# MEK4250

Exam preperation for Finite Elements in Computational Mechanics

August Femtehjell

Spring 2025

## Contents



**Abstract**

This document contains my preperation for the final oral exam for the course MEK4250–Finite Elements in Computational Mechanics, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: https://github.com/augustfe/MEK4250.

## Exam formalities

Six problems/topics are given for this exam. For each problem, the candidate must prepare a 20 minutes oral presentation. Try to communicate a good overview and understanding of the topic, but compose the talk so that you can demonstrate knowledge about details too. The student is expected to be able to stick to one subject for the 30 minutes for top grades. There are no aids besides a whiteboard and this document with the exam problems (experience with this type of exam and various aids tells that learning the content by heart gives by far the best delivery that demonstrates solid understanding).

We will throw a die and the number of eyes determines the topic to be presented. After your presentation, you will be given some questions, either about parts of your presentation or facts from the other topics. After each presentation, the next candidate can throw the die and thereby get about 10 minutes to collect the thoughts before presenting the assigned topic.

## 1 Weak formulation and finite element error estimation

### Problem description

Formulate a finite element method for the Poisson problem with a variable coefficient $\kappa : \Omega \to \mathbb{R}^{d\times d}$. Assume that $\kappa$ is positive and symmetric. Show that Lax–Milgram's theorem is satisfied. Consider extensions to e.g. convection-diffusion equation and the elasticity equation. Derive *a priori* error estimates in terms of Cea's lemma for the finite element method in the energy norm. Describe how to perform an estimation of convergence rates.

### 1.1 Weak formulation

The Poisson problem with a variable coefficient $\kappa$ is given by

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega_D, \\ \kappa \frac{\partial u}{\partial n} &= h \quad \text{on } \partial\Omega_N, \end{aligned} \tag{1.1}$$

with $\partial\Omega_D$ and $\partial\Omega_N$ disjoint parts of the boundary $\partial\Omega$. Here, $\partial\Omega_D$ denotes the Dirichlet boundary, while $\partial\Omega_N$ denotes the Neumann boundary.

Setting up the weak formulation roughly follows the following steps:

1. Multiply with a test function $v$ and integrate over the domain $\Omega$
2. Integrate by parts, and apply Green's lemma.
3. Apply the boundary conditions.

Multiplying with a test function $v$ and integrating over the domain $\Omega$ gives us

$$\int_\Omega -\nabla \cdot (\kappa \nabla u)\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x. \tag{1.2}$$

This is however not ideal, as we are now required to have $u \in H^2(\Omega)$, which is not ideal. We therefore apply Green's lemma to the left-hand side, which gives us

$$\int_\Omega -\nabla \cdot (\kappa \nabla u)\, v \,\mathrm{d}x = \int_\Omega \kappa \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \kappa \frac{\partial u}{\partial n}\, v \,\mathrm{d}s. \tag{1.3}$$

This eases the requirements on $u$, as we now only require $u \in H^1(\Omega)$, while strengthening the requirements on $v$ to $v \in H^1(\Omega)$.

We can now apply the boundary conditions. Splitting the boundary integral into two parts, we have

$$\int_{\partial\Omega} \kappa \frac{\partial u}{\partial n} v \, \mathrm{d}s = \int_{\partial\Omega_D} \kappa \frac{\partial u}{\partial n} v \, \mathrm{d}s + \int_{\partial\Omega_N} \kappa \frac{\partial u}{\partial n} v \, \mathrm{d}s. \tag{1.4}$$

As we have a section of Dirichlet boundary, we need not solve for $u$ here, as we know the value of $u$ on this section. We may therefore set $v = 0$ on $\partial\Omega_D$ by having $v \in H_0^1(\Omega)$, which gives us

$$\int_{\partial\Omega_D} \kappa \frac{\partial u}{\partial n} v \, \mathrm{d}s + \int_{\partial\Omega_N} \kappa \frac{\partial u}{\partial n} v \, \mathrm{d}s = \int_{\partial\Omega_N} h \, v \, \mathrm{d}s. \tag{1.5}$$

This gives us the weak formulation for the Poisson problem

$$\int_{\Omega} \kappa \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\Omega} f \, v \, \mathrm{d}x + \int_{\partial\Omega_N} h \, v \, \mathrm{d}s. \tag{1.6}$$

## 1.2 Lax–Milgram's theorem

Lax–Milgram's theorem states:

**Theorem 1.** *Let $V$ be a Hilbert space, $a(\cdot,\cdot)$ be a bilinear form, $L(\cdot)$ be a linear form, and let the following three conditions be satisfied:*

1. *$a(u,u) \geq \alpha \|u\|_V^2$ for all $u \in V$, where $\alpha > 0$ is a constant.*
2. *$a(u,v) \leq C\|u\|_V \|v\|_V$ for all $u, v \in V$, where $C > 0$ is a constant.*
3. *$L(v) \leq D\|v\|_V$ for all $v \in V$, where $D > 0$ is a constant.*

*Then, the problem of finding $u \in V$ such that*

$$a(u,v) = L(v) \quad \forall v \in V \tag{1.7}$$

*is well-posed in the sense that there exists a unique solution with the stability condition*

$$\|u\|_V \leq \frac{C}{\alpha} \|L\|_{V^*}. \tag{1.8}$$