MEK4250

Exercises for Finite Elements in Computational Mechanics

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Contents

Abstract

This document contains my solutions to the exercises for the course MEK4250–Finite Elements in Computational Mechanics, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MEK4250.

1 A glimpse at the finite element method

Exercise 1.1 Consider the problem $-u''(x) = x^2$ on the unit interval with $u(0) = u(1) = 0$. Let $u = \sum_{k=1}^{N} u_k \sin(\pi k x)$ and $v = \sin(\pi l x)$ for $l = 1, \ldots, N$, for e.g. $N = 10, 20, 40$ and solve (1.9). What is the error in L_2 and L_∞.

Solution 1.1 In this exercise, we use the Galerkin method to solve the problem, wishing to solve the problem as $Au = b$, where

$$A_{ij} = \int_\Omega k \nabla N_j \cdot \nabla N_i \, dx,$$

$$b_i = \int_\Omega f N_i \, dx + \int_{\partial \Omega_N} h N_i \, ds.$$

We begin by noting that

$$\nabla N_i = \frac{d}{dx} \sin(\pi i x) = \pi i \cos(\pi i x),$$

such that

$$\int_\Omega k \nabla N_j \cdot \nabla N_i \, dx = \int_0^1 k \pi^2 i j \cos(\pi j x) \cos(\pi i x) \, dx = \frac{\pi^2 i^2}{2} \delta_{ij}.$$

As we are given that the Dirichlet boundary conditions cover the entire boundary, and $\partial \Omega_D \cap \partial \Omega_N = \emptyset$, we have that the Neumann boundary integral is zero. The b vector is then given by

$$b_i = \int_\Omega f N_i \, dx = \int_0^1 x^2 \sin(\pi i x) \, dx = \frac{(2 - \pi^2 i^2)(-1)^i - 2}{\pi^3 i^3}.$$

Setting up and solving the system for varying N is then rather simples, implemented in `1_glimpse/ex1.py`. This gives the errors presented in Table 1, with the plotted solution in Figure 1a.

Table 1: Errors of approximations of u for varying N, with sine basis functions.

N	L_2	L_∞
10	0.001791	0.000224
20	0.000338	0.000059
40	0.000062	0.000015

(a) Approximated function.
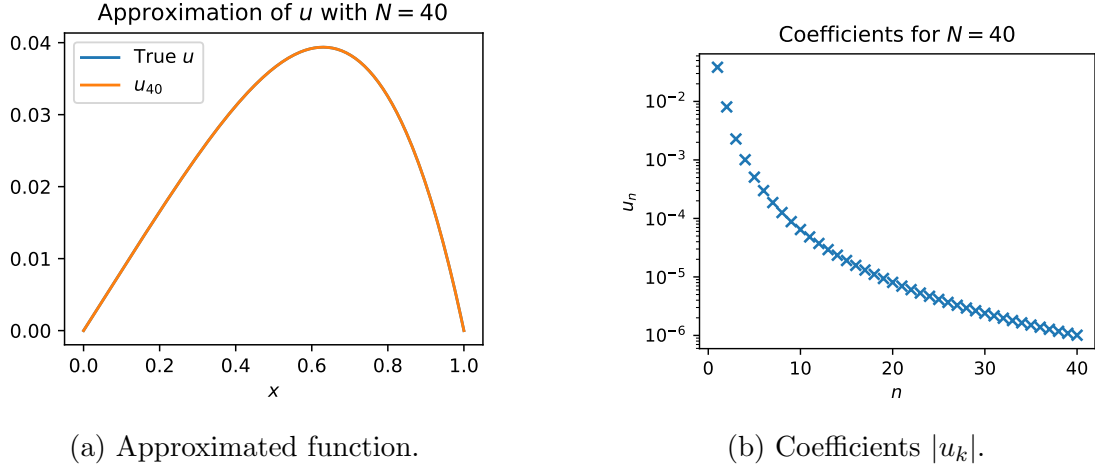
(b) Coefficients $|u_k|$.

Figure 1: Approximation of u with $N = 40$ sine basis functions.

Exercise 1.2 Consider the same problem as in the previous exercise, but using the Bernstein polynomials. That is, the basis for the Bernstein polynomial of order N on the unit inteval is $B_k^N(x) = x^k(1-x)^{N-k}$ for $k = 0, \ldots, N$. Let $u = \sum_{k=0}^{N} u_k B_k^N(x)$ and $v = B_l^N(x)$ for $l = 0, \ldots, N$ and solve (1.9). What is the error in L_2 and L_∞ in terms of N for $N = 1, 2, \ldots, 10$. Remark: Do the basis functions satisfy the boundary condtions? Should some of them be removed?

Solution 1.2 The Bernstein polynomials B_0^N and B_N^N both need to be removed, as they do not satisfy the Dirichlet boundary conditions, as $B_0^N(0) = 1 = B_N^N(1)$. We therefore need atleast $N = 3$, in order to get an atleast somewhat viable solution.

The Bernstein basis polynomials are defined as

$$B_k^N(x) = \binom{N}{k} x^k (1-x)^{N-k}.$$

Some useful properties which might come in handy are

1. The derivative of a basis polynomials is

$$\left(B_k^N(x)\right)' = N\left(B_{k-1}^{N-1}(x) - B_k^{n-1}(x)\right),$$

where we follow the convention of setting $B_{-1}^N(x) = 0 = B_{N+1}^N(x)$.

2. The definite integral on the unit line is given by

$$\int_0^1 B_k^N(x) = \frac{1}{N+1} \quad \text{for } k = 0, 1, \ldots, N.$$

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3. The multiple of two Bernstein polynomials is

$$
\begin{aligned}
B_k^N(x) \cdot B_q^M(x) &= \binom{N}{k} x^k (1-x)^{N-k} \binom{M}{q} x^q (1-x)^{M-q} \\
&= \binom{N}{k}\binom{M}{q} x^{k+q} (1-x)^{N+M-k-q} \\
&= \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} B_{k+q}^{N+M}(x)
\end{aligned}
$$

From these, we can gather that

$$
\int_0^1 B_k^N(x) B_q^M(x) \ dx = \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} \int_0^1 B_{k+q}^{N+M}(x) \ dx = \frac{\binom{N}{k}\binom{M}{q}}{(N+M+1)\binom{N+M}{k+q}}.
$$

The terms in the stiffness matrix are then given by

$$
\begin{aligned}
A_{ij} &= \int_0^1 \nabla B_i^N(x) \cdot \nabla B_j^N(x) \ dx \\
&= N^2 \int_0^1 \left(B_{i-1}^{N-1} - B_i^{N-1}\right)\left(B_{j-1}^{N-1} - B_j^{N-1}\right) \ dx \\
&= N^2 \int_0^1 B_{i-1}^{N-1} B_{j-1}^{N-1} - B_i^{N-1} B_{j-1}^{N-1} - B_{i-1}^{N-1} B_j^{N-1} + B_i^{N-1} B_j^{N-1} \ dx \\
&= N^2 \int_0^1 \alpha_{i-1,j-1} B_{i+j-2}^{2N-2} - (\alpha_{i,j-1} + \alpha_{i-1,j}) B_{i+j-1}^{2N-2} + \alpha_{i,j} B_{i+j}^{2N-2} \ dx \\
&= \frac{N^2}{2N-1}\left(\frac{\binom{N-1}{i-1}\binom{N-1}{j-1}}{\binom{2N-2}{i+j-2}} - \frac{\binom{N-1}{i}\binom{N-1}{j-1} + \binom{N-1}{i-1}\binom{N-1}{j}}{\binom{2N-2}{i+j-1}} + \frac{\binom{N-1}{i}\binom{N-1}{j}}{\binom{2N-2}{i+j}}\right)
\end{aligned}
$$

This can likely be written much nicer, however I cannot be bothered to do that right now.

Opting to take the easy way out instead, and utilizing `sympy` to solve the integrals, we can implement the solution in `1_glimpse/ex2.py`. The errors are presented in Table 2. As we can read from the table, the polynomial approximation is exact for $N > 3$, which is expected as the Bernstein polynomials are exact for polynomials of degree N.

Exercise 1.3 Consider the same problem as in the previous exercise, but with $-u''(x) = \sin(k\pi x)$ for $k = 1$ and $k = 10$.

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Table 2: Errors of approximations of u for varying N, with Bernstein basis functions.

N	L_2
2	$\frac{\sqrt{1330}}{6300}$
3	$\frac{\sqrt{70}}{12600}$
4–10	0

Solution 1.3 The approach for this is approximately the same, however we need to figure out the true solution in order to calculate the error.

$$u''(x) = -\sin(k\pi x)$$
$$u'(x) = \frac{1}{k\pi}\cos(k\pi x) + C_1$$
$$u(x) = \frac{1}{k^2\pi^2}\sin(k\pi x) + C_1 x + C_2.$$

As we have Dirichlet boundary conditions, we then set $C_1 = 0 = C_2$.

Exercise 1.4 Consider the same problem as in the previous exercise, but with the finite element method in for example FEniCS, FEniCSx or Firedrake, using Lagrange method of order 1, 2 and 3.

Solution 1.4 For this exercise, I will be using FEniCSx to solve the problem. The code is implemented in `doc/1_glimpse/ex4.py`, with the resulting approximations in Figure 2.

(a) $N = 5$.

(b) $N = 10$.

(c) $N = 20$.

(d) $N = 40$.

Figure 2: Approximation of u with varying N elements.

2 Crash course in Sobolev Spaces

Exercise 2.1 What is a norm? Show that

$$\|u\|_p = \left(\int_0^1 |u|^p \, dx \right)^{1/p}$$

defines a norm.

Solution 2.1 Following the definition in Spaces by Tom Lindstrøm, a norm is a function $\|\cdot\| : V \to \mathbb{R}$, where V is a vector space, such that

(i) $\|\boldsymbol{u}\| \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

(ii) $\|\alpha \boldsymbol{u}\| = |\alpha| \|\boldsymbol{u}\|$ for all $\alpha \in \mathbb{R}$ and all $\boldsymbol{u} \in V$.

(iii) (Triangle Inequality for Norms) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

Positivity is clear, as $|u|^p \geq 0$ for all $u \in L^p(0,1)$. The only way $\|u\|_p = 0$ is if $|u|^p = 0$. Homogeneity is also clear, as

$$\begin{aligned}
\|\alpha u\|_p &= \left(\int_0^1 |\alpha u|^p \, dx \right)^{1/p} \\
&= \left(\int_0^1 |\alpha|^p |u|^p \, dx \right)^{1/p} \\
&= |\alpha| \left(\int_0^1 |u|^p \, dx \right)^{1/p} \\
&= |\alpha| \|u\|_p.
\end{aligned}$$

The triangle inequality is a bit more involved, and not yet solved. Likely requires something like Minkowski's inequality.

Exercise 2.2 What is an inner product? Show that

$$(u, v)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x} \right)^i \left(\frac{\partial v}{\partial x} \right)^i \, dx$$

defines an inner product.

Solution 2.2 Again, Spaces by Tom Lindstrøm defines an inner product as a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$, where V is a vector space, such that

(i) $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

(ii) $(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{w}) + (\boldsymbol{v}, \boldsymbol{w})$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$.

(iii) $(\alpha \boldsymbol{u}, \boldsymbol{v}) = \alpha(\boldsymbol{u}, \boldsymbol{v})$ for all $\alpha \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in V$.

(iv) For all $\boldsymbol{u} \in V$, $(\boldsymbol{u}, \boldsymbol{u}) \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

Symmetry is clear, as

$$(u, v)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i dx = \sum_{i \leq k} \int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i dx = (v, u)_k.$$

Linearity in the first argument is also satisfied, as

$$
\begin{aligned}
(u + v, w)_k &= \sum_{i \leq k} \int_\Omega \left(\frac{\partial (u + v)}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i + \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i dx + \sum_{i \leq k} \int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i dx \\
&= (u, w)_k + (v, w)_k.
\end{aligned}
$$

Homogeneity in the first argument is also satisfied, as

$$
\begin{aligned}
(\alpha u, v)_k &= \sum_{i \leq k} \int_\Omega \left(\frac{\partial (\alpha u)}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i dx \\
&= \sum_{i \leq k} \int_\Omega \alpha \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i dx \\
&= \alpha \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i dx \\
&= \alpha(u, v)_k.
\end{aligned}
$$

Finally, positivity is also satisfied, as

$$(u, u)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i dx = \sum_{i \leq k} \int_\Omega \left(\frac{\partial^i u}{\partial x^i}\right)^2 dx \geq 0.$$

$(u, u)_k = 0$ only if $\frac{\partial^i u}{\partial x^i} = 0$ for all $i \leq k$, which implies $u = 0$. $(u, v)_k$ is therefore an inner product.

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Exercise 2.3 Compute the H^1 and L^2 norms of a random function with values in $(0, 1)$ on meshes representing the unit interval with 10, 100, and 1000 cells.

Solution 2.3 As a random function, I choose the Bernstein polynomial B_5^{10}, which is given by

$$B_5^{10}(x) = \binom{10}{5} x^5 (1-x)^5.$$

Exercise 2.4 Compute the H^1 and L^2 norms of the function $u(x) = \sin(k\pi x)$ on the unit interval analytically and compare with the values presented in Table 2.2.

Solution 2.4 The L^2 norm of $u(x) = \sin(k\pi x)$ is given by

$$
\begin{aligned}
\|u\|_2 &= \left(\int_0^1 \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left(\frac{1}{2} \int_0^1 1 - \cos(2k\pi x) \, dx \right)^{1/2} \\
&= \left(\frac{1}{2} \left[x - \frac{1}{2k\pi} \sin(2k\pi x) \right]_0^1 \right)^{1/2} \\
&= \left(\frac{1}{2} (1 - 0) \right)^{1/2} = \frac{\sqrt{2}}{2}.
\end{aligned}
$$

The H^1 norm is given by

$$
\begin{aligned}
\|u\|_1 &= \left(\int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 + u^2 \, dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi \cos(k\pi x))^2 + \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \cos^2(k\pi x) + \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \left(1 - \sin^2(k\pi x) \right) + \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 + (1 - (k\pi)^2) \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \int_0^1 \sin^2(k\pi x) \, dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \frac{1}{2} \right)^{1/2} = \sqrt{\frac{1 + (k\pi)^2}{2}}.
\end{aligned}
$$

The H^1 norm should then increase as k increases, while the L^2 norm should remain constant, and we do indeed see this behaviour in Table 2.2.

Exercise 2.5 Compute the H^1 and L^2 norms of the hat function in Picture 2.2.

Solution 2.5 The hat function in Picture 2.2 is given by

$$u(x) = \begin{cases} \frac{x+0.2}{0.2}, & x \in [-0.2, 0], \\ \frac{0.2-x}{0.2}, & x \in [0, 0.2], \\ 0, & \text{otherwise.} \end{cases}$$

The L^2 norm is given by

$$\|u\|_2 = \left(\int_0^1 u^2 \, dx \right)^{1/2}$$

$$= \left(\int_{-0.2}^0 \left(\frac{x+0.2}{0.2} \right)^2 dx + \int_0^{0.2} \left(\frac{0.2-x}{0.2} \right)^2 dx \right)^{1/2}$$

$$= \left(\int_{-0.2}^0 \left(\frac{x^2 + 0.4x + 0.04}{0.04} \right) dx + \int_0^{0.2} \left(\frac{0.04 - 0.4x + x^2}{0.04} \right) dx \right)^{1/2}$$

$$= \left(\frac{1}{0.02} \int_0^{0.2} x^2 - 0.4x + 0.04 \, dx \right)^{1/2}$$

$$= \left(\frac{1}{0.02} \left[\frac{1}{3}x^3 - 0.2x^2 + 0.04x \right]_0^{0.2} \right)^{1/2}$$

$$= \left(\frac{1}{0.02} \left(\frac{1}{3} \cdot 0.008 - 0.2 \cdot 0.04 + 0.04 \cdot 0.2 \right) \right)^{1/2}$$

$$= \sqrt{\frac{2}{15}}$$

The derivative of u is given by

$$\frac{\partial u}{\partial x} = \begin{cases} 5, & x \in [-0.2, 0], \\ -5, & x \in [0, 0.2], \\ 0 & \text{otherwise.} \end{cases}$$

Which gives the H^1 norm as

$$\|u\|_1 = \left(\|u\|_2^2 + |u|_1^2 \right)^{1/2} = \left(\frac{2}{15} + 25\frac{2}{5} \right)^{1/2} = \sqrt{\frac{152}{15}}.$$

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Exercise 2.6 Consider the following finite element function u defined as

$$u = \begin{cases} 1, & x = 0.5, \\ \frac{1}{h}x - \frac{1}{h}(0.5 - h), & x = (0.5 - h, 0.5), \\ -\frac{1}{h}x + \frac{1}{h}(0.5 - h), & x = (0.5, 0.5 + h), \\ 0, & \text{otherwise.} \end{cases}$$

That is, it corresponds to the hat function in Picture 2.2, where $u(0.5) = 1$ and the hat function is zero everywhere in $(0, 0.5 - h)$ and $(0.5 + h, 1)$. Compute the H^1 and L^2 norms of this function analytically, and the L^2, H^1, and H^{-1} norms numerically for $h = 10$, 100, and 1000.

Solution 2.6 Equivalently, we can write u as

$$u = \begin{cases} 1, & x = \frac{1}{2}, \\ \frac{1}{h}x - \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2} - h, \frac{1}{2}\right), \\ -\frac{1}{h}x + \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2}, \frac{1}{2} + h\right), \\ 0, & \text{otherwise.} \end{cases}$$

We begin by computing the L^2 norm of u analytically.

$$\|u\|_2 = \left(\int_0^1 u^2 \, dx \right)^{1/2}$$

$$= \left(\int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2} - h) \right)^2 dx + \int_{\frac{1}{2}}^{\frac{1}{2}+h} \left(-\frac{1}{h}x + \frac{1}{h}(\frac{1}{2} - h) \right)^2 dx \right)^{1/2}$$

$$= \left(2 \int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2} - h) \right)^2 dx \right)^{1/2}$$

$$= \left(\frac{2}{h^2} \int_{\frac{1}{2}-h}^{\frac{1}{2}} x^2 - 2x(\frac{1}{2} - h) + (\frac{1}{2} - h)^2 \, dx \right)^{1/2}$$

$$= \left(\frac{2}{h^2} \left[\frac{1}{3}x^3 - (\frac{1}{2} - h)x^2 + (\frac{1}{2} - h)^2 x \right]_{\frac{1}{2}-h}^{\frac{1}{2}} \right)^{1/2}$$

$$= \frac{\sqrt{2}}{h} \left(\left[\frac{1}{3} \cdot \frac{1}{8} - (\frac{1}{2} - h) \cdot \frac{1}{4} + (\frac{1}{2} - h)^2 \cdot \frac{1}{2} \right] \right.$$

$$\left. - \left[\frac{1}{3} \cdot \left(\frac{1}{2} - h \right)^3 - (\frac{1}{2} - h) \cdot \left(\frac{1}{2} - h \right)^2 + (\frac{1}{2} - h)^2 \cdot \left(\frac{1}{2} - h \right) \right] \right)^{1/2}$$

$$= \frac{\sqrt{2}}{h} \left(\frac{1}{24} - \frac{1}{8} + \frac{h}{4} + \frac{1}{8} - \frac{h}{2} + \frac{h^2}{2} - \frac{1}{3} \left(\frac{1}{2} - h \right)^3 \right)^{1/2}$$

Opting to instead use Wolfram Alpha to solve the integral, we find that

$$\|u\|_2 = \sqrt{\frac{2h}{3}}.$$

The H^1 semi-norm is hopefully simpler to compute, and is given by

$$|u|_1 = \left(\int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 dx \right)^{1/2}$$

$$= \left(\int_{0.5-h}^{0.5} \frac{1}{h^2} \, dx + \int_{0.5}^{0.5+h} \frac{1}{h^2} \, dx \right)^{1/2}$$

$$= \left(\frac{1}{h^2} (0.5 - (0.5 - h)) + \frac{1}{h^2} (0.5 + h - 0.5) \right)^{1/2}$$

$$= \left(\frac{2h}{h^2} \right)^{1/2} = \sqrt{\frac{2}{h}}.$$

This gives us the H^1 norm as

$$\|u\|_1 = \sqrt{\frac{2}{h} + \frac{2h}{3}} = \sqrt{\frac{6 + 2h^2}{3h}}.$$

3　The finite element method for elliptic problems

Exercise 3.1　Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega u\, v\ dx$$

is a bilinear form.

Solution 3.1　To show that $a(u, v)$ is a bilinear form, we need to show that it is linear in both arguments. We can firstly note that

$$a(u, v) = \int_\Omega u\, v\ dx = \int_\Omega v\, u\ dx = a(v, u),$$

showing that $a(u, v)$ is symmetric. We therefore only need to show that it is linear in one of the arguments.

Let $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
a(\alpha u + \beta v, w) &= \int_\Omega (\alpha u + \beta v)\, w\ dx \\
&= \int_\Omega \alpha u\, w + \beta v\, w\ dx \\
&= \alpha \int_\Omega u\, w\ dx + \beta \int_\Omega v\, w\ dx \\
&= \alpha a(u, w) + \beta a(v, w),
\end{aligned}$$

showing that $a(u, v)$ is linear in the first argument, and therefore a bilinear form.

Exercise 3.2　Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega u\, v\ dx$$

forms an inner product.

Solution 3.2　To show that $a(u, v)$ forms an inner product, we need to show that it is symmetric, positive definite, and linear in the first argument. We have already shown that $a(u, v)$ is symmetric and linear in the previous exercise. We can also see that

$$a(u, u) = \int_\Omega u\, u\ dx = \int_\Omega u^2\ dx \geq 0,$$

showing that $a(u, u)$ is positive definite. We have therefore shown that $a(u, v)$ forms an inner product.

Exercise 3.3 Let $\Omega = (0, 1)$, then for all functions in $H_0^1(\Omega)$, Poincaré's inequality states that

$$\|u\|_{L^2} \le C \left\| \frac{\partial u}{\partial x} \right\|_{L^2} = C|u|_{H^1},$$

Use this inequality to show that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$.

Solution 3.3 We can use Poincaré's inequality to show that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$. We have that

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + |u|_{H^1}^2 \right)^{1/2} \le \left(C^2 \|\nabla u\|_{L^2}^2 + |u|_{H^1}^2 \right)^{1/2} = \sqrt{1 + C^2}|u|_{H^1},$$

showing that the H^1 norm is bounded above by the H^1 semi-norm. We can also see that

$$|u|_{H^1} \le \left(\|u\|_{L^2}^2 + |u|_{H^1}^2 \right)^{1/2} = \|u\|_{H^1},$$

showing that the standard H^1 norm is bounded below by the H^1 semi-norm. We have therefore shown that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$.

Exercise 3.4 Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \ dx$$

forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product.

Solution 3.4 What does it mean for an inner product to be equivalent with another inner product? Assuming it means that two inner products are equivalent if we can bound one by the other, and vice versa.

To show that $a(u, v)$ forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product, we need to show that it is symmetric, positive definite, and linear in the first argument. We clearly have

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \ dx = \int_\Omega \nabla v \cdot \nabla u \ dx = a(v, u),$$

and

$$\begin{aligned} a(\alpha u + \beta v, w) &= \int_\Omega \nabla(\alpha u + \beta v) \cdot \nabla w \ dx \\ &= \int_\Omega \alpha \nabla u \cdot \nabla w + \beta \nabla v \cdot \nabla w \ dx \\ &= \alpha \int_\Omega \nabla u \cdot \nabla w \ dx + \beta \int_\Omega \nabla v \cdot \nabla w \ dx \\ &= \alpha a(u, w) + \beta a(v, w), \end{aligned}$$

meaning that we are only missing the positive definiteness. We can see that

$$a(u, u) = \int_\Omega \nabla u \cdot \nabla u \; dx = \int_\Omega |\nabla u|^2 \; dx = |u|_{H^1}^2 \geq 0,$$

showing that $a(u, u)$ is positive definite, and therefore forms an inner product.

$a(u, v)$ induces the H^1 semi-norm, which we have already shown to be equivalent with the standard H^1 norm. We have therefore shown that $a(u, v)$ forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product.

Exercise 3.5 Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega (k \nabla u) \cdot \nabla v \; dx$$

forms an inner product on $H_0^1(\Omega)$ given that $k \in \mathbb{R}^{n \times n}$ is strictly positive and bounded. The inner product is equivalent with the standard $H_0^1(\Omega)$ inner product.

Solution 3.5 We begin by showing that $a(u, v)$ is symmetric. We have that

$$\begin{aligned}
a(u, v) &= \int_\Omega (k \nabla u) \cdot \nabla v \; dx \\
&= \int_\Omega (\nabla u)^T k^T \nabla v \; dx \\
&= \int_\Omega \left((\nabla u)^T k^T \nabla v \right)^T \; dx \\
&= \int_\Omega (\nabla v)^T k \nabla u \; dx \\
&= \int_\Omega k^T \nabla v \cdot \nabla u \; dx \\
&= a(v, u),
\end{aligned}$$

assuming that it is implied that k is symmetric. Next, we show that $a(u, v)$ is linear in the first argument. Let $u, v, w \in H_0^1(\Omega)$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
a(\alpha u + \beta v, w) &= \int_\Omega (k \nabla (\alpha u + \beta v)) \cdot \nabla w \; dx \\
&= \int_\Omega (k \alpha \nabla u + k \beta \nabla v) \cdot \nabla w \; dx \\
&= \alpha \int_\Omega k \nabla u \cdot \nabla w \; dx + \beta \int_\Omega k \nabla v \cdot \nabla w \; dx \\
&= \alpha a(u, w) + \beta a(v, w),
\end{aligned}$$

showing that $a(u, v)$ is linear in the first argument. We are then just missing the positive definiteness. We can see that

$$a(u, u) = \int_\Omega (k\nabla u) \cdot \nabla u \; dx = \int_\Omega (\nabla u)^T k^T \nabla u \; dx \geq \int_\Omega k_0 |\nabla u|^2 \; dx \geq 0,$$

using that k is strictly positive and bounded by k_0.

Noting that

$$\int_\Omega k_0 \nabla u \cdot \nabla u \; dx \leq \int_\Omega (k\nabla u) \cdot \nabla u \; dx \leq \int_\Omega k_1 \nabla u \cdot \nabla u \; dx,$$

we see that the induced norm is equivalent with the H^1 semi-norm, and thus also the standard $H_0^1(\Omega)$ norm.

Exercise 3.6 Make a Python code that defines a Lagrange element of arbitrary order on the reference triangle.

Solution 3.6 In order to get a better grasp of the Lagrange elements, an illustration of the elements of order up to 4 on the reference triangle is shown in Figure 3, with the corresponding nodes marked in purple.
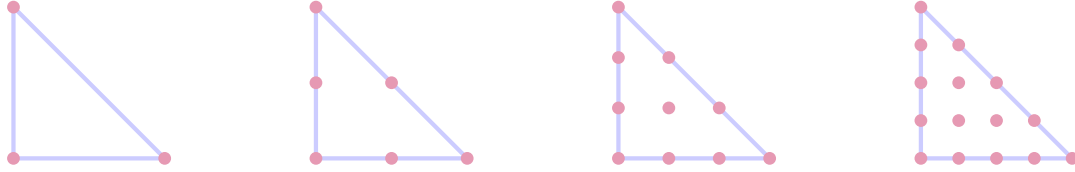

Figure 3: Lagrange elements of order up to 4 on the reference triangle.

In order to find the Lagrange basis of order n, we seek to find the coefficients a_{ij} such that

$$\ell_i = \sum_j a_{ij} \phi_j,$$

where ℓ_i is the ith Lagrange basis function, and ϕ_j is the monomials, satisfying

$$\ell_i(x_j) = \delta_{ij}.$$

We do this by solving

$$\ell_i(x_j) = \sum_k a_{ik} \phi_k(x_j) = \delta_{ij},$$

or in matrix form

$$A\Phi = I,$$

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where A is the matrix of coefficients, Φ is the matrix of monomials at the lagrange nodes, and I is the identity matrix. From this, we can find the coefficients by

$$A = \Phi^{-1}.$$

The Python code for this is available in `3_elliptic/lagrange_elements.py`