in fact four possible switching patterns

$$\{+U_1,\ +U_2,\ +U_3\}$$
$$\{-U_1,\ -U_2,\ +U_3\}$$
$$\{-U_2,\ +U_1,\ +U_3\}$$
$$\{+U_2,\ -U_1,\ +U_3\}$$

Hint: Notice that the selection of the first and second axes for this problem is not unique.

(c) For the special case of $\theta = 180$ deg, show that there are eight possible switching patterns, i.e., the four switching patterns described in Problem 7.11(b) plus $\{+U_1, -U_2, -U_3\}$, $\{-U_1, +U_2, -U_3\}$, $\{+U_2, +U_1, -U_3\}$, and $\{-U_2, -U_1, -U_3\}$.

7.12 Consider again the time-optimal reorientation problem of an inertially symmetric rigid body described by Eqs. (7.59), (7.60), and (7.62), but with the following control input constraints:

$$\|\mathbf{u}\|_{2\zeta} = \left\{|u_1|^{2\zeta} + |u_2|^{2\zeta} + |u_3|^{2\zeta}\right\}^{\frac{1}{2\zeta}} = 1$$

Note that with $\zeta = 1$, the control constraint surface is a sphere and with $\zeta \to \infty$ the surface approaches a cube. Minimizing the Hamiltonian subject to the preceding constraint, obtain the optimal control inputs as

$$u_i = \frac{-\mathrm{sgn}\{\lambda_{\omega_i}(t)\}\left|\lambda_{\omega_i}(t)\right|^{\frac{1}{2\zeta-1}}}{\left\{\sum_{j=1}^{3}\left|\lambda_{\omega_j}\right|^{\frac{2\zeta}{2\zeta-1}}\right\}^{\frac{1}{2\zeta}}}, \qquad i = 1, 2, 3$$

Hint: See Bilimoria and Wie [6,7].

7.3 Quaternion-Feedback Reorientation Maneuvers

Most three-axis stabilized spacecraft utilize a sequence of rotational maneuvers about each control axis. Many spacecraft also perform rotational maneuvers about an inertially fixed axis during an acquisition mode, e.g., sun acquisition or Earth acquisition, so that a particular sensor will pick up a particular target.

Spacecraft are sometimes required to maneuver as fast as possible within the physical limits of actuators and sensors. The X-ray Timing Explorer (XTE) spacecraft launched in 1996 is one of such spacecraft, and it is required to maneuver about an inertially fixed axis as fast as possible within the saturation limit of rate gyros [14]. The XTE spacecraft is controlled by a set of skewed reaction wheels; thus, the maximum available control torque also needs to be considered in such a near-minimum-time eigenaxis maneuver. As was studied in the preceding section, however, the eigenaxis rotation is, in general, not time optimal.

In this section we introduce a feedback control logic for three-axis, large-angle reorientation maneuvers, and we further extend such a simple feedback control logic to a case in which the spacecraft is required to maneuver about an inertially fixed axis as fast as possible within the saturation limits of rate gyros as well as reaction wheels. (This section is based on [15–17].)

7.3.1 Quaternion Feedback Control

Consider the attitude dynamics of a rigid spacecraft described by Euler's rotational equation of motion

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{u} \tag{7.65}$$

where $\mathbf{J}$ is the inertia matrix, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the angular velocity vector, and $\mathbf{u} = (u_1, u_2, u_3)$ the control torque input vector. The cross product of two vectors is represented in matrix notation as

$$\boldsymbol{\omega} \times \mathbf{h} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

where $\mathbf{h} = \mathbf{J}\boldsymbol{\omega}$ is the angular momentum vector. It is assumed that the angular velocity vector components ω_i along the body-fixed control axes are measured by rate gyros.

Euler's rotational theorem states that the rigid-body attitude can be changed from any given orientation to any other orientation by rotating the body about an axis, called the Euler axis, that is fixed to the rigid body and stationary in inertial space. Such a rigid-body rotation about an Euler axis is often called the eigenaxis rotation.

Let a unit vector along the Euler axis be denoted by $\mathbf{e} = (e_1, e_2, e_3)$ where e_1, e_2, and e_3 are the direction cosines of the Euler axis relative to either an inertial reference frame or the body-fixed control axes. The four elements of quaternions are then defined as

$$\begin{aligned} q_1 &= e_1 \sin(\theta/2) \\ q_2 &= e_2 \sin(\theta/2) \\ q_3 &= e_3 \sin(\theta/2) \\ q_4 &= \cos(\theta/2) \end{aligned}$$

where θ denotes the rotation angle about the Euler axis, and we have

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

The quaternion kinematic differential equations are given by

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{7.66}$$

Like the Euler-axis vector $\mathbf{e} = (e_1,\ e_2,\ e_3)$, defining a quaternion vector $\mathbf{q} = (q_1, q_2, q_3)$ as

$$\mathbf{q} = \mathbf{e} \sin \frac{\theta}{2}$$

we rewrite Eq. (7.66) as

$$2\dot{\mathbf{q}} = q_4 \boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{q} \tag{7.67a}$$

$$2\dot{q}_4 = -\boldsymbol{\omega}^T \mathbf{q} \tag{7.67b}$$

where

$$\boldsymbol{\omega} \times \mathbf{q} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Because quaternions are well suited for onboard real-time computation, spacecraft orientation is nowadays commonly described in terms of the quaternions, and a linear state feedback controller of the following form can be considered for real-time implementation:

$$\mathbf{u} = -\mathbf{K}\mathbf{q}_e - \mathbf{C}\boldsymbol{\omega} \tag{7.68}$$

where $\mathbf{q}_e = (q_{1e}, q_{2e}, q_{3e})$ is the attitude error quaternion vector and $\mathbf{K}$ and $\mathbf{C}$ are controller gain matrices to be properly determined. The attitude error quaternions $(q_{1e}, q_{2e}, q_{3e}, q_{4e})$ are computed using the desired or commanded attitude quaternions $(q_{1c}, q_{2c}, q_{3c}, q_{4c})$ and the current attitude quaternions (q_1, q_2, q_3, q_4), as follows:

$$\begin{bmatrix} q_{1e} \\ q_{2e} \\ q_{3e} \\ q_{4e} \end{bmatrix} = \begin{bmatrix} q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\ -q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\ q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\ q_{1c} & q_{2c} & q_{3c} & q_{4c} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{7.69}$$

If the commanded attitude quaternion vector is simply the origin defined as

$$(q_{1c}, q_{2c}, q_{3c}, q_{4c}) = (0, 0, 0, +1)$$

then the control logic (7.68) becomes

$$\mathbf{u} = -\mathbf{K}\mathbf{q} - \mathbf{C}\boldsymbol{\omega} \tag{7.70}$$

On the other hand, if the origin is chosen as (0, 0, 0, -1), then the control logic (7.68) becomes

$$\mathbf{u} = +\mathbf{K}\mathbf{q} - \mathbf{C}\boldsymbol{\omega} \tag{7.71}$$

Note, however, that both quaternions (0, 0, 0, $+1$) and (0, 0, 0, -1) correspond to the physically identical orientation.

Without loss of generality, we consider here the control logic of the form (7.70). As shown by Wie and Barba [15] and Wie et al., [16] the origin, either $(0, 0, 0, +1)$ or $(0, 0, 0, -1)$, of the closed-loop nonlinear systems of a rigid spacecraft with such control logic is globally asymptotically stable for the following gain selections.

Controller 1:

$$\mathbf{K} = k\mathbf{I}, \qquad \mathbf{C} = \operatorname{diag}(c_1, c_2, c_3) \tag{7.72a}$$

Controller 2:

$$\mathbf{K} = \frac{k}{q_4^3}\mathbf{I}, \qquad \mathbf{C} = \operatorname{diag}(c_1, c_2, c_3) \tag{7.72b}$$

Controller 3:

$$\mathbf{K} = k \operatorname{sgn}(q_4)\mathbf{I}, \qquad \mathbf{C} = \operatorname{diag}(c_1, c_2, c_3) \tag{7.72c}$$

Controller 4:

$$\mathbf{K} = [\alpha\mathbf{J} + \beta\mathbf{I}]^{-1}, \qquad \mathbf{K}^{-1}\mathbf{C} > 0 \tag{7.72d}$$

where k and c_i are positive scalar constants, $\mathbf{I}$ is a 3×3 identity matrix, $\operatorname{sgn}(\cdot)$ denotes the signum function, and α and β are nonnegative scalars.

Note that controller 1 is a special case of controller 4 with $\alpha = 0$, and that β can also be simply selected as zero when $\alpha \neq 0$. Controllers 2 and 3 approach the origin, either $(0, 0, 0, +1)$ or $(0, 0, 0, -1)$, by taking a shorter angular path.

Problems

7.13 Consider the rotational equations of motion of a rigid spacecraft about principal axes described by

$$\begin{aligned}
2\dot{q}_1 &= \omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2 \\
2\dot{q}_2 &= \omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1 \\
2\dot{q}_3 &= -\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4 \\
2\dot{q}_4 &= -\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3 \\
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}$$

The stability of the origin defined as

$$\begin{aligned}
\mathbf{x}^* &= (q_1, q_2, q_3, q_4, \omega_1, \omega_2, \omega_3) \\
&= (0,\ 0,\ 0,\ +1,\ 0,\ 0,\ 0)
\end{aligned}$$

is to be studied for the control torque inputs u_i of the form

$$
\begin{aligned}
u_1 &= -k_1 q_1 - c_1 \omega_1 \\
u_2 &= -k_2 q_2 - c_2 \omega_2 \\
u_3 &= -k_3 q_3 - c_3 \omega_3
\end{aligned}
$$

where k_i and c_i are positive constants.

(a) Show that for any positive constants c_i, the equilibrium point $\mathbf{x}^*$ is globally asymptotically stable if k_i are selected such that

$$\frac{J_2 - J_3}{k_1} + \frac{J_3 - J_1}{k_2} + \frac{J_1 - J_2}{k_3} = 0$$

Hint: Choose the following positive-definite function

$$E = \frac{J_1 \omega_1^2}{2k_1} + \frac{J_2 \omega_2^2}{2k_2} + \frac{J_3 \omega_3^2}{2k_3} + q_1^2 + q_2^2 + q_3^2 + (q_4 - 1)^2$$

as a Lyapunov function.

(b) Also determine whether or not the equilibrium point $\mathbf{x}^*$ is globally asymptotically stable for any positive values of k_i and c_i. (This is a much harder unsolved problem.)

7.14 Consider a rigid spacecraft with

$$\mathbf{J} = \begin{bmatrix} 1200 & 100 & -200 \\ 100 & 2200 & 300 \\ -200 & 300 & 3100 \end{bmatrix} \quad \mathrm{kg \cdot m^2}$$

It is assumed that $(q_1, q_2, q_3, q_4) = (0.5,\ 0.5,\ 0.5,\ -0.5)$ and $(\omega_1, \omega_2, \omega_3) = (0,\ 0,\ 0)$ at $t = 0$ and that the spacecraft needs to be reoriented within approximately 500 s to the origin $(q_{1c}, q_{2c}, q_{3c}, q_{4c}) = (0,\ 0,\ 0,\ \pm 1)$. This given initial orientation corresponds to an eigenangle-to-go of 240 deg or 120 deg depending on the direction of reorientation.

Neglecting the products of inertia of the spacecraft, synthesize quaternion feedback control logic of the form

$$\mathbf{u} = -\mathbf{K}\mathbf{q} - \mathbf{C}\boldsymbol{\omega}$$

with the four different types of gain matrices given by Eqs. (7.72)

For each controller, perform computer simulation of the closed-loop system including the products of inertia. In particular, plot the time history of the eigenangle θ and also plot q_i vs q_j $(i, j = 1, 2, 3)$.