6.10 Rigid Body in a Circular Orbit

The solution of most spacecraft dynamics and control problems requires a consideration of gravitational forces and moments. When a body is in a uniform gravitational field, its center of mass becomes the center of gravity and the gravitational torque about its center of mass is zero. The gravitational field is not uniform over a body in space, however, and a gravitational torque exists about the body's center of mass. This effect was first considered by D'Alembert and Euler in 1749. Later, in 1780, Lagrange used it to explain why the moon always has the same face toward the Earth. In this section, we derive the equations of motion of a rigid spacecraft in a circular orbit and study its stability.

6.10.1 Equations of Motion

Consider a rigid body in a circular orbit. A local vertical and local horizontal (LVLH) reference frame A with its origin at the center of mass of an orbiting spacecraft has a set of unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$, with $\vec{a}_1$ along the orbit direction, $\vec{a}_2$ perpendicular to the orbit plane, and $\vec{a}_3$ toward the Earth, as illustrated in Fig. 6.8. The angular velocity of A with respect to N is

$$\vec{\omega}^{A/N} = -n\vec{a}_2 \tag{6.138}$$

where n is the constant orbital rate. The angular velocity of the body-fixed reference frame B with basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is then given by

$$\vec{\omega}^{B/N} = \vec{\omega}^{B/A} + \vec{\omega}^{A/N} = \vec{\omega}^{B/A} - n\vec{a}_2 \tag{6.139}$$

where $\vec{\omega}^{B/A}$ is the angular velocity of B relative to A.

The orientation of the body-fixed reference frame B with respect to the LVLH reference frame A is in general described by the direction cosine matrix $\mathbf{C} = \mathbf{C}^{B/A}$ such that

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{6.140}$$

or

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \tag{6.141}$$

The gravitational force acting on a small mass element dm is given by

$$\mathrm{d}\vec{f} = -\frac{\mu \vec{R}\,\mathrm{d}m}{|\vec{R}|^3} = -\frac{\mu(\vec{R}_c + \vec{\rho})\,\mathrm{d}m}{|\vec{R}_c + \vec{\rho}|^3} \tag{6.142}$$

where μ is the gravitational parameter of the Earth, $\vec{R}$ and $\vec{\rho}$ are the position vectors of dm from the Earth's center and the spacecraft's mass center, respectively, and $\vec{R}_c$ is the position vector of the spacecraft's mass center from the Earth's center.

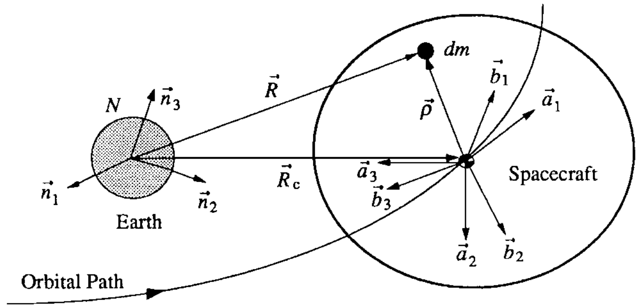

Fig. 6.8 Rigid body in a circular orbit.

The gravity-gradient torque about the spacecraft's mass center is then expressed as

$$\vec{M} = \int \vec{\rho} \times \mathrm{d}\vec{f} = -\mu \int \frac{\vec{\rho} \times \vec{R}_c}{|\vec{R}_c + \vec{\rho}|^3} \,\mathrm{d}m \tag{6.143}$$

and we have the following approximation:

$$\begin{aligned}|\vec{R}_c + \vec{\rho}|^{-3} &= R_c^{-3} \left\{1 + \frac{2(\vec{R}_c \cdot \vec{\rho})}{R_c^2} + \frac{\rho^2}{R_c^2}\right\}^{-\frac{3}{2}} \\ &= R_c^{-3} \left\{1 - \frac{3(\vec{R}_c \cdot \vec{\rho})}{R_c^2} + \text{higher-order terms}\right\}\end{aligned} \tag{6.144}$$

where $R_c = |\vec{R}_c|$ and $\rho = |\vec{\rho}|$. Because $\int \vec{\rho}\,\mathrm{d}m = 0$, the gravity-gradient torque neglecting the higher-order terms can be written as

$$\vec{M} = \frac{3\mu}{R_c^5} \int (\vec{R}_c \cdot \vec{\rho})\;(\vec{\rho} \times \vec{R}_c)\,\mathrm{d}m \tag{6.145}$$

This equation is further manipulated as follows:

$$\begin{aligned}\vec{M} &= -\frac{3\mu}{R_c^5} \vec{R}_c \times \int \vec{\rho}(\vec{\rho} \cdot \vec{R}_c)\,\mathrm{d}m \\ &= -\frac{3\mu}{R_c^5} \vec{R}_c \times \int \vec{\rho}\vec{\rho}\,\mathrm{d}m \cdot \vec{R}_c \\ &= -\frac{3\mu}{R_c^5} \vec{R}_c \times \left[\int \rho^2 \hat{I}\,\mathrm{d}m - \hat{J}\right] \cdot \vec{R}_c \\ &= -\frac{3\mu}{R_c^5} \vec{R}_c \times \int \rho^2 \hat{I}\,\mathrm{d}m \cdot \vec{R}_c + \frac{3\mu}{R_c^5} \vec{R}_c \times \hat{J} \cdot \vec{R}_c \\ &= \frac{3\mu}{R_c^5} \vec{R}_c \times \hat{J} \cdot \vec{R}_c\end{aligned}$$

because $\hat{J} = \int(\rho^2\hat{I} - \vec{\rho}\vec{\rho})\,\mathrm{d}m$ and $\vec{R}_c \times \hat{I}\cdot\vec{R}_c = \vec{R}_c \times \vec{R}_c = 0$.

Finally, the gravity-gradient torque is expressed in vector/dyadic form as

$$\vec{M} = 3n^2\vec{a}_3 \times \hat{J}\cdot\vec{a}_3 \tag{6.146}$$

where $n = \sqrt{\mu/R_c^3}$ is the orbital rate and $\vec{a}_3 \equiv -\vec{R}_c/R_c$.

The rotational equation of motion of a rigid body with an angular momentum $\vec{H} = \hat{J}\cdot\vec{\omega}^{B/N}$ in a circular orbit is then given by

$$\left\{\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\right\}_N \equiv \left\{\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\right\}_B + \vec{\omega}^{B/N} \times \vec{H} = \vec{M}$$

which can be written as

$$\hat{J}\cdot\dot{\vec{\omega}} + \vec{\omega} \times \hat{J}\cdot\vec{\omega} = 3n^2\,\vec{a}_3 \times \hat{J}\cdot\vec{a}_3 \tag{6.147}$$

where $\vec{\omega} \equiv \vec{\omega}^{B/N}$.

Because $\vec{\omega}$ and $\vec{a}_3$ can be expressed in terms of basis vectors of the body-fixed reference frame B as

$$\vec{\omega} = \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3 \tag{6.148a}$$

$$\vec{a}_3 = C_{13}\vec{b}_1 + C_{23}\vec{b}_2 + C_{33}\vec{b}_3 \tag{6.148b}$$

the equation of motion in matrix form becomes

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= 3n^2\begin{bmatrix} 0 & -C_{33} & C_{23} \\ C_{33} & 0 & -C_{13} \\ -C_{23} & C_{13} & 0 \end{bmatrix}\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}\begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix} \tag{6.149}$$

To describe the orientation of the body-fixed reference frame B with respect to the LVLH reference frame A in terms of three Euler angles θ_i $(i = 1, 2, 3)$, consider the rotational sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$ to B from A. For this sequence, we have

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 & \mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 & -\mathrm{s}\,\theta_2 \\ \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 - \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_3 & \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_3 & \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 \\ \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_3 & \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 - \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_3 & \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 \end{bmatrix}\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

where $\mathrm{c}\,\theta_i \equiv \cos\theta_i$ and $\mathrm{s}\,\theta_i \equiv \sin\theta_i$.

Also, for the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$, the angular velocity of B relative to A is represented as

$$\vec{\omega}^{B/A} = \omega_1' \vec{b}_1 + \omega_2' \vec{b}_2 + \omega_3' \vec{b}_3$$

where

$$\begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{s}\,\theta_2 \\ 0 & \mathrm{c}\,\theta_1 & \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 \\ 0 & -\mathrm{s}\,\theta_1 & \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{6.150}$$

Because

$$\vec{\omega} \equiv \vec{\omega}^{B/N} = \vec{\omega}^{B/A} + \vec{\omega}^{A/N} = \vec{\omega}^{B/A} - n\vec{a}_2$$

where $\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3$ and

$$\begin{aligned} \vec{a}_2 &= C_{12}\vec{b}_1 + C_{22}\vec{b}_2 + C_{32}\vec{b}_3 \\ &= \mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 \vec{b}_1 + (\mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_3)\vec{b}_2 + (\mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 - \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_3)\vec{b}_3 \end{aligned}$$

we have

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{s}\,\theta_2 \\ 0 & \mathrm{c}\,\theta_1 & \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 \\ 0 & -\mathrm{s}\,\theta_1 & \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} - n \begin{bmatrix} \mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 \\ \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_3 \\ \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 - \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_3 \end{bmatrix} \tag{6.151}$$

Finally, the kinematic differential equations of an orbiting rigid body can be found as

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\mathrm{c}\,\theta_2} \begin{bmatrix} \mathrm{c}\,\theta_2 & \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2 & \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2 \\ 0 & \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 & -\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 \\ 0 & \mathrm{s}\,\theta_1 & \mathrm{c}\,\theta_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \frac{n}{\mathrm{c}\,\theta_2} \begin{bmatrix} \mathrm{s}\,\theta_3 \\ \mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 \\ \mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 \end{bmatrix} \tag{6.152}$$

The dynamic equations of motion about body-fixed principal axes become

$$J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 = -3n^2(J_2 - J_3)C_{23}C_{33} \tag{6.153a}$$

$$J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 = -3n^2(J_3 - J_1)C_{33}C_{13} \tag{6.153b}$$

$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 = -3n^2(J_1 - J_2)C_{13}C_{23} \tag{6.153c}$$

where $C_{13} = -\sin\theta_2$, $C_{23} = \sin\theta_1\cos\theta_2$, and $C_{33} = \cos\theta_1\cos\theta_2$ for the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$. Furthermore, for small angles ($\sin\theta_i \approx \theta_i$ and $\cos\theta_i \approx 1$), these dynamic equations become

$$J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 = -3n^2(J_2 - J_3)\theta_1 \tag{6.154a}$$

$$J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 = 3n^2(J_3 - J_1)\theta_2 \tag{6.154b}$$

$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 = 0 \tag{6.154c}$$

Also, for small θ_i and $\dot{\theta}_i$, Eq. (6.151) can be linearized as

$$\omega_1 = \dot{\theta}_1 - n\theta_3 \tag{6.155a}$$

$$\omega_2 = \dot{\theta}_2 - n \tag{6.155b}$$

$$\omega_3 = \dot{\theta}_3 + n\theta_1 \tag{6.155c}$$

Substituting Eqs. (6.155) into Eqs. (6.154), we obtain the linearized equations of motion of a rigid body in a circular orbit, as follows, for roll, pitch, and yaw, respectively:

$$J_1\ddot{\theta}_1 - n(J_1 - J_2 + J_3)\dot{\theta}_3 + 4n^2(J_2 - J_3)\theta_1 = 0 \tag{6.156}$$

$$J_2\ddot{\theta}_2 + 3n^2(J_1 - J_3)\theta_2 = 0 \tag{6.157}$$

$$J_3\ddot{\theta}_3 + n(J_1 - J_2 + J_3)\dot{\theta}_1 + n^2(J_2 - J_1)\theta_3 = 0 \tag{6.158}$$

where θ_1, θ_2, and θ_3 are often called, respectively, the roll, pitch, and yaw attitude angles of the spacecraft relative to the LVLH reference frame A.

For these small angles θ_1, θ_2, and θ_3, the body-fixed reference frame B is, in fact, related to the LVLH frame A by

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \tag{6.159}$$

or

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \approx \begin{bmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \tag{6.160}$$

The angular velocity of B in N for this case of small relative angles of B with respect to A is also given by

$$\begin{aligned} \vec{\omega} \equiv \vec{\omega}^{B/N} &= \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3 \\ &\approx (\dot{\theta}_1 - n\theta_3)\vec{b}_1 + (\dot{\theta}_2 - n)\vec{b}_2 + (\dot{\theta}_3 + n\theta_1)\vec{b}_3 \end{aligned} \tag{6.161}$$

6.10.2 Linear Stability Analysis

Because the pitch-axis equation (6.157) is decoupled from the roll/yaw equations (6.156) and (6.158), consider first the characteristic equation of the pitch axis given by

$$s^2 + \left[3n^2(J_1 - J_3)/J_2\right] = 0 \tag{6.162}$$

If $J_1 > J_3$, then the characteristic roots are pure imaginary numbers and it is said to be (Lyapunov) stable. If $J_1 < J_3$, then one of the characteristic roots is a positive real number and it is said to be unstable. Therefore, the necessary and sufficient condition for pitch stability is

$$J_1 > J_3 \tag{6.163}$$

For the roll/yaw stability analysis, the roll/yaw equations (6.156) and (6.158) are rewritten as

$$\ddot{\theta}_1 + (k_1 - 1)n\dot{\theta}_3 + 4n^2 k_1 \theta_1 = 0 \tag{6.164a}$$

$$\ddot{\theta}_3 + (1 - k_3)n\dot{\theta}_1 + n^2 k_3 \theta_3 = 0 \tag{6.164b}$$

where

$$k_1 = (J_2 - J_3)/J_1, \qquad k_3 = (J_2 - J_1)/J_3 \tag{6.165}$$

Because of the physical properties of the moments of inertia ($J_1 + J_2 > J_3$, $J_2 + J_3 > J_1$, and $J_1 + J_3 > J_2$), k_1 and k_3 are, in fact, bounded as

$$|k_1| < 1, \qquad |k_3| < 1 \tag{6.166}$$

The roll/yaw characteristic equation can then be found as

$$s^4 + (1 + 3k_1 + k_1k_3)n^2 s^2 + 4k_1k_3n^4 = 0 \tag{6.167}$$

The roll/yaw characteristic roots become pure imaginary numbers if and only if

$$k_1k_3 > 0 \tag{6.168a}$$

$$1 + 3k_1 + k_1k_3 > 0 \tag{6.168b}$$

$$(1 + 3k_1 + k_1k_3)^2 - 16k_1k_3 > 0 \tag{6.168c}$$

which are the necessary and sufficient conditions for roll/yaw stability.

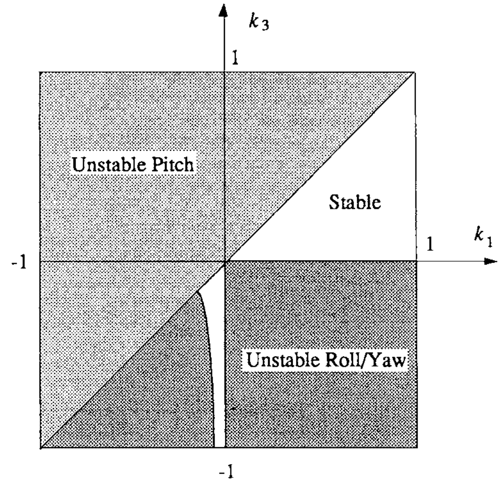

Fig. 6.9 Gravity-gradient stability plot.

The preceding results for linear stability of a rigid body in a circular orbit can be summarized using a stability diagram in the (k_1, k_3) plane, as shown in Fig. 6.9. For a further treatment of this subject, see Hughes [2].

Problems

6.10 Consider the sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2)$ from the LVLH reference frame A to a body-fixed reference frame B for a rigid spacecraft in a circular orbit.

(a) Verify the following relationship:

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 & \mathrm{s}\,\theta_3 & -\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 \\ -\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2 & \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_3 & \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 \\ \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2 & -\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_3 & -\mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

where $\mathrm{c}\,\theta_i = \cos\theta_i$ and $\mathrm{s}\,\theta_i = \sin\theta_i$.

(b) Derive the following kinematic differential equation:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_3} \begin{bmatrix} \cos\theta_3 & -\cos\theta_1 \sin\theta_3 & \sin\theta_1 \sin\theta_3 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 \cos\theta_3 & \cos\theta_1 \cos\theta_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ n \\ 0 \end{bmatrix}$$

(c) For small attitude deviations from LVLH orientation, show that the linearized dynamic equations of motion, including the products of inertia, can be written as

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = n \begin{bmatrix} J_{31} & 2J_{32} & J_{33} - J_{22} \\ -J_{32} & 0 & J_{12} \\ J_{22} - J_{11} & -2J_{12} & -J_{13} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$+ 3n^2 \begin{bmatrix} J_{33} - J_{22} & J_{21} & 0 \\ J_{12} & J_{33} - J_{11} & 0 \\ -J_{13} & -J_{23} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + n^2 \begin{bmatrix} -2J_{23} \\ 3J_{13} \\ -J_{12} \end{bmatrix}$$

(d) Verify that the linearized equations of motion can also be written in terms of Euler angles, as follows:

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}$$

$$= n \begin{bmatrix} 0 & 2J_{32} & J_{11} - J_{22} + J_{33} \\ -2J_{32} & 0 & 2J_{12} \\ -J_{11} + J_{22} - J_{33} & -2J_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$+ n^2 \begin{bmatrix} 4(J_{33} - J_{22}) & 3J_{21} & -J_{31} \\ 4J_{12} & 3(J_{33} - J_{11}) & J_{32} \\ -4J_{13} & -3J_{23} & J_{11} - J_{22} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + n^2 \begin{bmatrix} -4J_{23} \\ 3J_{13} \\ J_{12} \end{bmatrix}$$

(e) For large, pitch-axis angular motion but with small roll/yaw angles, derive the following equations of motion about principal axes:

$$J_1\ddot{\theta}_1 + \left(1 + 3\cos^2\theta_2\right)n^2(J_2 - J_3)\theta_1 - n(J_1 - J_2 + J_3)\dot{\theta}_3$$
$$+ 3(J_2 - J_3)n^2(\sin\theta_2\cos\theta_2)\theta_3 = 0$$
$$J_2\ddot{\theta}_2 + 3n^2(J_1 - J_3)\sin\theta_2\cos\theta_2 = 0$$
$$J_3\ddot{\theta}_3 + \left(1 + 3\sin^2\theta_2\right)n^2(J_2 - J_1)\theta_3 + n(J_1 - J_2 + J_3)\dot{\theta}_1$$
$$+ 3(J_2 - J_1)n^2(\sin\theta_2\cos\theta_2)\theta_1 = 0$$

Note: See Ref. 10 for additional information pertaining to Problem 6.10.

6.11 Consider a rigid body in a circular orbit possessing a body-fixed reference frame B with basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$. The LVLH reference frame A with its origin at the center of mass of an orbiting spacecraft has a set of unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$, with $\vec{a}_1$ along the orbit direction, $\vec{a}_2$ perpendicular to the orbit plane, and $\vec{a}_3$ toward the Earth, as illustrated in Fig. 6.8.

Basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of B and basis vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ of A are related to each other by

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

where q_1, q_2, q_3, and q_4 are quaternions that describe the orientation of B relative to A.

(a) Verify that the three Euler angles of the $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_3(\theta_3) \leftarrow \mathbf{C}_2(\theta_2)$ sequence from the LVLH reference frame A to a body-fixed reference frame B for a rigid spacecraft in a circular orbit are related to quaternions by

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathrm{c}(\theta_1/2)\,\mathrm{s}(\theta_2/2)\,\mathrm{s}(\theta_3/2) + \mathrm{s}(\theta_1/2)\,\mathrm{c}(\theta_2/2)\,\mathrm{c}(\theta_3/2) \\ \mathrm{c}(\theta_1/2)\,\mathrm{s}(\theta_2/2)\,\mathrm{c}(\theta_3/2) + \mathrm{s}(\theta_1/2)\,\mathrm{c}(\theta_2/2)\,\mathrm{s}(\theta_3/2) \\ \mathrm{c}(\theta_1/2)\,\mathrm{c}(\theta_2/2)\,\mathrm{s}(\theta_3/2) - \mathrm{s}(\theta_1/2)\,\mathrm{s}(\theta_2/2)\,\mathrm{c}(\theta_3/2) \\ \mathrm{c}(\theta_1/2)\,\mathrm{c}(\theta_2/2)\,\mathrm{c}(\theta_3/2) - \mathrm{s}(\theta_1/2)\,\mathrm{s}(\theta_2/2)\,\mathrm{s}(\theta_3/2) \end{bmatrix}$$

where $\mathrm{c}(\theta_i/2) = \cos(\theta_i/2)$ and $\mathrm{s}(\theta_i/2) = \sin(\theta_i/2)$.

(b) Derive the following inverse relationships:

$$\theta_1 = \tan^{-1}\left\{\frac{2(q_1 q_4 - q_2 q_3)}{1 - 2q_1^2 - 2q_3^2}\right\}$$

$$\theta_2 = \tan^{-1}\left\{\frac{2(q_2 q_4 - q_1 q_3)}{1 - 2q_2^2 - 2q_3^2}\right\}$$

$$\theta_3 = \sin^{-1}\{2(q_1 q_2 + q_3 q_4)\}$$

(c) Derive the following kinematic differential equation for a rigid spacecraft in a circular orbit:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 + n & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 + n \\ \omega_2 - n & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 - n & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

(d) Derive the following equations of motion for principal axes:

$$J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 + 6n^2(J_2 - J_3)(q_1 q_4 + q_2 q_3)(1 - 2q_1^2 - 2q_2^2) = 0$$

$$J_2\dot{\omega}_2 - (J_3 - J_1)\omega_1\omega_3 + 6n^2(J_3 - J_1)(q_1 q_3 - q_2 q_4)(1 - 2q_1^2 - 2q_2^2) = 0$$

$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 + 12n^2(J_1 - J_2)(q_1 q_3 - q_2 q_4)(q_1 q_4 + q_2 q_3) = 0$$

Note: See Ref. 11 for additional information pertaining to Problem 6.11.

6.12 Let the x, y, and z axes of the LVLH reference frame A with a set of basis vectors $\{\vec{a}_x, \vec{a}_y, \vec{a}_z\}$ be the roll, pitch, and yaw axes, respectively. The origin of the LVLH reference frame is fixed at the mass center, with the x axis in the flight direction, the y axis perpendicular to the orbit plane, and the z axis toward the Earth.

(a) Show that the equation of motion of a rigid body in a circular orbit can be written as

$$\left\{\frac{d\vec{H}}{dt}\right\}_N \equiv \left\{\frac{d\vec{H}}{dt}\right\}_A + \vec{\omega}^{A/N} \times \vec{H} = \vec{M}$$

where $\vec{H} = \hat{J} \cdot \vec{\omega}^{B/N}$ and $\vec{M}$ is the gravity-gradient torque vector.

(b) Show that the equation of motion can be expressed in the LVLH frame A and written in matrix form, as follows:

$$\dot{\mathbf{H}} + \boldsymbol{\omega}^{A/N} \times \mathbf{H} = \mathbf{M}$$

where $\mathbf{H} = (H_x, H_y, H_z)$ is the angular momentum vector of the spacecraft expressed in the LVLH reference frame; $\dot{\mathbf{H}} = (\dot{H}_x, \dot{H}_y, \dot{H}_z)$ is the rate of change of $\mathbf{H}$ as measured in the LVLH frame; $\boldsymbol{\omega}^{A/N} = (\omega_x, \omega_y, \omega_z) = (0, -n, 0)$ is the angular velocity vector of the LVLH frame that rotates with the orbital rate n; $\mathbf{M}$ is the gravity-gradient torque vector expressed in the LVLH frame; and $\boldsymbol{\omega}^{A/N} \times \mathbf{H}$ denotes the cross product of two column vectors $\boldsymbol{\omega}^{A/N}$ and $\mathbf{H}$.

Hint: Let $\vec{H} = H_x\vec{a}_x + H_y\vec{a}_y + H_z\vec{a}_z$, $\vec{\omega}^{A/N} = \omega_x\vec{a}_x + \omega_y\vec{a}_y + \omega_z\vec{a}_z$, and $\vec{M} = M_x\vec{a}_x + M_y\vec{a}_y + M_z\vec{a}_z$.

(c) Assuming that the orientation of the vehicle's principal axes with respect to the LVLH frame is described by small angles of θ_x, θ_y, and θ_z, show that, for small attitude deviations from the LVLH frame, the gravity-gradient torque can be approximated as

$$\mathbf{M} = 3n^2 \begin{bmatrix} (J_3 - J_2)\theta_x \\ (J_3 - J_1)\theta_y \\ 0 \end{bmatrix}$$

where J_1, J_2, and J_3 are the principal moments of inertia of the vehicle about the body-fixed, principal-axis reference frame B; i.e., $\hat{J} = J_1\vec{b}_1\vec{b}_1 + J_2\vec{b}_2\vec{b}_2 + J_3\vec{b}_3\vec{b}_3$.

(a) Finally, show that the equations of motion expressed in the LVLH frame become

$$\begin{bmatrix} \dot{H}_x \\ \dot{H}_y \\ \dot{H}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & n \\ 0 & 0 & 0 \\ -n & 0 & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} + 3n^2 \begin{bmatrix} (J_3 - J_2)\theta_x \\ (J_3 - J_1)\theta_y \\ 0 \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

where

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} J_1\dot{\theta}_x + n(J_2 - J_1)\theta_z \\ J_2(\dot{\theta}_y - n) \\ J_3\dot{\theta}_z + n(J_3 - J_2)\theta_x \end{bmatrix}$$

and T_x, T_y, and T_z are the components of any other external torque expressed in the LVLH frame.

Note: See Ref. 12 for additional information pertaining to Problem 6.12.

6.11 Gyrostat in a Circular Orbit

There are basically two different types of spacecraft: 1) a three-axis stabilized spacecraft and 2) a dual-spin stabilized spacecraft.

A three-axis stabilized spacecraft with a bias-momentum wheel is often called a bias-momentum stabilized spacecraft. INTELSAT V and INTELSAT VII satellites are typical examples of a bias-momentum stabilized spacecraft. In this kind of spacecraft configuration, a wheel is spun up to maintain a certain level of gyroscopic stiffness and the wheel is aligned along the pitch axis, nominally parallel to orbit normal.

A spacecraft with a large external rotor is called a dual-spinner or dual-spin stabilized spacecraft. INTELSAT IV and INTELSAT VI satellites are typical examples of a dual-spin stabilized spacecraft. The angular momentum, typically 2000 N·m·s, of a dual-spin stabilized spacecraft is much larger than that of a bias-momentum stabilized spacecraft. For example, INTELSAT V, a bias-momentum stabilized satellite, has an angular momentum of 35 N·m·s.

In this section we formulate the equations of motion of an Earth-pointing spacecraft equipped with reaction wheels. A rigid body, consisting of a main platform and spinning wheels, is often referred to as a gyrostat.

Consider a generic model of a gyrostat equipped with two reaction wheels aligned along roll and yaw axes and a pitch momentum wheel, as illustrated in Fig. 6.10. The pitch momentum wheel is nominally spun up along the negative pitch axis. Like Fig. 6.8 of the preceding section, a LVLH reference frame A with its origin at the center of mass of an orbiting gyrostat has a set of unit vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$, with $\vec{a}_1$ along the orbit direction, $\vec{a}_2$ perpendicular to the orbit plane, and $\vec{a}_3$ toward the Earth. Let $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a set of basis vectors of a body-fixed reference frame B, which is assumed to be aligned with principal axes of the gyrostat.

The total angular momentum vector of the spacecraft is then expressed as

$$\vec{H} = (J_1\omega_1 + h_1)\vec{b}_1 + (J_2\omega_2 - H_0 + h_2)\vec{b}_2 + (J_3\omega_3 + h_3)\vec{b}_3 \tag{6.169}$$

where J_1, J_2, and J_3 are the principal moments of inertia of the gyrostat spacecraft; ω_1, ω_2, and ω_3 are the body-fixed components of the angular velocity of the spacecraft, i.e., $\vec{\omega}^{B/N} \equiv \vec{\omega} = \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3$; h_1, $-H_0 + h_2$, and h_3 are the body-fixed components of the angular momentum of the three wheels; and H_0 is the nominal pitch bias momentum along the negative pitch axis.