# Notes Masters Thesis

Alexander Uhlmann*
Supervisor: Prof. Dr. Peter Hintz

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**Abstract**

Notes

*auhlmann@ethz.ch

# 1 Baby Case

((Name for cut point which is not conj point)) ((Quotient direction set up to rescaling or restrict to unit length vectors)) ((Null geodesics intersect backwards cone exactly once))

**Theorem 1.1** (Baby Case)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in $M_j$ we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from $p_j^+$ cut off at the intersection with the forwards light cone of $p_j^-$. We assume that there exist a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at $p_j^+$ have a cut point in $K_j$.*

*Now let $V_j$ be open sets such that $\overline{V_j} \subset \operatorname{int} J(p_j^-, p_j^+)$ is compact. We assume that no null geodesic starting in $V_j$ has a conjugate point on $K_j$.*

*Then, if*

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

*there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.*

## 1.1 Preliminary Constructions

((Intro))

**Lemma 1.2.** *Let $(M, g), K, V, p^+, p^-$ be as in the statement of theorem 1.1 (we suppress the indices to simplify notation) then the following holds:*

(1) $(J^-(p^+) \setminus I^-(p^+)) \cap K = \mathcal{L}_{p^+}^- \cap K$ *and thus* $K = \mathcal{L}_{p^+}^- \cap J^+(p^-)$.

(2) *There exists a surjective smooth map $\Psi : S^n \times [0, 1] \to K$ such that the curves $t \mapsto \Psi(v, t), v \in S^n$ are null geodesics and*

$$\Psi(S^n \times \{1\}) = \{p^+\}, \quad \Psi(S^n \times \{0\}) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$$

(3) *There exist $0 < t_- < t_+ < 1$ such that the restriction $\Psi|_{S^n \times [t_-, t_+]}$ is a diffeomorphism onto its image and that for all $v \in S^n$, we have*

$$\Psi(v, t_-) \notin \bigcup_{p \in \overline{V}} J^+(p), \quad \Psi(v, t_+) \in \bigcap_{p \in \overline{V}} J^+(p)$$

*Proof.* ((Overhaul)) As we have no cut point in $J(p_1^-, p_1^+)$, the exponential map at $p_1^+$ is a diffeomorphism onto $J(p_1^-, p_1^+)$. Thus the preimage $\exp_{p_1^+}^{-1}(R)$ of the smooth submanifold

$$R = (J^-(p_j^+) \setminus I^-(p_j^+)) \cap (J^+(p_j^-) \setminus I^+(p_j^-)) = \mathcal{L}_{p_1^+}^- \cap \mathcal{L}_{p_1^-}^+$$

is a smooth submanifold of $L^{-}_{p_1^+}M$. We then let $\mathcal{A} = R$ and denote by $\mu_a(s) = \gamma_{p_1^+,a}(1-s)$ for $a \in R$. It is then easily checked that this parameterization satisfies all requirements and we are done. $\square$

Note that this implies that $K$ is a smooth $n$-dimensional submanifold of $M$ at any point away from its boundary. We will often treat $K$ itself as a submanifold when it is clear that we are working away from the boundary. This is almost always the case since any light cone originating in $\overline{V}$ will intersect $K$ in $\Psi(S^n \times (t_-, t_+)) \subset K$.

The next proposition allows us to endow $K$ with a number of "laboratory frames" we will use to conveniently describe the light cone observations on $K$.

**Proposition 1.3** (Laboratory Frames)**.** *Let $(M_j, g_j), K_j, V_j, p_j^+, p_j^-, \Phi$ be as in the statement of theorem 1.1 Then there exists a family of future pointing, null geodesics $\mu_a^{(1)} : [0,1] \to K_1$ indexed by $a \in \mathcal{A}$ where $\mathcal{A}$ is a metric space. Furthermore we can require the map $[0,1] \times \mathcal{A} \to K_1; (s,a) \mapsto \mu_a^{(1)}(s)$ to be open ((almost, needed?)) and continuous. If we then take $\mu_a^{(2)} \coloneqq \Phi(\mu_a^{(1)})$ we can achieve*

$$K_j = \bigcup_{a \in \mathcal{A}} \mu_a^{(j)}([0,1]). \tag{1}$$

*Proof.* ((TODO)) $\square$

*Remark* 1.4. To simplify notation we will continue with the construction on just one Lorentzian manifold $(M, g)$ of dimension $1+n$ and assume that we are given the following data to construct the required conformal diffeomorphism ((explain better)) from theorem 1.1.

1. A the quasi-manifold $K$,
2. the conformal class of $g|_K$ (but not only restricted to tangent vectors in $K$ ((i think??)) ),
3. the paths $\mu_a : [0,1] \to K, a \in \mathcal{A}$,
4. the set $\mathcal{P}_K(V)$ where $V$ is open and $\overline{V} \subset \operatorname{int} J(p^-, p^+)$ is compact.

Note that these data are invariant under conformal diffeomorphism, and any map we construct from it will thus also be invariant. We also remark that $\overline{V} \subset \operatorname{int} J(p^-, p^+)$ implies that $q \notin K$ for any $q \in \overline{V}$. ((Also mention that any light observation must lie in $[t_-, t_+]$)).

### 1.1.1 Geometry of Light Observation Sets

**Lemma 1.5.** *For any $q \in \overline{V}$ the restriction of the exponential map to null vectors $\exp_q : L_q^+ M \to M$ is* transverse *to $K$, i.e. for all $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_p K$.*

*Proof.* ((Update proof)) Suppose that there exists a $q \in \overline{V}$ and a $w \in L_q^+ M$ such that with $v := \gamma_{q,w}(1) \in L_p K$. Now, by definition of $\overrightarrow{\mathcal{P}_K}(q)$ we have $\exp_p(-v) = q \in V$. Also since $v \in L_p K$ there exists a $w \in L_{p^+}^- M$ and a $t \in \mathbb{R}_{>0}$ such that $\gamma_{p^+,w}(t) = p$ and $\gamma'_{p^+,w}(t) = -v$. This implies that $\gamma_{p^+,w}(t+1) = q \in V$. We thus have $q \in \mathcal{L}^-{}_{p^+}$. Furthermore we note that $q \in J^+(p^-)$ since $\overline{V} \subset \operatorname{int} J(p^-, p^+)$. Thus by 1.2(1) we have that $q \in K$. But this is a contradiction to the fact that $q \notin K$. □

**Lemma 1.6** (Direction Reconstruction)**.** *((Adjust formulation)) Let $p \in K$ then there exists an isomorphism $\Phi$ between the space $\mathcal{S}$ of linear spacelike hypersurfaces $S \subset T_p K$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+ V \subset T_p M$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+ V \mapsto T_p K \cap V^\perp \in \mathcal{S}$.*

*Moreover there exists an isomorphism between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_p M$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \mapsto S \oplus \operatorname{span} \Phi(S) \in \mathcal{N}$.*

*Proof.* ((Works same as in timelike boundary case)) □

**Definition 1.7** (Observation Preimage)**.** For any $q \in \overline{V}$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* $L_q^K M$ to be the preimage of $K$ under the exponential map restricted to $L_q^+ M$, i.e.

$$L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K) \subset L_q^+ M$$

**Lemma 1.8.** *For any $q \in \overline{V}$, $L_q^K M$ is a $n-1$-dimensional submanifold of $T_p M$ and $\exp_q : L_q^K M \to \mathcal{P}_K(q)$ is a local diffeomorphism ((onto its image)).*

*Proof.* ((For first part use transversality. Use the fact that we have no conj points on $K$ for the second part)) □

**Proposition 1.9.** *For any $q \in \overline{V}$, $\mathcal{P}_K(q)$ is locally the the finite union of transversal dimension $n-1$ submanifolds.*

*Proof.* To prove this we will first show that for any $p \in \mathcal{P}_K(q)$, $\pi^{-1}(p) \cap \overrightarrow{\mathcal{P}_K}(q)$ only contains finitely many elements, i.e. $p$ can only be hit by finitely many light rays originating at $q$:

Let $q \in \overline{V}$ and $v \in L_q^+ M$ such that $p = \exp_q(v)$. Since we required that $p \in K$ cannot be a conjugate point of $q$, $\exp_q$ must be a local diffeomorphism around $v$. This means that there exist open sets $v \in \mathcal{O}_v \subset T_q M, p \in \mathcal{U}_v \subset M$ such that $\exp_q : \mathcal{O}_v \to \mathcal{U}_v$ is a diffeomorphism. But this means that there cannot exist another $v' \in \mathcal{O}_v$ with $\exp_q(v') = p$. We now restrict ourselves only to null *directions* at $q$ i.e. the quotient $L_q^+ M/\mathbb{R}_+ \simeq S^{n-1}$. Since any null vector $v$ with $\exp_q(v) = p$ has an open neighborhood where no other vector can have this property, the set of null *directions* in $S^{n-1}$ which hit $p$ is discrete and thus finite because $S^{n-1}$ is compact. Because we only have finitely many null directions which hit $p$, $\pi^{-1}(p) \cap \overrightarrow{\mathcal{P}_K}(q)$ can only have finitely many elements, as desired.

Let now $v_1, \ldots, v_n \in L_q^+ M$ be these finitely many vectors with $\exp_q(v_i) = p$. As $p$ cannot be a conjugate point there exists a neighborhood $d \exp_q$ is a local diffeomophs

Next we will prove that the restricted canonical projection $\pi : \overrightarrow{\mathcal{P}_K}(q) \to \mathcal{P}_K(q)$ is locally diffeomorphic. Since any $(p, v) \in \overrightarrow{\mathcal{P}_K}(q)$ has an open neighborhood where there exists no $(p, v' \neq v)$ we can construct a smooth local inverse to $\pi$ by attaching the appropriate direction vector.

((finish + transversality)) □

**Definition 1.10** (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{O} \subset M$ of $p$ such that $\mathcal{O} \cap \mathcal{P}_K(q)$ is a submanifold.

**Corollary 1.11.** *((Maybe false)) The subset of non-cut points is dense in* $\mathcal{P}_K(q)$*.*

*Proof.* It suffices to show that for every cut point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $\mathcal{O} \subset \mathcal{P}_K(q)$ contains a non-cut point. □

**Proposition 1.12.** $\overrightarrow{\mathcal{P}_K}(q)$ *is diffeomorphic to* $S^{n-1}$ *((Probably not necessary)).*

*Proof.* ((TODO)) □

### 1.1.2 Observation Time Functions

**Definition 1.13** (Observation Time Function)**.** For $a \in \mathcal{A}$ the *observation time function* is defined as

$$\begin{aligned} f_a : \overline{V} &\to [0,1] \\ q &\mapsto \inf(\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\} \cup \{1\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) \coloneqq \mu_a(f_a(q)) \in M$ be the earliest point where $\mu_a$ sees light from $q$.

**Lemma 1.14.** *Let* $a \in \mathcal{A}$ *and* $q \in \overline{V}$*. Then*

(1) *It holds that* $t_- \leq f_a(q) \leq t_+$.

(2) *We have* $\mathcal{E}_a(q) \in J^+(q)$ *and* $\tau(q, \mathcal{E}_a(q)) = 0$. *Moreover the function* $s \mapsto \tau(q, \mu_a(s))$ *is continuous, non-decreasing on* $[0,1]$ *and strictly increasing on* $[f_a(q), 1]$.

(3) *Let* $p \in K$. *Then* $p = \mathcal{E}_a(q)$ *with some* $a \in \mathcal{A}$ *if and only if* $p \in \mathcal{P}_K(q)$ *and* $\tau(p,q) = 0$. *Furthermore, these are equivalent to the fact that there are* $v \in L_q^+ M$ *and* $t \in [0, \rho(q,v)]$ *such that* $p = \gamma_{q,v}(t)$.

(4) *The function* $q \mapsto f_a(q)$ *is continuous on* $\overline{V}$.

*Proof.* Let $a \in \mathcal{A}$ and $q \in \overline{V}$.

We begin by showing (1): By lemma 1.2(3) we have that $\mu_a(t_-) \notin J^+(q)$ and $\mu_a(t_+) \in J^+(q)$. The second part immediately yields $f_a(q) \leq t_+$ as $f_a(q)$ is the infimum over all observation times. For the first part we assume by contradiction that there were to exist a $t_{-2} < t_-$ with $\mu_a(t_{-2}) \in J^+(q)$. This allows us to construct a causal path from $q$ to $\mu_a(t_-)$ by joining the causal path from $q \to \mu_a(t_{-2})$ and the null geodesic $\mu_a$ from $t_{-2}$ to $t_-$. Since this would imply that $\mu_a(t_-) \in J^+(q)$ this is a contradiction and $f_a(q)$ must be bigger than $t_-$ proving (1).

(2) By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all $t_n$ we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from $q$ to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such $t$.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[0,1]$ we first note that it is the composition of two continuous functions. Non-decreasing then follows from the reverse triangle inequality together with the fact that $\mu_a$ is a null path.

Finally to show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 1]$ we let $f_a \leq t_1 < t_2 \leq 1$. Now by ((REF)) there exists a causal geodesic $\gamma_1 : [0,1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect $\gamma_1$ to $\mu_a|_{[t_1,t_2]}$ we get a path $\gamma_2$ connecting $q$ to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as $\mu_a$ is a null geodesic. Next we argue that $\gamma_2$ must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If $\gamma_1$ is timelike this observation is trivial as $\mu_a$ is lightlike. If however, $\gamma_1$ is lightlike (which is exactly the case if $t_1 = f_a(1)$), this fact follows from the transversality of light cone observations as noted in proposition 1.5. This means that $\gamma_2$ is a broken causal geodesic, which by ((REF)) implies that there exists a strictly longer timelike path

$\gamma_3$ connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Next to prove (3): To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q, p) = 0$. But now, by ((REF)) there exists a null geodesic from $q$ to $p$ which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q, p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0, 1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q, p) = 0$.

The other equivalence follows the definition of $\mathcal{P}_K(q)$ together with the definition of cut points.

Finally we prove (4): Let $q_i \to q$ in $\overline{V}$, let $t_i = f_a(q_i)$ and $t = f_a(q)$. Since $\tau$ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(q_j, \mu_a(t + \varepsilon)) = \tau(q, \mu_a(t + \varepsilon)) > 0$. Thus for $j$ big enough we have $\tau(q_i, \mu_a(t + \varepsilon) > 0$. But by (3) this implies that $a$ must have observed $q_i$ before $t+\varepsilon$ i.e. $f_a(q_i) < t+\varepsilon = f_a(q)+\varepsilon$. As $\varepsilon$ was arbitrary we get $\limsup_{j\to\infty} t_j \leq t$.

We assume now that $\liminf_{j\to\infty} t_j = t' < t$. Let $(t_i)$ be a convergent subsequence such that $f_a(q_i) = t_i \to t' < f_a(q)$. Now by the continuity of $\tau$ and $\mu_a$ we have

$$0 = \tau(q_i, \mu_a(f_a(q_i))) \to \tau(q, \mu_a(t')).$$

Furthermore by ((REF)) $\mu(s_i) \in J^+(q_i)$ for all $i$ implies $\mu(s') \in J^+(q)$. But now we have $\mu(s') \in \mathcal{P}_K(q)$ and $\tau(q, \mu_a(s')) = 0$ which by (3) implies that $\mu_a(s') = \mathcal{E}_a(q) = \mu_a(f_a(q))$. But this is a contradiction as $s' < f_a(q)$. ((More in-detail?)) □

By (3) of the above lemma, for any $q \in \overline{V}$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 1.13 that the set of earliest observations $\mathcal{P}_K(q)$ and the path $\mu_a$ completely determine the functions

$$f_a(q) = \min\{s \in [-1, 1] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2}$$

### 1.1.3 Set of earliest observations

**Definition 1.15** (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned}\mathcal{D}_K(q) = \{(p, v) \in L^+K \mid &(p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } w \in L^+_q M, 0 \leq t \leq \rho(q, w)\}, \\ \mathcal{D}^{reg}_K(q) = \{(p, v) \in L^+K \mid &(p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } w \in L^+_q M, 0 < t < \rho(q, w)\},\end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of $q$ and $\mathcal{D}_K^{reg}(q)$ is the *regular direction set* of $q$.

Let $\mathcal{E}_U(q) = \pi(\mathcal{D}_U(q))$ and $\mathcal{E}_U^{reg}(q) = \pi(\mathcal{D}_U^{reg}(q))$, where $\pi : TU \to U$ is the canonical projection. We say that $\mathcal{E}_U(q)$ is the set of earliest observations and $\mathcal{E}reg_U(q)$ is the set of earliest regular observations of $q$ in $U$. We denote the collection of earliest observation sets by $\mathcal{E}_U(V) = \{\mathcal{E}_U(q) \mid q \in V\}$.

Note that $\mathcal{E}_U(q) = \{\mathcal{E}_a(q) \mid a \in \mathcal{A}\}$.

**Proposition 1.16.** *For any $q \in \overline{V}$ it holds that*

(1) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points,*

(2) *$\mathcal{E}_K^{reg}(q)$ is a $n-1$-dimensional nonempty spacelike submanifold of $K$ which is open relative to $\mathcal{P}_K(q)$ and has $\overline{\mathcal{E}_K^{reg}(q)} = \mathcal{E}_K(q)$ and,*

(3) *$\mathcal{D}_K^{reg}$ is a nonempty submanifold of $\overrightarrow{K} := \pi^{-1}(K)$ ((...)) which is open*

*Proof.* We begin by proving (1):

Let $p$

$\square$

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

**Proposition 1.17.** *For any $q \in \overline{V}$, $\mathcal{E}reg_K(q) \subset K$ and $\mathcal{D}_K^{reg}(q) \subset TU$ are smooth submanifolds of dimension $n-1$ ((D has dim n)).*

*Proof.* ((TODO))

We will focus our attention to the case of $\mathcal{E}reg_U(q)$ as the argument for $\mathcal{D}_U^{reg}(q)$ is analogous Note first that $\mathcal{E}reg_U(q)$ can be rewritten as

$$\{\exp_q(w) \mid w \in L_q^+ M \text{ with } 1 < \rho(q,w)\}.$$

Next by lower semi-continuity of $\rho$ we get that $R = \{w \in L_q^+ M \mid 1 < \rho(q,w)$ is an open set and thus a dimension $(n-1)$ submanifold (this is because $L_q^+ M$ itself is of dimension $(n-1)$). But since $\rho(q,w)$ describes where $\exp_q$ first fails to be a diffeomorphism we get that the surjection $\exp_p : R \to \mathcal{E}reg_U(q)$ is a diffeomorphism. Thus, since $R$ was a manifold of dimension $(n-1)$, $\mathcal{E}reg_U(q)$ is also a manifold and has the required dimension. $\square$

Finally in this section we will prove

**Proposition 1.18.** *Let $q \in \overline{V}$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

*Proof.* ((Still True?)) For the left inclusion assume $p \in \mathcal{E}_U(q)$, i.e. there exists an $a \in \mathcal{A}$ such that $\mathcal{E}_a(q) = p$. Then lemma 1.14(3) immediately yields, $p \in \mathcal{P}_U(q)$ and $\tau(q,p) = 0$. Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. By as $\mathcal{P}_U(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p,q) > 0$, a contradiction.

For the other direction we assume we have $p \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 1.14(3) we only need to prove that $\tau(p,q) = 0$. Suppose that $\tau(p,q) > 0$. By equation 1 there exists an $a \in \mathcal{A}$ and a $s \in [-1,1]$ such that $\mu_a(s) = p$. Now since $\tau(p,q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since $\mu_a$ is timelike, which is a contradiction. $\square$

Thus $\mathcal{E}_U(q)$ truly deserves to be called the "set of earliest observations".

## 1.2 Constructive Solution of the Inverse Problem

((Intro))

### 1.2.1 Reconstruction ...

**Lemma 1.19.** *Thing with dir set reconstruction Also intersection is spacelike somewhere*

**Proposition 1.20.** *((Given data)) The light observations $\mathcal{P}_K(q)$ uniquely determines the light* direction *observation set $\mathcal{C}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$.*

*Proof.* 2nd part: from formula

1st part: from lemma + only finite nonconj cut points + we can parameterize $\mathcal{P}_K(q)$ by a spacelike submanifold of the forwards lightcone $\square$

**Proposition 1.21.** *((Given data)) Given the light direction observation set $\mathcal{P}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$, we can determine the sets $\mathcal{E}reg_K(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

*Proof.* ((Take $\pi^-1(\mathcal{E}_K(q)) \cap \mathcal{C}_K(q)$ for $\mathcal{D}_U(q)$, then remove all cut points (in this case points with equal $p$ but different $v$) in $\mathcal{D}_U(q)$ to obtain $\mathcal{D}_K^{reg}(q)$ and project again)) $\square$

### 1.2.2 Construction of $V$ as a topological manifold

((Intro))

Next we aim to reconstruct the topological and differential data of $V$. To that end we define the following functions.

For $q \in \overline{V}$ we define the function $F_q : \mathcal{A} \to \mathbb{R}$ by $a \mapsto f_a(q)$. We can then define the function

$$\begin{aligned} \mathcal{F} : \overline{V} &\to \mathbb{R}^{\mathcal{A}} \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in \overline{V}$ to the function $F_q : \mathcal{A} \to \mathbb{R}$. We endow the set $\mathbb{R}^{\mathcal{A}} = \{f : \mathcal{A} \to \mathbb{R}\}$ with the product topology.

((...))

We begin by establishing the topological structure:

**Lemma 1.22.** *(($V$ or $\overline{V}$?)) The map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

*Proof.* ((Works the same, use direction set reconstruction)) □

### 1.2.3 Construction of $V$ as a smooth manifold

Having established the topological structure of $V$ we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to $V$.

**Definition 1.23** (Coordinates on $V$)**.** We first define

$$\mathcal{Z} = \{(q, p) \in V \times K \mid p \in \mathcal{E}_U^{reg}(q)\}.$$

Then for every $(q, p) \in \mathcal{Z}$ there is a unique $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q, w) > 1$. Existence follows from lemma 1.14 while uniqueness follows from the fact that $p \in \mathcal{E}_U^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Theta : \mathcal{Z} &\mapsto L^+ V \\ (q, p) &\mapsto (q, w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0, w_0) \subset TM$ be a $\varepsilon$-neighborhood of $(q_0, w_0)$ with respect to the Sasaki-metric induced on $TM$ by $g^+$.

**Lemma 1.24.** *Let $(q_0, p_0) \in \mathcal{Z}$ and $(q_0, w_0) = \Theta(q_0, p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0, w_0) &\to M \times M \\ (q, w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

*is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) := X(\mathcal{W}_\varepsilon(q_0, w_0))$. When $\varepsilon$ is small enough, $\Theta$ coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0, p_0)$ with the inverse map of $X$. Moreover $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and the map $\Theta : \mathcal{Z} \to L^+ M$ is smooth.*

*Proof.* ((Works the same with minor adjustments?)) $\square$

((Explain what we're doing now))

**Proposition 1.25.** *Let $q \in \overline{V}$ and $(q_0, p_j) \in \mathcal{Z}$, $j = 1, \ldots, n$ and $w_j \in L^+_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j$, $j = 1, \ldots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of $q_0$ such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=1}^n$$

*define smooth coordinates on $V_1$. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of $f_{a_j}$ with respect to $q$ at $q_0$, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.*

*Proof.* ((Works almost the same, maybe clarify implicit function theorem stuff)) $\square$

**Definition 1.26** (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=1}^n \subset \mathcal{A}^n$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_U^{reg}(q)$ for all $j = 1, \ldots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are $C^0$-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are $C^\infty$-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, the above $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in \mathcal{A}^n$ there might be several sets $W$ for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form $C^0$-observation coordinates to clarify the notation we will sometimes denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of $V$ and makes $\mathcal{F}$ a diffeomorphism.

**Proposition 1.27.** *Let $\widehat{q} \in \widehat{V}$. Then there exist $C^\infty$-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$.*

*Furthermore, given the data from 1.4 we can determine all $C^0$-observation coordinates around $\widehat{q}$.*

*Finally given any $C^0$-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data 1.4, allows us to determine whether they are $C^\infty$-observation coordinates around $\widehat{q}$.*

*Proof.* ((Works the same way)) $\square$

### 1.2.4 Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths $\mu_a$ and the conformal class of the metric on $U$ determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

**Lemma 1.28.** *The data given in 1.4 determine a metric $G$ on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

*Proof.* ((Works the same)) □

# References

[Bee81] Easley Beem Ehrlich. *Global Lorenzian Geometry*. Pure and Applied Mathematics, 1981.

[Hal08] Rodney G Halburd. “Partial Differential Equations in General Relativity”. In: *Journal of Physics A: Mathematical and Theoretical* 41.47 (2008), p. 479003.

[KLU17] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*. 2017. arXiv: `1405.3386 [math.DG]`.

[One83] Barrett O’neill. *Semi-Riemannian geometry with applications to relativity*. Academic press, 1983.