where (J_1, J_2, J_3) are the principal moments of inertia of the spacecraft including the spherical, dissipative fuel slug of inertia J; $(\omega_1, \omega_2, \omega_3)$ are the body rates about the principal axes; $(\sigma_1, \sigma_2, \sigma_3)$ are the relative rates between the rigid body and the fuel slug about the principal axes; and μ is the viscous damping coefficient of the fuel slug. It is assumed that $J_1 > J_2 > J_3$ without loss of generality.

(a) Show that a necessary condition for the equilibrium points is $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0$, i.e., $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

(b) Show that an equilibrium point $(\omega_1, \omega_2, \omega_3, \sigma_1, \sigma_2, \sigma_3) = (\Omega, 0, 0, 0, 0, 0)$ is stable; i.e., a pure spinning motion about the major axis is stable.

(c) Show that an equilibrium point $(0, 0, \Omega, 0, 0, 0)$ is unstable; i.e., a pure spinning motion about the minor axis is unstable, whereas it is Lyapunov stable for a rigid body without energy dissipation.

(d) Show that an equilibrium point $(0, \Omega, 0, 0, 0, 0)$ is also unstable.

(e) Consider a spacecraft with the following numerical values: $(J_1, J_2, J_3, J) = (2000, 1500, 1000, 18)$ kg·m^2 and $\mu = 30$ N·m·s. Performing computer simulation, verify that the trajectory starting from an initial condition (0.1224, 0, 2.99, 0, 0, 0) rad/s ends up at (−1.5, 0, 0, 0, 0, 0) rad/s.
Note: The kinetic energy T and the angular momentum H of the system are

$$
\begin{aligned}
H^2 &= (J_1\omega_1 + J\sigma_1)^2 + (J_2\omega_2 + J\sigma_2)^2 + (J_3\omega_3 + J\sigma_3)^2 \\
2T &= (J_1 - J)\omega_1^2 + (J_2 - J)\omega_2^2 + (J_3 - J)\omega_3^2 \\
&\quad + J\left\{(\omega_1 + \sigma_1)^2 + (\omega_2 + \sigma_2)^2 + (\omega_3 + \sigma_3)^2\right\}
\end{aligned}
$$

During computer simulation of this case, the angular momentum H needs to be checked to see whether or not it is maintained at a constant value of 3000 N·m·s.

(f) Also perform computer simulation with a slightly different initial condition (0.125, 0, 2.99, 0, 0, 0) and verify that the trajectory ends up at (+1.5, 0, 0, 0, 0, 0).
Note: For such a spinning spacecraft with energy dissipation, a small change in initial conditions can lead to a change in the final spin polarity for ω_1. Such sensitive dependence on initial conditions is the property characterizing a chaotic dynamic system.

6.8 Spinning Axisymmetric Body with Constant Body-Fixed Torque

A simple solution to the problem of maintaining a desired orientation of a space vehicle during thrusting maneuvers is to spin the vehicle in the fashion of a football or a spinning rocket about its longitudinal axis. A thrust vector misalignment with the longitudinal axis will cause the vehicle to tumble in the absence of spinning. A spinning axisymmetric body possesses a gyroscopic stiffness to external disturbances, however, and its motion under the influence of disturbances is characterized by the precession and nutation of the longitudinal axis about the desired direction of the longitudinal axis.

Consider an axisymmetric rigid body possessing a body-fixed reference frame B with basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and with its origin at the center of mass. The reference frame B coincides with principal axes. It is assumed that the first and second axes are the transverse axes and that the third axis is the axis of symmetry. A longitudinal thrust vector is nominally aligned along $\vec{b}_3$ through the center of mass of the spacecraft.

Euler's rotational equations of motion of an axisymmetric spacecraft with $J_1 = J_2 = J$ are

$$J\dot{\omega}_1 - (J - J_3)\omega_3\omega_2 = M_1 \tag{6.90}$$

$$J\dot{\omega}_2 + (J - J_3)\omega_3\omega_1 = 0 \tag{6.91}$$

$$J_3\dot{\omega}_3 = 0 \tag{6.92}$$

where $\omega_i \equiv \vec{b}_i \cdot \vec{\omega}$ are the body-fixed components of the angular velocity of the spacecraft in an inertial reference frame and M_1 is the transverse torque component due to misalignment of the thrust vector.

From Eq. (6.92), we have

$$\omega_3 = \text{const} = n \tag{6.93}$$

where the constant n is called the spin rate of the spacecraft about its symmetry axis $\vec{b}_3$. Equations (6.90) and (6.91) then become

$$\dot{\omega}_1 = \lambda\omega_2 + \mu \tag{6.94}$$

$$\dot{\omega}_2 = -\lambda\omega_1 \tag{6.95}$$

where

$$\lambda = \frac{(J - J_3)n}{J}$$

and $\mu \equiv M_1/J$ denotes the constant disturbance acceleration resulting from misalignment of the thrust vector. Note that λ can be either positive or negative depending on whether the third axis is the minor or major axis.

To describe the rotational motion of the spinning spacecraft as seen from an inertial reference frame, we consider the body-fixed rotational sequence of $\mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_1(\theta_1)$. For this rotational sequence, we have the following kinematic differential equations:

$$\dot{\theta}_1 = (\omega_1 \cos\theta_3 - \omega_2 \sin\theta_3)/\cos\theta_2 \tag{6.96a}$$

$$\dot{\theta}_2 = \omega_1 \sin\theta_3 + \omega_2 \cos\theta_3 \tag{6.96b}$$

$$\dot{\theta}_3 = (-\omega_1 \cos\theta_3 + \omega_2 \sin\theta_3)\tan\theta_2 + \omega_3 \tag{6.96c}$$

For small θ_2, the kinematic differential equations become

$$\dot{\theta}_1 = \omega_1 \cos\theta_3 - \omega_2 \sin\theta_3 \tag{6.97a}$$

$$\dot{\theta}_2 = \omega_1 \sin\theta_3 + \omega_2 \cos\theta_3 \tag{6.97b}$$

$$\dot{\theta}_3 = -\theta_2\dot{\theta}_1 + \omega_3 \tag{6.97c}$$

Assuming $\theta_2\dot{\theta}_1 \ll \omega_3$, we can further approximate $\dot{\theta}_3$ as

$$\dot{\theta}_3 \approx \omega_3 = n = \text{const}$$

and $\theta_3 \approx nt$. Finally, we have a set of linearized equations of motion

$$\dot{\omega}_1 = \lambda\omega_2 + \mu \tag{6.98}$$

$$\dot{\omega}_2 = -\lambda\omega_1 \tag{6.99}$$

$$\dot{\theta}_1 = \omega_1 \cos nt - \omega_2 \sin nt \tag{6.100}$$

$$\dot{\theta}_2 = \omega_1 \sin nt + \omega_2 \cos nt \tag{6.101}$$

The solutions of Eqs. (6.98) and (6.99) for a constant μ can be found as

$$\omega_1(t) = \omega_1(0)\cos\lambda t + \omega_2(0)\sin\lambda t + (\mu/\lambda)\sin\lambda t \tag{6.102a}$$

$$\omega_2(t) = \omega_2(0)\cos\lambda t - \omega_1(0)\sin\lambda t - (\mu/\lambda)(1 - \cos\lambda t) \tag{6.102b}$$

For a case with $\omega_1(0) = \omega_2(0) = 0$, Eqs. (6.100) and (6.101) become

$$\dot{\theta}_1 = \frac{\mu}{\lambda}\left\{-\sin\frac{J_3}{J}nt + \sin nt\right\} \tag{6.103a}$$

$$\dot{\theta}_2 = \frac{\mu}{\lambda}\left\{\cos\frac{J_3}{J}nt - \cos nt\right\} \tag{6.103b}$$

because $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ and $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$. Integrating these equations with respect to time for the initial conditions $\theta_1(0) = \theta_2(0) = 0$, we obtain

$$\theta_1(t) = -A_p(1 - \cos\omega_p t) + A_n(1 - \cos\omega_n t) \tag{6.104a}$$

$$\theta_2(t) = A_p\sin\omega_p t - A_n\sin\omega_n t \tag{6.104b}$$

where

$$A_p = \frac{\mu J}{\lambda n J_3} = \text{precessional amplitude}$$

$$A_n = \frac{\mu}{\lambda n} = \text{nutational amplitude}$$

$$\omega_p = \frac{J_3 n}{J} = \text{precessional frequency}$$

$$\omega_n = n = \text{nutational frequency}$$

The path of the tip of the axis of symmetry in space is an *epicycloid* formed by a point on a circle of radius A_n rolling on the outside of a circle of radius A_p, centered at $\theta_1 = -A_p$ and $\theta_2 = 0$, when $J > J_3$.

Problem

6.9 Consider a spinning axisymmetric rocket with misaligned longitudinal thrust described by a set of differential equations of the form

$$\dot{\omega}_1 = \lambda\omega_2 + \mu$$
$$\dot{\omega}_2 = -\lambda\omega_1$$
$$\dot{\theta}_1 = (\omega_1 \cos\theta_3 - \omega_2 \sin\theta_3)/\cos\theta_2$$
$$\dot{\theta}_2 = \omega_1 \sin\theta_3 + \omega_2 \cos\theta_3$$
$$\dot{\theta}_3 = (-\omega_1 \cos\theta_3 + \omega_2 \sin\theta_3)\tan\theta_2 + n$$

in which the kinematic differential equations are not linearized yet.

(a) For the following parameter values and initial conditions:

$$J_3/J = 0.05, \qquad n = 15 \text{ rad/s}$$
$$\lambda = n(J - J_3)/J = 14.25 \text{ rad/s}$$
$$\mu = M_1/J = 0.1875 \text{ rad/s}^2$$
$$\omega_1(0) = \omega_2(0) = 0 \text{ rad/s}$$
$$\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$$

perform computer simulations of both the nonlinear and linear models. In particular, plot the paths of the tip of the axis of symmetry in the (θ_1, θ_2) plane. Compare the computer simulation results with the linear analysis results given by Eqs. (6.104).

(b) For the same parameter values and initial conditions as given in (a), but with $\omega_2(0) = 0.025$ rad/s, perform computer simulations of both the nonlinear and linear models, and compare the results in terms of the numerical values of A_p, A_n, ω_p, and ω_n.

Note: In Jarmolow [6] and Kolk [7], ω_n was modified as

$$\omega_n \approx n(J - J_3)/J$$

based on their computer simulation results. The discrepancy was attributed to the linearizing assumption: $\theta_2\dot{\theta}_1 \ll \omega_3$. On the contrary, however, the nonlinear and linear simulation results agree very well, as is verified in this problem.

6.9 Asymmetric Rigid Body with Constant Body-Fixed Torques

In the preceding section, we studied the problem of a spinning axisymmetric body under the influence of a constant torque along one of the transverse axes. The rotational motion of such an axisymmetric body was characterized by the precession and nutation of the longitudinal axis.

In this section, based on Refs. 8 and 9, we consider the general motion of an asymmetric rigid body under the influence of constant body-fixed torques.