MAT4170

Exercises for Spline Methods

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1 Bernstein-Bézier polynomials

Exercise 1.1 It is sometimes necessary to convert a polynomial in BB form to monomial form. Consider a quadratic BB polynomial,

$$p(x) = c_0(1-x)^2 + 2c_1 x(1-x) + c_2 x^2.$$

Express p in the monomial form

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

Solution 1.1 Rather than using the explicit formula for conversion, we can just expand the coefficients and collect terms.

$$\begin{aligned}
p(x) &= c_0(1-x)^2 + 2c_1 x(1-x) + c_2 x^2 \\
&= c_0(1 - 2x + x^2) + 2c_1(x - x^2) + c_2 x^2 \\
&= c_0 - 2c_0 x + c_0 x^2 + 2c_1 x - 2c_1 x^2 + c_2 x^2 \\
&= c_0 + (-2c_0 + 2c_1)x + (c_0 - 2c_1 + c_2)x^2.
\end{aligned}$$

Exercise 1.2 Consider a polynomial $p(x)$ of degree $\leq d$, for arbitrary d. Show that if

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x),$$

then
$$a_j = \binom{d}{j} \Delta^j c_0.$$

Hint: Use a Taylor approximation to p to show that $a_j = p^{(j)}(0)/j!$.

Solution 1.2 We have that
$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x).$$

By the Taylor approximation, we have that
$$p(x) = p(x+0) = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} x^j.$$

We thus have that
$$a_j = \frac{p^{(j)}(0)}{j!}.$$

By properties of the Bézier curves, we have that
$$p^{(j)}(x) = \frac{d!}{(d-j)!} \sum_{i=0}^{d-j} \Delta^j c_i B_i^{d-j}(x),$$

and specifically for $x = 0$,
$$p^{(j)}(0) = \frac{d!}{(d-j)!} \Delta^j c_0.$$

Combining these results, we have that
$$a_j = \frac{p^{(j)}(0)}{j!} = \frac{d!}{(d-j)!j!} \Delta^j c_0 = \binom{d}{j} \Delta^j c_0,$$

as we wanted to show.