MEK4250

Exercises for Finite Elements in Computational Mechanics

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Contents

Abstract

This document contains my solutions to the exercises for the course MEK4250–Finite Elements in Computational Mechanics, taught at the University of Oslo in the spring of 2025. The code for everything, as well as this document, can be found at my GitHub repository: `https://github.com/augustfe/MEK4250`.

1 A glimpse at the finite element method

Exercise 1.1 Consider the problem $-u''(x) = x^2$ on the unit interval with $u(0) = u(1) = 0$. Let $u = \sum_{k=1}^{N} u_k \sin(\pi k x)$ and $v = \sin(\pi l x)$ for $l = 1, \ldots, N$, for e.g. $N = 10, 20, 40$ and solve (1.9). What is the error in L_2 and L_∞.

Solution 1.1 In this exercise, we use the Galerkin method to solve the problem, wishing to solve the problem as $Au = b$, where

$$\begin{aligned} A_{ij} &= \int_\Omega k \nabla N_j \cdot \nabla N_i \, dx, \\ b_i &= \int_\Omega f N_i \, dx + \int_{\partial\Omega_N} h N_i \, ds. \end{aligned}$$

We begin by noting that

$$\nabla N_i = \frac{d}{dx} \sin(\pi i x) = \pi i \cos(\pi i x),$$

such that

$$\int_\Omega k \nabla N_j \cdot \nabla N_i \, dx = \int_0^1 k \pi^2 i j \cos(\pi j x) \cos(\pi i x) \, dx = \frac{\pi^2 i^2}{2} \delta_{ij}.$$

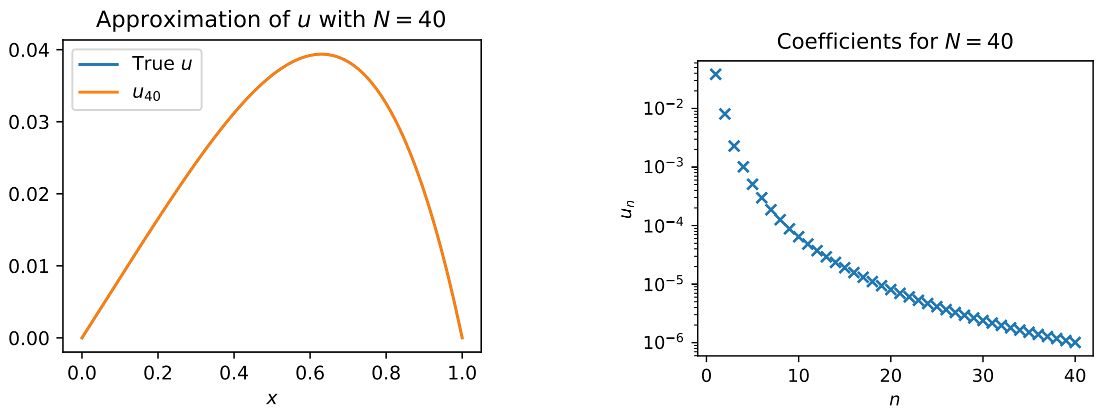
As we are given that the Dirichlet boundary conditions cover the entire boundary, and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, we have that the Neumann boundary integral is zero. The b vector is then given by

$$b_i = \int_\Omega f N_i \, dx = \int_0^1 x^2 \sin(\pi i x) \, dx = \frac{(2 - \pi^2 i^2)(-1)^i - 2}{\pi^3 i^3}.$$

Setting up and solving the system for varying N is then rather simples, implemented in `1_elliptic/ex1.py`. This gives the errors presented in Table 1, with the plotted solution in Figure 1a.

Table 1: Errors of approximations of u for varying N, with sine basis functions.

N	L_2	L_∞
10	0.001791	0.000224
20	0.000338	0.000059
40	0.000062	0.000015



(a) Approximated function.

(b) Coefficients $|u_k|$.

Figure 1: Approximation of u with $N = 40$ sine basis functions.

Exercise 1.2 Consider the same problem as in the previous exercise, but using the Bernstein polynomials. That is, the basis for the Bernstein polynomial of order N on the unit inteval is $B_k^N(x) = x^k(1-x)^{N-k}$ for $k = 0, \dots, N$. Let $u = \sum_{k=0}^{N} u_k B_k^N(x)$ and $v = B_l^N(x)$ for $l = 0, \dots, N$ and solve (1.9). What is the error in L_2 and L_∞ in terms of N for $N = 1, 2, \dots, 10$. Remark: Do the basis functions satisfy the boundary condtions? Should some of them be removed?

Solution 1.2 The Bernstein polynomials B_0^N and B_N^N both need to be removed, as they do not satisfy the Dirichlet boundary conditions, as $B_0^N(0) = 1 = B_N^N(1)$. We therefore need atleast $N = 3$, in order to get an atleast somewhat viable solution.

The Bernstein basis polynomials are defined as

$$B_k^N(x) = \binom{N}{k} x^k (1-x)^{N-k}.$$

Some useful properties which might come in handy are

1. The derivative of a basis polynomials is

$$\left(B_k^N(x)\right)' = N\left(B_{k-1}^{N-1}(x) - B_k^{n-1}(x)\right),$$

   where we follow the convention of setting $B_{-1}^N(x) = 0 = B_{N+1}^N(x)$.

2. The definite integral on the unit line is given by

$$\int_0^1 B_k^N(x) = \frac{1}{N+1} \quad \text{for } k = 0, 1, \dots, N.$$

3. The multiple of two Bernstein polynomials is

$$\begin{aligned}
B_k^N(x) \cdot B_q^M(x) &= \binom{N}{k} x^k (1-x)^{N-k} \binom{M}{q} x^q (1-x)^{M-q} \\
&= \binom{N}{k}\binom{M}{q} x^{k+q}(1-x)^{N+M-k-q} \\
&= \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} B_{k+q}^{N+M}(x)
\end{aligned}$$

From these, we can gather that

$$\int_0^1 B_k^N(x) B_q^M(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} \int_0^1 B_{k+q}^{N+M}(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{(N+M+1)\binom{N+M}{k+q}}.$$

The terms in the stiffness matrix are then given by

$$\begin{aligned}
A_{ij} &= \int_0^1 \nabla B_i^N(x) \cdot \nabla B_j^N(x)\ dx \\
&= N^2 \int_0^1 \left(B_{i-1}^{N-1} - B_i^{N-1}\right)\left(B_{j-1}^{N-1} - B_j^{N-1}\right)\ dx \\
&= N^2 \int_0^1 B_{i-1}^{N-1} B_{j-1}^{N-1} - B_i^{N-1} B_{j-1}^{N-1} - B_{i-1}^{N-1} B_j^{N-1} + B_i^{N-1} B_j^{N-1}\ dx \\
&= N^2 \int_0^1 \alpha_{i-1,j-1} B_{i+j-2}^{2N-2} - (\alpha_{i,j-1} + \alpha_{i-1,j}) B_{i+j-1}^{2N-2} + \alpha_{i,j} B_{i+j}^{2N-2}\ dx \\
&= \frac{N^2}{2N-1}\left(\frac{\binom{N-1}{i-1}\binom{N-1}{j-1}}{\binom{2N-2}{i+j-2}} - \frac{\binom{N-1}{i}\binom{N-1}{j-1} + \binom{N-1}{i-1}\binom{N-1}{j}}{\binom{2N-2}{i+j-1}} + \frac{\binom{N-1}{i}\binom{N-1}{j}}{\binom{2N-2}{i+j}}\right)
\end{aligned}$$

This can likely be written much nicer, however I cannot be bothered to do that right now.

Opting to take the easy way out instead, and utilizing `sympy` to solve the integrals, we can implement the solution in `1_elliptic/ex2.py`. The errors are presented in Table 2. As we can read from the table, the polynomial approximation is exact for $N > 3$, which is expected as the Bernstein polynomials are exact for polynomials of degree N.

Exercise 1.3 Consider the same problem as in the previous exercise, but with $-u''(x) = \sin(k\pi x)$ for $k = 1$ and $k = 10$.

Table 2: Errors of approximations of u for varying N, with Bernstein basis functions.

N	L_2
2	$\frac{\sqrt{1330}}{6300}$
3	$\frac{\sqrt{70}}{12600}$
4–10	0

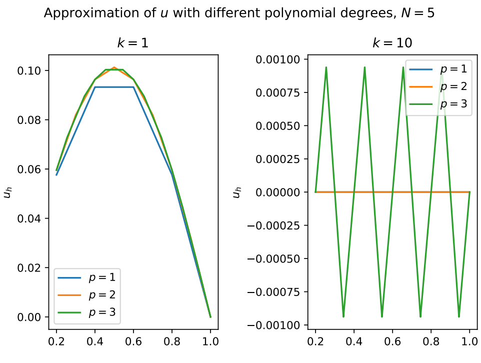
Solution 1.3 The approach for this is approximately the same, however we need to figure out the true solution in order to calculate the error.

$$\begin{aligned} u''(x) &= -\sin(k\pi x) \\ u'(x) &= \frac{1}{k\pi}\cos(k\pi x) + C_1 \\ u(x) &= \frac{1}{k^2\pi^2}\sin(k\pi x) + C_1 x + C_2. \end{aligned}$$

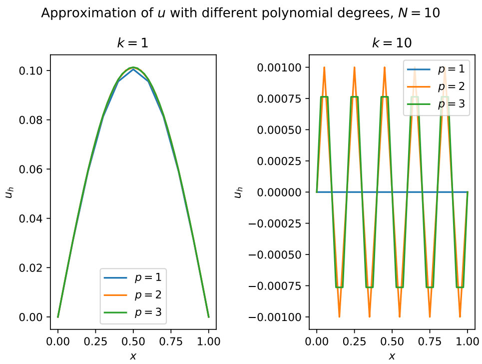
As we have Dirichlet boundary conditions, we then set $C_1 = 0 = C_2$.

Exercise 1.4 Consider the same problem as in the previous exercise, but with the finite element method in for example FEniCS, FEniCSx or Firedrake, using Lagrange method of order 1, 2 and 3.

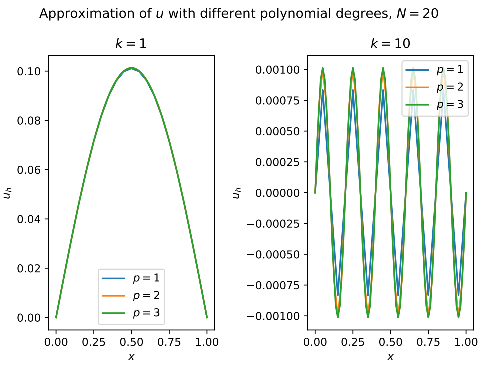
Solution 1.4 For this exercise, I will be using FEniCSx to solve the problem. The code is implemented in `doc/1_elliptic/ex4.py`, with the resulting approximations in Figure 2.



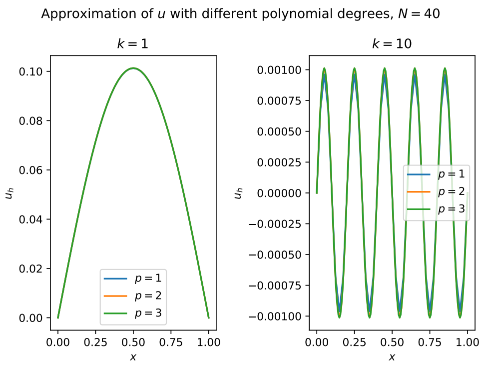
(a) $N = 5$.



(b) $N = 10$.



(c) $N = 20$.



(d) $N = 40$.

Figure 2: Approximation of u with varying N elements.

2 Crash course in Sobolev Spaces

Exercise 2.1 What is a norm? Show that

$$\|u\|_p = \left(\int_0^1 |u|^p \; dx \right)^{1/p}$$

defines a norm.

Solution 2.1 Following the definition in Spaces by Tom Lindstrøm, a norm is a function $\|\cdot\| : V \to \mathbb{R}$, where V is a vector space, such that

(i) $\|\boldsymbol{u}\| \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

(ii) $\|\alpha \boldsymbol{u}\| = |\alpha| \|\boldsymbol{u}\|$ for all $\alpha \in \mathbb{R}$ and all $\boldsymbol{u} \in V$.

(iii) (Triangle Inequality for Norms) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

Positivity is clear, as $|u|^p \geq 0$ for all $u \in L^p(0,1)$. The only way $\|u\|_p = 0$ is if $|u|^p = 0$. Homogeneity is also clear, as

$$\begin{aligned}
\|\alpha u\|_p &= \left(\int_0^1 |\alpha u|^p \; dx \right)^{1/p} \\
&= \left(\int_0^1 |\alpha|^p |u|^p \; dx \right)^{1/p} \\
&= |\alpha| \left(\int_0^1 |u|^p \; dx \right)^{1/p} \\
&= |\alpha| \|u\|_p .
\end{aligned}$$

The triangle inequality is a bit more involved, but we have

$$\begin{aligned}
\|u + v\|_p^p &= \int_0^1 |u + v|^p \; dx \\
&\leq \int_0^1 \left(|u| + |v| \right)^p \; dx \\
&\leq \int_0^1 |u|^p + |v|^p \; dx \\
&= \|u\|_p^p + \|v\|_p^p ,
\end{aligned}$$

which implies

$$\|u + v\|_p \leq \left(\|u\|_p^p + \|u\|_p^p \right)^{1/p} \leq \|u\|_p + \|u\|_p .$$

Exercise 2.2 What is an inner product? Show that

$$(u, v)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx$$

defines an inner product.

Solution 2.2 Again, Spaces by Tom Lindstrøm defines an inner product as a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$, where V is a vector space, such that

(i) $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

(ii) $(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{w}) + (\boldsymbol{v}, \boldsymbol{w})$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$.

(iii) $(\alpha \boldsymbol{u}, \boldsymbol{v}) = \alpha(\boldsymbol{u}, \boldsymbol{v})$ for all $\alpha \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in V$.

(iv) For all $\boldsymbol{u} \in V$, $(\boldsymbol{u}, \boldsymbol{u}) \geq 0$ with equality if and only if $\boldsymbol{u} = \boldsymbol{0}$.

Symmetry is clear, as

$$(u, v)_k = \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx = \sum_{i \leq k} \int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i \, dx = (v, u)_k.$$

Linearity in the first argument is also satisfied, as

$$\begin{aligned}
(u + v, w)_k &= \sum_{i \leq k} \int_\Omega \left(\frac{\partial (u + v)}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i + \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= \sum_{i \leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx + \sum_{i \leq k} \int_\Omega \left(\frac{\partial v}{\partial x}\right)^i \left(\frac{\partial w}{\partial x}\right)^i \, dx \\
&= (u, w)_k + (v, w)_k.
\end{aligned}$$

Homogeneity in the first argument is also satisfied, as

$$\begin{aligned}(\alpha u, v)_k &= \sum_{i\leq k} \int_\Omega \left(\frac{\partial(\alpha u)}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\ &= \sum_{i\leq k} \int_\Omega \alpha \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\ &= \alpha \sum_{i\leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial v}{\partial x}\right)^i \, dx \\ &= \alpha(u, v)_k.\end{aligned}$$

Finally, positivity is also satisfied, as

$$(u, u)_k = \sum_{i\leq k} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^i \left(\frac{\partial u}{\partial x}\right)^i \, dx = \sum_{i\leq k} \int_\Omega \left(\frac{\partial^i u}{\partial x^i}\right)^2 \, dx \geq 0.$$

$(u, u)_k = 0$ only if $\frac{\partial^i u}{\partial x^i} = 0$ for all $i \leq k$, which implies $u = 0$. $(u, v)_k$ is therefore an inner product.

Exercise 2.3 Compute the H^1 and L^2 norms of a random function with values in $(0, 1)$ on meshes representing the unit interval with 10, 100, and 1000 cells.

Solution 2.3 As a random function, I choose the Bernstein polynomial B_5^{10}, which is given by

$$B_5^{10}(x) = \binom{10}{5} x^5 (1 - x)^5.$$

Exercise 2.4 Compute the H^1 and L^2 norms of the function $u(x) = \sin(k\pi x)$ on the unit interval analytically and compare with the values presented in Table 2.2.

Solution 2.4 The L^2 norm of $u(x) = \sin(k\pi x)$ is given by

$$\begin{aligned}\|u\|_2 &= \left(\int_0^1 \sin^2(k\pi x) \, dx\right)^{1/2} \\ &= \left(\frac{1}{2}\int_0^1 1 - \cos(2k\pi x) \, dx\right)^{1/2} \\ &= \left(\frac{1}{2}\left[x - \frac{1}{2k\pi}\sin(2k\pi x)\right]_0^1\right)^{1/2} \\ &= \left(\frac{1}{2}(1 - 0)\right)^{1/2} = \frac{\sqrt{2}}{2}.\end{aligned}$$

The H^1 norm is given by

$$
\begin{aligned}
\|u\|_1 &= \left(\int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 + u^2 \; dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi \cos(k\pi x))^2 + \sin^2(k\pi x) \; dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \cos^2(k\pi x) + \sin^2(k\pi x) \; dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 \left(1 - \sin^2(k\pi x)\right) + \sin^2(k\pi x) \; dx \right)^{1/2} \\
&= \left(\int_0^1 (k\pi)^2 + (1 - (k\pi)^2) \sin^2(k\pi x) \; dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \int_0^1 \sin^2(k\pi x) \; dx \right)^{1/2} \\
&= \left((k\pi)^2 + (1 - (k\pi)^2) \frac{1}{2} \right)^{1/2} = \sqrt{\frac{1 + (k\pi)^2}{2}}.
\end{aligned}
$$

The H^1 norm should then increase as k increases, while the L^2 norm should remain constant, and we do indeed see this behaviour in Table 2.2.

Exercise 2.5 Compute the H^1 and L^2 norms of the hat function in Picture 2.2.

Solution 2.5 The hat function in Picture 2.2 is given by

$$
u(x) = \begin{cases} \frac{x+0.2}{0.2}, & x \in [-0.2, 0], \\ \frac{0.2-x}{0.2}, & x \in [0, 0.2], \\ 0, & \text{otherwise}. \end{cases}
$$

The L^2 norm is given by

$$
\begin{aligned}
\|u\|_2 &= \left(\int_0^1 u^2\ dx\right)^{1/2} \\
&= \left(\int_{-0.2}^{0} \left(\frac{x+0.2}{0.2}\right)^2\ dx + \int_0^{0.2} \left(\frac{0.2-x}{0.2}\right)^2\ dx\right)^{1/2} \\
&= \left(\int_{-0.2}^{0} \left(\frac{x^2+0.4x+0.04}{0.04}\right)\ dx + \int_0^{0.2} \left(\frac{0.04-0.4x+x^2}{0.04}\right)\ dx\right)^{1/2} \\
&= \left(\frac{1}{0.02}\int_0^{0.2} x^2 - 0.4x + 0.04\ dx\right)^{1/2} \\
&= \left(\frac{1}{0.02}\left[\frac{1}{3}x^3 - 0.2x^2 + 0.04x\right]_0^{0.2}\right)^{1/2} \\
&= \left(\frac{1}{0.02}\left(\frac{1}{3}\cdot 0.008 - 0.2\cdot 0.04 + 0.04\cdot 0.2\right)\right)^{1/2} \\
&= \sqrt{\frac{2}{15}}
\end{aligned}
$$

The derivative of u is given by

$$
\frac{\partial u}{\partial x} = \begin{cases} 5, & x\in[-0.2,0], \\ -5, & x\in[0,0.2], \\ 0 & \text{otherwise.} \end{cases}
$$

Which gives the H^1 norm as

$$
\|u\|_1 = \left(\|u\|_2^2 + |u|_1^2\right)^{1/2} = \left(\frac{2}{15} + 25\frac{2}{5}\right)^{1/2} = \sqrt{\frac{152}{15}}.
$$

Exercise 2.6 Consider the following finite element function u defined as

$$
u = \begin{cases} 1, & x = 0.5, \\ \frac{1}{h}x - \frac{1}{h}(0.5-h), & x = (0.5-h, 0.5), \\ -\frac{1}{h}x + \frac{1}{h}(0.5-h), & x = (0.5, 0.5+h), \\ 0, & \text{otherwise.} \end{cases}
$$

That is, it corresponds to the hat function in Picture 2.2, where $u(0.5) = 1$ and the hat function is zero everywhere in $(0, 0.5-h)$ and $(0.5+h, 1)$. Compute the H^1 and L^2 norms of this function analytically, and the L^2, H^1, and H^{-1} norms numerically for $h = 10$, 100, and 1000.

Solution 2.6 Equivalently, we can write u as

$$u = \begin{cases} 1, & x = \frac{1}{2}, \\ \frac{1}{h}x - \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2} - h, \frac{1}{2}\right), \\ -\frac{1}{h}x + \frac{1}{h}\left(\frac{1}{2} - h\right), & x \in \left(\frac{1}{2}, \frac{1}{2} + h\right), \\ 0, & \text{otherwise.} \end{cases}$$

We begin by computing the L^2 norm of u analytically.

$$\begin{aligned}
\|u\|_2 &= \left(\int_0^1 u^2 \, dx\right)^{1/2} \\
&= \left(\int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2} - h)\right)^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2}+h} \left(-\frac{1}{h}x + \frac{1}{h}(\frac{1}{2} - h)\right)^2 \, dx\right)^{1/2} \\
&= \left(2\int_{\frac{1}{2}-h}^{\frac{1}{2}} \left(\frac{1}{h}x - \frac{1}{h}(\frac{1}{2} - h)\right)^2 \, dx\right)^{1/2} \\
&= \left(\frac{2}{h^2}\int_{\frac{1}{2}-h}^{\frac{1}{2}} x^2 - 2x(\frac{1}{2} - h) + (\frac{1}{2} - h)^2 \, dx\right)^{1/2} \\
&= \left(\frac{2}{h^2}\left[\frac{1}{3}x^3 - (\frac{1}{2} - h)x^2 + (\frac{1}{2} - h)^2 x\right]_{\frac{1}{2}-h}^{\frac{1}{2}}\right)^{1/2} \\
&= \frac{\sqrt{2}}{h}\Bigg(\left[\frac{1}{3}\cdot\frac{1}{8} - (\frac{1}{2} - h)\cdot\frac{1}{4} + (\frac{1}{2} - h)^2\cdot\frac{1}{2}\right] \\
&\quad - \left[\frac{1}{3}\cdot\left(\frac{1}{2} - h\right)^3 - (\frac{1}{2} - h)\cdot\left(\frac{1}{2} - h\right)^2 + (\frac{1}{2} - h)^2\cdot\left(\frac{1}{2} - h\right)\right]\Bigg)^{1/2} \\
&= \frac{\sqrt{2}}{h}\left(\frac{1}{24} - \frac{1}{8} + \frac{h}{4} + \frac{1}{8} - \frac{h}{2} + \frac{h^2}{2} - \frac{1}{3}\left(\frac{1}{2} - h\right)^3\right)^{1/2}
\end{aligned}$$

Opting to instead use Wolfram Alpha to solve the integral, we find that

$$\|u\|_2 = \sqrt{\frac{2h}{3}}.$$

The H^1 semi-norm is hopefully simpler to compute, and is given by

$$
\begin{aligned}
|u|_1 &= \left(\int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 \, dx \right)^{1/2} \\
&= \left(\int_{0.5-h}^{0.5} \frac{1}{h^2} \, dx + \int_{0.5}^{0.5+h} \frac{1}{h^2} \, dx \right)^{1/2} \\
&= \left(\frac{1}{h^2} \left(0.5 - (0.5 - h)\right) + \frac{1}{h^2} \left(0.5 + h - 0.5\right) \right)^{1/2} \\
&= \left(\frac{2h}{h^2} \right)^{1/2} = \sqrt{\frac{2}{h}}.
\end{aligned}
$$

This gives us the H^1 norm as

$$
\|u\|_1 = \sqrt{\frac{2}{h} + \frac{2h}{3}} = \sqrt{\frac{6 + 2h^2}{3h}}.
$$