MAT4170

Exercises for Spline Methods

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Contents

1 Bernstein-Bézier polynomials

Exercise 1.1 It is sometimes necessary to convert a polynomial in BB form to monomial form. Consider a quadratic BB polynomial,

$$p(x) = c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2.$$

Express p in the monomial form

$$p(x) = a_0 + a_1x + a_2x^2.$$

Solution 1.1 Rather than using the explicit formula for conversion, we can just expand the coefficients and collect terms.

$$\begin{aligned}
p(x) &= c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2 \\
&= c_0(1-2x+x^2) + 2c_1(x-x^2) + c_2x^2 \\
&= c_0 - 2c_0x + c_0x^2 + 2c_1x - 2c_1x^2 + c_2x^2 \\
&= c_0 + (-2c_0 + 2c_1)x + (c_0 - 2c_1 + c_2)x^2.
\end{aligned}$$

Exercise 1.2 Consider a polynomial $p(x)$ of degree $\leq d$, for arbitrary d. Show that if

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x),$$

then

$$a_j = \binom{d}{j} \Delta^j c_0.$$

Hint: Use a Taylor approximation to p to show that $a_j = p^{(j)}(0)/j!$.

Solution 1.2 We have that

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x).$$

By the Taylor approximation, we have that

$$p(x) = p(x+0) = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} x^j.$$

We thus have that

$$a_j = \frac{p^{(j)}(0)}{j!}.$$

By properties of the Bézier curves, we have that

$$p^{(j)}(x) = \frac{d!}{(d-j)!} \sum_{i=0}^{d-j} \Delta^j c_i B_i^{d-j}(x),$$

and specifically for $x = 0$,

$$p^{(j)}(0) = \frac{d!}{(d-j)!} \Delta^j c_0.$$

Combining these results, we have that

$$a_j = \frac{p^{(j)}(0)}{j!} = \frac{d!}{(d-j)!j!} \Delta^j c_0 = \binom{d}{j} \Delta^j c_0,$$

as we wanted to show.

Exercise 1.3 We might also want to convert a polynomial from monomial form to BB form. Using Lemma 1.2, show that in the notation of the previous question,

$$c_i = \frac{i!}{d!} \sum_{j=0}^{i} \frac{(d-j)!}{(i-j)!} a_j.$$

Solution 1.3 Lemma 1.2 states that for $j = 0, 1, \ldots, d$,

$$x^j = \frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x).$$

We have that

$$\sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x)$$

$$\sum_{j=0}^{d} a_j \left[\frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x) \right] = \sum_{i=0}^{d} c_i B_i^d(x)$$

As we have $i \geq j$, we can reorder the summation to the form $j \leq i$, by using

$$\sum_{j=0}^{d} \sum_{i=j}^{d} (\ldots) = \sum_{i=0}^{d} \sum_{j=0}^{i} (\ldots).$$

This gives us

$$\sum_{i=0}^{d} \left[\sum_{j=0}^{i} a_j \frac{(d-j)!}{d!} \frac{i!}{(i-j)!} \right] B_i^d(x) = \sum_{i=0}^{d} c_i B_i^d(x).$$

Which by isolating the coefficients, gives us

$$c_i = \frac{i!}{d!} \sum_{j=0}^{i} \frac{(d-j)!}{(i-j)!} a_j,$$

as we wanted to show.