MAT4170

Exercises for Spline Methods

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Contents

Abstract

This document contains my solutions to the exercises for the course MAT4170–Spline Methods, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MAT4170.

1 Bernstein-Bézier polynomials

Exercise 1.1 It is sometimes necessary to convert a polynomial in BB form to monomial form. Consider a quadratic BB polynomial,

$$p(x) = c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2.$$

Express p in the monomial form

$$p(x) = a_0 + a_1x + a_2x^2.$$

Solution 1.1 Rather than using the explicit formula for conversion, we can just expand the coefficients and collect terms.

$$\begin{aligned} p(x) &= c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2 \\ &= c_0(1-2x+x^2) + 2c_1(x-x^2) + c_2x^2 \\ &= c_0 - 2c_0x + c_0x^2 + 2c_1x - 2c_1x^2 + c_2x^2 \\ &= c_0 + (-2c_0 + 2c_1)x + (c_0 - 2c_1 + c_2)x^2. \end{aligned}$$

Exercise 1.2 Consider a polynomial $p(x)$ of degree $\leq d$, for arbitrary d. Show that if

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x),$$

then

$$a_j = \binom{d}{j} \Delta^j c_0.$$

Hint: Use a Taylor approximation to p to show that $a_j = p^{(j)}(0)/j!$.

Solution 1.2 We have that

$$p(x) = \sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x).$$

By the Taylor approximation, we have that

$$p(x) = p(x+0) = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} x^j.$$

We thus have that

$$a_j = \frac{p^{(j)}(0)}{j!}.$$

By properties of the Bézier curves, we have that

$$p^{(j)}(x) = \frac{d!}{(d-j)!} \sum_{i=0}^{d-j} \Delta^j c_i B_i^{d-j}(x),$$

and specifically for $x = 0$,

$$p^{(j)}(0) = \frac{d!}{(d-j)!} \Delta^j c_0.$$

Combining these results, we have that

$$a_j = \frac{p^{(j)}(0)}{j!} = \frac{d!}{(d-j)!j!} \Delta^j c_0 = \binom{d}{j} \Delta^j c_0,$$

as we wanted to show.

Exercise 1.3 We might also want to convert a polynomial from monomial form to BB form. Using Lemma 1.2, show that in the notation of the previous question,

$$c_i = \frac{i!}{d!} \sum_{j=0}^{i} \frac{(d-j)!}{(i-j)!} a_j.$$

Solution 1.3 Lemma 1.2 states that for $j = 0, 1, \ldots, d$,

$$x^j = \frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x).$$

We have that

$$\sum_{j=0}^{d} a_j x^j = \sum_{i=0}^{d} c_i B_i^d(x)$$

$$\sum_{j=0}^{d} a_j \left[\frac{(d-j)!}{d!} \sum_{i=j}^{d} \frac{i!}{(i-j)!} B_i^d(x) \right] = \sum_{i=0}^{d} c_i B_i^d(x)$$

As we have $i \geq j$, we can reorder the summation to the form $j \leq i$, by using

$$\sum_{j=0}^{d} \sum_{i=j}^{d} (\ldots) = \sum_{i=0}^{d} \sum_{j=0}^{i} (\ldots).$$

This gives us

$$\sum_{i=0}^{d}\left[\sum_{j=0}^{i} a_j \frac{(d-j)!}{d!}\frac{i!}{(i-j)!}\right] B_i^d(x) = \sum_{i=0}^{d} c_i B_i^d(x).$$

Which by isolating the coefficients, gives us

$$c_i = \frac{i!}{d!}\sum_{j=0}^{i}\frac{(d-j)!}{(i-j)!}a_j,$$

as we wanted to show.

Exercise 1.4 Implement the de Casteljau algorithm for cubic Bézier curves in Matlab or Python (or some other programming language), taking repeated convex combinations. Choose a sequence of four control points and plot both the control polygon and the Bézier curve, like those in Figure 1.3.

Solution 1.4 The de Casteljau algorithm uses recursion to compute the value of a point along a Bézier curve by the following formula:

1. Initialize by setting $c_i^0 = c_i$ for $i = 0, 1, \ldots, d$.
2. Then, for each $r = 1, 2, \ldots, d$, let
$$c_i^r = (1-x)c_i^{r-1} + xc_{i+1}^{r-1}, \quad i = 0, 1, \ldots, d-r.$$
3. The last value c_0^d is the value of the Bézier curve at x.

This is implemented using Jax in Python in `de_casteljau.py`, and the result is shown in Figure 1, bearing a striking resemblance to the figure in the book.

Exercise 1.5 Show that the graph, $g(x) = (x, p(x))$ of the BB polynomial p in (1.6) is a Bézier curve in $\mathbb{R}^2$, with control points (ξ_i, c_i), $i = 0, 1, \ldots, d$, where $\xi_i = i/d$. *Hint:* Express x as a linear combination of $B_0^d(x), \ldots, B_d^d(x)$.

Solution 1.5 We can again utilize Lemma 1.2 to express x as a linear combination of the Bernstein polynomials. We have that, writing $x = x^1$ for clarity,

$$x^1 = \frac{(d-1)!}{d!}\sum_{i=1}^{d}\frac{i!}{(i-1)!}B_i^d(x) = \sum_{i=1}^{d}\frac{i}{d}B_i^d(x) = \sum_{i=0}^{d}\frac{i}{d}B_i^d(x) = \sum_{i=0}^{d}\xi_i B_i^d(x).$$

We can now express the graph of p as a Bézier curve in $\mathbb{R}^2$ by

$$g(x) = (x, p(x)) = \left(\sum_{i=0}^{d}\xi_i B_i^d(x), \sum_{i=0}^{d} c_i B_i^d(x)\right) = \sum_{i=0}^{d}(\xi_i, c_i)B_i^d(x) = \sum_{i=0}^{d}\boldsymbol{c}_i B_i^d(x),$$

where $\boldsymbol{c}_i = (\xi_i, c_i)$ are the control points of the Bézier curve.

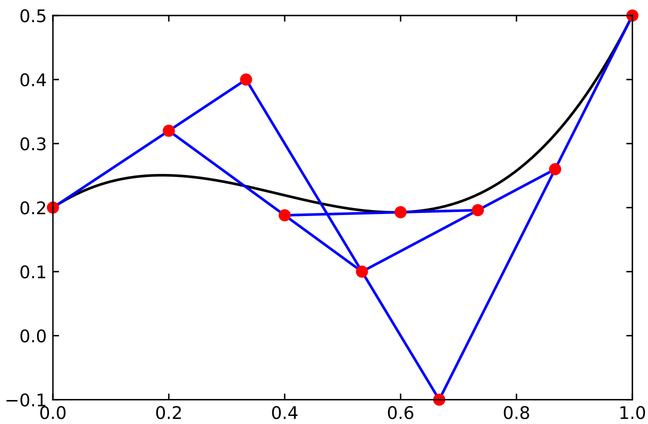

Figure 1: The de Casteljau algorithm applied to a cubic Bézier curve, with control points $(0.2, 0.4, -0.1, 0.5)$, illustrated at the point $x = 0.6$.

Exercise 1.6 Show that the tangent vector $\boldsymbol{p}'(x)$ of the Bézier curve in (1.6) lies in the convex cone of the vectors $\Delta \boldsymbol{c}_i$, i.e., in

$$\operatorname{cone}(\Delta \boldsymbol{c}_0, \ldots, \Delta \boldsymbol{c}_{d-1}) = \left\{ \sum_{i=0}^{d-1} \mu_i \Delta \boldsymbol{c}_i : \mu_1, \ldots, \mu_{d-1} \geq 0 \right\}.$$

Solution 1.6 The derivative (or perhaps *gradient* is the correct term) of the Bézier curve $\boldsymbol{p}(x)$ is given by

$$\boldsymbol{p}'(x) = d \sum_{i=0}^{d-1} (\boldsymbol{c}_{i+1} - \boldsymbol{c}_i) B_i^{d-1}(x) = d \sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x).$$

As $B_i^{d-1}(x) \geq 0$ for $x \in [0, 1]$, we can set $\mu_i = d B_i^{d-1}(x)$, and we have that

$$\boldsymbol{p}'(x) = \sum_{i=0}^{d-1} \mu_i \Delta \boldsymbol{c}_i \in \operatorname{cone}(\Delta \boldsymbol{c}_0, \ldots, \Delta \boldsymbol{c}_{d-1}),$$

as we wanted to show.

Exercise 1.7 Show that the first derivative of p in (1.6) can be expressed (and computed) as

$$p'(x) = d(c_1^{d-1} - c_0^{d-1}),$$

where c_1^{d-1}, c_0^{d-1} are the points of order $d - 1$ in de Casteljau's algorithm (1.10).

Solution 1.7 We have that

$$p(x) = c_0^d = (1 - x) c_0^{d-1} + x c_1^{d-1},$$

and thus by differentiating with respect to x, we have that

$$p'(x) = c_1^{d-1} - c_0^{d-1}.$$

This tells us that we cannot be as naive as this, as c_0^d is actually a function of x, and not simply a constant.

What we might instead need to note is that

$$c_i^r = \sum_{j=0}^{r} c_{i+j} B_j^r(x),$$

and combining this with the fact that

$$\left(B_i^d\right)'(x) = d\left(B_{i-1}^{d-1} - B_i^{d-1}\right)(x),$$

we have that

$$\begin{aligned} p'(x) &= d\sum_{i=0}^{d-1}\left(c_{i+1} - c_i\right)B_i^{d-1}(x) = d\left[\sum_{i=0}^{d-1} c_{i+1}B_i^{d-1}(x) - \sum_{i=0}^{d-1} c_i B_i^{d-1}(x)\right] \\ &= d(c_1^{d-1} - c_0^{d-1}), \end{aligned}$$

as we wanted to show.

Exercise 1.8 Show that the Bernstein basis polynomial $B_i^d(x)$ has only one maximum in $[0, 1]$, namely at $x = i/d$.

Solution 1.8 We do this by firstly computing the derivative of $B_i^d(x)$, which is given by

$$\left(B_i^d\right)'(x) = d\left(B_{i-1}^{d-1}(x) - B_i^{d-1}(x)\right).$$

A maximum or minimum of a function occurs where the derivative is zero, so we set

$$\begin{aligned} B_{i-1}^{d-1}(x) &= B_i^{d-1}(x) \\ \frac{(d-1)!}{(i-1)!(d-i)!}x^{i-1}(1-x)^{d-i} &= \frac{(d-1)!}{i!(d-1-i)!}x^i(1-x)^{d-i-1} \\ \frac{\cancel{(d-1)!}i!(d-1-i)!}{(i-1)!(d-i)!\cancel{(d-1)!}}\cancel{x^{i-1}}(1-x)^{\cancel{d-i}} &= x^{\cancel{i}}\cancel{(1-x)^{d-i-1}} \\ \frac{i}{d-i}(1-x) &= x \\ i - ix &= dx - ix \\ x &= \frac{i}{d}. \end{aligned}$$

We have thus shown that the Bernstein basis polynomials only have one extremal point.

We can use the second derivative to test if this is a maximum or a minimum, however we can instead note that $B_i^d(x)$ is a non-negative polynomial, which is only zero at either $x = 0$ or $x = 1$, and thus $x = i/d$ must be a maximum.

Exercise 1.9 Give a proof of the forward difference formula, (1.15).

Solution 1.9 The forward difference formula (1.15) is given by

$$\Delta^r c_0 = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} c_i.$$

The forward difference operator is defined by the recursion

$$\Delta^r c_i = \Delta^{r-1} c_{i+1} - \Delta^{r-1} c_i,$$

where $\Delta^0 c_i = c_i$.

We prove this by induction on r. For the base case $r = 1$, we have that

$$\Delta c_0 = c_1 - c_0 = \binom{1}{0}(-1)^{1-0} c_0 + \binom{1}{1}(-1)^{1-1} c_1.$$

For the induction step, we assume that the formula holds for $r = k$, and show that it holds for $r = k + 1$. We have that

$$\begin{aligned}
&\Delta^{k+1} c_0 \\
&= \Delta^k c_1 - \Delta^k c_0 \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} c_{i+1} - \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} c_i \\
&= \sum_{i=1}^{k+1} \binom{k}{i-1} (-1)^{k-i+1} c_i - \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} c_i \\
&= \binom{k}{k} c_{k+1} + \sum_{i=1}^{k} \left(\binom{k}{i-1} (-1)^{k-i+1} - \binom{k}{i} (-1)^{k-i} \right) c_i - \binom{k}{0} (-1)^k c_0 \\
&= \binom{k+1}{k+1} c_{k+1} + \sum_{i=1}^{k} (-1)^{(k+1)-i} \left(\binom{k}{i-1} + \binom{k}{i} \right) c_i + \binom{k+1}{0} (-1)^{k+1} c_0 \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^{k+1-i} c_i,
\end{aligned}$$

as we wanted to show.

Exercise 1.10 The Bernstein approximation to a function $f : [0, 1] \to \mathbb{R}$ of order d is the polynomial $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \sum_{i=0}^{d} f\left(\frac{i}{d}\right) B_i^d(x).$$

Show that if f is a polynomial of degree $m \le d$, then g has degree m.

Solution 1.10 Let q be the polynomial defined by

$$q(x) = f(x) - g(x).$$

We have that f is a polynomial of degree $m \le d$. As

$$q\left(\frac{i}{d}\right) = f\left(\frac{i}{d}\right) - g\left(\frac{i}{d}\right) = 0,$$

we have that q has $d+1$ roots, and thus q is either a polynomial of degree $d+1$, or $q = 0$. However, as q is the sum of two polynomials of degree m and d, respectively, we have that q is a polynomial of degree $\max(m, d)$. As $m \le d$, we have that q is at most a polynomial of degree d, and thus $q = 0$. q being the zero polynomial implies that $g = f$, and thus g has degree m.

Exercise 1.11 Show that the length of the Bézier curve p in (1.9) is bounded by the length of its control polygon,

$$\text{length}(p) \le \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\|.$$

Solution 1.11 The length of a curve is given by the integral of the norm of the derivative of the curve, i.e.,

$$\text{length}(p) = \int_0^1 \|\boldsymbol{p}'(x)\| \, dx.$$

We have that

$$\|\boldsymbol{p}'(x)\| = \|d \sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x)\| = d \|\sum_{i=0}^{d-1} \Delta \boldsymbol{c}_i B_i^{d-1}(x)\| \le d \sum_{i=0}^{d-1} \|\Delta \boldsymbol{c}_i\| B_i^{d-1}(x),$$

where we in the last inequality use the fact that $B_i^{d-1}(x)$ is a non-negative scalar for $x \in [0,1]$. We then have

$$
\begin{aligned}
\operatorname{length}(p) &= \int_0^1 \|\boldsymbol{p}'(x)\|\,dx \\
&\le \int_0^1 d\sum_{i=0}^{d-1}\|\Delta\boldsymbol{c}_i\| B_i^{d-1}(x)\,dx \\
&= \sum_{i=0}^{d-1}\|\Delta\boldsymbol{c}_i\| \int_0^1 dB_i^{d-1}(x)\,dx \\
&= \sum_{i=0}^{d-1}\|\Delta\boldsymbol{c}_i\|,
\end{aligned}
$$

using the property $\int_0^1 B_i^{d-1}(x)\,dx = 1/d$ in the last step, as we wanted to show.

2 Splines in Bernstein-Bézier Form

Exercise 2.1 Prove equation (2.8).

Solution 2.1 Equation (2.8) states that the condition for C^2 continuity of s can be expressed as the two equations in (2.7), namely

$$e_0 = c_d \quad \text{and} \quad e_1 = (1-\alpha)c_d + \alpha c_{d-1}$$

where

$$\alpha = \frac{c-a}{b-a},$$

plus the equation

$$e_2 = (1-\alpha)^2 c_d + 2(1-\alpha)\alpha c_{d-1} + \alpha^2 c_{d-2}.$$

In the original theorem, we have that a spline s has C^r continuity, for $r = 0, 1, \ldots, d$, if and only if

$$\frac{1}{(b-a)^k}\Delta^k c_{d-k} = \frac{1}{(c-b)^k}\Delta^k e_0, \qquad k = 0, 1, \ldots, r.$$

For $r = 2$, we must then solve for $k = 0, 1, 2$. In the first case we simply have

$$\frac{1}{(b-a)^0}\Delta^0 c_d = \frac{1}{(c-b)^0}\Delta^0 e_0,$$
$$c_d = e_0.$$

For $k = 1$ we have then have

$$\begin{aligned}
\frac{1}{(b-a)^1}\Delta^1 c_{d-1} &= \frac{1}{(c-b)^1}\Delta^1 e_0,\\
\frac{c_d - c_{d-1}}{b-a} &= \frac{e_1 - e_0}{c-b} = \frac{e_1 - c_d}{c-b},\\
e_1 - c_d &= \frac{c-b}{b-a}(c_{d-1} - c_d),\\
e_1 &= (1-\alpha)c_d + \alpha c_{d-1}.
\end{aligned}$$

Finally, for $k = 2$ we have

$$\begin{aligned}
\frac{1}{(c-b)^2}\Delta^2 e_0, &= \frac{1}{(b-a)^2}\Delta^2 c_{d-2}\\
\frac{e_2 - 2e_1 + e_0}{(c-b)^2} &= \frac{c_d - 2c_{d-1} + c_{d-2}}{(b-a)^2},\\
e_2 - 2\left((1-\alpha)c_d + \alpha c_{d-1}\right) + c_d &= \alpha^2\left(c_{d-2} - 2c_{d-1} + c_d\right)\\
e_2 &= \left(\alpha^2 - 2\alpha + 1\right)c_d + 2\left(\alpha - \alpha^2\right)c_{d-1} + \alpha^2 c_{d-2}\\
e_2 &= (1-\alpha)^2 c_d + 2(1-\alpha)\alpha c_{d-1} + \alpha^2 c_{d-2},
\end{aligned}$$

which is the desired result.

Exercise 2.2 Implement the de Casteljau algorithm for a planar Bézier curve of arbitrary degree d over a general interval $[a, b]$. Use the routine to make a program to plot the quadratic spline curve $\mathbf{s} : [0, 2] \to \mathbb{R}$, with pieces

$$\mathbf{p}(t) = \sum_{i=0}^{2} \mathbf{c}_i B_i^2(t), \qquad 0 \le t \le 1,$$

$$\mathbf{q}(t) = \sum_{i=0}^{2} \mathbf{d}_i B_i^2(t-1), \qquad 1 < t \le 2,$$

where $\mathbf{c}_0 = (-1, 1)$, $\mathbf{c}_1 = (-1, 0)$, $\mathbf{c}_2 = (0, 0)$, and $\mathbf{d}_0 = (0, 0)$, $\mathbf{d}_1 = (1, 0)$, $\mathbf{d}_2 = (2, 1)$.

Solution 2.2 The de Casteljau algorithm is implemented in `spline_cj.py`, and is practically identical to the one implemented in the previous section, due to the vectorization possibilities in `JAX`. The resulting figure is shown in Figure 2.

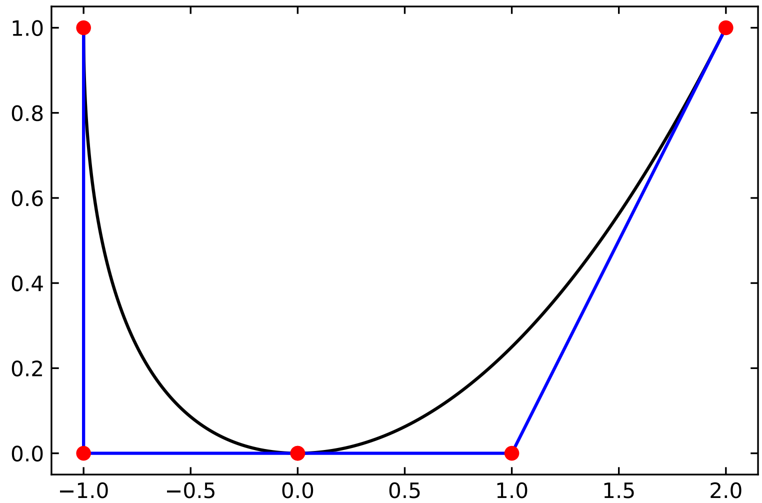

Figure 2: The quadratic spline curve $\mathbf{s} : [0, 2] \to \mathbb{R}$, with pieces $\mathbf{p}(t)$ and $\mathbf{q}(t)$.

Exercise 2.3 What is the order of continuity of $\mathbf{s}$ in Exercise 2.2 at the breakpoint $t = 1$?

Solution 2.3 We clearly have C^0 continuity at the breakpoint $t = 1$, as

$$\mathbf{c}_2 = (0, 0) = \mathbf{d}_0.$$

For C^1 continuity, we must have

$$\frac{\mathbf{c}_2 - \mathbf{c}_1}{1-0} = \frac{\mathbf{d}_1 - \mathbf{d}_0}{2-1}$$
$$(0,0) - (-1,0) = (1,0) - (0,0),$$

which also holds. Finally for C^2 continuity, we must have

$$\frac{\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0}{1^2} \stackrel{?}{=} \frac{\mathbf{d}_2 - 2\mathbf{d}_1 + \mathbf{d}_0}{1^2}$$
$$(0,0) - 2(-1,0) + (-1,1) = (1,1) \neq (0,1) = (2,1) - 2(1,0) + (0,0),$$

which it seems like we do not have. Thus, the order of continuity of $\mathbf{s}$ at the breakpoint $t = 1$ is C^1.

Exercise 2.4 The curvature of a parametric curve $\mathbf{r}(t)$ in $\mathbb{R}^2$ can be expressed as

$$\kappa(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3},$$

where $(a_1, a_2) \times (b_1, b_2) := a_1 b_2 - a_2 b_1$. What are the curvatures of $\mathbf{p}$ and $\mathbf{q}$ in Exercise 2.2 at the breakpoint $t = 1$? What can you say about the smoothness of $\mathbf{s}$?

Solution 2.4 We firstly begin by tabulating the derivatives of $\mathbf{p}$ and $\mathbf{q}$, based on the general formula

$$p^{(r)}(x) = \frac{d!}{(d-r)!} \frac{1}{h^r} \sum_{i=0}^{d-r} \Delta^r c_i B_i^{d-r}(\lambda).$$

In both cases here, we have $h = 1$, $d = 2$, and $\lambda = t$ for $\mathbf{p}$ and $\lambda = t - 1$ for $\mathbf{q}$. The derivatives for $\mathbf{p}$ are then

$$\begin{aligned} \mathbf{p}'(t) &= 2\left((\mathbf{c}_1 - \mathbf{c}_0)\, B_0^1(\lambda) + (\mathbf{c}_2 - \mathbf{c}_1)\, B_1^1(\lambda)\right) & \mathbf{p}''(t) &= 2\,(\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0)\, B_0^0(\lambda) \\ &= 2\left((\mathbf{c}_1 - \mathbf{c}_0)(1-t) + (\mathbf{c}_2 - \mathbf{c}_1)\, t\right) & &= 2\,(1,1) \\ &= 2\left((\mathbf{c}_2 - 2\mathbf{c}_1 + \mathbf{c}_0)t + (\mathbf{c}_1 - \mathbf{c}_0)\right) & &= (2,2), \\ &= 2\left((1,1)t + (0,-1)\right) & & \\ &= (2t, 2t-2), & & \end{aligned}$$

while we for $\mathbf{q}$ have

$$\begin{aligned} \mathbf{q}'(t) &= 2\left((\mathbf{d}_1 - \mathbf{d}_0)\, B_0^1(\lambda) + (\mathbf{d}_2 - \mathbf{d}_1)\, B_1^1(\lambda)\right) & \mathbf{q}''(t) &= 2\,(\mathbf{d}_2 - 2\mathbf{d}_1 + \mathbf{d}_0)\, B_0^0(\lambda) \\ &= 2\left((\mathbf{d}_1 - \mathbf{d}_0)\, t + (\mathbf{d}_2 - \mathbf{d}_1)(1-t)\right) & &= 2\,(0,1) \\ &= 2\left((-\mathbf{d}_2 + 2\mathbf{d}_1 - \mathbf{d}_0)t + (\mathbf{d}_2 - \mathbf{d}_1)\right) & &= (0,2), \\ &= 2\left((0,-1)t + (1,1)\right) & & \\ &= (2, -2t+2), & & \end{aligned}$$

where we have used that $B_0^1(1-t) = t$ and $B_1^1(1-t) = 1-t$. The curvatures are then

$$\kappa_{\mathbf{p}}(t) = \frac{(2t, 2t-2) \times (2,2)}{\sqrt{(2t)^2 + (2t-2)^2}^3} = \frac{4}{(8t^2 - 8t + 4)^{3/2}},$$
$$\kappa_{\mathbf{q}}(t) = \frac{(2, -2t+2) \times (0,2)}{\sqrt{2^2 + (-2t+2)^2}^3} = \frac{4}{(4t^2 - 8t + 8)^{3/2}}.$$

These give, at the breakpoint $t = 1$, the curvatures

$$\kappa_{\mathbf{p}}(1) = \frac{4}{2^3} = \frac{1}{2} \quad \text{and} \quad \kappa_{\mathbf{q}}(1) = \frac{4}{2^3} = \frac{1}{2}.$$

As we see, the curvatures are equal at the breakpoint $t = 1$. I'm not sure what this means...

Exercise 2.5 Show that the minimization property of Theorem 2.9 also holds for the natural spline of Section 2.7.

Solution 2.5 The natural spline of Section 2.7 is defined as the spline g defined for $x_1 < x_2 < \cdots < x_m$ by the piecewise cubic polynomials g_i such that $g(x) = g_i(x)$ for $x \in [x_i, x_{i+1}]$. In addition, we have the conditions

$$\begin{aligned} g_i(x_i) &= y_i & \qquad & i = 1, 2, \ldots, m, \\ g''_{i-1}(x_i) &= g''_i(x_i) & & i = 2, \ldots, m-1. \end{aligned}$$

combined with the natural end conditions

$$g''_1(x_1) = 0 \quad \text{and} \quad g''_m(x_m) = 0.$$

To prove the minimization property of Theorem 2.9, let h be any C^2 function satisfying the interpolation conditions. Let $e = g - h$. Then $e \in C^2[x_1, x_m]$ and

$$e(x_i) = g(x_i) - h(x_i) = 0 \quad \text{and} \quad e''(x_1) = e''(x_m) = 0$$

for $i = 1, 2, \ldots, m$. Continuing as in the proof of Theorem 2.9, we have $h = g + e$ so

$$\begin{aligned} \int (h'')^2 &= \int (g'' + e'')^2 \\ &= \int (g'')^2 + 2\int g''e'' + \int (e'')^2 \\ &\geq \int (g'')^2 + 2\int g''e''. \end{aligned}$$

Let $\phi = g''$, which then is piecewise linear. We then rewrite the equation as

$$\int (h'')^2 - \int (g'')^2 \geq 2 \int \phi e'',$$

where the goal is now to show that the right-hand side is nonnegative.

We have

$$\int \phi e'' = \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} \phi e'' = \sum_{i=1}^{m-1} [\phi e']_{x_i}^{x_{i+1}} - \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} \phi' e'$$

where we have used integration by parts. By the natural end conditions, the first term vanishes, as

$$\sum_{i=1}^{m-1} [\phi e']_{x_i}^{x_{i+1}} = \phi(x_m) e'(x_m) - \phi(x_1) e'(x_1) = 0.$$

The second term also vanishes, because as ϕ is piecewise linear, ϕ' is piecewise constant. This allows us to write

$$\int_{x_i}^{x_{i+1}} \phi' e' = \phi' \Big|_{[x_i, x_{i+1}]} \int_{x_i}^{x_{i+1}} e' = \phi' \Big|_{[x_i, x_{i+1}]} \Big(e(x_{i+1}) - e(x_i) \Big) = 0,$$

which vanishes as $e(x_i) = 0$ for all i. Thus, we have shown that the right-hand side is nonnegative, and the minimization property of Theorem 2.9 also holds for the natural spline of Section 2.7.