MEK4250

Exercises for Finite Elements in Computational Mechanics

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Contents

1 Elliptic equations and the finite element method

Exercise 1.1 Consider the problem $-u''(x) = x^2$ on the unit interval with $u(0) = u(1) = 0$. Let $u = \sum_{k=1}^N u_k \sin(\pi k x)$ and $v = \sin(\pi l x)$ for $l = 1, \dots, N$, for e.g. $N = 10, 20, 40$ and solve (1.9). What is the error in L_2 and L_∞.

Solution 1.1 In this exercise, we use the Galerkin method to solve the problem, wishing to solve the problem as $Au = b$, where

$$\begin{aligned} A_{ij} &= \int_\Omega k \nabla N_j \cdot \nabla N_i \, dx, \\ b_i &= \int_\Omega f N_i \, dx + \int_{\partial\Omega_N} h N_i \, ds. \end{aligned}$$

We begin by noting that

$$\nabla N_i = \frac{d}{dx} \sin(\pi i x) = \pi i \cos(\pi i x),$$

such that

$$\int_\Omega k \nabla N_j \cdot \nabla N_i \, dx = \int_0^1 k \pi^2 i j \cos(\pi j x) \cos(\pi i x) \, dx = \frac{\pi^2 i^2}{2} \delta_{ij}.$$

As we are given that the Dirichlet boundary conditions cover the entire boundary, and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, we have that the Neumann boundary integral is zero. The b

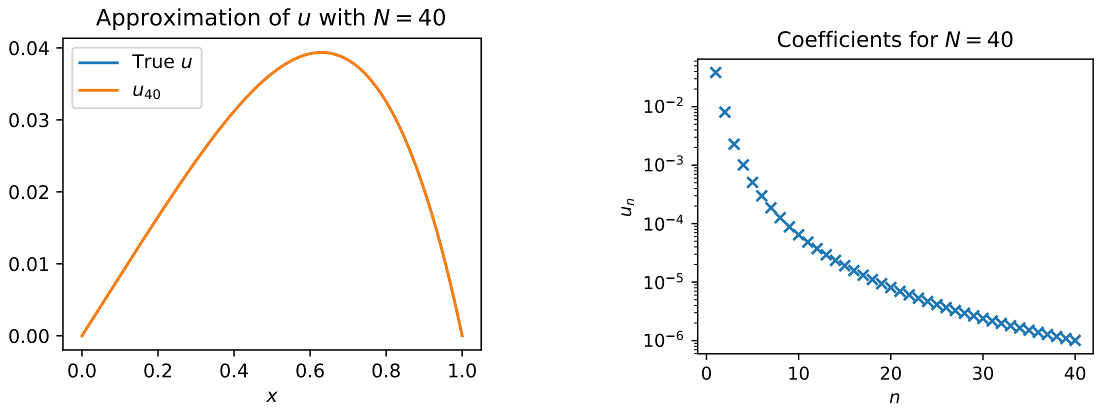
vector is then given by

$$b_i = \int_\Omega f N_i \, dx = \int_0^1 x^2 \sin(\pi i x) \, dx = \frac{(2 - \pi^2 i^2)(-1)^i - 2}{\pi^3 i^3}.$$

Setting up and solving the system for varying N is then rather simples, implemented in `exercise_1_1.py`. This gives the errors presented in Table 1, with the plotted solution in Figure 1a.

Table 1: Errors of approximations of u for varying N, with sine basis functions.

N	L_2	L_∞
10	0.001791	0.000224
20	0.000338	0.000059
40	0.000062	0.000015



(a) Approximated function.

(b) Coefficients $|u_k|$.

Figure 1: Approximation of u with $N = 40$ sine basis functions.

Exercise 1.2 Consider the same problem as in the previous exercise, but using the Bernstein polynomials. That is, the basis for the Bernstein polynomial of order N on the unit inteval is $B_k^N(x) = x^k(1-x)^{N-k}$ for $k = 0, \ldots, N$. Let $u = \sum_{k=0}^N u_k B_k^N(x)$ and $v = B_l^N(x)$ for $l = 0, \ldots, N$ and solve (1.9). What is the error in L_2 and L_∞ in terms of N for $N = 1, 2, \ldots, 10$. Remark: Do the basis functions satisfy the boundary condtions? Should some of them be removed?

Solution 1.2 The Bernstein polynomials B_0^N and B_N^N both need to be removed, as they do not satisfy the Dirichlet boundary conditions, as $B_0^N(0) = 1 = B_N^N(1)$. We therefore need atleast $N = 3$, in order to get an atleast somewhat viable solution.

The Bernstein basis polynomials are defined as

$$B_k^N(x) = \binom{N}{k} x^k (1-x)^{N-k}.$$

Some useful properties which might come in handy are

1. The derivative of a basis polynomials is
$$\left(B_k^N(x)\right)' = N\left(B_{k-1}^{N-1}(x) - B_k^{n-1}(x)\right),$$
where we follow the convention of setting $B_{-1}^N(x) = 0 = B_{N+1}^N(x)$.
2. The definite integral on the unit line is given by
$$\int_0^1 B_k^N(x) = \frac{1}{N+1} \quad \text{for } k = 0, 1, \ldots, N.$$
3. The multiple of two Bernstein polynomials is
$$\begin{aligned} B_k^N(x) \cdot B_q^M(x) &= \binom{N}{k} x^k (1-x)^{N-k} \binom{M}{q} x^q (1-x)^{M-q} \\ &= \binom{N}{k}\binom{M}{q} x^{k+q}(1-x)^{N+M-k-q} \\ &= \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} B_{k+q}^{N+M}(x) \end{aligned}$$

From these, we can gather that

$$\int_0^1 B_k^N(x) B_q^M(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{\binom{N+M}{k+q}} \int_0^1 B_{k+q}^{N+M}(x)\ dx = \frac{\binom{N}{k}\binom{M}{q}}{(N+M+1)\binom{N+M}{k+q}}.$$

The terms in the stiffness matrix are then given by

$$\begin{aligned} A_{ij} &= \int_0^1 \nabla B_i^N(x) \cdot \nabla B_j^N(x)\ dx \\ &= N^2 \int_0^1 \left(B_{i-1}^{N-1} - B_i^{N-1}\right)\left(B_{j-1}^{N-1} - B_j^{N-1}\right)\ dx \\ &= N^2 \int_0^1 B_{i-1}^{N-1} B_{j-1}^{N-1} - B_i^{N-1} B_{j-1}^{N-1} - B_{i-1}^{N-1} B_j^{N-1} + B_i^{N-1} B_j^{N-1}\ dx \\ &= N^2 \int_0^1 \alpha_{i-1,j-1} B_{i+j-2}^{2N-2} - (\alpha_{i,j-1} + \alpha_{i-1,j}) B_{i+j-1}^{2N-2} + \alpha_{i,j} B_{i+j}^{2N-2}\ dx \\ &= \frac{N^2}{2N-1}\left(\frac{\binom{N-1}{i-1}\binom{N-1}{j-1}}{\binom{2N-2}{i+j-2}} - \frac{\binom{N-1}{i}\binom{N-1}{j-1} + \binom{N-1}{i-1}\binom{N-1}{j}}{\binom{2N-2}{i+j-1}} + \frac{\binom{N-1}{i}\binom{N-1}{j}}{\binom{2N-2}{i+j}}\right) \end{aligned}$$

This can likely be written much nicer, however I cannot be bothered to do that right now.

Opting to take the easy way out instead, and utilizing `sympy` to solve the integrals, we can implement the solution in `exercise_1_2.py`. The errors are presented in Table 2. As we can read from the table, the polynomial approximation is exact for $N > 3$, which is expected as the Bernstein polynomials are exact for polynomials of degree N.

Table 2: Errors of approximations of u for varying N, with Bernstein basis functions.

N	L_2
2	$\frac{\sqrt{1330}}{6300}$
3	$\frac{\sqrt{70}}{12600}$
4–10	0

Exercise 1.3 Consider the same problem as in the previous exercise, but with $-u''(x) = \sin(k\pi x)$ for $k = 1$ and $k = 10$.

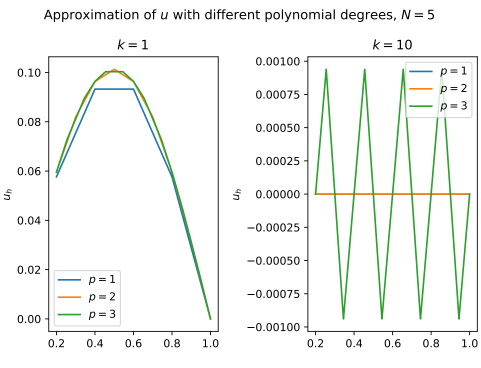
Solution 1.3 The approach for this is approximately the same, however we need to figure out the true solution in order to calculate the error.

$$\begin{aligned} u''(x) &= -\sin(k\pi x) \\ u'(x) &= \frac{1}{k\pi}\cos(k\pi x) + C_1 \\ u(x) &= \frac{1}{k^2\pi^2}\sin(k\pi x) + C_1 x + C_2. \end{aligned}$$

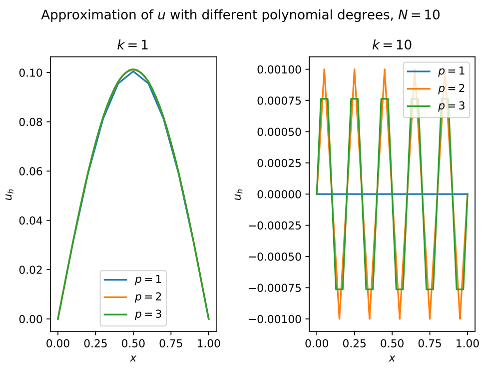
As we have Dirichlet boundary conditions, we then set $C_1 = 0 = C_2$.

Exercise 1.4 Consider the same problem as in the previous exercise, but with the finite element method in for example FEniCS, FEniCSx or Firedrake, using Lagrange method of order 1, 2 and 3.

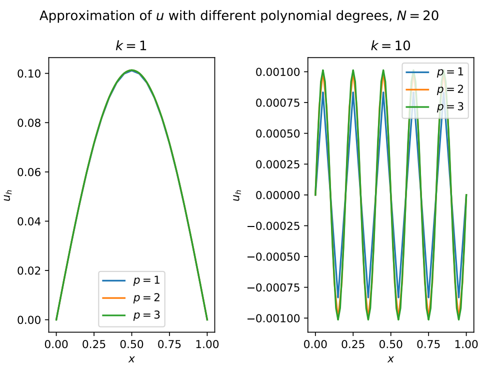
Solution 1.4 For this exercise, I will be using FEniCSx to solve the problem. The code is implemented in `doc/1_elliptic/ex4.py`, with the resulting approximations in Figure 2.



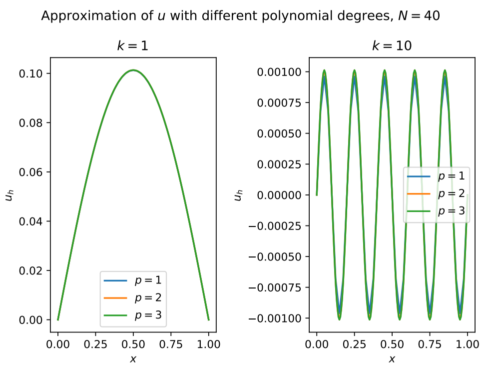
(a) $N = 5$.



(b) $N = 10$.



(c) $N = 20$.



(d) $N = 40$.

Figure 2: Approximation of u with varying N elements.