7

The Three-Body Problem

7.1 Aims and Objectives

- To model the problem of three bodies in mutual gravitational attaction and present its general solution by Lagrange.
- To develop the restricted three-body problem for modeling the flight of a spacecraft under the gravitational influence of two massive bodies.
- To discuss the solvability, equilibrium points, and numerical solutions of the restricted three-body problem with examples of earth–moon–spacecraft trajectories.

7.2 Equations of Motion

The *three-body problem* refers to the dynamical system comprising the motion of three masses under mutual gravitational attraction. It usually arises when we are interested in either the two-body orbital perturbations caused by a distant third body, or the motion of a smaller body in the gravitational field formed by two larger bodies. The analysis of the former kind are undertaken when considering either the orbital motion of a moon around a planet perturbed by the sun or that of a satellite around the earth perturbed by the sun or moon. The latter model is generally applied to a comet under the combined gravity of the sun and Jupiter, a spacecraft under mutual gravity of the earth–moon system, or an interplanetary probe approaching (or departing) a planet. The three-body problem is thus a higher-order gravitational model compared to the two-body problem considered in Chapters 4 and 5 and is a much better approximation of the actual motion of a spacecraft in the solar system.

The three-body problem has attracted attention of mathematicians and physicists over the past 300 years, primarily due to its promise of modeling the erratic behavior of the moon. The first analyses of the three-body problem were undertaken by Newton and Euler, but the first systematic study of its equilibrium points was presented by *Lagrange*, both in the 18th century. One could write the equations of motion for the three-body problem using the N-body equations derived in Chapter 4 with $N = 3$ as follows:

$$\frac{\mathrm{d}^2\mathbf{R_i}}{\mathrm{d}t^2} = G\sum_{j\neq i}^{3} \frac{m_j}{R_{ij}^3}(\mathbf{R_j} - \mathbf{R_i}) \qquad i = 1, 2, 3, \tag{7.1}$$

where G is the universal gravitational constant, $\mathbf{R_i}$ denotes the position of the center of mass of body i, and $R_{ij} \doteq \mid \mathbf{R_j} - \mathbf{R_i} \mid$ denotes the relative separation of the centers of mass of bodies i,j. The potential energy of the three-body system is given by

$$\begin{aligned} V &\doteq \frac{1}{2}G\sum_{i=1}^{3} m_i \sum_{j\neq i}^{3} \frac{m_j}{R_{ij}} \\ &= -G\left(\frac{m_1m_2}{r_{12}} + \frac{m_2m_3}{r_{23}} + \frac{m_1m_3}{r_{13}}\right) , \end{aligned} \tag{7.2}$$

while its kinetic energy is the following:

$$T \doteq \frac{1}{2}\sum_{i=1}^{3}\sum_{j\neq i}^{3} m_i \left(\frac{\mathrm{d}R_{ij}}{\mathrm{d}t}\right)^2 . \tag{7.3}$$

Since no external force acts upon the system, the total energy, $E = T + V$, is conserved and represents a scalar constant. Another six scalar constants are obtained by considering the motion of the center of mass, $\mathbf{r_c}$, which follows a straight line at constant velocity, $\mathbf{v_{c_0}}$, beginning from a constant initial position, $\mathbf{r_{c_0}}$, due to Newton's first law of motion (Chapter 4):

$$\mathbf{r_c} \doteq \frac{\sum_{i=1}^{3} m_i\mathbf{R_i}}{\sum_{i=1}^{3} m_i} = \mathbf{v_{c_0}}t + \mathbf{r_{c_0}} , \tag{7.4}$$

while three more scalar constants arise out of the conserved angular momentum $\mathbf{H}$, of the system about the center of mass. There are thus a total of only 10 scalar constants of the three-body problem, whereas we need 18 for a complete analytical solution. The three-body problem has therefore resisted attempts at a closed-form, general solution. However, Lagrange showed that certain particular solutions of the problem exist when the motion of the three bodies is confined to a single plane. Such a coplanar motion of bodies is the most common occurrence in the universe, such as the solar system. Before attempting Lagrange's particular solutions, let us rewrite the equations of motion in the following form:

$$\mathbf{f_i} = Gm_i\sum_{j\neq i}^{3} \frac{m_j}{R_{ij}^3}(\mathbf{R_j} - \mathbf{R_i}) \qquad i = 1, 2, 3, \tag{7.5}$$

where $\mathbf{f_i}$ is the net force experienced by the mass m_i due to the other two masses.

7.3 Lagrange's Solution

The particular solution by Lagrange is confined to a coplanar motion of the three masses. Let us choose a rotating coordinate frame fixed to the common center of mass such that the three bodies appear to be moving radially in the rotating frame. At any particular instant of time, the angle made by the rotating frame, $(\mathbf{i}, \mathbf{j})$, with an inertial frame, $(\mathbf{I}, \mathbf{J})$, is $\theta(t)$. Then the coordinate transformation between the two frames at that instant is given by

$$\begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \end{Bmatrix} = \mathsf{C}(t) \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \end{Bmatrix} , \tag{7.6}$$

where the rotation matrix, $\mathsf{C}(t)$, is the following (Chapter 2):

$$\mathsf{C}(t) \doteq \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} . \tag{7.7}$$

Now, in the rotating frame, the location of the mass m_i at time t is given by

$$\mathbf{R_i}(t) = a(t)\mathbf{R_i}(0) , \tag{7.8}$$

where $\mathbf{R_i}(0)$ is the initial location of the mass at $t = 0$. At any given instant, all three bodies share the same values of $\theta(t)$ and $a(t)$. However, due to the radial movement of the bodies, the angular speed, $\dot{\theta}$, of the frame keeps changing with time due to conservation of angular momentum. By substituting Eq. (7.8) into Eq. (7.5), we can express the net force experienced by m_i as

$$\mathbf{f_i}(t) = \frac{\mathbf{f_i}(0)}{a^2} , \tag{7.9}$$

and the skew-symmetric matrix $\mathsf{S}(\boldsymbol{\omega})$ of the frame's angular velocity, $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, is the following (Chapter 2):

$$\mathsf{S}(\boldsymbol{\omega}) \doteq \mathsf{C}^T\dot{\mathsf{C}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{\theta} . \tag{7.10}$$

The net acceleration of m_i expressed in the rotating frame is simply the following (Chapter 4):

$$\begin{aligned} \frac{\mathrm{d}^2\mathbf{R_i}}{\mathrm{d}t^2} = \mathsf{C}\bigg(& \frac{\partial^2\mathbf{R_i}}{\partial t^2} + 2\boldsymbol{\omega} \times \frac{\partial \mathbf{R_i}}{\partial t} \\ & + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\mathbf{R_i} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{R_i}] \bigg) , \end{aligned} \tag{7.11}$$

which can be written as follows (Chapter 2):

$$\begin{aligned} \frac{\mathrm{d}^2\mathbf{R_i}}{\mathrm{d}t^2} = \mathsf{C}\bigg(& \frac{\partial^2\mathbf{R_i}}{\partial t^2} + 2\mathsf{S}(\boldsymbol{\omega})\frac{\partial \mathbf{R_i}}{\partial t} \\ & + \dot{\mathsf{S}}(\boldsymbol{\omega})\mathbf{R_i} + \mathsf{S}^2(\boldsymbol{\omega})\mathbf{R_i} \bigg) , \end{aligned} \tag{7.12}$$

or, substituting Eqs. (7.8) and (7.9), we have

$$m_i\left[\left(\frac{\mathrm{d}^2a}{\mathrm{d}t^2}-a\dot{\theta}^2\right)\mathsf{I}+\frac{1}{a}\frac{\mathrm{d}(a^2\dot{\theta})}{\mathrm{d}t}\mathsf{J}\right]\mathbf{R_i}(0)=\frac{\mathbf{f_i}(0)}{a^2}\ , \tag{7.13}$$

where I is the identity matrix and

$$\mathsf{J}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ . \tag{7.14}$$

Since the net force experienced by m_i is toward the common center of mass (leading to its radial and rotary motion), we can express the force by

$$\mathbf{f_i}(t)=-m_i b^2\mathbf{R_i}(t)\ , \tag{7.15}$$

where b is a constant. In a planar motion with radial acceleration, we have

$$R_i^2\dot{\theta}=\text{constant}\ , \tag{7.16}$$

or,

$$\frac{\mathrm{d}a^2\dot{\theta}}{\mathrm{d}t}=0\ . \tag{7.17}$$

Therefore, it follows from Eq. (7.13) that

$$\frac{\mathrm{d}^2a}{\mathrm{d}t^2}-a\dot{\theta}^2=-\frac{b^2}{a^2}\ . \tag{7.18}$$

Clearly, Eqs. (7.17) and (7.18) represent a conic section (Chapter 4) in polar coordinates, which is the equation of the relative motion of two bodies. Hence, each mass in the coplanar three-body problem traces a conic section about the common center of mass. This simple and elegant solution possesses several interesting cases and offers a valuable insight into an otherwise intractable problem.

From the general, coplanar motion given above, certain stationary solutions in terms of fixed locations of the masses in the rotating frame can be derived. These solutions represent equilibrium points of the three-body problem and describe the three bodies moving in concentric, coplanar circles. Obviously, for a stationary solution with respect to the rotating frame, we require a constant value of $a=1$, which leads to a constant angular speed, $\omega=\dot{\theta}$, by angular momentum conservation. The equations of motion in such a case are written as follows:

$$\begin{aligned}\left(\frac{\omega^2}{G}-\frac{m_2}{R_{12}^3}-\frac{m_3}{R_{13}^3}\right)\mathbf{R_1}+\frac{m_2}{R_{12}^3}\mathbf{R_2}+\frac{m_3}{R_{13}^3}\mathbf{R_3}&=\mathbf{0},\\ \frac{m_1}{R_{12}^3}\mathbf{R_1}+\left(\frac{\omega^2}{G}-\frac{m_1}{R_{12}^3}-\frac{m_3}{R_{23}^3}\right)\mathbf{R_2}+\frac{m_3}{R_{23}^3}\mathbf{R_3}&=\mathbf{0},\\ m_1\mathbf{R_1}+m_2\mathbf{R_2}+m_3\mathbf{R_3}&=\mathbf{0}.\end{aligned} \tag{7.19}$$

Particular equilibrium solutions of Eq. (7.19) include the *equilateral triangle* configurations of the three masses, wherein the masses are at the same constant distance from the common center of mass ($R_{12} = R_{13} = R_{23} = \rho$). Hence, the angular speed is given by

$$\dot{\theta}^2 = \frac{G(m_1 + m_2 + m_3)}{\rho^3} \ . \tag{7.20}$$

Another set of equilibrium points contains the *colinear* solutions, where the three masses share a straight line. Let the axial location of the three masses be given by x_1, x_2, x_3, where we assume $x_1 < x_2 < x_3$. Then the equations of motion, written in terms of the distance ratio $\alpha \doteq R_{23}/R_{12}$, yield the following [11]:

$$\begin{aligned} x_1 &= -R_{12}\frac{m_2 + (1+\alpha)m_3}{m_1 + m_2 + m_3}, \\ \omega^2 &= \frac{G(m_1 + m_2 + m_3)}{R_{12}^3(1+\alpha)^2}\frac{m_2(1+\alpha)^2 + m_3}{m_2 + (1+\alpha)m_3} \\ &(m_1 + m_2)\alpha^5 + (3m_1 + 2m_2)\alpha^4 + (3m_1 + m_2)\alpha^3 \\ &\quad -(m_2 + 3m_3)\alpha^2 - (2m_2 + 3m_3)\alpha - (m_2 + m_3) = 0 \ . \end{aligned} \tag{7.21}$$

The last of Eq. (7.21), called the *quintic equation of Lagrange*, has only one positive root, α, which can be seen by examining the signs of the coefficients (they change sign only once). However, for each specific configuration of the three colinear masses, the positive value of α is different.

Example 7.1. Numerically determine the values of α for the colinear

(a) earth–moon–spacecraft system,
(b) sun–earth–moon system.

Assume the ratio of the moon's mass to the earth's mass is 1/81.3 and that of the earth's mass to the solar mass is 1/333,400. The spacecraft's mass is negligible in comparison with the masses of heavenly bodies.

The required numerical computation is easily performed using the intrinsic MATLAB function *root* and the last of Eq. (7.21) as follows:

```
>> m2=1/81.3; %moon between earth and spacecraft (m3=0,m1=1)
>> C=[1+m2  3+2*m2 3+m2 -m2 -2*m2 -m2];
>> roots(C)
   ans =     -1.4932 + 0.8637i
        -1.4932 - 0.8637i
         0.1678
        -0.0846 + 0.1310i
        -0.0846 - 0.1310i
>> m3=1/81.3; %spacecraft between earth and moon (m1=1,m2=0)
>> C=[1  3 3 -3*m3 -3*m3 -m3];
>> roots(C)
   ans =     -1.5014 + 0.8731i
        -1.5014 - 0.8731i
         0.1778
        -0.0875 + 0.1236i
        -0.0875 - 0.1236i
```

```
>> m3=1/81.3; %earth between spacecraft and moon (m1=0, m2=1)
>> C=[1  2 1 -1-3*m3 -2-3*m3 -1-m3];
>> roots(C)
   ans =      1.0071
        -0.5082 + 0.8660i
        -0.5082 - 0.8660i
        -0.9953 + 0.0785i
        -0.9953 - 0.0785i
```

Thus, for the earth–moon–spacecraft problem, $\alpha = 0.1678, 0.1778$, or 1.0071, depending upon whether the moon, the spacecraft, or the earth, respectively, lies between the other two bodies. For the sun–earth–moon system, the calculations of α are similarly carried out:

```
>> m3=333400; m2=1/81.3; %moon between earth and sun (m1=1)
>> C=[1+m2 3+2*m2 3+m2 -m2-3*m3 -2*m2-3*m3 -m2-m3];
>> roots(C)
   ans =    -50.4637 +86.2542i
        -50.4637 -86.2542i
         98.9396
        -0.5000 + 0.2887i
        -0.5000 - 0.2887i
>> m3=333400*81.3; m2=81.3; %earth between moon and sun (m1=1)
>> C=[1+m2 3+2*m2 3+m2 -m2-3*m3 -2*m2-3*m3 -m2-m3];
>> roots(C)
   ans =    -50.1385 +86.2542i
        -50.1385 -86.2542i
         99.2648
        -0.5000 + 0.2887i
        -0.5000 - 0.2887i
```

Thus, for the sun–earth–moon problem, $\alpha = 98.9396$ or 99.2648, depending upon whether the moon or the earth, respectively, lies between the other two bodies. The sun is so much more massive compared to the earth and the moon that the two values of α are very close together, the relative position of the earth and moon with respect to the sun makes little difference.

7.4 Restricted Three-Body Problem

When the mass of one of the three bodies, say m_3, is negligible in comparison with that of the other two bodies (called *primaries*), a simplification occurs in the three-body problem, where we neglect the gravitational pull of m_3 on both m_1 and m_2. In such a case, the motion of m_3 relative to the the primaries executing circular orbits about the common center of mass is referred to as the *restricted three-body problem*. The equations of motion of the restricted problem are usually nondimensionalized by dividing the masses by the total mass of the primaries, $m_1 + m_2$, and the distances by the constant separation between the primaries, R_{12}.

$$\begin{aligned} \mu &\doteq \frac{m_2}{m_1 + m_2}, \\ r_1 &\doteq \frac{R_{13}}{R_{12}}, \\ r_2 &\doteq \frac{R_{23}}{R_{12}}. \end{aligned} \tag{7.22}$$

Furthermore, the value of gravitational constant is nondimensionalized such that the angular velocity of the primaries, Eq. (7.20), becomes unity. Using the methods of Chapter 4, we can write the equations of motion of m_3 in terms of its position, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, relative to a rotating frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ with origin at the common center of mass (Fig. 7.1), and rotating with the nondimensional velocity, $\boldsymbol{\omega} = \mathbf{k}$ (Eq. (7.20)):

$$
\begin{aligned}
\ddot{x} - 2\dot{y} - x &= -\frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(1-\mu+x)}{r_2^3}, \\
\ddot{y} + 2\dot{x} - y &= -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}, \\
\ddot{z} &= -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}.
\end{aligned} \tag{7.23}
$$

Note that we have not assumed that m_3 lies in the same plane as the primaries. The set of coupled, nonlinear, ordinary differential equations has proved unsolvable in a closed form over the past two centuries. However, we can obtain certain important analytical insights into the problem without actually solving it.

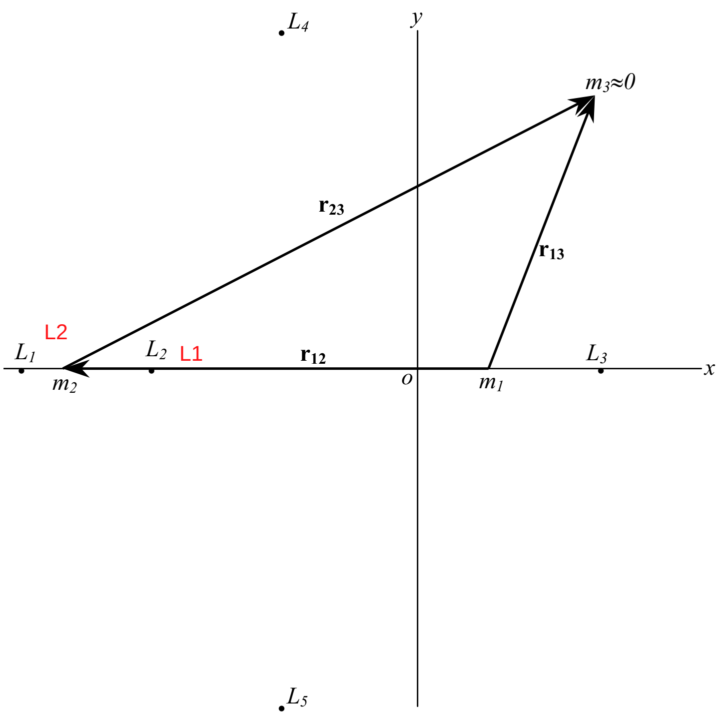

Fig. 7.1. Geometry of the restricted three-body problem.

7.4.1 Lagrangian Points and Their Stability

We have just seen that certain equilibrium solutions are possible to the general three-body problem. The equilibrium points for the restricted problem are called *Lagrangian points* (or *libration points*) and can be obtained by equating the time derivatives in Eq. (7.23) to zero, resulting in the following algebraic equations:

$$
\begin{aligned}
x &= \frac{(1-\mu)(x-\mu)}{r_1^3} + \frac{\mu(1-\mu+x)}{r_2^3}, \\
y &= \frac{(1-\mu)y}{r_1^3} + \frac{\mu y}{r_2^3}, \\
0 &= \frac{(1-\mu)z}{r_1^3} + \frac{\mu z}{r_2^3}.
\end{aligned}
\tag{7.24}
$$

From the last of Eq. (7.24), we have the result $z = 0$ for all Lagrangian points. Therefore, the equilibrium points always lie in the same plane as the primaries. As we have discovered above, the coplanar equilibrium solutions consist of the three possible colinear positions of m_3, called the *colinear Lagrangian points*, L_1, L_2, L_3, as well as the two equilateral triangle positions, called *triangular Lagrangian points*, L_4, L_5, with the two primaries (Fig. 7.1). The points L_1, L_2, L_3 are obtained from the solution of the quintic equation of Lagrange (the last of Eq. (7.21)), which for $m_3 = 0$ is written as follows:

$$
r_2^5 + (3-\mu)r_2^4 + (3-2\mu)r_2^3 - \mu r_2^2 - 2\mu r_2 - \mu = 0\,. \tag{7.25}
$$

As seen above, for the small values of μ that are typical in the solar system, we have only one real root of Eq. (7.25). The triangular Lagrangian points, L_4, L_5, correspond to $r_1 = r_2 = 1$ and are given by the coordinates $(\mu - \frac{1}{2}, \pm\sqrt{3}/2)$.

In order to investigate the stability of the Lagrangian points, we consider infinitesimal displacements, $\delta x, \delta y, \delta z$, from each equilibrium position, $x_0, y_0, 0$. If the displacements remain small, the equilibrium point is said to be stable, otherwise, unstable. Substituting $x = x_0 + \delta x, y = y_0 + \delta y, z = \delta z$ into the equations of motion, and neglecting second- and higher-order terms involving the small displacements, we have the following equation of motion for out-of-plane motion:

$$
\delta\ddot{z} + \left[\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3}\right]\delta z\,. \tag{7.26}
$$

Now, for small displacements the denominator terms can be approximated through binomial expansion as follows:

$$
\begin{aligned}
r_1^{-3} &\approx [(x_0-\mu)^2 + y_0^2]^{-\frac{3}{2}} - 3r_{1_0}^{-5}[(x_0-\mu)\delta x + y_0\delta y], \\
r_2^{-3} &\approx [(x_0+1-\mu)^2 + y_0^2]^{-\frac{3}{2}} - 3r_{2_0}^{-5}[(x_0+1-\mu)\delta x + y_0\delta y],
\end{aligned}
\tag{7.27}
$$

where r_{1_0}, r_{2_0} are the equilibrium values of r_1, r_2. Upon substitution of Eq. (7.27) into Eq. (7.26), we have

$$\delta\ddot{z} + c\delta z = 0 \,, \tag{7.28}$$

where

$$c \doteq (1-\mu)[(x_0-\mu)^2 + y_0^2]^{-\frac{3}{2}} + \mu[(x_0+1-\mu)^2 + y_0^2]^{-\frac{3}{2}} \tag{7.29}$$

is a constant. Thus, the out-of plane motion is decoupled from that occurring within the plane of the primaries. The general solution to Eq. (7.28) is written as

$$\delta z = \delta z_0 e^{\sqrt{-c}t} \,, \tag{7.30}$$

which represents a stable motion (constant amplitude oscillation) as long as $c \geq 0$, which is always satisfied. Hence, the out-of-plane small disturbance motion is unconditionally stable and can be safely ignored in a further analysis.[1]

For the small displacement, coplanar motion about a triangular Lagrangian point, say L_4, we can write

$$\begin{aligned} \delta\ddot{x} - 2\delta\dot{y} - \frac{3}{4}\delta x - \frac{3\sqrt{3}(\mu-\frac{1}{2})}{2}\delta y &= 0, \\ \delta\ddot{y} + 2\delta\dot{x} - \frac{3\sqrt{3}(\mu-\frac{1}{2})}{2}\delta x - \frac{9}{4}\delta y &= 0, \end{aligned} \tag{7.31}$$

whose general solution (Chapter 14) to initial displacement $\delta x_0, \delta y_0$ can be written as

$$\delta\mathbf{r} = \delta\mathbf{r_0} e^{\lambda t} \,, \tag{7.32}$$

where $\delta\mathbf{r} = (\delta x, \delta y)^T$, $\delta\mathbf{r_0} = (\delta x_0, \delta y_0)^T$, and λ is the *eigenvalue* [4] of the following matrix:

$$\mathsf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4} & \frac{3\sqrt{3}(\mu-\frac{1}{2})}{2} & 0 & 2 \\ \frac{3\sqrt{3}(\mu-\frac{1}{2})}{2} & \frac{9}{4} & -2 & 0 \end{pmatrix} . \tag{7.33}$$

The resulting characteristic equation for λ is thus the following:

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0 \,, \tag{7.34}$$

whose roots are

$$\lambda^2 = \frac{1}{2}[\pm\sqrt{1-27\mu(1-\mu)} - 1] \,. \tag{7.35}$$

[1] Battin [11] presents the general solution to the large displacement, out-of-plane, rectilinear motion about the center of mass ($x_0 = y_0 = 0$).

For stability, the values of λ must be purely imaginary, representing a harmonic oscillation about the equilibrium point. If the quantity in the square-root is negative, there is at least one value of λ with a positive real part (unstable system). Therefore, the critical values of μ representing the boundary between stable and unstable behavior of L_4 are those that correspond to

0<1-27mu(1-mu)<1
É preciso olhar a parte real das soluções complexas

$$1 - 27\mu(1-\mu) = 0 \,. \tag{7.36}$$

Hence, for the stability of the triangular Lagrangian points, we require either $\mu \leq 0.0385209$ or $\mu \geq 0.9614791$, which is always satisfied in the solar system. Therefore, we expect that the triangular Lagrangian points would provide stable locations for smaller bodies in the solar system. The existence of such bodies for the sun–Jupiter system has been verified in the form of *Trojan asteroids*. For the earth–moon system, $\mu = 0.01215$ is well within the stable region, hence one could expect the triangular points to be populated. However, the earth–moon system is not a good example of the restricted three-body problem because of an appreciable influence of the sun's gravity, which renders the triangular points of earth–moon system unstable. Hence, L_4, L_5 for the earth–moon system are empty regions in space, where one cannot park a spacecraft.

For a given system, one can determine the actual small displacement dynamics about the stable triangular Lagrangian points in terms of the eigenvalues, λ. For example, the earth–moon system has nondimensional natural frequencies 0.2982 and 0.9545, which indicate *long-period* and *short-period* modes of the small displacement motion. However, it is still left for us to investigate how much energy expenditure is necessary to reach the stable triangular points.

For the stability of the colinear Lagrangian points, we employ $y_0 = 0$, and the resulting equations of small displacement can be written as follows:

$$\begin{aligned}
\delta\ddot{x} - 2\delta\dot{y} - \left[\frac{2(1-\mu)}{(x_0-\mu)^3} + \frac{2\mu}{(x_0+1-\mu)^3} + 1\right]\delta x &= 0, \\
\delta\ddot{y} + 2\delta\dot{x} + \left[\frac{1-\mu}{(x_0-\mu)^3} + \frac{\mu}{(x_0+1-\mu)^3} - 1\right]\delta y &= 0 \,.
\end{aligned} \tag{7.37}$$

The eigenvalues of this system are the eigenvalues of the following square matrix:

$$\mathsf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 2 \\ 0 & -\beta & -2 & 0 \end{pmatrix} , \tag{7.38}$$

where

$$\begin{aligned}
\alpha &= \frac{2(1-\mu)}{(x_0-\mu)^3} + \frac{2\mu}{(x_0+1-\mu)^3} + 1, \\
\beta &= \frac{1-\mu}{(x_0-\mu)^3} + \frac{\mu}{(x_0+1-\mu)^3} - 1 \,.
\end{aligned} \tag{7.39}$$

The resulting characteristic equation for the eigenvalues, λ, is the following:

$$\lambda^4 + (4 + \beta - \alpha)\lambda^2 - \alpha\beta = 0 \; . \tag{7.40}$$

For the colinear Lagrangian points, $\alpha > 0$ and $\beta < 0$, which implies that the constant term in the characteristic equation is always positive. Therefore, there is always at least one eigenvalue with a positive real part, hence the system is unconditionally unstable. Thus, the colinear Lagrangian points of any restricted three-body system are always unstable. However, a spacecraft can successfully orbit a colinear point with small energy expenditure. Such an orbit around a Lagrangian point is termed a *halo orbit*, and several spacecraft have been designed to take advantage of halo orbits around the sun–earth L_1 and L_2 points for monitoring the interplanetary zone ahead of and behind the earth. For instance, a spacecraft near the sun–earth L_1 point can provide useful data about an approaching solar wind, and another about L_2 can explore the extent of the earth's magnetic field. Examples of spacecraft in halo orbits include the *International Sun-Earth Explorer (ISEE-3)*, the *Microwave Anisotropy Probe (MAP)*, the *Advanced Composition Explorer (ACE)*, and the *GEOTAIL* missions of NASA.

7.4.2 Jacobi's Integral

Jacobi derived the only additional scalar constant of the restricted three-body motion (out of the six necessary for a complete, closed-form solution), called *Jacobi's integral*. He defined a scalar function, J, by

$$J \doteq \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \; . \tag{7.41}$$

The equation of motion of the infinitesimal mass can then be expressed as follows in the frame rotating with angular velocity $\boldsymbol{\omega} = \mathbf{k}$:

$$\ddot{\mathbf{r}} + \boldsymbol{\omega} \times \dot{\mathbf{r}} = \frac{\partial J}{\partial \mathbf{r}}^T \; , \tag{7.42}$$

where $\dot{\mathbf{r}} \doteq \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$ is the velocity of the mass relative to the rotating frame, and $(\partial J/\partial \mathbf{r})^T$ represents the gradient of J with respect to $\mathbf{r}$ (Chapter 3). By taking the scalar product of Eq. (7.42) with $\dot{\mathbf{r}}$, we have

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}\dot{r}^2}{\mathrm{d}t} = \frac{\partial J}{\partial \mathbf{r}}^T \cdot \dot{\mathbf{r}} = \frac{\mathrm{d}J}{\mathrm{d}t} \; , \tag{7.43}$$

which is an exact differential and can be integrated to obtain

$$C = \frac{1}{2}v^2 - \frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \; . \tag{7.44}$$

Here, C is the constant of integration, called *Jacobi's constant*, and $v \doteq \dot{r}$ is the relative speed of the mass in the rotating frame. The Jacobi constant can thus be thought to represent a pseudo-energy of the mass m_3, which is a sum of the relative kinetic energy, the gravitational potential energy, and an additional pseudo-potential energy, $-1/2(x^2+y^2)$. The value of C at any point is a measure of its relative energy. The most useful interpretation of Jacobi's integral lies in the contours of zero relative speed, $v = 0$, which represent the boundary between a motion toward higher potential, or the return trajectory to the lower potential. On such a boundary, the mass must stop and turn back. Hence, the zero relative speed curves demarcate the regions accessible to a spacecraft and cannot be crossed.

Example 7.2. Plot the zero relative speed contours of constant C for $-1.5 \leq C \leq -2.5$ in the neighborhood of the Lagrangian points L_1, L_2, and L_4 for the earth–moon system with $\mu = 0.01215$. Analyze the energy requirements for reaching the given Lagrangian points.

The contours are plotted in Figs. 7.2 and 7.3, with the use of MATLAB statements such as

```
>> m=0.01215; %mu for earth-moon system
>> x=-1:0.02:1;y=0.5:0.005:1;
>> for i=1:size(x,2);for j=1:size(y,2);X(i,:)=x(i);
Y(:,j)=y(j);r1=sqrt((x(i)-m)^2+y(j)^2);
r2=sqrt((x(i)+1-m)^2+y(j)^2);
C(i,j)=-0.5*(x(i)^2+y(j)^2)-(1-m)/r1-m/r2;end;end
>> surfc(Y,X,C) %surface contours of constant C in nbd. of L4
```

The contours of C are seen in Fig. 7.2 to enclose a small spherical space in the vicinity of m_2, and a large cylindrical region around the origin with axis perpendicular to the orbital plane. We expect a similar spherical contour around the other primary, m_1. The closed zero-speed contours around a primary indicate that a flight from one primary to the other is impossible. As the value of C becomes less negative, the contours expand in size and may touch one another for some special values of C. For example, the zero-speed contours for $C = -1.594$ [which corresponds to $C(L_2)$] touch at the Lagrangian point L_2, thereby indicating the possibility of crossing over from one primary to the other through L_2 in a closed flight path, called *free-return trajectory*. At a larger value of C corresponding to $C(L_1)$, the region around L_2 opens up, indicating flight at nonzero relative velocity between the primaries, and the contours touch at L_1. For still larger C, corresponding to $C(L_3)$, the zero speed contours touch at L_3, while the region around L_1 is opened up, indicating an escape from the gravitational influence of the primaries. In such a case, contours enclose the triangular Lagrangian points, as depicted in Fig. 7.3, indicating that a flight to reach them is impossible from any of the primaries. As the value of C becomes very large, the forbidden regions around L_4, L_5 shrink, but never actually vanish, thereby denoting that the stable triangular Lagrangian points can be approached (but never reached) only with a very high initial energy.

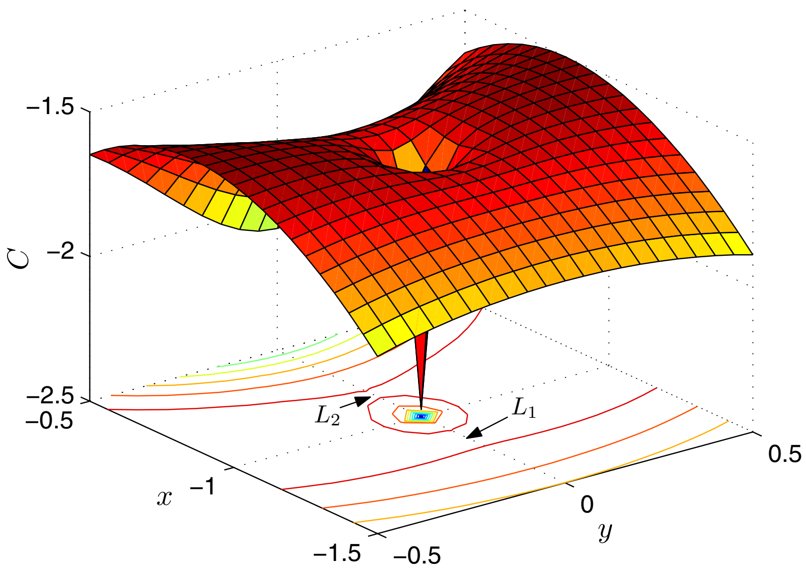

Fig. 7.2. Zero-speed contours of Jacobi's constant in the neighborhood of earth–moon L_1 and L_2.

7.4.3 Numerical Solution of the Restricted Problem

The restricted problem of three bodies appears more amenable to a solution than the general problem, due to the availability of an additional integral of motion, namely Jacobi's integral. However, the restricted three-body problem is unsolvable in a closed form because it does not possess the adequate number of scalar constants. With Jacobi's integral, the total number of scalar constants is only 13 whereas 18 are required for a closed-form solution. The unsolvability of the restricted three-body problem was also demonstrated in the 19th century by *Poincaré* using phase-space surfaces. The problem is also a member of a select group of mechanical systems, called *chaotic systems*, where a small change in the initial condition results in an arbitrarily large change in the response. Such systems are quite difficult to model, and their behavior is studied by a special branch of physics.

Numerical solutions to the problem have become possible since the availability of digital computers. When searching for a numerical solution, one desires a periodic behavior, which allows numerical integration of the equations of motion for only one time period, and a repetitive trajectory thereafter. Such solutions are encountered in the two-body orbits. However, in the restricted three-body problem, such periodic orbits are rarely encountered, and generally we have solutions changing forever with time. Numerical integration

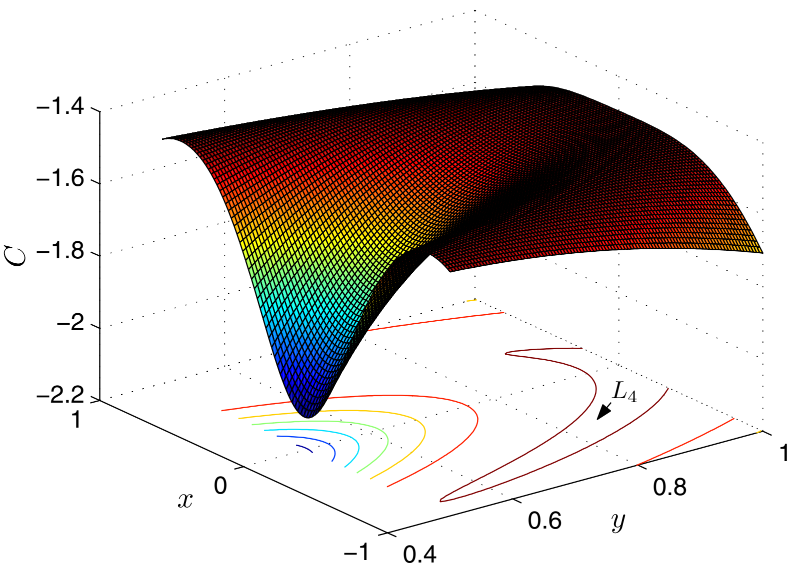

Fig. 7.3. Zero-speed contours of Jacobi's constant in the neighborhood of earth–moon L_4.

for such aperiodic solutions is frought with inaccuracies, due to truncation errors (Appendix A) that accumulate over time, and may grow to unacceptable magnitudes within a few orbital time periods of the primaries. Therefore, an accurate time-integration scheme, such as a higher-order Runge–Kutta technique (Appendix A), is employed with close tolerances to achieve reasonably accurate solutions.

Example 7.3. Simulate the trajectory of a spacecraft passing through the point $(0.1, 0)$ in the earth–moon system, with the following relative velocity components:

(a) $\dot{x} = 0,\ \dot{y} = 0.5,$
(b) $\dot{x} = -4,\ \dot{y} = 1,$
(c) $\dot{x} = -3.35,\ \dot{y} = 3,$
(d) $\dot{x} = -3.37,\ \dot{y} = 3,$
(e) $\dot{x} = -3.4,\ \dot{y} = 3,$
(f) $\dot{x} = -3.5,\ \dot{y} = 3,$
(g) $\dot{x} = -3.6,\ \dot{y} = 3.$

We begin by writing a MATLAB program for the restricted three-body equations of motion, called *res3body.m*, tabulated in Table 7.1. This code pro-

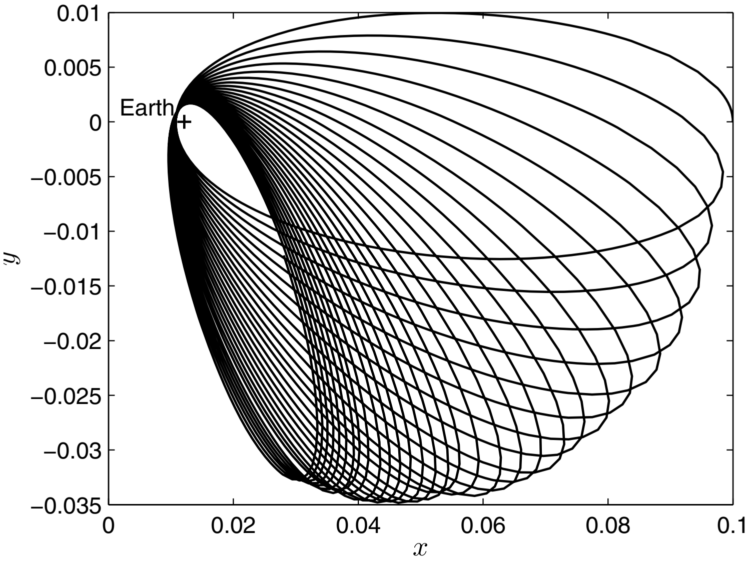

Fig. 7.4. Spacecraft trajectory with initial conditions $\dot{x}(0) = 0, \dot{y}(0) = 0.5$ [case (a)].

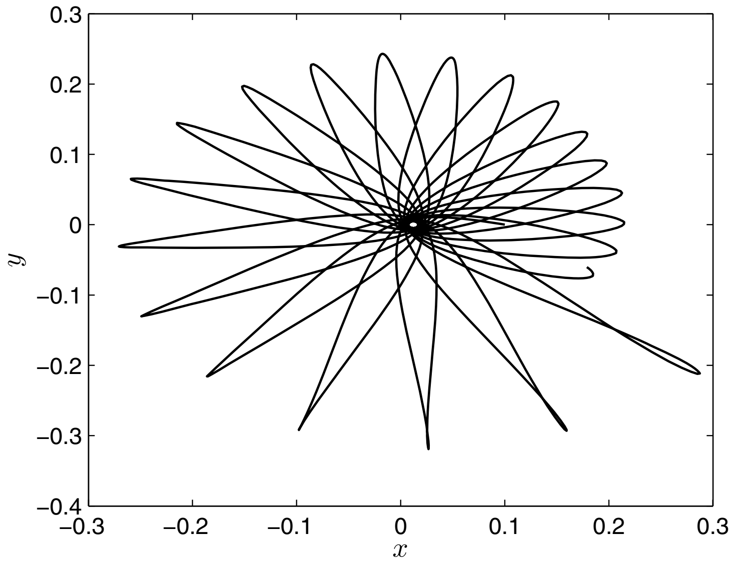

Fig. 7.5. Spacecraft trajectory with $\dot{x}(0) = -4, \dot{y}(0) = 1$ [case (b)].

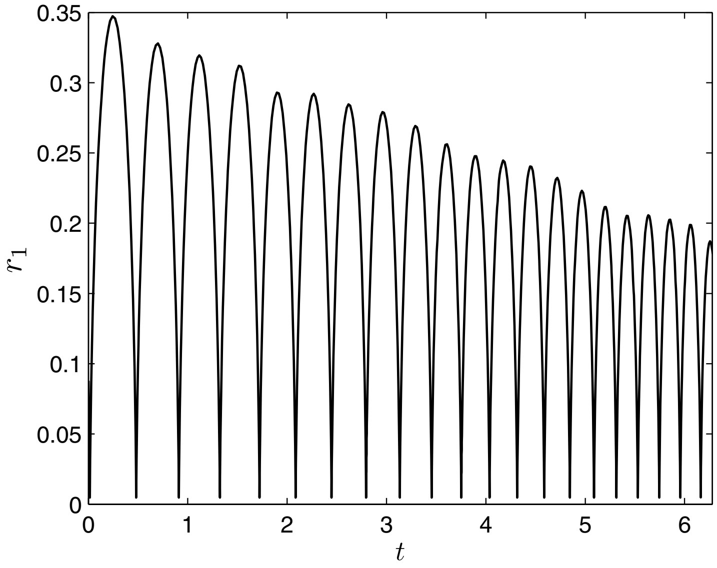

Fig. 7.6. Time variation of the orbital radius of spacecraft with $\dot{x}(0) = -4, \dot{y}(0) = 1$ [case (b)].

vides the nonlinear differential equations to the intrinsic Runge–Kutta solver, *ode45.m*. The trajectory for case (a) is simulated for $t = 1$ (one complete revolution of the primaries in the nondimensional time is $t = 2\pi$, which is about 30 days) and is plotted in Fig. 7.4. The orbits about the earth $(\mu, 0)$, with both apsidal and nodal rotation of the orbital plane caused by moon's gravity, are evident. As the initial velocity is increased in case (b), the orbits around the earth change into more energetic, highly eccentric trajectories, but the vehicle is still unable to cross the zero velocity contour of C for a lunar journey (Fig. 7.5). The time history of orbital radius, r_1, over one complete revolution of the primaries ($t = 2\pi$) in case (b) is shown in Fig. 7.6, indicating a decay of the orbit with time due to the moon's gravitation.

Figure 7.7 shows three free-return trajectories around the moon, arising out of cases (c)–(e). These trajectories can be employed for lunar exploration, without fuel expenditure for the return journey.[2] In case (c), the initial velocity is sufficiently large for a free-return trajectory around the moon and back to the starting point in about $t = 3.57$. In this case, the spacecraft passes slightly below the moon's orbit around the earth, but beyond L_2. The total flight time is reduced significantly in case (d) to about $t = 2.8$, when the space-

[2] The free-return concept formed the basis of Jules Verne's famous novel *From the Earth to the Moon* (1866), which was ultimately realized about a century later by the manned lunar exploration through the *Apollo* program.

craft passes around the moon, between L_1 and L_2. In the process, the return trajectory has a slightly larger kinetic energy due to the boost provided by the lunar *swing-by*. Lunar swing-by trajectories have been employed in cheaply boosting several spacecraft to the sun–earth Lagrangian points, such as the *ISEE-3, MAP*, and *ACE* missions of NASA [12]. In case (e), the round-trip flight time increases to about $t = 3.45$, as the spacecraft swings-by the moon at a greater distance, passing beyond L_1. For cases (f) and (g), plotted in Fig. 7.8, qualitatively different trajectories are observed. Case (f) is simulated for a long time ($t = 35$), demonstrating that the spacecraft makes a first pass of the moon at a large distance from L_1, but is unable escape the earth's gravity, which brings it closer to the moon in the next pass—crossing the earth–moon line near L_2—and ultimately brings the spacecraft into earth's orbit of ever-decreasing radius [somewhat similar to case (b)]. Case (f) illustrates a novel method of cheaply launching a satellite into the geosynchronous orbit ($r_1 \approx 0.1$) with the use of multiple lunar swing-bys. A similar approach of multiple planetary swing-bys has been found useful in reducing the mission cost of spacecraft bound for distant planets. As the initial energy is increased in case (g), the spacecraft does not return to earth but embarks on an escape trajectory from the earth–moon system, as depicted in Fig. 7.8. In such a trajectory, advantage is derived of gravity assist from the moon, thereby reducing the fuel expenditure, and hence, total cost of an interplanetary mission.

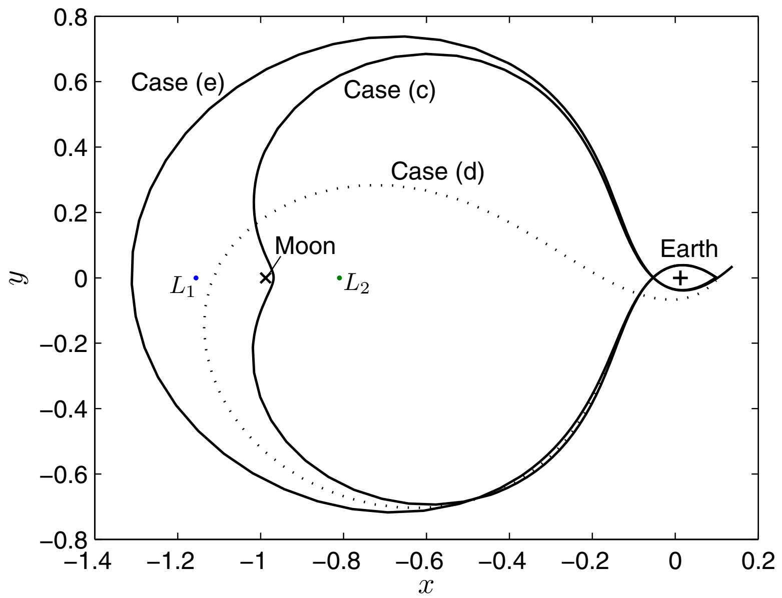

Fig. 7.7. Free lunar return trajectories [cases (c)–(e)].

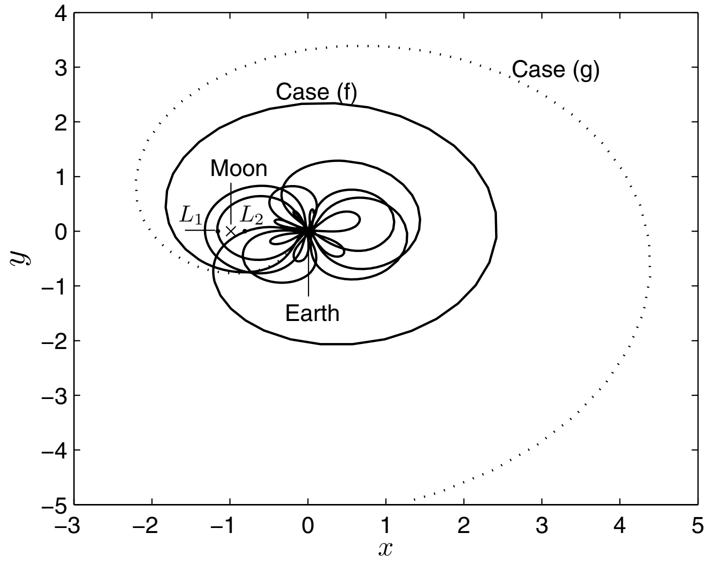

Fig. 7.8. Multiple lunar swing-by and escape trajectories [cases (f) and (g)].

Table 7.1. M-file *res3body.m* for Equations of Motion of the Restricted Three-Body Problem

```
function Xdot=res3body(t,X)
%Equations of motion for the restricted three-body problem.
%(c) 2006 Ashish Tewari
m=0.01215; %nondimensional mass of second primary
x=X(1); %x-coordinate
y=X(2); %y-coordinate
r1=sqrt((x-m)^2+y^2); %distance from first primary
r2=sqrt((x+1-m)^2+y^2); %distance from second primary
%Equations of motion:
Xdot(1,1)=X(3);
Xdot(2,1)=X(4);
Xdot(3,1)=x+2*Xdot(2,1)-(1-m)*(x-m)/r1^3-m*(x+1-m)/r2^3;
Xdot(4,1)=y-2*Xdot(1,1)-(1-m)*y/r1^3-m*y/r2^3;
```

The general mission design in a restricted three-body transfer involves the solution of a two-point boundary value problem, which requires special analytical and numerical techniques, similar to Lambert's guidance problem for two-body orbits (Chapter 5). These approaches range from the analytical search for particular periodic solutions passing through the given points [13], to the numerical patching of two-body Lambert solutions [14]. Such approximate solutions are indispensible and have served well in practical mission designs, keeping in mind the impossibility of solving the restricted three-body problem in a closed form (or even an approximate analytical form), thereby excluding the two-point boundary-value solutions of the actual problem.

7.5 Summary

The three-body problem arises when we are interested in either the two-body orbital perturbations caused by a distant third body or the motion of a smaller body in the gravitational field formed by two larger bodies. The physical conservation laws yield only 10 scalar constants of the three-body problem, whereas 18 are required for a complete analytical solution. Lagrange obtained particular solutions of the problem for a coplanar motion of the three bodies, including the colinear and equilateral triangle equilibrium point solutions with reference to a rotating frame. When the mass of one of the bodies is negligible compared to that of the other two bodies, the restricted three-body problem arises. Although the restricted problem is also unsolvable in a closed form, the stability of its equilibrium points—called Lagrangian points—can be analyzed using small disturbance approximation. While the colinear Lagrangian points of any restricted three-body system are always unstable, the equilateral points are conditionally stable (and the condition is always satisfied in the solar system). There exists an additional scalar constant of the restricted three-body motion, called *Jacobi's integral*, the zero relative speed contours of which provide an analysis of the accessible regions for the smallest body (spacecraft). Careful numerical integration of the restricted three-body equations of motion yields interesting trajectories and can be utilized to design and analyze lunar and interplantery missions.

Exercises

7.1. A spacecraft is located 6600 km away from the earth's center. What should be the minimum relative velocity of the spacecraft at this point in order to reach a triangular Lagrangian point of the earth–moon system?

7.2. Determine the locations of the Lagrangian points for the sun–Jupiter system with $\mu = 0.00095369$.

7.3. For the sun–Jupiter system, determine the nondimensional frequencies of small displacement motion of the *Trojan* asteroids about the triangular Lagrangian point.

7.4. Plot the zero relative speed contours of constant C for $-1.5 \leq C \leq -2.5$ in the neighborhood of the Lagrangian points L_1, L_2, and L_4 for the sun–Jupiter system.