MAT4170

Exercises for Spline Methods

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Abstract

This document contains my summary for the course MAT4170 – Spline Methods, taught at the University of Oslo in the spring of 2025, in preperation for the final oral exam. This summary is based on the lecture notes for the course, draft dated 25th of March, 2025. The code for everything, as well as this document, can be found at my GitHub repository: `https://github.com/augustfe/MAT4170`.

1 Bernstein polynomials

A Bernstein polynomial denoted B_i^d is defined by

$$B_i^d(x) = \binom{d}{i} x^i (1-x)^{d-i}, \tag{1.1}$$

where d is the degree of the polynomial, and i is the index of the polynomial. These polynomials satisfy a number of interesting properties.

Firstly, they are all non-negative on the interval $[0, 1]$. We can clearly see this as then both $x \geq 0$ and $1 - x \geq 0$ (and of course the binomial coefficient as well). In addition, for all $x \in \mathbb{R}$ we have that

$$\sum_{i=0}^{d} B_i^d(x) = 1, \tag{1.2}$$

and the polynomials therefore form a partition of unity. We can see this by noting

$$1 = 1^d = (x + (1-x))^d = \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i} = \sum_{i=0}^{d} B_i^d(x), \tag{1.3}$$

by the binomial theorem.

In order to compute the value of a Bernstein polynomial efficiently, we note that

$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i}, \tag{1.4}$$

most easily recalled by thinking about Pascals triangle. With this, we have that

$$\begin{aligned}
B_i^d &= \binom{d}{i} x^i (1-x)^{d-i} \\
&= \left(\binom{d-1}{i-1} + \binom{d-1}{i} \right) x^i (1-x)^{d-i} \\
&= x \binom{d-1}{i-1} x^{i-1} (1-x)^{(d-1)-(i-1)} + (1-x) \binom{d-1}{i} x^i (1-x)^{(d-1)-i} \\
&= x B_{i-1}^{d-1}(x) + (1-x) B_i^{d-1},
\end{aligned}$$

forming the basis for de Casteljau's algorithm.

2 B-Splines