Problem

6.9 Consider a spinning axisymmetric rocket with misaligned longitudinal thrust described by a set of differential equations of the form

$$\dot{\omega}_1 = \lambda\omega_2 + \mu$$
$$\dot{\omega}_2 = -\lambda\omega_1$$
$$\dot{\theta}_1 = (\omega_1 \cos\theta_3 - \omega_2 \sin\theta_3)/\cos\theta_2$$
$$\dot{\theta}_2 = \omega_1 \sin\theta_3 + \omega_2 \cos\theta_3$$
$$\dot{\theta}_3 = (-\omega_1 \cos\theta_3 + \omega_2 \sin\theta_3)\tan\theta_2 + n$$

in which the kinematic differential equations are not linearized yet.

(a) For the following parameter values and initial conditions:

$$J_3/J = 0.05, \qquad n = 15 \text{ rad/s}$$
$$\lambda = n(J - J_3)/J = 14.25 \text{ rad/s}$$
$$\mu = M_1/J = 0.1875 \text{ rad/s}^2$$
$$\omega_1(0) = \omega_2(0) = 0 \text{ rad/s}$$
$$\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$$

perform computer simulations of both the nonlinear and linear models. In particular, plot the paths of the tip of the axis of symmetry in the (θ_1, θ_2) plane. Compare the computer simulation results with the linear analysis results given by Eqs. (6.104).

(b) For the same parameter values and initial conditions as given in (a), but with $\omega_2(0) = 0.025$ rad/s, perform computer simulations of both the nonlinear and linear models, and compare the results in terms of the numerical values of A_p, A_n, ω_p, and ω_n.

Note: In Jarmolow [6] and Kolk [7], ω_n was modified as

$$\omega_n \approx n(J - J_3)/J$$

based on their computer simulation results. The discrepancy was attributed to the linearizing assumption: $\theta_2\dot{\theta}_1 \ll \omega_3$. On the contrary, however, the nonlinear and linear simulation results agree very well, as is verified in this problem.

6.9 Asymmetric Rigid Body with Constant Body-Fixed Torques

In the preceding section, we studied the problem of a spinning axisymmetric body under the influence of a constant torque along one of the transverse axes. The rotational motion of such an axisymmetric body was characterized by the precession and nutation of the longitudinal axis.

In this section, based on Refs. 8 and 9, we consider the general motion of an asymmetric rigid body under the influence of constant body-fixed torques.

6.9.1 Linear Stability of Equilibrium Points

Euler's rotational equations of motion are given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + M_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + M_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + M_3
\end{aligned}
$$

where J_i are the principal moments of inertia and M_i are the constant torque components along the body-fixed principal axes. It is assumed that $J_1 > J_2 > J_3$ without loss of generality. The equations of motion at steady state become

$$
\begin{aligned}
-(J_2 - J_3)\Omega_2\Omega_3 &= M_1 \\
+(J_1 - J_3)\Omega_3\Omega_1 &= M_2 \\
-(J_1 - J_2)\Omega_1\Omega_2 &= M_3
\end{aligned}
$$

where $(\Omega_1, \Omega_2, \Omega_3)$ is an equilibrium point. Combining these equations, we obtain

$$
(J_2 - J_3)(J_1 - J_3)(J_1 - J_2)\Omega_1^2\Omega_2^2\Omega_3^2 = M_1M_2M_3 \tag{6.105}
$$

which indicates that equilibrium points exist if and only if $M_1M_2M_3 > 0$ (only if $M_1M_2M_3 \geq 0$).

Given a constant torque vector (M_1, M_2, M_3) with $M_1M_2M_3 > 0$, eight equilibrium points $(\pm\Omega_1, \pm\Omega_2, \pm\Omega_3)$ exist where

$$
\begin{aligned}
\Omega_1 &= \sqrt{\frac{J_2 - J_3}{(J_1 - J_2)(J_1 - J_3)}\frac{M_2M_3}{M_1}} \\
\Omega_2 &= \sqrt{\frac{J_1 - J_3}{(J_1 - J_2)(J_2 - J_3)}\frac{M_3M_1}{M_2}} \\
\Omega_3 &= \sqrt{\frac{J_1 - J_2}{(J_1 - J_3)(J_2 - J_3)}\frac{M_1M_2}{M_3}}
\end{aligned}
$$

Equilibrium points associated with a torque vector (M_1, M_2, M_3) with $M_1M_2M_3 = 0$ are

$$
\begin{aligned}
&(M_1, 0, 0) \Rightarrow \{(0, \Omega_2, \Omega_3) \ : \ -(J_2 - J_3)\Omega_2\Omega_3 = M_1\} \\
&(0, M_2, 0) \Rightarrow \{(\Omega_1, 0, \Omega_3) \ : \ +(J_1 - J_3)\Omega_3\Omega_1 = M_2\} \\
&(0, 0, M_3) \Rightarrow \{(\Omega_1, \Omega_2, 0) \ : \ -(J_1 - J_2)\Omega_1\Omega_2 = M_3\} \\
&(0, 0, 0) \Rightarrow \{(0, 0, 0),\ (\Omega_1, 0, 0),\ (0, \Omega_2, 0),\ (0, 0, \Omega_3)\}
\end{aligned}
$$

where M_i and Ω_i are nonzero constants. For other cases of $(0, M_2, M_3)$, $(M_1, 0, M_3)$, and $(M_1, M_2, 0)$, no equilibrium points exist.

Let $(\Omega_1, \Omega_2, \Omega_3)$ be such an equilibrium point of a rigid body with $M_1 M_2 M_3 \geq 0$ and also let

$$\omega_1 = \Omega_1 + \Delta\omega_1$$
$$\omega_2 = \Omega_2 + \Delta\omega_2$$
$$\omega_3 = \Omega_3 + \Delta\omega_3$$

then the linearized equations of motion can be obtained as

$$\begin{bmatrix} \Delta\dot{\omega}_1 \\ \Delta\dot{\omega}_2 \\ \Delta\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1\Omega_3 & k_1\Omega_2 \\ -k_2\Omega_3 & 0 & -k_2\Omega_1 \\ k_3\Omega_2 & k_3\Omega_1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \end{bmatrix} \tag{6.106}$$

where k_i are all positive constants defined as

$$k_1 = \frac{J_2 - J_3}{J_1}, \qquad k_2 = \frac{J_1 - J_3}{J_2}, \qquad k_3 = \frac{J_1 - J_2}{J_3} \tag{6.107}$$

The characteristic equation is then obtained as

$$s^3 + \left(k_2 k_3 \Omega_1^2 - k_1 k_3 \Omega_2^2 + k_1 k_2 \Omega_3^2\right)s + 2k_1 k_2 k_3 \Omega_1 \Omega_2 \Omega_3 = 0 \tag{6.108}$$

and the linear stability of different types of equilibrium points can be summarized as in Table 6.1, where $\Omega_i \neq 0$.

As discussed in Chapter 1, however, the Lyapunov stability of a dynamic system linearized about an equilibrium point does not guarantee the Lyapunov stability of the equilibrium point of the nonlinear system. Furthermore, the linear stability analysis does not provide any information about the domains of attraction. Consequently, a nonlinear analysis is needed and will be discussed next.

Table 6.1 Linear stability of equilibrium points

Equilibrium points	Characteristic equations	Stability
$(0, 0, 0)$	$s^3 = 0$	Unstable
$(\Omega_1, 0, 0)$	$s\left(s^2 + k_2 k_3 \Omega_1^2\right) = 0$	Stable
$(0, \Omega_2, 0)$	$s\left(s^2 - k_1 k_3 \Omega_2^2\right) = 0$	Unstable
$(0, 0, \Omega_3)$	$s\left(s^2 + k_1 k_2 \Omega_3^2\right) = 0$	Stable
$(0, \Omega_2, \Omega_3)$	$s\left(s^2 - k_1 k_3 \Omega_2^2 + k_1 k_2 \Omega_3^2\right) = 0$	Stable for $k_3\Omega_2^2 < k_2\Omega_3^2$
$(\Omega_1, 0, \Omega_3)$	$s\left(s^2 + k_2 k_3 \Omega_1^2 + k_1 k_2 \Omega_3^2\right) = 0$	Stable
$(\Omega_1, \Omega_2, 0)$	$s\left(s^2 + k_2 k_3 \Omega_1^2 - k_1 k_3 \Omega_2^2\right) = 0$	Stable for $k_2\Omega_1^2 > k_1\Omega_2^2$
$(\Omega_1, \Omega_2, \Omega_3)$	Eq. (6.108)	Unstable

6.9.2 Constant Torque About the Major or Minor Axis

Consider a case in which a constant body-fixed torque acts along the major axis; i.e., $M_1 \neq 0$ and $M_2 = M_3 = 0$. For such a case, the equations of motion are simply given by

$$\frac{d\omega_1}{dt} - k_1\omega_2\omega_3 = M_1/J_1 \tag{6.109a}$$

$$\frac{d\omega_2}{dt} + k_2\omega_3\omega_1 = 0 \tag{6.109b}$$

$$\frac{d\omega_3}{dt} - k_3\omega_1\omega_2 = 0 \tag{6.109c}$$

For the convenience of mathematical derivations, we will employ the equations of motion in nondimensional form in the subsequent analysis and consider only the positive torque case ($M_1 > 0$) without loss of generality.

The equations of motion in nondimensional form for constant $M_1 > 0$ can be obtained as

$$\frac{dx_1}{d\tau} - x_2x_3 = 1 \tag{6.110a}$$

$$\frac{dx_2}{d\tau} + x_3x_1 = 0 \tag{6.110b}$$

$$\frac{dx_3}{d\tau} - x_1x_2 = 0 \tag{6.110c}$$

where $\tau = t\sqrt{\mu k_1k_2k_3}$, and

$$\mu = \frac{M_1}{J_1k_1\sqrt{k_2k_3}}, \qquad x_1 = \frac{\omega_1}{\sqrt{\mu k_1}}, \qquad x_2 = \frac{\omega_2}{\sqrt{\mu k_2}}, \qquad x_3 = \frac{\omega_3}{\sqrt{\mu k_3}}$$

Equilibrium curves (or manifolds) are then described by

$$\{(0, x_2, x_3) \ : \ -x_2x_3 = 1\}$$

From linear stability analysis, it has been shown that the equilibrium manifold is Lyapunov stable when $x_2^2 < x_3^2$ and unstable when $x_2^2 \geq x_3^2$. A stability diagram of equilibrium manifolds in the (x_2, x_3) plane is illustrated in Fig. 6.5. Also shown in this figure is a circle that touches the equilibrium manifolds at $(1, -1)$ and $(-1, 1)$.

Introducing a new variable θ_1 such that

$$\frac{d\theta_1}{d\tau} = x_1 \tag{6.111}$$

and $\theta_1(0) = 0$, we rewrite Eqs. (6.110b) and (6.110c) as

$$\frac{dx_2}{d\theta_1} + x_3 = 0 \tag{6.112a}$$

$$\frac{dx_3}{d\theta_1} - x_2 = 0 \tag{6.112b}$$

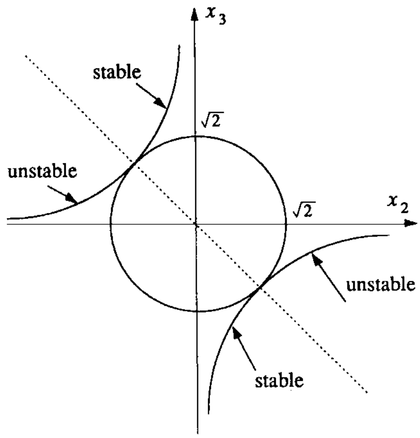


Fig. 6.5 Linear stability diagram of equilibrium manifolds in the (x_2,x_3) plane (μ > *0*).

The solution of these equations is simply given by

$$x_2 = x_2(0)\cos\theta_1 - x_3(0)\sin\theta_1 \tag{6.113a}$$

$$x_3 = x_2(0)\sin\theta_1 + x_3(0)\cos\theta_1 \tag{6.113b}$$

which can be rewritten as

$$x_2 = A\cos(\theta_1 + \phi) \tag{6.114a}$$

$$x_3 = A\sin(\theta_1 + \phi) \tag{6.114b}$$

where

$$A = \sqrt{x_2^2(0) + x_3^2(0)}$$

$$\phi = \tan^{-1}\left\{\frac{x_3(0)}{x_2(0)}\right\}$$

and we have

$$x_2^2 + x_3^2 = A^2 \qquad \text{for all } \tau \geq 0 \tag{6.115}$$

That is, the projection of the tip of the nondimensional angular velocity vector $\vec{x}$ onto the (x_2, x_3) plane normal to the major axis is a complete circle or a portion of a circle, although the body is acted upon by a constant torque along the major axis. Consequently, the endpoint of the angular velocity vector always lies on the surface of a circular cylinder defined by Eq. (6.115). Note that the end point of the angular velocity vector $\vec{\omega}$ actually lies on the surface of an elliptic cylinder.

Substituting Eqs. (6.114) into Eq. (6.110a), we obtain

$$\frac{d^2\theta_1}{d\tau^2} - \frac{A^2}{2}\sin 2(\theta_1 + \phi) = 1 \tag{6.116}$$

which can be rewritten as

$$\frac{d^2\theta}{d\tau^2} - A^2 \sin\theta = 2 \tag{6.117}$$

where $\theta = 2(\theta_1 + \phi)$. Note that Eq. (6.117) is similar to the equation of motion of an inverted pendulum with a constant external torque, but with a specified initial condition of

$$\theta(0) = 2\phi = 2\tan^{-1}\left\{\frac{x_3(0)}{x_2(0)}\right\}$$

and

$$-2\pi < \theta(0) < 2\pi$$

because we define the arctangent function such that

$$-\pi < \tan^{-1}\left\{\frac{x_3(0)}{x_2(0)}\right\} < \pi$$

For the phase-plane analysis, Eq. (6.117) is rewritten as

$$\frac{d\theta}{d\tau} = x \tag{6.118a}$$

$$\frac{dx}{d\tau} = A^2 \sin\theta + 2 \tag{6.118b}$$

which can be combined as

$$\frac{dx}{d\theta} = \frac{A^2 \sin\theta + 2}{x} \tag{6.119}$$

Integrating this equation after separation of variables, we obtain the trajectory equation in the (x, θ) plane for given initial conditions $x_2(0)$ and $x_3(0)$, as follows:

$$\tfrac{1}{2}x^2 + A^2\cos\theta - 2\theta = E \tag{6.120}$$

where E is the constant integral of the system.

Because $x = 2x_1, \theta = 2(\theta_1 + \phi)$, and $\theta_1(0) = 0$, the trajectory equation (6.120) at $\tau = 0$ becomes

$$2x_1^2(0) + A^2 \cos 2\phi - 4\phi = E \tag{6.121}$$

where

$$A = \sqrt{x_2^2(0) + x_3^2(0)}$$

$$\phi = \tan^{-1}\left\{\frac{x_3(0)}{x_2(0)}\right\}$$

Noting that

$$\cos 2\phi = \frac{1 - \tan^2\phi}{1 + \tan^2\phi} = \frac{x_2^2(0) - x_3^2(0)}{x_2^2(0) + x_3^2(0)}$$

one can rewrite the trajectory equation (6.121) at $\tau = 0$ as

$$2x_1^2(0) + x_2^2(0) - x_3^2(0) - 4\tan^{-1}\left\{\frac{x_3(0)}{x_2(0)}\right\} = E \tag{6.122}$$

Finally, we obtain the general trajectory equation in the (x_1, x_2, x_3) space, as follows:

$$2x_1^2 + x_2^2 - x_3^2 - 4\tan^{-1}\left\{\frac{x_3}{x_2}\right\} = E \tag{6.123}$$

The rotational motion of an asymmetric rigid body when a constant body-fixed torque acts along its major axis can now be analyzed using the phase-plane method as follows. The equilibrium points of Eq. (6.117) or Eqs. (6.118) are first determined by the equation

$$A^2 \sin\theta + 2 = 0 \tag{6.124}$$

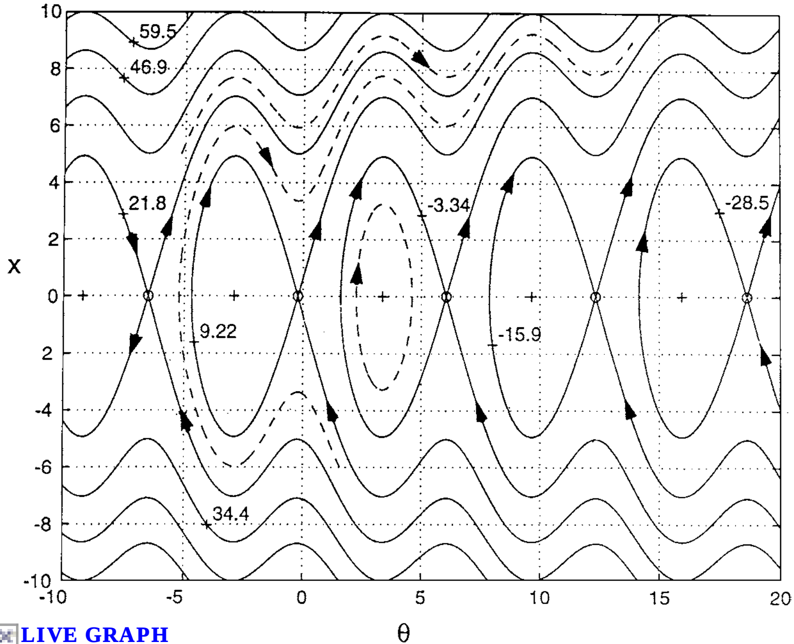
and we shall consider the following three cases.

1) If $A < \sqrt{2}$, then there exist no equilibrium points and all trajectories in the (x, θ) plane approach infinity. This corresponds to a case in which $x_2(0)$ and $x_3(0)$ lie inside the circle shown in Fig. 6.5.

2) If $A = \sqrt{2}$, then there exists an equilibrium point that is unstable and all trajectories in the (x, θ) plane approach infinity. This corresponds to a case in which $x_2(0)$ and $x_3(0)$ lie on the circle shown in Fig. 6.5.

3) If $A > \sqrt{2}$, then there exists an infinite number of stable and unstable equilibrium points along the θ axis. This corresponds to a case in which $x_2(0)$ and $x_3(0)$ lie outside the circle shown in Fig. 6.5. Furthermore, a stable periodic motion does exist (i.e., x is bounded and does not approach infinity) for certain initial conditions, as illustrated in Fig. 6.6. The separatrix that passes through an unstable equilibrium point θ^* is described by

$$\tfrac{1}{2}x^2 + A^2 \cos\theta - 2\theta = E^* \tag{6.125}$$



LIVE GRAPH
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Fig. 6.6 Phase-plane trajectories in the (x, θ) plane.

where

$$E^* = A^2 \cos\theta^* - 2\theta^*$$
$$A^2 = x_2^2(0) + x_3^2(0)$$
$$\theta^* = \sin^{-1}\left(\frac{-2}{A^2}\right) \pm 2n\pi, \qquad n = 0, 1, 2, \ldots$$

and the arcsine function is defined as $-\pi/2 \leq \sin^{-1}(\cdot) \leq \pi/2$. Such separatrices, which separate the stable and unstable domains in the (x, θ) plane, are indicated by solid lines in Fig. 6.6 for the case of $x_2(0) = 0$, $x_3(0) = 3$, $\theta(0) = 2\phi = \pi$, and $\theta^* = -0.224 \pm 2n\pi$.

Using the general trajectory equation (6.123), we can also find the equation of the *separatrix surface* in the (x_1, x_2, x_3) space of the form:

$$2x_1^2 + x_2^2 - x_3^2 - 4\tan^{-1}\left\{\frac{x_3}{x_2}\right\} = E^* \tag{6.126}$$

Such separatrix surfaces, which separate the stable and unstable domains in the (x_1, x_2, x_3) space, are shown in Fig. 6.7. Further details of such separatrix surfaces can be found in Refs. 8 and 9.

If a constant body-fixed torque acts along the minor axis, the resulting motion is also characterized as similar to the preceding case of a constant body-fixed torque

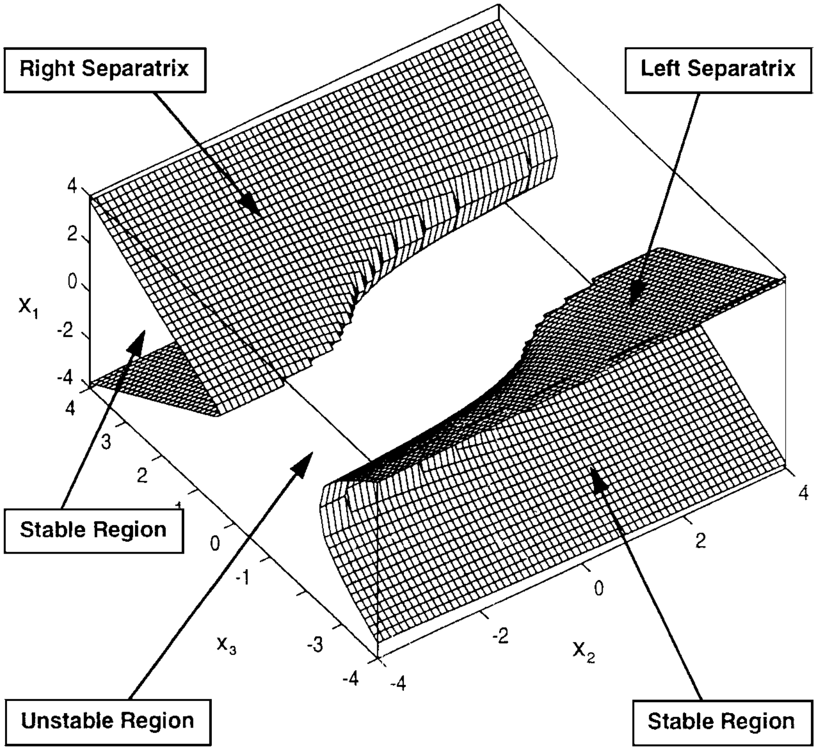

Fig. 6.7 Separatrix surfaces in the (x_1, x_2, x_3) space.

along the major axis. If a constant body-fixed torque acts along the intermediate axis, however, the resulting motion is quite different from the preceding case, as will be discussed next.

6.9.3 Constant Torque About the Intermediate Axis

Consider a case in which a constant body-fixed torque acts along the intermediate axis; i.e., $M_2 \neq 0$ and $M_1 = M_3 = 0$. For such a case, Euler's rotational equations are given by

$$
\begin{aligned}
\frac{d\omega_1}{dt} - k_1\omega_2\omega_3 &= 0 \\
\frac{d\omega_2}{dt} + k_2\omega_3\omega_1 &= M_2/J_2 \\
\frac{d\omega_3}{dt} - k_3\omega_1\omega_2 &= 0
\end{aligned}
$$

Similar to the preceding case, we will employ the following equations of motion in nondimensional form for $M_2 > 0$ in the subsequent analysis

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} - x_2 x_3 = 0 \tag{6.127a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}\tau} + x_3 x_1 = 1 \tag{6.127b}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}\tau} - x_1 x_2 = 0 \tag{6.127c}$$

where $\tau = t\sqrt{\mu k_1 k_2 k_3}$, and

$$\mu = \frac{M_2}{J_2 k_2 \sqrt{k_1 k_3}}, \qquad x_1 = \frac{\omega_1}{\sqrt{\mu k_1}}, \qquad x_2 = \frac{\omega_2}{\sqrt{\mu k_2}}, \qquad x_3 = \frac{\omega_3}{\sqrt{\mu k_3}}$$

Much like the new variable θ_1 introduced earlier, define a new variable θ_2 such that

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}\tau} = x_2 \tag{6.128}$$

and $\theta_2(0) = 0$, then we rewrite Eqs. (6.127a) and (6.127c) as

$$\frac{\mathrm{d}x_1}{\mathrm{d}\theta_2} - x_3 = 0 \tag{6.129a}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}\theta_2} - x_1 = 0 \tag{6.129b}$$

The solution of these equations is given by

$$x_1 = x_1(0)\cosh\theta_2 + x_3(0)\sinh\theta_2 \tag{6.130a}$$

$$x_3 = x_1(0)\sinh\theta_2 + x_3(0)\cosh\theta_2 \tag{6.130b}$$

where $x_1(0)$ and $x_3(0)$ are initial conditions at $\tau = 0$. Note that $\cosh x = (e^x + e^{-x})/2$, $\sinh x = (e^x - e^{-x})/2$, and $\cosh^2 x - \sinh^2 x = 1$. And we have

$$x_1^2 - x_3^2 = \text{const} = x_1^2(0) - x_3^2(0) \qquad \text{for all } \tau \geq 0 \tag{6.131}$$

That is, the projection of the tip of the nondimensional angular velocity vector $\vec{x}$ onto the (x_1, x_3) plane normal to the intermediate axis is a hyperbola, although the body is acted upon by a constant torque along the intermediate axis. Consequently, the end point of the nondimensional angular velocity vector always lies on the surface of one of the hyperbolic cylinders defined by Eq. (6.131). For particular initial conditions such that $|x_1(0)| = |x_3(0)|$, the resulting motion is along the separatrices described by

$$x_3 = \pm x_1$$

Substituting Eqs. (6.130) into Eq. (6.127b), and using the relationships $\sinh^2 x = (\cosh 2x - 1)/2$, $\cosh^2 x = (\cosh 2x + 1)/2$, and $\sinh 2x = 2\sinh x \cosh x$, we obtain

$$\frac{d^2\theta_2}{d\tau^2} = 1 - A\sinh 2\theta_2 - B\cosh 2\theta_2 \tag{6.132}$$

where

$$A = \tfrac{1}{2}\left\{x_1^2(0) + x_3^2(0)\right\}$$
$$B = x_1(0)x_3(0)$$

Because $A > B$ when $|x_1(0)| \neq |x_3(0)|$, Eq. (6.132) can be rewritten using $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$, as follows:

$$\frac{d^2\theta_2}{d\tau^2} = 1 - \sqrt{A^2 - B^2}\,\sinh(2\theta_2 + \phi) \tag{6.133}$$

where $\phi = \tanh^{-1}(B/A)$.

Defining $\theta = 2\theta_2 + \phi$, we rewrite Eq. (6.133) as

$$\frac{d\theta}{d\tau} = x \tag{6.134a}$$

$$\frac{dx}{d\tau} = -2\sqrt{A^2 - B^2}\,\sinh\theta + 2 \tag{6.134b}$$

which can be combined as

$$\frac{dx}{d\theta} = \frac{-2\sqrt{A^2 - B^2}\,\sinh\theta + 2}{x} \tag{6.135}$$

Integrating this equation after separation of variables, we obtain the trajectory equation on the (x, θ) plane, as follows:

$$\tfrac{1}{2}x^2 + 2\sqrt{A^2 - B^2}\,\cosh\theta - 2\theta = E \tag{6.136}$$

where E is the integral constant.

It can be shown that all the equilibrium points of Eq. (6.133) or Eqs. (6.134) are stable. Consequently, x is bounded and does not approach infinity for any values of the constant torque along the intermediate axis if $|x_1(0)| \neq |x_3(0)|$. However, for certain particular initial conditions such that $x_1(0) = x_3(0)$, x approaches $-\infty$ if $\mu < 0$ but it is bounded for $\mu \geq 0$. When $x_1(0) = -x_3(0)$, x approaches $+\infty$ if $\mu > 0$ but it is bounded for $\mu \leq 0$.

Like the preceding case of a constant torque about the major axis, the general trajectory equation in the (x_1, x_2, x_3) space can be found as

$$x_1^2 + 2x_2^2 + x_3^2 - 2\tanh^{-1}\left\{\frac{2x_1x_3}{x_1^2 + x_3^2}\right\} = E \tag{6.137}$$

For a more detailed treatment of this subject, the reader is referred to Leimanis [1] and Livneh and Wie [8, 9].