}^{n} (x_j^k - x_j)^2 \to 0 = 0, \tag{2.12}$$

showing that $x^k \to x$. This is all rather informal, but the details are easy to fill in.

Exercise 2.14. Show that every simplex cone is closed.

Solution 2.14. Let $x_1, \ldots, x_t \in \mathbb{R}^n$ be linearly independent vectors, and consider the simplex cone spanned by these vectors. Let X be the matrix with columns $x_1, \ldots, x_t$. Then any point in the simplex cone can be written as $X\boldsymbol{\lambda}$ for some $\boldsymbol{\lambda} \geq 0$. X then has full column rank, and thus $X^T X$ is invertible, so we define the pseudo-inverse $X^\dagger = (X^T X)^{-1} X^T$. We then obtain $\boldsymbol{\lambda} = X^\dagger x$, such that the simplex cone can be written as the inverse image of the closed set $\{\boldsymbol{\lambda} : \boldsymbol{\lambda} \geq 0\}$ under the continuous map $x \mapsto X^\dagger x$. The simplex cone is therefore closed.

Boundary of a set. The *boundary* $\mathrm{bd}(S)$ of S is defined by $\mathrm{bd}(S) = \mathrm{cl}(S) \setminus \mathrm{int}(S)$. For instance, we have that $\mathrm{bd}(B(a, r)) = \{x \in \mathbb{R}^n : \|x - a\| = r\}$.

Exercise 2.15. Prove that $x \in \mathrm{bd}(S)$ if and only if each ball with center x intersects both S and the complement of S.

Solution 2.15. For $x \in \mathrm{bd}(S)$, we have that $x \in \mathrm{cl}(S)$ and $x \notin \mathrm{int}(S)$. Since $x \in \mathrm{cl}(S)$, every ball centred at x intersects S. Since $x \notin \mathrm{int}(S)$, every ball centred at x also intersects the complement of S.

Affine hull. The *affine hull* of a set S, denoted by $\mathrm{aff}(S)$, is the smallest affine set containing S.

Relative interior and relative boundary. We say that x is a *relative interior point* of S if there is an $r > 0$ such that

$$B^\circ(x, r) \cap \operatorname{aff}(S) \subseteq S. \tag{2.13}$$

This means that x is the center of some open ball whose intersection with $\operatorname{aff}(S)$ is contained in S. We let the *relative interior* of S, denoted by $\operatorname{int}(S)$, be the set of all such relative interior points of S. Similarly, we define the *relative boundary* of S by $\operatorname{rbd}(S) = \operatorname{cl}(S) \setminus \operatorname{rint}(S)$.

Exercise 2.16. Consider again the set $C = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\}$. Verify that:

(*i*) C is closed,

(*ii*) $\dim(C) = 2$,

(*iii*) $\operatorname{int}(C) = \emptyset$,

(*iv*) $\operatorname{bd}(C) = C$,

(*v*) $\operatorname{rint}(C) = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$,

(*vi*) $\operatorname{rbd}(C) = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$.

Solution 2.16. (*i*): Assume that $x^k \to x$ where $x^k = (x_k, y_k, 0) \in C$ for all k. As $x_k^2 + y_k^2 \leq 1$ for all k, we have that $x^2 + y^2 \leq 1$ by taking limits, showing that $x \in C$. Similarly, as $z_k = 0$ for all k, we have that $z = 0$, and thus $x \in C$. C is therefore closed.

(*ii*): We find only two linearly independent vectors in C, for instance $(1, 0, 0)$ and $(0, 1, 0)$. Thus, $\dim(C) = 2$. A similar argument is possible through the affinely independent points $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$.

(*iii*): Any ball centred around a point $x = (x_1, x_2, 0) \in C$ will contain points of the form (x_1, x_2, ε) for $\varepsilon > 0$, which are not in C. Thus, $\operatorname{int}(C) = \emptyset$.

(*iv*): As $\operatorname{int}(C) = \emptyset$, we have that $\operatorname{bd}(C) = \operatorname{cl}(C) \setminus \operatorname{int}(C) = C \setminus \emptyset = C$.

(*v*): We have that $\operatorname{aff}(C)$ is the plane $z = 0$. For a point $x = (x_1, x_2, 0) \in C$ with $x_1^2 + x_2^2 < 1$, we can find a ball $B^\circ(x, r) \subset B^\circ(0, 1)$. Then

$$\begin{aligned} B^\circ(x, r) \cap \operatorname{aff}(C) &= \{(y_1, y_2, 0) : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\} \\ &= (B^\circ((x_1, x_2), r), 0) \subseteq C, \end{aligned} \tag{2.14}$$

so $\{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\} \subseteq \operatorname{rint}(C)$. If we have a point $x = (x_1, x_2, 0)$ with $x_1^2 + x_2^2 = 1$, any ball centred at x will contain points of the form (x_1, x_2, ε) for $\varepsilon > 0$, which are not in C. Thus, $\operatorname{rint}(C) = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$.

(v): We have simply

$$\begin{aligned}\mathrm{rbd}(C) &= \mathrm{cl}(C) \setminus \mathrm{rint}(C) = C \setminus \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\} \\ &= \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}.\end{aligned} \tag{2.15}$$

Exercise 2.17. Show that every polytope in $\mathbb{R}^n$ is bounded. (Hint: use the properties of the norm: $\|x + y\| \leq \|x\| + \|y\|$ and $\|\lambda x\| = \lambda \|x\|$ for $\lambda \geq 0$.)

Solution 2.17. For a polytope $P = \mathrm{conv}(\{x_1, \dots, x_t\})$, we can express any point $x \in P$ as

$$x = \sum_{i=1}^{t} \lambda_i x_i, \tag{2.16}$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{t} \lambda_i = 1$. We then have that

$$\begin{aligned}\|x\| &= \left\| \sum_{i=1}^{t} \lambda_i x_i \right\| \leq \sum_{i=1}^{t} \|\lambda_i x_i\| = \sum_{i=1}^{t} \lambda_i \|x_i\| \\ &\leq \max_{i=1,\dots,t} \|x_i\| \sum_{i=1}^{t} \lambda_i = \max_{i=1,\dots,t} \|x_i\|.\end{aligned} \tag{2.17}$$

Thus, P is bounded.

Exercise 2.18. Consider the standard simplex S_t. Show that it is compact, i.e., closed and bounded.

Solution 2.18. The standard simplex S_t is given by

$$S_t = \{x \in \mathbb{R}^t : x_i \geq 0, \sum_{i=1}^{t} x_i = 1\}, \tag{2.18}$$

or equivalently, $S_t = \mathrm{conv}(\{e_1, e_2, \dots, e_t\})$ where e_i are the standard basis vectors in $\mathbb{R}^t$. As S_t is a polytope, it is bounded by the previous exercise. It is closed as it is the inverse of the closed set $\{1\}$ under the continuous map $x \mapsto \sum_{i=1}^{t} x_i$ intersected with the closed set $\{x : x_i \geq 0 \text{ for all } i\}$. Thus, S_t is compact.

Exercise 2.19. Give an example of a convex cone which is not closed.

Solution 2.19. Consider the set $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. This set is a convex cone because for any $x, y \in C$ and any $\alpha, \beta \geq 0$, we have $\alpha x + \beta y \in C$. However, C is not closed because it does not contain the boundary line $x_2 = 0$. Thus, C is a convex cone that is not closed.

Exercise 2.20. Let $S \subseteq \mathbb{R}^2$ and let W be the set of all convex combinations of points in S. Prove that W is convex.

Solution 2.20. Let $u, v \in W$ be two convex combinations of points in S, i.e.,

$$u = \sum_{i=1}^{m} \lambda_i x_i \quad \text{and} \quad v = \sum_{j=1}^{n} \mu_j y_j. \tag{2.19}$$

where $x_i, y_j \in S$, $\lambda_i, \mu_j \geq 0$, and $\sum_{i=1}^{m} \lambda_i = 1$, $\sum_{j=1}^{n} \mu_j = 1$. For any $0 \leq \theta \leq 1$, we have that

$$\begin{aligned} \theta u + (1-\theta)v &= \theta \sum_{i=1}^{m} \lambda_i x_i + (1-\theta) \sum_{j=1}^{n} \mu_j y_j \\ &= \sum_{i=1}^{m} (\theta \lambda_i) x_i + \sum_{j=1}^{n} ((1-\theta)\mu_j) y_j. \end{aligned} \tag{2.20}$$

The new coefficients are non-negative, and furthermore we have

$$\sum_{i=1}^{m} (\theta \lambda_i) + \sum_{j=1}^{n} ((1-\theta)\mu_j) = \theta \cdot 1 + (1-\theta) \cdot 1 = 1, \tag{2.21}$$

showing that $\theta u + (1-\theta)v$ is a convex combination of points in S, and therefore lies in W. W is thus convex.

Proposition 2.1.1 (Convex sets). A set $C \subseteq \mathbb{R}^n$ is convex if and only if it contains all convex combinations of its points. A set $C \subseteq \mathbb{R}^n$ is a convex cone if and only if it contains all non-negative combinations of its points.

Exercise 2.21. Prove the second statement of Proposition 2.1.1.

Solution 2.21. This is the definition of a convex cone?

Exercise 2.22. Give a geometrical interpretation of the induction step in the proof of Proposition 2.1.1.

Solution 2.22. This essentially traces out a Bezier curve.

Exercise 2.23. Let $S = \{(0,0), (1,0), (0,1)\}$. Show that $\text{conv}(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$.

Solution 2.23. A point in $\text{conv}(S)$ can be written as

$$x = \lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,1) = (\lambda_2, \lambda_3), \tag{2.22}$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Thus, we have that $\lambda_2, \lambda_3 \geq 0$ and $\lambda_2 + \lambda_3 \leq 1$, showing that $\text{conv}(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$.

Exercise 2.24. Let S consist of the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$. Show that $\mathrm{conv}(S) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \text{ for } i = 1,2,3\}$. Also determine $\mathrm{conv}(S \setminus \{(1,1,1)\})$ as the solution set of a system of linear inequalities. Illustrate these cases geometrically.

Solution 2.24. Consider instead the points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ in $\mathbb{R}^2$, as the argument is similar and easier to visualize. A point in $\mathrm{conv}(S)$ can be written as

$$x = \lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,1) + \lambda_4(1,1) = (\lambda_2+\lambda_4, \lambda_3+\lambda_4), \tag{2.23}$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^4 \lambda_i = 1$. Thus we are bounded by the constraints

$$\begin{aligned} 0 \leq x_1 = \lambda_2 + \lambda_4 \leq 1, \\ 0 \leq x_2 = \lambda_3 + \lambda_4 \leq 1, \end{aligned}$$

showing that $\mathrm{conv}(S) = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_i \leq 1 \text{ for } i = 1,2\}$. The same argument holds in $\mathbb{R}^3$, where we end up with the unit cube. For $\mathrm{conv}(S \setminus \{(1,1)\})$, we end up in the same scenario as the previous exercise, spanning out the unit triangle, as in Fig. 4

Exercise 2.25. Let $A, B \subseteq \mathbb{R}^n$. Prove that $\mathrm{conv}(A+B) = \mathrm{conv}(A) + \mathrm{conv}(B)$. (Hint: it is useful to consider the sum $\sum_{j,k} \lambda_j \mu_k (a_j + b_k)$ where $a_j \in A$, $b_k \in B$ and $\lambda_j, \mu_k \geq 0$ with $\sum_j \lambda_j = 1$ and $\sum_k \mu_k = 1$.)

Solution 2.25. We have that

$$\begin{aligned} \sum_{j,k} \lambda_j \mu_k (a_j + b_k) &= \sum_{j,k} \lambda_j \mu_k a_j + \sum_{j,k} \lambda_j \mu_k b_k \\ &= \sum_k \mu_k \left(\sum_j \lambda_j a_j \right) + \sum_j \lambda_j \left(\sum_k \mu_k b_k \right) \\ &= \sum_j \lambda_j a_j + \sum_k \mu_k b_k. \end{aligned}$$

As $\sum_{j,k} \lambda_j \mu_k = 1$, the left-hand side is a convex combination of points in $A+B$, while the right-hand side is an element in $\mathrm{conv}(A) + \mathrm{conv}(B)$. Therefore, $\mathrm{conv}(A+B) \subseteq \mathrm{conv}(A) + \mathrm{conv}(B)$. As $\mathrm{conv}(A) + \mathrm{conv}(B)$ is a convex set, it equals its convex hull, and

$$\mathrm{conv}(A+B) \subseteq \mathrm{conv}(\mathrm{conv}(A) + \mathrm{conv}(B)) = \mathrm{conv}(A) + \mathrm{conv}(B). \tag{2.24}$$

Exercise 2.37. Let $S \subseteq \mathbb{R}^n$. Show that either $\mathrm{int}(S) = \mathrm{rint}(S)$ or $\mathrm{int}(S) = \emptyset$.

Solution 2.37. If $\mathrm{int}(S) \neq \emptyset$, then there exists a point $x \in \mathrm{int}(S)$ and an $r > 0$ such that $B^\circ(x, r) \subseteq S$. As $B^\circ(x, r) \subseteq \mathrm{aff}(S)$, we have that $x \in \mathrm{rint}(S)$. Thus, $\mathrm{int}(S) \subseteq \mathrm{rint}(S)$.

Conversely, if $x \in \mathrm{rint}(S)$, then there exists an $r > 0$ such that $B^\circ(x, r) \cap \mathrm{aff}(S) \subseteq S$. As $B^\circ(x, r) \cap \mathrm{aff}(S)$ is open in $\mathrm{aff}(S)$, we have that $x \in \mathrm{int}(S)$. Thus, $\mathrm{rint}(S) \subseteq \mathrm{int}(S)$.

Therefore, if $\mathrm{int}(S) \neq \emptyset$, we have $\mathrm{int}(S) = \mathrm{rint}(S)$. If $\mathrm{int}(S) = \emptyset$, then the statement holds trivially.

Exercise 2.38. Prove Theorem 2.4.3. (Hint: To prove that $\mathrm{rint}(C)$ is convex, use Theorem 2.4.2. Concerning $\mathrm{int}(C)$, use Exercise 2.37. Finally, to show that $\mathrm{cl}(C)$ is convex, let $x, y \in \mathrm{cl}(C)$ and consider the two point sequences that converge to x and y respectively. Then, look at a convex combination of x and y and construct a suitable sequence!)

Solution 2.38. Theorem 2.4.2 states that a convex set has a "thin boundary", i.e., for $C \subseteq \mathbb{R}^n$ non-empty and convex, and $x_1 \in \mathrm{rint}(C)$, $x_2 \in \mathrm{cl}(C)$, we have

$$(1 - \lambda)x_1 + \lambda x_2 \in \mathrm{rint}(C) \tag{2.25}$$

for all $0 \leq \lambda < 1$.

Theorem 2.4.3 on the other hand states as follows: If $C \subseteq \mathbb{R}^n$ is a convex set, then all sets $\mathrm{rint}(C)$, $\mathrm{int}(C)$ and $\mathrm{cl}(C)$ are convex. Therefore, assume $C \subseteq \mathbb{R}^n$ is convex.

Let $x, y \in \mathrm{rint}(C)$. As $\mathrm{rint}(C) \subseteq C \subseteq \mathrm{cl}(C)$, we can simply apply Theorem 2.4.2 to show that $\mathrm{rint}(C)$ is convex. By Exercise 2.37, we have that either $\mathrm{int}(C) = \mathrm{rint}(C)$ or $\mathrm{int}(C) = \emptyset$. In both cases, $\mathrm{int}(C)$ is convex.

Finally, let $x, y \in \mathrm{cl}(C)$, and let $x^k, y^k \in C$ be sequences such that $x^k \to x$ and $y^k \to y$. As C is convex, $(1 - \lambda)x^k + \lambda y^k \in C$ for all $0 \leq \lambda \leq 1$, which converges to $(1 - \lambda)x + \lambda y$. Thus, $(1 - \lambda)x + \lambda y \in \mathrm{cl}(C)$, showing that $\mathrm{cl}(C)$ is convex.