MEK4250

Exercises for Finite Elements in Computational Mechanics

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Contents

1 Elliptic equations and the finite element method

Exercise 1.1 Consider the problem $-u''(x) = x^2$ on the unit interval with $u(0) = u(1) = 0$. Let $u = \sum_{k=1}^{N} u_k \sin(\pi k x)$ and $v = \sin(\pi l x)$ for $l = 1, \dots, N$, for e.g. $N = 10, 20, 40$ and solve (1.9). What is the error in L_2 and L_∞.

Solution 1.1 In this exercise, we use the Galerkin method to solve the problem, wishing to solve the problem as $Au = b$, where

$$\begin{aligned} A_{ij} &= \int_\Omega k \nabla N_j \cdot \nabla N_i \, dx, \\ b_i &= \int_\Omega f N_i \, dx + \int_{\partial\Omega_N} h N_i \, ds. \end{aligned}$$

We begin by noting that

$$\nabla N_i = \frac{d}{dx} \sin(\pi i x) = \pi i \cos(\pi i x),$$

such that

$$\int_\Omega k \nabla N_j \cdot \nabla N_i \, dx = \int_0^1 k \pi^2 i j \cos(\pi j x) \cos(\pi i x) \, dx = \frac{\pi^2 i^2}{2} \delta_{ij}.$$

As we are given that the Dirichelt boundary conditions cover the entire boundary, and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, we have that the Neumann boundary integral is zero. The b

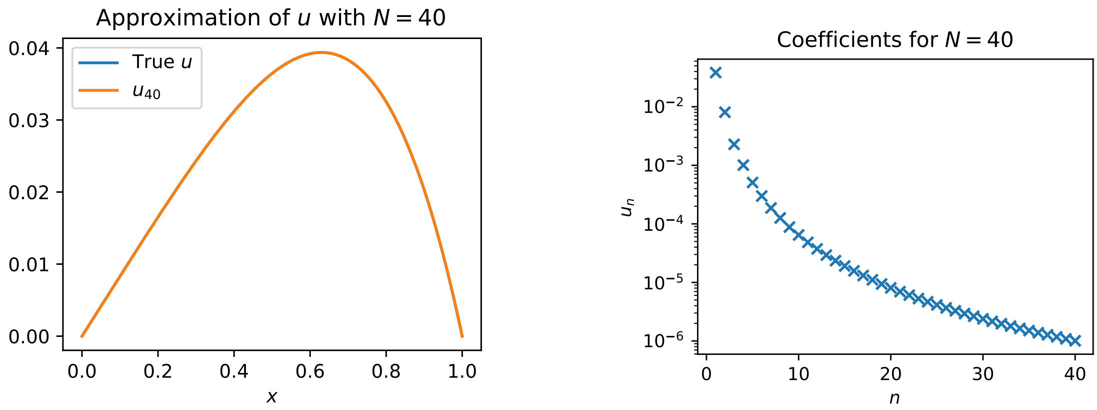
vector is then given by

$$b_i = \int_\Omega f N_i \, dx = \int_0^1 x^2 \sin(\pi i x) \, dx = \frac{(2 - \pi^2 i^2)(-1)^i - 2}{\pi^3 i^3}.$$

Setting up and solving the system for varying N is then rather simples, implemented in `exercise_1_1.py`. This gives the errors presented in Table 1, with the plotted solution in Figure 1a.

Table 1: Errors of approximations of u for varying N, with sine basis functions.

N	L_2	L_∞
10	0.001791	0.000224
20	0.000338	0.000059
40	0.000062	0.000015



(a) Approximated function.

(b) Coefficients $|u_k|$.

Figure 1: Approximation of u with $N = 40$ sine basis functions.

Exercise 1.2 Consider the same problem as in the previous exercise, but using the Bernstein polynomials. That is, the basis for the Bernstein polynomial of order N on the unit inteval is $B_k^N(x) = x^k(1-x)^{N-k}$ for $k = 0, \dots, N$. Let $u = \sum_{k=0}^N u_k B_k^N(x)$ and $v = B_l^N(x)$ for $l = 0, \dots, N$ and solve (1.9). What is the error in L_2 and L_∞ in terms of N for $N = 1, 2, \dots, 10$. Remark: Do the basis functions satisfy the boundary condtions? Should some of them be removed?