Mandatory assignment 1
MAT-MEK4270

October 8, 2024

1 1.2.1 The Dirichlet problem

The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \tag{1.2}$$

where Δ is the Laplace operator. The Dirichlet problem admits the solution

$$u(t, x, y) = \sin(k_x x) \sin(k_y y) \cos(\omega t). \tag{1.4}$$

Find the dispersion coefficient ω as a function of c, k_x and k_y.

1.1 Solution

We begin by finding the second derivatives of u. As

$$\frac{d^2}{dx^2} \sin(ax) = -a^2 \sin(ax),$$
$$\frac{d^2}{dx^2} \cos(ax) = -a^2 \cos(ax),$$

we have

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 u, \quad \frac{\partial^2 u}{\partial y^2} = -k_y^2 u, \quad \frac{\partial^2 u}{\partial t^2} = -\omega^2 u.$$

Inserting these into the Eq. (1.2) gives

$$-\omega^2 u = c^2(-k_x^2 u - k_y^2 u), \omega^2 = c^2(k_x^2 + k_y^2).$$

Taking the principal square root, we find $\omega = c\sqrt{k_x^2 + k_y^2}$. With $k_x = k_y = k$, we have $\omega = \sqrt{2}ck$.

2 1.2.3 Exact solution

The two stationary solutions to the wave equations are real and imaginary components of the waves

$$u(t, x, y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{1.6}$$

where the imaginary unit $\imath = \sqrt{-1}$. Show that Eq. (1.6) satisfies the wave equation.

2.1 Solution

We begin by rewriting (1.6), getting

$$u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)} = e^{\imath k_x x} e^{\imath k_y y} e^{-\imath \omega t} \tag{1}$$

The second derivatives are of u are then

$$\frac{\partial^2 u}{\partial x^2} = (\imath k_x)^2 u = -k_x^2 u, \quad \frac{\partial^2 u}{\partial y^2} = (\imath k_y)^2 u = -k_y^2 u, \quad \frac{\partial^2 u}{\partial t^2} = (-\imath \omega)^2 u = -\omega^2 u.$$

Combining these with the equations at hand, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \Delta u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ -\omega^2 u &= c^2 \left(-k_x^2 u - k_y^2 u \right) \\ \omega^2 &= c^2 \left(k_x^2 + k_y^2 \right) \end{aligned}$$

which satisfies the definition of ω. Thus, the wave equation is satisfied. Note that we are able to cancel out the u terms in the equation, as the exponential is always non-zero.

3 1.2.4 Dispersion coefficient

Assume that \$ m_x = m_y \$ such that \$ k_x = k_y = k \$. A discrete version of Eq. (1.6) will then read

$$u_{ij}^n = e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \tag{1.7}$$

where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω. Insert Eq. (1.7) into the discretized Eq. (1.3) and show that for CFL number $C = 1/\sqrt{2}$ we get $\tilde{\omega} = \omega$.

3.1 Solution

Eq. (1.3) is given by

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2} \right). \tag{1.3}$$

Inserting Eq. (1.7) into the left hand side of Eq. (1.3), we get

$$\begin{aligned} \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} &= e^{\imath kh(i+j)} \frac{e^{-\imath\tilde{\omega}(n+1)\Delta t} - 2e^{-\imath\tilde{\omega} n \Delta t} + e^{-\imath\tilde{\omega}(n-1)\Delta t}}{\Delta t^2} \\ &= e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \frac{e^{-\imath\tilde{\omega}\Delta t} - 2 + e^{\imath\tilde{\omega}\Delta t}}{\Delta t^2} \end{aligned}$$

The first term on the right hand side of Eq. (1.3) (and very similarly the second term) is

$$\begin{aligned} \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} &= e^{-\imath\tilde{\omega} n \Delta t} \frac{e^{\imath kh(i+1+j)} - 2e^{\imath kh(i+j)} + e^{\imath kh(i-1+j)}}{h^2} \\ &= e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \frac{e^{\imath kh} - 2 + e^{-\imath kh}}{h^2} \end{aligned}$$

Combining these, we get

$$\begin{aligned}
\frac{e^{-\imath\tilde{\omega}\Delta t}-2+e^{\imath\tilde{\omega}\Delta t}}{\Delta t^2} &= c^2\left(\frac{e^{\imath kh}-2+e^{-\imath kh}}{h^2}+\frac{e^{\imath kh}-2+e^{-\imath kh}}{h^2}\right)\\
\frac{e^{-\imath\tilde{\omega}\Delta t}-2+e^{\imath\tilde{\omega}\Delta t}}{\Delta t^2} &= 2c^2\left(\frac{e^{\imath kh}-2+e^{-\imath kh}}{h^2}\right)\\
e^{-\imath\tilde{\omega}\Delta t}-2+e^{\imath\tilde{\omega}\Delta t} &= 2\frac{c^2\Delta t^2}{h^2}\left(e^{\imath kh}-2+e^{-\imath kh}\right)\\
e^{-\imath\tilde{\omega}\Delta t}-2+e^{\imath\tilde{\omega}\Delta t} &= 2C^2\left(e^{\imath kh}-2+e^{-\imath kh}\right)\\
2\cos(\tilde{\omega}\Delta t)-2 &= 2C^2\left(2\cos(kh)-2\right)\\
\cos(\tilde{\omega}\Delta t)-1 &= 2C^2\left(\cos(kh)-1\right)\\
\cos(\tilde{\omega}\Delta t) &= 1+2C^2\cos(kh)-2C^2
\end{aligned}$$

Which, with $C=1/\sqrt{2}$, gives

$$cos(\tilde{\omega}\Delta t)=\cos(kh)$$

which has a trivial solution $\tilde{\omega}=\frac{kh}{\Delta t}$. With $C=1/\sqrt{2}$, we also get

$$\begin{aligned}
\frac{1}{\sqrt{2}} &= \frac{c\Delta t}{h}\\
h &= \sqrt{2}c\Delta t
\end{aligned}$$

such that

$$\tilde{\omega}=\frac{kh}{\Delta t}=\frac{k\sqrt{2}c\Delta t}{\Delta t}=\sqrt{2}kc=\omega$$

as desired.