Reconstruction of Lorentzian manifolds from null boundary light observation sets

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Abstract

Let (M, g) be a globally hyperbolic Lorentzian manifold and $p^+ \gg p^-$ be points in M separated by a timelike curve. Let V be an open subset of $J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$. We show that the topological, differentiable and conformal structure of V can be uniquely reconstructed from the *light observation sets* on the future null boundary K of $J(p^-, p^+)$, i.e. the sets $\mathcal{P}_K(q) := \mathcal{L}_q^+ \cap K$ for $q \in V$. Furthermore we show that we can reconstruct the topological data of V also if it extends to include the boundary K, even though the light observation sets are degenerate in this case. ((TODO: -use coloneqq for definitions, -finish and properly embed applications chapter, -do more with conclusion chapter, -finish $dH- > 0$ on bounday proof, -do graphics, -clean up appendices , -explain plots more))

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Chapter 1

Introduction

We aim to show that on a globally hyperbolic $1+n$-dimensional Lorentzian manifold (M, g) with two points suitable $p^- \ll p^+ \in M$ separated by a timelike path and a suitable *source set* $V \subset M$ which is and contained within the intersection of the causal future of p^+ and the causal past of p^- denoted $J(p^-, p^+)$, we can reconstruct the topological, differential and conformal structure of V using the light cone observations on the future boundary of $J(p^-, p^+)$.

This thesis will be structured as follows: In this chapter we will introduce some concepts in Lorentzian geometry needed to fully state our main results and give some outline as to their proof. We will also give an overview of related results.

The chapter 2 contains a lot of the technical results on light cones and their observation on a null surface. This then enables us in chapter 3 to prove the main reconstruction result where the source set is an open subset of the interior. In 4 we will then extend this result to settings where the source set extends up to be boundary. ((Applications and Conclusion and Appendix))

1.1 Setting

In the following (M, g) will always a globally hyperbolic Lorentzian manifold. We beging by introducing some definitions which will be very useful for describing the causality relations on M:

Definition 1.1.1. We write

1. $p \ll q$ if $p \neq q$ and there exist a future-pointing timelike curve from p to q,
2. $p < q$ if $p \neq q$ and there exist a future-pointing causal curve from p to q,
3. $p \leq q$ if $p = q$ or $p < q$.

We then define the *chronological future* and *causal future* of a point $p \in M$ as

$$I^+(p) := \{q \in M \mid p \ll q\}$$
$$J^+(p) := \{q \in M \mid p \leq q\}.$$

Chronological and causal past are defined analogously. We can extend these definitions to arbitrary sets by setting $I^\pm(A) := \bigcup_{p\in A} I^\pm(p)$ and $J^\pm(A)$ analogously. For two points $p^- \ll p^+$ resp. $p^- \leq p^+$ we denote $I(p^-, p^+) := I^+(p^-) \cap I^-(p^+)$ resp. $J(p^-, p^+) := J^+(p^-) \cap J^-(p^+)$.

For a point $p \in M$ we now look at the the null vectors in T_pM and null geodesics starting at p:

Definition 1.1.2 (Light Cones)**.** Let

$$L_pM := \{v \in T_pM \setminus \{0\} \mid g(v, v) = 0\}$$

be the set of null vectors at $p \in M$. We can split L_pM into L_p^+M and L_p^-M the future- and past-pointing null vectors. Furthermore we can define the bundle $LV := \bigcup_{p\in V} L_pV \subset TM$.

We now define the *future light cone* of $p \in M$ to be

$$\mathcal{L}_p^+ := \exp_p(L_p^+M) \cup \{p\}.$$

$\mathcal{L}_p^-$ is defined analogously.

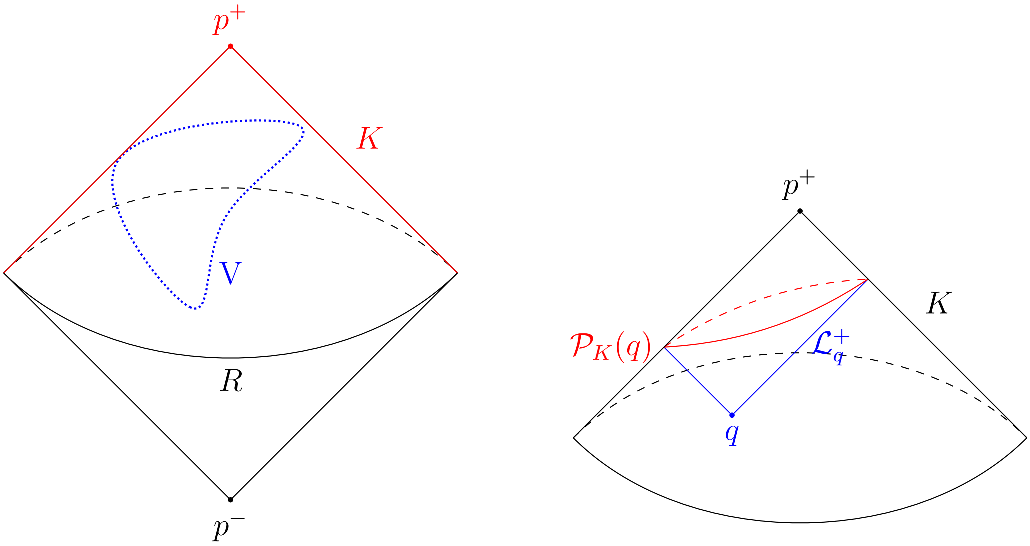
Note that for $p \in M$ we have $\mathcal{L}_p^+ \subset J^+(p)$ and $\mathcal{L}^+{}_p \supset J^+(p) \setminus I^+(p)$ if M is globally hyperbolic.

For a point $p \in M$ and a vector $v \in T_pM$, we will often write $\gamma_{p,v}$ for the unique geodesic starting at p with velocity v; we have $\gamma_{p,v}(t) = \exp_p(tv)$. Let $p \in M$ and $v \in T_pM$, we say that $\gamma_{p,v}$ has a *conjugate point* at $p' = \gamma_{p,v}(t)$ if $d\exp_p|_{tv}$ does not have full rank. If v is a null geodesic, i.e. $v \in L_q^\pm M$, we say that $\gamma_{p,v}$ has a *cut point* at $p' = \gamma_{p,v}$ if $\tau(p, p') = 0$ but $\tau(p, \gamma_{p,v}(t')) > 0$ for all $t' > t$; here $\tau : M \times M \to \mathbb{R}$ is the *time separation function* mapping two point to the length of the maximal geodesic joining them.

For a more in-depth introduction to causal relations, light cones, cut and conjucate points as well as an overview of all relevant results we refer the reader to appendix B.

Equipepd with this language we can state the regularity condition necessary for our reconstruction results to apply:

Definition 1.1.3 (Suitable)**.** We call $p^- \ll p^+ \in M$ *suitable* if p^+ has no past cut points in $\mathcal{L}_{p^+}^- \cap J^+(p^-)$. Furthermore we call $p^- \ll p^+ \in M$ and $V \subset J(p^-, p^+)$ *suitable*, if p^- and p^+ are *suitable* and no null geodesic starting in V has a conjugate point in $\mathcal{L}_{p^+}^- \cap J^+(p^-)$.



(a) Compact causal diamond $J(p^-, p^+)$ (b) Light cone observation set of a single point

Figure 1.1: Illustrations of $J(p^-, p^+)$, K, V and $\mathcal{P}_K(q)$ in the Minkowski case.

We will refer to the future boundary of $J(p^-, p^+)$, i.e. the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-, as the *observation set* $K := \mathcal{L}^-_{p^+} \cap J^+(p^-)$. We can then make formal the notion of light observations on the observation set K:

Definition 1.1.4 (Light Observation Set)**.** The *light observation set* of a point $q \in J(p^-, p^+)$ is defined as

$$\mathcal{P}_K(q) := \mathcal{L}_q^+ \cap K.$$

The collection of these sets is $\mathcal{P}_K(V) := \{\mathcal{P}_K(q) \mid q \in V\} \subset \mathcal{P}(K)$.

Note that $\mathcal{P}_K(V)$ is an unindexed set and we thus have a priori no information which observation set $\mathcal{P}_K(q)$ belongs to which point $q \in V$.

Now we are ready to state the main theorem concerning the reconstruction in the case where the source set V is contained within the interior of $J(p^-, p^+)$:

Theorem 1.1.5 (Interior Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+, V_j \subset M_j$* suitable *in M_j we denote $K_j := \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

If we assume that the source sets $V_j \subset J(p_j^-, p_j^+)^\circ$ are open subsets of the interior of $J(p^-, p^+)$. Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \{\Phi(\mathcal{P}_K(q)) \mid q \in V\} = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Essentially, the theorem states that on two Lorentzian manifolds with conformally equivalent light cone observations the source sets must be conformally equivalent. This implies that the observations uniquely determine the source set V up to conformal diffeomorphism, which is exactly what we want when we say that we can reconstruct V from the observations.

The statement for the boundary reconstruction is essentially the same. The one notable difference (and also the reason why it is called boundary construction) is that V is no longer required to be a subset of the interior of $J(p^-, p^+)$, but can now be a subset of $J(p^-, p^+) \setminus \{p^+\}$. We will only prove the topological reconstruction in this case as the differential and conformal reconstructions were beyond the scope of this thesis:

Theorem 1.1.6 (Boundary Reconstruction)**.** *Let $(M_j, g_j), p_j^{\pm}, V_j, K_j$ as in the previous theorem. If we now assume that the source sets $V_j \subset J(p^-, p^+) \setminus \{p^+\}$ are open subset and*

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \{\Phi(\mathcal{P}_K(q)) \mid q \in V\} = \mathcal{P}_{K_2}(V_2)$$

there exists a homeomorphism $\Phi : V_1 \to V_2$.

We will later conjecture that Φ can also be made to be a conformal diffeomorphism that preserves causality as in the previous theorem.

1.2 Proof Outline

Remark 1.2.1 (Data). In the following we will use an equivalent formulation to Theorems 1.1.5 and 1.1.6: Namely we will show that if $(M, g), K, V, p^+, p^-$ are as in Theorem 1.1.5 resp. 1.1.6, then given the *data*

(1) The smooth manifold K,

(2) the conformal class of $g|_K$ and

(3) the set of light cone observations $\mathcal{P}_K(V)$

we can construct a space $\widehat{V}$ which is conformally equivalent to V. In Theorems 1.1.5 and 1.1.6, the assumptions assure that for both $(M_i, g_i), K_i, V_i, p_i^+, p_i^-$ we have the same data. Therefore the reconstruction will yield the same $\widehat{V}$ which will then be conformally equivalent to both V_1 and V_2. This in turn implies that V_1 and V_2 are conformally equivalent.

In light of this we will from here on restrict ourselves to only one globally hyperbolic Lorentzian manifold (M, g) with p^+, p^-, V *suitable* and show how given the *data* we can construct $\widehat{V}$.

1.2.1 Interior Case

A core idea is to cover K with a family of observers $\mu_a : [-T_a, 0] \to K$, $a \in S^{n-1}$ travelling along future-pointing null geodesics. For a fixed observer $a \in S^{n-1}$ we can then define the *observation time function* $f_a : J(p^-, p^+) \to \mathbb{R}$ to be the time at which the observer μ_a first "sees" light emitted by q, i.e.

$$f_a(q) := \inf(\{t \in [-T_a, 0] \mid \mu_a(t) \in \mathcal{P}_K(q)\} \cup \{1\}).$$

For a fixed $q \in J(p^-, p^+)$ we then denote $F_q(a) := f_a(q)$. These functions have many desirable properties, i.e. $q \mapsto F_q$ is continuous, $F_q = F_{q'}$ implies $q = q'$ and $F_{q_n} \to F_{q_0}$ implies $q_n \to q_0$ on V. And importantly the light observation set $\mathcal{P}_K(q)$ for some $q \in J(p^-, p^+)$ fully determines F_q.

We can then define the map $\mathcal{F} : V \to \widehat{V} := \mathcal{F}(V) \subset \mathcal{C}^\infty(S^{n-1}); q \mapsto F_q$. We can fully determine $\widehat{V}$ from the light observation sets $\mathcal{P}_K(V)$. Because we have a topology on $\mathcal{C}^\infty(S^{n-1})$ we can determine the subspace topology on $\widehat{V}$ and the nice properties of F_q ensure that the map $\mathcal{F}$ is a homeomorphism, allowing us to determine the topology on V.

To determine the differential structure of V we let $q \in V$, and pick $1 + n$ observers $a_0, \ldots, a_n \in S^{n-1}$. We denote $w_i \in L_q^+ M$ for the null vectors pointing from q to $p_i := \mu_{a_i}(f_{a_i}(q))$ the points of earliest observation. We then show that if the set $(w_0, \ldots, w_n)$ is linearly independent, the map $q' \mapsto (f_{a_0}(q'), \ldots, f_{a_n}(q'))$ defines smooth coordinates around q. We furthermore show that such coordinates always exists and we can determine them using the light observation sets, allowing us to determine the differential structure of V.

Finally to reconstruct the conformal type of the metric on V we show that the light observation sets allow us to determine all lightlike geodesics around a point $q \in V$, this allows us to determine the light cones for all points $q \in V$, which is equivalent to knowing the conformal type of the metric, finishing the proof.

1.2.2 Boundary Case

Recall that in this case the source set V can also intersect the boundary of $J(p^-, p^+)$ and thus the observation set K. This poses some difficulties because as the points $q \in V$ get closer to K, the observations become increasingly degenerate and lose many of their nice properties in the limit case $q \in K$. To solve this issue we will reconstruct the topology in two parts: The interior reconstruction allows us to reconstruct the topology on $V_2 := V \cap J(p^-, p^+)^{\circ}$, i.e. the interior part of V. The main challenge will now be to develop a reconstruction procedure on an open neighborhood of the boundary $K \setminus p^+$:

To that end we define the *unique minimum domain* $D := \{q \in J(p^-, p^+)^{\circ} \cup K \mid F_q \text{ has a unique minimum}\}$. We then show that D is indeed an open neighborhood of K, because as q approaches K in must have a unique minimum on S^{n-1}. An illustrative example for this behavior is the case of Minkowski space ((Do actual example?)). We then show that the unique minimum of F_q is well behaved on D.

Next we use these unique minima to "smooth out" the observation time functions F_q as they approach the boundary: For $q \in D$ with unique minimum at a_q we use a smooth bump function χ_{a_q} with $\chi_{a_q}(a_q) = 0$ and $\chi_{a_q}(a') = 1$ for a' far enough away from a_q. We then define *smoothed observation time functions* $H_q(a) := \chi_{a_q}(a)F_q(a)$ which have $H_q(a_q) = 0$. These functions are well behaved even if $q \in K$ and we can, analogously to the interior reconstruction, define a map $\mathcal{H} : D \to \widehat{D} = \mathcal{H}(D) \subset \mathcal{C}^\infty(S^{n-1}); q \mapsto H_q$, which can then be proven to be a homeomorphism, with respect to the canonical subspace topology. This allows us to recover the topology on $V_1 := V \cap D$.

Because all these constructions again only require the light observations sets $\mathcal{P}_K(V)$ and we have $V_1, V_2 \subset V$ open with $V = V_1 \cup V_2$ we can then combine the topologies on V_1 and V_2 to reconstruct the topology of V.

1.3 Related work

Both the setting as well as many of the techniques used in this thesis are mainly inspired by the work of Kurylev, Lassas, and Uhlmann [KLU17] and Hintz and Uhlmann [HU17]: [KLU17] treats the conformal reconstruction of a source spacetime V in the case where instead of a null hypersurface K we observe the light cones on an open set U. The reconstruction is then carried out by endowing U with a set of observers and measuring their observation times, similar similar to our approach. [HU17] contains a similar result but in case where the observations take place on a timelike boundary which reflects null geodesics. A lot of the techniques used in this thesis for dealing with the "slimness" of the observation sets are are employed in our approach as well.

For further results on reconstruction in the Lorentzian case we mention the work of Lassas, Oksanen, and Yang [LOY16] showing that the knowledge of the time separation function on a timelike hypersurface Σ allows one to the C^∞-jet of the metric on Σ and Larsson [Lar15] proving that the isometric structure of a compact Lorentzian manifold with boundary is completely determined by geodesic data at the boundary.

We also mention work of Wang and Zhou [WZ19] and Lassas, Uhlmann, and Wang [LUW18] which treats the isometric or conformal reconstruction of a spacetime given the source-to-solution map of appropriate nonlinear wave equations on a Lorentzan manifold.

There are also many related results in the Riemannian case: Lassas, Saksala, and Zhou [LSZ17] treat the reconstruction of a compact Riemannian manifold using the scattering data on the boundary, Stefanov, Uhlmann, and Vasy [SUV17] show on a Riemannian manifold with convex boundary then knowledge of the distance function on the boundary allows the reconstruction of the metric on some neighborhood of the boundary, and finally Pestov and Uhlmann [PU05] show a similar result which allows the reconstruction of the metric on the whole manifold in the two-dimensional case.

1.4 Notation

((Smaller Header? Even necessary?)) The notation used in thesis should be very close to the one used commonly in differential geometry and topology, but for the sake of completenes and clarity we note that for some subset $A \subset X$ of a topological space X, $\overline{A}$ denotes the closure of A, ∂A the boundary and A^o the interior.

If M is a smooth manifold and $p \in M$ a point then T_pM denotes the tangent space at p and T_p^*M the cotangent space. If $f : M \to N$ is a map between smooth manifolds $df : TM \to TN; [\gamma] \mapsto [f \circ \gamma]$ denotes the differential. When it is clear from the context $\pi : TM \to M; (p, v) \to v$ will be the canonical projection from the tangent bundle to the manifold, and for a product $A \times B$, $\pi_a : A \times B \to A$ and $\pi_b : A \times B \to B$ will be the canonical product projections.

For the sake of distinguishing the one timelike dimension from the spacelike dimensions on Lorentzian manifolds we usually let a Lorentzian manifold (M, g) be of dimension $1 + n$. And often, when it is clear from the context on what interval a geodesic γ is defined we will simply write $p \in \gamma$ to denote a point on the geodesic.

Chapter 2

Geometric Preliminaries

As noted in the introduction, in this chapter we will set up most of the machinery and prove most of the properties needed to prove theorems 3 and 4

2.1 Geometry of the light cone observations

We will begin by proving some relevant properties of K and $\mathcal{P}_K(q)$.

2.1.1 Observer Set

This lemma is important because characterization (1) shows that on $K = \mathcal{L}^-_{p^+} \cap J^+(p^-)$ all points p have $\tau(p, p^+) = 0$. (2-3) are also very useful because they show the light cones of points q in the interior of $J(p^-, p^+)$ will only intersect K in its relative interior, often allowing us to ignore the points where K fails to be a submanifold.

Lemma 2.1.1. *Let $p^-, p^+ \in M$ suitable and $R := K \setminus I^+(p^-)$ the past boundary of K then:*

(1) $K = J(p^-, p^+) \setminus I^-(p^+)$,

(2) $\mathcal{L}^-_{p^+} \cap J(p^-, p^+)^{\circ} = \emptyset$, *and*

(3) $\mathcal{L}^-_{p_0} \cap J(p^-, p^+)^{\circ} = J^-(p_0) \cap J(p^-, p^+)^{\circ} = \emptyset \quad \forall p_0 \in R$.

Proof. (1) We first rewrite $J(p^-, p^+) \setminus I^-(p^+) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ and immediately get $(J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-) \subset \mathcal{L}^-_{p^+} \cap J^+(p^-) = K$ as $J^-(p^+) \setminus I^-(p^+) \subset \mathcal{L}^-_{p^+}$. For the other inclusion we note that because we assumed $p^- \ll p^+$ to be suitable, we have we have $\tau(p, p^+) = 0$ for all $p \in K$. Together with $p \in \mathcal{L}^-_{p^+}$,

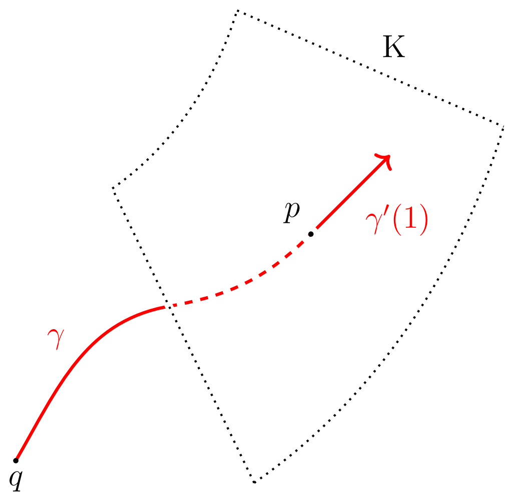


Figure 2.1: Illustration of a null geodesic γ from a point $q = \gamma(0) \in J(p^-, p^+)^{\circ}$ to a point $p = \gamma(1) \in K$ on the observations set, where γ must be transversal.

this implies $p \in J^-(p^+) \setminus I^-(p^+)$. Furthermore $p \in K$ also implies $p \in J^+(p^-)$. Putting this together we get $p \in (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ proving the equality.

For part (2) we assume there exists a $p \in \mathcal{L}^-_{p^+} \cap J(p^-, p^+)^{\circ}$. Note that $J(p^-, p^+)^{\circ} = I^+(p^-) \cap I^-(p^+)$. We thus have $p \in I^+(p^-) \subset J^+(p^-)$, which together with $p \in \mathcal{L}^-_{p^+}$ implies $p \in K$. But now we have $p \in I^-(p^+)$ and $p \in K$, a contradiction to (1).

Finally for part (3) we assume that there exists a $p_0 \in R$ and $p \in J^-(p_0) \cap J(p^-, p^+)^{\circ}$. Because $J(p^-, p^+)^{\circ} \subset I^+(p^-)$ there exists a timelike path from p^- to p. Because $p \in J^-(p_0)$ we can use proposition C.3.13 to construct a timelike path from p^- to p_0 implying $p_0 \in I^+(p^-)$. But because $p \in R = K \setminus I^+(p^-)$ this is a contradiction. $\mathcal{L}^-_{p_0} \subset J^-(p_0)$ then yields the first equality. $\square$

The following lemma will be essential in showing many of the desirable properties of the *observation time* functions, because it ensures that for any $q \in J(p^-, p^+)^{\circ}$ the intersection of the forwards light cone $\mathcal{L}^+_q \cap K$ is not degenerate. The fact that this fails if the source point q is in K is precisely why many of the nice properties of the observation time functions dont carry over to points on K.

Lemma 2.1.2. *For any $q \in J(p^-, p^+)^{\circ}$ the restriction of the exponential map to null vectors $\exp_q : L^+_q M \to M$ is* transverse *to K, i.e. for all $w \in L^+_q M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_p K$.*

Proof. In order to achieve a contradiction we assume that there exists a $q \in J(p^-, p^+)^{\circ}$ and a $w \in L^+_q M$ such that with $p = \gamma_{q,w}(1) \in K$ and $v := \gamma'_{q,w}(1) \in L_p K$. Since K is generated by backwards null geodesics originating at p^+ there exists a

$u \in L^-_{p^+}M$ and a $t \in \mathbb{R}_+$ with $\gamma_{p^+,u}(t) = p, \gamma'_{p^+,u}(t) = -v$. We can thus obtain an unbroken past-pointing null geodesic from p^+ to q by connecting $\gamma_{p^+,u}$ and $\gamma_{p,-v}$. But this implies that $q \in \mathcal{L}^-_{p^+} \cap J^+(p^-)$ which is a contradiction to 2.1.1(2).

We now prove that this implies that $\exp_q : L^+_q M \to M$ is transverse to K. We need to show that for every $w \in L^+_q M$ with $\exp_q(w) = p \in K$ we have

$$\mathrm{im}(d\exp_q|_w) \oplus T_pK = T_pM.$$

As T_pK is a null hypersurface we only need to prove that $\mathrm{im}(d\exp_q|_w)$ contains a null vector which is not a multiple of the null vector $v \in T_pK$ generating $T_pK = v^\perp$. But by the properties of the exponential map, $\mathrm{im}(d\exp_q|_w)$ contains $v' = \gamma'_{q,w}(1) \in T_pM$. And since we just proved that $v' \notin T_pK$, v' cannot be a multiple of v, as desired. □

This next lemma will also be very useful. The core idea is that K is a subset of $J^-(p^+)$. $J^-(p^+)$ has, by definition, the useful property that once a lightlike curve leaves it it can never return.

Lemma 2.1.3. *For $q \in J(p^-, p^+)^{\circ}$ and $w \in L^+_q M$ there exists a unique $t_w \in (0, \infty)$ such that $\gamma_{q,w}(t_w) \in K$.*

Proof. Let $q \in J(p^-, p^+)^{\circ}$ and $w \in L^+_q M$, by lemma B.2.2 any geodesic starting in the compact set $J(p^-, p^+)$ must eventually leave it, intersecting the boundary. As K is the future boundary of $J(p^-, p^+)$ there exists at least one $t_w \in (0, \infty)$ with $p = \gamma_{q,w}(t_w) \in K$. We now show $\gamma_{q,w}(t') \notin K$ for any other $t' \neq t_w$.

First let us consider the case $t' < t_w$. We can then append $\gamma_{q,w}|_{[t',t_w]}$ to the null geodesic $\sigma \subset K$, which has $\sigma(0) = p$ and $\sigma(1) = p^+$, to get a broken lightlike path from $\gamma_{q,w}(t')$ to p^+. The fact that this path must be broken follows from the transversality proven in the previous lemma. But the existence of this broken path implies $\tau(\gamma_{q,w}(t'), p^+) > 0$ and thus $\gamma_{q,w}(t') \in I^-(p^+)$. But as $K = J(p^-, p^+) \backslash I^-(p^+)$ we have $\gamma_{q,w}(t') \notin K$

Conversely we now assume $t' > t_w$. Again by the transversality of $\gamma_{q,w}$ to K we get that for $t' - t_w > \varepsilon > 0$ small enough we have $\gamma_{q,w}(t_w + \varepsilon) \notin J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$ because K is the future boundary of $J(p^-, p^+)$. As any point on $\gamma_{q,w}$ is in $J^+(p^-)$ we must have have $\gamma_{q,w}(t_w + \varepsilon) \notin J^-(p^+)$, i.e. there exists no lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+. But if $\gamma_{q,w}(t') \in J^-(p^+)$ there exists a path σ from $\gamma_{q,w}(t')$ to p^+ and we could construct a lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+ by appending $\gamma_{q,w}|_{[t_w+\varepsilon,t']}$ to σ, a contradiction. We thus have $\gamma_{q,w}(t') \notin J^-(p^+) \supset J(p^-, p^+) \supset K$, completing the proof. □

2.1.2 Parametrization of the observer set

We will now exploit the fact that no past null geodesic starting at p^+ has a cut point in K to construct a smooth parametrization of K, equivalent to a smoothly parameterized family of geodesics, generating K.

We will need to use the Lorentzian splitting theorem as stated in chapter 14 of [Bee81]:

Theorem 2.1.4 (Lorentzian Splitting Theorem)**.** *Let (M, g) be a globally hyperbolic Lorentzian manifold. Then (M, g) splits isometrically as a product $(\mathbb{R} \times H, -dt^2 \oplus h)$, where (V, h) is a complete Riemannian manifold.*

We will often denote the smooth projection from M to the timelike component as $\mathcal{T} : M \to \mathbb{R}$. Note also that such a splitting also induces a canonical Riemannian metric g^+ on M given by $g^+ := dt^2 \oplus h$. For such a metric we denote $CL_p^{\pm}M := \{v \in L_p^{\pm}M \mid \|v\|_{g^+} = \sqrt{2}\}$ for some $p \in M$, $\sqrt{2}$ is convenience as we will see soon and actual constant does not matter. We can then construct isometries such that

$$S^{n-1} \simeq CT_{\pi_h(p)}H := \{v_H \in T_{\pi_h(p)} \mid \|v_H\|_h = 1\} \simeq CL_p^{\pm}M. \tag{2.1}$$

$S^{n-1} \simeq CT_{\pi_H(p)}H$ follows by picking coordinates around $\pi_h(p)$. And the map $a_H \in CT_{\pi_h(p)}H \mapsto \partial_t + a_H \in CT_{\pi_h(p)}H$, is an isometry as desired; here we used the fact vecors in $CL_p^{\pm}M$ have length $\sqrt{(2)}$. From here on we will thus often use S^{n-1} and $CL_p^{\pm}M$ interchangably.

Note that this construction implies

$$\frac{d\mathcal{T}(\gamma_{p^+,v}(t))}{dt} = -1 \quad \text{for all } t \in [0, \infty), v \in CL_{p^+}^{-}M. \tag{2.2}$$

The following lemma closely resembles lemma 2.1.3:

Lemma 2.1.5. *Let $v \in CL_{p^+}^{-}M$ then there exists a unique $T_v \in \mathbb{R}_+$ such that $\gamma_{p^+,v}([0, T_v]) \in K$ and $\gamma_{p^+,v}(t') \notin K$ for all $t' > T_v$.*

Proof. Importantly we have $p^- \notin \mathcal{L}_{p^+}^{-}$ because $\tau(p^-, p^+) > 0$ and p^+ does not have any past cut points in $K = \mathcal{L}_{p^+}^{-} \cap J^+(p^-)$. The rest of the proof is essentially analogous to the proof of lemma 2.1.3. □

Lemma 2.1.6. *The map $v \in CL_{p^+}^{-}M \mapsto T_v \in \mathbb{R}_+$ is continuous.*

Proof. We will first show that it is bounded. Note that we have $\mathcal{T}(p) \geq \mathcal{T}(p^-)$ for all $p \in J^+(p^-)$ because $\mathcal{T}$ is strictly increasing along causal geodesics. Now equation 2.2 guarantees $\mathcal{T}(\gamma_{p^+,v}(2(t^+ - t^-))) = \mathcal{T}(p^+) - 2(t^+ - t^-) < t^-$, where $t^{\pm} := \mathcal{T}(p^{\pm})$. We thus have $\gamma_{p^+,v}(2(t^+ - t^-)) \notin J^+(p^-)$ and $T_v < t^+ - t^-$ for all $v \in CL_{p^+}^{-}M$.

To show that the map is continuous we assume $v_n \to v_0 \in CL_{p^+}^{-}M$ but T_{v_n} does not converge to T_{v_0}. Because T_{v_n} is bounded there must exist a convergent subsequence $T_{v_j} \to T' \neq T_{v_0}$. We denote $p_j := \gamma_{p^+,v_j}(T_{v_j})$ and have $p_j \to p' := \gamma_{p^+,v_0}(T')$. Furthermore because $p_j \in J^+(p^-) \setminus I^+(p^-)$ closed we also have $p' \in J^+(p^-) \setminus I^+(p^-)$, but this is a contradiction to the previous lemma. □

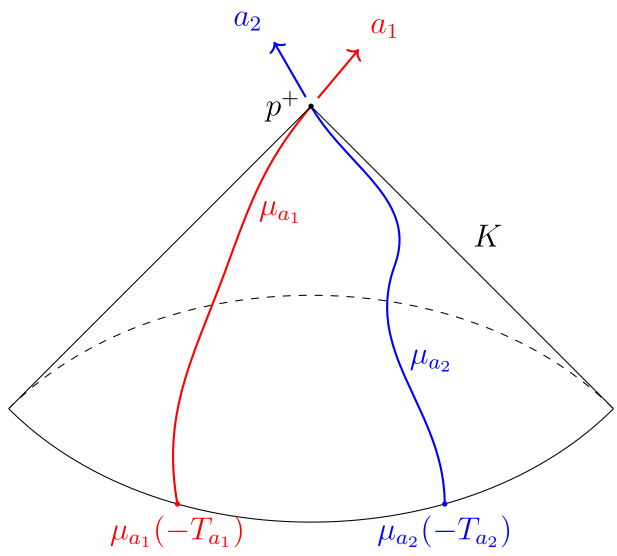


Figure 2.2: Illustration of two observers on the observation set as constructed in proposition 2.1.7; $a_1, a_2 \in S^{n-1}$ correspond to unit null directions at p^+. The observer μ_{a_i} is then taken to be the null geodesic which hits p^+ with the repsective null direction.

For some $a \in S^{n-1}$ often write $T_a := T_v$ where $v \in CL^-_{p^+}M$ is obtained via the equivalence from equation 2.1. Because $a \mapsto T_a$ is a continuous map on a compact set there exists a maximum $T_{S^{n-1}}$.

With this setup we can now parameterize K:

Proposition 2.1.7. *Let $\mathcal{S} := \{(a,t) \in S^{n-1} \times (-\infty, 0] \mid t \in [-T_a, 0]\}$. Then the map*

$$\begin{aligned} \Theta : \mathcal{S} &\to K \\ (a,t) &\mapsto \exp_{p^+}(-tv_a) \end{aligned}$$

where $v_a \in CL^-_{p^+}M$ is again given by equation 2.1, has the following properties:

(1) *$\Theta : \mathcal{S} \to K$ is a surjective smooth map such that the curves*

$$\mu_a := t \mapsto \Theta(a,t) \quad a \in S^{n-1}, t \in [-T_a, 0]$$

are null geodesics,

(2) *$\Theta(S^{n-1} \times \{0\}) = \{p^+\}, \quad \Theta(\{(a, T_a) \mid a \in S^{n-1}\}) = R$ and*

(3) *$\Theta : \mathcal{S}^\times := \{(a,t) \in S^{n-1} \times (-\infty, 0] \mid t \in (-T_a, 0)\} \to K \setminus (p^+ \cup R)$ is a diffeomorphism.*

Proof. To show (1) we first note that the fact that Θ is surjective follows from lemma 2.1.5, while smoothness and the geodesic property follows from the fact that $\exp_{p^+}$ and $a \in S^{n-1} \mapsto v_a \in v_a \in CL^-_{p^+}M$ are smooth. The two equalities in (2) follow from definition together with lemma 2.1.5. Finally for (3) we note that because $a \mapsto T_a$ is a continuous map, $\mathcal{S}^\times$ is a open submanifold of $S^{n-1} \times (-\infty, 0]$. Furthermore because by assumption p^- and p^+ are *suitable*, past null geodesics originating at p^+ have no cut points in K, i.e. $\rho(p^+, v_a) > T_a$ for all $a \in S^{n-1}$. But this means that Θ is a diffeomorphism on $\mathcal{S}^\times$ and by (2) we have $\Theta(\mathcal{S}^\times) = K \setminus (\{p^+\} \cup R)$, as desired. □

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will treat K itself as a submanifold when it is clear that we are working away from the boundary. This is often the case as by 2.1.1(2-3) no null geodesic originating from the interior of $J(p^-, p^+)$ can reach p^+ or R, i.e. the boundary of K.

Furthermore by the properties of Θ we have

$$\mu_a([-T_a, 0]) \cap \mu_{a'}([-T_a, 0]) = \{p^+\} \text{ for } a \neq a' \in S^{n-1} \text{ and} \tag{2.3}$$

$$\bigcup_{a \in S^{n-1}} \mu_a([-T_a, 0]) = K \tag{2.4}$$

which implies that for every point $p \in K \setminus \{p^+\}$ there exist unique (a, t) such that $\mu_a(t) = p$. Owing to equation 2.2 also have

$$\frac{d\mathcal{T}(\mu_a(t))}{dt} = 1 \quad \text{for all } a \in S^{n-1}, t \in [-T_a, 0] \tag{2.5}$$

which implies

$$\mathcal{T}(\mu_a(t)) = \mathcal{T}(p^+) + t \quad \text{for all } a \in S^{n-1}, t \in [-T_a, 0] \tag{2.6}$$

making $\mathcal{T}(\mu_a(t))$ independent of $a \in S^{n-1}$.

And finally we can see that we can construct the map Θ and thus the geodesics μ_a using only the data outlined in remark 1.2.1, because we have the smooth structure of K and $g|_K$ determines all null geodesics on K.

2.1.3 Differential Constructions

Having parameterized K we can now outline some differential properties of K and $\mathcal{P}_K(q)$:

The first lemma closely resembles lemma 2.5 in [HU17] with only minor adjustments to adapt it to our case. It is reproduced here for the sake of completeness. It will allow us to reconstruct the direction of incoming light rays at point in $\mathcal{P}_K(q)$ which will locally correspond to the spacelike hypersurface.

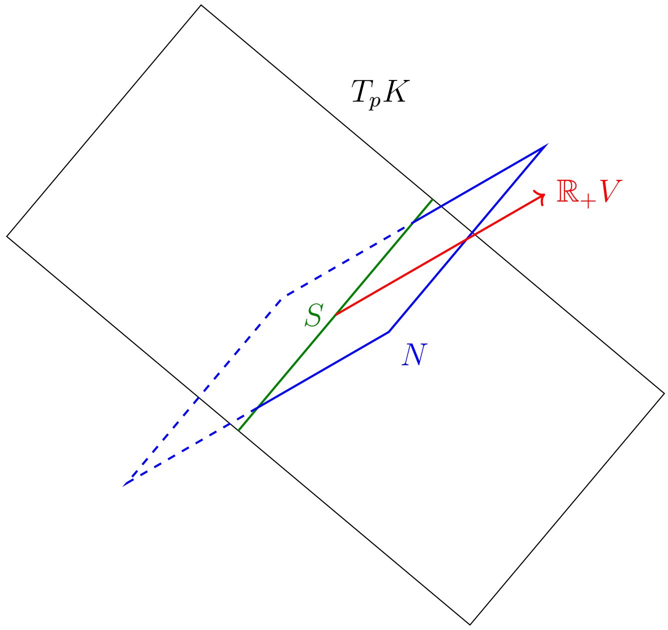


Figure 2.3: Illustration of lemma 2.1.8; S, N and $\mathbb{R}_+V$ are as below.((more))

Lemma 2.1.8 (Direction Reconstruction)**.** *Let $p \in K$ then there exists a bijection Φ between the space $\mathcal{S}$ of spacelike hyperplanes $S \subset T_pK$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+V \subset T_pM$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto T_pK \cap V^\perp \in \mathcal{S}$.*

Moreover there exists a bijection between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_pM$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \ni S \mapsto S \oplus \operatorname{span}\Phi(S) \in \mathcal{N}$.

Proof. Let $p \in K$, and $S \subset T_pK$ be a spacelike hyperplane. The orthogonal complement $S^\perp \subset T_pM$ then is a two-dimensional lorentzian subspace. Hence there exist four light rays which are multiples of the vectors $V, -V, W, -W$ in $S^\perp$, where we assume without loss of generality that V and W are future-pointing. Since $T_pK = v^\perp$ for some future-pointing null vector $v \in T_pK$, we have $v \in S^\perp$ and can WLOG assume $\mathbb{R}_+W = \mathbb{R}_+v$, i.e. $\mathbb{R}_+W$ is the ray pointing along the null hypersurface K. This leaves $\mathbb{R}_+V$ as the unique future-pointing outward null ray which is perpendicular to S, and we can thus set $\Phi(S) = \mathbb{R}_+V$.

For to prove Φ is a bijection, we let $0 \neq V \in T_pM$ be an outward future-pointing null vector. In particular this means that $V \notin T_pK$. Thus $S = V^\perp \cap T_pK$ is a spacelike hyperplane in T_pK which satisfies $S = \Phi^{-1}(V)$.

For the last claim we note that the map $\mathcal{N} \ni N \mapsto N^\perp \cap L_p^+M \in \mathcal{V}$ maps a null hypersurface N to the unique ray along a future-pointing outward null generator of N. The inverse of this map is given by $\mathcal{V} \ni \mathbb{R}_+ V \mapsto V^\perp \in \mathcal{N}$. Composition of these maps with Φ yields the desired bijection $\mathcal{N} \to \mathcal{S}$. □

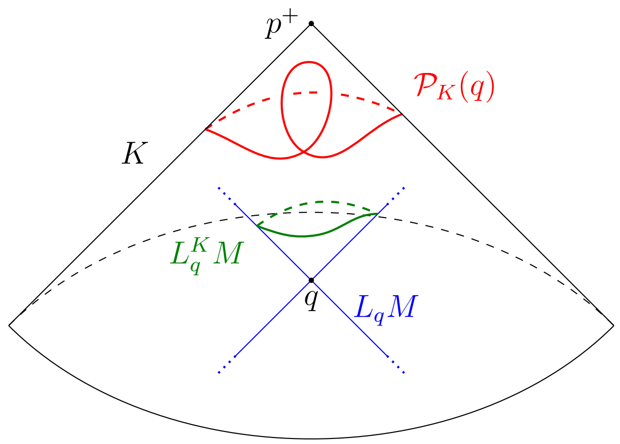


Figure 2.4: Illustration of the observation preimage $L_q^K M$ as defined in 2.1.9; The observation preimage $L_q^K M \subset L_q M$ is obtained by taking the preimage under $\exp_q$ restricted to $L_q^+ M$ of $\mathcal{P}_K(q)$. Note that $L_q^K M$ is a submanifold even if $\mathcal{P}_K(q)$ fails to be one.

The next part can be seen as an extension of lemma 2.1.3 we will show that for a $q \in V$, the set of null vectors hitting K is actually a smooth submanifold of the future light cone and that on $L_q^K M$, $\exp_q$ is a local diffeomorphism. The fact that no geodesic starting in V can have a conjugate point in K and the transversality of $\exp_q$ to K are curcial here.

Definition 2.1.9 (Observation Preimage)**.** For any $q \in V$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* $L_q^K M$ to be the preimage of K under the exponential map restricted to $L_q^+ M$, i.e.

$$L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K) \subset L_q^+ M$$

Lemma 2.1.10. *For any $q \in V$, the observation preimage $L_q^K M$ is a $n-1$-dimensional submanifold of $L_q^+ M$.*

Furthermore, for any $w \in L_q^K M$ there exist a relatively open neighborhood $O_w \subset L_q^K M$ such that $\exp_q : O_w \to U_w := \exp_q(O_w) \subset \mathcal{P}_K(q)$ is a diffeomorphism.

Proof. By lemma 2.1.2, $\exp_q : L_q^+ M \to M$ is transverse to K (here we treat $L_q^+ M$ and K as submanifolds, because by lemma 2.1.1(2-3) we can disregard the boundary points). Thus by the preimage lemma A.0.1 $L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K)$ is a $n-1$-dimensional submanifold of $L_q^+ M$.

For the second part let $w \in L_q^K M$, since $p := \exp_q(w) \in K$ and we assumed that such a p cannot be a null conjugate point, we know that $\exp_q : L_q^+ M \to M$ has an invertible differential at w. Thus, by the implicit function theorem, there exists an open neighborhood $O'_w \subset L_q^+ M$ of w such that $\exp_q : O'_w \to \exp_q(O'_w)$ is a diffeomorphism. If we then restrict $\exp_q$ to $O_w := O'_w \cap L_q^K M$ the map is still a diffeomorphism as desired. $\square$

Note that by the invariance of domain theorem U_w is an open submanifold of $\mathcal{P}_K(q)$

Corollary 2.1.11. *The map*

$$\begin{aligned} S^{n-1} \simeq CL_q^+ M &\to L_q^K M \\ w &\mapsto t_w w \end{aligned}$$

where t_w is as in 2.1.3, is a diffeomorphism.

Proof. This result follows immediately from lemma 2.1.3 together with the fact that since K is (away from its boundary) a smooth submanifold, the map $w \mapsto t_w$ is smooth. $\square$

We now aim to show that for $q \in V$, $\mathcal{P}_K(q)$ is localy the finite union of submanifold. This will make use of a lot of the machinery developed so far.

Lemma 2.1.12. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $w_1, \ldots, w_N \in L_q^K M$ such that $\exp_q(w_i) = p$. Furthermore with O_{w_i} as in the previous lemma such that $\exp_q : O_{w_i} \to U_{w_i}$ are a diffeomorphisms, there exists an open neighborhood $U \subset \mathcal{P}_K(q)$ of p such that*

$$\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^{N} O_{w_i}$$

Proof. Note that the previous corollary immediately yields that $L_q^K M$ is compact. Let $q \in V$, $p \in \mathcal{P}$. We first remark that, by the previous lemma, for any $w \in \exp_q^{-1}(p) \cap L_q^K M$ there exist open neighborhoods $w \in O_w \subset L_q^K M$ and $p \in U_w = \exp_q(O_w) \subset \mathcal{P}_K(q)$ making $\exp_q : O_w \to U_w$ a diffeomorphism.

To show that there can only be finitely many $w \in L_q^K M$ with $\exp_q(w) = p$ we let

$$C := \exp_q^{-1}(p) \cap L_q^K M.$$

As M is hausdorff, p is closed and $\exp_q$ is continuous, C must be closed. $C \subset L_q^K M$ is thus a closed subset of a compact space, making C itself compact as well. Now the family $\{O_w \mid w \in \exp_q^{-1}(p) \cap L_q^K M\}$ is an open cover of C. But because C is compact there must exist a finite subcover such that

$$C \subset O := \bigcup_{i=1}^{N} O_{w_i}.$$

We can now make some observations: By definition, for any $w \in L_q^K M \setminus C$ we have $\exp_q(w) \neq p$. And as $\exp_q$ is a diffeomorphism on O_{w_i} for all $i = 1, \ldots, N$, it must be injective and we get $\exp_q^{-1}(p) \cap O_{w_i} = \{w_i\}$. We thus have

$$\exp_q^{-1}(p) \cap O = \{w_1, \ldots, w_N\}.$$

Furthermore, as $C \subset O$ for any $p \in L_q^K M \setminus O \subset L_q^K M \setminus C$ we still have $\exp_q(w) \neq p$. In other words:

$$\exp_q^{-1}(p) \cap (L_q^K M \setminus O) = \emptyset.$$

Putting these two observations together we get

$$\exp_q^{-1}(p) \cap L_q^K M = \{w_1, \ldots, w_N\},$$

as desired.

To show the second part we denote

$$L^\times := L_q^K M \setminus O \quad \text{ and have } L^\times \cap \exp_q^{-1}(p) = \emptyset.$$

Note that $L^\times$ is a closed and thus compact subset of L_q^K. We then endow M the geodesic metric d induced by g^+. This lets us define the continuous function

$$\begin{aligned} g : L^\times &\to \mathbb{R} \\ w &\mapsto d(\exp_q(w), p). \end{aligned}$$

Because $L^\times \cap \exp_q^{-1}(p) = \emptyset$ we have $g(w) > 0$ for all $w \in L^\times$. But now, as $L^\times$ is compact there exists a $\varepsilon > 0$ such that $g(w) = d(\exp_q(w), p) > \varepsilon$ for all $w \in L^\times$. We can now choose

$$U := B_\varepsilon(p) \cap \mathcal{P}_K(q)$$

and get an open neighborhood of p in $\mathcal{P}_K(q)$ with $\exp^{-1}(U) \cap L^\times = \emptyset$. But this means

$$\exp_q^{-1}(U) \cap L_q^K M \subset O = \bigcup_{i=1}^{N} O_{w_i}$$

completing the proof. $\square$

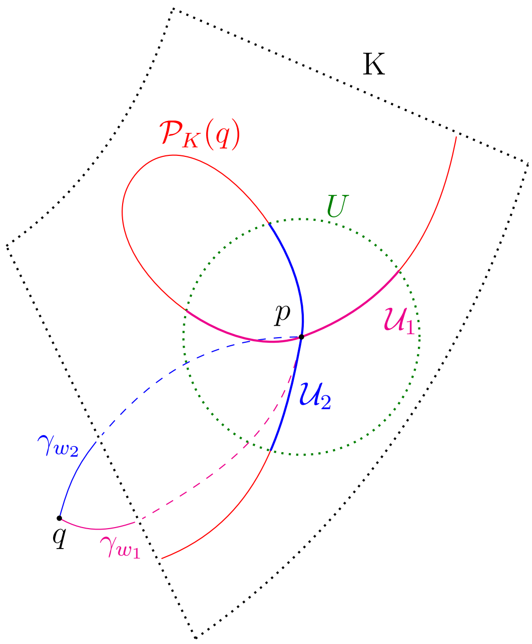


Figure 2.5: Illustration of proposition 2.1.13; even if $\mathcal{P}_K(q)$ fails to be a submanifold, locally it is the union of only finitely many submanifolds. ((Explain U and other stuff))

Armed with the previous lemma we can now show:

Proposition 2.1.13. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$. There exists an open neighborhood $p \in U \subset \mathcal{P}_K(q)$, a positive integer N and N pairwise transversal, spacelike, codimension 1 submanifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ and $p \in \mathcal{U}_i$ for $i = 1, \ldots, N$.*

Proof. Let $q \in V$ and $p \in \mathcal{P}_K(q)$. By the previous lemma we know that there can only be finitely many $w_1, \ldots, w_n \in L_q^K M$ with $\exp_q(w_i) = p$.

By lemma 2.1.10, for each w_i there exists a neighborhood $O_{w_i} \subset L_q^K M$ of w_i such that $\exp_q : O_{w_i} \to U_{w_i} := \exp_q(O_{w_i})$ is a diffeomorphism. Thus $U_{w_i} \subset \mathcal{P}_K(q)$ is a codimension 1 submanifold of K and we have $\bigcup_{i=1}^N U_{w_i} \subset \mathcal{P}_K(q)$.

Now we use the second part of the previous lemma to obtain an open neighborhood $U \subset \mathcal{P}_K(q)$ of p, such that $\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$. Thus any point $p \in \mathcal{P}_K(q) \cap U$ is contained in some $\mathcal{V}_i$ and we have $\bigcup_{i=1}^N U_{w_i} \supset \mathcal{P}_K(q) \cap U$. We then define

$$\mathcal{U}_i := U \cap U_{w_i}$$

and have

$$\bigcup_{i=1}^N \mathcal{U}_i = \mathcal{P}_K(q) \cap U$$

as desired. Furthermore, because U is an open neighborhood of p, $\mathcal{U}_i$ is still a codimension 1 submanifold of K and $p \in \mathcal{U}_i$.

We show that $\mathcal{U}_i$ is spacelike. To that end let $p \in \mathcal{U}_i$. Note that we have $\mathcal{U}_i \subset K$ and $\mathcal{U}_i \subset U'_{w_i} = \exp_q(O'_{w_i})$, where $w_i \in O'_{w_i} \subset L_q^+M$ is an open neighborhood of w_i in L_q^+M such that on O'_{w_i}, $\exp_q$ is a diffeomorphism onto its image. Both K and U'_{w_i} are null hypersurfaces around p but by proposition 2.1.2 they are transversal and thus cannot be generated by the same null rays. Thus $T_p\mathcal{U}_i = T_pK \cap T_pU'_{w_i}$ can only contain spacelike vectors.

Finally to prove that they are transversal at p, we assume by contradiction that there exist $i \neq j$ such that $T_p\mathcal{U}_i = T_p\mathcal{U}_j$. But by lemma 2.1.8 this would imply that $v_i = cv_j$ for a $c \in \mathbb{R}_+$, where $v_i = \gamma'(1)_{q,w_i}, v_i = \gamma'(1)_{q,w_i}$. This would imply $w_i = w_j$, a contradiction. □

Definition 2.1.14 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{U} \subset M$ of p such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a $n-1$ dimensional submanifold of M.

Note that $p \in \mathcal{P}_K(q)$ is regular if and only if $N = 1$ for p in the previous proposition.

Corollary 2.1.15. *The subset of regular points $\mathcal{P}_K^{reg}(q) \subset \mathcal{P}_K(q)$ is open and dense in $\mathcal{P}_K(q)$.*

Proof. The fact that it is open follows immediately from the definition: Let $p \in \mathcal{P}_K(q)$ be regular. There thus exists an open neighborhood $p \in \mathcal{U} \subset M$ such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a submanifold. But now for every point $p' \in \mathcal{U} \cap \mathcal{P}_K(q)$, $\mathcal{U}$ also makes p' a regular point making $\mathcal{U} \cap \mathcal{P}_K(q)$ an open neighborhood of regular points of p. Thus every regular point has an open neighborhood of regular points making the set of regular points itself open.

To prove the set of regular points is dense in $\mathcal{P}_K(q)$ we to show that for every point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $U' \subset \mathcal{P}_K(q)$ contains a regular point. By the previous proposition, for U' small enough we have $\mathcal{P}_K(q) \cap U' = \bigcup_{i=1}^N \mathcal{U}_i$, where $\mathcal{U}_i$ are pairwise transversal. This means their intersection is of lower dimension and

$$\mathcal{U}_i \setminus \bigcup_{j \neq i} \mathcal{U}_j \quad \text{is open and nonempty for every } i = 1, \dots N.$$

We can find a $p' \in \mathcal{U}_i$ for some $i \in 1, \dots, N$ such that $p' \notin \mathcal{U}_j$ for $j \neq i$. Then because $\mathcal{U}_i \setminus \bigcup_{j \neq i}$ is open can find an open neighborhood $\mathcal{O}'$ around p' such that $\mathcal{O}' \cap \mathcal{P}_K(q) \subset \mathcal{U}_i$ which means p' is a regular point, as desired. □

2.2 Observation Time Functions

In this sections we will use the fact that we can cover K with null geodesics μ_a to give structure to the light observation sets $\mathcal{P}_K(q)$.

Definition 2.2.1 (Observation Time Function)**.** For $a \in S^{n-1}$ the *observation time function* is defined as

$$\begin{aligned} f_a : J(p^-, p^+) &\to [-T_a, 0] \\ q &\mapsto \inf(\{s \in [-T_a, 0] \mid \mu_a(s) \in J^+(q)\} \cup \{0\}). \end{aligned}$$

Moreover, we define the *earliest observation point* $\mathcal{E}_a(q) := \mu_a(f_a(q)) \in M$.

Intuititively, $f_a(q)$ corresponds to the earliest time at which the observer μ_a measures light from q and $\mathcal{E}_a(q)$ is the point at which this happens.

We will first give some useful properties of these observation time functions: (1) corrensponds to the fact that future light cones from points $q \in J(p^-, p^+)^\circ$ only intersect the interior of K, as show in lemma 2.1.1(2-3).

Lemma 2.2.2. *Let $a \in S^{n-1}$ and $q \in J(p^-, p^+)^\circ$. Then*

(1) *It holds that $f_a(q) \in (-T_a, 0)$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[-T_a, 0]$ and strictly increasing on $[f_a(q), 0]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p, q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L^+_q M$ and $t \in [0, \rho(q, v)]$ such that $p = \gamma_{q,v}(t)$.*

Proof. Let $a \in \mathcal{A}$ and $q \in V$.

We begin by showing (1): Because $q \in J(p^-, p^+)^\circ = I^+(p^-) \cap I^-(p^+)$ we have $q \in I^-(p^+)$ and conversely $p^+ \in I^+(q)$. By lemma B.1.3 we know that $I^+(q)$ is open and thus it forms an open neighborhood of p^+. But as μ_a is a continuous path with $\mu_a(1) = p^+$ there must exist a $t < 0$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. Hence we have $f_a(q) < 0$.

To show $f_a(q) > -T_a$ we assume $f_a(q) = -T_a$ to achieve a contradiction. We thus have $-T_a = \inf\{s \in [-T_a, 0] \mid \mu_a(s) \in J^+(q)\}$. This means that there exists a convergent sequence $t_n \searrow 0$ as $n \to \infty$ such that $\mu_a(t_n) \in J^+(q)$ for all n. Because μ_a is continuous and $J^+(q)$ closed we have $p_0 := \mu_a(-T_a) \in J^+(q)$. But $p_0 = \mu_a(-T_a) \in R$ by prop 2.1.7(2). Hence we get $p_0 \in J^+(q) \cap R$ for $q \in V$, which is a contradiction to 2.1.1(3).

To show (2) we proceed as follows: By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all t_n we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from q to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such t.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[-T_a, 0]$ we first note that it is the composition of two continuous functions. Monotony then follows from the reverse triangle inequality for τ (see B.4.6(2)) together with the fact that μ_a is a future pointing null geodesic.

To show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 0]$ we let $f_a \leq t_1 < t_2 \leq 0$. Now by B.4.6(4) there exists a causal geodesic $\gamma_1 : [0,1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect γ_1 to $\mu_a|_{[t_1,t_2]}$ we get a path γ_2 connecting q to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as μ_a is a null geodesic. Next we argue that γ_2 must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If γ_1 is timelike this observation is trivial as μ_a is lightlike. If however, γ_1 is lightlike (which is only the case if $t_1 = f_a(q)$), this fact follows from the transversality of light cone observations as noted in proposition 2.1.2. This means that γ_2 is a broken causal geodesic, which implies that there exists a strictly longer timelike geodesic γ_3 connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Finally we can take on (3) To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q,p) = 0$. But now, by B.4.6 there exists a null geodesic from q to p which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q,p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0,1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q,p) = 0$.

The other equivalence follows from the definition of $\mathcal{P}_K(q)$ together with the definition of cut points. □

By (3) of the above lemma, for any $q \in V$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.2.1 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-T_a, 0] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2.7}$$

Note that we can use $f_a(q)$ to define the related functions $f : J(p^-, p^+) \times \times S^{n-1} \to [-T_{S^{n-1}}, 0]; (q, a) \mapsto f_a(q)$ and $F_q : S^{n-1} \to [-T_{S^{n-1}}, 0]; a \mapsto f_a(q)$.

Proposition 2.2.3. *The function $f : J(p^-, p^+)^{\circ} \times S^{n-1} \to [-T_{S^{n-1}}, 0]$; $(q, a) \mapsto f_a(q)$ is continuous.*

Proof. We want to show that for every convergent sequence $(q_n, a_n) \to (q_0, a_0) \in J(p^-, p^+)^{\circ} \times S^{n-1}$ we have $t_n := f_{a_n}(q_n) \to f_{a_0}(q_0) =: t_0$ as $n \to \infty$. Because the sequence t_n lives in $[-T_{S^{n-1}}, 0]$ and is thus bounded it suffices to show that for every convergent subsequence $t_j = f_{a_j}(q_j) \to t'$ we have $t' = t_0$. Note that still $(q_j, a_j) \to (q_0, a_0)$ because they are the subsequence of a convergent sequence. The points of earliest observation converge:

$$\mathcal{E}_{a_j}(q_i) = \mu_{a_j}(f_{a_j}(q_j)) = \mu_{a_j}(t_j) = \Theta(a_j, t_j) \to \Theta(a_0, t') = \mu_{a_0}(t') = p'$$

because $(a_j, t_j) \to (a_0, t')$ and Θ is continuous. The first key observation is that because $q_j \to q_0$ and $J^+(q_i) \ni \mathcal{E}_{a_j}(q_j) \to p'$ lemma B.4.5 implies $p' \in J^+(q_0)$.

Furthermore we have

$$0 = \tau(q_j, \mathcal{E}_{a_j}(q_j)) = \tau(q_j, \Theta(a_j, t_j)) \to \tau(q_0, \Theta(a_0, t')) = \tau(q_0, p') = 0$$

because τ and Φ are continuous.

We can now combine these observations and get: $p' \in \mathcal{L}^+_{q_0}$ because $p' \in J^+(q_0)$ and $\tau(q_0, p') = 0$ imply that there exist a null geodesic from q_0 to p'. $p' \in \mathcal{P}_K(q_0)$ because $p' \in \mu_{a_0}([0, 1]) \subset K$ and $p' \in \mathcal{L}^+_{q_0}$. But now lemma 2.2.2(3) yields that $p' = \mathcal{E}_{a_0}(q_0)$ and we get

$$\mu_{a_0}(t') = p' = \mathcal{E}_{a_0}(q_0) = \mu_{a_0}(f_{a_0}(q_0)) = \mu_{a_0}(t_0).$$

Because μ_a is injective we get $t' = t_0$, as desired. Hence every convergent subsequence of t_n goes to t_0 which, by compactness of $[-T_{S^{n-1}}, 0]$, implies that also $f_{a_n}(q_n) = t_n \to t_0 = f_{a_0}(q_0)$, proving that f is continuous. □

Proposition 2.2.4. *If $q_n \to q_0 \in J(p^-, p^+)^{\circ}$ as $n \to \infty$ and we denote $F_q : S^{n-1} \to \mathbb{R}; a \mapsto f_a(q)$. Then $F_{q_n} \to F_{q_0}$ uniformly over S^{n-1} as $n \to \infty$.*

Proof. Let $q_n \to q_0 \in V$ be a convergent sequence. We can endow M with a metric d, which is induced by g^+. Then there exists an $\varepsilon > 0$ and a $N \in \mathbb{N}$ such that $q_n \in \overline{B_\varepsilon(q_0)}$ for all $n \geq N$. After discarding the first N points of the sequence we may assume that $q_n \in \overline{B_\varepsilon(q_0)}\ \forall n$.

By the previous proposition

$$f : (\overline{B_\varepsilon(q_0)}, d) \times (S^{n-1}, d_{S^{n-1}}) \to ([0, 1], d_{[0,1]})$$

is a continuous function from and to compact spaces. Now we can apply lemma A.0.2 to find that $F_{q_n} \to F_{q_0}$ uniformly.

□

2.2.1 Set of earliest observations

After defining the observation time functions and the earliest observation point for a $q \in V$ and observer $\mu_a, a \in S^{n-1}$. We can now at the set of all earliest observation points together:

Definition 2.2.5 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned}\mathcal{D}_K(q) = \{(p,v) \in L^+M \mid &(p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L^+_q M, 0 \leq t \leq \rho(q,w)\},\\ \mathcal{D}^{reg}_K(q) = \{(p,v) \in L^+M \mid &(p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L^+_q M, 0 < t < \rho(q,w)\},\end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}^{reg}_K(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_K(q) := \pi(\mathcal{D}_K(q))$ and $\mathcal{E}^{reg}_K(q) := \pi(\mathcal{D}^{reg}_K(q))$, where $\pi : TM \to M$ is the canonical projection. We say that $\mathcal{E}_K(q)$ is the *set of earliest observations* and $\mathcal{E}^{reg}_K(q)$ is the *set of earliest regular observations* of q in K. We denote the collection of earliest observation sets by $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$.

Note that $\mathcal{E}_K(q) = \{\mathcal{E}_a(q) \mid a \in S^{n-1}\}$, and thus is really the set of all earliest observation points of q. The intuition for the direction set $\mathcal{D}_K(q)$ is that they contain all points $p \in \mathcal{E}_K(q)$ together with the directions of the corresponding null geodesics going from q to p, at p.

We can now characterize these observation sets and see that regular points in $\mathcal{E}^{reg}_K(q)$ resp. $\mathcal{D}^{reg}_K(q)$ correspond to regular observation points $p \in \mathcal{P}^{reg}_K(q)$:

Proposition 2.2.6. *For any $q \in V$ it holds that*

(1) *Let $T = \{p \in \mathcal{L}^+_q \mid \tau(q,p) = 0\}$ then*

$$\mathcal{E}_K(q) = \mathcal{P}_K(q) \cap T \quad \text{and} \quad \mathcal{E}^{reg}_K(q) = \mathcal{P}^{reg}_K(q) \cap T,$$

(2) *$\mathcal{E}^{reg}_K(q)$ is an open subset of $\mathcal{P}^{reg}_K(q)$, and is thus also a $n-1$-dimensional spacelike submanifold of K,*

(3) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points,*

(4) *$\overline{\mathcal{E}^{reg}_K(q)}$ is open and dense in $\mathcal{E}_K(q)$ and*

(5) *$\mathcal{D}^{reg}_K(q)$ is a nonempty open n-dimensional submanifold of $\pi^{-1}(K)$.*

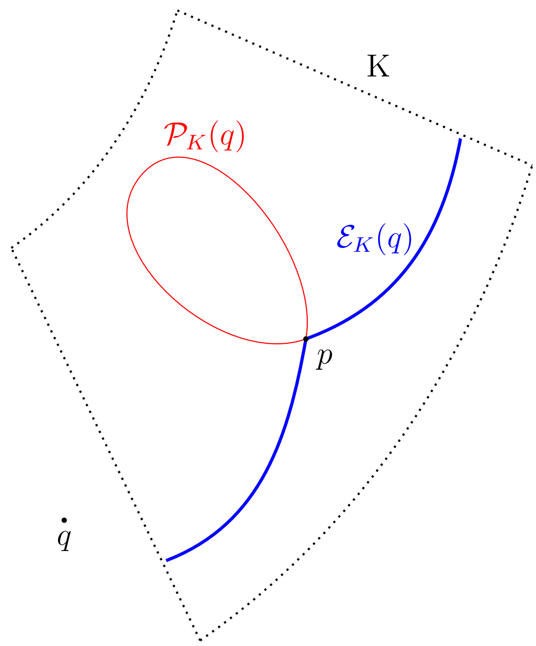


Figure 2.6: Illustration of the *set of earliest observations* $\mathcal{E}_K(q)$; we can see that at points $p \in K$ where pq fails to be a submanifold, $\mathcal{E}_K(q)$ incurrs a "sharp edge".

Proof. Let $q \in V$. We first look at a useful relation of the exponential map to cut points: We define $\mathcal{V} := \{w \in L_q^+ M \mid \rho(q,w) > 1\}$. By B.5.5, $\rho(q,w)$ is lower semicontinuous and $\mathcal{V}$ is thus open. Furthermore, by the definition of cut points, $\mathcal{V}$ is star-shaped around $0 \in L_q^+ M$. Because by B.5.4 cut points are exactly the points where $\exp_q$ first fails to be a diffeomorphism, $\exp_q : \mathcal{V} \to \mathcal{W} := \exp_q(\mathcal{V})$ is a diffeomorphism. Furthermore by the invarance of domain theorem we get that $\mathcal{W} \subset \mathcal{L}_q^+$ is relatively open. Note that this implies that for any $p \in \mathcal{W}$, there exists a $p \in U \subset M$ open such that $p \in \mathcal{L}_q^+ \cap U$ is a n-dimensional submanifold of M.

We can now move on to proving (1): $p \in \mathcal{E}_K(q) \iff p \in \mathcal{P}_K(q) \cap T$ follows immediately from lemma 2.2.2(3).

Now let $p \in \mathcal{E}_K^{reg}(q)$. By definition this implies that $p \in \mathcal{W}$ and we get an $p \in U \subset M$ open such that $p \in \mathcal{L}_q^+ \cap U$ is a dimension n submanifold. Now, around p, K is also a dimension n submanifold, transversal to $\mathcal{L}_q^+$ and thus $K \cap \mathcal{L}_q^+ \cap U = \mathcal{P}_K(q) \cap U$ is a dimension $n-1$ submanifold around p. Thus p is a regular point, i.e. $p \in \mathcal{P}_K^{reg}(q)$. $\tau(q,p) = 0$ follows immediately from the fact that $\rho(q,w) > 1$, proving the first direction.

To show the reverse direction we assume $p \in \mathcal{P}_K^{reg}(q)$ with $\tau(q,p) = 0$. Because $p \in \mathcal{P}_K^{reg}(q)$ by definition 2.1.14 there exists exactly one $w \in L_q^K M$ such that $\exp_q(w) = p$. From $\tau(q,w) = 0$ we get $\rho(q,p) \geq 1$. Now if $\rho(q,w) = 1$, p would be a cut point. By theorem B.5.4 this would mean that either $p \in K$ is a conjugate point to q or there exists a $w \neq w' \in L_q^K M$ with $\exp_q(w') = p$. The first option is impossible because we assumed V to be suitable which means that no $q \in V$ can have a conjugate point on K. The second option is also impossible because we assumed p to be a regular point in $\mathcal{P}_K(q)$. We thus must have $\rho(q, w) > 1$, implying $p \in \mathcal{E}_K^{reg}(q)$.

We now move on to (2): To prove that $\mathcal{E}_K^{reg}(q)$ is open in $\mathcal{P}_K^{reg}(q)$ we claim that $\mathcal{E}_K^{reg}(q) = \mathcal{P}_K^{reg}(q) \cap \mathcal{W}$. To that end we first note that $\mathcal{E}_K^{reg}(q) \subset \mathcal{W} \subset T$. Recall that by (1) we have $\mathcal{E}_K^{reg}(q) = \mathcal{P}_K^{reg}(q) \cap T$. Applying $\cap \mathcal{W}$ to both sides yields

$$\mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q) \cap \mathcal{W} = \mathcal{P}_K^{reg}(q) \cap T \cap \mathcal{W} = \mathcal{P}_K^{reg}(q) \cap \mathcal{W}$$

as desired.

Proposition 2.1.13 implies that $\mathcal{P}_K^{reg}(q)$ is a $n-1$ dimensional spacelike submanifold of M. Because $\mathcal{W} \subset \mathcal{L}_q^+$ is open and $\mathcal{P}_K^{reg}(q) \subset \mathcal{L}_q^+$, $\mathcal{E}_K^{reg}(q)$ is a relatively open subset of $\mathcal{P}_K^{reg}(q)$, as desired. This also means that as an open subset of a submanifold, $\mathcal{E}_K^{reg}(q)$ is a $n-1$-dimensional spacelike submanifold of K as well.

We can now tackle (3): Let $p \in \mathcal{E}_K(q)$ be a cut point, then by proposition 2.1.13, there exists an open neighborhood $p \in U \subset M$ and N codimension 1 pairwise transversal manifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$. Because $\tau(q, p) = 0$ and the manifolds are pairwise transversal and intersect at p ((SEE FIGURE)), $\mathcal{E}_K(q)$ must have a sharp edge at p and cannot be a submanifold. For the other direction we assume that $p \in \mathcal{E}_K(q)$ is not a cut point. Then, by definition we have $p \in \mathcal{E}_K^{reg}(q)$ which is a submanifold.

Moving on to (4), the fact fact that $\mathcal{E}_K^{reg}(q)$ is dense in $\mathcal{E}_K(q)$ follows by an argument which is analogous to the one used in the proof of corollary 2.1.15. To show that it is relatively open in $\mathcal{E}_K(q)$ we use that $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K(q) \cap \mathcal{W}$ with $\mathcal{W}$ open in $\mathcal{L}_q^+$.

Finally the proof of (5) is analogous to (2) with the difference in submanifold dimension originating from the face that for any $(p, v) \in \mathcal{D}_K^{reg}(q)$ we also have $(p, cv) \in \mathcal{D}_K^{reg}(q)$ for all $c \in \mathbb{R}_+$ ((explain more)). □

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, (3) implies that it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

We can now give an alternative characterization of $\mathcal{E}_K(q)$ which is intuitively more in line with "the set of earliest observations"

Proposition 2.2.7. *Let $q \in V$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \text{there are no } p' \in \mathcal{P}_K(q) \text{ such that } p' < p\}.$$

Proof. For the left inclusion assume $p \in \mathcal{E}_K(q)$, i.e. there exists an $a \in S^{n-1}$ such that $\mathcal{E}_a(q) = p$. Then lemma 2.2.2(3) immediately yields, $p \in \mathcal{P}_K(q)$ and $\tau(q, p) = 0$.

Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. Because $\mathcal{P}_K(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p,q) > 0$, a contradiction.

For the other direction we assume we have $p = \mu_a(t) \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 2.2.2(3) we only need to prove that $\tau(p,q) = 0$. Suppose that $\tau(p,q) > 0$. Now since $\tau(p,q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since μ_a is timelike, which is a contradiction. □

2.2.2 Observation Reconstruction

Crucially all observation sets defined in the previous sections can be fully determined by the data, the core idea is that we can use lemma 2.1.8 to determine the direction of incoming geodesics on $\mathcal{P}_K(q)$ allowing us to reconstruct the direction sets.

Proposition 2.2.8. *Given the data outlined in remark 1.2.1 we can uniquely determine $\mathcal{E}_K(q)$ and $\mathcal{E}_K^{reg}(q)$, as well as $\mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Proof. What we want to show is that given K, the conformal class of $g|_K$ and the set $\{\mathcal{P}_K(q) \mid q \in V\}$ we can reconstruct the sets stated above. Note that as described in proposition 2.1.7 ((Move to own remark?)) this data allows us to construct $\Theta : \mathcal{S} \to K$ and μ_a.

We first show that for a given $\mathcal{P}_K(q)$ we can determine $\mathcal{E}_K(q)$: By equation 2.7, for any $a \in S^{n-1}$ we can determine $f_a(q)$ and thus $\mathcal{E}_a(q) = \mu_a(f_a(q))$ using only $\mathcal{P}_K(q)$. We can then construct $\mathcal{E}_K(q) = \bigcup_{a \in S^{n-1}} \mathcal{E}_a(q)$. Furthermore, by proposition 2.2.6, $\mathcal{E}_K^{reg}(q)$ contains exactly the points $p \in \mathcal{E}_K(q)$ where $\mathcal{E}_K(q)$ is locally a submanifold of M and thus K. But because we know K we can determine all points where this is the case and reconstruct $\mathcal{E}_K^{reg}(q)$.

To reconstruct the direction set we first note that by lemma 2.1.13 for any $p \in \mathcal{P}_K(q)$ such that $\exp_q^{-1}(p) = \{w_1, \ldots, w_N\} \subset L_q^K M$, we have $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ where $p \in U \subset M$ open and $p \in \mathcal{U}_i$ are pairwise transversal spacelike hypersurfaces of K. For each w_i we let $v_i = \gamma'_{q,w_i}(1)$ be the outbound velocity vector of the null geodesic which starts at q with velocity w_i, once it hits K. To find $\mathcal{D}_K(q)$ we must reconstruct all such v_i.

To that end, note that we have $T_p\mathcal{P}_K(q) = \bigcup_{i=1}^N T_p\mathcal{U}_i$ where $T_p\mathcal{U}_i$ are spacelike hyperplanes. For each such hypersurface, using lemma 2.1.8 we can then find the outward pointing orthogonal null ray $\mathbb{R}_+ v_i$ which must contain the outbound velocity vector v_i at p. Thus for any $p \in \mathcal{P}_K(q)$ we can reconstruct $\mathbb{R}_+ v_i$ for all geodesics γ_{q,w_i} from q to p.

Now by definition for any $p \in \mathcal{E}_K(q)$, we have

$$\mathcal{D}_p := \pi^{-1}(p) \cap \mathcal{D}_K(q) = \{(p, \mathbb{R}_+ v_1), \ldots, (p, \mathbb{R}_+ v_N)\}$$

where $\pi : TM \to M$ is the canonical projection. As we saw for any $p \in \mathcal{E}_K(q) \subset \mathcal{P}_K(q)$ we can reconstruct $\mathcal{D}_p$ which allows us to reconstruct $\mathcal{D}_K(q) = \bigcup_{p \in \mathcal{E}_K(q)} \mathcal{D}_p$.

Finally we can reconstruct $\mathcal{D}_K^{reg}(q)$ by using $\mathcal{D}_K^{reg}(q) = \pi^{-1}(\mathcal{E}_K^{reg}(q)) \cap \mathcal{D}_K(q)$. $\square$

Note that we can adapt this proof to show that $\mathcal{E}_K(q)$ uniquely determines $\mathcal{E}_K^{reg}(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.

Because if the earliest observation set are equal also the direction sets must be equat we get:

Proposition 2.2.9. *Let $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$. Then $q = q'$.*

Proof. We assume by contradiction that $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$ and $q \neq q'$ Let $p_1, p_2 \in \mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ with $p_1 \neq p_2$. Because p_1 and p_2 cannot be cut points there must exist unique $w_1, w_2 \in L_q^K M$ and $w_1', w_2' \in L_{q'}^K M$ such that $\gamma_{q,w_i}(1) = p_i$ and $\gamma_{q',w_i'}(1) = p_i$. Because $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ we can use lemma 2.1.8 to show that

$$v_i = \gamma'_{q,w_i}(1) = c_i \gamma_{q',w_i'}(1) = c_i v_i'$$

for some $c_i > 0$.

Now $\gamma_{p_i,-v_i}$ are two past-pointing null geodesics going from p_i through q and q'. Hence there either exists a null geodesic from q to q' or from q' to q. We will WLOG assume $q' \in J^+(q)$. Now there must exist $t_1, t_2 \in (0,1)$ such that $\gamma_{q,w_i}(t_i) = q'$. But this would make q' a cut point of q which is impossible as we assumed $p_i \in \mathcal{E}_K^{reg}(q)$. $\square$

2.3 Smooth Constructions

Finally in this chapter we show some important differential properties of the observation time functions:

2.3.1 Coordinate Construction

We will now show that for $q \in V$ and well-chosen $a_0, \ldots a_n$, the map $q' \mapsto (f_{a_0}(q'), \ldots, f_{a_n}(q'))$ defines local coordinates, as discussed in the introduction. To prepare we look at tuples (q,p) of the form $q \in V, p \in \mathcal{E}_K^{reg}(q)$ and show that they form a manifold:

Definition 2.3.1 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_K^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w(q,p) \in L_q^K M$ such that $\gamma_{q,w(q,p)}(1) = p$ and $\rho(q, w(q,p)) > 1$. Existence follows from lemma 2.2.2 while uniqueness follows from the fact that $p \in \mathcal{E}_K^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Omega : \mathcal{Z} &\mapsto L^K V \\ (q,p) &\mapsto (q, w(q,p)) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0, w_0) \subset TM$ be a ε-neighborhood of (q_0, w_0) with respect to the Sasaki-metric induced on TM by g^+.

This lemma will be useful because it will allow us to show that Ω is bounded

Lemma 2.3.2. *((Move to appendix?))The function*

$$\begin{aligned} T_+ : L^+ J(p^-, p^+) &\to \mathbb{R} \\ (q,w) &\mapsto \sup\{t \geq 0 \mid \gamma_{q,w}(t) \in J^-(p^+)\} \end{aligned}$$

is finite and upper semicontinuous.

Proof. Finiteness follows from lemma B.2.2. We now want to show that T_+ is upper semicontinuous. To that end let $(q_n, w_n) \to (q_0, w_0) \in L^+J(p^-, p^+)$, we want to show that $\limsup_{n\to\infty} T_+(q_n, w_n) \leq T_+(q_0, w_0)$: Let $\varepsilon > 0$ and set $t_0 = T_+(q_0, w_0)$. Then by definition we have $\gamma_{q_0,w_0}(t_0) \in M \setminus J^-(p^+)$. Because $\gamma_{q_n,w_n}(t_0) \to \gamma_{q_0,w_0}(t_0)$ and $M \setminus J^-(p^+)$ open, there exists a $N \in \mathbb{N}$ such that $\gamma_{q_n,w_n}(t_0) \in M \setminus J^-(p^+)$ for all $n \geq N$. Note that if $\gamma_{q_n,w_n}(t_0) \notin J^-(p^+)$ then for any $t' \geq t_0$ we also have $\gamma_{q_n,w_n}(t') \notin J^-(p^+)$ because otherwise we could obtain a lightlike path from $\gamma_{q_n,w_n}(t_0)$ to p^+, a contradiction. Thus, by definition $T_+(q_n, w_n) \leq t_0$ and $\limsup_{n\to\infty} T_+(q_n, w_n) \leq t_0 = T_+(q_0, w_0) + \varepsilon$. Finally because $\varepsilon > 0$ was arbitrary we get $\limsup_{n\to\infty} T_+(q_n, w_n) \leq T_+(q_0, w_0)$ as desired. $\square$

This lemma will be very useful whenever we are dealing with $\mathcal{Z}$:

Lemma 2.3.3. *Let $(q_0, p_0) \in \mathcal{Z}$ and $(q_0, w_0) = \Omega(q_0, p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0, w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) := X(\mathcal{W}_\varepsilon(q_0, w_0))$. When ε is small enough, Ω coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0, p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $2n$-dimensional manifold and the map $\Omega : \mathcal{Z} \to L^K M$ is smooth.

Proof. Because $p_0 \in \mathcal{P}_K(q_0)$ and $q_0 \in V$ we have, by assumption in theorem 1.1.5 that p_0 cannot be a conjugate point of q_0. Hence for $\varepsilon > 0$ small enough

$X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) = X(\mathcal{W}_\varepsilon(q_0, w_0))$ is a diffeomorphism with $\mathcal{U}_\varepsilon(q_0, p_0)$ open in $M \times M$ by the invariance of of domain theorem.

Next we aim to show that $\Omega : \mathcal{Z} \to L^K V$ is continuous at $(q_0, p_0) \in \mathcal{Z}$. We proceed by assuming there exists a sequence $(q_n, p_n) \in \mathcal{Z}$ converging to (q_0, p_0) such that $\Theta(q_n, p_n) = (q_n, w_n) \in L^+V$ does not converge to $\Theta(q_0, p_0) = (q_0, w_0)$.

First of all we aim to show that the sequence (q_n, w_n) is bounded and thus has a convergent subsequence: Because $q_n \to q_0$ we only need to show that w_n is bounded. To that end we introduce an arbitrary riemannian metric consistent with the topology on M and can write $w_n = t_n \overline{w_n}$ where $\|\overline{w_n}\|_{g^+} = 1$. To show that t_n is bounded we first define

$$C := \{(q, w) \in L^+M \mid q \in J(p^-, p^+) \text{ and } \|w\|_{g^+} = 1\}$$

and C is compact and because T_+ is upper semicontinuous on C, there exists a $c_0 > 0$ such that $T_+(q, w) \leq c_0$ for all $(q, w) \in C$. Recall that we have $\gamma_{q_n, \overline{w_n}}(t_n) = \exp_{q_n}(w_n) = p_n \in K \subset J(p^-, p^+)$. Together with $(q_n, \overline{w_n}) \in C$ this yields

$$\|w_n\|_{g^+} = t_n \|\overline{w_n}\|_{g^+} = t_n \leq T_+(q_n, \overline{w_n}) < c_0,$$

proving $(q_n, w_n) \in L^K V$ is bounded.

We can thus obtain a convergent subsequence $(q_k, w_k) = \Theta(q_k, p_k) \to (q_0, w')$ with $w' \neq w_0$. Since the exponential map is continuous, we would have

$$\exp_{q_n}(w') = \lim_{n\to\infty} \exp_{q_n}(w_n) = \lim_{n\to\infty} p_n = p_0 = \exp_{q_n}(w_0).$$

with $w' \neq w_0$. But since $p_0 \in \mathcal{E}_K^{reg}(q)$ cannot be a cut point this is a contradiction and $\Omega : \mathcal{Z} \to L^K V$ must be continuous.

Next we use the fact that Ω is continuous and get $\Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0)) \subset \mathcal{Z}$ is open. We can thus find a $\varepsilon_1 \in (0, \varepsilon)$ such that for the open ball $\mathcal{U}_{\varepsilon_1}(q_0, w_0) \subset M$ we have

$$\mathcal{Y}_{\varepsilon_1} := \mathcal{U}_{\varepsilon_1}(q_0, w_0) \cap \mathcal{Z} \subset \Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0))$$

implying $\Omega(\mathcal{Y}_{\varepsilon_1}) \subset \mathcal{W}_\varepsilon(q_0, w_0)$. Then for $(q, p) \in \mathcal{Y}_{\varepsilon_1}$ and $(q, w) = \Omega(q, p) \in \mathcal{W}_\varepsilon(p_0, w_0)$ we have $\exp_q(w) = p$. Hence $X(\Omega(q, p)) = (q, p)$. But now since $(q, p) \in \mathcal{U}_\varepsilon(p_0, q_0)$ we can apply X^{-1} to both sides and get $\Omega(q, p) = X^{-1}(q, p)$. Thus on $\mathcal{Y}_{\varepsilon_1}$ the function $\Omega : \mathcal{Y}_{\varepsilon_1} \to TM$ coincides with the smooth function $X^{-1} : \mathcal{Y}_{\varepsilon_1} \to TM$, which implies that Ω is smooth with full rank differential on $\mathcal{Y}_{\varepsilon_1}$ as well.

Now since $(q_0, p_0) \in \mathcal{Z}$ was arbitrary we get that $\Theta : \mathcal{Z} \to L^+V$ is smooth everywhere, injective and locally diffeomorphic with full rank. Thus $\mathcal{Z}$ diffeomorphic to an open subset of $L^K V$. This makes it a manifold with dimension $(n+1) + (n-1) = 2n$. $\square$

We can now construct local coordinates:

Proposition 2.3.4. *Let $q_0 \in V$ and $(q_0, p_j) \in \mathcal{Z}, j = 0, \dots, n$ and $w_j \in L^K_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \dots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=0}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

Proof. First we need some setup: Let $(q_0, p_0) \in \mathcal{Z}$ and $w_0 \in L^+_{q_0}M$ such that $\gamma_{q_0,w_0}(1) = p_0$. Furthermore let $\varepsilon > 0$ be small enough such that the map $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ is a diffeomorphism (see the previous lemma). We will denote this inverse by $X^{-1}(q, p) = (q, w(q, p))$ and write $\mathcal{W} = \mathcal{W}_\varepsilon(q_0, w_0), \mathcal{U} = \mathcal{U}_\varepsilon(q_0, p_0)$.

We associate with any $(q, p) \in \mathcal{U}$ the energy $E(q, p) = E(\gamma_{q,w(q,p)}([0, 1]))$ of the geodesic segment connecting q to p. The energy of a piecewise smooth curve $\alpha : [0, l] \to M$ is defined as

$$E(\alpha) = \frac{1}{2}\int_0^l g(\alpha'(t), \alpha'(t))dt.$$

Note that the sign of $E(\alpha)$ depends on the causal nature of $\gamma_{q,w(q,p)}$. In particular $E(q, p) = 0$ if and only if $w(q, p)$ is light-like. Moreover, as X^{-1} is smooth on $\mathcal{U}$, so is $E(p, q)$.

We now return to consider $(q_0, p_0) \in \mathcal{Z}$ and let $a \in S^{n-1}$ be such that $p_0 \in \mu_a$. Then $p_0 = \mu_a(s_0)$ with $s_0 = f_a(q_0)$ as $p_0 \in \mathcal{E}^{reg}_K(q_0)$ and $s_0 \in (0, 1)$ by lemma 2.2.2(1).

Let $V_0 \subset V$ be an open neighborhood of q_0 and $t_1, t_2 \in (-T_a, 0), t_1 < s_0 < t_2$, such that $V_0 \times \mu_a([t_1, t_2]) \subset \mathcal{U}$, which exist because $\mathcal{U}$ is open. Then for any $q \in V_0, s \in (t_1, t_2)$ the function $\mathbf{E}_a(q, s) \coloneqq E(q, \mu_a(s))$ is well defined and smooth.

We want to use first variation formula for $\mathbf{E}_a(q, s)$ ((E Reference)) to calculate $\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\Big|_{s=s_0}$ and $\nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0}$.

For the first part we define the variation $\mathbf{x}(t, s) = \gamma_{q_0,w(s)}(t), t \in [0, 1]$ where $w(s) := w(q_0, \mu_a(s + s_0)), s \in [t_1 - s_0, t_2 - s_0]$. Note that $\mathbf{x}(t, 0) = \gamma_{q_0,w_0}(t)$. We can then use the equation from proposition C.3.11 to get

$$\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\bigg|_{s=s_0} = E'_{\mathbf{x}}(0) = g(V, \gamma'_{q_0,w_0})\big|_0^1$$

since γ_{q_0,w_0} is a geodesic and $\mathbf{x}$ has no breaks. If we now further notice that $V(0) = 0$ as $\mathbf{x}(0, s) = q_0$ for all $s \in [t_1, t_2]$ and $V(1) = \mu_a'(s_0) = \mu_a'(f_a(q_0))$ as $\mathbf{x}(1, s) = \mu_a(s + s_0)$ we can conclude

$$
\begin{aligned}
\left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} &= g(V(1), \gamma_{q_0,w_0}'(1)) - g(V(0), \gamma_{q_0,w_0}'(0)) \\
&= g(\mu_a'(f_a(q_0)), \gamma_{q_0,w_0}'(1))
\end{aligned}
$$

For the second part we will introduce coordinates $\mathbf{q} = (q_0, \ldots, q_n)$ around q_0. Then the gradient can be written as

$$
\nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q, s_0)}{\partial q_i}\right|_{q=q_0} \partial_j.
$$

To calculate $\left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0}$ we now introduce variations $\mathbf{x}_i(t, s) = \gamma_{q(s),w(s)}(t)$ where $w(s) := w(q(s), \mu_a(s_0))$ and $q(s) := q^{-1}(q_0(q_0), \ldots, q_i(q_0) + s, \ldots q_n(q_0))$ is obtained by increasing the i-th coordinate by s. Note that these variations all have $\mathbf{x}_i(t, 0) = \gamma_{q_0,w_0}(t)$, $\mathbf{x}_i(1, s) = \mu_a(s_0)$ thus $V_{\mathbf{x}_i}(1) = 0$ and $V_{\mathbf{x}_i}(0) = \frac{\partial}{\partial s}\mathbf{x}_i(0, s)|_{s=0} = \partial_i$. After again applying proposition C.3.11

$$
\left.\frac{\partial \mathbf{E}_a(q, s_0)}{\partial q_i}\right|_{q=q_0} = E_{\mathbf{x}_i}'(0) = -g(V(0), \gamma_{q_0,w_0}'(0)) = -g(\partial_i, w_0).
$$

Combining this with coordinate representation of the gradient we get

$$
\begin{aligned}
\nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q, s_0)}{\partial q_i}\right|_{q=q_0} \partial_j &= -g^{ij}(g_{\alpha\beta}\partial_i^\alpha w_0^\beta)\partial_j \\
&= -g^{ij} g_{i\beta} w_0^\beta \partial_j = -\delta_\beta^j w_0^\beta \partial_j \\
&= -w_0^j \partial_j = -w_0.
\end{aligned}
$$

We thus managed to calculate what we wanted and can summarize as

$$
\left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} = g(v, \mu_a'(f_a(q_0))), \quad \nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0} = -w_0 \tag{2.8}
$$

where $w_0 = w(q_0, p_0)$ and $v = \gamma_{q_0,w_0}'(1)$. Since $\mu_a'(f_a(q_0))$ and v are both future-pointing null vectors, which by lemma 2.1.2 must be transversal we have $\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(v, \mu_a'(f_a(q))) < 0$.

We can now use the implicit function theorem on $V_0 \times [t_1, t_2]$ with equation $E_a(q, s) = 0$ and single solution $E_a(q_0, s_0) = 0$. This yields an open neighborhood

$V_a \subset V_0$ and a smooth function $q \mapsto s_a(q)$ such that $E_a(q, s_a(q)) = 0$ for all $q \in V_a$. Now $E_a(q, s_a(q)) = E(q, \mu_a(s_a(q))) = 0$, implies $\mu_a(s_a(q)) \in \mathcal{P}_K(q)$. This together with $(q, s_a(q)) \in \mathcal{U}$ implies that $\mu_a(s_a(q)) \in \mathcal{E}_K^{reg}(q)$ and thus $s_a(q) = f_a(q)$ on V_a. Hence we have $\nabla f_a(q)|_{q=q_0} = \nabla s_a(q)|_{q=q_0}$ and from equation 2.8 together with the implicit function theorem it follows that

$$\nabla f_a(q)|_{q=q_0} = \frac{1}{c(q_0, a)} w_0, \quad c(q_0, a) = \left. \frac{\partial \mathbf{E}_a(q_0, s)}{\partial s} \right|_{s=s_0} < 0, \tag{2.9}$$

where $p_0 = \mu_a(s_0) = \mathcal{E}_a(q_0)$, $s_0 = f_a(q_0)$ and $w_0 = w(q_0, p_0)$.

Next we choose $p_0, \dots, p_n \in \mathcal{E}_K^{reg}(q_0)$ and let $w_0, \dots, w_n \in L_{q_0}^K M$ such that $p_i = \gamma_{q_0, w_i}(1)$, i.e. $w_i = w(q_0, p_i)$. We assume that $w_0, \dots, w_n$ are linearly independent. Moreover let $a_j \in S^{n-1}$ such that $p_i \in \mu_{a_j}$ and $\overrightarrow{a} = (a_j)_{j=1}^n$. Finally we denote by $q \mapsto s_{a_j}(q) = f_{a_j}(q)$ the above constructed smooth functions which are defined on some neighborhoods $V_{a_j} \subset V$ of q_0.

Let $V_{\overrightarrow{a}} = \bigcap_{j=1}^n V_{a_j}$ and consider the map

$$\begin{aligned} \mathbf{f}_{\overrightarrow{a}} : V_{\overrightarrow{a}} &\to \mathbb{R}^n \\ q &\mapsto (f_{a_1}(q), \dots, f_{a_n}(q)). \end{aligned}$$

Because all of its components are smooth, $\mathbf{f}_{\overrightarrow{a}}$ itself is smooth as well. By equation 2.9 each component has gradient $\nabla f_{a_j}(q)\big|_{q=q_0} = \frac{1}{c(q_0, a_j)} w_i$ with $c(q_0, a_j) \neq 0$. Since we assumed that $w_0, \dots, w_n$ be independent, $\mathbf{f}_{\overrightarrow{a}}$ is non-degenerate at q_0 and thus defines a smooth coordinate system in some neighborhood V_1 of q_0. $\square$

The results from this subsection will form the main prerequesites for the reconstruction of the differential structure of V because they allow us to understand $\mathcal{Z}$ in terms of $L^K V$ and to construct local coordinates from observation times.

2.3.2 Observation Time Smoothness

Now we show that the observation time function varies smoothly in the observer a and the observerd point p almost everywhere, and that its derivatives are well behaved. This culminates in proof that if F_{q_n} converges suitably to F_{q_0} we have $q_n \to q_0$ which will be crucial in the reconstruction of the topology of V.

Definition 2.3.5 (Regular Observer). Let $q \in V$ we call $a \in S^{n-1}$ a *regular observer* of q if $\mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)$ and write

$$\mathcal{A}^{reg}(q) := \{a \in S^{n-1} \mid \mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)\} \subset S^{n-1}$$

for the set of regular observers. Note that because $\mathcal{E}_K^{reg}(q)$ open and dense in $\mathcal{E}_K(q)$, $\mathcal{A}^{reg}(q)$ is open and dense in S^{n-1}

For the next proposition we again endow M with the geodesic metric d induced by g^+. This allows us to define open balls. To show that the observation time function f is smooth we will again construct it as a function supplied by the implicit function theorem in a very similar fashion to proposition 2.3.4:

Proposition 2.3.6. *Let $q_0 \in V$ and $a_0 \in \mathcal{A}^{reg}(q_0)$ a regular observer of q_0. Then there exists a $\varepsilon > 0$ such that $f : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$; $(q, a) \mapsto f_a(q)$ is smooth.*

Proof. Let $p_0 \in \mathcal{E}_K^{reg}(q_0)$ with $p_0 = \gamma_{q_0,w_0}(1)$ and $p_0 = \mu_{a_0}(t_0)$. Then we have $(q_0, p_0) \in \mathcal{Z}$ and by lemma 2.3.3 there exists a $\delta > 0$ such that $X : \mathcal{W}_\delta(q_0, w_0) \to \mathcal{U}_\delta(q_0, p_0)$ is a diffeomorphism. Note that we can choose $\delta > 0$ such that $\rho(q, w(q, p)) > 1$ for all $(q, p) \in \mathcal{U}_\delta(q_0, p_0) \cap L^+M$.

Then

$$\begin{aligned} Y : \mathcal{W}_\delta(q_0, w_0) \cap L^K M &\to M \times S^{n-1} \times [0, 1] \\ (q, w) &\mapsto (q, \Theta^{-1}(X(q, w))) \end{aligned}$$

is a diffeomorphism onto its image $\mathcal{V}_\delta := Y(\mathcal{W}_\delta(q_0, w_0))$ which is an open neighborhood of (q_0, a_0, t_0) in $M \times S^{n-1} \times [0, 1]$. There thus exists a $\delta > \lambda > 0$ such that $\overline{B_\lambda(q_0)} \times \overline{B_\lambda(a_0)} \times \overline{B_\lambda(t_0)} \subset \mathcal{V}_\delta$.

On this space we can then define the function $\mathbf{E}(q, a, s) := E(q, \Theta(a, s))$ with $E(q, p)$ as in the previous lemma. This function is well defined and smooth with $E(q_0, a_0, t_0) = 0$ and $\left.\frac{\partial \mathbf{E}(q_0,a_0,t)}{\partial s}\right|_{t=t_0} < 0$ by the same argument as in the previous proof. We can thus apply the implicit function theorem to get a $\varepsilon > 0$ and a smooth function

$$\begin{aligned} s : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} &\to \overline{B_\lambda(t_0)} \\ (q, a) &\mapsto (t) \end{aligned}$$

with $s(q_0, a_0) = t_0$ and $\mathbf{E}(q, a, s(q, a)) = 0$.

Let $(q, a) \in \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ then

$$\mathbf{E}(q, a, s(q, a)) = E(q, \Theta(a, s(q, a))) = E(q, \mu_a(s(q, a))) = 0$$

implies that $p = \mu_a(s(q, a)) \in \mathcal{P}_K(q)$. Furthermore by definition we have $p \in \mathcal{U}_\delta(q_0, p_0)$ which implies $\rho(q, w(q, p)) > 1$ and thus $p = \mu_a(s(q, a)) \in \mathcal{E}_K^{reg}(q)$. Thus we have that $s(q, a) = f_a(q)$, making $f_a(q)$ a smooth function on $\overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ as desired. $\square$

Note that this result implies that $\mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)$ for all $(q, a) \in \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ and $f : V \times S^{n-1} \to [0, 1]$ smooth around all (q, a) such that $q \in V, a \in \mathcal{A}^{reg}(q)$.

Because this implies that also the derivative is a smooth function we can apply lemma A.0.2 to get

Proposition 2.3.7. *Let $q_n \to q_0 \in V$ and $A \subset S^{n-1}$ open, such $\overline{A} \subset \mathcal{A}^{reg}(q_0)$. Then for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $F_{q_n}|_{\overline{A}}$ is smooth and $\|dF_{q_n}|_a - dF_{q_0}|_a\|_{g_{S^{n-1}}} < \varepsilon$ for all $a \in \overline{A}$.*

Proof. By the previous proposition for all $a \in \overline{A}$ there exists a $\varepsilon_a > 0$ such that $f : \overline{B_{\varepsilon_a}(q_0)} \times \overline{B_{\varepsilon_a}(a)} \to [0,1]$ is smooth. Then $\bigcup_{a \in \overline{A}} B_{\varepsilon_a}(a)$ is an open cover of the compact $\overline{A} \subset S^{n-1}$. Hence there exist $(a_1, \varepsilon_1), \dots (a_N, \varepsilon_N)$ such that $\bigcup_{j=1}^N B_{\varepsilon_j}(a_j) \supset \overline{A}$. We then let $\varepsilon_0 := \min_{j=1,\dots,N} \varepsilon_j$ and get $B_{\varepsilon_0}(q_0) = \bigcap_{j=1}^N B_{\varepsilon_j}(q_0)$ is open.

Let now $(q,a) \in B_{\varepsilon_0}(q_0) \times \overline{A}$ then there exists a $j \in 1, \dots N$ such that $a \in B_{\varepsilon_j}(a_j)$ and we have $q \in B_{\varepsilon_0}(q_0) \subset B_{\varepsilon_j}(q_0)$. Thus by construction, f is smooth at (q,a). As the choice (q,a) was arbitrary f is smooth on $B_{\varepsilon_0}(q_0) \times \overline{A}$. Because $q_n \to q_0$ there exists a $N_1 \in \mathbb{N}$ such that $n \geq N$ implies $q_n \in B_{\varepsilon_0}(q_0)$ and we have $F_{q_n}|_{\overline{A}}$ is smooth.

We now want to show that also the derivatives of F_{q_n} wrt. $a \in S^{n-1}$ converge uniformly on $\overline{A}$: By the above argument

$$\begin{aligned} f' : \overline{B_{\frac{\varepsilon_0}{2}}(q_0)} \times \overline{A} &\to T^*S^{n-1} \\ (q,a) &\mapsto dF_q|_a \end{aligned}$$

is a continuous function on a compact metric spaces to a metric space (here we endow T^*S^{n-1} with some metric compatible with its topology). But now we can apply lemma A.0.2 to find that there exists a $N_2 > N_1$ such that $n \geq N_2$ implies $\|dF_{q_n}|_a - dF_{q_n}|_a\|_{g_{S^{n-1}}}$ for all $a \in \overline{A}$. ☐

Corollary 2.3.8. *Let $q_n \to q_0 \in V$ and $a_0 \in \mathcal{A}^{reg}(q_0)$. Then $dF_{q_n}|_{a_0} \to dF_{q_0}|_{a_0}$.*

In the following we will for any $(q,p) \in \mathcal{Z}$ denote $v(q,p) := \gamma'_{q,w(q,p)}(1)$, i.e. the velocity vector of the unique geodesic from q to p at p. Sometimes we can only recover the direction of $v(q,p)$ and will denote $\overline{v}(q,p) = \frac{v(q,p)}{\|v(q,p)\|}$.

This corollary follows from lemma 2.1.8 ((extended to show that it is homeo))

Corollary 2.3.9. *Let $(q_n)_{n=1}^\infty, q_0 \in V$ and $a_0 \in \mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_0} \to dF_{q_0}|_{a_0}$. Then $\overline{v}_n := \overline{v}(q_n, \mathcal{E}_{a_0}(q_n)) \to \overline{v}_0 := \overline{v}(q_0, \mathcal{E}_{a_0}(q_0))$.*

We can now prove that convergence in F_q implies convergence in q as desired:

Proposition 2.3.10. *Let $(q_n)_{n=1}^\infty, q_0 \in V$ and $a_1, a_2 \in \mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_i} \to dF_{q_0}|_{a_i}$. Then $q_n \to q_0$.*

Proof. We denote $p_n^i = \mathcal{E}_{a_i}(q_n)$ and $p_0^i = \mathcal{E}_{a_i}(q_0)$. By the previous corollary we have $\overline{v}_n^i := \overline{v}(q_n, p_n^i) \to \overline{v}_0^i := \overline{v}(q_0, p_0^i)$ for $i = 1,2$ in $CTM = \{(p,v) \in TM \mid g^+(v,v) = 1\}$. Note that by definition there exist $t_n^i, t_0^i \in \mathbb{R}_+$ such that

$$q_0 = \gamma_{p_0^i, v_0^i}(-t_0^i) \quad \text{and} \quad q_n = \gamma_{p_n^i, v_n^i}(-t_n^i) \quad \text{for } i = 1,2.$$

We now want to show that $t_n^i \to t_0^i$. By contradiction we assume that t_n^i does not converge to t_0^i. By a similar argument to the one employed in the proof of lemma 2.3.3 we find that t_n^i must be bounded. t_n^i has thus a convergent subsequence $t_j^i \to t_\times^i \neq t_0^i$ for $i = 1, 2$. Now we let d be a metric on M compatible with the topology and note that because $(q, w, t) \mapsto \gamma_{q,w}(t)$ is continuous we have

$$0 = \lim_{n\to\infty} d(\gamma_{p_j^1,v_j^1}(-t_j^1), \gamma_{p_j^2,v_j^2}(-t_j^2)) = d(\gamma_{p_0^1,v_0^1}(-t_\times^1), \gamma_{p_0^2,v_0^2}(-t_\times^2)),$$

i.e. $q_\times := \gamma_{p_0^i,v_0^i}(-t_\times^i)$ for $i = 1, 2$. But this is a contradiction because p_0^1 and p_0^2 are in $\mathcal{E}_K^{reg}(q_0)$ and thus cannot be cut points of q_0. $\square$

And lastly we show that F also has bounded derivatives.

Proposition 2.3.11. *Let $q_0 \in V$, $\varepsilon > 0$ such that $\overline{B_\varepsilon(q_0)} \subset V$ and define*

$$D_\varepsilon := \{(q, a) \in V \times S^{n-1} \mid q \in \overline{B_\varepsilon(q_0)}, a \in \mathcal{A}^{reg}(q)\}.$$

Then

$$\begin{aligned} f' : D_\varepsilon &\to T^*S^{n-1} \\ (q, a) &\mapsto dF_q|_a \end{aligned}$$

is bounded.

Proof. ((Make more rigorous / shorter / more understandable. The main idea here is that dF must be bounded because at the points where it is not defined i.e. points where $\mathcal{E}_a(q) \notin \mathcal{E}_K^{reg}(q)$, dF does not go to infinity but has multiple conflicting values (see prop 2.1.13), we try to show that by expressing dF as in terms of dh and dY which are well defined at the points where dF fails to be so))

We begin by defining the map

$$\begin{aligned} Y : L^K\overline{B_\varepsilon(q_0)} &\to \overline{B_\varepsilon(q_0)} \times S^{n-1} \\ (q, w) &\mapsto (q, \pi_a(\Theta^{-1}(X(q, w)))) \end{aligned}$$

mapping a null direction to the observer which sees the resulting geodesic. This map is smooth, surjective and locally diffeomorphic by lemma 2.3.3.

We also define the map

$$\begin{aligned} h : L^K\overline{B_\varepsilon(q_0)} &\to [-T_{sn}, 0] \\ (q, w) &\mapsto \pi_t(\Theta^{-1}(X(q, w))) \end{aligned}$$

mapping a null direction to its observation time. This map is also smooth.

We then define

$$\begin{aligned} P :=& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \rho(q,w) \geq 1\} \\ =& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \exp_q(w) \in \mathcal{E}_K(q)\} \end{aligned}$$

which is closed by the lower semicontinuity of ρ and thus compact. Now the following diagramm commutes:

$$\begin{tikzcd} P \subset L^K\overline{B_\varepsilon(q_0)} \arrow[r, "h"] \arrow[d, "Y"] & {[0,1]} \\ \overline{B_\varepsilon(q_0)} \times S^{n-1} \arrow[ur, "f"'] & \end{tikzcd}$$

Let now g^+ be the Riemannian metric induced on M by the splitting and $\widehat{g}^+$ the corresponding Sasaki metric induced on TM. Let also $g^\times := g^+ \oplus g_{S^{n-1}}$ be the product metric on $M \times S^{n-1}$ with $g_{S^{n-1}}$ the standard Riemannian metric on S^{n-1}. Because $h : P \to [-T_{S^{n-1}}, 0]$ is smooth, $dh : TP \to \mathbb{R}$ is smooth as well. Furthermore because P is compact and $dh_{(q,w)}$ is linear for all $(q,w) \in P$ we get that dh is bounded on TP, i.e. there exists a $c_1 > 0$ such that for all $(q,w) \in P$ and $(q',w') \in T_{(q,w)}P$ we have

$$|dh_{(q,w)}(q',w')| \leq c_1 \|(q',w')\|_{\widehat{g}^+}.$$

Similarly because Y is smooth as well as locally diffeomorphic its derivative is bounded from below, i.e. there exists a $c_2 > 0$ such that for all $(q,w) \in P$ and $(q',w') \in T_{(q,w)}P$ we have

$$\|dY_{(q,w)}(q',w')\|_{g^\times} \geq c_2 \|(q',w')\|_{\widehat{g}^+}.$$

We now define

$$\begin{aligned} P^{reg} :=& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \rho(q,w) > 1\} \\ =& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \exp_q(w) \in \mathcal{E}_K^{reg}(q)\} \text{ and} \end{aligned}$$

and the following again diagramm commutes:

$$\begin{tikzcd} P^{reg} \subset P \arrow[r, "h"] \arrow[d, "Y"] & {[0,1]} \\ D_\varepsilon \subset \overline{B_\varepsilon(q_0)} \times S^{n-1} \arrow[ur, "f"'] & \end{tikzcd}$$

Additionally in this case Y is a diffeomorphism and f is smooth. Let $(q,a) \in D_\varepsilon$ and $(q',a') \in T_{(q,a)}D_\varepsilon$, then there exists a unique $(q,w) = Y^{-1}(q,a) \in P^{reg}$ and $(q',w') = dY^{-1}(q',a') \in T_{(q,w)}P^{reg}$ and we have

$$\begin{aligned} |df_{(q,a)}(q',a')| &= |dh_{(q,w)} \circ dY^{-1}_{(q,a)}(q',a')| \leq c_1 \|dY^{-1}_{(q,a)}(q',a')\|_{\widehat{g}^+} \\ &= c_1 \|(q',w')\|_{\widehat{g}^+} \leq c_1 c_2 \|dY_{q,w}(q',w')\|_{g^\times} = c_1 c_2 \|(q',a')\|_{g^\times}. \end{aligned}$$

This implies that df and thus also dF is bounded on D_ε as desired. $\square$

Chapter 3

Interior Reconstruction

In this chapter we will use the observation time functions to reconstruct the topological, differential and conformal data of V.

3.1 Construction of the topology

The central idea in this section is to find a subspace of the space of all functions from S^{n-1} to $\mathbb{R}$, $\mathbb{R}^{S^{n-1}}$ such that for all $q \in V$, F_q is contained in this subspace and convergence in this subspace is equivalent to convergenge in V. A suitable space for this task turns out to be $\mathcal{C}^\infty(S^{n-1})$, the space of continuous function $F : S^{n-1} \to [-T_{S^{n-1}}, 0]$ which are smooth on a dense open set in S^{n-1} and bounded derivative, endowed with the metric

$$d(F,G) := d_\infty(F,G) + \int_{S^{n-1}} \|dF_a - dG_a\|_{g_{S^{n-1}}} da,$$

where $d_\infty(F,G) := \max_{a \in S^{n-1}} |F(a) - G(a)|$. Note that by definition of $\mathcal{C}^\infty(S^{n-1})$ the subset of S^{n-1} where F or G are not smooth is a null set, making the integral well-defined.

We can then define the function

$$\begin{aligned} \mathcal{F} : V &\to (\mathcal{C}^\infty, d) \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in V$ to the function $F_q : S^{n-1} \to \mathbb{R}$.

The following argument establishes that the canonical topological structure on $\mathcal{F}(V)$, i.e. the topology obtained by taking the subspace topology with respect to the topology induced by d on $\mathcal{C}^\infty$, is the same as the pushforward under $\mathcal{F}$ of the topology on V, making $\mathcal{F}$ a homeomorphism.

Lemma 3.1.1. *The map $\mathcal{F} : V \to \widehat{V} := \mathcal{F}(V)$ is a well-defined continuous and bijective.*

Proof. First of all we show that $\mathcal{F} : V \to (\mathcal{C}^\infty, d)$ is well-defined. Let $q \in V$, then F_q is continuous by proposition 2.2.3, smooth on a dense open set (namely $\mathcal{A}^{reg}(q)$) by proposition 2.3.6, with bounded derivative by proposition 2.3.11.

To prove that $\mathcal{F}$ is continuous we let $q_n \to q_0 \in V$. By proposition 2.2.4 $F_{q_n} \to F_{q_0}$ uniformly and thus $d_\infty(F_{q_n}, F_{q_0}) \to 0$. Now we need to show that

$$\int_{S^{n-1}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \to 0.$$

To that end let $\varepsilon > 0$ and δ_1 such that $\overline{B_{\delta_1}(q_0)} \subset V$. Because $q_n \to q_0$, after possibly discarding finitely many q_n we may assume $q_n \in B_{\delta_1}(q_0)$. Because $dF_q|_a$ is bounded on D_{δ_1} by proposition 2.3.11 there exists a $c > 0$ such that $\| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} < c$ for all $n \in \mathbb{N}$ and $a \in \mathcal{A}^{reg}(q_n) \cap \mathcal{A}^{reg}(q_0)$.

On the other hand because $\mathcal{A}^{reg}(q_0)$ is dense and open in S^{n-1} we have $\int_{\mathcal{A}^{reg}(q_0)} da = \int_{S^{n-1}} da$. Hence we can find an open set $A \in S^{n-1}$ such that $\overline{A} \subset \mathcal{A}^{reg}(q_0)$ and

$$\int_{S^{n-1} \setminus \overline{A}} da < \frac{\varepsilon}{2c}.$$

Applying proposition 2.3.7 to A yields a $N \in \mathbb{N}$ such that for all $n \geq N$ we have $F_{q_n}|_{\overline{A}}$ smooth and $\|dF_{q_n}|_a - dF_{q_0}|_a\|_{g_{S^{n-1}}} < \frac{\varepsilon}{2}$ for all $a \in \overline{A}$.

We can now write

$$\begin{aligned}
\int_{S^{n-1}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da &= \int_{S^{n-1} \setminus \overline{A}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \\
&\quad + \int_{\overline{A}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \\
&< \varepsilon
\end{aligned}$$

because

$$\int_{S^{n-1} \setminus \overline{A}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \leq \int_{S^{n-1} \setminus \overline{A}} c da < \frac{\varepsilon}{2} \quad \text{and}$$
$$\int_{\overline{A}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da < \int_{\overline{A}} \frac{\varepsilon}{2} da \leq \frac{\varepsilon}{2}.$$

Because $\varepsilon > 0$ was arbitrary we get $\int_{S^{n-1}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \to 0$ and thus $d(F_{q_n}, F_{q_0}) \to 0$ proving that $\mathcal{F}$ is continuous.

Finally, injectivity follows from the fact that for any $q, q' \in V$ we have $\mathcal{F}(q) = \mathcal{F}(q') \implies F_q = F_{q'} \implies \mathcal{E}_K(q) = \mathcal{E}_K(q')$ which implies $q = q'$ by proposition 2.2.9. $\square$

However there is still some work required to show that $\mathcal{F}^{-1}$ is continuous on $\widehat{V}$:

Lemma 3.1.2. *Let $F_n \to F_0$ in $\widehat{V}$ then $q_n := \mathcal{F}^{-1}(F_n) \to q_0 := \mathcal{F}^{-1}(F_0)$.*

Proof. Note that by the previous result $\mathcal{F} : V \to \widehat{V}$ is a bijection and thus q_n and q_0 are well defined and we have $F_n = F_{q_n}$ resp. $F_0 = F_{q_0}$. We now aim to find $a_1, a_2 \in \mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_i} \to dF_{q_0}|_{a_i}$, allowing us to apply proposition 2.3.10: Let for some set $S \subset S^{n-1}$ we let $\mu(S) := \int_S da$ be the standard set measure and $S^c = S^{n-1} \setminus S$ the complement.

Let

$$A := \mathcal{A}^{reg}(q_0) \cap \bigcap_{n=1}^{\infty} \mathcal{A}^{reg}(q_n)$$

and

$$C = A^c, \quad C_n = \mathcal{A}^{reg}(q_n)^c, \quad C_0 = \mathcal{A}^{reg}(q_n)^c.$$

Because $\mu(\mathcal{A}^{reg}(q_n)) = \mu(\mathcal{A}^{reg}(q_0)) = \mu(S^{n-1}) < \infty$, we have $\mu(C_n) = \mu(C_0) = 0$. This yields

$$\mu(C) = \mu\left(C_0 \cup \bigcup_{n=1}^{\infty} C_n\right) \leq \mu(C_0) + \sum_{n=1}^{\infty} \mu(C_n) = 0$$

and thus $\mu(A) = \mu(S^{n-1}) - \mu(C) = \mu(S^{n-1}) > 0$.

We then define the set of *stragglers* as

$$S(A) := \{a \in A \mid \lim_{n\to\infty} dF_{q_n}|_a \neq dF_{q_0}|_a\}.$$

Because $F_{q_n} \to F_{q_0}$ with respect to d we must have $\int_{S^{n-1}} \| dF_{q_n}|_a - dF_{q_0}|_a\|_{g_{S^{n-1}}} da \to 0$ which implies $\mu(S(A)) = 0$. But now we have $\mu(A \setminus S(A)) > 0$ which implies that there exist two $a_1, a_2 \in A \setminus S(A)$. By definition F_{q_n} is smooth at a_i for all $n \in \mathbb{N}$ and $dF_{q_n}|_{a_i} \to dF_{q_0}|_{a_i}$. Now we can apply proposition 2.3.10 and get $q_n \to q_0$ as desired. $\square$

And we get:

Corollary 3.1.3. *$\mathcal{F} : V \to \widehat{V}$ is a homeomorphism.*

Note that because $\mathcal{P}_K(V)$ determines all F_q and thus the set $\widehat{V}$, it is uniquely determined by the data 1.2.1. We can thus reconstruct the topology of V because we can determine the topology of $(\mathcal{C}^\infty(S^{n-1}), d)$ and the subspace topology on $\widehat{V}$ which must be equivalent to the topology on V.

3.2 Smooth Reconstruction

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

3.2.1 Construction of smooth coordinates

In line with the last section we will consider $\widehat{V} := \mathcal{F}(V)$ a topological space. We denote the points of this manifold by $\widehat{q} := \mathcal{F}(q) = F_q$. This means that points in $\widehat{V}$ are functions from S^{n-1} to $\mathbb{R}$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

This definition is related to the coordinates constructed in proposition 2.3.4, i.e. obervation coordinates consist of a tuple of observation times from $n+1$ observers.

Definition 3.2.1 (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=0}^n \subset (S^{n-1})^{n+1}$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_K^{reg}(q)$ for all $j = 0, \dots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in (S^{n-1})^{n+1}$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will often denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

Crucially for a tuple $\overrightarrow{a} \in (S^{n-1})^{1+n}$ we can determine $\mathbf{s}_{\overrightarrow{a}} : \widehat{V} \to \mathbb{R}^{n+1}$ using only the data from 1.2.1. This is because for a given $\widehat{q} = F_q \in \widehat{V}$ we have $\mathbf{s}_{\overrightarrow{a}}(\widehat{q}) = (F_q(a_0), \dots, F_q(a_n))$ requiring no external information.

We can now determine the differential structure of V:

Proposition 3.2.2. *Let $\widehat{q} \in \widehat{V}$ then the following holds:*

(1) *Given the data from 1.2.1 we can determine all C^0-observation coordinates around $\widehat{q}$,*

(2) *there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$ and*

(3) *given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data 1.2.1, allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.*

Proof. We begin with some setup: Let $q \in V$. We say that $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ are *associated* with respect to q if $p \in \mu_a$, i.e. $p = \mathcal{E}_a(q)$.

To prove part (1), we let $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. We want to show that for any choice of observers $\overrightarrow{a} = (a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ we can determine if they form C^0-observation coordinates. First of all we need to check whether the associated $p_j = \mathcal{E}_{a_j}(q)$ are regular points, i.e. $p_j \in \mathcal{E}_K^{reg}(q)$. But as $\widehat{q} = \mathcal{F}(q) = F_q$ we can recover $\mathcal{E}_K q = \bigcup_{a\in S^{n-1}} \mu_a(F_q(a))$ and also the associated points $p_j = \mu_{a_j}(F_q(a_j))$. By proposition 2.2.8 this allows us to determine $\mathcal{E}_K^{reg}(q)$ and for all p_j we can then simply check whether they lie in $\mathcal{E}_K^{reg}(q)$.

We now need to check whether there exists an open neighborhood W of $\widehat{q}$ such that the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is injective. By definition we have

$$\mathbf{s}_{\overrightarrow{a}}(\widehat{q}) = (\widehat{q}(a_1), \dots, \widehat{q}(a_n)) = (F_q(a_0), \dots, F_q(a_n))$$

which means that the data allows us to fully determine $\mathbf{s}_{\overrightarrow{a}}$ on $\widehat{V}$. But since by corollary 3.1.3, the data allows us do construct the topology on $\widehat{V}$ we can determine whether there exists an open neighborhood W of $\widehat{q}$ such that $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is injective and thus open by the invariance of domain theorem.

To show (2) we let again $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. Let $(a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ such that the associated $p_j \in \mathcal{E}_K^{reg}(q)$ and the vectors $\{w_j = w(q, p_j) \mid j = 0, \dots n\}$ are linearly independent. We can find such a set of linearly independent vectors because by proposition 2.2.6 $\mathcal{E}_K^{reg}(q)$ is an open subset of $\mathcal{E}_K(q)$. Now by proposition 2.3.4 the observation time functions $\mathbf{f}_{\overrightarrow{a}}$ define smooth coordinates on a neighborhood V_1 of q. Thus $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F}$ are smooth local coordinates as well making $(\mathbf{s}_{\overrightarrow{a}}, \mathcal{F}(V_1))$ C^∞-observation coordinates.

Moving on to part (3): We begin by proving that the set of points in $(\mathcal{E}_K^{reg}(q))^{n+1}$ which yield C^∞-observation coordinates is open and dense in $(\mathcal{E}_K^{reg}(q))^{n+1}$. We consider $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ which are associated. Let

$$K(q) = \{(w_j)_{j=0}^n \mid w_j \in L_q^K M, \rho(q, w_j) > 1, \gamma_{q,w_j}(1) \in K\}$$

and define on $K(q)$ the map

$$\begin{aligned} H : K(q) &\to K^{n+1} \\ (w_j)_{j=0}^n &\mapsto (\gamma_{q,w_j}(1))_{j=0}^n. \end{aligned}$$

We will denote $p_j = \gamma_{q,w_j}(1) = \exp_q(w_j)$. Then by definition $p_j \in \mathcal{E}_K^{reg}(q)$ and $w_j = \Omega(q, p_j)$. As ρ is lower semi-continuous, we see that $K(q) \subset (L_q^K M)^n$ is open by an analogous argument to the one in the proof of 2.2.6. As the exponential map is continuous, H is also continuous. Furthermore as $\Omega : \mathcal{Z} \to L^+V$ is continuous and injective, we can construct a continuous inverse to H, making $H : K(q) \to H(K(q)) = (\mathcal{E}_K^{reg}(q))^{n+1}$ a homeomorphism. We will denote $Y(q) \coloneqq (\varepsilon_U^{reg}(q))^{n+1}$. Note that for all $\widehat{q} \in \widehat{V}$, the data 1.2.1 determine $\mathcal{E}_K^{reg}(q)$ and thus also the set $Y(q) \subset K^n$, where $q = \mathcal{F}^{-1}(\widehat{q})$.

Let us now consider the set

$$K_0(q) = \{(w_j)_{j=1}^n \in K(q) \mid w_1, \dots, w_n \text{ are linearly independent}\}.$$

As linear independence is an open and non-degenerate property $K_0(q)$ is open and dense in $K(q)$. Since H is a homeomorphism, $Y_0(q) = H(K_0(q))$ is open and dense in $Y(q)$ as well.

We can now prove the final part of the proposition: Recall that given C^0-observation coordinates around $\widehat{q}$, we want to determine if they are also C^∞-observation coordinates $\widehat{q}$. To that end, let $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ be C^0-observation coordinates around $\widehat{q} \in W_{\overrightarrow{a}}$ with $q = \mathcal{F}^{-1}(\widehat{q})$. By definition we have $p_j \in \mathcal{E}_K^{reg}(q)$ where $p_j = \mathcal{E}_{a_j}(q)$ are associated with a_j and hence $(p_j)_{j=0}^n \subset Y(q)$. In the case where $(p_j)_{j=0}^n \in Y_0(q)$, by proposition 2.3.4, q has a neighborhood $V_1 \subset M$ on which the function $\mathbf{f}_{\overrightarrow{a}} : V_1 \to \mathbb{R}^n$ gives smooth local coordinates. Thus, after possibly restricting $W_{\overrightarrow{a}}$, $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$. We then let $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$, $\overrightarrow{b} \in (S^{n-1})^{n+1}$ be different C^0-observation coordinates around $\widehat{q}$ and let $(\widetilde{p}_j)_{j=0}^n \in Y(q)$ be such that $\widetilde{p}_j$ is associated to b_j. Since all smooth coordinates must be compatible, then $(\widetilde{p}_j)_{j=0}^n \in Y_0(q)$ if and only if

$$\text{The function } \mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1} \text{ is smooth at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ and the Jacobian determinant } \det(D(\mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1})) \text{ at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ is non-zero.} \tag{3.1}$$

Here the "only if"-direction follows from the fact that the nondegeneracy of the Jacobian ensures that the linear independence of the spanning vectors is preserved.

For some $\overrightarrow{p} = (p_j)_{j=0}^n \in Y(q)$ with $\overrightarrow{a}$ associated we define $\mathcal{X}_{\overrightarrow{p}} \subset Y(q)$ to be the set of $(\widetilde{p}_j)_{j=0}^n \in Y(q)$, such that for the associated $\overrightarrow{b}$ there exists $W_{\overrightarrow{b}}$ such that $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$ are C^0-coordinates around $\widehat{q}$ and condition 3.1 is satisfied.

If $\overrightarrow{p} \in Y_0(q)$ we see that $Y_0(q) \subset \mathcal{X}_{\overrightarrow{p}}$. On the other hand $\overrightarrow{p} \notin Y_0(q)$ we have $Y_0(q) \cap \mathcal{X}_{\overrightarrow{p}} = \emptyset$. Since the set $Y_0(q)$ is open and dense in $Y(q)$, we see that $\overrightarrow{p} \in Y_0(q)$ if and only if the interior of $\mathcal{X}_{\overrightarrow{p}}$ is dense subset of $Y(q)$. Since the data 1.2.1 is sufficient to determine $Y(q)$ and $\mathcal{X}_{\overrightarrow{p}}$, we can determine whether $\overrightarrow{p} \in Y_0(q)$ or not. And since, by proposition 2.3.4, the C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q} = \mathcal{F}(q)$ are C^∞-observation coordinates if and only if $\overrightarrow{p} \in Y_0(q)$, where $\overrightarrow{p}$ are associated to $\overrightarrow{a}$ wrt. q, we can determine all C^0-observation coordinates around $\widehat{q}$ which are also C^∞-observation coordinates. □

3.3 Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. The next lemma will allow us to determine a time-orientation on $\widehat{V}$ making $\mathcal{F} : V \to \widehat{V}$ a causal map, and a metric G which is conformally equivalent to $\widehat{g}$. The key ideas are that we can use some equations from proposition 2.3.4 to determine a timelike, future-pointing vector field on $\widehat{V}$, and that for a given $\widehat{q} \in \widehat{V}$ we can determine all null geodesics through $\widehat{q}$ allowing us to determine all null cones $L_{\widehat{q}}\widehat{V}$ in $\widehat{V}$.

Lemma 3.3.1. *The data given in 1.2.1 allows us to determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Proof. Let $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^∞-observation coordinates on $\widehat{V}$ and $\widehat{q} \in W_{\vec{a}}$. We begin by constructing a time orientation on $\widehat{V}$: Let $a_1, a_2 \in \overrightarrow{a}$ and $p_1, p_2 \in U$ be associated wrt. the point $q = \mathcal{F}^{-1}(\widehat{q})$, i.e. $p_i = \mathcal{E}_{a_i}(q)$. Because $\mathbf{f}_{\vec{a}} = \mathbf{s}_{\vec{a}} \circ \mathcal{F}$ are smooth coordinates we have that the vectors $w(q, p_1)$ and $w(q, p_2)$ pointing from q to p_i must be non-parallel. Therefore, by equation 2.9 we see that the gradient vectors $\nabla f_{a_i}(q)$ are non-parallel, lightlike and past-pointing. Thus the co-vectors $-ds_{a_1}|_{\widehat{q}}$ and $-ds_{a_2}|_{\widehat{q}}$ are non-parallel lightlike and future-pointing. This follows from the fact that $\mathcal{F}$ is an isometry and the co-vector df_a is the image of ∇f_a under the canonical isomophism. Moreover because the data allows us to fully determine s_{a_1} and s_{a_2} on $\widehat{V}$ (see previous proof) we can also determine ds_{a_1} resp. ds_{a_2}.

The co-vector field $X = (-ds_{a_1}) + (-ds_{a_2})$ is timelike and future-pointing and forms a local time-orientation on $W_{\vec{a}}$. Using bump functions and a partition of unity we can then obtain a time-orientation on the whole of $\widehat{V}$ since all orientations agree where they overlap.

Now we turn our attention to the construction of a metric G which is conformal to $\widehat{g}$: Let again $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^∞-observation coordinates on $\widehat{V}$ with $\widehat{q}_0 \in W_{\vec{a}}$ and $q_0 \in V$ such that $\widehat{q}_0 = \mathcal{F}(q_0)$. As in the previous proof, using the data given in 1.2.1 and the function $\widehat{q}_0 = F_{q_0}$ we can determine $\mathcal{E}_K(q_0), \mathcal{E}_K^{reg}(q_0), \mathcal{D}_K(q_0)$ and $\mathcal{D}_K^{reg}(q_0)$ by 2.2.8.

We then fix the point $\widehat{q}_0 = \mathcal{F}(q_0)$ and the tuple $(p, v) \in \mathcal{D}_K^{reg}(q_0)$. Let $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{p,v}((-\widehat{t}, 0]) \subset M$ is defined and has no cut point. For $q \in V$, we have that $q \in \gamma_{p,v}((-\widehat{t}, 0))$ if and only if $(p, v) \in \mathcal{D}_K^{reg}(q)$. Hence for a fixed $(p, v) \in \mathcal{D}_K^{reg}(q_0)$ the data allows us to whether some $\widehat{q} \in W_{\vec{a}}$ has $q = \mathcal{F}^{-1}(\widehat{q}) \in \gamma_{p,v}((-\widehat{t}, 0))$ by checking if $(p, v) \in \mathcal{D}^{reg}(q)$. This allows us to determine

$$\beta = \{\widehat{q} \in W_{\vec{a}} \mid \widehat{q} = \mathcal{F}(q), \mathcal{D}_K^{reg}(q) \ni (p, v)\} = \mathcal{F}(\gamma_{p,v}((-\widehat{t}, 0))) \cap W_{\vec{a}}.$$

Therefore, on $W_{\vec{a}} \subset \widehat{V}$ we can find the image, under the map $\mathcal{F}$, of the light-like geodesic segment $\gamma_{p,v}((-\widehat{t}, 0)) \cap \mathcal{F}^{-1}(W_{\vec{a}})$ that contains $q_0 = \gamma_{p,v}(-t_1)$. Let $\alpha(s), s \in (-s_0, s_0)$ be a smooth path on $W_{\vec{a}}$ such that $\partial_s \alpha(s)$ is never zero, $\alpha((-s_0, s_0)) \subset \beta$ and $\alpha(0) = \widehat{q}_0$. Such a smooth path can, for example be obtained by endowing $\widehat{V}$ with some arbitrary Riemannian metric and parameterizing by arc-length. Then $\widehat{w} = \partial_s \alpha(s)|_{s=0} \in T_{\widehat{q}_0}\widehat{V}$ has the form $\widehat{w} = c\mathcal{F}(\gamma'_{p,v}(-t_1))$ where $c \neq 0$.

Since we can do the above construction for all points $(p, v) \in \mathcal{D}_U^{reg}(q_0)$, we can determine in the tangent space $T_{\widehat{q}_0}\widehat{V}$ the set

$$\Gamma = \mathcal{F}_*(\{cw \in L_{q_0}M \mid \exp_{q_0}(w) \in \mathcal{E}_K^{reg}(q_0), c \in \mathbb{R} \setminus \{0\}\})$$

which is an open, non-empty subset of the light cone at $\widehat{q}_0$ wrt. the metric $\widehat{g}$. But now, since the light cone is determined by a quadratic equation in the tangent space, having an open set Γ determines the whole light cone. By repeating this construction for all points $\widehat{q} \in \widehat{V}$, we can uniquely determine $L\widehat{V}$. Using proposition B.6.3 we can then determine the conformal class of the tensor $\widehat{g} = \mathcal{F}_*g$ in the manifold $\widehat{V}$.

The above shows that the data 1.2.1 determine the conformal class of the metric tensor $\widehat{g}$. And in particular we can construct a metric G on $\widehat{V}$ that is conformal to $\widehat{g}$ and satisfies $G(X, X) = -1$. $\square$

3.4 Reconstruction overview

We have gone through all the steps necessary to reconstruct the conformal, differential and topological data of V and will now tie this all together to give a detailed account of the actual reconstruction.

As mentioned in remark 1.2.1 we want to prove the following theorem which implies theorem 1.1.5:

Theorem 3.4.1. *Let (M, g) be a globally hyperbolic Lorentzian manifold and $p^+, p^- \in M, V \subset J(p^-, p^+)$ suitable such that V is an open subset of $J(p^-, p^+)^{\circ}$. Then given*

(1) *The smooth manifold K,*

(2) *the conformal class of $g|_K$ and*

(3) *the set of light cone observations $\mathcal{P}_K(V)$*

we can construct a globally hyperbolic Lorentzian manifold $\widehat{V}$ such that there exists a conformal diffeomorphism $\mathcal{F} : V \to \widehat{V}$, which preserves causality.

Proof. To construct the space $\widehat{V}$ which is conformally diffeomorphic to V we follow these steps:

- As $f_a(q) = \min\{s \in [-T_a, 0] \mid \mu_a(s) \in \mathcal{P}_K(q)\}$ we can determine $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$ from $\mathcal{P}_K(V)$.
- Proposition 2.2.8 then allows us to determine $\mathcal{D}_K^{reg}(q)$, $\mathcal{D}_K(q)$ and $\mathcal{E}_K^{reg}(q)$ for a given $\mathcal{E}_K(q) \in \mathcal{E}_K(V)$. We can thus construct $\mathcal{D}_K^{reg}(V)$, $\mathcal{D}_K(V)$ and $\mathcal{E}_K^{reg}(V)$.

- We define the function

$$\begin{aligned}\mathcal{F} : V &\to \mathcal{F}(V) = \widehat{V} \subset (\mathcal{C}^\infty(S^{n-1}), d) \\ q &\mapsto \widehat{q} = F_q = (a \mapsto f_a(q)).\end{aligned}$$

  For a given $\mathcal{E}_K(q)$ we can construct $\widehat{q}$ by $\widehat{q}(a) = f_a(q) = s$ such that $\mu_a(s) \in \mathcal{E}_K(q)$. This allows us to construct the map

$$\begin{aligned}\widetilde{\mathcal{F}} : \mathcal{E}_K(V) &\to \widehat{V} \\ \mathcal{E}_K(q) &\mapsto \widehat{q}.\end{aligned}$$

  And we can thus determine the set $\widehat{V} = \widetilde{\mathcal{F}}(\mathcal{E}_K(V))$.

- By taking the subspace topology with respect to the topology on $\mathcal{C}^\infty(S^{n-1})$ induced by d we can determine a topology on $\widehat{V}$. By corollary 3.1.3 this topology is homeomorphic to the topology on V, making $\mathcal{F}$ a homeomorphism.

- For a given point $\widehat{q} \in \widehat{V}$ we can use proposition 3.2.2 and the data to determine all C^0-observation coordinates around $\widehat{q}$. We can then determine for each of these coordinates if they are also C^∞-observation coordinates, and find at least one such coordinate system since existence is guaranteed. We can repeat that step for each $\widehat{q} \in \widehat{V}$ to find smooth coordinates on $\widehat{V}$, making $\mathcal{F}$ a diffeomorphism.

- Finally we can use lemma 3.3.1 to construct a metric G and time-orientation X on $\widehat{V}$ which is conformal to $\widehat{g} = \mathcal{F}_* g$ and makes $\mathcal{F}$ causal. $\mathcal{F} : (V, g|_U) \to (\widehat{V}, G)$ is thus a causal conformal diffeomorphism as desired.

□

Remark 3.4.2. In the statement of theorem 1.1.5 we required that V be a subset of the interior $J(p^-, p^+)^{\mathrm{o}}$. This is because as we approach the observation set K, the light cone observation sets get increasingly degenerate and loose many of their nice properties for points on the boundary, i.e. if we had a $q \in V \cap K$. This issue will be adressed in the next chapter by smoothing the observation time functions at the boundary.

However if $q \in V$ approaches the past boundary $K^- := \mathcal{L}^+_{p^-} \cap I^-(p^+)$ the situation is simpler: Because we are always away from the set of observers K, the light cone observation sets remain transverse to K and thus well behaved even for $q \in \mathcal{L}^+_{p^-} \cap I^-(p^+)$. It is thus possible to relax the condition $V \subset J(p^-, p^+)^{\mathrm{o}}$ to $V \subset J(p^-, p^+) \setminus K$ in theorem 1.1.5 with only minor modifications to the proofs.

Chapter 4

Boundary Reconstruction

4.1 Setting

In this section we will examine how we can extend our reconstruction result to the case where the source set V is no longer contained within the interior of $J(p^-, p^+)$ but is now allowed to extend up to the boundary $K \setminus \{p^+\}$.

This is complicated by the fact that as $q \in J(p^-, p^+)$ approaches the observation set K, the light observation sets $\mathcal{P}_K(q)$ get increasingly warped and are degenerate if $q \in K$.

Analogous to the interior reconstruction case we will again prove the modified version outlined in remark 1.2.1, and let (M, g) be a globally hyperbolic Lorentzian manifold, with $p^+, p^- \in M, V \in J(p^-, p^+)$ suitable such that $V \in J(p^-, p^+)$ is relatively open.

4.2 Preliminaries

To extend the reconstruction up to the edge of $J(p^-, p^+)$ we will essentially split up the reconstruction into two steps: We will split up V into $V \cap (J(p^-, p^+) \setminus K)$ and $V \cap D$ where D is the set of all points such that F_q has a unique minimum. On $V \cap (J(p^-, p^+) \setminus K)$ we can use the reconstruction result from the previous chapter and on $V \cap D$ we will use the fact that the observation time functions have unique minima to smooth them on the boundary K and allow a reconstruction.

4.2.1 Definitions

We first need to introduce some new concepts:

Definition 4.2.1 (Unique minimum domain)**.** We define the *unique minimum domain* $D \subset J(p^-, p^+)$ to be

$$D := \{q \in J(p^-, p^+)^\circ \cup K \mid F_q \text{ has a unique minimum}\}. \tag{4.1}$$

We will often describe this minimum with

$$(a_q, t_q) = (\arg\min_{a \in S^{n-1}} F_q, \min_{a \in S^{n-1}} F_q).$$

We will see that D an open neighborhood of $K \setminus \{p^+\}$ allowing us to reconstruct boundary points. As mentioned remark 3.4.2 the reconstruction from the past chapter can be applied to the whole $J(p^-, p^+) \setminus K$, because D is an open neighborhood of K in $J(p^-, p^+)$ this will allow us to reconstruct set in all of $J(p^-, p^+) \setminus \{p^+\}$.

Definition 4.2.2 (Constant observation time domain)**.** For some $t_0 \in (T_{S^{n-1}}, 0)$ we define the *constant observation time domain* as

$$T_{t_0} = \{p \in K \mid p = \mu_a(t_0), a \in S^{n-1}\} = K \cap \mathcal{T}^{-1}(\mathcal{T}(p^+) + t_0) \subset K. \tag{4.2}$$

Where the second characterization follows from equation 2.6. Because $\mathcal{T}^{-1}(\mathcal{T}(p^+) + t_0)$ is a cauchy hypersurface and thus a spacelike submanifold, T_{t_0} is a $n-1$ dimensional spacelike submanifold of K (away from its boundary). Thus for every $a \in S^{n-1}$ such that $T_a > t_0$ we can use lemma 2.1.8 to find the unique future-pointing outward null ray $\mathbb{R}_+ \nu_{a,t_0} \in L^+_{\Theta(a,t_0)} M$ such that $T_{\Theta(a,t_0)} T_{t_0} = \nu^\perp_{a,t_0} \cap T_{\Theta(a,t_0)} K$.

Note that for every $q \in J(p^-, p^+)^\circ$ and $p = \Theta(a, t) \in \mathcal{P}_K(q)$ we have $t > -T_a$ which implies that p is in the relative interior of T_t, i.e. there exists an open neighborhood $p \in U \subset M$ such that $T_t \cap U$ is a submanifold.

This lemma is often useful because is shows that for F_q with $q \in J(p^-, p^+)^\circ$ local minima must be regular points and the unique null geodesic from q to the minimum must have a specific direction at the minimum related to the constant observation time domain:

Lemma 4.2.3. *Let $q \in J(p^-, p^+)^\circ$ with (a_q, t_q) a local minimum of F_q and $p_q := \mu_{a_q}(t_q)$. Then we have $p_q \in \mathcal{E}^{reg}_K(q)$ and $v(q, p_q) \in \mathbb{R}_+ \nu_{a_q,t_q}$, i.e. if $w_q \in L^K_q M$ is the unique null vector such that $\gamma_{q,w_q}(1) = p$ we have $\gamma'_{q,w_q}(1) \in \mathbb{R}_+ \nu_{a_q,t_q}$.*

Proof. Note that we have $t_q = f_{a_q}(q)$ and thus

$$p_q = \mu_{a_q}(t_q) = \mu_{a_q}(f_{a_q}(q)) = \mathcal{E}_{a_q}(q) \in \mathcal{P}_K(q),$$

proving that there exists a $w_q \in L^K_q M$ such that $p_q = \gamma_{q,w_q}(1)$.

Now we need to show that indeed $p_q \in \mathcal{E}^{reg}_K(q)$. We recall that by proposition 2.1.13 there exists an open neighborhood $p_q \in U \subset \mathcal{P}_K(q)$ such that $\mathcal{P}_K(q) \cap U$ is the union of N pairwise transversal, spacelike, dimension $n-1$ submanifolds $\mathcal{V}_i$. Because t_q is the minimum of F_q we must have $T_{p_q}\mathcal{U}_i = T_{p_q}T_{t_q}$ for all $i = 1, \dots N$. But because the manifolds must be pairwise transversal, we must have $N = 1$, implying that p_q is a regular point of $\mathcal{P}_K(q)$. Together with $p_q \in \mathcal{E}_K(q)$ this yields $p_q \in \mathcal{E}_K^{reg}(q)$.

Finally $\gamma'_{q,w_q}(1) = \mathbb{R}_+\nu_{a_q,t_q}$ follows from the fact that $T_{p_q}\mathcal{V}_1 = T_{p_q}T_{t_q}$. □

4.2.2 Observation time Functions at the boundary

In this subsection we will study how the observation time functions behave for point $q \in K$ as well as for sequences $q_n \in V \to q_0 \in K$:

We can explicitly characterize observation time functions at the boundary.

Lemma 4.2.4. *For $q_0 = \mu_{a_0}(t_0) \in K$ we have*

$$F_{q_0}(a) = \begin{cases} t_0 & \text{if } a = a_0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We begin with the case $a = a_0$ then $F_{q_0}(a_0) = t_0$ follows immediately from the definition of $f_{a_0}(q_0)$. Note that this also covers the case where $q_0 = p^+$. For the case where $a \neq a_0$ and $q_0 \neq p^+$ we suppose that $F_{q_0}(a) = f_a(q_0) < 0$ by contradiction. Then we have $\tau(q_0, \mathcal{E}_a(q_0)) = 0$ which implies that there exists a null geodesic γ with $\gamma(0) = q_0$ and $p := \mathcal{E}_a(q_0) = \gamma(1)$. If $\gamma'(1) = \mu'_a(f_a(q_n))$ we would have $q_0 \in \mu_a([0,1)) \cap \mu_{a_0}([0,1))$ which is a contradiction to lemma 2.1.1. We must thus have $\gamma'(1) \neq \mu'_a(f_a(q_n))$ but this means there exists a broken null geodesic from q_0 to p^+ which is also a contradiction because $q_0 \in K$ by assumption and $K \cap I^-(p^+) = \emptyset$ by lemma 2.1.1. □

Remark 4.2.5. The previous lemma shows that the observation time functions F_q for $q \in K$ lose many nice properties they had when $q \neq K$. In particular if $q \in K$, then F_q is not continuous at a_0. Furthermore let $q_n = \Theta(a_n, t_0) \to q_0 = \Theta(a_0, t_0)$ with $a_n \notin a_0$, then $F_{q_0}(a_0) = t_0$ but $F_{q_n}(a_0) = 0$ for all $n \in \mathbb{N}$, implying F_{q_n} fails to even converge pointwise to F_{q_0}. Later on we will fix some of these issues by multiplying F_q with a smoothing bump function.

This lemma shows that even though obervation time functions dont retain many of their nice properties at the boundary they at least behave somewhat regularly away from the minimum.

Lemma 4.2.6. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$ and $A \subset S^{n-1}$ an open neighborhood of a_0 then we have $F_{q_n}|_{S^{n-1}\setminus A} \to 0$ uniformly.*

Proof. Because any q_n can either lie in the boundary K or in the interior $J(p^-,p^+)^o$ we can instead look at the subsequences $(q_n)_{n=1}^\infty \cap K$, $(q_n)_{n=1}^\infty \cap J(p^-,p^+)^o$. If we can prove that both subsequences converge to q_0 then we have also proven that q_n itself converges to q_0.

Hence let now $q_n \to q_0 \in K \setminus p^+$ with $q_n = \mu_{a_n}(t_n) \in K \setminus p^+$. We then have $a_n \to a_0$ and thus $a_n \in A$ for all $n \geq N$ for some $N \in \mathbb{N}$. But by the previous lemma this implies that $F_{q_n}|_{S^{n-1}\setminus A} = F_{q_0}|_{S^{n-1}\setminus A} = 0$ and we are done.

For the other part $q_n \to q_0 \in K \setminus p^+$ with $q_n \in J(p^-,p^+)^o$. We suppose by contradiction that there exists a $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists a $n \geq N$ and a $a \in S^{n-1} \setminus A$ such that $f_a(q_n) < -\varepsilon$. We can thus construct a sequence (a_k, q_k) such that $f_{a_k}(q_k) < -\varepsilon$ for all $k \in \mathbb{N}$. Because f is bounded and S^{n-1} compact there exists a convergent subsequence (a_j, q_j) such that $t_j := f_{a_j}(q_j) \to t' \leq -\varepsilon$, $a_j \to a' \in S^{n-1}\mathcal{A}$ and $q_j \to q_0$. Now we have $\mu_{a_j}(t_j) = \Theta(a_j, t_j) \to \Theta(a', t') = \mu_{a'}(t')$ and

$$0 = \lim_{j\to\infty} \tau(q_j, \mathcal{E}_{a_j}(q_j)) = \lim_{j\to\infty} \tau(q_j, \mu_{a_j}(t_j)) = \tau(q_0, \mu_{a'}(t')).$$

Furthermore because $\mu_{a_j}(t_j) = \mathcal{E}_{a_j}(q_j)$ we have $\mu_{a_j}(t_j) \in J^+(q_j)$. By ((REF)) this implies $\mu_{a'}(t') \in J^+(q_0)$. But this together with $\tau(q_0, \mu_{a'}(t'))$ implies that $\mu_{a'}(t') = \mathcal{E}_{a'}(q_0)$ and $f_{a'}(q_0) = t' < -\varepsilon$. Finally because $a_0 \in A$ and $a' \in S^{n-1} \setminus A$ we have $a' \neq a_0$ and $f_{a'}(q_0) < 0$, a contradiction to the previous lemma. $\square$

Lemma 4.2.7. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$. Then*

$$\liminf_{n\to\infty} \min_{a\in S^{n-1}} F_{q_n}(a) \geq t_0.$$

Proof. Suppose by contradiction that there exists a convergent subsequence q_k of q_n such that $\min_{a\in S^{n-1}} F_{q_k}(a) \to t' < t_0$. There thus exists a sequence of a_k such that $F_{q_k}(a_k) \to t' < t_0$. Taking subsequences again we get $a_j \to a'$ and $t_j := F_{q_j}(a_j) \to t' < t_0$. Then we have

$$J^+(q_j) \ni \mu_{a_j}(F_{q_j}(a_j)) \to \mu_{a'}(t') \in J^+(q_0)$$

by continuity of μ and lemma B.4.5. We also have

$$0 = \lim_{n\to\infty} \tau(q_j, \mu_{a_j}(F_{q_j}(a_j))) = \tau(q_0, \mu_{a'}(t'))$$

which implies $\mu_{a'}(t') = \mathcal{E}_{a'}(q_0)$ and $F_{q_0}(a') = t' < t_0$. A contradiction because $F_{q_0} \geq t_0$ by lemma 4.2.4. $\square$

This proposition is important because it shows that D is an open neighborhood of the boundary, which means a reconstruction on D is also a reconstruction of the boundary. The key idea here is that as $q_n \to q_0 = \mu_{a_0}(t_0) \in K \setminus \{p^+\}$ we can use lemma 4.2.6 to control the converge away from the minimum a_0 and to introduce local coordinates around $a_0 \in S^{n-1}$ to study the behavior close to a_0

Proposition 4.2.8. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$ then there exists a $\varepsilon > 0$ and a $N \in \mathbb{N}$ such that for all $n \geq N$, F_{q_n} has a unique minimum (a_n, t_n) and $(a_n, t_n) \to (a_0, t_0)$.*

Proof. As in a previous proof we can again separately prove the statement for the cases $q_n \in K$ for all $n \in \mathbb{N}$ and $q_n \notin K$ for all $n \in \mathbb{N}$. If $q_n \in K$ the statement follows immediately. We can thus from now on assume $q_n \notin K$.

First of all we let $O \subset M$ be a open convex neighborhood of q_0. Because $q_n \to q_0$ there exists a N_1 such that $n \geq N$ implies $q_n \in O$.

Recall that the Lorentzian splitting induced a Riemannian metric g^+ on M. For $a \in S^{n-1}, t \in [-T_a, 0]$ let $\nu_{a,t} \in CL^+_{\Theta(a,t)}M$ be the unique outward future pointing null vector orthogonal to T_t at a with $\|\nu_{a,t}\|_{g^+} = 1$, as in definition 4.2.2. We define the map

$$\begin{aligned} X : \mathbb{R}_+ \times \mathcal{S} &\to M \\ (c, a, t) &\mapsto \exp_{\Theta(a,t)}(-c\nu_{a,t}) \end{aligned}$$

which is smooth because $\nu_{a,t}$ varies smoothly in (a, t). We have $X(0, a_0, t_0) = q_0$ and X has invertible differential at $(0, a_0, t_0)$. Therefore there exists a $\varepsilon > 0$ such that $B_\varepsilon(a_0) \times B_\varepsilon(t_0) \subset \mathcal{S}$ and $X : B_\varepsilon(0) \cap \mathbb{R}_+ \times B_\varepsilon(a_0) \times B_\varepsilon(t_0) \to O_\varepsilon$ is a diffeomorphism. Because $-\nu_{a,t}$ is inward pointing we have $O_\varepsilon \subset J(p^-, p^+)$ for $\varepsilon > 0$ small enough. In this case, by the invariance of domain theorem, $O_\varepsilon \subset J(p^-, p^+)$ is a relatively open neighborhood of q_0. After further reducing ε, we can achieve that no two rays intersect in O_ε, i.e.

$$\gamma_{\nu_{a_1,t_1}} \cap \gamma_{\nu_{a_2,t_2}} \cap O_\varepsilon = \emptyset \quad \text{for all } a_1, a_2 \in B_\varepsilon(a_0), t_1, t_2 \in B_\varepsilon(t_0).$$

This possible because around $\Theta(a_0, t_0)$, K is a smooth submanifold. Finally we can reduce $\varepsilon > 0$ to get $O_\varepsilon \subset O$.

Because O_ε is open there exists a $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $q_n \in O_\varepsilon \subset O$. In this case we can write $q_n = X(c_n, a_n, t_n)$. We want to show that there exists a $N_3 \geq N_2$ such that for all $n \geq N_3$, F_{q_n} must have a global minimum in $B_\varepsilon(a_0)$. First of all because $q_n \in J(p^-, p^+)^{\circ}$, F_{q_n} is a continuous function an a compact set. There must thus exists at least one $a'_n \in S^{n-1}$ such that $t'_n := F_{q_n}(a'_n) \leq F_{q_n}(a)$ for all $a \in S^{n-1}$. Note that because t'_n is a minimum, the same argument as in lemma 4.2.3 yields that $\Theta(a'_n, t'_n) \in \mathcal{E}^{reg}_K(q_n)$ and $v(q_n, \Theta(a'_n, t'_n)) \in \mathbb{R}_+\nu_{a'_n,t'_n}$.

Next we want to show that if n is big enough, any such a'_n must lie in $B_\varepsilon(a_0)$. To that end we first note that $\Theta(a_n, t_n) \in \mathcal{E}^{reg}_K(q_n) \subset \mathcal{P}_K(q_n)$ because q_n and $\Theta(a_n, t_n)$ both lie in the convex neighborhood O. This implies $F_{q_n}(a_n) = t_n$. Because $t_n \in B_\varepsilon(t_0)$ we know that $\min_{a \in S^{n-1}} F_{q_n}(a) = t'_n \leq t_n < t_0 + \varepsilon < 0$. By lemma 4.2.6 we can then find a $N_3 \in \mathbb{N}$ such that $n \geq N_3$ implies $F_{q_n}(a) > t_0 + \varepsilon$ for all

$a \in S^{n-1} \setminus B_\varepsilon(a_0)$. But this means that F_{q_n} cannot have a minimum outside of $B_\varepsilon(a_0)$.

Next we want to show that $a_n' = a_n$ and $t_n' = t_n$ implying F_{q_n} has a unique minimum. We have $a_n' \in B_\varepsilon(a_0)$ for $n \geq N_3$. By the previous lemma there exists a N_4 such that $\min_{a \in S^{n-1}} F_{q_n}(a) = t_n' > t_0 - \varepsilon$ for all $n \geq N$. Combining this with $t_n' \leq t_n < t_0 + \varepsilon$ we have $t_n' \in B_\varepsilon(t_0)$. Now $\gamma_{\nu_{a_n,t_n}}$ and $\gamma_{\nu_{a_n',t_n'}}$ both contain $q_n \in O_\varepsilon$, and have $a_n, a_n' \in B_\varepsilon(a_0)$ and $t_n, t_n' \in B_\varepsilon(t_0)$ this is a contradiction if $a_n \neq a_n'$ or $t_n' \neq t_n$.

Finally $(a_n, t_n) \to (a_0, t_0)$ follows from the fact that X is a diffeomorphism and thus has a continuous inverse. $\square$

By lemma A.0.3 we immediately get:

Corollary 4.2.9. *There exists an open neighborhood $K \setminus p^+ \subset O \subset J(p^-, p^+)$ such that $O \subset D$, i.e. for every $q \in O$, F_q has a unique minimum.*

We can now show that D is an open subset of $J(p^-, p^+)$:

Proposition 4.2.10. *Let $q_0 \in D$ and $q_n \to q_0$ in V. Then there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies F_{q_n} has a unique minimum (a_n, t_n) and $(a_n, t_n) \to (a_0, t_0)$ where (a_0, t_0) is the unique minimum of F_{q_0}.*

Proof. We may assume $q_0, q_n \in J(p^-, p^+)^{\circ}$ because the case $q_0 \in K \setminus \{p^+\}$ is covered by the previous proposition and $q_0 \in J(p^-, p^+)^{\circ}$ implies $q_n \in J(p^-, p^+)^{\circ}$ eventually because the interior is open.

First we write $t_0 = F_{q_0}(a_0) < F_{q_0}(a'), a_0 \neq a' \in S^{n-1}$ for the unique minimum of F_{q_0}. Let $p_0 = \mathcal{E}_{a_0}(q_0)$, then by lemma 4.2.3, $p_0 \in \mathcal{E}_K^{reg}(q_0)$ and $a_0 \in \mathcal{A}^{reg}(q_0)$. By prop 2.3.6 there exists a $\varepsilon > 0$ such that $f : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$ is smooth. In particular $F_{q_0} = f(q_0, \cdot\,)$ is smooth on $\overline{B_\varepsilon(a_0)} \subset \mathcal{A}^{reg}(q_0)$. Furthermore because a_0 is a local minimum of the smooth F_{q_n} we must have $dF_{q_0}|_{a_0} = 0$ and its hessian $\mathcal{H}_{F_{q_0}}(a_0)$ must be positive definite. Because positive definiteness of the hessian is equivalent to it having only positive eigenvalues, there exists $\delta > 0$ such that every eigenvalue of $\mathcal{H}_{F_{q_0}}(a)$ is bigger than $c_0 > 0$ for all $a \in \overline{B_\delta(a_0)} \subset \overline{B_\varepsilon(a_0)}$. Hence $\mathcal{H}_{F_{q_0}}$ is positive definite and F_{q_0} is convex on all of $\overline{B_\delta(a_0)}$.

By an analogous argument to the one employed in the proof of 2.3.7 we can prove that $F_{q_n}|_{\overline{B_\delta(a_0)}}$ is smooth for n big enough and $\mathcal{H}_{F_{q_n}} \to \mathcal{H}_{F_{q_0}}$ uniformly on $\overline{B_\delta(a_0)}$. Because every eigenvalue of $\mathcal{H}_{F_{q_0}}(a)$ is bigger than $c_0 > 0$ for all $a \in \overline{B_\delta(a_0)}$, there must exist a $N_1 \geq \in \mathbb{N}$ such that $F_{q_n}|_{\overline{B_\delta(a_0)}}$ is smooth and $\mathcal{H}_{F_{q_n}}$ has only positive eigenvalues on $\overline{B_\delta(a_0)}$ for all $n \geq N_1$. Therefore F_{q_n} is convex on $\overline{B_\delta(a_0)}$ as well.

Next we prove that F_{q_n} must have all its minima in $B_{\frac{\delta}{2}}(a_0)$. We first note that because a_0 is the unique minimum of F_{q_0} we have $F_{q_0}(a) - F_{q_0}(a_0) > 0$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. Because F_{q_0} is continuous and $S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$ compact there exists a $c_1 > 0$ such that $F_{q_0}(a) > F_{q_0}(a_0) + c_1$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. By proposition 2.2.4, $F_{q_n} \to F_{q_0}$ uniformly. Hence there exists a $N_2 \geq N_1$ such that $F_{q_n}(a_0) \leq F_{q_0}(a_0) + \frac{c_1}{2}$. Thus we have $\min_{a\in S^{n-1}} F_{q_n}(a) \leq F_{q_n}(a_0) \leq F_{q_0}(a_0) + \frac{c_1}{2}$ for all $n \geq N_3$. But again by uniform convergence there exists a $N_3 \geq N_2$ such that $F_{q_n}(a) > F_{q_0}(a_0) + \frac{c_1}{2}$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. Hence F_{q_n} has no global minima in $S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$ for all $n \geq N_3$.

But because F_{q_n} is a continuous function on a compact space there must exist a minimum $a_n \in S^{n-1}$ such that $t_n := F_{q_n}(a_n) \leq F_{q_n}(a')$ for all $a' \in S^{n-1}$. As we just saw we must have $a_n \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. But we also proved that F_{q_n} is convex on $B_\delta(a_0)$ which means that a_0 must be the unique minimum of F_{q_n} on $B_\delta(a_0)$. Because F_{q_n} cannot have another minimum outside of $B_{\frac{\delta}{2}}(a_0)$, a_0 must be the unique minimum of F_{q_n} and thus $q_n \in D$.

Finally we prove that $a_n \to a_0$. We suppose by contradiction that a_n does not converge to a_0. Because S^{n-1} is compact there exists a convergent subsequence q_j such that $a_j \to a' \neq a_0$ and $q_j \in D$ for all $j \in \mathbb{N}$. Because $q_0, q_j \in J(p^-, p^+)^{\mathrm{o}}$, we have $F_{q_j}(a_j) \to F_{q_0}(a')$. Furthermore we have $F_{q_j}(a_j) = \min_{a\in S^{n-1}} F_{q_j} \to \min_{a\in S^{n-1}} F_{q_0}$ because $F_{q_j} \to F_{q_0}$ uniformly. But this implies $F_{q_0}(a') = \min_{a\in S^{n-1}} F_{q_0} = F_{q_0}(a_0)$, a contradiction because F_{q_0} was assumed to have a unique minimum. Note that because f is continuous on $J(p^-, p^+)^{\mathrm{o}} \times S^{n-1}$ this implies $t_n = f(q_n, a_n) \to t_0 = f(q_0, a_0)$ as well. $\square$

Note that by lemma A.0.3 this shows that D is open.

4.3 Smoothed Observation Time Functions

In this section we will define "smoothed" observation time functions which will be regular enough at the boundary K to carry out a similar reconstruction as in the previous chapter.

To that end we define

Definition 4.3.1 (Observation Bump Function)**.** For $a_0 \in S^{n-1}$ want to define the *observation bump function* $\chi_{a_0} : S^{n-1} \to [0, 1]$ to be a smooth function which varies smoothly in a_0, has $\chi_{a_0}(a') = 0$ if and only if $a' = a_0$, is symmetric around a_0 and there exist $\varepsilon_1 > 0$ such that $\chi_{a_0}(a') = 1$ for all $a' \in S^{n-1} \setminus B_{\varepsilon_1}(a_0)$ and a $\varepsilon_1 > \varepsilon_2 > 0$ such that $\max_{a'\in B_\varepsilon(a_0)} \chi_{a_0}(a') < \frac{\varepsilon}{T_{S^{n-1}}}$ for all $\varepsilon < \varepsilon_2$.

To that end, for a fixed $a_0 \in S^{n-1}$ we introduce local radial coordinates $a \in B_1(a_0) \mapsto (r(a), \omega(a)) \in [0, 1] \times S^{n-2}$ such that $r(a_0) = 0$. For $a \in S^{n-1}$ we then define

$$\chi_{a_0}(a) := \begin{cases} 1 - \exp(-\frac{r(a)^2}{1-r(a)^2}) & \text{if } a \in B_1(a_0) \\ 1 & \text{otherwise} \end{cases}$$

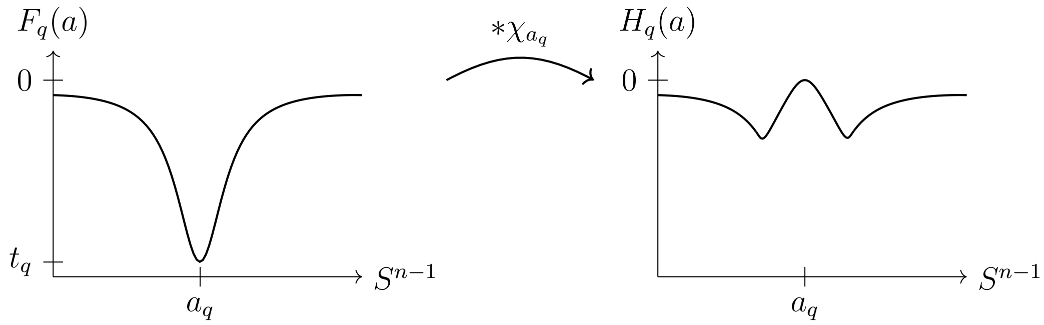


Figure 4.1: Plots of F_q and H_q in the Minkowski case; we can see that multiplying F_q by χ_{a_q} reduces its diminishes its spike at a_q.

Because $\chi_{a_0}(a)$ only depends on $r(a)$ it is symmetric, by construction is is smooth and also varies smoothly in a_0. We can see that $\varepsilon_1 = 1$ any because we have $\frac{\chi_{a_0}(r)}{r}|_{r=0} = 0$ there exists a suitable $\varepsilon_1 > \varepsilon_2 > 0$.

Equipped with these functions we can now define the

Definition 4.3.2 (Smoothed Observation Time Function)**.** We define the *smoothed observation time function* as

$$\begin{aligned} h : D \times S^{n-1} &\to [-T_{S^{n-1}}, 0] \\ (q, a) &\mapsto \chi_{a_q}(a) f(q, a) \end{aligned}$$

where a_q is the location of the unique minimum of F_q. Note that h is well defined because $\chi_{a_q}(a) \in [0, 1]$ for all $a \in S^{n-1}$. Analogous to the previous observation time functions we define $h_a(q) := h(q, a)$ and $H_q(a) := h(q, a)$.

Remark 4.3.3. Note that for $q \in K$ we have $H_q(a) = 0$ for all $a \in S^{n-1}$. Furthermore, because χ_{a_q} is smooth and by proposition 2.3.6 we get that for any $q \in D$, H_q is continuous on S^{n-1} and smooth on $\mathcal{A}^{reg}(q)$ (where for any $q \in K$ we define $\mathcal{A}^{reg}(q) = S^{n-1}$).

We will now show that H_q is regular even at the boundary: First of all it converges uniformly for any limit point $q_0 \in D$.

Proposition 4.3.4. *Let $q_n \in D \to q_0 \in D$. Then $H_{q_n} \to H_{q_0}$ uniformly.*

Proof. Let a_n resp. a_0 be the location of the minimum of F_{q_n} resp. F_{q_0}. We will again treat the cases $q_0 \in K$ and $q_0 \in J(p^-, p^+)^o$ seperately: If $q_0 \in J(p^-, p^+)^o$ there exists a $N_1 \in \mathbb{N}$ such that $q_n \in J(p^-, p^+)^o$ for all $n \geq N_1$. We claim that $h : D \cap J(p^-, p^+)^o \times S^{n-1}$ is a continuous function . This is because $f(q, a)$ is continuous and $q_n \to q_0$ implies $a_n \to a_0$ by the previous lemma, which implies

$\chi_{a_n} \to \chi_{a_0}$ because χ_a varies smoothly in $a \in S^{n-1}$. But now we can apply lemma A.0.2 to get $H_{q_n} \to H_{q_0}$ uniformly.

Now we treat the case $q_0 \in K$. We can again split up q_n into two subsequences $q_{i_n} \in J(p^-, p^+)^{\mathrm{o}}$ and $q_{j_n} \in K$. Since we have $H_{q_{j_n}}(a) = 0$ for all $a \in S^{n-1}$, $H_{q_{j_n}} \to H_{q_0}$ follows immediately since we have $H_{q_0}(a) = 0$ for all $a \in S^{n-1}$.

It remains to prove that $H_{q_{i_n}} \to H_{q_0}$ uniformly. To simplify notation we will denote $q_k := q_{i_n}$, and a_k for the location of the unique minimum of F_{q_k}. We want to show that for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $H_{a_k}(a) > -\varepsilon$ for all $a \in S^{n-1}$:

To that end let $\varepsilon > 0$. Because $q_k \to q_0$ implies $a_k \to a_0$ by proposition 4.2.8, there exists a $N_1 \in \mathbb{N}$ such that $a_k \in B_{\frac{\varepsilon}{2}}(a_0)$ for all $k \geq N_1$. Hence we have $B_{\frac{\varepsilon}{2}}(a_0) \subset B_\varepsilon(a_k)$ and we have $\chi_{a_k}(a) < \varepsilon$ for all $a \in B_{\frac{\varepsilon}{2}}(a_0) \subset B_\varepsilon(a_n)$. For any $a \in B_{\frac{\varepsilon}{2}}(a_0)$ we thus have $H_{q_k}(a) = \chi_{a_k}(a) f(q_k, a) > -\varepsilon$ because $f(q_k, a) \in [-T_{S^{n-1}}, 0]$.

It remains to show that there exists a $N_2 \in \mathbb{N}$ such that $H_{q_k}(a) > -\varepsilon$ for all $a \in S^{n-1} \setminus B_{\frac{\varepsilon}{2}}(a_0)$ and $n \geq N_2$. Because $B_{\frac{\varepsilon}{2}}(a_0)$ is an open neighborhood of we can apply 4.2.6 to find a $N_2 \in \mathbb{N}$ with $F_{q_k}(a) > -\varepsilon$ for all $q \in S^{n-1} \setminus B_{\frac{\varepsilon}{2}}(a_0)$ and $n \geq N_2$. Because $\chi_a < 1$ this implies $H_{q_k}(a) > -\varepsilon$ and we are done after setting $N := \max\{N_1, N_2\}$. □

Corollary 4.3.5. $h : D \times S^{n-1} \to [0,1]$ *is continuous.*

Proof. Let $(q_n, a_n) \to (q_0, a_0) \in D \times S^{n-1}$. The case where $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ was treated in the proof of the previous proposition. We can thus assume $q_0 \in K$. Furthermore we assume $q_n \in J(p^-, p^+)^{\mathrm{o}}$ because if q_n has a subsequence in K it is trivial to show that h converges on this subsequence. Because $h(q_0, a_0) = 1$ for any $a_0 \in S^{n-1}$ it remains to show that $h(q_n, a_n) = H_{q_n}(a_n) \to 1$, which follows immediately from the previous proposition. □

We now want to show that h is even smooth on a suitable subset of $D \times S^{n-1}$, i.e. the analogue of proposition 2.3.6. To that end we must first show that the unique minimum of F_q varies smoothly with q:

Lemma 4.3.6. *For every* $q \in D \cap J(p^-, p^+)^{\mathrm{o}}$ *there exists a* $\lambda > 0$ *such that the map*

$$\begin{aligned} a : B_\lambda(q_0) &\to S^{n-1} \\ q &\mapsto \operatorname*{arg\,min}_{a \in S^{n-1}} F_q \end{aligned}$$

is smooth.

Proof. Let $q_0 \in D \cap J(p^-, p^+)^{\mathrm{o}}$ with minimum at $a_0 \in S^{n-1}$. Recall that this implies $a_0 \in \mathcal{A}^{reg}(q_0)$. By proposition 2.3.6 there exists a $\varepsilon > 0$ such that f :

$\overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$ is smooth. Following an analogous argument to the one used in proposition 4.2.8 we can show that there exists a $\varepsilon > \delta > 0$ such that the map $f : \overline{B_\delta(q_0)} \times \overline{B_\delta(a_0)} \to [-T_{S^{n-1}}, 0]$ has positive definite hessian with respect to a and for every $q \in \overline{B_\delta(q_0)}$ we have $\arg\min_{a \in S^{n-1}} F_q(a) \in \overline{B_\delta(a_0)}$.

We then define the function

$$\begin{aligned} f' : \overline{B_\delta(q_0)} \times \overline{B_\delta(a_0)} &\to T^*S^{n-1} \\ (q, a) &\mapsto dF_q|_a \end{aligned}$$

which is smooth because f is smooth on its domain and has $f'(q_0, a_0) = 0$. Furthermore because f is has a positive definite hessian with respect to a, the non-degeneracy condition of the implicit function theorem is satisfied and can find a $\lambda > 0$ and a smooth map $q \in B_\lambda(q_0) \mapsto a(q) \in B_\delta(a_0)$ such that $f'(q, a(q)) = 0$. Because f is positive definite with respect to a on $B_\delta(a_0)$ and by choice of δ, F_q must have its minimum in $B_\delta(a_0)$ and $a(q)$ must be the location of this minimum as desired. □

Corollary 4.3.7. *Let* $\mathcal{C} := \{(q, a) \in V \times S^{n-1} \mid q \in D \cap J(p^-, p^+)^{\circ}, a \in \mathcal{A}^{reg}(q)\}$ *then* $h : \mathcal{C} \to [-T_{S^{n-1}}, 0]$ *is a smooth and* $dH_q|(a)$ *is bounded for all* $(q, a) \in \mathcal{C}$.

Proof. As shown in proposition 2.3.6, f is smooth on $\mathcal{C}$. By the previous lemma $q \in D \cap J(p^-, p^+)^{\circ} \mapsto a(q)$ is smooth as well. Hence the map $h(q, a) = \chi_{a(q)}(a) f(q, a)$ is the product of smooth functions making it smooth itself.

The boundedness of $dH_q|(a)$, follows because $dF_q|(a)$ is bounded by proposition 2.3.11 together with the fact that χ_a has bounded derivative because it is smooth on a compact set. □

Lemma 4.3.8. *We can choose* χ_a *such that for all* $q_n \to q_0 \in K$ *we have*

$$\max_{a \in \mathcal{A}^{reg}(q_n)} \|dH_{q_n}|_{a_n}\|_{g_{S^{n-1}}} \to 0.$$

Proof. ((Todo split up in close to q_0 and far away use that dF grows at most polynomially maybe also prove that $q \mapsto a(q)$ is smooth at boundary as well)) □

4.4 Reconstruction

We can now reconstruct the topological structure of V

Analogous to the reconstruction in the previous chapter we let $\mathcal{C}^\infty(S^{n-1})$ be the space of continuous functions $H : S^{n-1} \to [-T_{S^{n-1}}, 0]$ which are smooth on a dense open set in S^{n-1}. We again endow this space with the metric

$$d(H_1, H_2) := d_\infty(H_1, H_2) + \int_{S^{n-1}} \|dH_1|_a - dH_2|_a\|_{g_{S^{n-1}}} da,$$

where $d_\infty(H_1, H_2) := \max_{a \in S^{n-1}} |H_1(a) - H_2(a)|$. Note that by definition of $\mathcal{C}^\infty(S^{n-1})$ the subset of S^{n-1} where H_1 or H_2 are not smooth is a null set, making the integral well-defined.

For $q \in D$ with minimum $t_q \in [-T_{a_q}, 0]$ at $a_q \in S^{n-1}$ we define

$$\begin{aligned} \mathcal{H} : D &\to \mathcal{S} \times (\mathcal{C}^\infty, d) \\ q &\mapsto (a_q, t_q, H_q) \end{aligned}$$

where $H_q(a) = h(q, a)$ is the smoothed observation time function.

The follow lemma assures that no information is lost when passing from F_q to H_q:

Lemma 4.4.1. *For any $q \in D$ we can recover F_q given only $\mathcal{H}(q)$.*

Proof. First of all, given $\mathcal{H}(q) = (a_q, t_q, H_q)$ we can determine whether $q \in K$ or $q \in J(p^-, p^+)^{\mathrm{o}}$, because $q \in K$ if and only if $\min_{a \in S^{n-1}} H_q(a) = 0$. We can thus treat the cases seperately: If $q \in K$ we have $q = \Theta(a_q, t_q)$ and lemma 4.2.4 allows us to fully reconstruct F_q.

Now for the case where $q \in J(p^-, p^+)^{\mathrm{o}}$: We have $H_q(a) = \chi_{a_q}(a) F_q(a)$ and thus $F_q(a) = \frac{1}{\chi_{a_q}(a)} H_q(a)$. This allows us to reconstruct $F_q(a)$ for all $a \neq a_q$ because $\chi_{a_q}(a) \neq 0$ for all $a \neq a_q$. But by definition we have $F_q(a_q) = t_q$ and we have fully reconstructed F_q. $\square$

We denote $V_1 := V \cap D$

Lemma 4.4.2. $\mathcal{H} : V_1 \to \widehat{V_1} := \mathcal{H}(V_1) \subset \mathcal{S} \times (\mathcal{C}^\infty, d)$ *is well-defined, continuous and bijective.*

Proof. We begin by proving that $\mathcal{H}$ is well-defined. Because $\mathcal{H}$ is defined on D, any $q \in D$ must have a unique minimum making $q \mapsto (a_q, t_q, H_q)$ well-defined. Furthermore we have $H_q \in \mathcal{C}^\infty(S^{n-1})$ by corollary 4.3.7, together with the fact that $H_q = 1$ is also smooth for $q \in K$.

Now we want to prove that for any $q_n \to q_0 \in V_1$ with unique minima at (a_n, t_n) resp. (a_0, t_0) we have $(a_n, t_n) \to (a_0, t_0)$ and $d(q_n, q_0) \to 0$. By proposition 4.2.10 we have $(a_n, t_n) \to (a_0, t_0)$.

By proposition 4.3.4 we have $H_{q_n} \to H_{q_0}$ uniformly, which implies $d_\infty(H_{q_n}, H_{q_0}) \to 0$. It remains to show $\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da \to 0$. We again treat the cases $q_0 \in K$ and $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ seperately: If $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ we can assume without loss of generality that $q_n \in J(p^-, p^+)^{\mathrm{o}}$ as well. Then we can use corollary 4.3.7 and an analogous argument to the one used in lemma 3.1.1 to show that $\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da \to 0$.

It remains to show that

$$\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da = \int_{S^{n-1}} \|dH_{q_n}|_a - 0\|_{g_{S^{n-1}}} da \to 0$$

for $q_0 \in K$. But this follows immediately from lemma 4.3.8.

Finally we show that $\mathcal{H}$ is injective. Note that we proved in the previous lemma that $\mathcal{H}(q) = (a_q, t_q, H_q)$ allows us to determine whether $q \in K$ of $q \in J(p^-, p^+)^{\circ}$. If $q \in K$ we have $q = \Theta(a_q, t_q)$ making $\mathcal{H}$ injective on the boundary. If $q \in J(p^-, p^+)^{\circ}$, the previous lemma allows us to reconstruct F_q and thus $\mathcal{E}_K(q)$. Because $q \in V$ we can apply proposition 2.2.9 proving that $\mathcal{H}$ is injective. □

Lemma 4.4.3. *Let $q_n \in V_1$ such that $\mathcal{H}(q_n) \to \mathcal{H}(q_0)$ in $\widehat{V}_1$ for some $q_0 \in V_1$. Then also $q_n \to q_0$.*

Proof. By definition we have $\mathcal{H}(q_n) = (a_n, t_n, H_{q_n}) \to (a_0, t_0, H_{q_0}) = \mathcal{H}(q_0)$. Because we can determine from $\mathcal{H}(q_0)$ whether $q_0 \in K$ or $q_0 \in J(p^-, p^+)^{\circ}$ we can treat the two cases seperately. If $q_0 \in J(p^-, p^+)^{\circ}$ we have $\min_{a \in S^{n-1}} H_{q_0}(a) < 1$, then by uniform convergence there exists a $N_1 \in \mathbb{N}$ such that $\min_{a \in S^{n-1}} H_{q_n}(a) < 0$, implying $q_n \in J(p^-, p^+)^{\circ}$ for all $n \geq N_0$. We can then apply an analogous argument to the one used in lemma 3.1.2 to get $q_n \to q_0$.

For the case $q_0 \in K$ we can again split up $\mathcal{H}(q_n)$ into two subsequences, $\mathcal{H}(q_{i_n})$ where $q_{i_n} \in K$ and $\mathcal{H}(q_{j_n})$ where $q_{j_n} \in J(p^-, p^+)^{\circ}$ for all $n \in \mathbb{N}$. We thus have $q_{i_n} = \Theta(a_{i_n}, t_{i_n})$ which implies $q_{i_n} \to q_0$ because $(a_{i_n}, t_{i_n}) \to (a_0, t_0)$. For the other case ((explain more in-depth)) we denote $q_k := q_{j_n} \in J(p^-, p^+)^{\circ}$ and $(a_k, t_k) := (a_{j_n}, t_{j_n})$ to simplify notation. Because $H_{q_k} \to H_{q_0} = 0$ uniformly and there exists a $\varepsilon > 0$ such that $H_{q_k}|_{S^{n-1} \setminus B_\varepsilon(a_k)} = F_{q_k}|_{S^{n-1} \setminus B_\varepsilon(a_k)}$ we also have $\max_{a \in S^{n-1}} F_{q_k} \to 0$. This implies that $d(q_k, K) \to 0$. Now we suppose by contradiction that q_k does not converge to q_0. We thus have a convergent subsequence $q_j \to q' \neq q_0 \in K$ and $a_j \to a_0, t_j \to t_0$. We can the apply proposition 4.2.10 to q_j and q' to find that (a_0, t_0) is also the unique minimum of $F_{q'}$. But because $q' \in K$ we have $q' = \Theta(a_0, t_0) = q_0$, a contradiction. □

Corollary 4.4.4. *$\mathcal{H} : V_1 \to \widehat{V}_1$ is a homeomorphism.*

Remark 4.4.5. Note that we can use 3.1.3 on $V_2 := V \cap (J(p^-, p^+) \setminus K)$ to get a homeomorphism $\mathcal{F} : V_2 \to \widehat{V}_2 := \mathcal{F}(V_2)$. By proposition 4.2.10 we know that D is an open neighborhood of $K \setminus \{p^+\}$. Therefore $V_1 = D \cap V$ and $V_2 = (J(p^-, p^+) \setminus K) \cap V$ are both open subsets of $J(p^-, p^+)$ and we have $V_1 \cup V_2 = V$ because $V \subset J(p^-, p^+) \setminus \{p^+\}$.

((TODO: Explain more and include V_1 V_2 properties overview and that we given $\mathcal{H}(q_1)$ and $\mathcal{F}(q_2)$ we can determine if $q_1 = q_2$)) ((TODO: Include past boundary of $J(p^-, p^+)$ for now we assume $V \cap \partial J(p^-, p^+)^- = \emptyset$)) We can now reconstruct the topology on V:

Proposition 4.4.6. *A set $O \subset V$ is open if and only if $\mathcal{H}(O \cap V_1) \subset \widehat{V_1}$ and $\mathcal{F}(O \cap V_2) \subset \widehat{V_2}$ is open.*

Proof. For the first direction we suppose that $O \subset V$ is open. Because V_1 and V_2 are open so are $O \cap V_1$ and $O \cap V_2$. But because both $\mathcal{H}$ and $\mathcal{F}$ are homeomorphisms and thus open maps, $\mathcal{H}(O \cap V_1)$ and $\mathcal{F}(O \cap V_2)$ must be open as well.

For the other direction we assume that $\mathcal{H}(O \cap V_1)$ and $\mathcal{F}(O \cap V_2)$ are open. Because $\mathcal{H}$ and $\mathcal{F}$ are bijective and continuous, $O \cap V_1$ and $O \cap V_2$ must be open as well. Furthermore we have $V_1 \cup V_2 = V$ and thus

$$O = O \cap V = O \cap (V_1 \cup V_2) = (O \cap V_1) \cup (O \cap V_2)$$

must be open, as desired. □

Corollary 4.4.7. *Given data 1.2.1 we can determine if a set is open ((TODO))*

Proof. Suppose we are given the data from 1.2.1 and the light observation set $\mathcal{P}_K(\mathcal{O}) = \{\mathcal{P}_K(q) \mid q \in \mathcal{O}\}$ of a subset $\mathcal{O} \subset V$. For any $\mathcal{P}_K(q) \in \mathcal{P}_K(\mathcal{O})$ we can first evaluate if $q \in V \setminus K$ and $q \in D$ by checking if F_q is continuous in a and if it has a unique minimum. If $q \in D$ we can construct H_q; note that F_q and H_q are completely determined by $\mathcal{P}_K(q)$ and the data. This allows us to construct $\mathcal{H}(\mathcal{O} \cap V_1) = \{H_q \mid q \in V \cap D\}$ and $\mathcal{F}(\mathcal{O} \cap V_2) = \{F_q \mid q \in V \cap (J(p^-, p^+) \setminus K)\}$. Because we know the topology on $\widehat{V_1}$ and $\widehat{V_2}$ we can determine if $\mathcal{H}(\mathcal{O} \cap V_1) \subset \widehat{V_1}$ and $\mathcal{F}(\mathcal{O} \cap V_2) \subset \widehat{V_2}$ are open. By the previous proposition this allows us to determine whether $\mathcal{O}$ was open. Therefore we know all open subsets of V and with it the topology of V. □

4.5 Smooth and Conformal Reconstruction

As mentioned in the introduction the smooth and conformal reconstruction in the boundary case was beyond the scope of this paper. We do however conjecture that the reconstruction remains possible in this case as well, and want to sketch a possible proof:

Conjecture 4.5.1 (Full Boundary Reconstruction)**.** *Let $(M_j, g_j), p_j^\pm$ as in theorem 1.1.6. Let now $V_j \subset J(p_j^-, p_j^+) \setminus p_j^+$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.*

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

The main challenge here is the differential reconstruction: We can again proceed by looking at $V_1 = D \cap V$ and $V_2 = (J(p^-, p^+) \setminus K) \cap V$ seperately. For the second case concerning V_2 we should be able to again use the interior reconstruction to determine the differential structure around point away from K.

As in the topology reconstruction the recovery of the differential structure on D should be more challenging: We need to find locally smooth coordinates on D which extend smoothly to the boundary and can be determined by the data. But here we actually already laid some groundwork: In the proof of proposition 4.2.8 we defined a map

$$\begin{aligned} X : \mathbb{R}_+ \times \mathcal{S} &\to M \\ (c, a, t) &\mapsto \exp_{\Theta(a,t)}(-c\nu_{a,t}) \end{aligned}$$

which for any $q_0 \in K$, after suitably restricting the domain, was a diffeomorphism onto an open neighborhood $q_0 \in O \subset J(p^-, p^+)$. This implies that in a suitable neighborhood of the boundary K, the distance c to K as well as the location a and value t of the unique minimum define smooth coordinates. However, while for any $q \in V$ we can determine whether F_q has a unqiue minimum and if it does the location and value of it as well from the light observation sets, there is no a priori way do determine the *distance* of some $q \in V$ to K from the light observations.

To solve this issue we introduce a proxy for the distance which should work if $q \in V$ is close enough to the boundary: For $q \in D$ with unique minimum at $(a_q, t_q) \in \mathcal{S}$ we define the map

$$d(q) := \frac{1}{\det \mathcal{H}_{F_q}(a_q)} \text{ if } q \in J(p^-, p^+)^{\circ} \text{ and } d(q) := 0 \text{ if } q \in K$$

where $\mathcal{H}_{F_q}$ again is the hessian of F_q. This map is well-defined because a_q must be a regular point ensuring F_q is smooth at a_q and in the proof of proposition 4.2.10 we saw that $\mathcal{H}_{F_q}(a_q)$ must be positive definite making $\frac{1}{\det \mathcal{H}_{F_q}(a_q)}$ positive and finite. Furthermore we note that this definition is coordinate independent. Importantly, because we know F_q we can determine $d(q)$ from the light observation sets as desired. We now conjecture that

Conjecture 4.5.2. *For a suitable open neighborhood $\mathcal{O}$ of K, the map*

$$\begin{aligned} x : \mathcal{O} &\to \mathbb{R} \times S^{n-1} \times \mathbb{R} \\ q &\mapsto (d(q), a_q, t_q) \end{aligned}$$

is a diffeomorphism onto its image.

This is supported by lemma 4.3.6 which shows that the map $q \mapsto a_q$ is smooth on the interior, as required. Because the light observations allow us to determine x on all of $V \cap D$ we can presumably identify all points where x fails to be injective and so determine a suitable $\mathcal{O}$ from the data, allowing us to determine the differential structure of V in a neighborhood of K as desired.

Finally for the conformal reconstruction we note that for $q \in J(p^-, p^+) \setminus K$ we can determine the light cone just as in the interior reconstruction case. But if $q \in K$ we already know the light cone at q because we know the metric on all of K.

Chapter 5

Applications

In this chapter we will examine some useful applications of the results proven in the previous chapters. We first show that the suitability conditions required in theorems 1.1.5 and 1.1.6 are somewhat generic as they are preserved by small perturbations. Then we will use a conformal embedding of Minkowski space into the Einstein universe to show that theorem 1.1.5 also allows us to reconstruct the complete spacetime from observations at future null infinity.

5.1 Stability Results

The following lemma guarantees that theorem 1.1.5 still applies even if we deviate the metric slightly:

Lemma 5.1.1. *Let (M,g) be a globally hyperbolic manifold with $p^-, p^+ \in M$* suitable*. Furthermore let $V \in J(p^-,p^+)^{\mathrm{o}}$ such for any $q \in \overline{V}$, no null geodesic starting at q has a conjugate point in K.*

If we vary the metric g slightly to $\widetilde{g} := g + h$ such that

$$|h_{ij}|_q| < \varepsilon, \quad |h_{ij,\alpha}|_q| < \varepsilon, \quad |h_{ij,\alpha,\beta}|_q| < \varepsilon \quad \text{for } i,j,\alpha,\beta \in \{1,\ldots,1+n\}$$

and $\widetilde{g}$ is smooth and has $h_q = 0$ for all $q \in M \setminus V$. Then if $\varepsilon > 0$ is small enough, $(M,\widetilde{g})$ is globally hyperbolic and p^-, p^+, V are still suitable

Proof. To distinguish between objects defined in terms of g or $\widetilde{g}$ we will add a prescript of the respective metric, for example ${}_g\exp_q$ is the exponential map defined with respect to g and ${}_{\widetilde{g}}\exp_q$ with respect to $\widetilde{g}$. We begin by mentioning that for $\varepsilon > 0$, small enough, $(M,\widetilde{g})$ still globally hyperbolic. This follows from theorem 12 in [Ger70].

Next we want to show that we also have $p^- \,{}_{\widetilde{g}}\!\ll p^+$, i.e. there exists a timelike (wrt. $\widetilde{g}$) path from p^- to p^+. Because $p^- \,{}_g\!\ll p^+$, there exists a path σ with $\sigma(0) = p^-$

and $\sigma(1) = p^+$ which is timelike wrt. g. Because $[0,1]$ is compact we can find a $\varepsilon_2 > 0$ such that σ is still timelike wrt. $\widetilde{g}$, and thus $p^- \ll g^+$.

Because $p^- \ll p^+$ and $(M, \widetilde{g})$ globally hyperbolic, the map $a \in S^{n-1} \mapsto {}_{\widetilde{g}}T_a \in (0,\infty)$ as in lemma 2.1.5 is well-defined and continuous making ${}_{\widetilde{g}}\mathcal{S} := \{(a,t) \in S^{n-1} \times [0,\infty) \mid t \in [0, {}_{\widetilde{g}}T_a]\}$ a compact set.

Now because the geodesic equation on $(M, \widetilde{g})$ is a second order ODE with coefficients $\widetilde{g}_{ij}$ and $\widetilde{g}_{ij,\alpha}$ and $h = \widetilde{g} - g$ has compact support, ${}_{\widetilde{g}}\exp : TM \to M$ depends smoothly on $\widetilde{g}_{ij}$ and $\widetilde{g}_{ij,\alpha}$ while $d\,{}_{\widetilde{g}}\exp : T(TM) \to TM$ depends smoothly on $\widetilde{g}_{ij}$, $\widetilde{g}_{ij,\alpha}$ and $\widetilde{g}_{ij,\alpha,\beta}$ for some $\varepsilon_3 > 0$ small enough. This also implies that ${}_{\widetilde{g}}\rho(q,w)$ as well as ${}_{\widetilde{g}}T_a$ depend smoothly on $\widetilde{g}$ and its first and second derivatives. Because we have ${}_g\rho(p^+, a) > {}_gT_a$ for all $a \in S^{n-1}$ there exists a $\varepsilon_4 > 0$ such that ${}_{\widetilde{g}}\rho(p^+, a) > {}_{\widetilde{g}}T_a$ for all $a \in S^{n-1}$ and we have proved that p^-, p^+ are suitable.

Note that because $\widetilde{g} = g$ outside of V, we still have $V \subset {}_{\widetilde{g}}J(p^-, p^+)^{\circ}$. We can then see that p^-, p^+, V are still suitable with respect to $\widetilde{g}$ and some $\varepsilon_5 > 0$ after noting that ${}_{\widetilde{g}}L^K\overline{V}$ is compact and ${}_g\exp$ has no conjugate points in ${}_gL^K\overline{V}$ by assumption. □

We can extend the previous result slightly to show that if also the past boundary of $J(p^-, p^+)$ has no cut points, then $J(p^-, p^+)$ and K are conserved.

Corollary 5.1.2. *Let (M,g) be a globally hyperbolic manifold with $p^-, p^+ \in M$* suitable *and such that no geodesic starting at p^- has a cut point in $\mathcal{L}^+_{p^-} \cap J^-(p^+)$. Furthermore let $V \in J(p^-, p^+)^{\circ}$ such for any $q \in \overline{V}$, no null geodesic starting at q has a conjugate point in K.*

If we vary the metric g slightly to $\widetilde{g} := g + h$ such that

$$|h_{ij}|_q| < \varepsilon, \quad |h_{ij,\alpha}|_q| < \varepsilon, \quad |h_{ij,\alpha,\beta}|_q| < \varepsilon \quad \textit{for } i,j,\alpha,\beta \in \{1,\dots,1+n\}$$

$q \in J(p^-, p^+)^{\circ}$ and $\widetilde{g}$ is smooth and has $h = 0$ for all $q \in M \setminus J(p^-, p^+)^{\circ}$. Then if $\varepsilon > 0$ is small enough, $(M, \widetilde{g})$ is globally hyperbolic and p^-, p^+, V are still suitable*. Furthermore we have ${}_gJ(p^-, p^+) = {}_{\widetilde{g}}J(p^-, p^+)$ and ${}_gK = {}_{\widetilde{g}}K$.*

Proof. The proof follows from the observation that the fact that p^+ and p^- have no cut points in $\mathcal{L}^{\mp}_{p^\pm} \cap J^{\pm}(p^{\mp})$ implies ${}_g\exp_{p^\pm} = {}_{\widetilde{g}}\exp_{p^\pm}$, together with an analogous argument to the previous lemma. □

Example 5.1.3. Because Minkowski space $(\mathbb{R}^{1+n}, g_M := -dt^2 + \sum dx^2)$ has no cut points. We can pick any p^-, p^+, V *suitable* such that $V \in J(p^-, p^+)^{\circ}$ using the previous lemma we can see that for small deviatations from g_M with support on V, theorem 1.1.5 still applies and we can reconstruct V from the light cone observations on K.

However this example is somewhat limited in scope because such a deviation cannot be physical. To get a physical example we will use the reconstruction result on the Einstein universe:

5.2 Einstein Universe

Definition 5.2.1 (Einstein Universe)**.** Let $(\mathbb{R}, -dt^2)$ be the real line with negative definite metric $-dt^2$ and (S^n, h) the n-sphere with the canonical Riemannian metric. The $1+n$ dimensional *Einstein universe* is then defined as the product $(\mathbb{R} \times S^n, -dt^2 \oplus h)$. Note that by construction is a Lorentzian globally hyperbolic manifold.

To better describe the Einstein universe the following remark is very useful

Remark 5.2.2. We can parameterize S^n by an angle $\alpha \in [0, \pi]$ and a point $\omega \in S^{n-1}$ via the map

$$\begin{aligned} S : [0, \pi] \times S^{n-1} &\to S^n \\ (\alpha, \omega) &\mapsto (\cos\alpha, \sin\alpha\omega) \end{aligned}$$

If for a $X \in S^n$ we write $X = (X_0, \overrightarrow{X}), X_0 \in \mathbb{R}, \overrightarrow{X} \in \mathbb{R}^n$. We can invert S by

$$\alpha = \arccos X_0, \quad \omega = \frac{\overrightarrow{X}}{\|\overrightarrow{X}\|}.$$

S is surjective and smooth but we have

$$(1, 0, \ldots, 0) = S(0, \omega) \quad \text{and} \quad (-1, 0, \ldots, 0) = S(\pi, \omega) \quad \text{for all } \omega \in S^{n-1},$$

which means S fails to be injective if $\alpha = \{0, \pi\}$. Nontheless for every $X \in S^n$, $X \mapsto \alpha$ is well defined and smooth.

5.2.1 Conformal Embedding

We will now describe how Minkowsky space can be conformally embedded into the Einstein universe and how we can find null the corresponding future and past infinities.

Definition 5.2.3. We can first define M^M, the image of the conformal embedding and thus a conformal copy of Minkowski space within the Einstein universe:

$$M^M := \{(T, X) \in \mathbb{R} \times S^n \mid T \in (-\pi, \pi), \alpha < \pi - |T|\}$$

Next we can define the future and past null infinities of M^M to be (almost) its future and past boundaries:

$$\begin{aligned} \mathcal{J}^+ &:= \{(T, X) \in \mathbb{R} \times S^n \mid T \in (0, \pi), \alpha = \pi - T\} \\ \mathcal{J}^- &:= \{(T, X) \in \mathbb{R} \times S^n \mid T \in (\pi, 0), \alpha = \pi + T\} \end{aligned}$$

As mentioned $\mathcal{J}^{\pm}$ are only *almost* the full boundary of M^M; in fact $\{T = \pm\pi, \alpha = 0\}$ and $\{T = 0, \alpha = \pi\}$ are missing. But as remarked in 5.2.2, at these points the mapping $X \mapsto \alpha$ is degenerate and

$$\begin{aligned} i^{\pm} :=& \{T = \pm\pi, \alpha = 0\} \\ i^{S} :=& \{T = 0, \alpha = \pi\} \end{aligned}$$

are single points which correspond exactly to timelike future, timelike past and spacelike infinities.

If we set $p^{\pm} = i^{\pm}$ we can see that $\mathcal{J}^{+} = K \setminus (\{p^{+}\} \cup R)$, and that they are *almost* suitable. A slight issue is that past null geodesics starting at $i^{+} = (\pi, 0)$ have a conjugate point at $i^{s} = (0, \pi) \in R$. We even have $R = \{i^{S}\}$. Meaning the past light cone starting at i^{+} fully collapses into i^{s}. To fix this we can again argue that by lemma 2.1.1(3) we can disregard this boundary behaviour and because no geodesic starting at i^{+} has a conjugate point in $\mathcal{J}^{+}$ we can continue as if i^{-} and i^{+} were suitable.

We can now construct our conformal embedding:

Proposition 5.2.4. *Let $(M^E, g_E) = (\mathbb{R} \times S^n, -dt^2 \oplus h)$ be the $1+n$ dimensional Einstein universe and $(\mathbb{R}^{1+n}, h = dt^2 - dx_n dx^n)$ the $1+n$ dimensional Minkovski space. Then the map*

$$\Psi : M^M \to \mathbb{R}^{1+n} \tag{5.1}$$

$$(T, X) \mapsto \frac{1}{\cos T + X_0}(\sin T, \overrightarrow{X}) \tag{5.2}$$

is a conformal diffeomorphism from M^M to the whole Minkovski space $\mathbb{R}^{1+n}$ with conformal factor $\cos(T) + X_0$. Here X_0 denotes the first coordinate of X under the canonical embedding of S^n into $\mathbb{R}^{n+1}$.

Proof. This can be verified by calculation. An extensive treatment can be found in [Hör97](A.4). □

Note that this allows us to identify $J^{\pm}$ with the future resp. past null infinities of Minkowski space, whereas $i^{\pm}$ are the future resp. past timelike infinities. We can thus understand a reconstruction of a set $V \subset J(i^{-}, i^{+})^{\circ}$ from light observations on $\mathcal{J}^{+}$ as a reconstruction of a subset of Minkowsky space from observations at future null infinity. Notably this remains possible even if we vary g slightly:

Example 5.2.5. Let now $V := \{(T, X) \in M^M \mid T > 0\}$ and $p^{\pm} = i^{\pm}$ Then we we have $V \in J(p^{-}, p^{+})^{\circ}$ and no null geodesic starting in $\overline{V}$ has cut point on $K = \mathcal{L}^{-}_{p^{+}} \cap J^{+}(p^{-}) \approx \mathcal{J}^{+}$. This is because for any $\omega \in S^n$ and $\eta \in T_\omega S^n$,

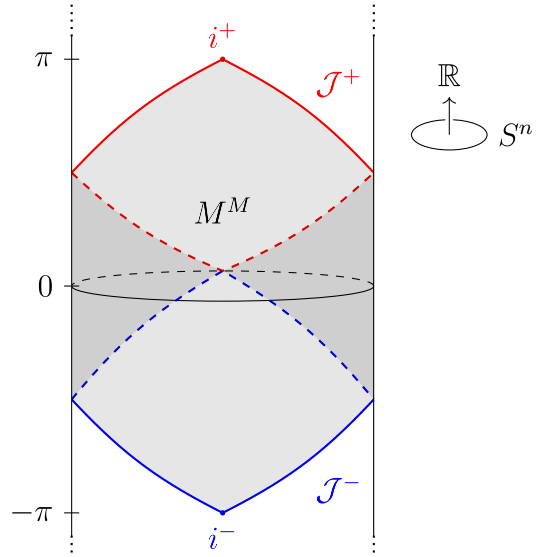


Figure 5.1: Illustration of the Einstein universe together with the conformal copy of Minkovski space M^M and the corresponding infinities $\mathcal{J}^{\pm}$ and $i^{\pm}$. We can see that $\mathcal{J}^{\pm}$ are codimension 1 null submanifolds of the Einstein universe and $i^{\pm}$ are single points.

$-\omega$ is a conjugate point of full dimension along $\gamma_{\omega,\eta}$ and in fact the first cut point. Furthermore by the same argument we can see that for all $v \in CL^-_{i^+}M$, $\gamma_{i^+,v}$ has a conjucate point of maximal dimension exactly where it intersects with $J^+(p^-) \setminus I^+(p^-)$, i.e. on the boundary R. Because of this R is now a single point and not a dimension $n-1$ submanifold anymore. By lemma 2.1.1(3) assures that this is no problem as all reconstruction only requires K to be regular on $K \cap I^+(p^-)$.

We can thus assume that p^-, p^+ and V are suitable and can apply corollary 5.1.2 to show that even if we vary g slightly to $\widetilde{g}$ on $J(p^-, p^+)^{\circ}$ we can still carry out the reconstruction.

If we then use Ψ to push the whole situation into $\mathbb{R}^{1+n}$ we can see that $\Psi_* g$ is a slight variation of the Minkowski metric which does not necessarily have compact support. If we view $^+$ as future null infinity we can see that we are able to reconstruct $\Psi(V) = \{(t, x) \in \mathbb{R}^{1+n} \mid t > 0\}$ from the light observations at null infinity.

Using theorem 3.5 in [Fri86] we can see that there even are families of physical spacetimes, i.e. satisfying the Einstein field equations, which can be reconstructed in this way.

Further directions

Finally outline some interesting further directions for study: First of all, analogously corollary 1.3 in [KLU17] it would be interesting to study whether in the boundary reconstruction case where V and K might overlap, it is possbible to even construct the metric g on V itself (and not only up to a conformal factor).

In line with the active reconstruction results by [WZ19] and [LUW18], it would be interesting to investigate whether such a result could also be obtained in our case, i.e. if for some nonlinear wave equations on (M, g) we could reconstruct $J(p^-, p^+)^{\circ}$ from knwoledge of the source-to-solution map, mapping sources on $K^- := (J^+(p^-) \setminus I^+(p^-)) \cap J^-(p^+)$ to observations on K.

Furthermore during our investiation of reconstruction on the Einstein universe the question arose whether a Lorentzian manifold $(\mathbb{R}^{1+n}, g)$ with no conjucate point must necessarily be flat. As shown by Guillarmou, Mazzucchelli, and Tzou [GMT19], this is the case for asymptotically flat Riemannian manifolds.

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Appendix A

Technical Lemmas

Lemma A.0.1 (Transverse Map)**.** *Let $f : M \to N$ be a smooth map transverse to the submanifold $L \subset N$ of codimension k and $f^{-1}(L)$ is nonempty. Then $f^{-1}(L)$ is a codimension k submanifold of M.*

Lemma A.0.2. *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces with X, Y compact. Let $f : X \times Y \to Z$ be a continuous functions and denote $f_x : Y \to Z; y \mapsto f_x(y) := f(x, y)$ for $x \in X$. Let $x_n \to x_0 \in X$ as $n \to \infty$ be a convergent sequence.*

Then $f_{x_n} \to f_{x_0}$ uniformly as $n \to \infty$.

Proof. Let $x_n \to x_0 \in X$ be a convergent sequence. We want to show that for any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon.$$

To that end let $\varepsilon > 0$. Then because X and Y are compact, we can use the Heine-Cantor theorem to get a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \wedge d_Y(y_1, y_2) < \delta \implies d_Z(f_{x_1}(y_1), f_{x_2}(x_2)) < \varepsilon.$$

Now if $N \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta \; \forall n \geq N$ and $y \in Y$ arbitrary we have $d_X(x_n, x_0) < \delta \wedge d_Y(y, y) < \delta$ which implies $d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$. Because $y \in Y$ was arbitrary we also get $max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$ and the proof is complete. □

Lemma A.0.3. *Let A be a first-countable topological space and $P : A \to \{false, true\}$ a property defined for all points $a \in A$. Suppose now that for any converging sequence $a_n \to a_0 \in A$ there exists a $N \in N$ such that $P(a_n)$ is true for all $n \geq N$.*

Then there exists an open neighborhood $O \in A$ of a_0 such that $P(a)$ is true for all $a \in O$.

Appendix B

Causality and Global Hyperbolicity

B.1 Causal Relations

In this first section we will establish which points in a Lorentzian manifold can be connected by timelike or lightlike paths under which circumstances.

Let (M, g) be a time-oriented Lorentzian manifold. We will first look at some properties of the causal and chronological future and past, as defined in 1.1.1.

Note that in the Minkowski case $\mathbb{R}^n_1$ the set $I^+(p)$ is open and $J^+(p) = \overline{I^+(p)}$ is closed. Furthermore $I^+(p)$ resp. $J^+(p)$ is the set of all $q \in R^n_1$ such that $\overrightarrow{pq}$ is timelike resp. causal. We will see that under sufficient conditions the first of the above facts also hold in the general case.

Corollary B.1.1. *If $x \ll y$ and $y \leq z$ or $x \leq y$ and $y \ll z$, then $x \ll z$.*

Proof. This follows immediately from proposition C.3.13 □

Let $\mathcal{U} \subset M$ be an open set. Then the *intrinsic* causality relations in $\mathcal{U}$ imply the ones in M. In particular, if we denote by $I^+(A, \mathcal{U})$ the chronological future in $\mathcal{U}$ of the set $A \subset \mathcal{U}$, we have that $I^+(A, \mathcal{U}) \subset I^+(A) \cap \mathcal{U}$.

With this in mind we will now consider the case of a convex set $\mathcal{C}$:

Lemma B.1.2. *Let $\mathcal{C}$ be a convex open set in M, then*

(1) *For $p \neq q$ in $\mathcal{C}$, $q \in J^+(p, \mathcal{C}) \iff \overrightarrow{pq}$ is future-pointing causal.*

(2) *$I^+(p, \mathcal{C})$ is open in $\mathcal{C}$ (hence also in M).*

(3) *$J^+(p, \mathcal{C})$ is the closure in $\mathcal{C}$ of $I^+(p, \mathcal{C})$.*

(4) *The relation $\leq$ is closed on $\mathcal{C}$, i.e. if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{C}$ then $q_n \in J^+(p_n, \mathcal{C})$ for all n implies $q \in J^+(p, \mathcal{C})$.*

(5) *A causal curve α contained in a compact $K \subset \mathcal{C}$ is continuously extendable.*

Proof. Properties (1-3) follow from the fact that the convex open set $\mathcal{C}$ is via the exponential map everywhere diffeomorphic to the tangent space $T_pM \simeq \mathbb{R}^n_1$ and thus the properties of the minkovski space also apply here.

To prove (4) we first note that by (1) we have that $q_n \in J^+(p_n, \mathcal{C})$ implies $\overrightarrow{p_nq_n}$ is future-pointing causal. Now by C.1.18 $(p_n, q_n) \mapsto \overrightarrow{p_nq_n}$ is continuous and thus $\overrightarrow{pq}$ is also future-pointing causal. Fact (4) then follows from again applying property (1).

To prove (5) we suppose that the domain of α is $[0, B)$ where $B < \infty$. As K is compact there exist a sequence $s_i \to B$ such that $\alpha(s_i)$ converges to a point $p \in K$. We must now prove that for any sequence $t_i \to B$ such that $\alpha(t_i) \to q$ we have $p = q$. Assume by contradiction that $p \neq q$. By possibly taking subsequences we can achieve that $s_i \leq t_i \leq s_{i+1}$. Then since α is causal we get $\alpha(s_i) \leq \alpha(t_i) \leq \alpha(s_{i+1})$ and thus $\alpha(t_i) \in J^+(\alpha(s_i), \mathcal{C})$ and $\alpha(s_{i+1}) \in J^+(\alpha(t_i), \mathcal{C})$. By (4) we now have $q \in J^+(p, \mathcal{C})$ and $p \in J^+(q, \mathcal{C})$ which by (1) implies that $\overrightarrow{pq}$ is at the same time, future and past pointing, a contradiction. $\square$

(2) can be generalized:

Lemma B.1.3. *The relation $\ll$ is open; that is if $p \ll q$ there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of p and q respectively such that for any $p' \in \mathcal{U}$ and $q' \in \mathcal{V}$ we still have $p \ll q$.*

Proof. Let σ be a timelike curve from p to q. Let $\mathcal{C}$ be a convex open neighborhood of q and q^- a point on σ which comes before q and still lies in $\mathcal{C}$. Then $I^+(q^-, \mathcal{C})$ is also an open neighborhood of q. If we proceed analogously for p with p^+ and $\mathcal{C}'$. Then we get that $I^-(p^+, \mathcal{C}')$ and $I^+(q^-, \mathcal{C})$ are the neighborhoods we were looking for. $\square$

Note that this lemma implies that $I^+(A)$ is open for any set A.

We can now further develop the topology of causality:

Lemma B.1.4. *For $A \subset M$ we have that:*

(1) $\operatorname{int} J^+(A) = I^+(A)$

(2) $J^+(A) \subset \overline{I^+(A)}$ *with equality iff* $J^+(A)$ *is closed.*

Proof. To prove (1) we first note that $I^+(A)$ is open as remarked above. Also $I^+(A) \subset J^+(A)$ by definition. Now if $q \in \operatorname{int} J^+(A)$, then for a convex neighborhood $\mathcal{C}$ of q, $I^-(q, \mathcal{C})$ contains a point of $J^+(A)$. Hence $q \in I^+J^+(A) = I^+(A)$.

Now to prove part (2): The equality assertion is clear, as $I^+(A) \subset J^+(A)$. Note that is suffices to consider only the case where $A = \{p\}$, since the general case then follows from

$$\bigcup_{p\in A} J^+(p) \subset \bigcup_{p\in A} \overline{I^+(p)} \subset \overline{\bigcup_{p\in A} I^+(p)}.$$

Let us thus consider the case of $\overline{I^+(p)}$. Clearly $p \in \overline{I^+(p)}$. Thus we only need to consider $p < q$. Let σ be a causal path from p to q. Let $\mathcal{C}$ be a convex neighborhood of q and q^- a point lying on γ in $\mathcal{C}$. Now by lemma B.1.2, $q^- \in J^+(p)$ and $I^+(J^+(p)) = I^+(p)$ we have

$$q \in J^+(q^-, \mathcal{C}) = \overline{I^+(q^-, \mathcal{C})} \subset \overline{I^+(J^+(p))} = \overline{I^+(p)}.$$

□

B.2 Causality Conditions

Definition B.2.1 (Strong Causality Condition)**.** We say that the *strong causality condition* holds at $p \in M$ if for any given neighborhood $\mathcal{U}$ of p there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that any causal curve with endpoints in $\mathcal{V}$ lies entirely within $\mathcal{U}$.

Intuitively this condition states that any causal curve which starts arbitrarily close to p and leaves some fixed neighborhood cannot return arbitrarily close to p. In particular this rules out closed causal loops.

The following lemma is in line with this intuition:

Lemma B.2.2. *Suppose the strong causality condition holds on a compact subset K of M. If α is a future-inextendable causal curve that starts in K, then α eventually permanently leaves K. That is, there exists a $s > 0$ such that $\alpha(t) \notin K$ for all $t \geq s$.*

Proof. Assume that the conclusion is false. Thus if the domain of α is $[0, B)$ for $B \leq \infty$, by the compactness of K, there exists a sequence $s_i \to B$ such that $\alpha(s_i) \to p \in K$. Since α has no future endpoint there must be some other sequence $t_j \to B$ such that $\alpha(t_j)$ does not converge to p. After taking further subsequences we can assume that some neighborhood $\mathcal{U}$ of p contains no $\alpha(t_j)$ and the sequences are alternating, i.e. $s_1 < t_1 < s_2 < t_2 < s_3 < \dots$. But now the curves $\alpha|_{[s_k, s_{k+1}}$ always leave the neighborhood $\mathcal{U}$ but return arbitrarily close and thus violated the strong causality condition. □

Under these conditions there exists a very useful lemma for constructing geodesics joining some $p < q$.

Lemma B.2.3. *Suppose the strong causality condition holds on a compact subset $K \subset M$. Let (α_n) be a sequence of future-pointing causal curve segments in K such that $\alpha_n(0) \to p$ and $\alpha_n(1) \to q \neq p$. Then there exists a future-pointing causal broken geodesic γ from p to q and a subsequence (α_m) of (α_n) such that $\lim_{m\to\infty} L(\alpha_m) \leq L(\gamma)$.*

This lemma is proven by leveraging the existence of quasi-limits together with the fact that given the strong causality condition, future inextendable curves must eventually leave a compact set K permanently. This proof can be found in detail in [One83, Lemma 14.14].

B.3 Time Separation Function

There is a natural way to generalize the notion of the separation of points $p \leq q$ in $\mathbb{R}^n_1$ to an arbitrary Lorentzian manifold M.

Definition B.3.1 (Time Separation)**.** Let $p, q \in M$, we define the *time separation* $\tau(p, q)$ from p to q as

$$\tau(p, q) \coloneqq \sup\{L(\alpha) \mid \alpha \text{ is a future-pointing causal curve segment from } p \text{ to } q\}.$$

We have $\tau(p, q) = \infty$ if the length is unbounded and $\tau(p, q) = 0$ if the separation is spacelike, i.e. $q \notin J^+(p)$. Note that for any causal path α the function $s \mapsto \tau(\alpha(0), \alpha(s))$ is monotonously increasing.

Lemma B.3.2. (1) *$\tau(p, q) > 0$ iff $p \ll q$.*

(2) *Reverse triangle inequality: If $p \leq q \leq r$, then $\tau(p, q) + \tau(q, r) \leq \tau(p, r)$.*

Proof. (1) If $\tau(p, q) > 0$ there exists a future-pointing causal curve α from p to q with $L(\alpha) > 0$. Thus α cannot be a null pregeodesic. By proposition C.3.13 there now exists a timelike curve from p to q. The converse follows immediately from the definition.

(2) If there are future-pointing causal curves from p to q and q to r we can pick causal curves α from p to q and β from q to r such that, for an arbitrarily small $\delta > 0$

$$L(\alpha) \geq \tau(p, q) - \delta/2, \quad L(\beta) \geq \tau(q, r) - \delta/2.$$

We then have

$$\tau(p, r) \geq L(\alpha + \beta) = L(\alpha) + L(\beta) \geq \tau(p, q) + \tau(q, r) - \delta$$

for any $\delta > 0$, as required. If there is no future-pointing causal path from WLOG p to q then $\tau(p, q) = 0$ and the result follows immediately. $\square$

Lemma B.3.3. *The time separation function $\tau : M \times M \to [0, \infty]$ is lower semicontinuous.*

Proof. If $\tau(p, q) = 0$ there is nothing to prove. Suppose $q \in I^+(p)$ and $0 < \tau(p, q) < \infty$.

Given $\delta > 0$ we must find neighborhoods $\mathcal{U}, \mathcal{V}$ such that for all $p' \in \mathcal{U}, q' \in \mathcal{V}$ we have $\tau(p', q') > \tau(p, q) - \delta$.

Let α be a timelike curve from p to q with $L(\alpha) > \tau(p, q) - \delta/3$. Let $\mathcal{C}$ be a convex neighborhood of q and q^- on α and in $\mathcal{C}$. Since in convex neighborhoods the map $q' \mapsto L(\sigma_{q^-q'})$, where $\sigma_{q^-q'}$ is the radial geodesic, is continuous there exists a neighborhood $\mathcal{V}$ of q such that for all $q' \in \mathcal{V}$ we have $L(\sigma_{q^-q'}) > L(\sigma_{q^-q}) - \delta/3$.

By analogous argument we get that there exists a p^+ and neighborhood $\mathcal{U}$ of p such that for all $p' \in \mathcal{U}$ we have $L(\sigma_{p'p^+}) > L(\sigma_{pp^+}) - \delta/3$.

Putting this together and using the fact that $L(\sigma_{q^-q}) \geq L(\alpha|_{[q^-,q]})$, resp $L(\sigma_{pp^+}) \geq L(\alpha|_{[p,p^+]})$ we have

$$\begin{aligned}
\tau(p', q') &\geq L(\sigma_{p'p^+}) + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q'}) \\
&> L(\sigma_{pp^+}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q}) - \delta/3 \\
&\geq L(\alpha|_{[p,p^+]}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\alpha|_{[q^-,q]}) - \delta/3 \\
&= L(\alpha) - 2\delta/3 > \tau(p, q) - \delta
\end{aligned}$$

as required. □

B.4 Globally Hyperbolic Manifolds

It is convenient to define

$$J(p, q) := J^+(p) \cap J^-(q)$$

Note that any future-pointing causal path from p to q must be contained in $J(p, q)$.

We can now give a powerful condition as to when the supremal path of $\tau(p, q)$ is actually achieved:

Proposition B.4.1. *For $p < q$, if the set $J(p, q)$ is compact and the strong causality condition holds on it, then there is a causal geodesic from p to q of length $\tau(p, q)$.*

Proof. Let (α_n) be a sequence of future-pointing curve segments from p to q whose lengths converge to $\tau(p, q)$ (the existence of such a sequence is guaranteed as $\tau(p, q)$ is the supremum of such curves). These curves are all in $J(p, q)$ which is compact. Hence, by lemma B.2.3, there exists a broken causal geodesic γ with

$$\tau(p, q) = \lim_{n\to\infty} L(\alpha_n) \leq L(\gamma) \leq \tau(p, q).$$

But now, if γ were to have any actual breaks, by corollary C.3.7 there would exist a longer curve, which is a contradiction. $\square$

Note that this implies in particular that $\tau(p,q)$ is always finite if $J(p,q)$ is compact.

This motivates the following definitions:

Definition B.4.2 (Globally Hyperbolic)**.** A subset $\mathcal{H} \subset M$ is called *globally hyperbolic* if (1) the strong causality conditions holds and (2) for all $p, q \in \mathcal{H}$ with $p < q$, $J(p,q)$ is compact.

Definition B.4.3. Let $\gamma : [0,T]$ be a causal geodesic from $p = \gamma(0)$ to $q = \gamma(T)$. We call γ *maximal* if we have $L(\gamma) = \tau(p,q)$ and hence $L(\gamma|_{[0,t]}) = \tau(p, \gamma(t))$ for all $0 \leq t \leq T$.

Lemma B.4.4. *If $\mathcal{U}$ is globally hyperbolic open set, then the time separation function $\tau : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ is continuous.*

Proof. We know from a previous lemma that τ is always lower semicontinuous. Suppose, for contradiction, that is is not upper semicontinuous at (p,q), i.e. there exists a number $\delta > 0$ and sequences $p_n \to p$ and $q_n \to q$ such that $\tau(p_n, q_n) \geq \tau(p,q) + \delta$ for all n.

Since $\tau(p_n, q_n) > 0$, there exists a causal curve α_n from p_n to q_n such that $L(\alpha_n) > \tau(p_n, q_n) - 1/n$. Because $\mathcal{U}$ is open it contains also the slightly earlier resp. later points $p^- \ll p$, $q^+ \gg q$. As $I^+(p^-)$ resp. $I^-(q^+)$ are open neighborhoods of p resp. q, p_n and q_n are eventually contained in them and we can WLOG assume that they always are. It follows that the curves α_n are all contained in the compact set $J(p^-, q^+)$. Now we can apply lemma B.2.3 to obtain a broken geodesic γ from $p = \lim p_n$ to $q = \lim q_n$ with

$$L(\gamma) \geq \lim_{n\to\infty} L(\alpha_n) \geq \lim_{n\to\infty} \tau(p_n, q_n) \geq \tau(p,q) + \delta.$$

But since δ itself is a curve from p to q this is a contradiction. $\square$

Lemma B.4.5. *If $\mathcal{U} \subset M$ is a globally hyperbolic open set, then the causality relation $\leq$ is closed on $\mathcal{U}$.*

Proof. We again have to show that if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{U}$ and $p_n \leq q_n$ for all n, then also $p \leq q$.

If $p = q$ the result follows immediately. We can thus assume $p \neq q$ and $p_n < q_q$ for all n. Let α_n then be a causal curve from p_n to q_n. As in the preceding proof, all α are in $J(p^-, q^+)$ and by lemma $B.2.3$, there exists a causal curve γ from p to q. This implies $p < q$. $\square$

Remark B.4.6. We can now summarize the results from this section for the case where (M, g) is a globally hyperbolic Lorentzian manifold:

For any $p \in M$, $I^{\pm}(p)$ is open and $J^{\pm}(p)$ is closed with $\operatorname{int} J^{\pm}(p) = I^{\pm}(p)$ and $\overline{I^{\pm}(p)} = J^{\pm}(p)$.

For the time separation function we can say the following:

(1) $\tau(p, q) > 0$ iff $p \ll q$.

(2) $\tau(x, y)$ satisfies the *reverse triangle inequality*:

$$\tau(x, y) + \tau(y, z) \leq \tau(x, z) \quad \text{for } x \leq y \leq z.$$

(3) $(x, y) \mapsto \tau(x, y)$ is continuous in $M \times M$.

(4) For $x < y$ there exists a causal geodesic γ from x to y such that $L(\gamma) = \tau(x, y)$.

B.5 Light Cones

In this section we will examine some relevant properties of the light cone as defined in 1.1.2

B.5.1 Null Cut Points

To better understand the behavior of null geodesics we will introduce so called *cut points* which intuitively are the points where a null geodesic stops being maximal. Such cut points are the product of curvature as in the minkovski case there are none.

For $(p, v) \in TM$ with $v \neq 0$ let $\mathcal{T}(x, v) \in (0, \infty]$ be the maximal value for which $\gamma_v : [0, \mathcal{T}(x, v))$ is defined.

Definition B.5.1 (Cut Locus Function and Cut Points)**.** For $(p, v) \in L^+M$ we define the *cut locus function*

$$\rho(p, v) \coloneqq \sup\{s \in [0, \mathcal{T}(p, v)) \mid \tau(x, \gamma_v(s)) = 0\}.$$

The points $x_1 = \gamma_v(t_1), x_2 = \gamma_v(t_2), t_1 < t_2 \in [0, t_0]$ are called *cut points* on $\gamma_v([0, t_0])$ if $t_2 - t_1 = \rho(x_1, v_1)$ for $v_1 = \gamma_v'(t_1)$. In particular, the point $p(x, v) = \gamma_v(s)|_{s=\rho(x,v)}$, if it exists, is called the *first cut point* on the geodesic γ_v.

Lemma B.5.2. *Let $p < q \in M$. Suppose there are two distinct future-pointed null geodesics $\alpha : [0, a) \to M, \beta : [0, b) \to M$ from $p = \alpha(0) = \beta(0)$ through $q = \alpha(1) = \beta(1)$. Then both geodesics have a cut point in* $[0, 1]$*, i.e. q comes on or after the first cut point.*

Proof. We will show that for any $s \in (1, a)$ we have $\tau(p, \alpha(s)) > 0$ since this implies that α must have a cut point at or before 1. Let $\gamma = \beta|_{[0,1]} + \alpha|_{(1,a)}$ be the broken null geodesic obtained by traveling from p to q on β and then continuing on α. Thus for any $s \in (1, a)$, $\gamma|_{[0,s]}$ is a broken null geodesic and by proposition C.3.13 there exists a timelike curve from p to $\gamma(s) = \alpha(s)$ which implies $\tau(p, \alpha(t)) > 0$ as required.

The proof for β follows analogously. $\square$

Lemma B.5.3. *Let now (M, g) be globally hyperbolic, and let $p < q \in M$ with $\tau(p, q) = 0$. Assume that $p_n \to p$ and $q_n \to q$ with $p_n \leq q_n$. Let γ_n be maximal geodesics joining p_n to q_n with initial direction v_n. Then the set (v_n) has a limit w and γ_w is a maximal null geodesic from p to q.*

Proof. As in the proof of lemma B.4.4 there exist $p^- \ll p$ $q^+ \gg q$ such that p_n, q_n, γ_n all lie in $J(p^-, q^+)$ which is compact. By lemma B.2.3 there exists a future-pointing broken geodesic λ which is the quasi-limit of γ_n (see [One83, Def. 14.7]). Thus there exists a convex neighborhood $\mathcal{C}$ of p and a sequence s_n such that $\lim_{n\to\infty} x_n := \gamma_n(s_n) \to x = \lambda(s) \in \mathcal{C}$ and $\gamma_n|_{[0,s_n]} \in \mathcal{C}$. Note that since γ_n is a maximal geodesic we have that $\gamma_n|_{[0,s_n]}$ is the unique radial geodesic from p_n to x_n and we have $v_n = \gamma_n'(0) = \overrightarrow{p_n x_n}$. Now by lemma B.1.2 $(p', q') \to \overrightarrow{p'q'}$ is continuous and we thus have that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \overrightarrow{p_n x_n} = \overrightarrow{px} =: w.$$

By construction, see [One83, Lemma 14.14], $\lambda|_{[0,s]}$ is the radial geodesic in $\mathcal{C}$ from p to x and thus also $\lambda'(0) = \overrightarrow{px} = w$.

It remains to show that λ is an actual unbroken geodesic. But since $L(\lambda) \leq \tau(p, q) = 0$ it follows from proposition C.3.13 that λ must be smooth null geodesic.

Thus also $\lambda = \gamma_w$ since λ is a geodesic with initial velocity w. $\square$

Theorem B.5.4 (Cut Point Characterization)**.** *Let (M, g) be globally hyperbolic. Then for $(x, p) \in L^+M$, $p(x, v)$ is either the first conjugate point on γ_v or the first point on γ_v where there exists another null geodesic γ_w from x to $p(x, v)$ where $v \neq cw$.*

Proof. Let $q = p(x, v) = \gamma_v(t)$ be the first cut point on the null geodesic γ_v. Let furthermore $t_n \to t$ be a monotonously decreasing sequence such that $\gamma_v(t_n)$ is well defined for all n. Now since M is globally hyperbolic there exist maximal geodesics γ_n from p to $q_n := \gamma_v(t_n)$. Note that since $q = \gamma_v(t)$ is the first cut point of γ_v we have $\tau(p, \gamma_v(t_n)) > 0$ for all n. But since γ_v is a null geodesic, it has zero length and cannot be maximal up until any of the t_n. Thus γ_n cannot equal γ_v and in particular $v_n := \gamma_n'(0) \neq v$ for all n. We can apply the previous lemma to obtain a geodesic γ_w and a null vector w such that $v_n \to w$ and γ_w is a maximal geodesic from p to q.

Now we can distinguish to cases: If $v \neq w$ there exist two distinct maximal geodesics, namely γ_v and γ_w joining p and q.

If however, $v = w$ we can view γ_n as a variation of γ_v through geodesics starting at p which additionally satisfy that the limiting variation at q is zero (since the q_n converge to q). q is thus a conjugate point of γ_v. $\square$

Proposition B.5.5. *For (M, g) globally hyperbolic, $\rho(p, v)$ is lower semicontinuous.*

Proof. It suffices to prove that if $(p_n, v_n) \to (p, v)$ in TM and $\rho(p_n, v_n) \to A$ in $\mathbb{R} \cup \{\infty\}$, then $\rho(p, v) \leq A$. If $A = \infty$ there is nothing to prove we will thus assume that $A < \infty$. We further assume $\rho(p, v) > A$ to derive a contradiction.

We can choose a $\delta > 0$ such that $A + \delta < \rho(p, v)$ and $q \coloneqq \gamma_v(A + \delta)$ exists. We define $b_n = \rho(p_n, v_n) + \delta$ and can force for n large enough $b_n < \rho(p, v)$ and $\gamma_n \coloneqq \gamma_{v_n}$ defined past b_n. We then denote $q_n = \gamma_n(b_n)$.

Since $b_n > \rho(p_n, v_n)$, γ_n cannot be maximal from p_n to q_n. Now, since M is globally hyperbolic, by B.4.1 we can find maximal null geodesics σ_n from p_n to q_n with initial velocity w_n. By B.5.3 $w_n \to w$ with γ_w a maximal null geodesic from p to q.

Since q cannot be conjugate point (because this would make it a cut point) we cannot have $w_n \to w = v$. Thus we must have $w \neq v$, but this implies that there are two distinct maximal geodesics from p to q, namely γ_v and γ_w, thus $q = \gamma : v(A+\delta)$ must be a cut point of γ_v. This implies that $\rho(p, v) \leq A + \delta$, which is a contradiction since we assumed $A + \delta < \rho(p, v)$. $\square$

B.6 Conformal Structure

Definition B.6.1 (Conformal Diffeomorphism)**.** A map $\Psi : (M_1, g_1) \to (M_2, g_2)$ is called a *conformal diffeomorphism* or *homothety* if $\Psi : M_1 \to M_2$ is a diffeomorphism and $\Psi^* g_2 = e^{2\Omega} g_1$ where $\Omega \in C(M_1)$ and nowhere zero.

We further say that $\Psi : V_1 \to V_2$ *preserves causality* if $x < y$ implies $\Psi(x) < \Psi(y)$.

It can be calculated that the connections D on M_1 and $\widetilde{D}$ on M_2 are related by the following equation:

$$\widetilde{D}_{\Psi_* X} \Psi_* Y = f_* D_X Y + X(\Omega) \Psi_* Y + Y(\Omega) \Psi \tag{B.1}$$

Proposition B.6.2. *$\gamma : I \to M_1$ is a null geodesic if, and only if $\sigma \coloneqq \Psi \circ \gamma$ is also a null geodesic.*

Proof. By the symmetry of the situation (i.e. Ψ^{-1} is also a conformal diffeomorphism) is suffices to show only one direction. Suppose now $\gamma : I \to M_1$ is a null geodesic on M_1 and $\sigma = \Psi \circ \gamma$. By the previous equation we have

$$\widetilde{D}_{\sigma'}\sigma'(t) = 2\gamma'(t)(\Omega)\sigma'(t).$$

We can now reparameterize σ such that $2\gamma'(t)(\Omega)$ is always zero and σ is a null geodesic as desired. $\square$

The following proposition asserts that the conformal data of a metric can be reconstructed from knowledge of the null cones:

Proposition B.6.3. *Let M be a smooth manifold of dimension $n \geq 3$ with Lorentzian metrics g and h. Suppose that for any $v \in TM$ we have $g(v, v) = 0$ iff $h(v, v) = 0$. Then there exists a smooth nowhere zero function $\Omega \in C(M)$ such that $g = e^{2\Omega}h$.*

Proof. The proof follows from the fact that the nullcones are given by systems of quadratic equations and some linear algebra. It can be found in more detailed form at [Bee81, Theorem 2.3] $\square$

We can see that even the cut locus is conserved under conformal transformation:

Proposition B.6.4. *Let $\gamma : [0, a) \to (M_1, g_1)$ be a null geodesic with first cut point $q = \gamma(t_0)$. Then $q' = \Psi(q)$ is the first null cut point of $p' = \Psi(p)$ along the null pregeodesic $\Psi \circ \gamma$.*

Proof. We can WLOG (since Ψ either causal or anti-causal and the proof of the anti-causal case is analogous) assume that Ψ is causal and γ is future-pointing. $\Psi \circ \gamma$ is thus also a future-pointed pre-geodesic which can be reparameterized to a null geodesic σ with $p' = \sigma(0)$ and $q' = \sigma(t_1)$. We will denote by τ_j the time separation function on M_j.

We first show that $\tau_2(p', \sigma(t)) = 0$ for $t \in [0, t_1]$, i.e. that q', if it is a cut point, is indeed the first cut point. To obtain a contradiction we assume that there exists a $t \in [0, t_1]$ with $\tau_2(p', \sigma(t)) > 0$. We my thus find a future-pointing causal curve β from p' to $\sigma(t)$ with $L_{g_2}(\beta) > 0$. Now $\Psi^{-1} \circ \beta$ is a future-directed causal curve in M_1 from p to $\Psi^{-1}(\sigma(t))$ with $L_{g_1}(\Psi^{-1} \circ \beta) > 0$. But since $t \leq t_1$ we have $\Psi^{-1}(\sigma(t) = \gamma(t_2)$ with $t_2 \in [0, t_0]$ and thus $\tau_1(p, \gamma(t_2)) > 0$. This would mean that γ has a cut point at t_2, before t_0 which is a contradiction.

We will now show that $\tau_2(q', \sigma(t)) > 0$ for any $t > t_1$, as this would make $q' = \sigma(t_1)$ a future cut point of p along σ as required. Let thus $t > t_1$. There exists a $t_2 > t_0$ such that $\Psi^{-1}(\sigma(t)) = \gamma(t_2)$. Now since $\gamma(t_2)$ lies past the first cut point of γ, we have $\tau_1(p, \gamma(t_2)) > 0$ and there exists a future-pointing causal curve α in M_1 with $L_{g_1}(\alpha) > 0$. Now $\Psi \circ \alpha$ is also a future-pointing causal curve from p' to $\sigma(t)$ with $L_{g_2}(\Psi \circ \alpha) > 0$ and thus $\tau_2(p', \sigma(t)) \geq L_{g_2}(\Psi \circ \alpha) > 0$ as required. $\square$

B.7 Short Cut Argument

Theorem B.7.1. *Let* (M, g) *be globally hyperbolic and* $p < q$ *in* M*, then there exists a future-pointed null geodesic* $\gamma : [0, a) \to M$ *from* $p = \gamma(0)$ *to* $q = \gamma(t_0)$ *and we have* $\tau(p, q) = 0$ *if and only if* γ *has no cut points in* $[0, t_0)$*.*

Proof. The existence of γ is assured by proposition B.4.1. Now suppose we have $\tau(p, q) > 0$ by the continuity of τ there must be a cut point $\gamma(t)$ before q, i.e. $t < t_0$. Suppose on the other hand that γ has a cut point $\gamma(t)$ with $t < t_0$. Then by the definition of cut points we must have $\tau(p, q) > 0$ as $t < t_0$. $\square$

We can apply this theorem to the case of a path from p to q which is the union of the future pointing light-like pregeodesics $\gamma_{p,v}([0, t_0])$ and $\gamma_{x_1,w}([0, t_1])$ where $x_1 = \gamma_{p,v}(t_0), q = \gamma_{x_1,w}(t_1)$. Let $\zeta = \gamma'_{p,v}(t_0)$. If there are no $c > 0$ such that $\zeta = cw$ or equivalently, the union of these two paths is not also a light-like pregeodesic, then we have $\tau(p, q) > 0$. By B.4.6, this implies that there exists a time-like geodesic from p to q and thus $\tau(p, q) > 0$. This is called a *short-cut argument.*

Appendix C

Lorentzian Geometry

((Keep this chapter?)) This chapter contains some basics and and relevant results from Lorentzian geometry, often presented without proof. It closely follows chapters 3,8 and 10 of [One83].

C.1 Lorentzian Manifolds

C.1.1 Covariant Derivative and Levi-Civita Connection

Definition C.1.1 (Connection)**.** A *connection* is a map $D : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ such that

(C1) $D_V W$ is $C(M)$-linear in V

(C2) $D_V W$ is $\mathbb{R}$-linear in W

(C3) $D_V(fW) = V(f)W + fD_V W$ for $f \in C(M)$

Theorem C.1.2 (Levi-Civita Connection)**.** *On a semi-Riemannian manifold* (M, g) *there exists a unique connection* D *such that*

(C4) $[V, W] = D_V W - D_W V$

(C5) $Xg(V, W) = g(D_X V, W) + g(V, D_X W)$

Proposition C.1.3 (Covariant Derivative in Coordinates)**.** *In coordinates the covariant derivative for a vector field* X *is given by*

$$(D_{\partial_i} X)^k = X^k_{,i} + \Gamma^k_{ij} X^j$$

Definition C.1.4 (Covariant Differential)**.** The *covariant differential* of a tensor field $A \in T^r_s(M)$ is the $(r, s+1)$-tensor field DA given by

$$(DA)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s, V) \coloneqq (D_V A)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s)$$

C.1.2 Parallel Transport and Geodesics

Definition C.1.5 (Vector Field on a Curve)**.** Let $\gamma : I \to M$ be a smooth curve. The space of *vector fields along* γ, denoted by $\Gamma(\gamma)$, corresponds to smooth maps $Z : I \to TM$ such that $Z(t) = (\gamma(t), v \in T_{\gamma(t)M})$, i.e. vector fields along γ parameterized by I.

Note that any vector field on M is also a vector field along γ.
There is a natural way to define the vector rate of change Z' for any $Z \in \Gamma(\gamma)$

Definition C.1.6 (Induced Covariant Derivative)**.** Suppose that γ is regular, i.e. $\gamma' \neq 0$ everywhere. Then the *induced covariant derivative* on $\Gamma(\gamma)$ can be defined as

$$Z' = DZ/dt \coloneqq D_{\gamma'}Z \qquad \text{where } Z \in \Gamma(\gamma)$$

we then have that

$$(Z')^k = \frac{Z^k}{dt} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}Z^j$$

If $Z' = 0$, it is said to be *parallel.*

Proposition C.1.7 (Parallel Translation)**.** *For a curve $\gamma : I \to M$, let $a \in I$ and $z \in T_{\gamma(a)}M$.*
Then there exist a unique parallel vector field Z on γ such that $Z(a) = z$

Note that this induces a function

$$P = P^b_a(\gamma) : T_{\gamma(a)}M \to T_{\gamma(b)}M \qquad \text{where } a, b \in I$$

called the *parallel translation along* γ.

Lemma C.1.8. *Parallel translation is a linear isometry.*

C.1.3 Geodesics

Definition C.1.9 (Geodesic)**.** A curve γ is called a *geodesic* if its acceleration $\gamma'' = D_{\gamma'}\gamma'$ is zero.

Note that for a geodesic we thus have

$$\frac{d^2(\gamma^i)}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0 \qquad \text{for all } k \in \{1, \ldots, n\} \tag{C.1}$$

These equations are known as the *geodesic equations*

Proposition C.1.10. *Given a tangent vector $v \in T_pM$ there is a unique maximal geodesic γ_v such that its initial velocity is v; that is $\gamma_v'(0) = v$.*

C.1.4 Exponential Map

For any point $p \in M$, the *exponential map* is (where it is defined) given by

$$\begin{aligned} \exp_p : T_pM &\to M \\ v &\to \gamma_v(1) \end{aligned}$$

Note that we have

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t) \qquad \text{for } v \in T_pM, t \in \mathbb{R}$$

.

After restriction it is a well defined diffeomorphism:

Proposition C.1.11. *For every $p \in M$ there exists a neighborhood of zero $\widetilde{U} \subset T_pM$ on which the the exponential map $\exp_p$ is a diffeomorphism onto a neighborhood $U \subset M$ of p.*

If additionally, $\widetilde{U}$ is starshaped we call U normal *and for every point $p' \in U$ there exists a unique geodesic $\sigma : [0,1] \to U$ from p to p' in U. Furthermore we have $\sigma'(0) = \exp^{-1}(p) \in \widetilde{U}$.*

Note that if U is normal, the inverse exponent map $\exp^{-1} : U \to \widetilde{U}$ induces a so called *normal chart* on U. If we then pick a orthonormal basis $e_1, \dots, e_n$ of T_pM we a normal coordinate system with $(x^1, \dots, x^n)$, which assigns to each point $p' \in U$ the coordinates of $\exp_p^{-1}(p') \in \widetilde{U}$ relative to the basis $(e_1, \dots, e_n)$. I.e.

$$\exp_p^{-1}(p') = \sum x^i(p)e_i$$

Proposition C.1.12 (Normal Coordinates)**.** *If $(x^1, \dots, x^n)$ are a normal coordinate system at $p \in M$ we have that*

$$g_{ij}(p) = \delta_{ij}\varepsilon_j \qquad \textit{and} \qquad \Gamma^k_{ij}(p) = 0 \qquad \forall i, j, k$$

C.1.5 Gauss Lemma and Convex Sets

The Gauss lemma asserts that the exponential map is a radial isometry

Lemma C.1.13 (Gauss Lemma)**.** *Let $p \in M$ and $0 \neq x \in T_pM$, if $v_x, w_x \in T_x(T_pM)$ with v_x radial, then*

$$\langle d\exp_p(v_x), d\exp_p(w_x)\rangle = \langle v_x, w_x\rangle$$

We denote by $D \subset TM$ the largest domain of exp, namely the set of all vectors in $v \in TM$ such that the geodesic γ_v is defined on $[0,1]$. It follows that $D_p = D \cap T_pM$ is the largest domain of $\exp_p$.

Corollary C.1.14. *The domain D of* $\exp$ *is open in TM. The domain D_p of* $\exp_p$ *is an open subset of T_pM and starshaped about 0.*

Definition C.1.15 (Convex Set)**.** An open set C is *convex* if it is a normal neighborhood of each of its points.

Proposition C.1.16. *Every $p \in M$ has a convex neighborhood.*

Lemma C.1.17. *A geodesic $\gamma : [0, b) \to M$ is geodesically extendable if and only if it is continuously extendable.*

For $\mathcal{C}$ a convex open set with $p, q \in \mathcal{C}$. Suppose that σ_{pq} is the radial geodesic from p to q. We then call $\sigma'_{pq}(0) \in T_pM$ the *displacement vector* from p to q and denote it by $\overrightarrow{pq}$.

Lemma C.1.18. *Let $\mathcal{C}$ be a convex open set. The map $\Delta : \mathcal{C} \times \mathcal{C} \to TM$ given by $(p, q) \mapsto \overrightarrow{pq}$ is smooth and open.*

Lemma C.1.19. *Given any open covering $\mathcal{O}$ of M there exists a convex covering $\mathcal{C}$ such that each element of $\mathcal{C}$ is contained within some element of $\mathcal{O}$.*

C.1.6 Arc Length

Definition C.1.20 (Arc Length)**.** Let $\gamma : [a, b] \to M$ be a piecewise smooth curve. The *arc length* of γ is

$$L(\gamma) := \int_a^b |\gamma'(s)| ds$$

Note that this length is invariant under monotone reparameterization and if $|\gamma'(s)| > 0$ everywhere we can achieve $|\gamma'(s)| = 1$ by reparameterization.

Lemma C.1.21. $L(\sigma_{pq}) = |\overrightarrow{pq}|$ *where $\overrightarrow{pq} \in T_pM$ is the vector such that* $\exp_p(\overrightarrow{pq}) = q$.

Proposition C.1.22. *Let U be a normal neighborhood of $p \in M$. If there exists a timelike curve in U from p to p' then the radial geodesic segment σ form p to p' is the unique longest timelike curve in U from p to p'.*

This works because if the timelike curve strays from σ it incurs a spacelike velocity component which only serves to reduce its length.

C.1.7 Lorentz Vector Space

Lorentz Vector Spaces are Scalar Product Spaces of index 1 and dimension ≥ 2. They are abstractions of the tangent spaces to a Lorentz manifold. Let W be a subspace of a Lorentz vector space V, there are *three* possibilities:

1. $g|_W$ is positive definite; W is a inner product space and is said to be *spacelike*.

2. $g|_W$ is nondegenerate of index 1; Then W is *timelike*,

3. $g|_W$ is degenerate; Then W is *lightlike*.

Proposition C.1.23 (Backwards inequalities for causal vectors)**.** *Let v and w be causal vectors. Then*

1. $|\langle v, w\rangle| \geq |v||w|$

2. $|v| + |w| \leq |v + w|$

To identify the *future timecone* at every point of a Lorentzian manifold we need a *time orientation* (a smooth assignment of a timecone to each point $p \in M$)

Lemma C.1.24. *A Lorentz manifold is time-orientable iff there exists a timelike vector field.*

C.1.8 Curvature

The Riemannian curvature tensor is the $(1, 3)$ tensor field defined by

$$\begin{aligned} R : \Gamma(M)^3 &\to \Gamma(M) \\ R_{XY}Z &= D_{[X,Y]} - [D_X, D_Y]Z \end{aligned}$$

which measures the degree to which the covariant derivative fails to be "lie-like". Note that R is a tensor as it is $C(M)$-linear in all arguments.

The curvature tensor is highly symmetric:

Proposition C.1.25. *Let $x, y, z, v, w \in T_pM$, then*

1. $R_{xy} = -R_{yx}$

2. $\langle R_{xy}v, w\rangle = -\langle R_{xy}w, v\rangle$

3. $R_{xy}z + R_{yz}x + R_{zx}y = 0$

4. $\langle R_{xy}v, w\rangle = \langle R_{vw}x, y\rangle$

C.2 Two-Parameter Maps

Due to their usefulness we will give a brief introduction to *two-parameter maps*

$$\mathbf{x} : \mathcal{D} \to M \qquad \text{where } \mathcal{D} \subset \mathbb{R}^2 \text{ and satisfies the interval condition.}$$

$\mathbf{x}$ defines families of u- and v-parameter curves $\alpha_{v_0} \coloneqq u \mapsto \mathbf{x}(u, v_0)$, resp. $\beta_{u_0} \coloneqq v \mapsto \mathbf{x}(u_0, v)$.

The *partial velocities* are defined as

$$\mathbf{x}_u \coloneqq d\mathbf{x}(\partial_u) = \frac{d}{du}\alpha_{v_0}, \quad \mathbf{x}_v \coloneqq d\mathbf{x}(\partial_v) = \frac{d}{dv}\beta_{u_0}.$$

They are vector fields.

Let now Z be a vector field on $\mathbf{x}$, we can then define the *partial covariant derivatives*

$$Z_u = DZ/\partial u \coloneqq D_{\alpha_{v_0}} Z, \quad Z_v = DZ/\partial v \coloneqq D_{\beta_{u_0}} Z.$$

In coordinates:

$$(Z_u)^k = \frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma^k_{ij} Z^i \frac{\partial \mathbf{x}^j}{\partial u}$$

A very useful proposition:

Proposition C.2.1. *For a two-parameter map $\mathbf{x}$ and a vector field Z, we have that*

1. $\mathbf{x}_{uv} = \mathbf{x}_{vu}$
2. $Z_{uv} - Z_{vu} = R_{x_u, x_v} Z$

C.3 Calculus of Variation

We want to study the change in arc length of a curve segment under small displacements. To that end we will introduce the *variation* of a curve.

Definition C.3.1. A *variation* of a curve segment $\gamma : [a, b] \to M$ is a mapping

$$\mathbf{x} : [a, b] \times (-\delta, \delta) \to M$$

such that $\mathbf{x}(u, 0) = \gamma(u)$ for all $u \in [a, b]$.

The u-parameter curves (i.e. $\mathbf{x}(\cdot, v)$ for fixed $v \in (-\delta, \delta)$) are called *longitudinal*, the v-parameter curves are called *transverse*

The vector field V on γ given by $V(u) = \frac{d}{dv}\mathbf{x}(u, 0)$ is called the *variation vector field*. It is the initial velocity of the transverse curve at that point.

If the longitudinal curves of $\mathbf{x}$ are geodesic, $\mathbf{x}$ is called a *geodesic variation*.

C.3.1 Jacobi Fields

Definition C.3.2 (Jacobi differential)**.** If γ is geodesic, a vector field Y on γ that satisfies the *Jacobi differential equation*

$$Y'' = R_{Y\gamma'}(\gamma')$$

is called a *Jacobi vector field*

Proposition C.3.3. *The variation vector field of a geodesic variation is a Jacobi field.*

Lemma C.3.4 (Unique Jacobi Field)**.** *Let γ be a geodesic with $\gamma(0) = p$ and $v, w \in T_pM$. Then there exist a unique Jacobi field Y on γ such that $Y(0) = v, Y'(0) = w$.*

Since the jacobi equation is linear, the space of jacobi fields is thus a $2n$-dimensional vector space.

Proposition C.3.5 (Jacobi Fields and Exponential Maps)**.** *Let $p \in M$ and $x \in T_pM$. For $v_x \in T_x(T_pM)$,*

$$d\exp_p(v_x) = V(1)$$

where V is the unique Jacobi field on the geodesic γ_x such that

$$V(0) = 0, \quad V'(0) = v \in T_pM$$

C.3.2 First and Second Deviation of Arc Length

We denote by $L_{\mathbf{x}}(v)$ the arc length of $\mathbf{x}(\cdot, v)$. Under mild conditions the function $L = L_{\mathbf{x}}$ is smooth and we are interested in finding formulas for the *first and second variation of arc length* (i.e. $L'(0)$ and $L''(0)$).

We consider piecewise smooth geodesics γ. To measure discontinuities at break points (u_i) we define

$$\Delta\gamma'(u_i) \coloneqq \gamma'(u_i^+) - \gamma'(u_i^-) \in T_{\gamma(u_i)}M$$

Furthermore, to treat spacelike and timelike curves in a uniform fashion we will define the *sign* ε of a curve as $\varepsilon = \operatorname{sgn}\langle\gamma', \gamma'\rangle$.

We can now state the first variation formula

Proposition C.3.6 (First Variation Formula)**.** *Let $\gamma : [a, b] \to M$ be a piecewise smooth curve with constant speed $c > 0$ and sign ε. If $\mathbf{x}$ is a variation of γ, then*

$$L'(0) = \frac{\varepsilon}{c}\left[-\int_a^b \langle\gamma'', V\rangle\, du - \sum_{i=1}^{k} \langle\Delta\gamma'(u_i), V(u_i)\rangle + \langle\gamma', V\rangle\big|_a^b\right]$$

Note that for a smooth and fixed endpoint variation we have that

$$L'(0) = -\frac{\varepsilon}{c} \int_a^b \langle \gamma'', V \rangle \, du$$

Corollary C.3.7. *Let γ be a constant speed curve. We have that γ is an unbroken geodesic if and only if the first variation is zero for every fixed endpoint variation* $\mathbf{x}$.

By this corollary it is sufficient to study geodesics in the treatment of the second variation.

We define the *transverse acceleration vector field* $A(u) := \mathbf{x}_{vv}(u, 0)$.

Proposition C.3.8 (Synge's Formula for Second Variation)**.** *Let $\gamma : [a, b] \to M$ be a geodesic with constant speed $c > 0$ and sign ε. If $\mathbf{x}$ is a variation of γ, then*

$$L''(0) = \frac{\varepsilon}{c} \left[\int_a^b \left\{ \left\langle (V')^\perp, (V')^\perp \right\rangle - \langle R_{V\gamma'} V, \gamma' \rangle \right\} du + \langle \gamma', A \rangle \big|_a^b \right]$$

C.3.3 Conjugate Point

Definition C.3.9 (Conjugate Point)**.** Points $\sigma(a)$ and $\sigma(b)$ on a geodesic σ are *conjugate along* σ if there is a nonzero Jacobi field J on σ such that $J(a) = J(b) = 0$.

We define $\mathcal{J}_{ab}$ to be the set of all Jacobi fields on σ that vanish at a and b. They are perpendicular to σ since they vanish twice.

Proposition C.3.10 (Conjugate Characterization)**.** *Let $\sigma : [0, b] \to M$ be a geodesic starting at p. The following are equivalent:*

1. *$\sigma(b)$ is a conjugate point of $p = \sigma(0)$ along σ,*
2. *There is a nontrivial variation $\mathbf{x}$ of σ through geodesics starting at p such that $\mathbf{x}_v(b, 0) = 0$,*
3. *$\exp_p$ is singular at $b\sigma'(0)$, i.e. there exist $0 \neq x \in T_{b\sigma'(0)}(T_p M)$ with $d\exp_p(x) = 0$.*

C.3.4 Energy Variation

For a curve segment $\alpha : [0, b] \to M$ we define the energy as

$$E(\alpha) = \frac{1}{2} \int_0^b \langle \alpha', \alpha' \rangle \, du$$

And for a piecewise smooth variation $\mathbf{x}$ of α we define

$$E_{\mathbf{x}}(v) = \frac{1}{2}\int_0^b \langle \mathbf{x}_u, \mathbf{x}_u \rangle \, du$$

where v is fixed.

Note that E is always smooth and can thus be used to study null geodesics.

Proposition C.3.11 (First Variation Formula)**.** *Let $\mathbf{x}$ be a variation of $\alpha : [0, b] \to M$ with V and A the variation and transverse acceleration vector fields. Then*

$$E'_{\mathbf{x}}(0) = \int_0^b g(V', \alpha')du = -\int_0^b g(V, \gamma'') - \sum_{i=1}^{k} g(V, \Delta\alpha')(u_i) + g(V, \alpha')|_0^b$$

where $u_1 < \cdots < u_k$ are the breaks of $\mathbf{x}$ and α.

C.3.5 Smoothing

We will now show how given a path which fails to be a null geodesic, we can find a small variation such that it becomes timelike. This will be essential for establishing causal structure of Lorentzian manifolds.

Lemma C.3.12. *Let $\alpha : [0, 1] \to M$ be a causal curve in a Lorentz manifold (M, g) and $\mathbf{x}(u, v)$ a variation of α with variation field V. If $g(V'(u), \alpha'(u)) < 0$ for all $u \in [0, 1]$ then for all sufficiently small $v > 0$ the longitudinal curve α_v of $\mathbf{x}$ is timelike.*

Proof. Because α is causal we have

$$g(\mathbf{x}_u, \mathbf{x}_u)(u, 0) = g(\alpha'(u), \alpha'(u)) \le 0 \quad \text{for all } u \in [0, 1].$$

Furthermore we have $V'(u) = V_u(u) = \mathbf{x}_{vu}(u, 0) = \mathbf{x}_{uv}(u, 0)$ and for all $u \in [0, 1]$:

$$\begin{aligned} &\frac{\partial}{\partial v} g(\mathbf{x}_u, \mathbf{x}_u)(u, 0) = 2g(D_{\frac{\partial}{\partial v}}\mathbf{x}_u(u, 0), \mathbf{x}_u(u, 0)) \\ &= 2g(\mathbf{x}_{uv}(u, 0), \alpha'(u)) = 2g(V'(u), \alpha'(u)) < 0. \end{aligned}$$

But now, since α is defined on the closed interval $[0, 1]$ for $v > 0$ sufficiently small we have $g(\mathbf{x}_u, \mathbf{x}_u)(u, v) < 0$ for all $u \in [0, 1]$. Hence α_v is timelike. □

Proposition C.3.13 (Smoothing of causal curves)**.** *Let M be a Lorentz manifold and α a causal curve from p that is not a null pregeodesic, then there exist a timelike curve from p to q arbitrarily close to α.*

Proof. We can WLOG assume that the domain of α is $[0, 1]$. We will first consider two special cases.

Case 1. $\alpha'(0)$ or $\alpha'(1)$ is timelike. Assuming the latter let W be obtained by parallel translation of $\alpha'(1)$ along α. Then, as parallel translation is an isometry, W and α' are always in the same causal cone and since W is timelike, $g(W, \alpha') < 0$. As α' is continuous there exists a $\delta > 0$ such that $g(\alpha', \alpha') < -\delta$ on $[1-\delta, 1]$. Let f be any smooth function on $[0, 1]$ vanishing at the endpoints with $f' > 0$ on $[0, 1-\delta]$. We then set $V = fW$ and get $g(V', \alpha') = g(f'W + fW', \alpha') = fg(W, \alpha) < 0$ on $[0, 1-\delta]$ as $W' = 0$, $f' > 0$ and $g(W, \alpha') < 0$ on $[0, 1-\delta]$. Let $\mathbf{x}$ be a fixed endpoint variation of α with variation field V. By the above lemma there exists a $v > 0$ sufficiently small such that the longitudinal curve α_v has become timelike on $[0, 1-\delta]$ and is still timelike on $[1-\delta, 1]$.

Case 2. α is a smooth null curve. Differentiation of $g(\alpha', \alpha') = 0$ shows that $\alpha'' \perp \alpha'$. Now α'' cannot always be parallel to α', or α could be reparameterized to a null geodesic. Thus the function $g(\alpha'', \alpha'') \geq 0$ is not equal to zero as $g(\alpha', \alpha') = g(\alpha'', \alpha'') = g(\alpha', \alpha'') = 0$ would imply $g(\alpha'' + \alpha', \alpha'' + \alpha') = 0$ which can only be true if $\alpha'' = c\alpha'$. $g(\alpha'', \alpha'') \geq 0$ follows from the fact that $g(\alpha'', \alpha'') < 0$ would imply that α'' is timelike and thus $g(\alpha', \alpha'') > 0$.

Let W be a parallel timelike vector field on α in the same causal cone as α' at each point so $g(W, \alpha') < 0$. Let $V = fW + \widetilde{f}\alpha''$ where f and $\widetilde{f}$ vanish at the endpoints and are to be determined such that $g(V', \alpha') < 0$.

Since $g(\alpha'', \alpha') = 0$ implies $g(\alpha''', \alpha') + g(\alpha'', \alpha'') = 0$ and $W' = 0$ we compute

$$g(V', \alpha') = g(f'W + fW' + \widetilde{f}'\alpha'' + g\alpha''', \alpha') = f'g(W, \alpha') - \widetilde{f}g(\alpha'', \alpha'').$$

Because $h = g(\alpha'', \alpha'')/g(W, \alpha)$ is not identical to zero there exists a smooth $\widetilde{f}$ vanishing at endpoints such that

$$\int_0^1 \widetilde{f}h du = -1$$

Let $f(u) = \int_0^u (\widetilde{f}h + 1)du$. Then f vanishes at endpoints and $f' = \widetilde{f}h + 1 > \widetilde{f}h = \widetilde{f}g(\alpha'', \alpha'')/g(W, \alpha)$. Consequently, as $g(W, \alpha') < 0$

$$\begin{aligned} g(V', \alpha') &= f'g(W, \alpha') - \widetilde{f}g(\alpha'', \alpha'') \\ &< \widetilde{f}g(\alpha'', \alpha'')/g(W, \alpha')g(W, \alpha') - \widetilde{f}g(\alpha'', \alpha'') = 0. \end{aligned}$$

And we can again apply the lemma above.

To complete the proof, note that if γ' is timelike at a non-endpoint s then Case 1 applies on $[0, s]$ and $[s, 1]$ to give the required result. Thus we are left with the case of a piecewise smooth null curve α. Unless every smooth segment of α can be reparameterized as a null geodesic, then by Case 2 some one can be varied to become timelike on that segment. Then we can apply Case 1 again to get a timelike curve.

Thus there only remains the case of a broken null geodesic α. It suffices to assume there is a single break $0 < s < 1$. Let W on α be obtained by parallel translation of $\Delta\alpha'(s) = \alpha'(s^+) - \alpha'(s^-)$. Recall that these two velocities are by definition in the same causal cone, so using the reverse cauchy-schwarz inequality we get $g(W, \alpha')$ is negative on $[0, s^-]$ and positive on $[s^+, 1]$. Now we choose a piecewise smooth function f on $[0, 1]$ that vanishes at the endpoints and positive derivative on $[0, s^-]$ and negative derivative on $[s^+, 1]$. Then for $V = fW$ we have $g(V', \alpha') < 0$ and the lemma applies. $\square$