MAT4170

Second mandatory exercise

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Abstract

This document contains my solutions to the second mandatory assignment for the course MAT4170–Spline Methods, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MAT4170.

Exercise 2.2 Implement the de Casteljau algorithm for a planar Bézier curve of arbitrary degree d over a general interval $[a, b]$. Use the routine to make a program to plot the quadratic spline curve $\mathbf{s} : [0, 2] \to \mathbb{R}$, with pieces

$$\mathbf{p}(t) = \sum_{i=0}^{2} \mathbf{c}_i B_i^2(t), \qquad 0 \leq t \leq 1,$$
$$\mathbf{q}(t) = \sum_{i=0}^{2} \mathbf{d}_i B_i^2(t-1), \qquad 1 < t \leq 2,$$

where $\mathbf{c}_0 = (-1, 1)$, $\mathbf{c}_1 = (-1, 0)$, $\mathbf{c}_2 = (0, 0)$, and $\mathbf{d}_0 = (0, 0)$, $\mathbf{d}_1 = (1, 0)$, $\mathbf{d}_2 = (2, 1)$.

Solution 2.2 The de Casteljau algorithm is implemented in `spline_cj.py`[1], and is practically identical to the one implemented in the previous section, due to the vectorization possibilities in `JAX`. The resulting figure is shown in Fig. 1.

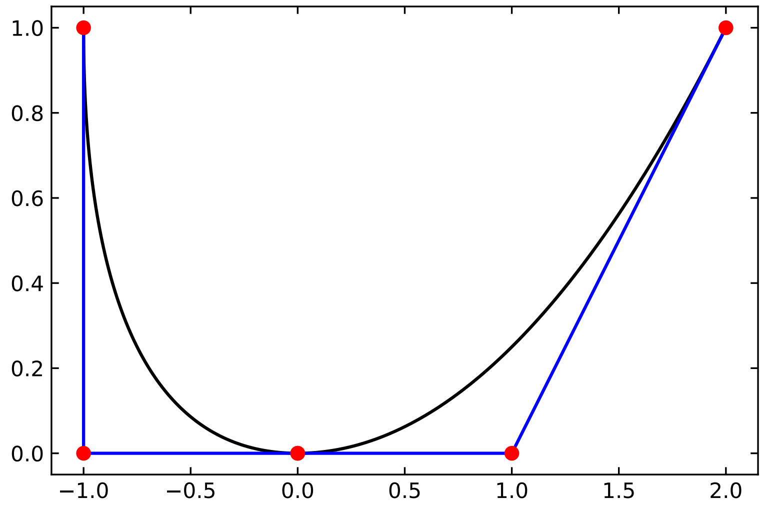

Figure 1: The quadratic spline curve $\mathbf{s} : [0, 2] \to \mathbb{R}$, with pieces $\mathbf{p}(t)$ and $\mathbf{q}(t)$.

Exercise 2.3 What is the order of continuity of $\mathbf{s}$ in Exercise 2.2 at the breakpoint $t = 1$?

Solution 2.3 We clearly have C^0 continuity at the breakpoint $t = 1$, as

$$\mathbf{c}_2 = (0, 0) = \mathbf{d}_0.$$

[1] Available here.

For C^1 continuity, we must have

$$\frac{\mathbf{c}_2 - \mathbf{c}_1}{1-0} = \frac{\mathbf{d}_1 - \mathbf{d}_0}{2-1}$$
$$(0,0) - (-1,0) = (1,0) - (0,0),$$

which also holds. Finally for C^2 continuity, we must have

$$\frac{\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0}{1^2} \stackrel{?}{=} \frac{\mathbf{d}_2 - 2\mathbf{d}_1 + \mathbf{d}_0}{1^2}$$
$$(0,0) - 2(-1,0) + (-1,1) = (1,1) \neq (0,1) = (2,1) - 2(1,0) + (0,0),$$

which it seems like we do not have. Thus, the order of continuity of $\mathbf{s}$ at the breakpoint $t = 1$ is C^1.

Exercise 3.1 Suppose that x_0, x_1, x_2 are distinct, and let $f_i = f(x_i)$, $i = 0, 1, 2$, for some function f. Show by direct calculation that the recursive formula

$$[x_0, x_1, x_2]f = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}$$

can be expressed as

$$[x_0, x_1, x_2]f = \sum_{i=0}^{2} \frac{f_i}{\prod_{j \neq i}(x_i - x_j)}.$$

Solution 3.1 We have

$$[x_0, x_1, x_2]f = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}$$

where we begin by expanding the top fractions.

$$\begin{aligned}
\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0} &= \frac{f_2}{x_2 - x_1} - f_1\left(\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right) + \frac{f_0}{x_1 - x_0} \\
&= \frac{f_2}{x_2 - x_1} - f_1 \frac{x_1 - x_0 + x_2 - x_1}{(x_2 - x_1)(x_1 - x_0)} + \frac{f_0}{x_1 - x_0} \\
&= \frac{f_2}{x_2 - x_1} - f_1 \frac{x_2 - x_0}{(x_2 - x_1)(x_1 - x_0)} + \frac{f_0}{x_1 - x_0} \\
&= \frac{f_2}{x_2 - x_1} + \frac{f_1(x_2 - x_0)}{(x_1 - x_2)(x_1 - x_0)} + \frac{f_0}{x_1 - x_0} \qquad (3.1.1)
\end{aligned}$$

Dividing (3.1.1) by $x_2 - x_0$ gives

$$\begin{aligned}
[x_0, x_1, x_2]f &= \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} \\
&= \frac{f_2}{(x_2 - x_1)(x_2 - x_0)} + \frac{f_1}{(x_1 - x_2)(x_1 - x_0)} + \frac{f_0}{(x_1 - x_0)(x_2 - x_0)} \\
&= \frac{f_2}{(x_2 - x_1)(x_2 - x_0)} + \frac{f_1}{(x_1 - x_2)(x_1 - x_0)} + \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} \\
&= \sum_{i=0}^{2} \frac{f_i}{\prod_{j \neq i}(x_i - x_j)},
\end{aligned}$$

as desired.

Exercise 3.3 Show that if $f(x) = 1/x$ and that $x_0, x_1, \ldots, x_k \neq 0$ then

$$[x_0, \ldots, x_k]f = (-1)^k \frac{1}{x_0 x_1 \cdots x_k}.$$

Solution 3.3 In the base case we have simply

$$[x_0]f = \frac{1}{x_0} = (-1)^0 \frac{1}{x_0}.$$

In the case of $x_0 = x_1 = \cdots = x_k$ we have

$$[\underbrace{x_0, x_0, \ldots, x_0}_{k+1}]f = \frac{f^{(k)}(x_0)}{k!} = (-1)^k \frac{1}{x_0^{k+1}} \frac{k!}{k!} = (-1)^k \frac{1}{x_0 x_1 \cdots x_k},$$

so mulitplicities are handled correctly. For two distinct points x_0, x_1 we have

$$[x_0, x_1]f = \frac{f_1 - f_0}{x_1 - x_0} = \frac{1/x_1 - 1/x_0}{x_1 - x_0} = \frac{x_0 - x_1}{x_0 x_1 (x_1 - x_0)} = (-1)^1 \frac{1}{x_0 x_1},$$

so the formula holds for $k = 1$. Assume that the formula holds for $k = n$, and consider $k = n + 1$. We have

$$\begin{aligned}
[x_0, \ldots, x_{n+1}]f &= \frac{[x_1, \ldots, x_{n+1}]f - [x_0, \ldots, x_n]f}{x_{n+1} - x_0} \\
&= \frac{(-1)^n \frac{1}{x_1 \cdots x_{n+1}} - (-1)^n \frac{1}{x_0 \cdots x_n}}{x_{n+1} - x_0} \\
&= (-1)^n \frac{\frac{1}{x_{n+1}} - \frac{1}{x_0}}{(x_{n+1} - x_0)x_1 \cdots x_n} \\
&= (-1)^n \frac{\frac{x_0 - x_{n+1}}{x_0 x_{n+1}}}{(x_{n+1} - x_0)x_1 \cdots x_n} \\
&= (-1)^{n+1} \frac{x_{n+1} - x_0}{(x_{n+1} - x_0)x_0 x_1 \cdots x_n x_{n+1}} \\
&= (-1)^{n+1} \frac{1}{x_0 x_1 \cdots x_{n+1}},
\end{aligned}$$

proving the formula by induction.

Exercise 3.5 Use the recursion formula (Theorem 3.4) to show that

a) $B[0, 0, 0, 1](x) = (1 - x)^2 B[0, 1](x)$,

b) $B[0, 0, 1, 2](x) = x(2 - 3x/2)B[0, 1](x) + \frac{1}{2}(2 - x)^2 B[1, 2](x)$,

c) $B[0, 1, 1, 2](x) = x^2 B[0, 1](x) + (2 - x)^2 B[1, 2](x)$.

Solution 3.5 The recursion formula states that for $d \geq 1$,

$$B_{i,d}(x) = \frac{x - t_i}{t_{i+d} - t_i} B_{i,d-1}(x) + \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}} B_{i+1,d-1}(x).$$

I'm not entirely sure which knots the B-splines are defined on, however I guess the exercise is to relate these ones to the cardinal B-splines.

a) In this case we have $t_i = t_d$, meaning that the first term in the recursion formula is zero. We then have

$$B[0, 0, 0, 1](x) = 0 + \frac{1 - x}{1 - 0} B[0, 0, 1](x).$$

Recursing on $B[0, 0, 1](x)$, again the first term is zero, so we have

$$(1 - x)B[0, 0, 1](x) = (1 - x)\frac{1 - x}{1 - 0} B[0, 1](x) = (1 - x)^2 B[0, 1](x).$$

b) Next, we have

$$
\begin{aligned}
B[0,0,1,2](x) &= \frac{x-0}{1-0}B[0,0,1](x) + \frac{2-x}{2-0}B[0,1,2](x) \\
&= x(1-x)B[0,1](x) + \frac{2-x}{2}\left(xB[0,1](x) + (2-x)B[1,2](x)\right) \\
&= x\left(1-x+\frac{2-x}{2}\right)B[0,1](x) + \frac{1}{2}(2-x)^2B[1,2](x) \\
&= x\left(2-\frac{3}{2}x\right)B[0,1](x) + \frac{1}{2}(2-x)^2B[1,2](x).
\end{aligned}
$$

c) Finally, for the last expression we have

$$B[0,1,1,2](x) = \frac{x-0}{1-0}B[0,1,1](x) + \frac{2-x}{2-1}B[1,1,2](x).$$

Considering the two terms separately, we have

$$B[0,1,1](x) = xB[0,1](x)$$

as the second term vanishes, and

$$B[1,1,2](x) = \frac{2-x}{2-1}B[1,2](x) = (2-x)^2B[1,2](x),$$

where the first term vanishes. Combining these, we find

$$B[0,1,1,2](x) = x^2B[0,1](x) + (2-x)^2B[1,2](x),$$

as desired.

Exercise B-splines. What are the smoothness properties of $B[0,0,0,0,1](x)$ and $B[0,1,1,1,2](x)$?

Solution B-splines. The smoothness of a B-spline of degree d at a knot with multiplicity m is C^{d-m}. Denoting $B[0,0,0,0,1](x)$ as $B^1(x)$, we that the degree of B^1 is $d=3$. 0 has a multiplicity of four, so the smoothness of B^1 at 0 is C^{-1}. At 1, the multiplicity is 1, so the smoothness is C^2. Between the knots, the smoothness is C^∞.

Denoting $B[0,1,1,1,2](x)$ as $B^2(x)$, we that the degree of B^2 is $d=3$. 0 and 2 both have a multiplicity of one, so the smoothness is C^2 at both points. 1 has a multiplicity of three, so the smoothness at this point is C^0.

Exercise 2.4 (*Optional*) The curvature of a parametric curve $\mathbf{r}(t)$ in $\mathbb{R}^2$ can be expressed as

$$\kappa(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3},$$

where $(a_1, a_2) \times (b_1, b_2) := a_1 b_2 - a_2 b_1$. What are the curvatures of $\mathbf{p}$ and $\mathbf{q}$ in Exercise 2.2 at the breakpoint $t = 1$? What can you say about the smoothness of $\mathbf{s}$?

Solution 2.4 (*Optional*) We firstly begin by tabulating the derivatives of $\mathbf{p}$ and $\mathbf{q}$, based on the general formula

$$p^{(r)}(x) = \frac{d!}{(d-r)!} \frac{1}{h^r} \sum_{i=0}^{d-r} \Delta^r c_i B_i^{d-r}(\lambda).$$

In both cases here, we have $h = 1$, $d = 2$, and $\lambda = t$ for $\mathbf{p}$ and $\lambda = t - 1$ for $\mathbf{q}$. The derivatives for $\mathbf{p}$ are then

$$\begin{aligned} \mathbf{p}'(t) &= 2\left((\mathbf{c}_1 - \mathbf{c}_0)\, B_0^1(\lambda) + (\mathbf{c}_2 - \mathbf{c}_1)\, B_1^1(\lambda)\right) \\ &= 2\left((\mathbf{c}_1 - \mathbf{c}_0)(1-t) + (\mathbf{c}_2 - \mathbf{c}_1)\, t\right) \\ &= 2\left((\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0)t + (\mathbf{c}_1 - \mathbf{c}_0)\right) \\ &= 2\left((1,1)t + (0,-1)\right) \\ &= (2t, 2t-2), \end{aligned} \qquad \begin{aligned} \mathbf{p}''(t) &= 2\left(\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0\right) B_0^0(\lambda) \\ &= 2\,(1,1) \\ &= (2,2), \end{aligned}$$

while we for $\mathbf{q}$ have

$$\begin{aligned} \mathbf{q}'(t) &= 2\left((\mathbf{d}_1 - \mathbf{d}_0)\, B_0^1(\lambda) + (\mathbf{d}_2 - \mathbf{d}_1)\, B_1^1(\lambda)\right) \\ &= 2\left((\mathbf{d}_1 - \mathbf{d}_0)\, t + (\mathbf{d}_2 - \mathbf{d}_1)(1-t)\right) \\ &= 2\left((-\mathbf{d}_2 + 2\mathbf{d}_1 - \mathbf{d}_0)t + (\mathbf{d}_2 - \mathbf{d}_1)\right) \\ &= 2\left((0,-1)t + (1,1)\right) \\ &= (2, -2t+2), \end{aligned} \qquad \begin{aligned} \mathbf{q}''(t) &= 2\left(\mathbf{d}_2 - 2\mathbf{d}_1 + \mathbf{d}_0\right) B_0^0(\lambda) \\ &= 2\,(0,1) \\ &= (0,2), \end{aligned}$$

where we have used that $B_0^1(1-t) = t$ and $B_1^1(1-t) = 1-t$. The curvatures are then

$$\begin{aligned} \kappa_{\mathbf{p}}(t) &= \frac{(2t, 2t-2) \times (2,2)}{\sqrt{(2t)^2 + (2t-2)^2}^3} = \frac{4}{(8t^2 - 8t + 4)^{3/2}}, \\ \kappa_{\mathbf{q}}(t) &= \frac{(2, -2t+2) \times (0,2)}{\sqrt{2^2 + (-2t+2)^2}^3} = \frac{4}{(4t^2 - 8t + 8)^{3/2}}. \end{aligned}$$

These give, at the breakpoint $t = 1$, the curvatures

$$\kappa_{\mathbf{p}}(1) = \frac{4}{2^3} = \frac{1}{2} \quad \text{and} \quad \kappa_{\mathbf{q}}(1) = \frac{4}{2^3} = \frac{1}{2}.$$

As we see, the curvatures are equal at the breakpoint $t = 1$. This means that while we do not have C^2 continuity, we have G^2 continuity / smoothness.