MAT4120

Mandatory assignment for Mathematical Optimization

August Femtehjell

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Contents

Abstract

This document contains my solutions to the mandatory assignment for the course MAT4120–Mathematical Optimization, taught at the University of Oslo in the autumn of 2024. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MAT4120.

Exercise 2.6. If S is convex, then $\operatorname{conv}(S) = S$. Show this!

Solution 2.6. Since S is convex, for any $u, v \in S$ and any $0 \leq \lambda \leq 1$, we have $\lambda u + (1-\lambda)v \in S$. By the definition of convex hull, $\operatorname{conv}(S)$ is the smallest convex set containing S. Since S is already convex and contains itself, it follows that $\operatorname{conv}(S) = S$.

Exercise 2.38. Prove Theorem 2.4.3. (Hint: To prove that $\operatorname{rint}(C)$ is convex, use Theorem 2.4.2. Concerning $\operatorname{int}(C)$, use Exercise 2.37. Finally, to show that $\operatorname{cl}(C)$ is convex, let $x, y \in \operatorname{cl}(C)$ and consider the two point sequences that converge to x and y respectively. Then, look at a convex combination of x and y and construct a suitable sequence!)

Solution 2.38. Theorem 2.4.2 states that a convex set has a "thin boundary", i.e., for $C \subseteq \mathbb{R}^n$ non-empty and convex, and $x_1 \in \operatorname{rint}(C)$, $x_2 \in \operatorname{cl}(C)$, we have

$$(1-\lambda)x_1 + \lambda x_2 \in \operatorname{rint}(C) \tag{2.1}$$

for all $0 \leq \lambda < 1$.

Theorem 2.4.3 on the other hand states as follows: If $C \subseteq \mathbb{R}^n$ is a convex set, then all sets $\operatorname{rint}(C)$, $\operatorname{int}(C)$ and $\operatorname{cl}(C)$ are convex. Therefore, assume $C \subseteq \mathbb{R}^n$ is convex.

Let $x, y \in \operatorname{rint}(C)$. As $\operatorname{rint}(C) \subseteq C \subseteq \operatorname{cl}(C)$, we can simply apply Theorem 2.4.2 to show that $\operatorname{rint}(C)$ is convex. By Exercise 2.37, we have that either $\operatorname{int}(C) = \operatorname{rint}(C)$ or $\operatorname{int}(C) = \emptyset$. In both cases, $\operatorname{int}(C)$ is convex.

Finally, let $x, y \in \operatorname{cl}(C)$, and let $x^k, y^k \in C$ be sequences such that $x^k \to x$ and $y^k \to y$. As C is convex, $(1-\lambda)x^k + \lambda y^k \in C$ for all $0 \leq \lambda \leq 1$, which converges to $(1-\lambda)x + \lambda y$. Thus, $(1-\lambda)x + \lambda y \in \operatorname{cl}(C)$, showing that $\operatorname{cl}(C)$ is convex.

Exercise 3.6. Let L be a line in $\mathbb{R}^n$. Find the nearest point in L to a point $x \in \mathbb{R}^n$. Use your result to find the nearest point on the line $L = \{(x, y) : x + 3y = 5\}$ to the point $(1, 2)$.

Solution 3.6. Let L be the defined by the line $\{b + td : t \in \mathbb{R}\}$, where b is a point on the line and d is a direction vector. The nearest point on L to a point $x \in \mathbb{R}^n$ can be found by projecting the vector $x - b$ onto the direction vector d. The projection is given by

$$\operatorname{proj}_d(x-b) = b + \frac{(x-b)^T d}{\|d\|^2} d. \tag{3.2}$$

With $b = (5, 0)$, we note that $(1, 3)$ is orthogonal to the line, so we choose the direction vector to be $d = (-3, 1)$. Then, the nearest point on the line L to the point $(1, 2)$ is given by

$$\begin{aligned} \operatorname{proj}_d((1,2)-(5,0)) = \operatorname{proj}_d((-4,2)) &= (5,0) + \frac{(-4,2)^T(-3,1)}{\|(-3,1)\|^2}(-3,1) \\ &= (5,0) + \frac{14}{10}(-3,1) = \left(\frac{25}{5}, 0\right) + \left(-\frac{21}{5}, \frac{7}{5}\right) \\ &= \left(\frac{4}{5}, \frac{7}{5}\right). \end{aligned} \tag{3.3}$$

Therefore, the nearest point on the line L to the point $(1, 2)$ is $\left(\frac{4}{5}, \frac{7}{5}\right)$.

Exercise 3.18. Consider the outer description of closed convex sets given in Corollary 3.2.4. What is this description for each of the following sets:

(*i*) $C_1 = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$,

(*ii*) $C_2 = \operatorname{conv}(\{0, 1\}^n)$,

(*iii*) C_3 is the convex hull of the points $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ in $\mathbb{R}^2$.

(*iv*) C_4 is the convex hull of all vectors in $\mathbb{R}^n$ having components that are either 1 or -1.

Solution 3.18. We find the outer descriptions as follows:

(i) The half-planes are the tangents to the unit ball, as used in a previous exercise.

(ii) With e_i being the ith standard basis vector in $\mathbb{R}^n$, the half-planes are given by $e_i^T x = 0$ and $e_i^T x = 1$ for $i = 1, \dots, n$.

(iii) Similarly, here the half-planes are given by $x_1 = 1$, $x_1 = -1$, $x_2 = 1$, and $x_2 = -1$, almost equivalently to (ii).

(iv) Again, the half-planes are given by $e_i^T x = 1$ and $e_i^T x = -1$ for $i = 1, \dots, n$, almost equivalently to (ii) and (iii).

Exercise 4.17. Consider a polyhedral cone $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$ (where, as usual A is a real $m \times n$-matrix). Show that 0 is the unique vertex of C.

Solution 4.17. Assume that $\operatorname{rank}(A) < n$. Then, there exists a non-zero vector z such that $Az = 0$. As such, we have that for any $x \in C$ and any $\lambda \in \mathbb{R}$,

$$A(x + \lambda z) = Ax + \lambda Az = Ax \leq 0,$$

showing that $x + \lambda z \in C$. Therefore, if $\operatorname{rank}(A) < n$, then C has no vertices. As such, the exercise is missing information, as it is not necessarily true that 0 is the unique vertex of C.

As a simple counterexample, consider

$$C = \{(x_0, x_1) \in \mathbb{R}^2 : x_0 \leq 0\}, \tag{4.4}$$

which contains the line $\{(0, x_1) : x_1 \in \mathbb{R}\}$, and thus has no vertices.

If on the other hand $\operatorname{rank}(A) = n$, then there exists an $n \times n$ submatrix A' of A such that $\det(A') \neq 0$. Then, the only solution to $A'x = 0$ is $x = 0$, regardless of which such submatrix we choose. Therefore, 0 is the only point that can be a vertex of C.

Exercise 4.22. Let $P \subset \mathbb{R}^2$ be the polyhedron being the solution set of the linear system

$$\begin{array}{rcrcl} x & - & y & \leq & 0 \\ -x & + & y & \leq & 1 \\ & & 2y & \geq & 5 \\ 8x & - & y & \leq & 16 \\ x & + & y & \geq & 4 \end{array} \tag{4.5}$$

Find all the extreme points of P.

Solution 4.22. Written in matrix form, we have

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -2 \\ 8 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 16 \\ -4 \end{bmatrix}. \tag{4.6}$$

I chose to solve this using Python, with the library SymPy. The code is freely available at `mandatory/extreme_points.py`, and briefly works by iterating over all 2×2 submatrices of A and checking if the corresponding equations yield a point in the feasible region. With this approach I found the extreme points to be

$$\left\{ \frac{1}{2}\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \frac{1}{7}\begin{bmatrix} 17 \\ 24 \end{bmatrix}, \frac{1}{16}\begin{bmatrix} 37 \\ 40 \end{bmatrix} \right\}. \tag{4.7}$$

The feasible region and all extreme points are illustrated in Figure 1.

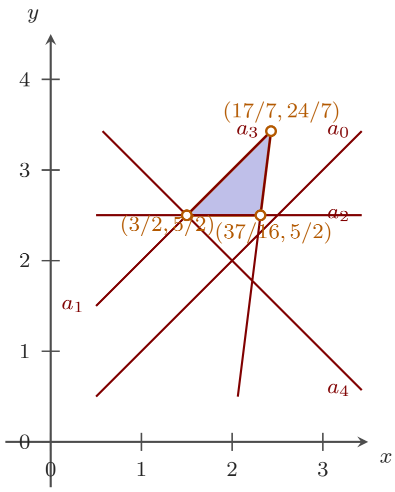

Figure 1: Feasible region of Exercise 4.22 with the constraint boundaries and extreme points.

Exercise 5.10. By the result above we have that if f and g are convex functions, then the function $\max\{f, g\}$ is also convex. Prove this result directly from the definition of a convex function.

Solution 5.10. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, and define $h(x) =$

$\max\{f(x), g(x)\}$. Then, for any $x, y \in \mathbb{R}^n$ and any $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \max\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\leq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\} \\ &\leq \lambda \max\{f(x), g(x)\} + (1-\lambda)\max\{f(y), g(y)\} \\ &= \lambda h(x) + (1-\lambda)h(y), \end{aligned}$$

showing that h is convex.

Proposition 5.1.2. (Increasing slopes) A function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if for each $x_0 \in \mathbb{R}$ the slope function

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0} \tag{5.8}$$

is increasing on $\mathbb{R} \setminus \{x_0\}$.

Exercise 5.22. Let $f : (0, \infty) \to \mathbb{R}$ and define the function $g : (0, \infty) \to \mathbb{R}$ by $g(x) = xf(\frac{1}{x})$. Prove that f is convex if and only if g is convex. Hint: Prove that

$$\frac{g(x) - g(x_0)}{x - x_0} = f(\tfrac{1}{x_0}) - \frac{1}{x_0} \cdot \frac{f(\frac{1}{x}) - f(\frac{1}{x_0})}{\frac{1}{x} - \frac{1}{x_0}} \tag{5.9}$$

and use Proposition 5.1.2. Why is the function $x \mapsto xe^{1/x}$ convex?

Solution 5.22. We have that

$$\begin{aligned} \frac{g(x) - g(x_0)}{x - x_0} &= \frac{xf(\frac{1}{x}) - x_0 f(\frac{1}{x_0})}{x - x_0} \\ &= \frac{xf(\frac{1}{x}) - xf(\frac{1}{x_0}) + xf(\frac{1}{x_0}) - x_0 f(\frac{1}{x_0})}{x - x_0} \\ &= \frac{x(f(\frac{1}{x}) - f(\frac{1}{x_0}))}{x - x_0} + \frac{(x - x_0)f(\frac{1}{x_0})}{x - x_0} \\ &= f(\tfrac{1}{x_0}) + \frac{x(f(\frac{1}{x}) - f(\frac{1}{x_0}))}{x - x_0} \\ &= f(\tfrac{1}{x_0}) + \frac{1}{x_0} \frac{xx_0(f(\frac{1}{x}) - f(\frac{1}{x_0}))}{x - x_0}. \end{aligned}$$

Then, as

$$\frac{xx_0}{x - x_0} = \frac{1}{\frac{x}{xx_0} - \frac{x_0}{xx_0}} = \frac{1}{\frac{1}{x_0} - \frac{1}{x}},$$

Equation (5.9) follows.

Assume f is convex. Let h be the slope function of f at x_0, i.e.,

$$h(x) = \frac{f(x) - f(x_0)}{x - x_0}. \tag{5.10}$$

As $1/x$ decreases as x increases, we have that $h(1/x)$ is decreasing. Therefore, $-h(1/x)$ is increasing, and offsetting this with the constant $f(1/x_0)$ maintains the increasing property. Thus, by Proposition 5.1.2, g is convex, as it's slope function is increasing.

With $f(x) = e^x$, we have $g(x) = xf(1/x) = xe^{1/x}$, which is therefore convex, as e^x is convex.