where

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} J_1\dot{\theta}_x + n(J_2 - J_1)\theta_z \\ J_2(\dot{\theta}_y - n) \\ J_3\dot{\theta}_z + n(J_3 - J_2)\theta_x \end{bmatrix}$$

and $T_x$, $T_y$, and $T_z$ are the components of any other external torque expressed in the LVLH frame.

*Note:* See Ref. 12 for additional information pertaining to Problem 6.12.

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## 6.11 Gyrostat in a Circular Orbit

There are basically two different types of spacecraft: 1) a three-axis stabilized spacecraft and 2) a dual-spin stabilized spacecraft.

A three-axis stabilized spacecraft with a bias-momentum wheel is often called a bias-momentum stabilized spacecraft. INTELSAT V and INTELSAT VII satellites are typical examples of a bias-momentum stabilized spacecraft. In this kind of spacecraft configuration, a wheel is spun up to maintain a certain level of gyroscopic stiffness and the wheel is aligned along the pitch axis, nominally parallel to orbit normal.

A spacecraft with a large external rotor is called a dual-spinner or dual-spin stabilized spacecraft. INTELSAT IV and INTELSAT VI satellites are typical examples of a dual-spin stabilized spacecraft. The angular momentum, typically 2000 N·m·s, of a dual-spin stabilized spacecraft is much larger than that of a bias-momentum stabilized spacecraft. For example, INTELSAT V, a bias-momentum stabilized satellite, has an angular momentum of 35 N·m·s.

In this section we formulate the equations of motion of an Earth-pointing spacecraft equipped with reaction wheels. A rigid body, consisting of a main platform and spinning wheels, is often referred to as a gyrostat.

Consider a generic model of a gyrostat equipped with two reaction wheels aligned along roll and yaw axes and a pitch momentum wheel, as illustrated in Fig. 6.10. The pitch momentum wheel is nominally spun up along the negative pitch axis. Like Fig. 6.8 of the preceding section, a LVLH reference frame $A$ with its origin at the center of mass of an orbiting gyrostat has a set of unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$, with $\vec{a}_1$ along the orbit direction, $\vec{a}_2$ perpendicular to the orbit plane, and $\vec{a}_3$ toward the Earth. Let $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a set of basis vectors of a body-fixed reference frame $B$, which is assumed to be aligned with principal axes of the gyrostat.

The total angular momentum vector of the spacecraft is then expressed as

$$\vec{H} = (J_1\omega_1 + h_1)\vec{b}_1 + (J_2\omega_2 - H_0 + h_2)\vec{b}_2 + (J_3\omega_3 + h_3)\vec{b}_3 \quad (6.169)$$

where $J_1$, $J_2$, and $J_3$ are the principal moments of inertia of the gyrostat spacecraft; $\omega_1$, $\omega_2$, and $\omega_3$ are the body-fixed components of the angular velocity of the spacecraft, i.e., $\vec{\omega}^{B/N} \equiv \vec{\omega} = \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3$; $h_1$, $-H_0 + h_2$, and $h_3$ are the body-fixed components of the angular momentum of the three wheels; and $H_0$ is the nominal pitch bias momentum along the negative pitch axis.

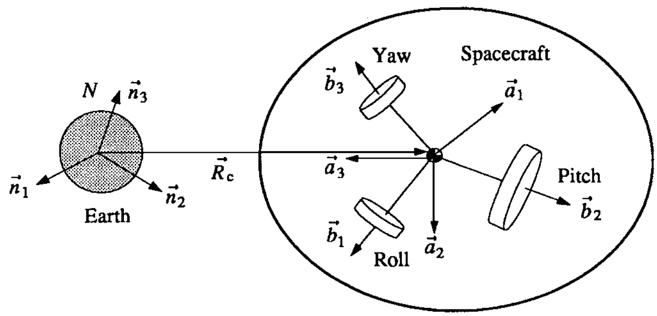


**Fig. 6.10 Gyrostat in a circular orbit.**

The rotational equation of motion is then simply given by

$$\dot{\vec{H}} = \left\{ \frac{\mathrm{d}\vec{H}}{\mathrm{d}t} \right\}_B + \vec{\omega}^{B/N} \times \vec{H} = \vec{M} \tag{6.170}$$

where $\vec{M}$ is the gravity-gradient torque acting on the vehicle. For the principal-axis frame $B$, the equations of motion can be written as

$$J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 + \dot{h}_1 + \omega_2 h_3 - \omega_3(-H_0 + h_2) = M_1 \tag{6.171a}$$

$$J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 + \dot{h}_2 + \omega_3 h_1 - \omega_1 h_3 = M_2 \tag{6.171b}$$

$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 + \dot{h}_3 + \omega_1(-H_0 + h_2) - \omega_2 h_1 = M_3 \tag{6.171c}$$

where $M_i = \vec{M} \cdot \vec{b}_i$.

For small relative angles between $B$ and $A$, we have

$$\omega_1 = \dot{\theta}_1 - n\theta_3 \tag{6.172a}$$

$$\omega_2 = \dot{\theta}_2 - n \tag{6.172b}$$

$$\omega_3 = \dot{\theta}_3 + n\theta_1 \tag{6.172c}$$

and

$$M_1 = -3n^2(J_2 - J_3)\theta_1 \tag{6.173a}$$

$$M_2 = 3n^2(J_3 - J_1)\theta_2 \tag{6.173b}$$

$$M_3 = 0 \tag{6.173c}$$

where $n$ is the orbital rate and $\theta_1$, $\theta_2$, and $\theta_3$ are called the roll, pitch, and yaw attitude angles of the spacecraft relative to the LVLH reference frame $A$.

The linearized equations of motion of a gyrostat spacecraft in a circular orbit in terms of small roll, pitch, and yaw angles can be obtained as

$$\begin{aligned} J_1\ddot{\theta}_1 &+ \left[4n^2(J_2 - J_3) + nH_0\right]\theta_1 \\ &+ [-n(J_1 - J_2 + J_3) + H_0]\dot{\theta}_3 + \dot{h}_1 - nh_3 = 0 \end{aligned} \tag{6.174}$$

$$J_2\ddot{\theta}_2 + 3n^2(J_1 - J_3)\theta_2 + \dot{h}_2 = 0 \tag{6.175}$$

$$\begin{aligned} J_3\ddot{\theta}_3 &+ \left[n^2(J_2 - J_1) + nH_0\right]\theta_3 \\ &- [-n(J_1 - J_2 + J_3) + H_0]\dot{\theta}_1 + \dot{h}_3 + nh_1 = 0 \end{aligned} \tag{6.176}$$

These linearized equations of motion will be used in Chapter 7 when we design attitude control systems of a bias-momentum stabilized spacecraft.

## 6.12 Dual-Spinner with a Platform Damper

A dual-spin spacecraft consisting of an external, axisymmetric rotor and a platform with a mass–spring–damper model is illustrated in Fig. 6.11. The so-called "despun platform" of dual-spin stabilized, geosynchronous communications satellites maintains continuous Earth-pointing, and thus it is actually spinning at orbital rate. The large external rotor with a much higher spin rate provides gyroscopic stiffness for attitude stability.

Most dual spinners such as INTELSAT III, launched in the early 1960s, had their rotor spin axis aligned with the major principal axis of the spacecraft. That is, the early, small dual spinners were disk shaped because of the major-axis stability condition of a spinning body with energy dissipation. In the mid-1960s, however, larger communications satellites had to be designed and dual spinners were no longer limited to the oblate (disk-shaped) configuration because of fairing constraints of launch vehicles. Consequently, some stability criteria for a prolate (rod-shaped) dual spinner were developed independently [13]. It was argued that the addition of energy dissipating devices on the despun platform would offset the destabilizing effect of energy dissipation in the rotor of a prolate dual spinner. Although such arguments were not rigorous at that time, an experimental prolate dual spinner, called the tactical communications satellite (TACSAT) was launched in 1969. Its successful mission led to many theoretical analysis results [14–16] and, furthermore, resulted in many prolate dual spinners such as the INTELSAT IV series starting in 1971.

In this section, we formulate the equations of motion of a dual-spin spacecraft with a despun platform damper, illustrated in Fig. 6.11. The problem of a rigid spacecraft with internal moving mass was first investigated independently by Roberson [17] in 1958 and then by Grubin [18, 19] in the early 1960s. For dynamic problems with internal moving mass, we may choose the composite center of mass of the total system as a reference point for the equations of motion. This formulation leads to a time-varying inertia matrix of the main rigid body, because the

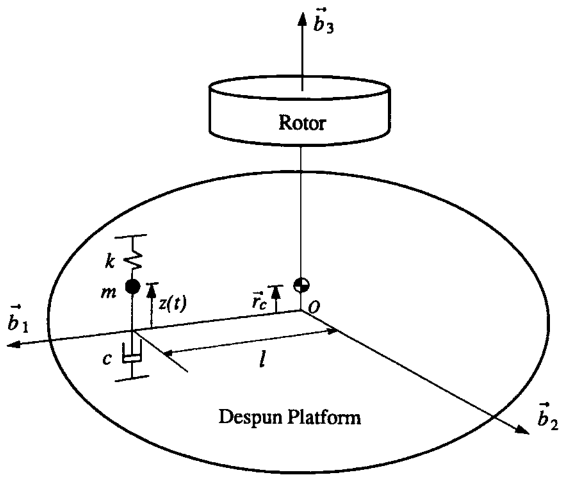


**Fig. 6.11 Dual-spinner with a despun platform damper.**

reference point is not fixed at the main body as the internal mass moves relative to the main body. On the other hand, we may choose the center of mass of the main body as the reference point, which leads to a constant inertia matrix of the main body relative to the reference point.

As illustrated in Fig. 6.11, a dual-spin spacecraft consists of an external rotor and a despun platform. The despun platform is considered to be the main body of the total system of mass $M$, and it has an internal moving part of mass $m$ connected by a spring of stiffness $k$ and a dashpot of damping coefficient $c$. The main body has a body-fixed reference frame $B$ with basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$. The point mass $m$ is located at a fixed distance $\ell$ from the reference point $O$ along $\vec{b}_1$ direction and has a relative displacement $z$ along $\vec{b}_3$ direction. The reference point $O$ of $B$ is assumed to be the center of mass of the main body when $z = 0$. The reference frame $B$ is also assumed to be aligned with principal axes of the main body when $z = 0$. The rotor has an angular momentum of $J\Omega$ relative to the main body.

The angular momentum equation of an external torque-free, dual spinner with respect to the reference point $O$ can be written as

$$\dot{\vec{h}}_o + M\vec{r}_c \times \vec{a}_o = 0 \tag{6.177}$$

where $M$ is the total mass, $\vec{r}_c$ is the position vector of the composite center of mass from the reference point $O$, and $\vec{a}_o$ is the inertial acceleration of the point $O$. The relative angular momentum of the total system about point $O$, denoted by $\vec{h}_o$, is given by

$$\vec{h}_o = \hat{J} \cdot \vec{\omega} + J\Omega\vec{b}_3 - m\ell\dot{z}\vec{b}_2 \tag{6.178}$$

where $\vec{\omega}$ is the angular velocity vector of the main body and

$$\hat{J} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} J_1 + mz^2 & 0 & -m\ell z \\ 0 & J_2 + mz^2 & 0 \\ -m\ell z & 0 & J_3 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}$$

is the inertia dyadic of the main body, including the damper mass, about the reference point $O$.

From the geometry of the system, we find the position vector of the composite center of mass from the reference point $O$, as follows:

$$\vec{r}_c = (m/M)\, z\vec{b}_3 \tag{6.179}$$

The inertial acceleration of the reference point $O$ is related to the inertial acceleration of the composite center of mass as

$$\vec{a}_c = \vec{a}_o + \ddot{\vec{r}}_c \tag{6.180}$$

Because $\vec{a}_c = 0$ for dynamic systems with zero external force, we have

$$\vec{a}_o = -\ddot{\vec{r}}_c = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{m}{M} z\vec{b}_3\right) \tag{6.181}$$

Expressing the angular velocity vector $\vec{\omega}$ as

$$\vec{\omega} = \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3 \tag{6.182}$$

we obtain

$$\begin{aligned} \vec{a}_o = -\frac{m}{M}\big[&(2\omega_2\dot{z} + \dot{\omega}_2 z + \omega_2\omega_3 z)\vec{b}_1 + (-2\omega_1\dot{z} - z\dot{\omega}_1 + z\omega_2\omega_3)\vec{b}_2 \\ &+ \left(\ddot{z} - \omega_1^2 z - \omega_2^2 z\right)\vec{b}_3\big] \end{aligned} \tag{6.183}$$

and

$$\begin{aligned} \vec{h}_o = &\left[\left(J_1 + mz^2\right)\omega_1 - m\ell\omega_3 z\right]\vec{b}_1 + \left[\left(J_2 + mz^2\right)\omega_2 - m\ell\dot{z}\right]\vec{b}_2 \\ &+ [-m\ell\omega_1 z + J_3\omega_3 + J\Omega]\vec{b}_3 \end{aligned} \tag{6.184}$$

Finally, the rotational equations of motion of the main body about the reference point $O$ can be obtained as

$$\begin{aligned} J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 + J\Omega\omega_2 + m(1 - m/M)\dot{\omega}_1 z^2 - m(1 - m/M)\omega_2\omega_3 z^2 \\ + 2m(1 - m/M)\omega_1 z\dot{z} - m\ell\dot{\omega}_3 z - m\ell\omega_1\omega_2 z = 0 \end{aligned} \tag{6.185}$$

$$\begin{aligned} J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 - J\Omega\omega_1 + m(1 - m/M)\dot{\omega}_2 z^2 + m(1 - m/M)\omega_1\omega_3 z^2 \\ + 2m(1 - m/M)\omega_2 z\dot{z} - m\ell\ddot{z} + m\ell\omega_1^2 z - m\ell\omega_3^2 z = 0 \end{aligned} \tag{6.186}$$

$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 + J\dot{\Omega} + m\ell\omega_2\omega_3 z - 2m\ell\omega_1\dot{z} - m\ell\dot{\omega}_1 z = 0 \tag{6.187}$$

The equation of motion for the rotor is given by

$$J(\dot{\omega}_3 + \dot{\Omega}) = T \tag{6.188}$$

where $T$ is the rotor spin control torque.

The dynamic equation of the internal moving mass itself can also be found by applying Newton's second law, as follows:

$$m\vec{a} = F_1\vec{b}_1 + F_2\vec{b}_2 - (c\dot{z} + kz)\vec{b}_3 \tag{6.189}$$

where $F_1$ and $F_2$ are constraint forces, and $\vec{a}$ is the inertial acceleration of the internal moving mass, which can be expressed as

$$\vec{a} = \vec{a}_c + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\ell\vec{b}_1 + \left(1 - \frac{m}{M}\right)z\vec{b}_3\right\} \tag{6.190}$$

Because $\vec{a}_c = 0$, we have

$$\vec{a} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\ell\vec{b}_1 + \left(1 - \frac{m}{M}\right)z\vec{b}_3\right\} \tag{6.191}$$

and the equation of motion of the damper mass along the $\vec{b}_3$ direction can be written as

$$m(1 - m/M)\ddot{z} + c\dot{z} + kz - m(1 - m/M)\left(\omega_1^2 + \omega_2^2\right)z + m\ell\omega_1\omega_3 - m\ell\dot{\omega}_2 = 0 \tag{6.192}$$

The rigorous stability analysis of a dual-spin spacecraft with energy dissipation is beyond the scope of this book and will not be pursued further in this text. For a more detailed treatment of this subject, the reader is referred to Hughes [2] and Kaplan [20], and also Refs. 14–16, and 21–26.

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