MEK4250

Exercises for Finite Elements in Computational Mechanics

August Femtehjell

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Contents

Abstract

This document contains my solutions to the exercises for the course MEK4250–Finite Elements in Computational Mechanics, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MEK4250.

1 A glimpse at the finite element method

Exercise 1.1 Consider the problem $-u''(x) = x^2$ on the unit interval with $u(0) = u(1) = 0$. Let $u = \sum_{k=1}^{N} u_k \sin(\pi k x)$ and $v = \sin(\pi l x)$ for $l = 1, \ldots, N$, for e.g. $N = 10, 20, 40$ and solve (1.9). What is the error in L_2 and L_∞.

Solution 1.1 In this exercise, we use the Galerkin method to solve the problem, wishing to solve the problem as $Au = b$, where

$$A_{ij} = \int_\Omega k \nabla N_j \cdot \nabla N_i \, dx,$$
$$b_i = \int_\Omega f N_i \, dx + \int_{\partial\Omega_N} h N_i \, ds.$$

We begin by noting that

$$\nabla N_i = \frac{d}{dx} \sin(\pi i x) = \pi i \cos(\pi i x),$$

such that

$$\int_\Omega k \nabla N_j \cdot \nabla N_i \, dx = \int_0^1 k \pi^2 ij \cos(\pi j x) \cos(\pi i x) \, dx = \frac{\pi^2 i^2}{2} \delta_{ij}.$$

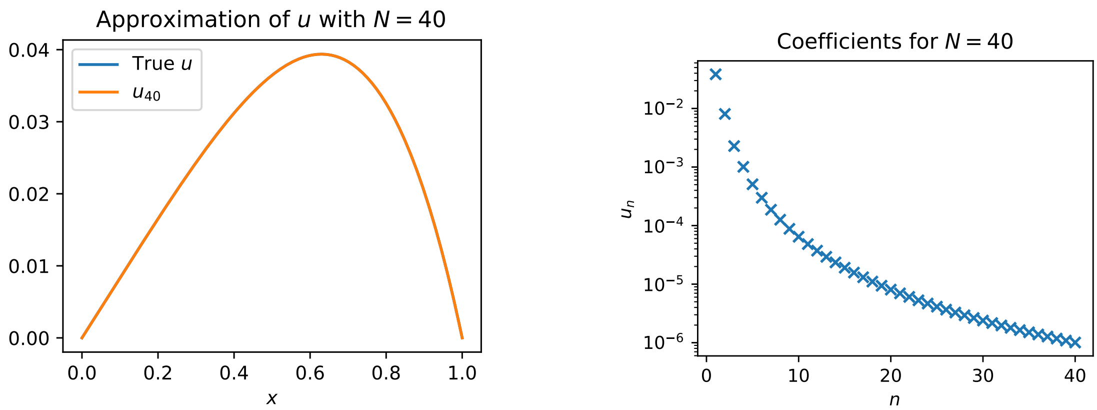
As we are given that the Dirichlet boundary conditions cover the entire boundary, and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, we have that the Neumann boundary integral is zero. The b vector is then given by

$$b_i = \int_\Omega f N_i \, dx = \int_0^1 x^2 \sin(\pi i x) \, dx = \frac{(2 - \pi^2 i^2)(-1)^i - 2}{\pi^3 i^3}.$$

Setting up and solving the system for varying N is then rather simples, implemented in `1_glimpse/ex1.py`. This gives the errors presented in Table 1, with the plotted solution in Figure 1a.

Table 1: Errors of approximations of u for varying N, with sine basis functions.

N	L_2	L_∞
10	0.001791	0.000224
20	0.000338	0.000059
40	0.000062	0.000015



(a) Approximated function.

(b) Coefficients $|u_k|$.

Figure 1: Approximation of u with $N = 40$ sine basis functions.

Exercise 1.2 Consider the same problem as in the previous exercise, but using the Bernstein polynomials. That is, the basis for the Bernstein polynomial of order N on the unit inteval is $B_k^N(x) = x^k(1-x)^{N-k}$ for $k = 0, \dots, N$. Let $u = \sum_{k=0}^N u_k B_k^N(x)$ and $v = B_l^N(x)$ for $l = 0, \dots, N$ and solve (1.9). What is the error in L_2 and L_∞ in terms of N for $N = 1, 2, \dots, 10$. Remark: Do the basis functions satisfy the boundary condtions? Should some of them be removed?

Solution 1.2 The Bernstein polynomials B_0^N and B_N^N both need to be removed, as they do not satisfy the Dirichlet boundary conditions, as $B_0^N(0) = 1 = B_N^N(1)$. We therefore need atleast $N = 3$, in order to get an atleast somewhat viable solution.

The Bernstein basis polynomials are defined as

$$B_k^N(x) = \binom{N}{k} x^k (1-x)^{N-k}.$$

Some useful properties which might come in handy are

1. The derivative of a basis polynomials is

$$\left(B_k^N(x)\right)' = N\left(B_{k-1}^{N-1}(x) - B_k^{n-1}(x)\right),$$

   where we follow the convention of setting $B_{-1}^N(x) = 0 = B_{N+1}^N(x)$.

2. The definite integral on the unit line is given by

$$\int_0^1 B_k^N(x) = \frac{1}{N+1} \quad \text{for } k = 0, 1, \dots, N.$$

3. The multiple of two Bernstein polynomials is

$$\begin{aligned}
B_k^N(x) \cdot B_q^M(x) &= \binom{N}{k} x^k (1-x)^{N-k} \binom{M}{q} x^q (1-x)^{M-q} \\
&= \binom{N}{k}\binom{M}{q} x^{k+q}(1-x)^{N+M-k-q} \\
&= \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} B_{k+q}^{N+M}(x)
\end{aligned}$$

From these, we can gather that

$$\int_0^1 B_k^N(x) B_q^M(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} \int_0^1 B_{k+q}^{N+M}(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{(N+M+1)\binom{N+M}{k+q}}.$$

The terms in the stiffness matrix are then given by

$$\begin{aligned}
A_{ij} &= \int_0^1 \nabla B_i^N(x) \cdot \nabla B_j^N(x)\ dx \\
&= N^2 \int_0^1 \left(B_{i-1}^{N-1} - B_i^{N-1}\right)\left(B_{j-1}^{N-1} - B_j^{N-1}\right)\ dx \\
&= N^2 \int_0^1 B_{i-1}^{N-1} B_{j-1}^{N-1} - B_i^{N-1} B_{j-1}^{N-1} - B_{i-1}^{N-1} B_j^{N-1} + B_i^{N-1} B_j^{N-1}\ dx \\
&= N^2 \int_0^1 \alpha_{i-1,j-1} B_{i+j-2}^{2N-2} - (\alpha_{i,j-1} + \alpha_{i-1,j}) B_{i+j-1}^{2N-2} + \alpha_{i,j} B_{i+j}^{2N-2}\ dx \\
&= \frac{N^2}{2N-1}\left(\frac{\binom{N-1}{i-1}\binom{N-1}{j-1}}{\binom{2N-2}{i+j-2}} - \frac{\binom{N-1}{i}\binom{N-1}{j-1} + \binom{N-1}{i-1}\binom{N-1}{j}}{\binom{2N-2}{i+j-1}} + \frac{\binom{N-1}{i}\binom{N-1}{j}}{\binom{2N-2}{i+j}}\right)
\end{aligned}$$

This can likely be written much nicer, however I cannot be bothered to do that right now.

Opting to take the easy way out instead, and utilizing `sympy` to solve the integrals, we can implement the solution in `1_glimpse/ex2.py`. The errors are presented in Table 2. As we can read from the table, the polynomial approximation is exact for $N > 3$, which is expected as the Bernstein polynomials are exact for polynomials of degree N.

Exercise 1.3 Consider the same problem as in the previous exercise, but with $-u''(x) = \sin(k\pi x)$ for $k = 1$ and $k = 10$.

Table 2: Errors of approximations of u for varying N, with Bernstein basis functions.

N	L_2
2	$\frac{\sqrt{1330}}{6300}$
3	$\frac{\sqrt{70}}{12600}$
4–10	0

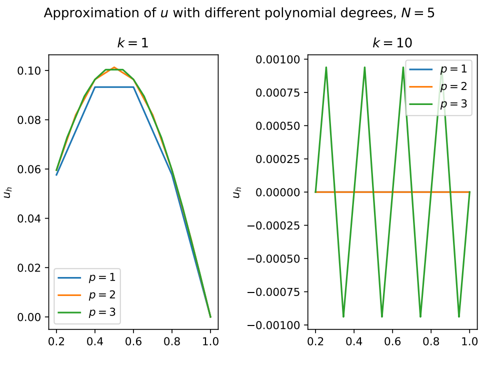
Solution 1.3 The approach for this is approximately the same, however we need to figure out the true solution in order to calculate the error.

$$\begin{aligned} u''(x) &= -\sin(k\pi x) \\ u'(x) &= \frac{1}{k\pi}\cos(k\pi x) + C_1 \\ u(x) &= \frac{1}{k^2\pi^2}\sin(k\pi x) + C_1 x + C_2. \end{aligned}$$

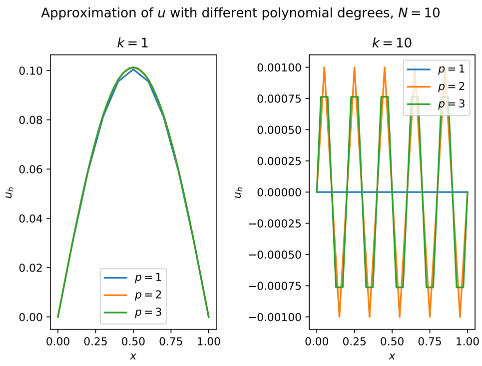
As we have Dirichlet boundary conditions, we then set $C_1 = 0 = C_2$.

Exercise 1.4 Consider the same problem as in the previous exercise, but with the finite element method in for example FEniCS, FEniCSx or Firedrake, using Lagrange method of order 1, 2 and 3.

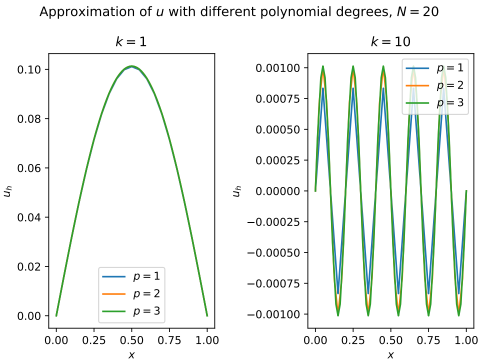
Solution 1.4 For this exercise, I will be using FEniCSx to solve the problem. The code is implemented in `doc/1_glimpse/ex4.py`, with the resulting approximations in Figure 2.



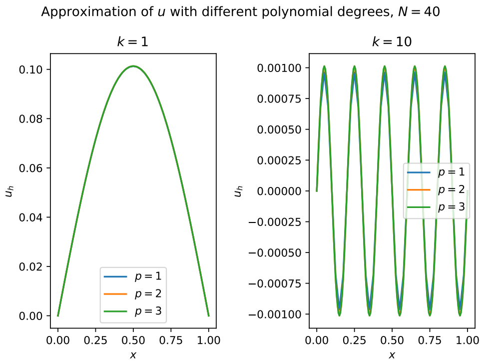
(a) $N = 5$.



(b) $N = 10$.



(c) $N = 20$.



(d) $N = 40$.

Figure 2: Approximation of u with varying N elements.

2 Crash course in Sobolev Spaces

Exercise 2.1 What is a norm? Show that

$$\|u\|_p = \left(\int_0^1 |u|^p \, dx \right)^{1/p}$$

defines a norm.

Solution 2.1 Following the definition in Spaces by Tom Lindstrøm, a norm is a function $\|\cdot\| : V \to \mathbb{R}$, where V is a vector space, such that

(i) $\|\boldsymbol{u}\| \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

(ii) $\|\alpha \boldsymbol{u}\| = |\alpha| \|\boldsymbol{u}\|$ for all $\alpha \in \mathbb{R}$ and all $\boldsymbol{u} \in V$.

(iii) (Triangle Inequality for Norms) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

Positivity is clear, as $|u|^p \geq 0$ for all $u \in L^p(0,1)$. The only way $\|u\|_p = 0$ is if $|u|^p = 0$. Homogeneity is also clear, as

$$\begin{aligned}
\|\alpha u\|_p &= \left(\int_0^1 |\alpha u|^p \, dx \right)^{1/p} \\
&= \left(\int_0^1 |\alpha|^p |u|^p \, dx \right)^{1/p} \\
&= |\alpha| \left(\int_0^1 |u|^p \, dx \right)^{1/p} \\
&= |\alpha| \|u\|_p.
\end{aligned}$$

The triangle inequality is a bit more involved, and not yet solved. Likely requires something like Minkowski's inequality.

Exercise 2.2 What is an inner product? Show that

$$(u, v)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x} \right)^i \left(\frac{\partial v}{\partial x} \right)^i \, dx$$

defines an inner product.

Solution 2.2 Again, Spaces by Tom Lindstrøm defines an inner product as a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$, where V is a vector space, such that

(i) $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

(ii) $(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{u},\boldsymbol{w}) + (\boldsymbol{v},\boldsymbol{w})$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V$.

(iii) $(\alpha\boldsymbol{u},\boldsymbol{v}) = \alpha(\boldsymbol{u},\boldsymbol{v})$ for all $\alpha \in \mathbb{R}$, $\boldsymbol{u},\boldsymbol{v} \in V$.

(iv) For all $\boldsymbol{u} \in V$, $(\boldsymbol{u},\boldsymbol{u}) \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

Symmetry is clear, as

$$(u,v)_k = \sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx = \sum_{i\leq k}\int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i \, dx = (v,u)_k.$$

Linearity in the first argument is also satisfied, as

$$\begin{aligned}
(u+v,w)_k &= \sum_{i\leq k}\int_\Omega \left(\frac{\partial(u+v)}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i + \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx + \sum_{i\leq k}\int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= (u,w)_k + (v,w)_k.
\end{aligned}$$

Homogeneity in the first argument is also satisfied, as

$$\begin{aligned}
(\alpha u,v)_k &= \sum_{i\leq k}\int_\Omega \left(\frac{\partial(\alpha u)}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\
&= \sum_{i\leq k}\int_\Omega \alpha\left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\
&= \alpha\sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\
&= \alpha(u,v)_k.
\end{aligned}$$

Finally, positivity is also satisfied, as

$$(u,u)_k = \sum_{i\leq k}\int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i \, dx = \sum_{i\leq k}\int_\Omega \left(\frac{\partial^i u}{\partial x^i}\right)^2 \, dx \geq 0.$$

$(u,u)_k = 0$ only if $\frac{\partial^i u}{\partial x^i} = 0$ for all $i \leq k$, which implies $u = 0$. $(u,v)_k$ is therefore an inner product.

Exercise 2.3 Compute the H^1 and L^2 norms of a random function with values in $(0,1)$ on meshes representing the unit interval with 10, 100, and 1000 cells.

Solution 2.3 As a random function, I choose the Bernstein polynomial B_5^{10}, which is given by

$$B_5^{10}(x) = \binom{10}{5} x^5 (1-x)^5.$$

Exercise 2.4 Compute the H^1 and L^2 norms of the function $u(x) = \sin(k\pi x)$ on the unit interval analytically and compare with the values presented in Table 2.2.

Solution 2.4 The L^2 norm of $u(x) = \sin(k\pi x)$ is given by

$$\begin{aligned}
\|u\|_2 &= \left(\int_0^1 \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left(\frac{1}{2} \int_0^1 1 - \cos(2k\pi x)\ dx \right)^{1/2} \\
&= \left(\frac{1}{2} \left[x - \frac{1}{2k\pi} \sin(2k\pi x) \right]_0^1 \right)^{1/2} \\
&= \left(\frac{1}{2} (1-0) \right)^{1/2} = \frac{\sqrt{2}}{2}.
\end{aligned}$$

The H^1 norm is given by

$$\begin{aligned}
\|u\|_1 &= \left(\int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 + u^2\ dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi \cos(k\pi x))^2 + \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \cos^2(k\pi x) + \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \left(1 - \sin^2(k\pi x)\right) + \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 + (1 - (k\pi)^2) \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \int_0^1 \sin^2(k\pi x)\ dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \frac{1}{2} \right)^{1/2} = \sqrt{\frac{1 + (k\pi)^2}{2}}.
\end{aligned}$$

The H^1 norm should then increase as k increases, while the L^2 norm should remain constant, and we do indeed see this behaviour in Table 2.2.

Exercise 2.5 Compute the H^1 and L^2 norms of the hat function in Picture 2.2.

Solution 2.5 The hat function in Picture 2.2 is given by

$$u(x) = \begin{cases} \frac{x+0.2}{0.2}, & x \in [-0.2, 0], \\ \frac{0.2-x}{0.2}, & x \in [0, 0.2], \\ 0, & \text{otherwise.} \end{cases}$$

The L^2 norm is given by

$$\begin{aligned}
\|u\|_2 &= \left(\int_0^1 u^2 \, dx \right)^{1/2} \\
&= \left(\int_{-0.2}^0 \left(\frac{x+0.2}{0.2} \right)^2 dx + \int_0^{0.2} \left(\frac{0.2-x}{0.2} \right)^2 dx \right)^{1/2} \\
&= \left(\int_{-0.2}^0 \left(\frac{x^2+0.4x+0.04}{0.04} \right) dx + \int_0^{0.2} \left(\frac{0.04-0.4x+x^2}{0.04} \right) dx \right)^{1/2} \\
&= \left(\frac{1}{0.02} \int_0^{0.2} x^2 - 0.4x + 0.04 \, dx \right)^{1/2} \\
&= \left(\frac{1}{0.02} \left[\frac{1}{3}x^3 - 0.2x^2 + 0.04x \right]_0^{0.2} \right)^{1/2} \\
&= \left(\frac{1}{0.02} \left(\frac{1}{3} \cdot 0.008 - 0.2 \cdot 0.04 + 0.04 \cdot 0.2 \right) \right)^{1/2} \\
&= \sqrt{\frac{2}{15}}
\end{aligned}$$

The derivative of u is given by

$$\frac{\partial u}{\partial x} = \begin{cases} 5, & x \in [-0.2, 0], \\ -5, & x \in [0, 0.2], \\ 0 & \text{otherwise.} \end{cases}$$

Which gives the H^1 norm as

$$\|u\|_1 = \left(\|u\|_2^2 + |u|_1^2 \right)^{1/2} = \left(\frac{2}{15} + 25\frac{2}{5} \right)^{1/2} = \sqrt{\frac{152}{15}}.$$

Exercise 2.6 Consider the following finite element function u defined as

$$u = \begin{cases} 1, & x = 0.5, \\ \frac{1}{h}x - \frac{1}{h}(0.5 - h), & x = (0.5 - h, 0.5), \\ -\frac{1}{h}x + \frac{1}{h}(0.5 - h), & x = (0.5, 0.5 + h), \\ 0, & \text{otherwise.} \end{cases}$$

That is, it corresponds to the hat function in Picture 2.2, where $u(0.5) = 1$ and the hat function is zero everywhere in $(0, 0.5 - h)$ and $(0.5 + h, 1)$. Compute the H^1 and L^2 norms of this function analytically, and the L^2, H^1, and H^{-1} norms numerically for $h = 10$, 100, and 1000.

Solution 2.6 Equivalently, we can write u as

$$u = \begin{cases} 1, & x = \frac{1}{2}, \\ \frac{1}{h}x - \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2} - h, \frac{1}{2}\right), \\ -\frac{1}{h}x + \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2}, \frac{1}{2} + h\right), \\ 0, & \text{otherwise.} \end{cases}$$

We begin by computing the L^2 norm of u analytically.

$$
\begin{aligned}
\|u\|_2 &= \left(\int_0^1 u^2\ dx\right)^{1/2} \\
&= \left(\int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2}-h)\right)^2\ dx + \int_{\frac{1}{2}}^{\frac{1}{2}+h} \left(-\frac{1}{h}x + \frac{1}{h}(\frac{1}{2}-h)\right)^2\ dx\right)^{1/2} \\
&= \left(2\int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2}-h)\right)^2\ dx\right)^{1/2} \\
&= \left(\frac{2}{h^2}\int_{\frac{1}{2}-h}^{\frac{1}{2}} x^2 - 2x(\frac{1}{2}-h) + (\frac{1}{2}-h)^2\ dx\right)^{1/2} \\
&= \left(\frac{2}{h^2}\left[\frac{1}{3}x^3 - (\frac{1}{2}-h)x^2 + (\frac{1}{2}-h)^2 x\right]_{\frac{1}{2}-h}^{\frac{1}{2}}\right)^{1/2} \\
&= \frac{\sqrt{2}}{h}\Bigg(\left[\frac{1}{3}\cdot\frac{1}{8} - (\frac{1}{2}-h)\cdot\frac{1}{4} + (\frac{1}{2}-h)^2\cdot\frac{1}{2}\right] \\
&\quad - \left[\frac{1}{3}\cdot\left(\frac{1}{2}-h\right)^3 - (\frac{1}{2}-h)\cdot\left(\frac{1}{2}-h\right)^2 + (\frac{1}{2}-h)^2\cdot\left(\frac{1}{2}-h\right)\right]\Bigg)^{1/2} \\
&= \frac{\sqrt{2}}{h}\left(\frac{1}{24} - \frac{1}{8} + \frac{h}{4} + \frac{1}{8} - \frac{h}{2} + \frac{h^2}{2} - \frac{1}{3}\left(\frac{1}{2}-h\right)^3\right)^{1/2}
\end{aligned}
$$

Opting to instead use Wolfram Alpha to solve the integral, we find that

$$
\|u\|_2 = \sqrt{\frac{2h}{3}}.
$$

The H^1 semi-norm is hopefully simpler to compute, and is given by

$$
\begin{aligned}
|u|_1 &= \left(\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2\ dx\right)^{1/2} \\
&= \left(\int_{0.5-h}^{0.5} \frac{1}{h^2}\ dx + \int_{0.5}^{0.5+h} \frac{1}{h^2}\ dx\right)^{1/2} \\
&= \left(\frac{1}{h^2}\left(0.5 - (0.5-h)\right) + \frac{1}{h^2}\left(0.5 + h - 0.5\right)\right)^{1/2} \\
&= \left(\frac{2h}{h^2}\right)^{1/2} = \sqrt{\frac{2}{h}}.
\end{aligned}
$$

This gives us the H^1 norm as

$$\|u\|_1 = \sqrt{\frac{2}{h} + \frac{2h}{3}} = \sqrt{\frac{6 + 2h^2}{3h}}.$$

3 Some common finite elements

Exercise 3.1 Consider two triangles T_0 and T_1 defined by the vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(3,4)$, $(4,2)$, $(4,4)$. Compute the mapping between them.

Solution 3.1 Let x and $\hat{x}$ be the coordinates in the reference element T_0 and the target element T_1, respectively. We seek then to find the mappings

$$\begin{aligned} x &= F_{T_0}(\hat{x}) = A_{T_0}\hat{x} + x_0, \\ \hat{x} &= F_{T_1}(x) = A_{T_1}x + \hat{x}_0, \end{aligned}$$

The Jacobian of the mapping is

$$\frac{\partial x}{\partial \hat{x}} = J(\hat{x}) = A_{T_0}. \tag{1}$$

Enumerating the vertices, we have

$$\begin{aligned} x_0 &= (0,0) & x_1 &= (1,0) & x_2 &= (0,1) \\ \hat{x}_0 &= (3,4) & \hat{x}_1 &= (4,2) & \hat{x}_2 &= (4,4). \end{aligned}$$

We have that

$$F_{T_1}(x_0) = A_{T_1}\begin{bmatrix}0\\0\end{bmatrix} + \hat{x}_0 = \hat{x}_0$$

Letting $A_{T_1} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, we have

$$\begin{aligned} \hat{x}_1 &= \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \hat{x}_0 \\ \begin{bmatrix}4\\2\end{bmatrix} &= \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} \\ \begin{bmatrix}1\\-2\end{bmatrix} &= \begin{bmatrix}a\\c\end{bmatrix} \end{aligned}$$

Similarly, we find that $(b,c) = \hat{x}_2 - \hat{x}_0 = (1,0)$, such that

$$A_{T_1} = \begin{bmatrix}1 & 1\\ -2 & 0\end{bmatrix}$$

Then,

$$\begin{aligned} \hat{x} &= A_{T_1}x + \hat{x}_0 \\ A_{T_1}x &= \hat{x} - \hat{x}_0 \\ x &= A_{T_1}^{-1}(\hat{x} - \hat{x}_0) \\ x &= A_{T_1}^{-1}\hat{x} - A_{T_1}^{-1}\hat{x}_0 \\ x &= A_{T_0}\hat{x} + x_0 \end{aligned}$$

Then, we need only invert A_{T_1}.

$$\begin{bmatrix}1 & 1\\ -2 & 0\end{bmatrix} \Bigg| \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} \Bigg| \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \Bigg| \begin{bmatrix}1 & 0\\ 1 & \frac{1}{2}\end{bmatrix} \to \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Bigg| \begin{bmatrix}0 & -\frac{1}{2}\\ 1 & \frac{1}{2}\end{bmatrix},$$

such that the inverse mapping is

$$A_{T_0} = \frac{1}{2}\begin{bmatrix}0 & -1\\ 2 & 1\end{bmatrix}$$

Exercise 3.2 Make a Python code that defines a Hermite element on the unit interval.

Solution 3.2 We consider Hermite interpolation onto $k+1$ points, with a single derivative at each node. That is, they satisfy

$$\begin{aligned} d_i(H_j) &= H_j(x_i) = \delta_{ij}, \quad \text{for even } i, j,\\ d_i(H_j) &= H_j'(x_i) = \delta_{ij}, \quad \text{for odd } i, j. \end{aligned} \tag{2}$$

Exercise 3.3 Make a Python code that defines a Lagrange element of arbitrary order on the reference triangle consisting of the vertices $(0,0)$, $(1,0)$, and $(0,1)$. Let $\mathbb{P}_k = \{x^i y^j\}$ for i, j such that $i + j \leq k$.

Solution 3.3 In order to illustrate the goal, the nodes of the first four Lagrange elements are shown in Fig. 3. We firstly need to decide how we are going to compute the polynomials. The easiest method is perhaps to set up the Vandermonde matrix

$$\begin{bmatrix} 1 & x_0 & y_0 & x_0^2 & x_0y_0 & y_0^2 & \dots\\ 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & \dots\\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is, with all the monomials of the form $x^i y^j$ for $i + j \leq k$, for each of the nodes. Inverting this matrix gives us the coefficients of the Lagrange polynomials.

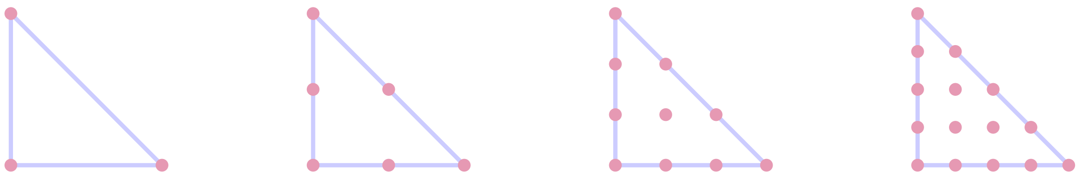


Figure 3: Lagrange elements of order up to 4 on the reference triangle.

We would however generally want to avoid inverting matrices, as this is both costly, and can lead to numerical instability. Instead, we can use a definition of the

Lagrange polynomials on triangles, analogously to the one-dimensional case. This process is illustrated in Fig. 4. The trick is to use the equations for lines in order to zero out the other nodes, in the same way as we use the points directly in the one-dimensional case.

We denote the points on the triangle by a triplet (i, j, k), in reference to the corners, in a manor such that $i + j + k = n$, where n is the order of the polynomial. In the example, we wish to find the Lagrange polynomial which is equal to 1 at the point $(\frac{1}{4}, \frac{1}{4})$, which corresponds to the triplet $(2, 1, 1)$. This already contains the information about which lines we need, namely the two diagonal lines above, the one vertical line to the left, and the one horizontal line below. These lines are marked with a thick line in the figure.

Multiplying the equations for these lines together gives us the polynomial

$$p(x, y) = (x + y - 1)\left(x + y - \tfrac{3}{4}\right) xy,$$

which indeed is zero at all other nodes. It does not however equal 1 at our desired point, which is easily fixed by normalizing the value, giving

$$\ell_{211}(x, y) = \frac{(x + y - 1)\left(x + y - \frac{3}{4}\right) xy}{\left(\frac{1}{4} + \frac{1}{4} - 1\right)\left(\frac{1}{4} + \frac{1}{4} - \frac{3}{4}\right)\left(\frac{1}{4}\right)^2} = 128xy(x + y - 1)\left(x + y - \tfrac{3}{4}\right).$$

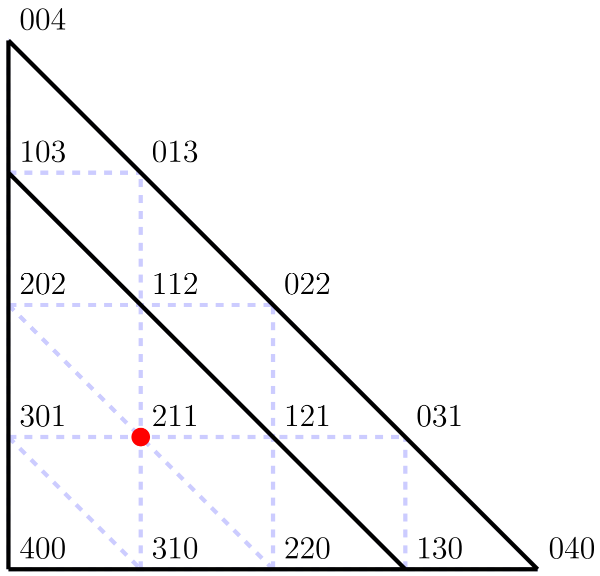


Figure 4: Lagrange elements of order up to 4 on the reference triangle.

This process is implemented in `lagrange.py`.

Exercise 3.4 Check that the interpolation result of the Bramble-Hilbert lemma 2.4 applies to the Lagrange interpolation on the unit line. Consider for example a function $f(x) = \sin(x)$ on the unit interval. The function f is a good example as it cannot be expressed as a polynomial of finite order, but can be approximated arbitrarily well.

Solution 3.4 For a polynomial $p_n(x)$ that interpolates a function f at $n+1$ distinct points $x_0, \ldots, x_n \in [0,1]$, we have that for each $x \in [0,1]$ there exists $\xi \in [0,1]$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

The Bramble-Hilbert lemma states

$$|u - P_m u|_{k,p} \leq C h^{m-k} |u|_{m,p} \quad \text{for } k = 2, \ldots, m \text{ and } p \geq 1.$$

In this exercise, we then have

$$|\sin - p_n|_{k,p} = \left(\int_0^1 \left| \frac{\partial^k}{\partial x^k} (\sin(x) - p_n(x)) \right|^p dx \right)^{\frac{1}{p}}$$

4 The finite element method for elliptic problems

Exercise 4.1 Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega u\, v \; dx$$

is a bilinear form.

Solution 4.1 To show that $a(u, v)$ is a bilinear form, we need to show that it is linear in both arguments. We can firstly note that

$$a(u, v) = \int_\Omega u\, v \; dx = \int_\Omega v\, u \; dx = a(v, u),$$

showing that $a(u, v)$ is symmetric. We therefore only need to show that it is linear in one of the arguments.

Let $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} a(\alpha u + \beta v, w) &= \int_\Omega (\alpha u + \beta v)\, w \; dx \\ &= \int_\Omega \alpha u\, w + \beta v\, w \; dx \\ &= \alpha \int_\Omega u\, w \; dx + \beta \int_\Omega v\, w \; dx \\ &= \alpha a(u, w) + \beta a(v, w), \end{aligned}$$

showing that $a(u, v)$ is linear in the first argument, and therefore a bilinear form.

Exercise 4.2 Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega u\, v \; dx$$

forms an inner product.

Solution 4.2 To show that $a(u, v)$ forms an inner product, we need to show that it is symmetric, positive definite, and linear in the first argument. We have already shown that $a(u, v)$ is symmetric and linear in the previous exercise. We can also see that

$$a(u, u) = \int_\Omega u\, u \; dx = \int_\Omega u^2 \; dx \geq 0,$$

showing that $a(u, u)$ is positive definite. We have therefore shown that $a(u, v)$ forms an inner product.

Exercise 4.3 Let $\Omega = (0, 1)$, then for all functions in $H_0^1(\Omega)$, Poincaré's inequality states that

$$\|u\|_{L^2} \le C\left\|\frac{\partial u}{\partial x}\right\|_{L^2} = C|u|_{H^1},$$

Use this inequality to show that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$.

Solution 4.3 We can use Poincaré's inequality to show that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$. We have that

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right)^{1/2} \le \left(C^2\|\nabla u\|_{L^2}^2 + |u|_{H^1}^2\right)^{1/2} = \sqrt{1 + C^2}|u|_{H^1},$$

showing that the H^1 norm is bounded above by the H^1 semi-norm. We can also see that

$$|u|_{H^1} \le \left(\|u\|_{L^2}^2 + |u|_{H^1}^2\right)^{1/2} = \|u\|_{H^1},$$

showing that the standard H^1 norm is bounded below by the H^1 semi-norm. We have therefore shown that the H^1 semi-norm defines a norm equivalent with the standard H^1 norm on $H_0^1(\Omega)$.

Exercise 4.4 Let $\Omega = (0, 1)$. Show that

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; dx$$

forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product.

Solution 4.4 What does it mean for an inner product to be equivalent with another inner product? Assuming it means that two inner products are equivalent if we can bound one by the other, and vice versa.

To show that $a(u, v)$ forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product, we need to show that it is symmetric, positive definite, and linear in the first argument. We clearly have

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega \nabla v \cdot \nabla u \; dx = a(v, u),$$

and

$$\begin{aligned}
a(\alpha u + \beta v, w) &= \int_\Omega \nabla(\alpha u + \beta v) \cdot \nabla w \; dx \\
&= \int_\Omega \alpha \nabla u \cdot \nabla w + \beta \nabla v \cdot \nabla w \; dx \\
&= \alpha \int_\Omega \nabla u \cdot \nabla w \; dx + \beta \int_\Omega \nabla v \cdot \nabla w \; dx \\
&= \alpha a(u, w) + \beta a(v, w),
\end{aligned}$$

meaning that we are only missing the positive definiteness. We can see that

$$a(u,u) = \int_\Omega \nabla u \cdot \nabla u \; dx = \int_\Omega |\nabla u|^2 \; dx = |u|_{H^1}^2 \geq 0,$$

showing that $a(u,u)$ is positive definite, and therefore forms an inner product.

$a(u,v)$ induces the H^1 semi-norm, which we have already shown to be equivalent with the standard H^1 norm. We have therefore shown that $a(u,v)$ forms an inner product on $H_0^1(\Omega)$ equivalent with the standard H^1 inner product.

Exercise 4.5 Let $\Omega = (0,1)$. Show that

$$a(u,v) = \int_\Omega (k\nabla u) \cdot \nabla v \; dx$$

forms an inner product on $H_0^1(\Omega)$ given that $k \in \mathbb{R}^{n\times n}$ is strictly positive and bounded. The inner product is equivalent with the standard $H_0^1(\Omega)$ inner product.

Solution 4.5 We begin by showing that $a(u,v)$ is symmetric. We have that

$$\begin{aligned}
a(u,v) &= \int_\Omega (k\nabla u) \cdot \nabla v \; dx \\
&= \int_\Omega (\nabla u)^T k^T \nabla v \; dx \\
&= \int_\Omega \left((\nabla u)^T k^T \nabla v\right)^T \; dx \\
&= \int_\Omega (\nabla v)^T k \nabla u \; dx \\
&= \int_\Omega k^T \nabla v \cdot \nabla u \; dx \\
&= a(v,u),
\end{aligned}$$

assuming that it is implied that k is symmetric. Next, we show that $a(u,v)$ is linear in the first argument. Let $u,v,w \in H_0^1(\Omega)$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
a(\alpha u + \beta v, w) &= \int_\Omega (k\nabla(\alpha u + \beta v)) \cdot \nabla w \; dx \\
&= \int_\Omega (k\alpha\nabla u + k\beta\nabla v) \cdot \nabla w \; dx \\
&= \alpha \int_\Omega k\nabla u \cdot \nabla w \; dx + \beta \int_\Omega k\nabla v \cdot \nabla w \; dx \\
&= \alpha a(u,w) + \beta a(v,w),
\end{aligned}$$

showing that $a(u, v)$ is linear in the first argument. We are then just missing the positive definiteness. We can see that

$$a(u, u) = \int_\Omega (k\nabla u) \cdot \nabla u \; dx = \int_\Omega (\nabla u)^T k^T \nabla u \; dx \geq \int_\Omega k_0 |\nabla u|^2 \; dx \geq 0,$$

using that k is strictly positive and bounded by k_0.

Noting that

$$\int_\Omega k_0 \nabla u \cdot \nabla u \; dx \leq \int_\Omega (k\nabla u) \cdot \nabla u \; dx \leq \int_\Omega k_1 \nabla u \cdot \nabla u \; dx,$$

we see that the induced norm is equivalent with the H^1 semi-norm, and thus also the standard $H_0^1(\Omega)$ norm.

Exercise 4.6 Make a Python code that defines a Lagrange element of arbitrary order on the reference triangle.

Solution 4.6 In order to get a better grasp of the Lagrange elements, an illustration of the elements of order up to 4 on the reference triangle is shown in Figure 5, with the corresponding nodes marked in purple.

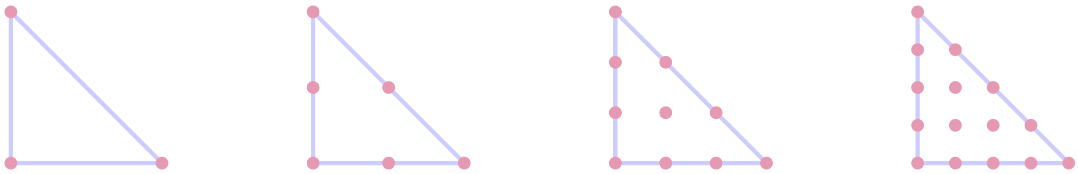


Figure 5: Lagrange elements of order up to 4 on the reference triangle.

In order to find the Lagrange basis of order n, we seek to find the coefficients a_{ij} such that

$$\ell_i = \sum_j a_{ij} \phi_j,$$

where ℓ_i is the ith Lagrange basis function, and ϕ_j is the monomials, satisfying

$$\ell_i(x_j) = \delta_{ij}.$$

We do this by solving

$$\ell_i(x_j) = \sum_k a_{ik} \phi_k(x_j) = \delta_{ij},$$

or in matrix form

$$A\Phi = I,$$

where A is the matrix of coefficients, Φ is the matrix of monomials at the lagrange nodes, and I is the identity matrix. From this, we can find the coefficients by

$$A = \Phi^{-1}.$$

The Python code for this is available in `4_elliptic/lagrange_elements.py`

5 Discretization of a convection-diffusion problem

Exercise 5.1 Show that the matrix obtained from a central difference scheme applied to the operator $Lu = u_x$ is skew-symmetric. Furthermore, show that the matrix obtained by linear continuous Lagrange elements are also skew-symmetric. Remark: The matrix is only skew-symmetric in the interior of the domain, not at the boundary.

Solution 5.1 We consider the operator $Lu = u_x$. The central difference scheme applied to this operator is

$$L_h u = \frac{u_{i+1} - u_{i-1}}{2h},$$

where h is the mesh size. The matrix representation of this operator is, considering the interior points only,

$$L_h = \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \end{bmatrix}.$$

This matrix is skew-symmetric, as can be seen by transposing it and negating it, ignoring the mismatched dimension of the matrix caused by ignoring the boundaries.

The matrix A for describing $Lu = u_x$ using linear continuous Lagrange elements is defined by the elements

$$A_{ij} = \int_\Omega (L\phi_i)\phi_j \, dx = \int_\Omega \phi_i' \phi_j \, dx.$$

As the basis functions are linear, the derivative of the basis functions are constant. If there is no overlap between the basis functions, the integral is zero, and we clearly have $A_{ij} = 0 = -A_{ji}$. Suppose then that they do contain some overlap, on an interval $[x_l, x_u]$. Then we have

$$A_{ij} = \int_\Omega \phi_i' \phi_j \, dx = \int_{x_l}^{x_u} \phi_i' \phi_j \, dx = -\int_{x_l}^{x_u} \phi_i \phi_j' \, dx + [\phi_i \phi_j]_{x_l}^{x_u} = -A_{ji} + [\phi_i \phi_j]_{x_l}^{x_u}.$$

Now, as the exercise hints to, for the interior points, the boundary term is zero, and we have $A_{ij} = -A_{ji}$.

Exercise 5.2 Estimate numerically the constant in Cea's lemma for various α and h for the Example 4.1.

Solution 5.2 Cea's lemma states:

> **Theorem 5.1** ***Cea's lemma.***
> *Suppose the conditions for Lax-Milgram's theorem are satisfied and that we solve the linear problem of finding $u_h \in V_{h,g}$ such that*
>
> $$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_{h,0}$$
>
> *on a finite element space of order t. Then,*
>
> $$\|u - u_h\|_V \le C_1 h^t \|u\|_{t+1}.$$
>
> *Here $C_1 = \frac{CB}{\alpha}$, where B comes from the approximation property and α and C are the constants of Lax-Milgram's theorem.*

I'm unsure about how to bound the constant from above, but we can at least bound it from below by

$$C_1 \ge \frac{\|u - u_h\|_V}{\|u\|_{t+1} h^t}.$$

In example 4.1 we consider the 1D convection diffusion problem, with $w = 1$, defined by

$$\begin{aligned} -u_x - \mu u_{xx} &= 0, \\ u(0) &= 0, \\ u(1) &= 1. \end{aligned}$$

The analytical solution to this is

$$u(x) = \frac{e^{-x/\mu} - 1}{e^{-1/\mu} - 1}.$$

This leads to the following estimates for C_1 for varying μ and h, as shown in Table 3.

Table 3: Estimates of C_1 for various μ and h.

$h \backslash \mu$	0.100	0.010	0.001
0.1000	0.525379	15.970240	149.093342
0.0100	0.054700	5.622143	162.507563
0.0010	0.005472	0.583342	56.752388
0.0001	0.000547	0.058356	5.884473