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Existence and Stability of Local Excitations in Homogeneous Neural Fields

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Summary. Dynamics of excitation patterns is studied in one-dimensional homogeneous lateral-inhibition type neural fields. The existence of a local excitation pattern solution as well as its waveform stability is proved by the use of the Schauder fixed-point theorem and a generalized version of the Perron–Frobenius theorem of positive matrices to the function space. The dynamics of the field is in general multi-stable so that the field can keep short-term memory.

Key words: Neural field — Waveform stability — Lateral inhibition — Dynamics of pattern formation — Perron–Frobenius theorem.

1. Introduction

Cortical neural tissues are highly developed in the mammalian nervous system. They are regarded mathematically as neural fields, in which information processing takes place in the form of excitation patterns. Since the pioneering work of Wiener and Rosenblueth (1946) and Beurle (1956), many researchers have remarked on the importance of propagation of neural excitation patterns as well as their interactions in neural fields (e.g. Farley and Clark, 1961; Griffith, 1963, 1965; Ahn and Freeman, 1973; Okuda, 1974; Amari, 1975; Ellias and Grossberg, 1975; Stanley, 1976; etc.). Wilson and Cowan (1973) proposed new equations of neural fields and found various interesting phenomena by computer simulation (see also Tokura and Morishita, 1977). Most of the neural fields treated so far have lateral-inhibition type mutual connections, which are found in many parts of the central nervous system (Ratliff, 1965). There have been presented a number of mathematical treatments on the existence, uniqueness and stability of equilibrium states in this type of neural net or field (see, e.g., Reichardt and MacGinitie, 1962; Walter, 1972; Hadeler, 1974; Oguztöreli, 1975; see also Coleman, 1971). However, most of the above mathematical analyses have treated the monostable case, in which a net or field has one and only one stable equilibrium state, although it has been pointed out that neural nets of recurrent connections are in general multi-stable.

Amari (1977) treated one-dimensional homogeneous neural fields of general lateral-inhibition type connections under the simplification that each portion of the fields is either excited or nonexcited, taking two-valued activity of 0 and 1 only. He

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gave a complete categorization of the dynamics of one-layer fields of the above type. Five types of dynamics are proved to exist. They are in general multi-stable, and some fields have an ability of keeping a localized excitation pattern at any location as a stable equilibrium, while the quiescent state is also a stable state. These fields can keep a localized excitation at a place where a stimulus came, even after it disappeared. This might have some relation with short-term memory. Two-layer fields admit oscillatory and travelling wave solution (see also Wilson and Cowan, 1973). Amari and Arbib (1977) made use of these multi-stable properties to yield an analysis of stereo perception by competition and cooperation mechanism, which is supposed to play a principal role in the parallel information processing in nerve systems (see also Marr et al., 1978). Amari (to appear) has also used the theory to analyze the mechanism of the formation of topographic connections between two neural fields, by modifying the Willshaw–Malsburg model (Willshaw and Malsburg, 1976).

It is expected that most of the results obtained in Amari (1977) under the above-mentioned simplification are valid for the case where the activity takes on analog values (continuous time case). The present paper shows this, by proving the existence and stability of localized excitation pattern solutions in the monostable and multi-stable cases in the sense of the waveform stability. The mono-stable case has been analyzed by the help of a contraction mapping (Walter, 1972; Hadeler, 1974; Oguztöreli, 1975 and others). This powerful method cannot in general be applied to the multi-stable case, which we are especially interested in. (Recent work of Cowan and Ermentrout [to appear] also treats this case.) We use the Schauder fixed-point theorem and a theorem on positive operators (which is a generalization of Perron–Frobenius theorem for positive matrices) to complete the proof.

2. Neural Field Equation

We consider a one-dimensional neural field. Let $u(x, t)$ be the average membrane potential of neurons located at position x at time t. The average is carried out over a short time around t and over a short interval around x so that $u(x, t)$ can be regarded as a differentiable function of x and t. We assume that the average activity, i.e., the pulse emission rate, of neurons at x at t is given by a function of $u(x, t)$ as

$$z(x, t) = f[u(x, t)] \tag{1}$$

The function f is called the output function, and is a monotonically increasing non-linear function saturating to a constant for large u.

Let $w(x, y)$ be the average intensity of connections from neurons at place y to those at place x. The activity $z(y, t)$ of neurons at y causes an increase in the potential $u(x, t)$ at x through the connections $w(x, y)$ such that the time rate of the increment is proportional to $w(x, y)z(y, t)$. We assume that the potential $u(x, t)$ decays to a constant $-v$ with time constant τ, while it increases in proportion to the sum of all the stimuli arriving at the neurons. We call v the threshold of the field. Let $s(x, t)$ be the intensity of applied stimulus from the outside of the field to the neurons at place

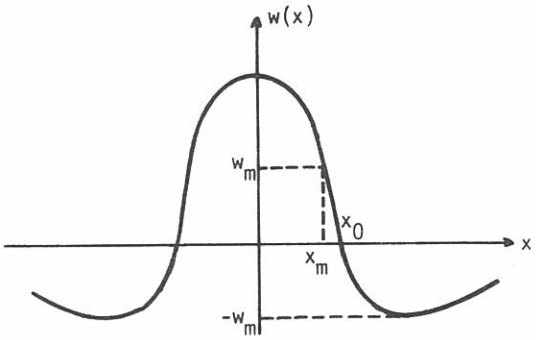

Fig. 1. Weight function $w(x)$ of connections

x at time t. We then have the following neural field equation:

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) f[u(y, t)]\, dy - v + s(x, t). \tag{2}$$

The field equation can easily be extended to the case where the field is two-dimensional and consists of many mutually connected sublayers.

We treat the case when $w(x, y)$ depends on the distance $|x - y|$ only. This means that the field is homogeneous and symmetric. We can put

$$w(x, y) = w(x - y),$$

where $w(x)$ is a continuous even function. We treat lateral-inhibition type fields, in which excitatory connections dominate for proximate neurons and inhibitory connections dominate at greater distances. In this case $w(x)$ has a shape shown in Figure 1, positive in a neighborhood of the origin and negative or equal to zero outside this neighborhood.[1] It is monotonically decreasing for $x > 0$ until it attains the minimum at $x = x_0$ and then becomes monotonically nondecreasing. Thus, we assume that

$$w(x) \begin{cases} > 0, & \text{for } |x| < x_0 \\ \leqslant 0, & \text{for } |x| \geqslant x_0 \end{cases}$$

and that $w(\infty) = 0$.

The output function treated here is of the following type (Fig. 2),

$$f(u) = \begin{cases} 0, & 0 > u \\ \varphi(u), & 0 \leqslant u \leqslant h \\ 1, & h < u \end{cases} \tag{3}$$

where $\varphi(u)$ is an arbitrary monotonically increasing differentiable function with $\varphi(0) = 0$, $\varphi(1) = 1$, and the maximum pulse emission rate is normalized to 1. Let f_0 and f_1 be, respectively, the following step-functions

$$f_0(u) = 1(u) = \begin{cases} 1, & u > 0 \\ 0, & u < 0 \end{cases} \tag{4}$$

[1] Since a neuron is either excitatory or else inhibitory, the neural field of lateral-inhibition type recurrent connections must include at least two types of neurons, one excitatory and the other inhibitory. Hence, we need to consider two simultaneous field equations for excitatory and inhibitory neurons instead of one mixed equation (2). However, so long as the equilibrium state is concerned, the mixed equation (2) gives a good approximation for them (see Amari, 1977).

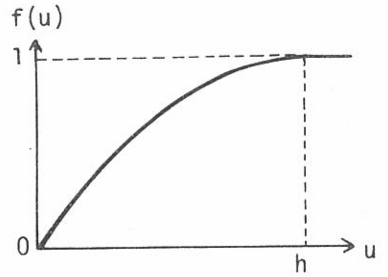

Fig. 2. Output function $f(u)$

$$f_h(u) = 1(u - h) = \begin{cases} 1, u > h \\ 0, u < h \end{cases} \tag{5}$$

In order to eliminate unnecessary complications, we do not define $1(u)$ for $u = 0$. Then, the output function f satisfies

$$f_0(u) \geqslant f(u) \geqslant f_h(u). \tag{6}$$

When the output function is a unit step-function f_0 or f_h, the dynamics of excitation patterns are fully studied in Amari (1977). We will show that similar results hold in the case with f between f_0 and f_h.

3. Dynamics of Fields with Step Output Function

We summarize here some of the results obtained by Amari (1977), where the step function f_0 is used. We treat the case where $s(x, t)$ is kept a constant s, which we call the stimulation level of the field. The level can be controlled from the outside. An initial excitation pattern given to the field changes according to the dynamics (2) of the field, and eventually converges to one of the equilibrium solutions. The equation is in general multi-stable, having a number of equilibrium solutions. Stable equilibrium solutions are the patterns which the field can retain persistently under a constant stimulation level. We show how the equilibrium solutions depend on s.

The step output function with which the output of a neuron takes only on 1 and 0 makes the analysis of the field extremely simple. Given a distribution $u(x)$ of the potential, the field is divided into two regions, the excited region where $u(x) > 0$ and the output $z(x) = 1$, and the quiescent region where $u(x) < 0$ and the output $z(x) = 0$. Let

$$R[u] = \{x \mid u(x) > 0\} \tag{7}$$

be the excited region of $u(x)$.

An equilibrium solution $u(x)$ is called a ϕ-solution, if $u(x) < 0$, i.e., if no region is excited. An equilibrium solution $u(x)$ is called an ∞-solution, if $u(x) > 0$, i.e., if the whole region is excited. These are rather trivial solutions. An equilibrium solution $u(x)$ is called a local excitation solution, when its excited region is an interval,

$$R[u] = (a_1, a_2).$$

We call a local excitation solution an a-solution, where $a = a_2 - a_1$ is the length of the excited region. It should be noted that, when $u(x)$ is an equilibrium solution, so is $u(x - c)$ for arbitrary c, because the field is homogeneous.

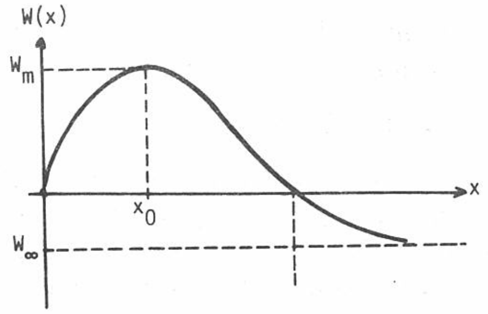

Fig. 3. Integral $W(x)$ of weight function $w(x)$

In order to show the characteristics of the field, we define the following quantities. Let

$$W(x) = \int_0^x w(y)\,dy \tag{8}$$

be the integral of $w(x)$. It is obvious that $W(x)$ is an odd function having only one peak for $x > 0$. A typical shape of $W(x)$ is shown in Figure 3. The following two quantities

$$W_m = \max_{x>0} W(x) = W(x_0) \tag{9}$$

$$W_\infty = \lim_{x\to\infty} W(x), \tag{10}$$

play an important role in the following theorems.

Theorem 1.

i) *The ϕ-solution exists, if and only if*

$$s < v.$$

ii) *The ∞-solution exists, if and only if*

$$s > -2W_\infty + v.$$

iii) *There exists an a-solution, if and only if the stimulation level s satisfies*

$$s = v - W(a)$$

and

$$s < v.$$

Theorem 2. *An a-solution is stable if* $w(a) < 0$, *and unstable if* $w(a) > 0$. *The ϕ- and ∞-solutions are stable.*

It should be noted that 'stable' implies 'wave-form stable' in the more strict sense. We treat the waveform stability in the present paper because of the homogeneity of the field. Mutual interactions of local excitations as well as those with non-homogeneous stimulations have also been studied in Amari (1977).

4. Existence of Local Excitation Solution

We generalize the results of Section 3 to the field with a continuous output function f of (3). It should be remarked that, when a field has the step output function f_h, it is equivalent to a field with the output function f_0 and with a new threshold $v + h$. This can be easily shown from

$$f_h(u) = f_0(u - h).$$

Hence, we know the characteristics of the dynamics of a field with output function f_0 and f_h. By virtue of $f_0 \geqslant f \geqslant f_h$, the field with output function f is expected to have characteristics similar to both of them. Now we extend previous definitions of the excited region as follows. Given a potential distribution $u(x)$, we call

$$R^*[u] = \{x \mid u(x) > h\}$$

a maximally excited region, in which the outputs are 1, and

$$R^-[u] = \{x \mid h > u(x) > 0\}$$

an incompletely excited region, in which the output is positive but less than 1.

An equilibrium solution $u(x)$ is called a ϕ-solution if $u(x) < 0$ for all x, and is called an ∞-solution if $u(x) > h$ for all x. An equilibrium solution $u(x)$ is called a local excitation solution, if $R^*[u]$ is an interval surrounded by an incompletely excited region R^-, i.e.,

$$R[u] = R^*[u] \cup R^-[u]$$

being another interval. Fields with output functions f, f_0 and f_h are called, respectively, f-, 0- and h-fields, where they are supposed to have the same connections $w(x)$.

Theorem 3. *An f-field has the ϕ-solution (∞-solution), if and only if the corresponding 0-field (h-field) has the ϕ-solution (∞-solution).*

We next consider the case where each of the 0- and h-field has a stable local excitation solution, or each has an unstable local excitation solution.

Lemma 1. *Both the 0- and h-field have a stable local excitation solution, if and only if*

$$h + v - W_m < s < v - \max\{0, W_\infty\}.$$

Both of them have an unstable local excitation solution, if and only if

$$h + v - W_m < s < v.$$

Proof. If the 0-field has a local excitation solution, Theorem 1 shows that the equation

$$W(a) + s - v = 0 \tag{11}$$

is satisfied by a positive number a. The solution is stable (unstable) when $w(a) = W'(a) < 0 (>0)$. Hence, from Figure 3, we see that eq. (11) is satisfied by a positive a with $W'(a) < 0$ when

$$W_\infty < -s + v < W_m \tag{12}$$

and by a positive a with $W'(a) > 0$ when

$$0 < -s + v < W_m. \tag{13}$$

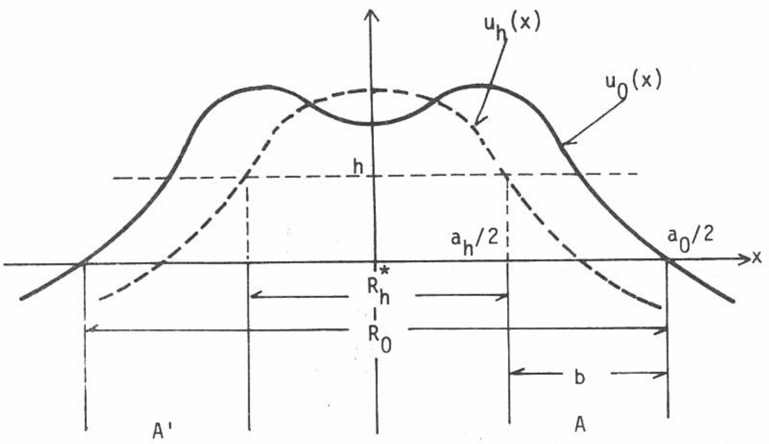

Fig. 4. Excited regions of $u_0(x)$ and $uh(x)$

The corresponding conditions for the h-field are obtained by replacing v in (12) and (13) by $v+h$. Taking into account that $s-v<0$ and $s-v-h<0$ must hold, we can prove the lemma by combining the above results.

We prove in the following that if the 0- and h-field have a stable local excitation equilibrium solution, the f-field also has a stable local excitation solution under some additional conditions. Let the 0- and h-fields have, respectively, local excitation solutions u_0 and u_h, and let a_0 and a_h be the lengths of their excited regions, $R_0 = R[u_0]$ and $R_h^* = R^*[u_h]$, respectively. Without loss of generality, we put the center of the excited regions at the origin.

It is clear that $a_0 > a_h$ and hence, R_h^* is included in R_0. The region $R_0 - R_h^*$ consists of two connected components A and A' (see Fig. 4) of length

$$b = \frac{a_0 - a_h}{2}. \tag{14}$$

Let w_m be

$$w_m = \left|\min_x w(x)\right|.$$

When $w(0) > w_m$ holds, we can determine x_m by $W(x_m) = w_m$ (Fig. 1) and $w(x) > w_m$ holds for $|x| < x_m$.

In order to simplify the discussion, we put the following assumptions, although they have no specific physiological or anatomical meaning.

Assumption 1. $w(0) > w_m$
Assumption 2. $b < x_m$
Assumption 3. $b < a_h/2$
Assumption 4. $w(x)$ is concave in the interval $|x| < b$.[2]

[2] Assumption 4 can be replaced by a different assumption, Assumption 4'. $b \leqq w_m/(M + M')$, where $M = \max_{x>0} w'(x)$, $M' = \max_{0<x<b} |w'(x)|$. Assumption 1 is satisfied when self excitatory recurrent connections are large (they may be narrow). Assumptions 2, 3 and 4' are satisfied when b is small.

Theorem 4. *If the* 0- *and h-field have stable local excitation equilibrium solutions* $u_0(x)$ *and* $u_h(x)$, *respectively, then the f-field has a local excitation equilibrium solution* $u(x)$ *satisfying*

$$f_h[u_h(x)] \leqslant f[u(x)] \leqslant f_0[u_0(x)]$$

under Assumptions 1, 2, 3 *and* 4.

Proof. An equilibrium solution of the f-field satisfies

$$u(x) = s - v + \int w(x-y) f[u(y)]\, dy.$$

If the above equation has a solution $u(x)$ such that

$$u(x) > h \quad \text{for} \quad x \in R_h^*$$

and

$$u(x) < 0 \quad \text{for} \quad x \notin R_0$$

then the equation can be rewritten as

$$u(x) = s - v + \int_{R_h^*} w(x-y)\, dy + \int_{R_0 - R_h^*} w(x-y) f[u(y)]\, dy$$

Taking account of

$$u_h(x) = s - v + \int_{R_h^*} w(x-y)\, dy$$

and presuming

$$u(x) = u(-x)$$

we have

$$u(x) = u_h(x) + \int_A r(x, y) f[u(y)]\, dy, \tag{15}$$

where

$$r(x, y) = w(x-y) + w(x+y). \tag{16}$$

We first show that Eq. (15) has a solution $u(x)$ defined on A, when it is restricted on A only.

Lemma 2. *Equation* (15) *has a solution* $u(x)$ *defined on* A *satisfying*

$$u_0(x) \geqslant u(x) \geqslant u_h(x). \tag{17}$$

Proof of the lemma. We first define a non-linear operator $T_g: L_2(A) \to L_2(A)$ associated with a non-negative function $g(x)$ by

$$(T_g u)(x) = u_h(x) + \int_A r(x, y) g[u(y)]\, dy, \tag{18}$$

where the domain and the range of T_g are functions defined on A. Here $L_2(A)$ is the space of square-integrable functions on A. Then u_0, u_h and u are fixed points of

T_{f0}, T_{fh} and T_f, respectively,

$$T_{f0}u_0 = u_0,$$
$$T_{fh}u_h = u_h,$$
$$T_f u = u.$$

Assumption 2 implies

$$r(x, y) > 0, \quad x, y \in A.$$

From this, we can easily prove that, for

$$g(x) \geqslant m(x),$$
$$(T_g u)(x) \geqslant (T_m u)(x).$$

Similarly from the monotonicity of f, we have, for

$$u_1(x) \geqslant u_2(x),$$
$$(T_f u_1)(x) \geqslant (T_f u_2)(x).$$

We now show the existence of a fixed point u of T_f by the use of the Schauder fixed point theorem (see Cronin, 1964):

Schauder Fixed Point Theorem. *Let K be a convex closed bounded set and T a compact transformation such that $T(K) \subset K$. Then T has a fixed point.*

We first note that

$$u_0(x) \geqslant u_h(x), \quad x \in A \tag{19}$$

since

$$u_0(x) - u_h(x) = \int_A r(x, y)\, dy.$$

Let K be the set of square-integrable functions R defined on the interval A, satisfying

$$u_0(x) \geqslant k(x) \geqslant u_h(x).$$

Obviously, K is a closed convex set in L_2. For any $k \in K$, we have

$$T_f k \geqslant T_f u_0 \geqslant T_{f0} u_0 = u_0,$$
$$T_f k \leqslant T_f u_h \leqslant T_{fh} u_h = u_h,$$

so that

$$T(K) \subset K$$

holds. Since T_f is a compact transformation, there exists a $u \in K$ which satisfies $T_f u = u$. This proves the lemma.

Lemma 3. *The fixed point $u(x)$ is a monotonically decreasing and differentiable function on A.*

The proof is given in Appendix 1. The equilibrium solution $u(x)$ defined on A can be extended over the whole region by (15). We show that the extended solution really satisfies

$$u(x) > h, \quad x \in R_h^*$$

and

$$u(x) < 0$$

outside R_0. Let us put

$$F(y) = \begin{cases} 1 & 0 \leqslant y \leqslant a_h/2, \\ f[u(y)] & a_h/2 \leqslant y \leqslant a_0/2. \end{cases}$$

Then, $F(y)$ is a monotonically decreasing function with $F(a_0/2) = 0$, $F(a_h/2) = 1$. We have from (15)

$$\begin{aligned} u(x) &= s - v + \int_0^{a_0/2} r(x, y) F(y)\, dy \\ &= s - v - \int_0^{a_0/2} r(x, y)\, dy \int_y^{a_0/2} F'(z)\, dz. \end{aligned}$$

By exchanging the order of integration, we have

$$\begin{aligned} u(x) &= s - v - \int_{a_h/2}^{a_0/2} \int_0^z r(x, y)\, dy F'(z)\, dz \\ &= \int_{a_h/2}^{a_0/2} \left\{ s - v + \int_0^z r(x, y)\, dy \right\} d(1 - F(z)) \\ &= \int_{a_h/2}^{a_0/2} p(x, z)\, d(1 - F(z)) \end{aligned}$$

where

$$p(x, z) = s - v + \int_{-z}^{z} w(x - y)\, dy = s - v + \int_0^z r(x, y)\, dy. \tag{21}$$

Hence, $u(x)$ can be written as a convex linear combination $p(x, z)$, $z \in A$. We prove in Appendix 2 that $p(x, z) > h$ for $x \in R_h^*$, $z \in A$, and $p(x, z) < 0$ for x outside R_0 and $z \in A$. Moreover, $u(x) = u(-x)$ is surely satisfied. Hence, the $u(x)$ is a local excitation solution of the f-field.

5. Stability of Local Excitation Solution

We show the stability of the local excitation equilibrium solution by the use of theorems stated in Krasnosel'skii (1964), which are an extension of the Perron–Frobenius theorem on positive matrices to positive operators of the functional space.

Theorem 5. *The f-field has a stable local excitation equilibrium solution, when both the 0- and h-field have a stable one.*

Proof. Let $v(x, t)$ be small disturbances from the equilibrium solution $u(x)$. Then, the variational equation of (2) is given by

$$\tau \frac{\partial v(x, t)}{\partial t} = -v + \int w(x - y)f'[u(y)]v(y, t)\, dy. \tag{22}$$

Since $f'[u(y)] = 0$ in the maximally excited region $R^*[u]$ as well as in the quiescent region (the complement of $R[u]$), the small disturbances $v(x, t)$ in these regions have no effects on the incompletely excited region $R^-[u]$. Hence, we consider the incompletely excited region only. Lemma 3 shows that the region is composed of two disjoint components $B \subset A$ and $B' \subset A'$. Let c be the length of B and B', and let us put

$$B = (a/2, a/2 + c), \qquad B' = (-a/2, -a/2 - c),$$

where a is the length of $R^*[u]$.

Let

$$\begin{aligned} v_1(x, t) &= v(x + a/2, t), \\ v_2(x, t) &= v(-x - a/2, t), \qquad 0 < x < c \end{aligned} \tag{23}$$

be, respectively, the disturbances in B and B'. Then, (22) can be decomposed into

$$\begin{aligned} \tau \frac{\partial v_1(x, t)}{\partial t} &= -v_1 + \int_0^c e_1(x, y)v_1(y, t)\, dy \\ &\quad + \int_0^c e_2(x, y)v_2(y, t)\, dy \\ \tau \frac{\partial v_2(x, t)}{\partial t} &= -v_2 + \int_0^c e_2(x, y)v_1(y, t)\, dy \\ &\quad + \int_0^c e_1(x, y)v_2(y, t)\, dy, \end{aligned} \tag{24}$$

where

$$\begin{aligned} e_1(x, y) &= w(x - y)f'[u(y + a/2)], \\ e_2(x, y) &= w(x + y + a)f'[u(y + a/2)]. \end{aligned} \tag{25}$$

Let

$$\begin{aligned} q_1(x, t) &= v_1(x, t) + v_2(x, t), \\ q_2(x, t) &= v_1(x, t) - v_2(x, t). \end{aligned} \tag{26}$$

Then, (24) can be rewritten as

$$\begin{aligned} \tau \frac{\partial q_1(x, t)}{\partial t} &= -q_1 + \int_0^c b_1(x, y)q_1(y, t)\, dy, \\ \tau \frac{\partial q_2(x, t)}{\partial t} &= -q_2 + \int_0^c b_2(x, y)q_2(y, t)\, dy, \end{aligned} \tag{27}$$

where

$$\begin{aligned} b_1(x, y) &= e_1(x, y) + e_2(x, y), \\ b_2(x, y) &= e_1(x, y) - e_2(x, y). \end{aligned} \tag{28}$$

Let us put

$$q_1(x, t) = e^{\lambda_1 t} q_1(x),$$
$$q_2(x, t) = e^{\lambda_2 t} q_2(x) \tag{29}$$

and substitute these in (27). We then have the eigenvalue problem

$$\int_0^c b_1(x, y) q_1(y)\, dy = (1 + \lambda_1 \tau) q_1(x),$$
$$\int_0^c b_2(x, y) q_2(y)\, dy = (1 + \lambda_2 \tau) q_2(x), \tag{30}$$

from which λ_i and $q_i(x)$ are determined. Here $\mu_1 = 1 + \lambda_1 \tau$ and $\mu_2 = 1 + \lambda_2 \tau$, respectively, are the eigenvalues of the integral operators (in the left-hand side) of (30). Equation (29) implies that the q_i-component of disturbances converges to 0 as t tends to infinity, if the real part of the corresponding λ_i is negative, i.e. if the corresponding eigenvalue μ_i of (30) has a real part smaller than 1.

Since the field is homogeneous, for an arbitrary constant c, $u(x - c)$, a parallel shift of $u(x)$, is also an equilibrium solution. Hence, we easily have that $u'(x)$ gives a solution of (22). It corresponds to a solution

$$\bar{q}_1(x) = 2u'(x + a/2), \tag{31}$$

of the former of (30) with the eigenvalue $\mu_1 = 1$. When all the other eigenvalues of (30) are less than 1 in magnitude, the real parts of λ_1, and λ_2 are negative (except the one corresponding to the parallel shift of $u(x)$), and the $u(x)$ is waveform stable.

We remark that

$$b_1(x, y) > b_2(x, y) > 0$$

holds for $0 < x < c$ and $0 < y < c$. Let us consider the eigenvalue problem

$$\int_0^c b_i(x, y) q_i(y)\, dy = \mu_i q_i(x), \qquad i = 1, 2. \tag{32}$$

Since $b_i(x, y) > 0$, this is a generalization of the eigenvalue problem of strictly positive matrices to the functional space. We use the following theorem (Krasnosel'skii, 1964).

Theorem on Positive Operators with Respect to a Reproducing cone K in a Banach Space. *Let P be a positive linear operator and $\phi_0 \in K$ be an eigenvector*

$$P\varphi_0 = \mu_0 \varphi_0. \tag{33}$$

Then, the eigenvalue μ_0 of P is real, positive and simple, φ_0 is unique in K, and μ_0 is greater than the absolute values of any other eigenvalues of P. Let P_1 and P_2 be two positive operators such that

$$P_1 \varphi > P_2 \varphi$$

for $\varphi \in K$. Then, the respective eigenvalues satisfy

$$\mu_{10} > \mu_{20}.$$

We can apply the theorem to our problem. Here we take the set of positive square-integrable functions over the interval $(0, c)$ as K, which plays the role of the set positive vectors. Then, the integral operators of (32) are g-positive, where we may choose $g = 1$, because the kernels $b_i(x, t)$ are positive and bounded.

The theorem shows that

$$\varphi_0 = -2u'(x + a/2) \in K$$

is the unique eigenfunction belonging to K, with $\mu_0 = 1$. Hence, all the other eigenvalues are less than 1 in magnitude. All the eigenvalues of (32) for $i = 2$ are smaller than 1 in magnitude because of $b_2 < b_1$. This proves the waveform stability of the $u(x)$.

Appendix 1. By the direct differentiation of (15), we have

$$\begin{aligned} u'(x) &= u_h'(x) + \int_A \partial_x r(x, y) f[u(y)]\, dy \\ &= w(x + a_h/2) - w(x - a_h/2) + \int_{a_h/2}^{x} w'(x - y) f\, dy \\ &\quad + \int_x^{a_0/2} w'(x - y) f\, dy + \int_{a_h/2}^{a_0/2} w'(x + y) f\, dy \end{aligned} \tag{34}$$

where $\partial_x = \partial/\partial x$. We note

$$\begin{aligned} &f[u(y)] \geqslant 0, \\ &w'(x - y) \leqslant 0, \quad \text{for} \quad a_h/2 \leqslant y \leqslant x, \\ &w'(x - y) \geqslant 0, \quad \text{for} \quad x \leqslant y \leqslant a_0/2. \end{aligned}$$

Further, from $w(a_h) < 0$, which is the stability condition of u_h, we have

$$w(x + a_h/2) < 0, \quad \text{for} \quad x > a_h/2.$$

Hence,

$$\int_x^{a_0/2} w'(x - y) f\, dy \leqslant \int_x^{a_0/2} w'(x - y)\, dy = w(0) - w(x - a_0/2),$$

$$\int_{a_h/2}^{x} w'(x - y) f\, dy \leqslant 0,$$

and

$$\int_{a_h/2}^{a_0/2} w'(x + y) f\, dy \leqslant \int_{w' > 0, y \in A} w'(x + y)\, dy < w_m. \tag{35}$$

We therefore have

$$u'(x) < w(0) - w(x - a_0/2) - w(x - a_h/2) + w_m. \tag{36}$$

Since $w(x)$ is concave in $|x| < b$ (Assumption 4),

$$w(x - a_0/2) + w(x - a_h/2) > w(0) + w(b) > w(0) + w_m.$$

This proves $u'(x) < 0$ on A.

We next prove this under Assumption 4′ instead of Assumption 4. To this end, we use

$$\int_{a_h/2}^{a_0/2} w'(x+y)f\,dy \leqslant \int_{w'>0,\,y\in A} w'(x+y)\,dy \leqslant Mb$$

instead of (35). We then have

$$u'(x) < w(0) - w(x - a_0/2) - w(x - a_h/2) + Mb.$$

From

$$w(0) - w(x - a_0/2) \leqslant M'b,$$
$$w(x - a_h/2) \geqslant w_m$$

($x \in A$), we have $u'(x) < 0$ on A.

Appendix 2. We have

$$\partial_x p(x, z) = w(x + z) - w(x - z)$$
$$\partial_z p(x, z) = w(x + z) + w(x - z)$$

by differentiating (21). We assume $z \in A$ throughout the discussion. We first show that $p(x, z)$ is negative for $x \notin R_0$ (i.e., for $x > a_0/2$). We have

$$p(a_0/2, z) \leqslant p(a_0/2, a_0/2) = u_0(a_0/2) = 0,$$

because $\partial_z p(x, z) > 0$ for $x = a_0/2$. Moreover, we have

$$\partial_x p(a_0/2, z) < 0.$$

Since $\partial_x p(x, z)$ changes its sign only once for $x > a_0/2$, and since $p(\infty, z) = s - v < 0$, $p(x, z)$ is negative for $x \notin R_0$.

We next prove that $p(x, z) > h$ for $0 \leqslant x < a_h/2$. From Assumption 3, we have

$$a_0/2 < a_h,$$

which implies $z < a_h$, because $z \in A$. The sign of $w(z)$, $z \in A$, is not definite. When $w(z) \geqslant 0$ (i.e., when $z \leqslant x_0$), we have

$$\partial_x p(x, z) = w(z + x) - w(z - x) < 0$$

for $0 \leqslant x < a_h/2$. Hence $p(x, z)$ is monotonically decreasing with respect to x for $0 \leqslant x < a_h/2$, and we get

$$p(x, z) > p(a_h/2, z) \geqslant p(a_h/2, a_h/2) = u_h(a_h/2) = h.$$

We next consider the case $w(z) < 0$ ($z > x_0$). We rewrite (21) as

$$p(x, z) = W(x + z) + W(z - x) + s - v.$$

From the definition of a_h.

$$W(a_h) + s - v = h.$$

Hence, it suffices to prove

$$W(z + x) + W(z - x) > W(a_h). \tag{37}$$

Since $w(z) < 0$, $W(z + x)$ is monotonically decreasing with respect to x, and

$$W(z + x) \geqslant W(a_0/2 + a_h/2) \geqslant W(a_0) > 0.$$

Hence, if $z - x$ is at a decreasing position of $W(z - x)$ (i.e., $z - x \geqslant x_0$), we have

$$W(z - x) \geqslant W(z) \geqslant W(a_0/2) \geqslant W(a_h)$$

and (37) is proved. On the contrary, if $z - x$ is at an increasing position of $W(z - x)$ (i.e., $z - x < x_0$), we have

$$W(z - x) > W(z - a_h/2) \geqslant (z - a_h/2)w_m \tag{38}$$

because of

$$z - x > z - a_h/2$$

and

$$z - a_h/2 < b$$

(Assumption 2). On the other hand,

$$W(z + x) > W(z + a_h/2) \geqslant W(a_h) - (z - a_h/2)w_m, \tag{39}$$

where the decomposition

$$z + a_h/2 = a_h + (z - a_h/2)$$

and

$$W'(y) = w(y) > -w_m$$

are used. By combining (38) and (39), we get (37), which proves the assertion.

References

Ahn, S. M., Freeman, W. J.: Steady-state and limit cycle activity of mass of neurons forming simple feedback loops (II): Distributed parameter model, Kybernetik **16**, 127–132 (1974)

Amari, S.: Homogeneous nets of neuron-like elements. Biol. Cybernetics **17**, 211–220 (1975)

Amari, S.: Dynamics of pattern formation in lateral-inhibition type neural fields. Biol. Cybernetics **27**, 77–87 (1977)

Amari, S.: Topographic organization of nerve fields, to appear

Amari, S., Arbib, M. A.: Competition and cooperation in neural nets. In: Systems Neuroscience (J. Metzler, ed.) pp. 119–165, New York–London: Academic Press, 1977

Beurle, R. L.: Properties of a mass of cells capable of regenerating pulses. Trans. Roy. Soc. London B-**240**, 55–94 (1956)

Coleman, B. D.: Mathematical theory of lateral sensory inhibition. Arch. Rational Mech. Anal., **43**, 79–100 (1971)

Cowan, J. D., Ermentrout, G. B.: Some aspects of the eigenbehaviour of neural nets. MAA Studies in Mathematical Biology, to appear

Cronin, J.: Fixed Points and Topological Degree in Non-Linear Analysis. Providence, Rhode Island, AMS (1964)

Ellias, S. A., Grossberg, S.: Pattern formation, contrast control, and oscillations in the short term memory of shunting on-center off-surround networks. Biol. Cybernetics **20**, 69–98 (1975)

Farley, B. G., Clark, W. A.: Activity in networks of neuron-like elements. Proc. 4th London Symp. on Inf. Theory (C. Cherry, ed.), London: Butterworths, 1961

Griffith, J. S.: A field theory of neural nets (I). Bull. Math. Biophys. **25**, 111–120 (1963)

Griffith, J. S.: A field theory of neural nets (II). Bull. Math. Biophys. **27**, 187–195 (1965)

Hadeler, K. P.: On the theory of lateral inhibition. Kybernetik **14**, 161–165 (1974)
Krasnosel'skii, M. A.: Positive Solutions of Operator Equations. (Translated by R. E. Flaherty.) Groningen, Noordhoff, 1964
Marr, D., Palm, G., Poggio, T.: Analysis of cooperative stereo algorithm. Biol. Cybernetics **28**, 223–239 (1978)
Oguztöreli, M. N.: On the activities in a continuous neural network. Biol. Cybernetics **18**, 41–48 (1975)
Okuda, M.: A dynamical behavior of active region in randomly connected neural network. J. Theor. Biol. **48**, 51–73 (1974)
Ratliff, F.: Mach Bands: Quantitative Studies on Neural Networks in the Retina. San Francisco, Holden-Day, 1975
Reichardt, W., MacGinitie, G.: Zur Theorie der lateralen Inhibition. Kybernetik **1**, 155–165 (1962)
Stanley, J. C.: Simulation studies of a temporal sequence memory model. Biol. Cybernetics **24**, 121–137 (1976)
Tanaka, A., Noguchi, S.: Topological properties of continuous neural networks with analog neurons. Trans. IECE, Japan, J61–D, 9–16 (1978)
Tokura, T., Morishita, I.: Analysis and simulations of double-layer neural networks with mutually inhibiting interconnections. Biol. Cybernetics **25**, 83–92 (1977)
Walter, H.: Inhibitionsfelder. Acta Informatica **1**, 253–269 (1972)
Wiener, N., Rosenblueth, A.: The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle. Arch. Inst. Cardiol. (Mexico) **16**, 205–265 (1946)
Wilson, H. R., Cowan, J. D.: A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue. Kybernetik **13**, 55–80 (1973)
Willshaw, D. J., Malsburg, C. von der: How patterned neural connections can be set up by self organization. Proc. Roy. Soc. London B**194**, 431–445 (1976)

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