MAT4170

Exercises for Spline Methods

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Exercise 1.1 It is sometimes necessary to convert a polynomial in BB form to monomial form. Consider a quadratic BB polynomial,

$$p(x) = c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2.$$

Express p in the monomial form

$$p(x) = a_0 + a_1x + a_2x^2.$$

Solution 1.1 Rather than using the explicit formula for conversion, we can just expand the coefficients and collect terms.

$$\begin{aligned}
p(x) &= c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2 \\
&= c_0(1-2x+x^2) + 2c_1(x-x^2) + c_2x^2 \\
&= c_0 - 2c_0x + c_0x^2 + 2c_1x - 2c_1x^2 + c_2x^2 \\
&= c_0 + (-2c_0 + 2c_1)x + (c_0 - 2c_1 + c_2)x^2.
\end{aligned}$$

Exercise 1.2 Consider a polynomial $p(x)$ of degree $\le d$, for arbitrary d. Show that if

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x),$$

then

$$a_j = \binom{d}{j} \Delta^j c_0.$$

Hint: Use a Taylor approximation to p to show that $a_j = p^{(j)}(0)/j!$.

Solution 1.2 We have that

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x).$$

By the Taylor approximation, we have that

$$p(x) = p(x+0) = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} x^j.$$

We thus have that

$$a_j = \frac{p^{(j)}(0)}{j!}.$$

By properties of the Bézier curves, we have that

$$p^{(j)}(x) = \frac{d!}{(d-j)!} \sum_{i=0}^{d-j} \Delta^j c_i B_i^{d-j}(x),$$

and specifically for $x = 0$,

$$p^{(j)}(0) = \frac{d!}{(d-j)!} \Delta^j c_0.$$

Combining these results, we have that

$$a_j = \frac{p^{(j)}(0)}{j!} = \frac{d!}{(d-j)!j!} \Delta^j c_0 = \binom{d}{j} \Delta^j c_0,$$

as we wanted to show.

Exercise 1.3 We might also want to convert a polynomial from monomial form to BB form. Using Lemma 1.2, show that in the notation of the previous question,

$$c_i = \frac{i!}{d!} \sum_{j=0}^{i} \frac{(d-j)!}{(i-j)!} a_j.$$

Solution 1.3 Lemma 1.2 states that for $j = 0, 1, \ldots, d$,

$$x^j = \frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x).$$

We have that

$$\sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x)$$

$$\sum_{j=0}^{d} a_j \left[\frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x) \right] = \sum_{i=0}^{d} c_i B_i^d(x)$$

As we have $i \geq j$, we can reorder the summation to the form $j \leq i$, by using

$$\sum_{j=0}^{d} \sum_{i=j}^{d} (\ldots) = \sum_{i=0}^{d} \sum_{j=0}^{i} (\ldots).$$

This gives us

$$\sum_{i=0}^{d} \left[\sum_{j=0}^{i} a_j \frac{(d-j)!}{d!} \frac{i!}{(i-j)!} \right] B_i^d(x) = \sum_{i=0}^{d} c_i B_i^d(x).$$

Which by isolating the coefficients, gives us

$$c_i = \frac{i!}{d!} \sum_{j=0}^{i} \frac{(d-j)!}{(i-j)!} a_j,$$

as we wanted to show.

Exercise 1.4 Implement the de Casteljau algorithm for cubic Bézier curves in Matlab or Python (or some other programming language), taking repeated convex combinations. Choose a sequence of four control points and plot both the control polygon and the Bézier curve, like those in Figure 1.3.

Solution 1.4 The de Casteljau algorithm uses recursion to compute the value of a point along a Bézier curve by the following formula:

1. Initialize by setting $c_i^0 = c_i$ for $i = 0, 1, \ldots, d$.

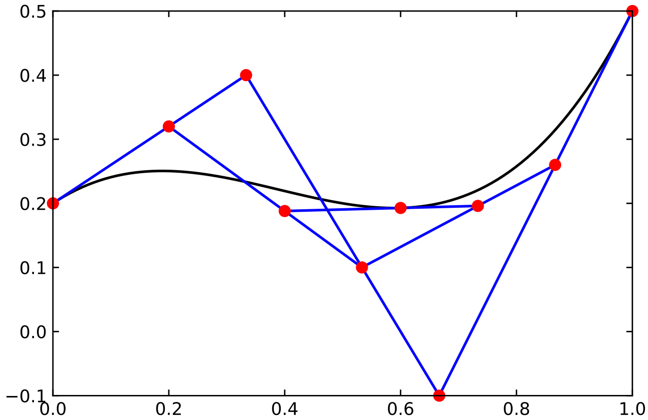

Figure 1: The de Casteljau algorithm applied to a cubic Bézier curve, with control points $(0.2, 0.4, -0.1, 0.5)$, illustrated at the point $x = 0.6$.

2. Then, for each $r = 1, 2, \dots, d$, let

$$c_i^r = (1-x)c_i^{r-1} + xc_{i+1}^{r-1}, \quad i = 0, 1, \dots, d-r.$$

3. The last value c_0^d is the value of the Bézier curve at x.

This is implemented using Jax in Python in `de_casteljau.py`, and the result is shown in Figure 1, bearing a striking resemblance to the figure in the book.

Exercise 1.5 Show that the graph, $g(x) = (x, p(x))$ of the BB polynomial p in (1.6) is a Bézier curve in $\mathbb{R}^2$, with control points (ξ_i, c_i), $i = 0, 1, \dots, d$, where $\xi_i = i/d$. *Hint:* Express x as a linear combination of $B_0^d(x), \dots, B_d^d(x)$.

Solution 1.5 We can again utilize Lemma 1.2 to express x as a linear combination of the Bernstein polynomials. We have that, writing $x = x^1$ for clarity,

$$x^1 = \frac{(d-1)!}{d!} \sum_{i=1}^{d} \frac{i!}{(i-1)!} B_i^d(x) = \sum_{i=1}^{d} \frac{i}{d} B_i^d(x) = \sum_{i=0}^{d} \frac{i}{d} B_i^d(x) = \sum_{i=0}^{d} \xi_i B_i^d(x).$$

We can now express the graph of p as a Bézier curve in $\mathbb{R}^2$ by

$$g(x) = (x, p(x)) = \left(\sum_{i=0}^{d} \xi_i B_i^d(x), \sum_{i=0}^{d} c_i B_i^d(x) \right) = \sum_{i=0}^{d} (\xi_i, c_i) B_i^d(x) = \sum_{i=0}^{d} \boldsymbol{c}_i B_i^d(x),$$

where $\boldsymbol{c}_i = (\xi_i, c_i)$ are the control points of the Bézier curve.

Exercise 1.6 Show that the tangent vector $\boldsymbol{p}'(x)$ of the Bézier curve in (1.6) lies in the convex cone of the vectors $\Delta \boldsymbol{c}_i$, i.e., in

$$\text{cone}(\Delta \boldsymbol{c}_0, \ldots, \Delta \boldsymbol{c}_{d-1}) = \left\{ \sum_{i=0}^{d-1} \mu_i \Delta \boldsymbol{c}_i : \mu_1, \ldots, \mu_{d-1} \geq 0 \right\}.$$

Solution 1.6 The derivative (or perhaps *gradient* is the correct term) of the Bézier curve $\boldsymbol{p}(x)$ is given by

$$\boldsymbol{p}'(x) = d \sum_{i=0}^{d-1} \left(\boldsymbol{c}_{i+1} - \boldsymbol{c}_i\right) B_i^{d-1}(x) = d \sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x).$$

As $B_i^{d-1}(x) \geq 0$ for $x \in [0,1]$, we can set $\mu_i = dB_i^{d-1}(x)$, and we have that

$$\boldsymbol{p}'(x) = \sum_{i=0}^{d-1} \mu_i \Delta \boldsymbol{c}_i \in \text{cone}(\Delta \boldsymbol{c}_0, \ldots, \Delta \boldsymbol{c}_{d-1}),$$

as we wanted to show.

Exercise 1.7 Show that the first derivative of p in (1.6) can be expressed (and computed) as

$$p'(x) = d(c_1^{d-1} - c_0^{d-1}),$$

where c_1^{d-1}, c_0^{d-1} are the points of order $d-1$ in de Casteljau's algorithm (1.10).

Solution 1.7 We have that

$$p(x) = c_0^d = (1-x)c_0^{d-1} + xc_1^{d-1},$$

and thus by differentiating with respect to x, we have that

$$p'(x) = c_1^{d-1} - c_0^{d-1}.$$

This tells us that we cannot be as naive as this, as c_0^d is actually a function of x, and not simply a constant.

What we might instead need to note is that

$$c_i^r = \sum_{j=0}^{r} c_{i+j} B_j^r(x),$$

and combining this with the fact that

$$\left(B_i^d\right)'(x) = d\left(B_{i-1}^{d-1} - B_i^{d-1}\right)(x),$$

we have that

$$\begin{aligned} p'(x) &= d\sum_{i=0}^{d-1}\left(c_{i+1}-c_i\right)B_i^{d-1}(x) = d\left[\sum_{i=0}^{d-1} c_{i+1}B_i^{d-1}(x) - \sum_{i=0}^{d-1} c_i B_i^{d-1}(x)\right] \\ &= d(c_1^{d-1}-c_0^{d-1}), \end{aligned}$$

as we wanted to show.

Exercise 1.8 Show that the Bernstein basis polynomial $B_i^d(x)$ has only one maximum in $[0,1]$, namely at $x = i/d$.

Solution 1.8 We do this by firstly computing the derivative of $B_i^d(x)$, which is given by

$$\left(B_i^d\right)'(x) = d\left(B_{i-1}^{d-1}(x) - B_i^{d-1}(x)\right).$$

A maximum or minimum of a function occurs where the derivative is zero, so we set

$$\begin{aligned} B_{i-1}^{d-1}(x) &= B_i^{d-1}(x) \\ \frac{(d-1)!}{(i-1)!(d-i)!}x^{i-1}(1-x)^{d-i} &= \frac{(d-1)!}{i!(d-1-i)!}x^i(1-x)^{d-i-1} \\ \frac{\cancel{(d-1)!}\,i!(d-1-i)!}{(i-1)!(d-i)!\cancel{(d-1)!}}\cancel{x^{i-1}}(1-x)^{\cancel{d-i}} &= x^{\cancel{i}}\cancel{(1-x)^{d-i-1}} \\ \frac{i}{d-i}(1-x) &= x \\ i - ix &= dx - ix \\ x &= \frac{i}{d}. \end{aligned}$$

We have thus shown that the Bernstein basis polynomials only have one extremal point.

We can use the second derivative to test if this is a maximum or a minimum, however we can instead note that $B_i^d(x)$ is a non-negative polynomial, which is only zero at either $x = 0$ or $x = 1$, and thus $x = i/d$ must be a maximum.

Exercise 1.9 Give a proof of the forward difference formula, (1.15).

Solution 1.9 The forward difference formula (1.15) is given by

$$\Delta^r c_0 = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}c_i.$$

The forward difference operator is defined by the recursion

$$\Delta^r c_i = \Delta^{r-1} c_{i+1} - \Delta^{r-1} c_i,$$

where $\Delta^0 c_i = c_i$.

We prove this by induction on r. For the base case $r = 1$, we have that

$$\Delta c_0 = c_1 - c_0 = \binom{1}{0}(-1)^{1-0} c_0 + \binom{1}{1}(-1)^{1-1} c_1.$$

For the induction step, we assume that the formula holds for $r = k$, and show that it holds for $r = k + 1$. We have that

$$\begin{aligned}
&\Delta^{k+1} c_0 \\
&= \Delta^k c_1 - \Delta^k c_0 \\
&= \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} c_{i+1} - \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} c_i \\
&= \sum_{i=1}^{k+1} \binom{k}{i-1}(-1)^{k-i+1} c_i - \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} c_i \\
&= \binom{k}{k} c_{k+1} + \sum_{i=1}^{k} \left(\binom{k}{i-1}(-1)^{k-i+1} - \binom{k}{i}(-1)^{k-i} \right) c_i - \binom{k}{0}(-1)^k c_0 \\
&= \binom{k+1}{k+1} c_{k+1} + \sum_{i=1}^{k} (-1)^{(k+1)-i} \left(\binom{k}{i-1} + \binom{k}{i} \right) c_i + \binom{k+1}{0}(-1)^{k+1} c_0 \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i}(-1)^{k+1-i} c_i,
\end{aligned}$$

as we wanted to show.

Exercise 1.10 The Bernstein approximation to a function $f : [0, 1] \to \mathbb{R}$ of order d is the polynomial $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \sum_{i=0}^{d} f\left(\frac{i}{d}\right) B_i^d(x).$$

Show that if f is a polynomial of degree $m \leq d$, then g has degree m.

Solution 1.10 Let q be the polynomial defined by

$$q(x) = f(x) - g(x).$$

We have that f is a polynomial of degree $m \leq d$. As

$$q\left(\frac{i}{d}\right) = f\left(\frac{i}{d}\right) - g\left(\frac{i}{d}\right) = 0,$$

we have that q has $d+1$ roots, and thus q is either a polynomial of degree $d+1$, or $q = 0$. However, as q is the sum of two polynomials of degree m and d, respectively, we have that q is a polynomial of degree $\max(m, d)$. As $m \leq d$, we have that q is at most a polynomial of degree d, and thus $q = 0$. q being the zero polynomial implies that $g = f$, and thus g has degree m.

Exercise 1.11 Show that the length of the Bézier curve p in (1.9) is bounded by the length of its control polygon,

$$\text{length}(p) \leq \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\|.$$

Solution 1.11 The length of a curve is given by the integral of the norm of the derivative of the curve, i.e.,

$$\text{length}(p) = \int_0^1 \|\boldsymbol{p}'(x)\| \, dx.$$

We have that

$$\|\boldsymbol{p}'(x)\| = \|d \sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x)\| = d\|\sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x)\| \leq d \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\|,$$

and thus

$$\text{length}(p) = \int_0^1 \|\boldsymbol{p}'(x)\| \, dx \leq \int_0^1 d \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\| \, dx = \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\|.$$