Reconstruction of Lorentzian manifolds from null light observation sets

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Abstract

Let (M, g) be a globally hyperbolic Lorentzian manifold and $p^+ \gg p^-$ be points in M separated by a timelike curve. And let V be an open subset of $J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$. We show that the topological, differentiable and conformal structure of V can be uniquely reconstructed from the *light observation sets* on the future null boundary K of $J(p^-, p^+)$, i.e. the sets $\mathcal{P}_K(q) := \mathcal{L}_q^+ \cap K$ for $q \in V$. Furthermore we show that we can reconstruct the topological data of V even if it extends to include the boundary K, even though the light observation sets are degenerate in this case.

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Chapter 1

Introduction

1.1 Main Results

((Introduce Notation etc.))

Definition 1.1.1 (Suitable)**.** We call $p^- \ll p^+ \in M$ *suitable* if p^+ has no past cut points in $\mathcal{L}^-_{p^+} \cap J^+(p^-)$. Furthermore we call $p^- \ll p^+ \in M$ and $V \subset J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$ *suitable*, if p^- and p^+ are *suitable* and no null geodesic starting in V has a conjugate point in $\mathcal{L}^-_{p^+} \cap J^+(p^-)$.

Theorem 1.1.2 (Interior Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+, V_j$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

We assume that $V_j \subset J(p_j^-, p_j^+)^\circ$ are open sets. Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Theorem 1.1.3 (Boundary Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

Now let $V_j \subset J(p_j^-, p_j^+) \setminus p_j^+$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a ((homeomorphism)) conformal diffeomorphism $\Phi : V_1 \to V_2$ *that preserves causality.*

Chapter 2

Geometric Preliminaries

2.1 Null Conjugate Points

((TODO Connection to cut points)) ((Leave this here?))

Definition 2.1.1 (Null Conjugate Point)**.** Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. We then call $p = \gamma_{q,w}(b)$ a *null conjugate point* if there exists a nontrivial variation $\mathbf{x} : [0, b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma_{q,w}$ through null geodesics such that $\mathbf{x}_v(b, 0) = 0$.

We have the following useful characterization:

Proposition 2.1.2. *Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. Then $p = \gamma_{q,w}(b)$ is a null conjugate point if and only if $\exp_q : L_q M \to M$ is singular at bw, i.e. if there exists a nonzero $\xi \in T_{bw}(L_q M)$ such that $d\exp_q(\xi) = 0$.*

Proof. We begin by proving the backwards direction and to that end assume that there exist a nonzero $\xi \in T_{bw}(L_q M)$ such that $d\exp_q(\xi) = 0$. By the construction of the tangent space there thus exists a non-constant path $\xi : (-\varepsilon, \varepsilon) \to L_q M$ with $\xi(0) = bw$. This allows us to construct the variation $\mathbf{x}(u, v) = \exp_q(\frac{u}{b}\xi(v))$ which has $\mathbf{x}(t, 0) = \gamma_{q,w}(t)$ and is a variation through null geodesics. Finally we have $\mathbf{x}_v(b, 0) = d\exp_q(\xi) = 0$ by the chain rule.

For the other direction we first note that by definition $\mathbf{x}(u, v) = \exp_q(u\mathbf{x}_u(0, v))$ and $\mathbf{x}_u(0, v) \in L_q M$ as $\mathbf{x}$ is a variation through *null* geodesics. Now again by the chain rule we have $0 = \mathbf{x}_v(b, 0) = d\exp_q|_{bw} \circ \frac{d}{dv}(bx_u(0, v))|_{v=0}$. But since $\xi := \frac{d}{dv}(bx_u(0, v))|_{v=0} \in T_{bw}(L_q M)$ we are done. □

Null conjugate points are also conformal invariants:

Proposition 2.1.3. *Let $\Phi : (M, g) \to (N, h)$ be a conformal diffeomorphism and $\gamma : [0, b] \to M$ a null geodesic. Then $\gamma(b)$ is a null conjugate point of γ if and only if $\Psi(\gamma(b))$ is a null conjugate point of $\Psi \circ \gamma$.*

Proof. ((Cite relevant prop)) Because of the symmetry of the situation we only need to prove one direction and suppose that $\gamma(b)$ is a null conjugate point of γ. We thus have a variation $\mathbf{x}$ of γ through null geodesics. But since Φ maps null geodesics to null geodesics, $\Phi \circ \mathbf{x}$ is a variation of $\Phi \circ \gamma$ through null geodesics in N, which implies that $\Phi(\gamma(b))$ is a null conjugate point of $\Phi \circ \gamma$. $\square$

2.2 Geometry of the light cone observations

Remark 2.2.1 (Data). In the following we will use an equivalent formulation to Theorems 1.1.2 and 1.1.3: Namely we will show that if $(M, g), K, V, p^+, p^-$ are as in Theorem 1.1.2 resp. 1.1.3, then given the *data*

(1) The smooth manifold K,

(2) the conformal class of $g|_K$ and

(3) the set of light cone observations $\mathcal{P}_K(V)$

we can construct a space $\widehat{V}$ which is conformally equivalent to V. In Theorems 1.1.2 and 1.1.3, the assumptions assure that for both $(M_i, g_i), K_i, V_i, p_i^+, p_i^-$ we have the same data. Therefore the reconstruction will yield the same $\widehat{V}$ which will then be conformally equivalent to both V_1 and V_2. This in turn implies that V_1 and V_2 are conformally equivalent.

In light of this we will from here on restrict ourselves to only one globally hyperbolic Lorentzian manifold (M, g) with p^+, p^-, V *suitable* and show how given the *data* we can construct $\widehat{V}$.

2.2.1 Observer Set

We will first show some useful properties of the observer set K:

Lemma 2.2.2. *Let $p^-, p^+ \in M$ suitable and $R := K \setminus I^+(p^-)$ the past boundary of K then:*

(1) $K = J(p^-, p^+) \setminus I^-(p^+)$,

(2) $\mathcal{L}^-_{p^+} \cap J(p^-, p^+)^{\circ} = \emptyset$, *and*

(3) $\mathcal{L}^-_{p_0} \cap J(p^-, p^+)^{\circ} = J^-(p_0) \cap J(p^-, p^+)^{\circ} = \emptyset \quad \forall p_0 \in R$.

Proof. (1) We first rewrite $J(p^-, p^+) \setminus I^-(p^+) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ and immediately get $(J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-) \subset \mathcal{L}^-_{p^+} \cap J^+(p^-) = K$ as $J^-(p^+) \setminus I^-(p^+) \subset \mathcal{L}^-_{p^+}$. For the other inclusion we note that by assumption for $p \in K$ we have $\tau(p, p^+) = 0$ and $p \in \mathcal{L}^-_{p^-}$. This implies $p \in J^-(p^+) \setminus I^-(p^+)$. Furthermore $p \in K$ also implies $p \in J^+(p^-)$. Putting this together we get $p \in (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ proving the equality.

For part (2) we assume there exists a $p \in \mathcal{L}^-_{p^+} \cap J(p^-, p^+)^{\circ}$. Recall that $J(p^-, p^+)^{\circ} = I^+(p^-) \cap I^-(p^+)$. We thus have $p \in I^+(p^-) \subset J^+(p^-)$, which together with $p \in \mathcal{L}^-_{p^+}$ implies $p \in K$. But now we have $p \in I^-(p^+)$ and $p \in K$, a contradiction to (1).

Finally for part (3) we assume that there exists a $p_0 \in R$ and $p \in J^-(p_0) \cap J(p^-, p^+)^{\circ}$. Because $J(p^-, p^+)^{\circ} \subset I^+(p^-)$ there exists a timelike path from p^- to p. Because $p \in J^-(p_0)$ as well we can construct a timelike path ((REF)) from p^- to p_0 implying $p_0 \in I^+(p^-)$. But because $p \in R = K \setminus I^+(p^-)$ this is a contradiction. $\mathcal{L}^-_{p_0} \subset J^-(p_0)$ then yields the first equality. $\square$

Lemma 2.2.3. *For any $q \in J(p^-, p^+)^{\circ}$ the restriction of the exponential map to null vectors $\exp_q : L^+_q M \to M$ is* transverse *to K, i.e. for all $w \in L^+_q M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_p K$.*

Proof. In order to achieve a contradiction we assume that there exists a $q \in J(p^-, p^+)^{\circ}$ and a $w \in L^+_q M$ such that with $p = \gamma_{q,w}(1) \in K$ and $v := \gamma'_{q,w}(1) \in L_p K$. Since K is generated by backwards null geodesics originating at p^+ there exists a $u \in L^-_{p^+} M$ such that there exists a $t \in \mathbb{R}_+$ with $\gamma_{p^+,u}(t) = p, \gamma'_{p^+,u}(t) = -v$. We can thus obtain an unbroken past-pointing null geodesic from p^+ to q by connecting $\gamma_{p^+,u}$ and $\gamma_{p,-v}$. But this implies that $q \in \mathcal{L}^-_{p^+}$ which is a contradiction to 2.2.2(2).

We now prove that this implies that $\exp_q : L^+_q M \to M$ is transverse to K. We need to show that for every $w \in L^+_q M$ with $\exp_q(w) = p \in K$ we have

$$\operatorname{im}(d \exp_q|_w) \oplus T_p K = T_p M.$$

As $T_p K$ is a null hypersurface we only need to prove that $\operatorname{im}(d \exp_q|_w)$ contains a null vector which is not a multiple of the null vector $v \in T_p K$ generating $T_p K = v^{\perp}$. But by the properties of the exponential map, $\operatorname{im}(d \exp_q|_w)$ contains $v' = \gamma'_{q,w}(1) \in T_p M$. And since we just proved that $v' \notin T_p K$, $v + v'$ must be a timelike vector and $\operatorname{im}(d \exp_q|_w) \oplus T_p K = T_p M$, as desired. $\square$

Lemma 2.2.4. *For $q \in J(p^-, p^+)^{\circ}$ and $w \in L^+_q M$ there exists exactly one $t_w \in (0, \infty)$ such that $\gamma_{q,w}(t_w) \in K$.*

Proof. Let $q \in J(p^-, p^+)^{\circ}$ and $w \in L^+_q M$, by ((Leavescompact)) any geodesic starting in the compact set $J(p^-, p^+)$ must eventually leave it, intersecting the boundary. As K is the future boundary of $J(p^-, p^+)$ there exists at least one $t_w \in (0, \infty)$ with $p = \gamma_{q,w}(t_w) \in K$. We now show $\gamma_{q,w}(t') \notin K$ for any other $t' \neq t_w$.

First let us consider the case $t' < t_w$. We can then append $\gamma_{q,w}|_{[t',t_w]}$ to the path $\mu_a|_{[s,1]}$, whre $a \in S^{n-1}$, $s \in [0,1]$ such that $\mu_a(s) = p$, to get a broken lightlike path from $\gamma_{q,w}(t')$ to p^+. The fact that this path must be broken follows from the transversality proven in the previous lemma. But the existence of this broken path implies $\tau(\gamma_{q,w}(t'), p^+) > 0$ and thus $\gamma_{q,w}(t') \in I^-(p^+)$. But as $K = J(p^-, p^+) \backslash I^-(p^+)$ we have $\gamma_{q,w}(t') \notin K$

Conversely we now assume $t' > t_w$. Again by the transversality of $\gamma_{q,w}$ to K we get that for $t' - t_w > \varepsilon > 0$ small enough we have $\gamma_{q,w}(t_w + \varepsilon) \notin J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$ because K is the future boundary of $J(p^-, p^+)$. As any point on $\gamma_{q,w}$ is in $J^+(p^-)$ we must have have $\gamma_{q,w}(t_w + \varepsilon) \notin J^-(p^+)$, i.e. there exists no lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+. But if $\gamma_{q,w}(t') \in J^-(p^+)$ there exists a path σ from $\gamma_{q,w}(t')$ to p^+ and we could construct a lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+ by appending $\gamma_{q,w}|_{[t_w+\varepsilon,t']}$ to σ, a contradiction. We thus have $\gamma_{q,w}(t') \notin J^-(p^+) \supset J(p^-, p^+) \supset K$, completing the proof. $\square$

2.2.2 Parametrization of the observer set

We will now exploit the fact that no past null geodesic starting at p^+ has a cut point in K to construct a smooth parametrization of K, equivalent to a smoothly parameterized family of geodesics, generating K.

We will need to use the Lorentzian splitting theorem ((REF beem)):

Theorem 2.2.5 (Lorentzian Splitting Theorem)**.** *Let (M, g) be a globally hyperbolic Lorentzian manifold. Then (M, g) splits isometrically as a product $(\mathbb{R} \times H, -dt^2 \oplus h)$, where (V, h) is a complete Riemannian manifold.*

We will often denote the smooth projection from M to the timelike component as $\mathcal{T} : M \to \mathbb{R}$. Note also that such a splitting also induces a canonical Riemannian metric g^+ on M given by $g^+ := dt^2 \oplus h$. For such a metric we denote $CL_p^{\pm} M := \{v \in L_p^{\pm} M \mid \|v\|_{g^+} = \sqrt{2}\}$ for some $p \in M$, we use $\sqrt{2}$ for convenience as we will see later because the actual constant does not matter. We can then construct isometries such that

$$S^{n-1} \simeq CT_{\pi_H(p)} H := \{v_H \in T_{\pi_H(p)} \mid \|v_H\|_h = 1\} \simeq CL_p^{\pm} M. \tag{2.1}$$

$S^{n-1} \simeq CT_{\pi_H(p)} H$ follows by picking coordinates around $\pi_H(p)$. And the map $a_H \in CT_{\pi_H(p)} H \mapsto \partial_t + a_H \in CT_{\pi_H(p)} H$, is an isometry as desired. From here on we will thus often use S^{n-1} and $CL_p^{\pm} M$ interchangably.

Note that this construction implies

$$\frac{d\mathcal{T}(\gamma_{p^+,v}(t))}{dt} = -1 \quad \text{for all } t \in [0, \infty), v \in CL_{p^+}^{-} M. \tag{2.2}$$

The following lemma closely resembles lemma 2.2.4:

Lemma 2.2.6. *Let $v \in CL^-_{p^+}M$ then there exists a unique $T_v \in \mathbb{R}_+$ such that $\gamma_{p^+,v}([0,T_v]) \in K$ and $\gamma_{p^+,v}(t') \notin K$ for all $t' > T_v$.*

Proof. Importantly we have $p^- \notin \mathcal{L}^-_{p^+}$ because $\tau(p^-,p^+) > 0$ and p^+ does not have any past cut points in $K = \mathcal{L}^-_{p^+} \cap J^+(p^-)$. The rest of the proof is essentially analogous to the proof of lemma 2.2.4. $\square$

Lemma 2.2.7. *The map $v \in CL^-_{p^+}M \mapsto T_v \in \mathbb{R}_+$ is continuous.*

Proof. We will first show that it is bounded. Note that we have $\mathcal{T}(p) \geq \mathcal{T}(p^-)$ for all $p \in J^+(p^-)$ because $\mathcal{T}$ is strictly increasing along causal geodesics. Now equation 2.2 guarantees $\mathcal{T}(\gamma_{p^+,v}(2(t^+ - t^-))) = \mathcal{T}(p^+) - 2(t^+ - t^-) < t^-$, where $t^\pm := \mathcal{T}(p^\pm)$. We thus have $\gamma_{p^+,v}(2(t^+ - t^-)) \notin J^+(p^-)$ and $T_v < t^+ - t^-$ for all $v \in CL^-_{p^+}M$.

To show that the map is smooth we assume $v_n \to v_0 \in CL^-_{p^+}M$ but T_{v_n} does not converge to T_{v_0}. Because T_{v_n} is bounded there must exist a convergent subsequence $T_{v_j} \to T' \neq T_{v_0}$. We denote $p_j := \gamma_{p^+,v_j}(T_{v_j})$ and have $p_j \to p' := \gamma_{p^+,v_0}(T')$. Furthermore because $p_j \in J^+(p^-) \setminus I^+(p^-)$ closed we also have $p' \in J^+(p^-) \setminus I^+(p^-)$, but this is a contradiction to the previous lemma. $\square$

For some $a \in S^{n-1}$ often write $T_a := T_v$ where $v \in CL^-_{p^+}M$ is obtained via the equivalence from equation 2.1. Because $a \mapsto T_a$ is a continuous map on a compact set there exists a maximum $T_{S^{n-1}}$.

Proposition 2.2.8. *Let $\mathcal{S} := \{(a,t) \in S^{n-1} \times (-\infty, 0] \mid t \in [-T_a, 0]\}$. Then the map*

$$\begin{aligned} \Theta : \mathcal{S} &\to K \\ (a,t) &\mapsto \exp_{p^+}(-tv_a) \end{aligned}$$

where $v_a \in CL^-_{p^+}M$ is again given by equation 2.1, has the following properties:

(1) *$\Theta : \mathcal{S} \to K$ is a surjective smooth map such that the curves*

$$\mu_a := t \mapsto \Theta(a,t) \quad a \in S^{n-1}, t \in [-T_a, 0]$$

are null geodesics,

(2) *$\Theta(S^{n-1} \times \{0\}) = \{p^+\}$, $\quad \Theta(\{(a, T_a) \mid a \in S^{n-1}\}) = R$ and*

(3) *$\Theta : \mathcal{S}^\times := \{(a,t) \in S^{n-1} \times (-\infty, 0] \mid t \in (-T_a, 0)\} \to K \setminus (p^+ \cup R)$ is a diffeomorphism.*

Proof. To show (1) we first note that the fact that Θ is surjective follows from lemma 2.2.6, while smoothness and the geodesic property follows from the fact that $\exp_{p^+}$ is smooth and $v_a \in L^-_{p^+}M$ for $a \in S^{n-1}$. The two equalities in (2) follow from definition together with lemma 2.2.6. Finally for (3) we note that because $a \mapsto T_a$ is a continuous map, $\mathcal{S}^\times$ is a open submanifold of $S^{n-1} \times (-\infty, 0]$. Furthermore because by assumption p^- and p^+ are *suitable*, past null geodesics originating at p^+ have no cut points in K, i.e. $\rho(p^+, v_a) > T_a$ for all $a \in S^{n-1}$. But this means that Θ is a diffeomorphism on $\mathcal{S}^\times$ and by (2) we have $\Theta(\mathcal{S}^\times) = K \setminus (\{p^+\} \cup R)$, as desired. □

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will often treat K itself as a submanifold when it is clear that we are working away from the boundary. This is often the case as by 2.2.2(2-3) no null geodesic originating from the interior of $J(p^-, p^+)$ can reach p^+ or R, i.e. the boundary of K.

Furthermore by the properties of Θ we have

$$\mu_a([-T_a, 0]) \cap \mu_{a'}([-T_a, 0]) = \{p^+\} \text{ for } a \neq a' \in S^{n-1} \text{ and} \tag{2.3}$$

$$\bigcup_{a \in S^{n-1}} \mu_a([-T_a, 0]) = K \tag{2.4}$$

which implies that for every point $p \in K \setminus \{p^+\}$ there exist unique (a, t) such that $\mu_a(t) = p$. Owing to equation 2.2 also have

$$\frac{d\mathcal{T}(\mu_a(t))}{dt} = 1 \quad \text{for all } a \in S^{n-1}, t \in [-T_a, 0] \tag{2.5}$$

which implies

$$\mathcal{T}(\mu_a(t)) = \mathcal{T}(p^+) + t \quad \text{for all } a \in S^{n-1}, t \in [-T_a, 0] \tag{2.6}$$

making $mcT(\mu_a(t))$ independent of $a \in S^{n-1}$.

And finally we can see that we can construct the map Θ and thus the geodesics μ_a using only the data outlined in remark 2.2.1, because we know K and $g|_K$ determines all null geodesics on K.

2.2.3 Differential Constructions

This lemma closely resembles lemma 2.5 in [**hintzpaper**] with only minor adjustments to adapt it to our case. It is reproduced here for the sake of completeness. This lemma will allow us to reconstruct the direction of incoming light rays at point in $\mathcal{P}_K(q)$ which will locally correspond to the spacelike hypersurface.

Lemma 2.2.9 (Direction Reconstruction)**.** *Let $p \in K$ then there exists a bijection Φ between the space $\mathcal{S}$ of spacelike hyperplanes $S \subset T_pK$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+V \subset T_pM$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto T_pK \cap V^\perp \in \mathcal{S}$.*

Moreover there exists a bijection between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_pM$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \ni S \mapsto S \oplus \operatorname{span} \Phi(S) \in \mathcal{N}$.

Proof. Let $p \in K$, and $S \subset T_pK$ be a spacelike hyperplane. The orthogonal complement $S^\perp \subset T_pM$ then is a two-dimensional lorentzian subspace. Hence there exist four light rays which are multiples of the vectors $V, -V, W, -W$ in $S^\perp$, where we WLOG assume that V and W are future-pointing. Since $T_pK = v^\perp$ for some future-pointing null vector $v \in T_pK$, we have $v \in S^\perp$ and can WLOG assume $\mathbb{R}_+W = \mathbb{R}_+v$, i.e. $\mathbb{R}_+W$ is the ray pointing along the null hypersurface K. This leaves $\mathbb{R}_+V$ as the unique future-pointing outward null ray which is perpendicular to S, and we can thus set $\Phi(S) = \mathbb{R}_+V$.

For to prove Φ is a bijection, we let $0 \neq V \in T_pM$ be an outward future-pointing null vector. In particular this means that $V \notin T_pK$. Thus $S = V^\perp \cap T_pK$ is a spacelike hyperplane in T_pK which satisfies $S = \Phi^{-1}(V)$.

For the last claim we note that the map $\mathcal{N} \ni N \mapsto N^\perp \cap L_p^+M \in \mathcal{V}$ maps a null hypersurface N to the unique ray along a future-pointing outward null generator of N. The inverse of this map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto V^\perp \in \mathcal{N}$. Composition of these maps with Φ yields the desired bijection $\mathcal{N} \to \mathcal{S}$. $\square$

Definition 2.2.10 (Observation Preimage)**.** For any $q \in V$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* L_q^KM to be the preimage of K under the exponential map restricted to L_q^+M, i.e.

$$L_q^KM := (\exp_q|_{L_q^+M})^{-1}(K) \subset L_q^+M$$

Lemma 2.2.11. *For any $q \in V$, the observation preimage L_q^KM is a $n-1$-dimensional submanifold of L_q^+M.*

Furthermore, for any $w \in L_q^KM$ there exist a relatively open neighborhood $O_w \subset L_q^KM$ such that $\exp_q : O_w \to U_w := \exp_q(O_w) \subset \mathcal{P}_K(q)$ is a diffeomorphism.

Proof. By lemma 2.2.3, $\exp_q : L_q^+M \to M$ is transverse to K (here we treat L_q^+M and K as submanifolds, because by lemma 2.2.2(2-3) we can disregard the boundary points). Thus by the preimage lemma A.0.1 $L_q^KM := (\exp_q|_{L_q^+M})^{-1}(K)$ is a $n-1$-dimensional submanifold of L_q^+M.

For the second part let $w \in L_q^KM$, since $p := \exp_q(w) \in K$ and we assumed that such a p cannot be a null conjugate point, we know that $\exp_q : L_q^+M \to M$

has an invertible differential at w. Thus, by the implicit function theorem, there exists an open neighborhood $O'_w \subset L_q^+ M$ of w such that $\exp_q : O'_w \to \exp_q(O'_w)$ is a diffeomorphism. If we then restrict $\exp_q$ to $O_w := O'_w \cap L_q^K M$ the map is still a diffeomorphism as desired. $\square$

Note that by the invariance of domain theorem U_w is an open submanifold of $\mathcal{P}_K(q)$

Corollary 2.2.12. *The map*

$$\begin{aligned} S^{n-1} \simeq CL_q^+ M &\to L_q^K M \\ w &\mapsto t_w w \end{aligned}$$

where t_w is as in 2.2.4, is a diffeomorphism.

Proof. This result follows immediately from lemma 2.2.4 together with the fact that since K is (away from its boundary) a smooth submanifold, the map $w \mapsto t_w$ is smooth. $\square$

Lemma 2.2.13. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $w_1, \ldots, w_N \in L_q^K M$ such that $\exp_q(w_i) = p$. Furthermore for O_{w_i} as in the previous lemma such that $\exp_q : O_{w_i} \to U_{w_i}$ is a diffeomorphism, there exists an open neighborhood $U \subset \mathcal{P}_K(q)$ of p such that*

$$\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$$

Proof. Note that the previous corollary immediately yields that $L_q^K M$ is compact. Let $q \in V$, $p \in \mathcal{P}$. We first remark that, by the previous lemma, for any $w \in \exp_q^{-1}(p) \cap L_q^K M$ there exist open neighborhoods $w \in O_w \subset L_q^K M$ and $p \in U_w = \exp_q(O_w) \subset \mathcal{P}_K(q)$ making $\exp_q : O_w \to U_w$ a diffeomorphism.

To show that there can only be finitely many $w \in L_q^K M$ with $\exp_q(w) = p$ we let

$$C := \exp_q^{-1}(p) \cap L_q^K M.$$

As M is hausdorff, p is closed and because $\exp_q$ is continuous, so is C. Now $C \subset L_q^K M$ is a closed subset of a compact space, making C itself compact as well. Now the family $\{O_w \mid w \in \exp_q^{-1}(p) \cap L_q^K M\}$ is an open cover of C. But because C is compact there must exist a finite subcover such that

$$C \subset O := \bigcup_{i=1}^N O_{w_i}.$$

We can now make some observations: By definition, for any $w \in L_q^K M \setminus C$ we have $\exp_q(w) \neq p$. And as $\exp_q$ is a diffeomorphism on O_{w_i} for all $i = 1, \dots, N$, it must be injective and we get $\exp_q^{-1}(p) \cap O_{w_i} = \{w_i\}$. We thus have

$$\exp_q^{-1}(p) \cap O = \{w_1, \dots, w_N\}.$$

Furthermore, as $C \subset O$ for any $p \in L_q^K M \setminus O \subset L_q^K M \setminus C$ we still have $\exp_q(w) \neq p$. In other words:

$$\exp_q^{-1}(p) \cap L_q^K M \setminus O = \emptyset.$$

Putting these two observations together we get

$$\exp_q^{-1}(p) \cap L_q^K M = \{w_1, \dots, w_N\},$$

as desired.

To show the second part we denote

$$L^\times := L_q^K M \setminus O \quad \text{ and have } L^\times \cap \exp_q^{-1}(p) = \emptyset.$$

Note that $L^\times$ is a closed and thus compact subset of L_q^K. We then endow M with an arbitrary metric d compatible with its topology. This lets us define the continuous function

$$\begin{aligned} g : L^\times &\to \mathbb{R} \\ w &\mapsto d(\exp_q(w), p). \end{aligned}$$

Because $L^\times \cap \exp_q^{-1}(p) = \emptyset$ we have $g(w) > 0$ for all $w \in L^\times$. But now, as $L^\times$ is compact there exists a $\varepsilon > 0$ such that $g(w) = d(\exp_q(w), p) > \varepsilon$ for all $w \in L^\times$. We can now choose

$$U := B_\varepsilon(p) \cap \mathcal{P}_K(q)$$

and get an open neighborhood of p in $\mathcal{P}_K(q)$ with $\exp^{-1}(U) \cap L^\times = \emptyset$. But this means

$$\exp_q^{-1}(U) \cap L_q^K M = O = \bigcup_{i=1}^N O_{w_i}$$

completing the proof. $\square$

We can immediately put these lemmas to use and prove this proposition characterizing the light observation set.

Proposition 2.2.14. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$. There exists an open neighborhood $p \in U \subset \mathcal{P}_K(q)$, a positive integer N and N pairwise transversal, spacelike, codimension 1 submanifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ and $p \in \mathcal{U}_i$ for $i = 1, \dots, N$.*

Proof. Let $q \in V$ and $p \in \mathcal{P}_K(q)$. By the previous lemma we know that there can only be finitely many $w_1, \ldots, w_n \in L_q^K M$ with $\exp_q(w_i) = p$.

By lemma 2.2.11, for each w_i there exists a neighborhood $O_{w_i} \subset L_q^K M$ of w_i such that $\exp_q : O_{w_i} \to U_{w_i} := \exp_q(O_{w_i})$ is a diffeomorphism. Thus $U_{w_i} \subset \mathcal{P}_K(q)$ is a codimension 1 submanifold of K and we have $\bigcup_{i=1}^N U_{w_i} \subset \mathcal{P}_K(q)$.

Now we use the second part of the previous lemma to obtain an open neighborhood $U \subset \mathcal{P}_K(q)$ of p, such that $\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$. Thus any point $p \in \mathcal{P}_K(q) \cap U$ is contained in some $\mathcal{V}_i$ and we have $\bigcup_{i=1}^N U_{w_i} \supset \mathcal{P}_K(q) \cap U$. We then define

$$\mathcal{U}_i := U \cap U_{w_i}$$

and have

$$\bigcup_{i=1}^N \mathcal{U}_i = \mathcal{P}_K(q) \cap U$$

as desired. Furthermore, because U is an open neighborhood of p, $\mathcal{U}_i$ is still a codimension 1 submanifold of K and $p \in \mathcal{U}_i$.

We show that $\mathcal{U}_i$ is spacelike. To that end let $p \in \mathcal{U}_i$. Note that we have $\mathcal{U}_i \subset K$ and $\mathcal{U}_i \subset U'_{w_i} = \exp_q(O'_{w_i})$, where $w_i \in O'_{w_i} \subset L_q^+ M$ is an open neighborhood of w_i in $L_q^+ M$ such that on O'_{w_i}, exp_q is a diffeomorphism onto its image. Both K and U'_{w_i} are null hypersurfaces around p but by proposition **??** they are transversal and thus cannot be generated by the same null rays. Thus $T_p\mathcal{U}_i = T_pK \cap T_pU'_{w_i}$ can only contain spacelike vectors.

Finally to prove that they are transversal at p, we assume by contradiction that there exist $i \neq j$ such that $T_p\mathcal{U}_i = T_p\mathcal{U}_j$. But by lemma 2.2.9 this would imply that $v_i = c * v_j$ for a $c \in \mathbb{R}_+$, where $v_i = \gamma'(1)_{q,w_i}$. This would imply $w_i = w_j$, a contradiction. $\square$

Definition 2.2.15 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{U} \subset M$ of p such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a $n-1$ dimensional submanifold of M.

Note that $p \in \mathcal{P}_K(q)$ is regular if and only if $N = 1$ for p in the previous proposition.

Corollary 2.2.16. *The subset of regular points, $\mathcal{P}_K^{reg}(q) \subset \mathcal{P}_K(q)$ is open and dense in $\mathcal{P}_K(q)$.*

Proof. The fact that it is open follows immediately from the definition: Let $p \in \mathcal{P}_K(q)$ be regular. There thus exists an open neighborhood $p \in \mathcal{U} \subset M$ such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a submanifold. But now for every point $p' \in \mathcal{U} \cap \mathcal{P}_K(q)$, $\mathcal{U}$ also makes p' a regular point making $\mathcal{U} \cap \mathcal{P}_K(q)$ an open neighborhood of regular points of p. Thus every regular point has an open neighborhood of regular points making the set of regular points itself open.

To prove the set of regular points is dense in $\mathcal{P}_K(q)$ we to show that for every point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $U' \subset \mathcal{P}_K(q)$ contains a regular point. By the previous proposition, for U' small enough we have $\mathcal{P}_K(q) \cap U' = \bigcup_{i=1}^{N} \mathcal{U}_i$, where $\mathcal{U}_i$ are pairwise transversal. This means their intersection is of lower dimension and

$$\mathcal{U}_i \setminus \bigcup_{j \neq i} \mathcal{U}_j \quad \text{is open and nonempty for every } i = 1, \dots N.$$

((Give name and close to p)) and we can find a $p' \in \mathcal{V}_i$ for some $i \in 1, \dots, N$ such that $p' \notin \mathcal{V}_j$ for $j \neq i$. Thus we can find an open neighborhood $\mathcal{O}'$ around p' such that $\mathcal{O}' \cap \mathcal{P}_K(q) \subset \mathcal{V}_i$ which means p' is a regular point, as desired. □

2.3 Observation Time Functions

Definition 2.3.1 (Observation Time Function)**.** For $a \in S^{n-1}$ the *observation time function* is defined as

$$\begin{aligned} f_a : J(p^-, p^+) &\to [-T_a, 0] \\ q &\mapsto \inf(\{s \in [-T_a, 0] \mid \mu_a(s) \in J^+(q)\} \cup \{0\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) \coloneqq \mu_a(f_a(q)) \in M$ be the earliest point where μ_a sees light from q.

Lemma 2.3.2. *Let $a \in S^{n-1}$ and $q \in J(p^-, p^+)^{\circ}$. Then*

(1) *It holds that $f_a(q) \in (-T_a, 0)$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[-T_a, 0]$ and strictly increasing on $[f_a(q), 0]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p, q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L_q^+ M$ and $t \in [0, \rho(q, v)]$ such that $p = \gamma_{q,v}(t)$.*

Proof. Let $a \in \mathcal{A}$ and $q \in V$.

We begin by showing (1): Because $q \in J(p^-, p^+)^{\circ} = I^+(p^-) \cap I^-(p^+)$ we have $q \in I^-(p^+)$ and conversely $p^+ \in I^+(q)$. By ((REF)) we know that $I^+(q)$ is open and thus it forms an open neighborhood of p^+. But as μ_a is a continuous path with $\mu_a(1) = p^+$ there must exist a $t < 0$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. Hence we have $f_a(q) < 0$.

To show $f_a(q) > -T_a$ we assume $f_a(q) = -T_a$ to achieve a contradiction. We thus have $-T_a = \inf\{s \in [-T_a, 0] \mid \mu_a(s) \in J^+(q)\}$. This means that there exists a convergent sequence $t_n \searrow 0$ as $n \to \infty$ such that $\mu_a(t_n) \in J^+(q)$ for all n. Because μ_a is continuous and $J^+(q)$ closed we have $p_0 := \mu_a(-T_a) \in J^+(q)$. But $p_0 = \mu_a(-T_a) \in R$ by prop 2.2.8(2). Hence we get $p_0 \in J^+(q) \cap R$ for $q \in V$, which is a contradiction to 2.2.2(3).

To show (2) we proceed as follows: By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all t_n we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from q to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such t.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[-T_a, 0]$ we first note that it is the composition of two continuous functions. Monotony then follows from the reverse triangle inequality together with the fact that μ_a is a future pointing null geodesic.

To show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 0]$ we let $f_a \leq t_1 < t_2 \leq 0$. Now by ((REF)) there exists a causal geodesic $\gamma_1 : [0,1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect γ_1 to $\mu_a|_{[t_1,t_2]}$ we get a path γ_2 connecting q to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as μ_a is a null geodesic. Next we argue that γ_2 must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If γ_1 is timelike this observation is trivial as μ_a is lightlike. If however, γ_1 is lightlike (which is only the case if $t_1 = f_a(q)$), this fact follows from the transversality of light cone observations as noted in proposition 2.2.3. This means that γ_2 is a broken causal geodesic, which by ((REF)) implies that there exists a strictly longer timelike path γ_3 connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Finally we can take on (3) To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q,p) = 0$. But now, by ((REF)) there exists a null geodesic from q to p which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q,p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0,1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q,p) = 0$.

The other equivalence follows from the definition of $\mathcal{P}_K(q)$ together with the definition of cut points. $\square$

By (3) of the above lemma, for any $q \in V$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.3.1 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-T_a, 0] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2.7}$$

Proposition 2.3.3. *The function $f : J(p^-, p^+)^{\mathrm{o}} \times S^{n-1} \to [-T_{S^{n-1}}, 0]$; $(q, a) \mapsto f_a(q)$ is continuous.*

Proof. We want to show that for every convergent sequence $(q_n, a_n) \to (q_0, a_0) \in J(p^-, p^+)^{\mathrm{o}} \times S^{n-1}$ we have $t_n := f_{a_n}(q_n) \to f_{a_0}(q_0) =: t_0$ as $n \to \infty$. Because the sequence t_n lives in $[-T_{S^{n-1}}, 0]$ and is thus bounded it suffices to show that for every convergent subsequence $t_j = f_{a_j}(q_j) \to t'$ we have $t' = t_0$. Note that still $(q_j, a_j) \to (q_0, a_0)$ because they are the subsequence of a convergent sequence. The points of earliest observation converge:

$$\mathcal{E}_{a_j}(q_i) = \mu_{a_j}(f_{a_j}(q_j)) = \mu_{a_j}(t_j) = \Theta(a_j, t_j) \to \Theta(a_0, t') = \mu_{a_0}(t') = p'$$

because $(a_j, t_j) \to (a_0, t')$ and Θ is continuous. The first key observation is that because $q_j \to q_0$ and $J^+(q_i) \ni \mathcal{E}_{a_j}(q_j) \to p'$ ((REF)) implies $p' \in J^+(q_0)$.

Furthermore we have

$$0 = \tau(q_j, \mathcal{E}_{a_j}(q_j)) = \tau(q_j, \Theta(a_j, t_j)) \to \tau(q_0, \Theta(a_0, t')) = \tau(q_0, p') = 0$$

because τ and Φ are continuous.

We can now combine these observations and get: $p' \in \mathcal{L}^+_{q_0}$ because $p' \in J^+(q_0)$ and $\tau(q_0, p') = 0$ imply that there exist a null geodesic from q_0 to p'. $p' \in \mathcal{P}_K(q_0)$ because $p' \in \mu_{a_0}([0, 1]) \subset K$ and $p' \in \mathcal{L}^+_{q_0}$. But now lemma 2.3.2(3) yields that $p' = \mathcal{E}_{a_0}(q_0)$ and we get

$$\mu_{a_0}(t') = p' = \mathcal{E}_{a_0}(q_0) = \mu_{a_0}(f_{a_0}(q_0)) = \mu_{a_0}(t_0).$$

Because μ_a is injective we get $t' = t_0$, as desired. Hence every convergent subsequence of t_n goes to t_0 which, by compactness of $[-T_{S^{n-1}}, 0]$, implies that also $f_{a_n}(q_n) = t_n \to t_0 = f_{a_0}(q_0)$, proving that f is continuous. $\square$

Proposition 2.3.4. *If $q_n \to q_0 \in J(p^-, p^+)^{\mathrm{o}}$ as $n \to \infty$ and we denote $F_q : S^{n-1} \to \mathbb{R}; a \mapsto f_a(q)$. Then $F_{q_n} \to F_{q_0}$ uniformly over S^{n-1} as $n \to \infty$.*

Proof. Let $q_n \to q_0 \in V$ be a convergent sequence. We can endow M with a metric d, which is induced by g^+. Then there exists an $\varepsilon > 0$ and a $N \in \mathbb{N}$ such that $q_n \in \overline{B_\varepsilon(q_0)}$ for all $n \geq N$. After discarding the first N points of the sequence we may assume that $q_n \in \overline{B_\varepsilon(q_0)}$ $\forall n$.

By the previous proposition

$$f : (\overline{B_\varepsilon(q_0)}, d) \times (S^{n-1}, d_{S^{n-1}}) \to ([0,1], d_{[0,1]})$$

is a continuous function from and to compact spaces. Now we can apply lemma A.0.2 to find that $F_{q_n} \to F_{q_0}$ uniformly.

$\square$

2.3.1 Set of earliest observations

Definition 2.3.5 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned} \mathcal{D}_K(q) = \{(p,v) \in L^+M \mid & (p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } p \in K, w \in L^+_q M, 0 \le t \le \rho(q,w)\}, \\ \mathcal{D}^{reg}_K(q) = \{(p,v) \in L^+M \mid & (p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } p \in K, w \in L^+_q M, 0 < t < \rho(q,w)\}, \end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}^{reg}_K(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_K(q) = \pi(\mathcal{D}_K(q))$ and $\mathcal{E}^{reg}_K(q) = \pi(\mathcal{D}^{reg}_K(q))$, where $\pi : TM \to M$ is the canonical projection. We say that $\mathcal{E}_K(q)$ is the set of earliest observations and $\mathcal{E}^{reg}_K(q)$ is the set of earliest regular observations of q in K. We denote the collection of earliest observation sets by $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$.

Note that $\mathcal{E}_K(q) = \{\mathcal{E}_a(q) \mid a \in S^{n-1}\}$.

Proposition 2.3.6. *For any $q \in V$ it holds that*

(1) *Let $T = \{p \in \mathcal{L}^+_q \mid \tau(q,p) = 0\}$ then*

$$\mathcal{E}_K(q) = \mathcal{P}_K(q) \cap T \quad \textit{and} \quad \mathcal{E}^{reg}_K(q) = \mathcal{P}^{reg}_K(q) \cap T,$$

(2) *$\mathcal{E}^{reg}_K(q)$ is an open subset of $\mathcal{P}^{reg}_K(q)$, and is thus also a $n-1$-dimensional spacelike submanifold of K,*

(3) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points*

(4) *$\overline{\mathcal{E}^{reg}_K(q)}$ is open and dense in $\mathcal{E}_K(q)$,*

(5) *$\mathcal{D}^{reg}_K(q)$ is a nonempty open n-dimensional submanifold of $\pi^{-1}(K)$.*

Proof. Let $q \in V$. We first look at a useful relation of the exponential map to cut points: We define $\mathcal{V} := \{w \in L^+_q M \mid \rho(q,w) > 1\}$. By B.5.6, $\rho(q,w)$ is lower semicontinuous and $\mathcal{V}$ is thus open. Furthermore by the definition of cut points, $\mathcal{V}$ is star-shaped around $0 \in L^+_q M$. Because by B.5.5 cut points are exactly the points where $\exp_q$ first fails to be a diffeomorphism, $\exp_q : \mathcal{V} \to \mathcal{W} := \exp_q(\mathcal{V})$ is a diffeomorphism. Furthermore by the invarance of domain theorem we get that $\mathcal{W} \subset \mathcal{L}^+_q$ is relatively open. Note that this also implies that for any $p \in \mathcal{W}$, there exists a $p \in U \subset M$ open such that $p \in \mathcal{L}^+_q \cap U$ is a n-dimensional submanifold of M.

We can now move on to proving (1): $p \in \mathcal{E}_K(q) \iff p \in \mathcal{P}_K(q) \cap T$ follows immediately lemma 2.3.2(3).

Let $p \in \mathcal{E}^{reg}_K(q)$. By definition this implies that $p \in \mathcal{W}$ and we get an $p \in U \subset M$ open such that $p \in \mathcal{L}^+_q \cap U$ is a dimension n submanifold. Now, around p, K is also a dimension n submanifold, transversal to $\mathcal{L}^+_q$ and thus $K \cap \mathcal{L}^+_q \cap U = \mathcal{P}_K(q) \cap U$ is a dimension $n-1$ submanifold around p. Thus p is a regular point, i.e. $p \in \mathcal{P}^{reg}_K(q)$. $\tau(q,p) = 0$ follows immediately from the fact that $\rho(q,w) > 1$, proving the first direction.

To show the reverse direction we assume $p \in \mathcal{P}^{reg}_K(q)$ with $\tau(q,p) = 0$. Because $p \in \mathcal{P}^{reg}_K(q)$ by definition 2.2.15 there exists exactly one $w \in L^K_q M$ such that $\exp_q(w) = p$. From $\tau(q,w) = 0$ we get $\rho(q,p) \geq 1$. Now if $\rho(q,w) = 1$, p would be a cut point. By theorem B.5.5 this would mean that either $p \in K$ is a conjugate point to q or there exists a $w \neq w' \in L^K_q M$ with $\exp_q(w') = p$. The first option is impossible because in the statement of theorem 1.1.2 we assumed that no $q \in V$ can have a conjugate point on K. The second option is also impossible because we assumed p to be a regular point in $\mathcal{P}_K(q)$. We thus must have $\rho(q,w) > 1$, implying $p \in \mathcal{E}^{reg}_K(q)$.

We now move on to (2): To prove that $\mathcal{E}^{reg}_K(q)$ is open in $\mathcal{P}^{reg}_K(q)$ we claim that $\mathcal{E}^{reg}_K(q) = \mathcal{P}^{reg}_K(q) \cap \mathcal{W}$. To that end we first note that $\mathcal{E}^{reg}_K(q) \subset \mathcal{W} \subset T$. Recall that by (1) we have $\mathcal{E}^{reg}_K(q) = \mathcal{P}^{reg}_K(q) \cap T$. Applying $\cap\mathcal{W}$ to both sides yields

$$\mathcal{E}^{reg}_K(q) = \mathcal{E}^{reg}_K(q) \cap \mathcal{W} = \mathcal{P}^{reg}_K(q) \cap T \cap \mathcal{W} = \mathcal{P}^{reg}_K(q) \cap \mathcal{W}$$

as desired.

Proposition 2.2.14 implies that $\mathcal{P}^{reg}_K(q)$ is a $n-1$ dimensional spacelike submanifold of M. Because $\mathcal{W} \subset \mathcal{L}^+_q$ is open and $\mathcal{P}^{reg}_K(q) \subset \mathcal{L}^+_q$, $\mathcal{E}^{reg}_K(q)$ is a relatively open subset of $\mathcal{P}^{reg}_K(q)$, as desired. This also means that $\mathcal{E}^{reg}_K(q)$ itself is a open subset of a $n-1$-dimensional spacelike submanifold of M as well.

We can now tackle (3): Let $p \in \mathcal{E}_K(q)$ be a cut point, then by proposition 2.2.14, there exists an open neighborhood $p \in U \subset M$ and N codimension 1 pairwise transversal manifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$. Because $\tau(q,p) = 0$ and the manifolds are pairwise transversal and intersect at p ((SEE FIGURE)),

$\mathcal{E}_K(q)$ must have a sharp edge at p meaning it cannot be a submanifold. For the other direction we assume that $p \in \mathcal{E}_K(q)$ is not a cut point. Then, by definition we have $p \in \mathcal{E}_K(q)$ which is a submanifold.

Moving on to (4), the fact fact that $\mathcal{E}_K^{reg}(q)$ is dense in $\mathcal{E}_K(q)$ follows by an argument which is analogous to the one used in the proof of corollary 2.2.16. To show that it is relatively open in $\mathcal{E}_K(q)$ we use that $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K(q) \cap \mathcal{W}$ with $\mathcal{W}$ open in $\mathcal{L}_q^+$.

Finally the proof of (5) is analogous to (2) with the difference in submanifold dimension originating from the face that for any $(p, v) \in \mathcal{D}_K^{reg}(q)$ we also have $(p, cv) \in \mathcal{D}_K^{reg}(q)$ for all $c \in \mathbb{R}_+$ (explain more). $\square$

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

Proposition 2.3.7. *Let $q \in V$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

Proof. For the left inclusion assume $p \in \mathcal{E}_K(q)$, i.e. there exists an $a \in S^{n-1}$ such that $\mathcal{E}_a(q) = p$. Then lemma 2.3.2(3) immediately yields, $p \in \mathcal{P}_K(q)$ and $\tau(q, p) = 0$. Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. Because $\mathcal{P}_K(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p, q) > 0$, a contradiction.

For the other direction we assume we have $p = \mu_a(t) \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 2.3.2(3) we only need to prove that $\tau(p, q) = 0$. Suppose that $\tau(p, q) > 0$. Now since $\tau(p, q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since μ_a is timelike, which is a contradiction. $\square$

Thus $\mathcal{E}_K(q)$ truly deserves to be called the "set of earliest observations".

2.3.2 Observation Reconstruction

Proposition 2.3.8. *Given the data outlined in remark 2.2.1 we can uniquely determine $\mathcal{E}_K(q)$ and $\mathcal{E}_K^{reg}(q)$, as well as $\mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Proof. What we want to show is that given K, the conformal class of $g|_K$ and the set $\{\mathcal{P}_K(q) \mid q \in V\}$ we can reconstruct the sets stated above. Note that as described in proposition 2.2.8 ((Move to own remark?)) this data allows us to construct $\Theta : \mathcal{S} \to K$ and μ_a.

We first show that for a given $\mathcal{P}_K(q)$ we can determine $\mathcal{E}_K(q)$: By equation 2.7, for any $a \in S^{n-1}$ we can determine $f_a(q)$ and thus $\mathcal{E}_a(q) = \mu_a(f_a(q))$ using only $\mathcal{P}_K(q)$. We can then construct $\mathcal{E}_K(q) = \bigcup_{a \in S^{n-1}} \mathcal{E}_a(q)$. Furthermore, by proposition 2.3.6, $\mathcal{E}_K^{reg}(q)$ contains exactly the points $p \in \mathcal{E}_K(q)$ where $\mathcal{E}_K(q)$ is locally a submanifold of M and thus K. But because we know K we can determine all points where this is the case and reconstruct $\mathcal{E}_K^{reg}(q)$.

To reconstruct the direction set we first note that by lemma 2.2.14 for any $p \in \mathcal{P}_K(q)$ such that $\exp_q^{-1}(p) = \{w_1, \dots, w_N\} \subset L_q^K M$, we have $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ where $p \in U \subset M$ open and $p \in \mathcal{U}_i$ are pairwise transversal spacelike hypersurfaces of K. For each w_i we let $v_i = \gamma'_{q,w_i}(1)$ be the outbound velocity vector of the null geodesic which starts at q with velocity w_i, once it hits K. To find $\mathcal{D}_K(q)$ we must reconstruct all such v_i.

To that end, note that we have $T_p\mathcal{P}_K(q) = \bigcup_{i=1}^N T_p\mathcal{U}_i$ where $T_p\mathcal{U}_i$ are spacelike hyperplanes. For each such hypersurface, using lemma 2.2.9 we can then find the outward pointing orthogonal null ray $\mathbb{R}_+ v_i$ which must contain the outbound velocity vector v_i at p. Thus for any $p \in \mathcal{P}_K(q)$ we can reconstruct $\mathbb{R}_+ v_i$ for all geodesics γ_{q,w_i} from q to p.

Now by definition for any $p \in \mathcal{E}_K(q)$, we have

$$\mathcal{D}_p := \pi^{-1}(p) \cap \mathcal{D}_K(q) = \{(p, \mathbb{R}_+ v_1), \dots, (p, \mathbb{R}_+ v_N)\}$$

where $\pi : TM \to M$ is the canonical projection. As we saw for any $p \in \mathcal{E}_K(q) \subset \mathcal{P}_K(q)$ we can reconstruct $\mathcal{D}_p$ which allows us to reconstruct $\mathcal{D}_K(q) = \bigcup_{p \in \mathcal{E}_K(q)} \mathcal{D}_p$.

Finally we can reconstruct $\mathcal{D}_K^{reg}(q)$ by using $\mathcal{D}_K^{reg}(q) = \pi^{-1}(\mathcal{E}_K^{reg}(q)) \cap \mathcal{D}_K(q)$. $\square$

Note that we can adapt this proof to show that $\mathcal{E}_K(q)$ uniquely determines $\mathcal{E}_K^{reg}(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.

Proposition 2.3.9. *Let $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$. Then $q = q'$.*

Proof. We assume by contradiction that $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$ and $q \neq q'$ Let $p_1, p_2 \in \mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ with $p_1 \neq p_2$. Because p_1 and p_2 cannot be cut points there must exist unique $w_1, w_2 \in L_q^K M$ and $w'_1, w'_2 \in L_{q'}^K M$ such that $\gamma_{q,w_i}(1) = p_i$ and $\gamma_{q',w'_i}(1) = p_i$. Because $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ we can use lemma 2.2.9 to show that

$$v_i = \gamma'_{q,w_i}(1) = c_i \gamma_{q',w'_i}(1) = c_i v'_i$$

for some $c_i > 0$.

Now $\gamma_{p_i,-v_i}$ are two past-pointing null geodesics going from p_i through q and q'. Hence there either exists a null geodesic from q to q' or from q' to q. We will WLOG assume $q' \in J^+(q)$. Now there must exist $t_1, t_2 \in (0,1)$ such that $\gamma_{q,w_i}(t_i) = q'$. But this would make q' a cut point of q which is impossible as we assumed $p_i \in \mathcal{E}_K^{reg}(q)$. $\square$

2.4 Smooth Constructions

Definition 2.4.1 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_K^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w \in L_q^K M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q,w) > 1$. Existence follows from lemma 2.3.2 while uniqueness follows from the fact that $p \in \mathcal{E}_K^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Omega : \mathcal{Z} &\mapsto L^K V \\ (q,p) &\mapsto (q,w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0,w_0) \subset TM$ be a ε-neighborhood of (q_0,w_0) with respect to the Sasaki-metric induced on TM by g^+.

Lemma 2.4.2. *((Move to appendix?))The function*

$$\begin{aligned} T_+ : L^+J(p^-,p^+) &\to \mathbb{R} \\ (q,w) &\mapsto \sup\{t \geq 0 \mid \gamma_{q,w}(t) \in J^-(p^+)\} \end{aligned}$$

is finite and upper semicontinuous.

Proof. Finiteness follows from lemma B.2.2. We now want to show that T_+ is upper semicontinuous. To that end let $(q_n,w_n) \to (q_0,w_0) \in L^+J(p^-,p^+)$, we want to show that $\limsup_{n\to\infty} T_+(q_n,w_n) \leq T_+(q_0,w_0)$: Let $\varepsilon > 0$ and set $t_0 = T_+(q_0,w_0)$. Then by definition we have $\gamma_{q_0,w_0}(t_0) \in M \setminus J^-(p^+)$. Because $\gamma_{q_n,w_n}(t_0) \to \gamma_{q_0,w_0}(t_0)$ and $M \setminus J^-(p^+)$ open, there exists a $N \in \mathbb{N}$ such that $\gamma_{q_n,w_n}(t_0) \in M \setminus J^-(p^+)$ for all $n \geq N$. Note that if $\gamma_{q_n,w_n}(t_0) \notin J^-(p^+)$ then for any $t' \geq t_0$ we also have $\gamma_{q_n,w_n}(t') \notin J^-(p^+)$ because otherwise we could obtain a lightlike path from $\gamma_{q_n,w_n}(t_0)$ to p^+, a contradiction. Thus, by definition $T_+(q_n,w_n) \leq t_0$ and $\limsup_{n\to\infty} T_+(q_n,w_n) \leq t_0 = T_+(q_0,w_0) + \varepsilon$. Finally because $\varepsilon > 0$ was arbitrary we get $\limsup_{n\to\infty} T_+(q_n,w_n) \leq T_+(q_0,w_0)$ as desired. ☐

Lemma 2.4.3. *Let $(q_0,p_0) \in \mathcal{Z}$ and $(q_0,w_0) = \Omega(q_0,p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0,w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0,w_0) \to \mathcal{U}_\varepsilon(q_0,p_0) := X(\mathcal{W}_\varepsilon(q_0,w_0))$. When ε is small enough, Ω coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0,p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $2n$-dimensional manifold and the map $\Omega : \mathcal{Z} \to L^K M$ is smooth.

Proof. Because $p_0 \in \mathcal{P}_K(q_0)$ and $q_0 \in V$ we have, by assumption in theorem 1.1.2 that p_0 cannot be a conjugate point of q_0. Hence for $\varepsilon > 0$ small enough $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) = X(\mathcal{W}_\varepsilon(q_0, w_0))$ is a diffeomorphism with $\mathcal{U}_\varepsilon(q_0, p_0)$ open in $M \times M$ by the invariance of of domain theorem.

Next we aim to show that $\Omega : \mathcal{Z} \to L^K V$ is continuous at $(q_0, p_0) \in \mathcal{Z}$. We proceed by assuming there exists a sequence $(q_n, p_n) \in \mathcal{Z}$ converging to (q_0, p_0) such that $\Theta(q_n, p_n) = (q_n, w_n) \in L^+V$ does not converge to $\Theta(q_0, p_0) = (q_0, w_0)$.

First of all we aim to show that the sequence (q_n, w_n) is bounded and thus has a convergent subsequence: Because $q_n \to q_0$ we only need to show that w_n is bounded. To that end we introduce an arbitrary riemannian metric consistent with the topology on M and can write $w_n = t_n \overline{w_n}$ where $\|\overline{w_n}\|_{g^+} = 1$. To show that t_n is bounded we first define

$$C := \{(q, w) \in L^+M \mid q \in J(p^-, p^+) \text{ and } \|w\|_{g^+} = 1\}$$

and C is compact and because T_+ is upper semicontinuous on C, there exists a $c_0 > 0$ such that $T_+(q, w) \le c_0$ for all $(q, w) \in C$. Recall that we have $\gamma_{q_n, \overline{w_n}}(t_n) = \exp_{q_n}(w_n) = p_n \in K \subset J(p^-, p^+)$. Together with $(q_n, \overline{w_n}) \in C$ this yields

$$\|w_n\|_{g^+} = t_n \|\overline{w_n}\|_{g^+} = t_n \le T_+(q_n, \overline{w_n}) < c_0,$$

proving $(q_n, w_n) \in L^K V$ is bounded.

We can thus obtain a convergent subsequence $(q_k, w_k) = \Theta(q_k, p_k) \to (q_0, w')$ with $w' \neq w_0$. Since the exponential map is continuous, we would have

$$\exp_{q_n}(w') = \lim_{n\to\infty} \exp_{q_n}(w_n) = \lim_{n\to\infty} p_n = p_0 = \exp_{q_n}(w_0).$$

with $w' \neq w_0$. But since $p_0 \in \mathcal{E}_K^{reg}(q)$ cannot be a cut point this is a contradiction and $\Omega : \mathcal{Z} \to L^K V$ must be continuous.

Next we use the fact that Ω is continuous and get $\Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0)) \subset \mathcal{Z}$ is open. We can thus find a $\varepsilon_1 \in (0, \varepsilon)$ such that for the open ball $\mathcal{U}_{\varepsilon_1}(q_0, w_0) \subset M$ we have

$$\mathcal{Y}_{\varepsilon_1} := \mathcal{U}_{\varepsilon_1}(q_0, w_0) \cap \mathcal{Z} \subset \Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0))$$

implying $\Omega(\mathcal{Y}_{\varepsilon_1}) \subset \mathcal{W}_\varepsilon(q_0, w_0)$. Then for $(q, p) \in \mathcal{Y}_{\varepsilon_1}$ and $(q, w) = \Omega(q, p) \in \mathcal{W}_\varepsilon(p_0, w_0)$ we have $\exp_q(w) = p$. Hence $X(\Omega(q, p)) = (q, p)$. But now since $(q, p) \in \mathcal{U}_\varepsilon(p_0, q_0)$ we can apply X^{-1} to both sides and get $\Omega(q, p) = X^{-1}(q, p)$. Thus on $\mathcal{Y}_{\varepsilon_1}$ the function $\Omega : \mathcal{Y}_{\varepsilon_1} \to TM$ coincides with the smooth function $X^{-1} : \mathcal{Y}_{\varepsilon_1} \to TM$, which implies that Ω is smooth with full rank differential on $\mathcal{Y}_{\varepsilon_1}$ as well.

Now since $(q_0, p_0) \in \mathcal{Z}$ was arbitrary we get that $\Theta : \mathcal{Z} \to L^+V$ is smooth everywhere, injective and locally diffeomorphic with full rank. Thus $\mathcal{Z}$ diffeomorphic to an open subset of $L^K V$. This makes it a manifold with dimension $(n+1) + (n-1) = 2n$. $\square$

Proposition 2.4.4. *Let $q_0 \in V$ and $(q_0, p_j) \in \mathcal{Z}, j = 0, \ldots, n$ and $w_j \in L^K_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \ldots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=0}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

Proof. First we need some setup: Let $(q_0, p_0) \in \mathcal{Z}$ and $w_0 \in L^+_{q_0}M$ such that $\gamma_{q_0,w_0}(1) = p_0$. Furthermore let $\varepsilon > 0$ be small enough such that the map $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ is a diffeomorphism (see the previous lemma). We will denote this inverse by $X^{-1}(q, p) = (q, w(q, p))$ and write $\mathcal{W} = \mathcal{W}_\varepsilon(q_0, w_0), \mathcal{U} = \mathcal{U}_\varepsilon(q_0, p_0)$.

We associate with any $(q, p) \in \mathcal{U}$ the energy $E(q, p) = E(\gamma_{q,w(q,p)}([0, 1]))$ of the geodesic segment connecting q to p. The energy of a piecewise smooth curve $\alpha : [0, l] \to M$ is defined as

$$E(\alpha) = \frac{1}{2} \int_0^l g(\alpha'(t), \alpha'(t)) dt.$$

Note that the sign of $E(\alpha)$ depends on the causal nature of $\gamma_{q,w(q,p)}$. In particular $E(q, p) = 0$ if and only if $w(q, p)$ is light-like. Moreover, as X^{-1} is smooth on $\mathcal{U}$, so is $E(p, q)$.

We now return to consider $(q_0, p_0) \in \mathcal{Z}$ and let $a \in S^{n-1}$ be such that $p_0 \in \mu_a$. Then $p_0 = \mu_a(s_0)$ with $s_0 = f_a(q_0)$ as $p_0 \in \mathcal{E}^{reg}_K(q_0)$ and $s_0 \in (0, 1)$ by lemma 2.3.2(1).

Let $V_0 \subset V$ be an open neighborhood of q_0 and $t_1, t_2 \in (-T_a, 0), t_1 < s_0 < t_2$, such that $V_0 \times \mu_a([t_1, t_2]) \subset \mathcal{U}$, which exist because $\mathcal{U}$ is open. Then for any $q \in V_0, s \in (t_1, t_2)$ the function $\mathbf{E}_a(q, s) := E(q, \mu_a(s))$ is well defined and smooth.

We want to use first variation formula for $\mathbf{E}_a(q, s)$ ((E Reference)) to calculate $\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0}$ and $\nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0}$.

For the first part we define the variation $\mathbf{x}(t, s) = \gamma_{q_0,w(s)}(t), t \in [0, 1]$ where $w(s) := w(q_0, \mu_a(s + s_0)), s \in [t_1 - s_0, t_2 - s_0]$. Note that $\mathbf{x}(t, 0) = \gamma_{q_0,w_0}(t)$. We then get

$$\left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} = E'_{\mathbf{x}}(0) = g(V, \gamma'_{q_0,w_0})\big|_0^1$$

since γ_{q_0,w_0} is a geodesic and $\mathbf{x}$ has no breaks. If we now further notice that $V(0) = 0$ as $\mathbf{x}(0, s) = q_0$ for all $s \in [t_1, t_2]$ and $V(1) = \mu'_a(s_0) = \mu'_a(f_a(q_0))$ as

$\mathbf{x}(1,s) = \mu_a(s+s_0)$ we can conclude

$$\begin{aligned}\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} &= g(V(1),\gamma'_{q_0,w_0}(1)) - g(V(0),\gamma'_{q_0,w_0}(0)) \\ &= g(\mu'_a(f_a(q_0)),\gamma'_{q_0,w_0}(1))\end{aligned}$$

For the second part we will introduce coordinates $\mathbf{q} = (q_0,\dots,q_n)$ around q_0. Then the gradient can be written as

$$\nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} \partial_j.$$

To calculate $\left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0}$ we now introduce variations $\mathbf{x}_i(t,s) = \gamma_{q(s),w(s)}(t)$ where $w(s) := w(q(s),\mu_a(s_0))$ and $q(s) := q^{-1}(q_0(q_0),\dots,q_i(q_0)+s,\dots q_n(q_0))$ is obtained by increasing the i-th coordinate by s. Note that these variations all have $\mathbf{x}_i(t,0) = \gamma_{q_0,w_0}(t)$, $\mathbf{x}_i(1,s) = \mu_a(s_0)$ thus $V_{\mathbf{x}_i}(1) = 0$ and $V_{\mathbf{x}_i}(0) = \frac{\partial}{\partial s}\mathbf{x}_i(0,s)|_{s=0} = \partial_i$. After again applying proposition ((E REF))

$$\left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} = E'_{\mathbf{x}_i}(0) = -g(V(0),\gamma'_{q_0,w_0}(0)) = -g(\partial_i,w_0).$$

Combining this with coordinate representation of the gradient we get

$$\begin{aligned}\nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} \partial_j &= -g^{ij}(g_{\alpha\beta}\partial_i^\alpha w_0^\beta)\partial_j \\ &= -g^{ij}g_{i\beta}w_0^\beta\partial_j = -\delta^j_\beta w_0^\beta \partial_j \\ &= -w_0^j\partial_j = -w_0.\end{aligned}$$

We thus managed to calculate what we wanted and can summarize as

$$\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(v,\mu'_a(f_a(q_0))), \quad \nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = -w_0 \tag{2.8}$$

where $w_0 = w(q_0,p_0)$ and $v = \gamma'_{q_0,w_0}(1)$. Since $\mu'_a(f_a(q_0))$ and v are both future-pointing null vectors, which by lemma 2.2.3 must be transversal we have $\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(v,\mu'_a(f_a(q))) < 0$.

We can now use the implicit function theorem on $V_0 \times [t_1,t_2]$ with equation $E_a(q,s) = 0$ and single solution $E_a(q_0,s_0) = 0$. This yields an open neighborhood $V_a \subset V_0$ and a smooth function $q \mapsto s_a(q)$ such that $E_a(q,s_a(q)) = 0$ for all $q \in V_a$. Now $E_a(q,s_a(q)) = E(q,\mu_a(s_a(q))) = 0$, implies $\mu_a(s_a(q)) \in \mathcal{P}_K(q)$. This together with $(q, s_a(q)) \in \mathcal{U}$ implies that $\mu_a(s_a(q)) \in \mathcal{E}_K^{reg}(q)$ and thus $s_a(q) = f_a(q)$ on V_a. Hence we have $\nabla f_a(q)|_{q=q_0} = \nabla s_a(q)|_{q=q_0}$ and from equation 2.8 together with the implicit function theorem it follows that

$$\nabla f_a(q)|_{q=q_0} = \frac{1}{c(q_0, a)} w_0, \quad c(q_0, a) = \left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} < 0, \tag{2.9}$$

where $p_0 = \mu_a(s_0) = \mathcal{E}_a(q_0), s_0 = f_a(q_0)$ and $w_0 = w(q_0, p_0)$.

Next we choose $p_0, \ldots, p_n \in \mathcal{E}_K^{reg}(q_0)$ and let $w_0, \ldots, w_n \in L_{q_0}^K M$ such that $p_i = \gamma_{q_0, w_i}(1)$, i.e. $w_i = w(q_0, p_i)$. We assume that $w_0, \ldots, w_n$ are linearly independent. Moreover let $a_j \in S^{n-1}$ such that $p_i \in \mu_{a_j}$ and $\overrightarrow{a} = (a_j)_{j=1}^n$. Finally we denote by $q \mapsto s_{a_j}(q) = f_{a_j}(q)$ the above constructed smooth functions which are defined on some neighborhoods $V_{a_j} \subset V$ of q_0.

Let $V_{\overrightarrow{a}} = \bigcap_{j=1}^n V_{a_j}$ and consider the map

$$\begin{aligned} \mathbf{f}_{\overrightarrow{a}} : V_{\overrightarrow{a}} &\to \mathbb{R}^n \\ q &\mapsto (f_{a_1}(q), \ldots, f_{a_n}(q)). \end{aligned}$$

Because all of its components are smooth, $\mathbf{f}_{\overrightarrow{a}}$ itself is smooth as well. By equation 2.9 each component has gradient $\nabla f_{a_j}(q)\big|_{q=q_0} = \frac{1}{c(q_0, a_j)} w_i$ with $c(q_0, a_j) \neq 0$. Since we assumed that $w_0, \ldots, w_n$ be independent, $\mathbf{f}_{\overrightarrow{a}}$ is non-degenerate at q_0 and thus defines a smooth coordinate system in some neighborhood V_1 of q_0. $\square$

Definition 2.4.5 (Regular Observer)**.** Let $q \in V$ we call $a \in S^{n-1}$ a *regular observer* of q if $\mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)$ and write

$$\mathcal{A}^{reg}(q) := \{a \in S^{n-1} \mid \mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)\} \subset S^{n-1}$$

for the set of regular observers. Note that because $\mathcal{E}_K^{reg}(q)$ open and dense in $\mathcal{E}_K(q)$, $\mathcal{A}^{reg}(q)$ is open and dense in S^{n-1}

For the next proposition we again endow M a metric d which is induced by g^+. This allows us to define open balls.

Proposition 2.4.6. *Let $q_0 \in V$ and $a_0 \in \mathcal{A}^{reg}(q_0)$ a regular observer of q_0. Then there exists a $\varepsilon > 0$ such that $f : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$; $(q, a) \mapsto f_a(q)$ is smooth.*

Proof. Let $p_0 \in \mathcal{E}_K^{reg}(q_0)$ with $p_0 = \gamma_{q_0, w_0}(1)$ and $p_0 = \mu_{a_0}(t_0)$. Then we have $(q_0, p_0) \in \mathcal{Z}$ and by lemma 2.4.3 there exists a $\delta > 0$ such that $X : \mathcal{W}_\delta(q_0, w_0) \to \mathcal{U}_\delta(q_0, p_0)$ is a diffeomorphism. Note that we can choose $\delta > 0$ such that $\rho(q, w(q, p)) > 1$ for all $(q, p) \in \mathcal{U}_\delta(q_0, p_0) \cap L^+ M$.

Then

$$\begin{aligned} Y : \mathcal{W}_\delta(q_0, w_0) \cap L^K M &\to M \times S^{n-1} \times [0,1] \\ (q,w) &\mapsto (q, \Theta^{-1}(X(q,w))) \end{aligned}$$

is a diffeomorphism onto its image $\mathcal{V}_\delta := Y(\mathcal{W}_\delta(q_0, w_0))$ which is an open neighborhood of (q_0, a_0, t_0) in $M \times S^{n-1} \times [0,1]$. There thus exists a $\delta > \lambda > 0$ such that $\overline{B_\lambda(q_0)} \times \overline{B_\lambda(a_0)} \times \overline{B_\lambda(t_0)} \subset \mathcal{V}_\delta$.

On this space we can then define the function $\mathbf{E}(q,a,s) := E(q, \Theta(a,s))$ with $E(q,p)$ as in the previous lemma. This function is well defined and smooth with $E(q_0,a_0,t_0) = 0$ and $\left.\frac{\partial \mathbf{E}(q_0,a_0,t)}{\partial s}\right|_{t=t_0} < 0$ by the same argument as in the previous proof. We can thus apply the implicit function theorem to get a $\varepsilon > 0$ and a smooth function

$$\begin{aligned} s : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} &\to \overline{B_\lambda(t_0)} \\ (q,a) &\mapsto (t) \end{aligned}$$

with $s(q_0,a_0) = t_0$ and $\mathbf{E}(q,a,s(q,a)) = 0$.

Let $(q,a) \in \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ then

$$\mathbf{E}(q,a,s(q,a)) = E(q, \Theta(a, s(q,a))) = E(q, \mu_a(s(q,a))) = 0$$

implies that $p = \mu_a(s(q,a)) \in \mathcal{P}_K(q)$. Furthermore by definition we have $p \in \mathcal{U}_\delta(q_0,p_0)$ which implies $\rho(q, w(q,p)) > 1$ and thus $p = \mu_a(s(q,a)) \in \mathcal{E}_K^{reg}(q)$. Thus we have that $s(q,a) = f_a(q)$, making $f_a(q)$ a smooth function on $\overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ as desired. □

Note that this result implies that $\mathcal{E}_a(q) \in \mathcal{E}_K^{reg}(q)$ for all $(q,a) \in \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)}$ and $f : V \times S^{n-1} \to [0,1]$ smooth around all (q,a) such that $q \in V, a \in \mathcal{A}^{reg}(q)$.

We can use previous result to get

Proposition 2.4.7. *Let $q_n \to q_0 \in V$ and $A \subset S^{n-1}$ open, such $\overline{A} \subset \mathcal{A}^{reg}(q_0)$. Then for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $F_{q_n}|_{\overline{A}}$ is smooth and $\|dF_{q_n}|_a - dF_{q_0}|_a\|_{g_{S^{n-1}}} < \varepsilon$ for all $a \in \overline{A}$.*

Proof. By the previous proposition for all $a \in \overline{A}$ there exists a $\varepsilon_a > 0$ such that $f : \overline{B_{\varepsilon_a}(q_0)} \times \overline{B_{\varepsilon_a}(a)} \to [0,1]$ is smooth. Then $\bigcup_{a \in \overline{A}} B_{\varepsilon_a}(a)$ is an open cover of the compact $\overline{A} \subset S^{n-1}$. Hence there exist $(a_1, \varepsilon_1), \ldots (a_N, \varepsilon_N)$ such that $\bigcup_{j=1}^N B_{\varepsilon_j}(a_j) \supset \overline{A}$. We then let $\varepsilon_0 := \min_{j=1,\ldots,N} \varepsilon_j$ and get $B_{\varepsilon_0}(q_0) = \bigcap_{j=1}^N B_{\varepsilon_j}(q_0)$ is open.

Let now $(q,a) \in B_{\varepsilon_0}(q_0) \times \overline{A}$ then there exists a $j \in 1, \ldots N$ such that $a \in B_{\varepsilon_j}(a_j)$ and we have $q \in B_{\varepsilon_0}(q_0) \subset B_{\varepsilon_j}(q_0)$. Thus by construction, f is smooth at (q,a).

As the choice (q,a) was arbitrary f is smooth on $B_{\varepsilon_0}(q_0)\times\overline{A}$. Because $q_n\to q_0$ there exists a $N_1\in\mathbb{N}$ such that $n\geq N$ implies $q_n\in B_{\varepsilon_0}(q_0)$ and we have $F_{q_n}|_{\overline{A}}$ is smooth.

We now want to show that also the derivatives of F_{q_n} wrt. $a\in S^{n-1}$ converge uniformly on $\overline{A}$: By the above argument

$$f':\overline{B_{\frac{\varepsilon_0}{2}}(q_0)}\times\overline{A}\to T^*S^{n-1}$$
$$(q,a)\mapsto dF_q|_a$$

is a continuous function on a compact metric spaces to a metric space (here we endow T^*S^{n-1} with some metric compatible with its topology). But now we can apply lemma A.0.2 to find that there exists a $N_2>N_1$ such that $n\geq N_2$ implies $\|dF_{q_n}|_a-dF_{q_n}|_a\|_{g_{S^{n-1}}}$ for all $a\in\overline{A}$. □

Corollary 2.4.8. *Let $q_n\to q_0\in V$ and $a_0\in\mathcal{A}^{reg}(q_0)$. Then $dF_{q_n}|_{a_0}\to dF_{q_0}|_{a_0}$.*

In the following we will for any $(q,p)\in\mathcal{Z}$ denote $v(q,p):=\gamma'_{q,w(q,p)}(1)$, i.e. the velocity vector of the unique geodesic from q to p at p. Additionally, because sometimes we can only recover the direction of $v(q,p)$ we denote $\overline{v}(q,p)=\frac{v(q,p)}{\|v(q,p)\|}$.

This corollary follows from lemma 2.2.9 ((extended to show that it is homeo))

Corollary 2.4.9. *Let $(q_n)_{n=1}^{\infty},q_0\in V$ and $a_0\in\mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_0}\to dF_{q_0}|_{a_0}$. Then $\overline{v}_n:=\overline{v}(q_n,\mathcal{E}_{a_0}(q_n))\to\overline{v}_0:=\overline{v}(q_0,\mathcal{E}_{a_0}(q_0))$.*

Finally we can prove

Proposition 2.4.10. *Let $(q_n)_{n=1}^{\infty},q_0\in V$ and $a_1,a_2\in\mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_i}\to dF_{q_0}|_{a_i}$. Then $q_n\to q_0$.*

Proof. We denote $p_n^i=\mathcal{E}_{a_i}(q_n)$ and $p_0^i=\mathcal{E}_{a_i}(q_0)$. By the previous corollary we have $\overline{v}_n^i:=\overline{v}(q_n,p_n^i)\to\overline{v}_0^i:=\overline{v}(q_0,p_0^i)$ for $i=1,2$ in $CTM=\{(p,v)\in TM\mid g^+(v,v)=1\}$. Note that by definition there exist $t_n^i,t_0^i\in\mathbb{R}_+$ such that

$$q_0=\gamma_{p_0^i,v_0^i}(-t_0^i)\quad\text{and}\quad q_n=\gamma_{p_n^i,v_n^i}(-t_n^i)\quad\text{for } i=1,2.$$

We now want to show that $t_n^i\to t_0^i$. By contradiction we assume that t_n^i does not converge to t_0^i. By a similar argument to the one employed in the proof of lemma 2.4.3 we find that t_n^i must be bounded. t_n^i has thus a convergent subsequence $t_j^i\to t_\times^i\neq t_0^i$ for $i=1,2$. Now we let d be a metric on M compatible with the topology and note that because $(q,w,t)\mapsto\gamma_{q,w}(t)$ is continuous we have

$$0=\lim_{n\to\infty}d(\gamma_{p_j^1,v_j^1}(-t_j^1),\gamma_{p_j^2,v_j^2}(-t_j^2))=d(\gamma_{p_0^1,v_0^1}(-t_\times^1),\gamma_{p_0^2,v_0^2}(-t_\times^2)),$$

i.e. $q_\times:=\gamma_{p_0^i,v_0^i}(-t_\times^i)$ for $i=1,2$. But this is a contradiction because p_0^1 and p_0^2 are in $\mathcal{E}_K^{reg}(q_0)$ and thus cannot be cut points of q_0. □

Proposition 2.4.11. *Let $q_0 \in V$, $\varepsilon > 0$ such that $\overline{B_\varepsilon(q_0)} \subset V$ and define*

$$D_\varepsilon := \{(q,a) \in V \times S^{n-1} \mid q \in \overline{B_\varepsilon(q_0)}, a \in \mathcal{A}^{reg}(q)\}.$$

Then

$$\begin{aligned} f' : D_\varepsilon &\to T^*S^{n-1} \\ (q,a) &\mapsto dF_q|_a \end{aligned}$$

is bounded.

Proof. ((Make more rigorous / shorter / more understandable. The main idea here is that dF must be bounded because at the points where it is not defined i.e. points where $\mathcal{E}_a(q) \notin \mathcal{E}_K^{reg}(q)$, dF does not go to infinity but has multiple conflicting values (see prop 2.2.14), we try to show that by expressing dF as in terms of dh and dY which are well defined at the points where dF fails to be so))

We begin by defining the map

$$\begin{aligned} Y : L^K\overline{B_\varepsilon(q_0)} &\to \overline{B_\varepsilon(q_0)} \times S^{n-1} \\ (q,w) &\mapsto (q, \pi_a(\Theta^{-1}(X(q,w)))) \end{aligned}$$

which is smooth surjective and locally diffeomorphic by ((REF)).

We also define the map

$$\begin{aligned} h : L^K\overline{B_\varepsilon(q_0)} &\to [0,1] \\ (q,w) &\mapsto \pi_t(\Theta^{-1}(X(q,w))) \end{aligned}$$

which is also smooth.

We then define

$$\begin{aligned} P : &= \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \rho(q,w) \geq 1\} \\ &= \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \exp_q(w) \in \mathcal{E}_K(q)\} \end{aligned}$$

which is closed by the lower semicontinuity of ρ and thus compact. Now the following diagramm commutes:

$$\begin{tikzcd} P \subset L^K\overline{B_\varepsilon(q_0)} \arrow[r, "h"] \arrow[d, "Y"] & {[0,1]} \\ \overline{B_\varepsilon(q_0)} \times S^{n-1} \arrow[ur, "f"'] & \end{tikzcd}$$

Let now g^+ a riemannian metric on M and $\widehat{g}^+$ the corresponding Sasaki metric induced on TM. Let also $g^\times := g^+ + g_{S^{n-1}}$ be the product metric on $M \times S^{n-1}$

with $g_{S^{n-1}}$ the standard riemannian metric on S^{n-1}. Now because $h : P \to [0,1]$ is smooth, $dh : TP \to \mathbb{R}$ is smooth as well. Furthermore because P is compact and $dh_{(q,w)}$ is linear for all $(q,w) \in P$ we get that dh is bounded on TP, i.e. there exists a $c_1 > 0$ such that for all $(q,w) \in P$ and $(q',w') \in T_{(q,w)}P$ we have

$$|dh_{(q,w)}(q',w')| \leq c_1 \|(q',w')\|_{\widehat{g}^+}.$$

Similarly because Y is also smooth and also locally diffeomorphic its derivative is also bounded from below, i.e. there exists a $c_2 > 0$ such that for all $(q,w) \in P$ and $(q',w') \in T_{(q,w)}P$ we have

$$\|dY_{(q,w)}(q',w')\|_{g^\times} \geq c_2 \|(q',w')\|_{\widehat{g}^+}.$$

We now define

$$\begin{aligned} P^{reg} :=& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \rho(q,w) > 1\} \\ =& \{(q,w) \in L^K\overline{B_\varepsilon(q_0)} \mid \exp_q(w) \in \mathcal{E}_K^{reg}(q)\} \text{ and} \end{aligned}$$

and the following again diagramm commutes:

$$\begin{tikzcd} P^{reg} \subset P \arrow[r, "h"] \arrow[d, "Y"] & {[0,1]} \\ D_\varepsilon \subset \overline{B_\varepsilon(q_0)} \times S^{n-1} \arrow[ur, "f"'] & \end{tikzcd}$$

Additionally in this case Y is a diffeomorphism and f is smooth. Let $(q,a) \in D_\varepsilon$ and $(q',a') \in T_{(q,a)}D_\varepsilon$, then there exists a unique $(q,w) = Y^{-1}(q,a) \in P^{reg}$ and $(q',w') = dY^{-1}(q',a') \in T_{(q,w)}P^{reg}$ and we have

$$\begin{aligned} |df_{(q,a)}(q',a')| &= |dh_{(q,w)} \circ dY^{-1}_{(q,a)}(q',a')| \leq c_1 \|dY^{-1}_{(q,a)}(q',a')\|_{\widehat{g}^+} \\ &= c_1 \|(q',w')\|_{\widehat{g}^+} \leq c_1 c_2 \|dY_{q,w}(q',w')\|_{g^\times} = c_1 c_2 \|(q',a')\|_{g^\times}. \end{aligned}$$

This implies that df and thus also dF is bounded on D_ε as desired. □

Chapter 3

Interior Reconstruction

3.1 Construction of the topology

We aim to reconstruct the topological and differential data of V. To that end we define the following functions.

For $q \in V$ we define the function $F_q : S^{n-1} \to [-T_{S^{n-1}}, 0]$ by $a \mapsto f_a(q)$ as in the previous chapter. We let $\mathcal{C}^\infty(S^{n-1})$ be the space of continuous function $F : S^{n-1} \to [-T_{S^{n-1}}, 0]$ which are smooth on a dense open set in S^{n-1}. We can then endow this space with the metric

$$d(F, G) := d_\infty(F, G) + \int_{S^{n-1}} \|dF_a - dG_a\|_{g_{S^{n-1}}} da,$$

where $d_\infty(F, G) := \max_{a \in S^{n-1}} |F(a) - G(a)|$. Note that by definition of $\mathcal{C}^\infty(S^{n-1})$ the subset of S^{n-1} where F or G are not smooth is a null set, making the integral well-defined.

We can then define the function

$$\begin{aligned} \mathcal{F} : V &\to (\mathcal{C}^\infty, d) \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in V$ to the function $F_q : S^{n-1} \to \mathbb{R}$.

The following argument establishes that the canonical topological structure on $\mathcal{F}(V)$, i.e. the topology obtained by taking the subspace topology wrt. the topology induced by d on $\mathcal{C}^\infty$, is the same as the pushforward under $\mathcal{F}$ of the topology on V, making $\mathcal{F}$ a homeomorphism. ((Explain that data allows us to determine d_∞))

Lemma 3.1.1. *The map $\mathcal{F} : V \to \widehat{V} := \mathcal{F}(V)$ is a well-defined continuous and bijective.*

Proof. First of all we show that $\mathcal{F} : V \to (\mathcal{C}^\infty, d)$ is well-defined. Let $q \in V$, then F_q is continuous by proposition 2.3.3, and smooth on a dense open set (namely $\mathcal{A}^{reg}(q)$) by proposition 2.4.6.

To prove that $\mathcal{F}$ is continuous we let $q_n \to q_0 \in V$. By proposition 2.3.4 $F_{q_n} \to F_{q_0}$ uniformly and thus $d_\infty(F_{q_n}, F_{q_0}) \to 0$. Now we need to show that

$$\int_{S^{n-1}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \to 0.$$

To that end let $\varepsilon > 0$ and δ_1 such that $\overline{B_{\delta_1}(q_0)} \subset V$. Because $q_n \to q_0$, after possibly discarding finitely many q_n we may assume $q_n \in B_{\delta_1}(q_0)$. Because $dF_q|_a$ is bounded on D_{δ_1} by proposition 2.4.11 there exists a $c > 0$ such that $\| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} < c$ for all $n \in \mathbb{N}$ and $a \in \mathcal{A}^{reg}(q_n) \cap \mathcal{A}^{reg}(q_0)$.

On the other hand because $\mathcal{A}^{reg}(q_0)$ is dense and open in S^{n-1} we have $\int_{\mathcal{A}^{reg}(q_0)} da = \int_{S^{n-1}} da$. Hence we can find an open set $A \in S^{n-1}$ such that $\overline{A} \subset \mathcal{A}^{reg}(q_0)$ and

$$\int_{S^{n-1} \setminus \overline{A}} da < \frac{\varepsilon}{2c}.$$

Applying proposition 2.4.7 to A yields a $N \in \mathbb{N}$ such that for all $n \geq N$ we have $F_{q_n}|_{\overline{A}}$ smooth and $\|dF_{q_n}|_a - dF_{q_0}|_a\|_{g_{S^{n-1}}} < \frac{\varepsilon}{2}$ for all $a \in \overline{A}$.

We can now write

$$\begin{aligned}
\int_{S^{n-1}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da &= \int_{S^{n-1} \setminus \overline{A}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \\
&\quad + \int_{\overline{A}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \\
&< \varepsilon
\end{aligned}$$

because

$$\int_{S^{n-1} \setminus \overline{A}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \leq \int_{S^{n-1} \setminus \overline{A}} c da < \frac{\varepsilon}{2} \quad \text{and}$$
$$\int_{\overline{A}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da < \int_{\overline{A}} \frac{\varepsilon}{2} da \leq \frac{\varepsilon}{2}.$$

Because $\varepsilon > 0$ was arbitrary we get $\int_{S^{n-1}} \| \, dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \to 0$ and thus $d(F_{q_n}, F_{q_0}) \to 0$ proving $\mathcal{F}$ is continous.

Finally, injectivity follows from the fact that for any $q, q' \in V$ we have $\mathcal{F}(q) = \mathcal{F}(q') \implies F_q = F_{q'} \implies \mathcal{E}_K(q) = \mathcal{E}_K(q')$ which implies $q = q'$ by proposition 2.3.9. $\square$

However there is still some work required to show that $\mathcal{F}^{-1}$ is continuous on $\widehat{V}$:

Lemma 3.1.2. *Let $F_n \to F_0$ in $\widehat{V}$ then $q_n := \mathcal{F}^{-1}(F_n) \to q_0 := \mathcal{F}^{-1}(F_0)$.*

Proof. Note that by the previous result $\mathcal{F} : V \to \widehat{V}$ is a bijection and thus q_n and q_0 are well defined and we have $F_n = F_{q_n}$ resp. $F_0 = F_{q_0}$. We now aim to find $a_1, a_2 \in \mathcal{A}^{reg}(q_0)$ such that $dF_{q_n}|_{a_i} \to dF_{q_0}|_{a_i}$, allowing us to apply proposition 2.4.10: Let for some set $S \subset S^{n-1}$ we let $\mu(S) := \int_S da$ be the standard set measure and $S^c = S^{n-1} \setminus S$ the complement.

Let

$$A := \mathcal{A}^{reg}(q_0) \cap \bigcap_{n=1}^{\infty} \mathcal{A}^{reg}(q_n)$$

and

$$C = A^c, \quad C_n = \mathcal{A}^{reg}(q_n)^c, \quad C_0 = \mathcal{A}^{reg}(q_n)^c.$$

Because $\mu(\mathcal{A}^{reg}(q_n)) = \mu(\mathcal{A}^{reg}(q_0)) = \mu(S^{n-1}) < \infty$, we have $\mu(C_n) = \mu(C_0) = 0$. This yields

$$\mu(C) = \mu\left(C_0 \cup \bigcup_{n=1}^{\infty} C_n\right) \leq \mu(C_0) + \sum_{n=1}^{\infty} \mu(C_n) = 0$$

and thus $\mu(A) = \mu(S^{n-1}) - \mu(C) = \mu(S^{n-1}) > 0$.

We then define the set of *stragglers* as

$$S(A) := \{a \in A \mid \lim_{n\to\infty} dF_{q_n}|_a \neq dF_{q_0}|_a\}.$$

Because $F_{q_n} \to F_{q_0}$ with respect to d we must have $\int_{S^{n-1}} \| dF_{q_n}|_a - dF_{q_0}|_a \|_{g_{S^{n-1}}} da \to 0$ which implies $\mu(S(A)) = 0$. But now we have $\mu(A \setminus S(A)) > 0$ which implies that there exist two $a_1, a_2 \in A \setminus S(A)$. By definition F_{q_n} is smooth at a_i for all $n \in \mathbb{N}$ and $dF_{q_n}|_{a_i} \to dF_{q_0}|_{a_i}$. Now we can apply proposition 2.4.10 and get $q_n \to q_0$ as desired. □

And we get:

Corollary 3.1.3. *$\mathcal{F} : V \to \widehat{V}$ is a homeomorphism.*

3.2 Smooth Reconstruction

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

3.2.1 Construction of smooth coordinates

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

Definition 3.2.1 (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=0}^n \subset (S^{n-1})^{n+1}$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_K^{reg}(q)$ for all $j = 0, \dots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in (S^{n-1})^{n+1}$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will often denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

Proposition 3.2.2. *Let $\widehat{q} \in \widehat{V}$ then the following holds:*

(1) *Given the data from 2.2.1 we can determine all C^0-observation coordinates around $\widehat{q}$,*

(2) *there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$ and*

(3) *given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data 2.2.1, allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.*

Proof. We begin with some setup: Let $q \in V$. We say that $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ are *associated* with respect to q if $p \in \mu_a$, i.e. $p = \mathcal{E}_a(q)$.

To prove part (1), we let $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. We want to show that for any choice of observers $\overrightarrow{a} = (a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ we can determine if they form C^0-observation coordinates. First of all we need to check whether the associated $p_j = \mathcal{E}_{a_j}(q)$ are regular points, i.e. $p_j \in \mathcal{E}_K^{reg}(q)$. But as $\widehat{q} = \mathcal{F}(q) = F_q$ we can recover $\mathcal{E}_K q = \bigcup_{a \in S^{n-1}} \mu_a(F_q(a))$ and also the associated points $p_j = \mu_{a_j}(F_q(a_j))$. By proposition 2.3.8 this allows us to determine $\mathcal{E}_K^{reg}(q)$ and for all p_j we can then simply check whether they lie in $\mathcal{E}_K^{reg}(q)$.

We now need to check whether there exists an open neighborhood W of $\widehat{q}$ such that the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is injective. By definition we have

$$\mathbf{s}_{\overrightarrow{a}}(\widehat{q}) = (\widehat{q}(a_1), \dots, \widehat{q}(a_n)) = (F_q(a_0), \dots, F_q(a_n))$$

which means that the data allows us to fully determine $\mathbf{s}_{\vec{a}}$ on $\widehat{V}$. But since by corollary 3.1.3, the data allows us do construct the topology on $\widehat{V}$ we can determine whether there exists an open neighborhood W of $\widehat{q}$ such that $\mathbf{s}_{\vec{a}} : W \to \mathbb{R}^n$ is injective and thus open by the invariance of domain theorem.

To show (2) we let again $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. Let $(a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ such that the associated $p_j \in \mathcal{E}_K^{reg}(q)$ and the vectors $\{w_j = w(q, p_j) \mid j = 0, \dots n\}$ are linearly independent. We can find such a set of linearly independent vectors because by proposition 2.3.6 $\mathcal{E}_K^{reg}(q)$ is an open subset of $\mathcal{E}_K(q)$. Now by proposition 2.4.4 the observation time functions $\mathbf{f}_{\vec{a}}$ define smooth coordinates on a neighborhood V_1 of q. Thus $\mathbf{s}_{\vec{a}} \circ \mathcal{F}$ are smooth local coordinates as well making $(\mathbf{s}_{\vec{a}}, \mathcal{F}(V_1))$ C^∞-observation coordinates.

Moving on to part (3): We begin by proving that the set of points in $(\mathcal{E}_K^{reg}(q))^{n+1}$ which yield C^∞-observation coordinates is open and dense in $(\mathcal{E}_K^{reg}(q))^{n+1}$. We consider $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ which are associated. Let

$$K(q) = \{(w_j)_{j=0}^n \mid w_j \in L_q^K M, \rho(q, w_j) > 1, \gamma_{q,w_j}(1) \in K\}$$

and define on $K(q)$ the map

$$\begin{aligned} H : K(q) &\to K^{n+1} \\ (w_j)_{j=0}^n &\mapsto (\gamma_{q,w_j}(1))_{j=0}^n. \end{aligned}$$

We will denote $p_j = \gamma_{q,w_j}(1) = \exp_q(w_j)$. Then by definition $p_j \in \mathcal{E}_K^{reg}(q)$ and $w_j = \Omega(q, p_j)$. As ρ is lower semi-continuous, we see that $K(q) \subset (L_q^K M)^n$ is open by an analogous argument to the one in the proof of 2.3.6. As the exponential map is continuous, H is also continuous. Furthermore as $\Omega : \mathcal{Z} \to L^+V$ is continuous and injective, we can construct a continuous inverse to H, making $H : K(q) \to H(K(q)) = (\mathcal{E}_K^{reg}(q))^{n+1}$ a homeomorphism. We will denote $Y(q) \coloneqq (\varepsilon_U^{reg}(q))^{n+1}$. Note that for all $\widehat{q} \in \widehat{V}$, the data 2.2.1 determine $\mathcal{E}_K^{reg}(q)$ and thus also the set $Y(q) \subset K^n$, where $q = \mathcal{F}^{-1}(\widehat{q})$.

Let us now consider the set

$$K_0(q) = \{(w_j)_{j=1}^n \in K(q) \mid w_1, \dots, w_n \text{ are linearly independent}\}.$$

As linear independence is an open and non-degenerate property $K_0(q)$ is open and dense in $K(q)$. Since H is a homeomorphism, $Y_0(q) = H(K_0(q))$ is open and dense in $Y(q)$ as well.

We can now prove the final part of the proposition: Recall that given C^0-observation coordinates around $\widehat{q}$, we want to determine if they are also C^∞-observation coordinates $\widehat{q}$. To that end, let $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^0-observation coordinates around $\widehat{q} \in W_{\vec{a}}$ with $q = \mathcal{F}^{-1}(\widehat{q})$. By definition we have $p_j \in \mathcal{E}_K^{reg}(q)$ where $p_j = \mathcal{E}_{a_j}(q)$ are associated with a_j and hence $(p_j)_{j=0}^n \subset Y(q)$. In the case where $(p_j)_{j=0}^n \in Y_0(q)$, by proposition 2.4.4, q has a neighborhood $V_1 \subset M$ on which the function $\mathbf{f}_{\overrightarrow{a}} : V_1 \to \mathbb{R}^n$ gives smooth local coordinates. Thus, after possibly restricting $W_{\overrightarrow{a}}$, $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$. We then let $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$, $\overrightarrow{b} \in (S^{n-1})^{n+1}$ be different C^0-observation coordinates around $\widehat{q}$ and let $(\widetilde{p}_j)_{j=0}^n \in Y(q)$ be such that $\widetilde{p}_j$ is associated to b_j. Since all smooth coordinates must be compatible, then $(\widetilde{p}_j)_{j=0}^n \in Y_0(q)$ if and only if

$$\text{The function } \mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1} \text{ is smooth at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ and the Jacobian determinant } \det(D(\mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1})) \text{ at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ is non-zero.} \quad (3.1)$$

Here the "only if"-direction follows from the fact that the nondegeneracy of the Jacobian ensures that the linear independence of the spanning vectors is preserved.

For some $\overrightarrow{p} = (p_j)_{j=0}^n \in Y(q)$ with $\overrightarrow{a}$ associated we define $\mathcal{X}_{\overrightarrow{p}} \subset Y(q)$ to be the set of $(\widetilde{p}_j)_{j=0}^n \in Y(q)$, such that for the associated $\overrightarrow{b}$ there exists $W_{\overrightarrow{b}}$ such that $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$ are C^0-coordinates around $\widehat{q}$ and condition 3.1 is satisfied.

If $\overrightarrow{p} \in Y_0(q)$ we see that $Y_0(q) \subset \mathcal{X}_{\overrightarrow{p}}$. On the other hand $\overrightarrow{p} \notin Y_0(q)$ we have $Y_0(q) \cap \mathcal{X}_{\overrightarrow{p}} = \emptyset$. Since the set $Y_0(q)$ is open and dense in $Y(q)$, we see that $\overrightarrow{p} \in Y_0(q)$ if and only if the interior of $\mathcal{X}_{\overrightarrow{p}}$ is dense subset of $Y(q)$. Since the data 2.2.1 is sufficient to determine $Y(q)$ and $\mathcal{X}_{\overrightarrow{p}}$, we can determine whether $\overrightarrow{p} \in Y_0(q)$ or not. And since, by proposition 2.4.4, the C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q} = \mathcal{F}(q)$ are C^∞-observation coordinates if and only if $\overrightarrow{p} \in Y_0(q)$, where $\overrightarrow{p}$ are associated to $\overrightarrow{a}$ wrt. q, we can determine all C^0-observation coordinates around $\widehat{q}$ which are also C^∞-observation coordinates. □

3.3 Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric on U determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

Lemma 3.3.1. *The data given in 2.2.1 allows us to determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Proof. Let $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ be C^∞-observation coordinates on $\widehat{V}$ and $\widehat{q} \in W_{\overrightarrow{a}}$. We begin by constructing a time orientation on $\widehat{V}$: Let $a_1, a_2 \in \overrightarrow{a}$ and $p_1, p_2 \in U$ be associated wrt. the point $q = \mathcal{F}^{-1}(\widehat{q})$, i.e. $p_i = \mathcal{E}_{a_i}(q)$. Because $\mathbf{f}_{\overrightarrow{a}} = \mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F}$ are smooth coordinates we have that the vectors $w(q, p_1)$ and $w(q, p_2)$ pointing from q to p_i must be non-parallel. Therefore, by equation 2.9 we see that the gradient vectors $\nabla f_{a_i}(q)$ are non-parallel, lightlike and past-pointing. Thus the co-vectors $-ds_{a_1}|_{\widehat{q}}$ and $-ds_{a_2}|_{\widehat{q}}$ are non-parallel lightlike and future-pointing. This follows from the fact that $\mathcal{F}$ is an isometry and the co-vector df_a is the image of ∇f_a under the canonical isomophism. Moreover because the data allows us to fully determine s_{a_1} and s_{a_2} on $\widehat{V}$ (see previous proof) we can also determine ds_{a_1} resp. ds_{a_2}.

The co-vector field $X = (-ds_{a_1}) + (-ds_{a_2})$ is timelike and future-pointing and forms a local time-orientation on $W_{\vec{a}}$. Using bump functions and a partition of unity we can then obtain a time-orientation on the whole of $\widehat{V}$ since all orientations agree where they overlap.

Now we turn our attention to the construction of a metric G which is conformal to $\widehat{g}$: Let again $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ be C^∞-observation coordinates on $\widehat{V}$ with $\widehat{q}_0 \in W_{\vec{a}}$ and $q_0 \in V$ such that $\widehat{q}_0 = \mathcal{F}(q_0)$. As in the previous proof, using the data given in 2.2.1 and the function $\widehat{q}_0 = F_{q_0}$ we can determine $\mathcal{E}_K(q_0), \mathcal{E}_K^{reg}(q_0), \mathcal{D}_K(q_0)$ and $\mathcal{D}_K^{reg}(q_0)$ by 2.3.8.

We then fix the point $\widehat{q}_0 = \mathcal{F}(q_0)$ and the tuple $(p,v) \in \mathcal{D}_K^{reg}(q_0)$. Let $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{p,v}((-\widehat{t},0]) \subset M$ is defined and has no cut point. For $q \in V$, we have that $q \in \gamma_{p,v}((-\widehat{t},0))$ if and only if $(p,v) \in \mathcal{D}_K^{reg}(q)$. Hence for a fixed $(p,v) \in \mathcal{D}_K^{reg}(q_0)$ the data allows us to whether some $\widehat{q} \in W_{\vec{a}}$ has $q = \mathcal{F}^{-1}(\widehat{q}) \in \gamma_{p,v}((-\widehat{t},0))$ by checking if $(p,v) \in \mathcal{D}^{reg}(q)$. This allows us to determine

$$\beta = \{\widehat{q} \in W_{\vec{a}} \mid \widehat{q} = \mathcal{F}(q), \mathcal{D}_K^{reg}(q) \ni (p,v)\} = \mathcal{F}(\gamma_{p,v}((-\widehat{t},0))) \cap W_{\vec{a}}.$$

Therefore, on $W_{\vec{a}} \subset \widehat{V}$ we can find the image, under the map $\mathcal{F}$, of the light-like geodesic segment $\gamma_{p,v}((-\widehat{t},0)) \cap \mathcal{F}^{-1}(W_{\vec{a}})$ that contains $q_0 = \gamma_{p,v}(-t_1)$. Let $\alpha(s), s \in (-s_0, s_0)$ be a smooth path on $W_{\vec{a}}$ such that $\partial_s \alpha(s)$ is never zero, $\alpha((-s_0,s_0)) \subset \beta$ and $\alpha(0) = \widehat{q}_0$. Such a smooth path can, for example be obtained by endowing $\widehat{V}$ with some arbitrary Riemannian metric and parameterizing by arc-length. Then $\widehat{w} = \partial_s \alpha(s)|_{s=0} \in T_{\widehat{q}_0}\widehat{V}$ has the form $\widehat{w} = c\mathcal{F}(\gamma'_{p,v}(-t_1))$ where $c \neq 0$.

Since we can do the above construction for all points $(p,v) \in \mathcal{D}_U^{reg}(q_0)$, we can determine in the tangent space $T_{\widehat{q}_0}\widehat{V}$ the set

$$\Gamma = \mathcal{F}_*(\{cw \in L_{q_0}M \mid \exp_{q_0}(w) \in \mathcal{E}_K^{reg}(q_0), c \in \mathbb{R} \setminus \{0\}\})$$

which is an open, non-empty subset of the light cone at $\widehat{q}_0$ wrt. the metric $\widehat{g}$. But now, since the light cone is determined by a quadratic equation in the tangent space, having an open set Γ determines the whole light cone. By repeating this construction for all points $\widehat{q} \in \widehat{V}$, we can uniquely determine $L\widehat{V}$. Using proposition B.6.3 we can then determine the conformal class of the tensor $\widehat{g} = \mathcal{F}_* g$ in the manifold $\widehat{V}$.

The above shows that the data 2.2.1 determine the conformal class of the metric tensor $\widehat{g}$. And in particular we can construct a metric G on $\widehat{V}$ that is conformal to $\widehat{g}$ and satisfies $G(X,X) = -1$. $\square$

3.4 Overview

We have now gone through all the steps necessary to reconstruct the conformal, differential and topological data of V and will now tie this all together to give a detailed account of the actual reconstruction.

As mentioned in remark 2.2.1 we want to prove the following theorem which implies theorem 1.1.2:

Theorem 3.4.1. *Let (M, g) be a globally hyperbolic Lorentzian manifold and $p^+, p^- \in M, V \subset J(p^-, p^+)$ suitable such that V is an open subset of $J(p^-, p^+)^{\mathrm{o}}$. Then given*

(1) *The smooth manifold K,*

(2) *the conformal class of $g|_K$ and*

(3) *the set of light cone observations $\mathcal{P}_K(V)$*

we can construct a globally hyperbolic Lorentzian manifold $\widehat{V}$ such that there exists a conformal diffeomorphism $\mathcal{F} : V \to \widehat{V}$, which preserves causality.

Proof. To construct the space $\widehat{V}$ which is conformally diffeomorphic to V we follow these steps:

- As $f_a(q) = \min\{s \in [-T_a, 0] \mid \mu_a(s) \in \mathcal{P}_K(q)\}$ we can determine $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$ from $\mathcal{P}_K(V)$.
- Proposition 2.3.8 then allows us to determine $\mathcal{D}_K^{reg}(q)$, $\mathcal{D}_K(q)$ and $\mathcal{E}_K^{reg}(q)$ for a given $\mathcal{E}_K(q) \in \mathcal{E}_K(V)$. We can thus construct $\mathcal{D}_K^{reg}(V)$, $\mathcal{D}_K(V)$ and $\mathcal{E}_K^{reg}(V)$.
- We define the function

$$\begin{aligned} \mathcal{F} : V &\to \mathcal{F}(V) = \widehat{V} \subset (\mathcal{C}^\infty(S^{n-1}), d) \\ q &\mapsto \widehat{q} = F_q = (a \mapsto f_a(q)). \end{aligned}$$

  For a given $\mathcal{E}_K(q)$ we can construct $\widehat{q}$ by $\widehat{q}(a) = f_a(q) = s$ such that $\mu_a(s) \in \mathcal{E}_K(q)$. This allows us to construct the map

$$\begin{aligned} \widetilde{\mathcal{F}} : \mathcal{E}_K(V) &\to \widehat{V} \\ \mathcal{E}_K(q) &\mapsto \widehat{q}. \end{aligned}$$

  And we can thus determine the set $\widehat{V} = \widetilde{\mathcal{F}}(\mathcal{E}_K(V))$.
- By taking the subspace topology with respect to the topology on $\mathcal{C}^\infty(S^{n-1})$ induced by d we can determine a topology on $\widehat{V}$. By corollary 3.1.3 this topology is homeomorphic to the topology on V, making $\mathcal{F}$ a homeomorphism.

- For a given point $\widehat{q} \in \widehat{V}$ we can use proposition 3.2.2 and the data to determine all C^0-observation coordinates around $\widehat{q}$. We can then determine for each of these coordinates if they are also C^∞-observation coordinates, and find at least one such coordinate system since existence is guaranteed. We can repeat that step for each $\widehat{q} \in \widehat{V}$ to find smooth coordinates on $\widehat{V}$, making $\mathcal{F}$ a diffeomorphism.

- Finally we can use lemma 3.3.1 to construct a metric G and time-orientation X on $\widehat{V}$ which is conformal to $\widehat{g} = \mathcal{F}_* g$ and makes $\mathcal{F}$ causal. $\mathcal{F} : (V, g|_U) \to (\widehat{V}, G)$ is thus a causal conformal diffeomorphism as desired.

$\square$

Remark 3.4.2. In the statement of theorem 1.1.2 we required that V be a subset of the interior $J(p^-, p^+)^{\mathrm{o}}$. This is because as we approach the observation set K, the light cone observation sets get increasingly degenerate and loose many of their nice properties for points on the boundary, i.e. if we had a $q \in V \cap K$. This issue will be adressed in the next chapter by smoothing the observation time functions at the boundary.

However if $q \in V$ approaches the past boundary $\mathcal{L}^+_{p^-} \cap I^-(p^+)$ the situation is much simpler: Because we are always away from the set of observers K, the light cone observation sets remain well behaved even for $q \in \mathcal{L}^+_{p^-} \cap I^-(p^+)$. It is thus possible to relax the condition $V \subset J(p^-, p^+)^{\mathrm{o}}$ to $V \subset J(p^-, p^+) \setminus K$ in theorem 1.1.2 with only minor modifications to the proofs.

Chapter 4

Boundary Reconstruction

4.1 Setting

In this section we will examine how we can extend our reconstruction result to the case where the observed set V is no longer contained within the interior of $J(p^-, p^+)$ but is now allowed to extend up to the boundary. In other words we want to recover the conformal structure of $J(p^-, p^+)$ from light cone observations made on the future null boundary $K = J(p^-, p^+) \setminus I^-(p^+)$.

This is complicated by the fact that as $q \in J(p^-, p^+)$ approaches the boundary, the light observation sets $\mathcal{P}_K(q)$ get increasingly warped and is degenerate if q is in the boundary.

Analogous to the interior reconstruction case we will again prove the modified version outlined in remark 2.2.1, and let (M, g) be a globally hyperbolic Lorentzian manifold, with $p^+, p^- \in M, V \in J(p^-, p^+)$ suitable such that $V \in J(p^-, p^+)$ is relatively open.

4.2 Preliminaries

To extend the reconstruction up to the edge of $J(p^-, p^+)$ we will essentially split up the reconstruction into two steps: We will split up V into $V \cap (J(p^-, p^+) \setminus K)$ and $V \cap D$ where D is the set of all points such that F_q has a unique minimum. On $V \cap (J(p^-, p^+) \setminus K)$ we will use the reconstruction result from the previous chapter and on $V \cap D$ we will use the fact that the observation time functions have unique minima to smooth them on the boundary K.

To that end we need to introduce some new concepts:

Definition 4.2.1 (Unique minimum domain)**.** We define the *unique minimum* *domain* $D \subset J(p^-, p^+)$ to be

$$D := \{q \in J(p^-, p^+)^{\mathrm{o}} \cup K \mid F_q \text{ has a unique minimum}\}. \tag{4.1}$$

We will often describe this minimum with

$$(a_q, t_q) = (\arg\min_{a \in S^{n-1}} F_q, \min_{a \in S^{n-1}} F_q).$$

This will be the domain where we will establish our reconstruct procedure. We will see that D an open neighborhood of the boundary K allowing us to reconstruct boundary points. As mentioned remark 3.4.2 the reconstruction from the past chapter can be applied to the whole $J(p^-, p^+) \setminus K$, because D is an open neighborhood of K in $J(p^-, p^+)$ this will allow us to reconstruct set in all of $J(p^-, p^+)$.

Definition 4.2.2 (Constant observation time domain)**.** For some $t_0 \in (T_{S^{n-1}}, 0)$ we define the *constant observation time domain* as

$$T_{t_0} = \{p \in K \mid p = \mu_a(t_0), a \in S^{n-1}\} = K \cap \mathcal{T}^{-1}(\mathcal{T}(p^+) + t_0) \subset K. \tag{4.2}$$

Where the second characterization follows from equation 2.6. Because $\mathcal{T}^{-1}(\mathcal{T}(p^+) + t_0)$ is a cauchy hypersurface and thus a spacelike submanifold, T_{t_0} is a $n-1$ dimensional spacelike submanifold of K (away from its boundary). Thus for every $a \in S^{n-1}$ such that $T_a > t_0$ we can use lemma 2.2.9 to find the unique future-pointing outward null ray $\mathbb{R}_+\nu_{a,t_0} \in L^+_{\Theta(a,t_0)}M$ such that $T_{\Theta(a,t_0)}T_{t_0} = \nu^{\perp}_{a,t_0} \cap T_{\Theta(a,t_0)}K$.

Note that for every $q \in J(p^-, p^+)^{\mathrm{o}}$ and $p = \Theta(a,t) \in \mathcal{P}_K(q)$ we have $t > -T_a$ which implies that p is in the relative interior of T_t. I.e. there exists an open neighborhood $p \in U \subset M$ such that $T_t \cap U$ is a submanifold.

Lemma 4.2.3. *Let $q \in J(p^-, p^+)^{\mathrm{o}}$ with (a_q, t_q) the minimum of F_q and $p_q := \mu_{a_q}(t_q)$. Then we have $p_q \in \mathcal{E}^{reg}_K(q)$ and $v(q, p_q) \in \mathbb{R}_+\nu_{a_q,t_q}$, i.e. if $w_q \in L^K_q M$ is the unique null vector such that $\gamma_{q,w_q}(1) = p$ we have $\gamma'_{q,w_q}(1) \in \mathbb{R}_+\nu_{a_q,t_q}$.*

Proof. Note that we have $t_q = f_{a_q}(q)$ and thus

$$p_q = \mu_{a_q}(t_q) = \mu_{a_q}(f_{a_q}(q)) = \mathcal{E}_{a_q}(q) \in \mathcal{P}_K(q),$$

proving that there exists a $w_q \in L^K_q M$ such that $p_q = \gamma_{q,w_q}(1)$.

Now we need to show that indeed $p_q \in \mathcal{E}^{reg}_K(q)$. We recall that by prop 2.2.14 there exists an open neighborhood $p_q \in U \subset M$ such that $\mathcal{P}_K(q) \cap U$ is the union of N pairwise transversal, spacelike, dimension $n-1$ submanifolds $\mathcal{V}_i$. Because t_q is the minimum of F_q we must have $T_{p_q}\mathcal{V}_i = T_{p_q}T_{t_q}$ for all $i = 1, \dots N$. But because the manifolds must be pairwise transversal, we must have $N = 1$, implying that p_q is a regular point. Together with $p_q \in \mathcal{E}_K(q)$ this yields $p_q \in \mathcal{E}^{reg}_K(q)$.

Finally $\gamma'_{q,w_q}(1) = \mathbb{R}_+\nu_{a_q,t_q}$ follows from the fact that $T_{p_q}\mathcal{V}_1 = T_{p_q}T_{t_q}$. $\square$

Lemma 4.2.4. *For $q_0 = \mu_{a_0}(t_0) \in K$ we have*

$$F_{q_0}(a) = \begin{cases} t_0 & \text{if } a = a_0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We begin with the case $a = a_0$ then $F_{q_0}(a_0) = t_0$ follows immediately from the definition of $f_{a_0}(q_0)$. Note that this also covers the case where $q_0 = p^+$. For the case where $a \neq a_0$ and $q_0 \neq p^+$ we suppose that $F_{q_0}(a) = f_a(q_0) < 0$ by contradiction. Then we have $\tau(q_0, \mathcal{E}_a(q_0)) = 0$ which implies that there exists a null geodesic γ with $\gamma(0) = q_0$ and $p := \mathcal{E}_a(q_0) = \gamma(1)$. If $\gamma'(1) = \mu_a'(f_a(q_n))$ we would have $q_0 \in \mu_a([0,1)) \cap \mu_{a_0}([0,1))$ which is a contradiction to lemma 2.2.2. We must thus have $\gamma'(1) \neq \mu_a'(f_a(q_n))$ but this means there exists a broken null geodesic from q_0 to p^+ which is also a contradiction because $q_0 \in K$ by assumption and $K \cap I^-(p^+) = \emptyset$ by lemma 2.2.2. □

Remark 4.2.5. The previous lemma shows that the observation time functions F_q for $q \in K$ lose many nice properties they had when $q \neq K$. In particular if $q \in K$, then F_q is not continuous at a_0. Furthermore let $q_n = \Theta(a_n, t_0) \to q_0 = \Theta(a_0, t_0)$ with $a_n \notin a_0$, then $F_{q_0}(a_0) = t_0$ but $F_{q_n}(a_0) = 0$ for all $n \in \mathbb{N}$, implying F_{q_n} fails to even converge pointwise to F_{q_0}. Later on we will fix some of these issues by multiplying F_q with a smoothing bump function.

Lemma 4.2.6. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$ and $A \subset S^{n-1}$ an open neighborhood of a_0 then we have $F_{q_n}|_{S^{n-1}\setminus A} \to 0$ uniformly.*

Proof. Because any q_n can either lie in the boundary K or in the interior $J(p^-, p^+)^{\mathrm{o}}$ we can instead look at the subsequences $(q_n)_{n=1}^{\infty} \cap K$, $(q_n)_{n=1}^{\infty} \cap J(p^-, p^+)^{\mathrm{o}}$. If we can prove that both subsequences converge to q_0 then we have also proven that q_n itself converges to q_0.

Hence let now $q_n \to q_0 \in K \setminus p^+$ with $q_n = \mu_{a_n}(t_n) \in K \setminus p^+$. We then have $a_n \to a_0$ and thus $a_n \in A$ for all $n \geq N$ for some $N \in \mathbb{N}$. But by the previous lemma this implies that $F_{q_n}|_{S^{n-1}\setminus A} = F_{q_0}|_{S^{n-1}\setminus A} = 0$ and we are done.

For the other part $q_n \to q_0 \in K \setminus p^+$ with $q_n \in J(p^-, p^+)^{\mathrm{o}}$. We suppose by contradiction that there exists a $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists a $n \geq N$ and a $a \in S^{n-1} \setminus A$ such that $f_a(q_n) < -\varepsilon$. We can thus construct a sequence (a_k, q_k) such that $f_{a_k}(q_k) < -\varepsilon$ for all $k \in \mathbb{N}$. Because f is bounded and S^{n-1} compact there exists a convergent subsequence (a_j, q_j) such that $t_j := f_{a_j}(q_j) \to t' \leq -\varepsilon$, $a_j \to a' \in S^{n-1}\mathcal{A}$ and $q_j \to q_0$. Now we have $\mu_{a_j}(t_j) = \Theta(a_j, t_j) \to \Theta(a', t') = \mu_{a'}(t')$ and

$$0 = \lim_{j\to\infty} \tau(q_j, \mathcal{E}_{a_j}(q_j)) = \lim_{j\to\infty} \tau(q_j, \mu_{a_j}(t_j)) = \tau(q_0, \mu_{a'}(t')).$$

Furthermore because $\mu_{a_j}(t_j) = \mathcal{E}_{a_j}(q_j)$ we have $\mu_{a_j}(t_j) \in J^+(q_j)$. By ((REF)) this implies $\mu_{a'}(t') \in J^+(q_0)$. But this together with $\tau(q_0, \mu_{a'}(t'))$ implies that

$\mu_{a'}(t') = \mathcal{E}_{a'}(q_0)$ and $f_{a'}(q_0) = t' < -\varepsilon$. Finally because $a_0 \in A$ and $a' \in S^{n-1} \setminus A$ we have $a' \neq a_0$ and $f_{a'}(q_0) < 0$, a contradiction to the previous lemma. □

Lemma 4.2.7. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$. Then*

$$\liminf_{n\to\infty} \min_{a \in S^{n-1}} F_{q_n}(a) \geq t_0.$$

Proof. Suppose by contradiction that there exists a convergent subsequence q_k of q_n such that $\min_{a\in S^{n-1}} F_{q_k}(a) \to t' < t_0$. There thus exists a sequence of a_k such that $F_{q_k}(a_k) \to t' < t_0$. Taking subsequences again we get $a_j \to a'$ and $t_j := F_{q_j}(a_j) \to t' < t_0$. Then we have

$$J^+(q_j) \ni \mu_{a_j}(F_{q_j}(a_j)) \to \mu_{a'}(t') \in J^+(q_0)$$

by continuity of μ and ((REF)). We also have

$$0 = \lim_{n\to\infty} \tau(q_j, \mu_{a_j}(F_{q_j}(a_j))) = \tau(q_0, \mu_{a'}(t'))$$

which implies $\mu_{a'}(t') = \mathcal{E}_{a'}(q_0)$ and $F_{q_0}(a') = t' < t_0$. A contradiction because $F_{q_0} \geq t_0$ by lemma 4.2.4. □

Proposition 4.2.8. *Let $q_n \in V \to q_0 = \mu_{a_0}(t_0) \in K \setminus p^+$ then there exists a $\varepsilon > 0$ and a $N \in \mathbb{N}$ such that for all $n \geq N$, F_{q_n} has a unique minimum (a_n, t_n) and $(a_n, t_n) \to (a_0, t_0)$.*

Proof. As in a previous proof we can again separately prove the statement for the cases $q_n \in K$ for all $n \in \mathbb{N}$ and $q_n \notin K$ for all $n \in \mathbb{N}$. If $q_n \in K$ the statement follows immediately. We can thus from now on assume $q_n \notin K$.

First of all we let $O \subset M$ be a open convex neighborhood of q_0. Because $q_n \to q_0$ there exists a N_1 such that $n \geq N$ implies $q_n \in O$.

Recall that the Lorentzian splitting induced a Riemannian metric g^+ on M. For $a \in S^{n-1}, t \in [-T_a, 0]$ let $\nu_{a,t} \in CL^+_{\Theta(a,t)}M$ be the unique outward future pointing null vector orthogonal to T_t at a with $\|\nu_{a,t}\|_{g^+} = 1$, as in definition 4.2.2. We define the map

$$\begin{aligned} X : \mathbb{R}_+ \times \mathcal{S} &\to M \\ (c, a, t) &\mapsto \exp_{\Theta(a,t)}(-c\nu_{a,t}) \end{aligned}$$

which is smooth because $\nu_{a,t}$ varies smoothly in (a, t). We have $X(0, a_0, t_0) = q_0$ and X has invertible differential at $(0, a_0, t_0)$. Therefore there exists a $\varepsilon > 0$ such that $B_\varepsilon(a_0) \times B_\varepsilon(t_0) \subset \mathcal{S}$ and $X : B_\varepsilon(0) \cap \mathbb{R}_+ \times B_\varepsilon(a_0) \times B_\varepsilon(t_0) \to O_\varepsilon$ is a diffeomorphism. Because $-\nu_{a,t}$ is inward pointing we have $O_\varepsilon \subset J(p^-, p^+)$ for $\varepsilon > 0$ small enough. In this case, by the invariance of domain theorem, $O_\varepsilon \subset J(p^-, p^+)$

is a relatively open neighborhood of q_0. After further reducing ε, we can achieve that no two rays intersect in O_ε, i.e.

$$\gamma_{\nu_{a_1,t_1}} \cap \gamma_{\nu_{a_2,t_2}} \cap O_\varepsilon = \emptyset \quad \text{for all } a_1, a_2 \in B_\varepsilon(a_0), t_1, t_2 \in B_\varepsilon(t_0).$$

This possible because around $\Theta(a_0, t_0)$, K is a smooth submanifold. Finally we can reduce $\varepsilon > 0$ to get $O_\varepsilon \subset O$.

Because O_ε is open there exists a $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $q_n \in O_\varepsilon \subset O$. In this case we can write $q_n = X(c_n, a_n, t_n)$. We want to show that there exists a $N_3 \geq N_2$ such that for all $n \geq N_3$, F_{q_n} must have a global minimum in $B_\varepsilon(a_0)$. First of all because $q_n \in J(p^-, p^+)^{\circ}$, F_{q_n} is a continuous function an a compact set. There must thus exists at least one $a'_n \in S^{n-1}$ such that $t'_n := F_{q_n}(a'_n) \leq F_{q_n}(a)$ for all $a \in S^{n-1}$. Note that because t'_n is a minimum, the same argument as in lemma 4.2.3 yields that $\Theta(a'_n, t'_n) \in \mathcal{E}_K^{reg}(q_n)$ and $v(q_n, \Theta(a'_n, t'_n)) \in \mathbb{R}_+ \nu_{a'_n, t'_n}$.

Next we want to show that if n is big enough, any such a'_n must lie in $B_\varepsilon(a_0)$. To that end we first note that $\Theta(a_n, t_n) \in \mathcal{E}_K^{reg}(q_n) \subset \mathcal{P}_K(q_n)$ because q_n and $\Theta(a_n, t_n)$ both lie in the convex neighborhood O. This implies $F_{q_n}(a_n) = t_n$. Because $t_n \in B_\varepsilon(t_0)$ we know that $\min_{a \in S^{n-1}} F_{q_n}(a) = t'_n \leq t_n < t_0 + \varepsilon < 0$. By lemma 4.2.6 we can then find a $N_3 \in \mathbb{N}$ such that $n \geq N_3$ implies $F_{q_n}(a) > t_0 + \varepsilon$ for all $a \in S^{n-1} \setminus B_\varepsilon(a_0)$. But this means that F_{q_n} cannot have a minimum outside of $B_\varepsilon(a_0)$.

Next we want to show that $a'_n = a_n$ and $t'_n = t_n$ implying F_{q_n} has a unique minimum. We have $a'_n \in B_\varepsilon(a_0)$ for $n \geq N_3$. By the previous lemma there exists a N_4 such that $\min_{a \in S^{n-1}} F_{q_n}(a) = t'_n > t_0 - \varepsilon$ for all $n \geq N$. Combining this with $t'_n \leq t_n < t_0 + \varepsilon$ we have $t'_n \in B_\varepsilon(t_0)$. Now $\gamma_{\nu_{a_n,t_n}}$ and $\gamma_{\nu_{a'_n,t'_n}}$ both contain $q_n \in O_\varepsilon$, and have $a_n, a'_n \in B_\varepsilon(a_0)$ and $t_n, t'_n \in B_\varepsilon(t_0)$ this is a contradiction if $a_n \neq a'_n$ or $t'_n \neq t_n$.

Finally $(a_n, t_n) \to (a_0, t_0)$ follows from the fact that X is a diffeomorphism and thus has a continuous inverse. □

By lemma A.0.3 we immediately get:

Corollary 4.2.9. *There exists an open neighborhood $K \setminus p^+ \subset O \subset J(p^-, p^+)$ such that $O \subset D$, i.e. for every $q \in O$, F_q has a unique minimum.*

Proposition 4.2.10. *Let $q_0 \in D$ and $q_n \to q_0$ in V. Then there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies F_{q_n} has a unique minimum (a_n, t_n) and $(a_n, t_n) \to (a_0, t_0)$ where (a_0, t_0) is the unique minimum of F_{q_0}.*

Proof. We may assume $q_0, q_n \in J(p^-, p^+)^{\circ}$ because the case $q_0 \in K \setminus \{p^+\}$ is covered by the previous proposition and $q_0 \in J(p^-, p^+)^{\circ}$ implies $q_n \in J(p^-, p^+)^{\circ}$ eventually because the interior is open.

First we write $t_0 = F_{q_0}(a_0) < F_{q_0}(a'), a_0 \neq a' \in S^{n-1}$ for the unique minimum of F_{q_0}. Let $p_0 = \mathcal{E}_{a_0}(q_0)$, then by lemma 4.2.3, $p_0 \in \mathcal{E}_K^{reg}(q_0)$ and $a_0 \in \mathcal{A}^{reg}(q_0)$. By prop 2.4.6 there exists a $\varepsilon > 0$ such that $f : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$ is smooth. In particular $F_{q_0} = f(q_0, \cdot\,)$ is smooth on $\overline{B_\varepsilon(a_0)} \subset \mathcal{A}^{reg}(q_0)$. Furthermore because a_0 is a local minimum of the smooth F_{q_n} we must have $dF_{q_0}|_{a_0} = 0$ and its hessian $H_{F_{q_0}}(a_0)$ must be positive definite. Because positive definiteness of the hessian is equivalent to it having only positive eigenvalues, there exists $\delta > 0$ such that every eigenvalue of $H_{F_{q_0}}(a)$ is bigger than $c_0 > 0$ for all $a \in \overline{B_\delta(a_0)} \subset \overline{B_\varepsilon(a_0)}$. Hence $H_{F_{q_0}}$ is positive definite and F_{q_0} is convex on all of $\overline{B_\delta(a_0)}$.

By an analogous argument to the one employed in the proof of 2.4.7 we can prove that $F_{q_n}|_{\overline{B_\delta(a_0)}}$ is smooth for n big enough and $H_{F_{q_n}} \to H_{F_{q_0}}$ uniformly on $\overline{B_\delta(a_0)}$. Because every eigenvalue of $H_{F_{q_0}}(a)$ is bigger than $c_0 > 0$ for all $a \in \overline{B_\delta(a_0)}$, there must exist a $N_1 \geq \in \mathbb{N}$ such that $F_{q_n}|_{\overline{B_\delta(a_0)}}$ is smooth and $H_{F_{q_n}}$ has only positive eigenvalues on $\overline{B_\delta(a_0)}$ for all $n \geq N_1$. Therefore F_{q_n} is convex on $\overline{B_\delta(a_0)}$ as well.

Next we prove that F_{q_n} must have all its minima in $B_{\frac{\delta}{2}}(a_0)$. We first note that because a_0 is the unique minimum of F_{q_0} we have $F_{q_0}(a) - F_{q_0}(a_0) > 0$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. Because F_{q_0} is continuous and $S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$ compact there exists a $c_1 > 0$ such that $F_{q_0}(a) > F_{q_0}(a_0) + c_1$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. By proposition 2.3.4, $F_{q_n} \to F_{q_0}$ uniformly. Hence there exists a $N_2 \geq N_1$ such that $F_{q_n}(a_0) \leq F_{q_0}(a_0) + \frac{c_1}{2}$. Thus we have $\min_{a \in S^{n-1}} F_{q_n}(a) \leq F_{q_n}(a_0) \leq F_{q_0}(a_0) + \frac{c_1}{2}$ for all $n \geq N_3$. But again by uniform convergence there exists a $N_3 \geq N_2$ such that $F_{q_n}(a) > F_{q_0}(a_0) + \frac{c_1}{2}$ for all $a \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. Hence F_{q_n} has no global minima in $S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$ for all $n \geq N_3$.

But because F_{q_n} is a continuous function on a compact space there must exist a minimum $a_n \in S^{n-1}$ such that $t_n := F_{q_n}(a_n) \leq F_{q_n}(a')$ for all $a' \in S^{n-1}$. As we just saw we must have $a_n \in S^{n-1} \setminus B_{\frac{\delta}{2}}(a_0)$. But we also proved that F_{q_n} is convex on $B_\delta(a_0)$ which means that a_0 must be the unique minimum of F_{q_n} on $B_\delta(a_0)$. Because F_{q_n} cannot have another minimum outside of $B_{\frac{\delta}{2}}(a_0)$, a_0 must be the unique minimum of F_{q_n} and thus $q_n \in D$.

Finally we prove that $a_n \to a_0$. We suppose by contradiction that a_n does not converge to a_0. Because S^{n-1} is compact there exists a convergent subsequence q_j such that $a_j \to a' \neq a_0$ and $q_j \in D$ for all $j \in \mathbb{N}$. Because $q_0, q_j \in J(p^-, p^+)^{\mathrm{o}}$, we have $F_{q_j}(a_j) \to F_{q_0}(a')$. Furthermore we have $F_{q_j}(a_j) = \min_{a \in S^{n-1}} F_{q_j} \to \min_{a \in S^{n-1}} F_{q_0}$ because $F_{q_j} \to F_{q_0}$ uniformly. But this implies $F_{q_0}(a') = \min_{a \in S^{n-1}} F_{q_0} = F_{q_0}(a_0)$, a contradiction because F_{q_0} was assumed to have a unique minimum. Note that because f is continuous on $J(p^-, p^+)^{\mathrm{o}} \times S^{n-1}$ this implies $t_n = f(q_n, a_n) \to t_0 = f(q_0, a_0)$ as well. $\square$

Note that by lemma A.0.3 this shows that D is open.

4.3 Smoothed Observation Time Functions

In this section we will define "smoothed" observation time functions have more regular properties at the boundary K then the previously used observation time functions.

To that end we define

Definition 4.3.1 (Observation Bump Function)**.** ((Write more neatly)) For $a \in S^{n-1}$ we define the *observation bump function* $\chi_a : S^{n-1} \to [0,1]$ to be a smooth function which varies smoothly in a, has $\chi_a(a') = 0$ if and only if $a' = a$, is symmetric around a and there exist $\varepsilon_1 > 0$ such that $\chi_a(a') = 1$ for all $a' \in S^{n-1} \setminus B_{\varepsilon_1}(a)$ and a $\varepsilon_1 > \varepsilon_2 > 0$ such that $\max_{a' \in B_\varepsilon(a)} \chi_a(a') < \frac{\varepsilon}{T_{S^{n-1}}}$ for all $\varepsilon < \varepsilon_2$.

((More in-depth construction))

Equipped with these functions we can now define the

Definition 4.3.2 (Smoothed Observation Time Function)**.** We define the *smoothed observation time function* as

$$\begin{aligned} h : D \times S^{n-1} &\to [-T_{S^{n-1}}, 0] \\ (q, a) &\mapsto \chi_{a_q}(a) f(q, a) \end{aligned}$$

where a_q is the location of the unique minimum of F_q. Analogous to the previous observation time functions we define $h_a(q) := h(q,a)$ and $H_q(a) := h(q,a)$.

Remark 4.3.3. Note that for $q \in K$ we have $H_q(a) = 0$ for all $a \in S^{n-1}$. Furthermore, because χ_{a_q} is smooth and by proposition 2.4.6 we get that for any $q \in D$, H_q is continuous on S^{n-1} and smooth on $\mathcal{A}^{reg}(q)$ where for any $q \in K$ we define $\mathcal{A}^{reg}(q) = S^{n-1}$.

Proposition 4.3.4. *Let $q_n \in D \to q_0 \in D$. Then $H_{q_n} \to H_{q_0}$ uniformly.*

Proof. Let a_n resp. a_0 be the location of the minimum of F_{q_n} resp. F_{q_0}. We will again treat the cases $q_0 \in K$ and $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ seperately: If $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ there exists a $N_1 \in \mathbb{N}$ such that $q_n \in J(p^-, p^+)^{\mathrm{o}}$ for all $n \geq N_1$. We claim that $h : D \cap J(p^-, p^+)^{\mathrm{o}} \times S^{n-1}$ is a continuous function . This is because $f(q,a)$ is continuous and $q_n \to q_0$ implies $a_n \to a_0$ by the previous lemma, which implies $\chi_{a_n} \to \chi_{a_0}$ because χ_a varies smoothly in $a \in S^{n-1}$. But now we can apply lemma A.0.2 to get $H_{q_n} \to H_{q_0}$ uniformly.

Now we treat the case $q_0 \in K$. We can again split up q_n into two subsequences $q_{i_n} \in J(p^-, p^+)^{\mathrm{o}}$ and $q_{j_n} \in K$. Since we have $H_{q_{j_n}}(a) = 0$ for all $a \in S^{n-1}$, $H_{q_{j_n}} \to H_{q_0}$ follows immediately since we have $H_{q_0}(a) = 0$ for all $a \in S^{n-1}$.

It remains to prove that $H_{q_{i_n}} \to H_{q_0}$ uniformly. To simplify notation we will denote $q_k := q_{i_n}$, and a_k for the location of the unique minimum of F_{q_k}. We want to show that for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $H_{a_k}(a) > -\varepsilon$ for all $a \in S^{n-1}$:

To that end let $\varepsilon > 0$. Because $q_k \to q_0$ implies $a_k \to a_0$ by proposition 4.2.8, there exists a $N_1 \in \mathbb{N}$ such that $a_k \in B_{\frac{\varepsilon}{2}}(a_0)$ for all $k \geq N_1$. Hence we have $B_{\frac{\varepsilon}{2}}(a_0) \subset B_\varepsilon(a_k)$ and we have $\chi_{a_k}(a) < \varepsilon$ for all $a \in B_{\frac{\varepsilon}{2}}(a_0) \subset B_\varepsilon(a_n)$. For any $a \in B_{\frac{\varepsilon}{2}}(a_0)$ we thus have $H_{q_k}(a) = \chi_{a_k}(a) f(q_k, a) > -\varepsilon$ because $f(q_k, a) \in [-T_{S^{n-1}}, 0]$.

It remains to show that there exists a $N_2 \in \mathbb{N}$ such that $H_{q_k}(a) > -\varepsilon$ for all $a \in S^{n-1} \setminus B_{\frac{\varepsilon}{2}}(a_0)$ and $n \geq N_2$. Because $B_{\frac{\varepsilon}{2}}(a_0)$ is an open neighborhood of we can apply 4.2.6 to find a $N_2 \in \mathbb{N}$ with $F_{q_k}(a) > -\varepsilon$ for all $q \in S^{n-1} \setminus B_{\frac{\varepsilon}{2}}(a_0)$ and $n \geq N_2$. Because $\chi_a < 1$ this implies $H_{q_k}(a) > -\varepsilon$ and we are done after setting $N := \max\{N_1, N_2\}$. $\square$

Corollary 4.3.5. *$h : D \times S^{n-1} \to [0, 1]$ is continuous.*

Proof. Let $(q_n, a_n) \to (q_0, a_0) \in D \times S^{n-1}$. The case where $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ was treated in the proof of the previous proposition. We can thus assume $q_0 \in K$. Furthermore we assume $q_n \in J(p^-, p^+)^{\mathrm{o}}$ because if q_n has a subsequence in K it is trivial to show that h converges on this subsequence. Because $h(q_0, a_0) = 1$ for any $a_0 \in S^{n-1}$ it remains to show that $h(q_n, a_n) = H_{q_n}(a_n) \to 1$, which follows immediately from the previous proposition. $\square$

Lemma 4.3.6. *For every $q \in D \cap J(p^-, p^+)^{\mathrm{o}}$ there exists a $\lambda > 0$ such that the map*

$$\begin{aligned} a : B_\lambda(q_0) &\to S^{n-1} \\ q &\mapsto \underset{a \in S^{n-1}}{\arg\min}\, F_q \end{aligned}$$

is smooth.

Proof. Let $q_0 \in D \cap J(p^-, p^+)^{\mathrm{o}}$ with minimum at $a_0 \in S^{n-1}$. Recall that this implies $a_0 \in \mathcal{A}^{reg}(q_0)$. By proposition 2.4.6 there exists a $\varepsilon > 0$ such that $f : \overline{B_\varepsilon(q_0)} \times \overline{B_\varepsilon(a_0)} \to [-T_{S^{n-1}}, 0]$ is smooth. Following an analogous argument to the one used in proposition 4.2.8 we can show that there exists a $\varepsilon > \delta > 0$ such that the map $f : \overline{B_\delta(q_0)} \times \overline{B_\delta(a_0)} \to [-T_{S^{n-1}}, 0]$ has positive definite hessian with respect to a and for every $q \in \overline{B_\delta(q_0)}$ we have $\arg\min_{a \in S^{n-1}} F_q(a) \in \overline{B_\delta(a_0)}$.

We then define the function

$$\begin{aligned} f' : \overline{B_\delta(q_0)} \times \overline{B_\delta(a_0)} &\to T^* S^{n-1} \\ (q, a) &\mapsto dF_q|_a \end{aligned}$$

which is smooth because f is smooth on its domain and has $f'(q_0, a_0) = 0$. Furthermore because f is has a positive definite hessian with respect to a, the non-degeneracy condition of the implicit function theorem is satisfied and can find a $\lambda > 0$ and a smooth map $q \in B_\lambda(q_0) \mapsto a(q) \in B_\delta(a_0)$ such that $f'(q, a(q)) = 0$. Because f is positive definite with respect to a on $B_\delta(a_0)$ and by choice of δ, F_q must have its minimum in $B_\delta(a_0)$ and $a(q)$ must be the location of this minimum as desired. □

Corollary 4.3.7. *Let* $\mathcal{C} := \{(q, a) \in V \times S^{n-1} \mid q \in D \cap J(p^-, p^+)^\circ, a \in \mathcal{A}^{reg}(q)\}$ *then* $h : \mathcal{C} \to [-T_{S^{n-1}}, 0]$ *is a smooth and* $dH_q|(a)$ *is bounded for all* $(q, a) \in \mathcal{C}$.

Proof. As shown in proposition 2.4.6, f is smooth on $\mathcal{C}$. By the previous lemma $q \in D \cap J(p^-, p^+)^\circ \mapsto a(q)$ is smooth as well. Hence the map $h(q, a) = \chi_{a(q)}(a) f(q, a)$ is the product of smooth functions making it smooth itself.

The boundedness of $dH_q|(a)$, follows because $dF_q|(a)$ is bounded by proposition 2.4.11 together with the fact that χ_a has bounded derivative because it is smooth on a compact set. □

Lemma 4.3.8. *We can choose* χ_a *such that for all* $q_n \to q_0 \in K$ *we have*

$$\max_{a \in \mathcal{A}^{reg}(q_n)} \|dH_{q_n}|_{a_n} - 0\|_{g_{S^{n-1}}} \to 0.$$

Proof. ((Todo split up in close to q_0 and far away use that dF grows at most polynomially maybe also prove that $q \mapsto a(q)$ is smooth at boundary as well)) □

4.4 Reconstruction

We can now reconstruct the topological structure of V

Analogous to the reconstruction in the previous chapter we let $\mathcal{C}^\infty(S^{n-1})$ be the space of continuous functions $H : S^{n-1} \to [-T_{S^{n-1}}, 0]$ which are smooth on a dense open set in S^{n-1}. We again endow this space with the metric

$$d(H_1, H_2) := d_\infty(H_1, H_2) + \int_{S^{n-1}} \|dH_1|_a - dH_2|_a\|_{g_{S^{n-1}}} da,$$

where $d_\infty(H_1, H_2) := \max_{a \in S^{n-1}} |H_1(a) - H_2(a)|$. Note that by definition of $\mathcal{C}^\infty(S^{n-1})$ the subset of S^{n-1} where H_1 or H_2 are not smooth is a null set, making the integral well-defined.

For $q \in D$ with minimum $t_q \in [-T_{a_q}, 0]$ at $a_q \in S^{n-1}$ we define

$$\begin{aligned} \mathcal{H} : D &\to \mathcal{S} \times (\mathcal{C}^\infty, d) \\ q &\mapsto (a_q, t_q, H_q) \end{aligned}$$

where $H_q(a) = h(q, a)$ is the smoothed observation time function.

Lemma 4.4.1. *For any $q \in D$ we can recover F_q given only $\mathcal{H}(q)$.*

Proof. First of all, given $\mathcal{H}(q) = (a_q, t_q, H_q)$ we can determine whether $q \in K$ or $q \in J(p^-, p^+)^{\mathrm{o}}$, because $q \in K$ if and only if $\min_{a \in S^{n-1}} H_q(a) = 0$. We can thus treat the cases seperately: If $q \in K$ we have $q = \Theta(a_q, t_q)$ and lemma 4.2.4 allows us to fully reconstruct F_q.

Now for the case where $q \in J(p^-, p^+)^{\mathrm{o}}$: We have $H_q(a) = \chi_{a_q}(a) F_q(a)$ and thus $F_q(a) = \frac{1}{\chi_{a_q}(a)} H_q(a)$. This allows us to reconstruct $F_q(a)$ for all $a \neq a_q$ because $\chi_{a_q}(a) \neq 0$ for all $a \neq a_q$. But by definition we have $F_q(a_q) = t_q$ and we have fully reconstructed F_q. □

We denote $V_1 := V \cap D$

Lemma 4.4.2. *$\mathcal{H} : V_1 \to \widehat{V_1} := \mathcal{H}(V_1) \subset \mathcal{S} \times (\mathcal{C}^\infty, d)$ is well-defined, continuous and bijective.*

Proof. We begin by proving that $\mathcal{H}$ is well-defined. Because $\mathcal{H}$ is defined on D, any $q \in D$ must have a unique minimum making $q \mapsto (a_q, t_q, H_q)$ well-defined. Furthermore we have $H_q \in \mathcal{C}^\infty(S^{n-1})$ by corollary 4.3.7, together with the fact that $H_q = 1$ is also smooth for $q \in K$.

Now we want to prove that for any $q_n \to q_0 \in V_1$ with unique minima at (a_n, t_n) resp. (a_0, t_0) we have $(a_n, t_n) \to (a_0, t_0)$ and $d(q_n, q_0) \to 0$. By proposition 4.2.10 we have $(a_n, t_n) \to (a_0, t_0)$.

By proposition 4.3.4 we have $H_{q_n} \to H_{q_0}$ uniformly, which implies $d_\infty(H_{q_n}, H_{q_0}) \to 0$. It remains to show $\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da \to 0$. We again treat the cases $q_0 \in K$ and $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ seperately: If $q_0 \in J(p^-, p^+)^{\mathrm{o}}$ we can assume without loss of generality that $q_n \in J(p^-, p^+)^{\mathrm{o}}$ as well. Then we can use corollary 4.3.7 and an analogous argument to the one used in lemma 3.1.1 to show that $\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da \to 0$.

It remains to show that

$$\int_{S^{n-1}} \|dH_{q_n}|_a - dH_{q_0}|_a\|_{g_{S^{n-1}}} da = \int_{S^{n-1}} \|dH_{q_n}|_a - 0\|_{g_{S^{n-1}}} da \to 0$$

for $q_0 \in K$. But this follows immediately from lemma 4.3.8.

Finally we show that $\mathcal{H}$ is injective. Note that we proved in the previous lemma that $\mathcal{H}(q) = (a_q, t_q, H_q)$ allows us to determine whether $q \in K$ of $q \in J(p^-, p^+)^{\mathrm{o}}$. If $q \in K$ we have $q = \Theta(a_q, t_q)$ making $\mathcal{H}$ injective on the boundary. If $q \in J(p^-, p^+)^{\mathrm{o}}$, the previous lemma allows us to reconstruct F_q and thus $\mathcal{E}_K(q)$. Because $q \in V$ we can apply proposition 2.3.9 proving that $\mathcal{H}$ is injective. □

Lemma 4.4.3. *Let $q_n \in V_1$ such that $\mathcal{H}(q_n) \to \mathcal{H}(q_0)$ in $\widehat{V_1}$ for some $q_0 \in V_1$. Then also $q_n \to q_0$.*

Proof. By definition we have $\mathcal{H}(q_n) = (a_n, t_n, H_{q_n}) \to (a_0, t_0, H_{q_0}) = \mathcal{H}(q_0)$. Because we can determine from $\mathcal{H}(q_0)$ whether $q_0 \in K$ or $q_0 \in J(p^-, p^+)^{\circ}$ we can treat the two cases seperately. If $q_0 \in J(p^-, p^+)^{\circ}$ we have $\min_{a \in S^{n-1}} H_{q_0}(a) < 1$, then by uniform convergence there exists a $N_1 \in \mathbb{N}$ such that $\min_{a \in S^{n-1}} H_{q_n}(a) < 0$, implying $q_n \in J(p^-, p^+)^{\circ}$ for all $n \geq N_0$. We can then apply an analogous argument to the one used in lemma 3.1.2 to get $q_n \to q_0$.

For the case $q_0 \in K$ we can again split up $\mathcal{H}(q_n)$ into two subsequences, $\mathcal{H}(q_{i_n})$ where $q_{i_n} \in K$ and $\mathcal{H}(q_{j_n})$ where $q_{j_n} \in J(p^-, p^+)^{\circ}$ for all $n \in \mathbb{N}$. We thus have $q_{i_n} = \Theta(a_{i_n}, t_{i_n})$ which implies $q_{i_n} \to q_0$ because $(a_{i_n}, t_{i_n}) \to (a_0, t_0)$. For the other case ((explain more in-depth)) we denote $q_k := q_{j_n} \in J(p^-, p^+)^{\circ}$ and $(a_k, t_k) := (a_{j_n}, t_{j_n})$ to simplify notation. Because $H_{q_k} \to H_{q_0} = 0$ uniformly and there exists a $\varepsilon > 0$ such that $H_{q_k}|_{S^{n-1} \setminus B_\varepsilon(a_k)} = F_{q_k}|_{S^{n-1} \setminus B_\varepsilon(a_k)}$ we also have $\max_{a \in S^{n-1}} F_{q_k} \to 0$. This implies that $d(q_k, K) \to 0$. Now we suppose by contradiction that q_k does not converge to q_0. We thus have a convergent subsequence $q_j \to q' \neq q_0 \in K$ and $a_j \to a_0, t_j \to t_0$. We can the apply proposition 4.2.10 to q_j and q' to find that (a_0, t_0) is also the unique minimum of $F_{q'}$. But because $q' \in K$ we have $q' = \Theta(a_0, t_0) = q_0$, a contradiction. □

Corollary 4.4.4. *$\mathcal{H} : V_1 \to \widehat{V_1}$ is a homeomorphism.*

Remark 4.4.5. Note that we can use 3.1.3 on $V_2 := V \cap (J(p^-, p^+) \setminus K)$ to get a homeomorphism $\mathcal{F} : V_2 \to \widehat{V_2} := \mathcal{F}(V_2)$.

((TODO: Explain more and include V_1 V_2 properties overview and that we given $\mathcal{H}(q_1)$ and $\mathcal{F}(q_2)$ we can determine if $q_1 = q_2$)) ((TODO: Include past boundary of $J(p^-, p^+)$ for now we assume $V \cap \partial J(p^-, p^+)^- = \emptyset$)) We can now reconstruct the topology on V:

Proposition 4.4.6. *A set $O \subset V$ is open if and only if $\mathcal{H}(O \cap V_1) \subset \widehat{V_1}$ and $\mathcal{F}(O \cap V_2) \subset \widehat{V_2}$ is open.*

Proof. For the first direction we suppose that $O \subset V$ is open. Because V_1 and V_2 are open so are $O \cap V_1$ and $O \cap V_2$. But because both $\mathcal{H}$ and $\mathcal{F}$ are homeomorphisms and thus open maps, $\mathcal{H}(O \cap V_1)$ and $\mathcal{F}(O \cap V_2)$ must be open as well.

For the other direction we assume that $\mathcal{H}(O \cap V_1)$ and $\mathcal{F}(O \cap V_2)$ are open. Because $\mathcal{H}$ and $\mathcal{F}$ are bijective and continuous, $O \cap V_1$ and $O \cap V_2$ must be open as well. Furthermore we have $V_1 \cup V_2 = V$ and thus

$$O = O \cap V = O \cap (V_1 \cup V_2) = (O \cap V_1) \cup (O \cap V_2)$$

must be open, as desired. □

Corollary 4.4.7. *Given data 2.2.1 we can determine if a set is open ((TODO))*

Chapter 5

Applications

5.1 Stability Results

The following lemma guarantees that theorem 1.1.2 still applies even if we deviate the metric slightly:

Lemma 5.1.1. *Let (M, g) be a globally hyperbolic manifold with $p^-, p^+ \in M$* suitable*. Furthermore let $V \in J(p^-, p^+)^{\circ}$ such for any $q \in \overline{V}$, no null geodesic starting at q has a conjugate point in K.*

If we vary the metric g slightly to $\widetilde{g} := g + h$ such that

$$|h_{ij}|_q| < \varepsilon, \quad |h_{ij,\alpha}|_q| < \varepsilon, \quad |h_{ij,\alpha,\beta}|_q| < \varepsilon \quad \text{for } i, j, \alpha, \beta \in \{1, \ldots, 1+n\}$$

and $\widetilde{g}$ is smooth and has $h_q = 0$ for all $q \in M \setminus V$. Then if $\varepsilon > 0$ is small enough, $(M, \widetilde{g})$ is globally hyperbolic and p^-, p^+, V are still suitable

Proof. To distinguish between objects defined in terms of g or $\widetilde{g}$ we will add a prescript of the respective metrice, for example ${}_g\exp_q$ is the exponential map defined with respect to g and ${}_{\widetilde{g}}\exp_q$ with respect to $\widetilde{g}$. We begin by showing that for $\varepsilon_1 > 0$, small enough, $(M, \widetilde{g})$ still globally hyperbolic. This follows from ((GEROCHPAPER))

Next we want to show that we also have $p^- \,{}_{\widetilde{g}}\!\ll p^+$, i.e. there exists a timelike (wrt. $\widetilde{g}$) path from p^- to p^+. Because $p^- \,{}_g\!\ll p^+$, there exists a path σ with $\sigma(0) = p^-$ and $\sigma(1) = p^+$ which is timelike wrt. g. Because $[0, 1]$ is compact we can find a $\varepsilon_2 > 0$ such that σ is still timelike wrt. $\widetilde{g}$, and thus $p^- \ll g^+$.

Because $p^- \ll p^+$ and $(M, \widetilde{g})$ globally hyperbolic, the map $a \in S^{n-1} \mapsto {}_{\widetilde{g}}T_a \in (0, \infty)$ as in lemma 2.2.6 is well-defined and continuous making ${}_{\widetilde{g}}\mathcal{S} := \{(a, t) \in S^{n-1} \times [0, \infty) \mid t \in [0, {}_{\widetilde{g}}T_a]\}$ a compact set.

Now because the geodesic equation on $(M, \widetilde{g})$ is a second order ODE with coefficients $\widetilde{g}_{ij}$ and $\widetilde{g}_{ij,\alpha}$ and $h = \widetilde{g} - g$ has compact support, ${}_{\widetilde{g}}\exp : TM \to M$

depends smoothly on $\widetilde{g}_{ij}$ and $\widetilde{g}_{ij,\alpha}$ while $d_{\widetilde{g}}\exp : T(TM) \to TM$ depends smoothly on $\widetilde{g}_{ij}$, $\widetilde{g}_{ij,\alpha}$ and $\widetilde{g}_{ij,\alpha,\beta}$ for some $\varepsilon_3 > 0$ small enough. This also implies that ${}_{\widetilde{g}}\rho(q, w)$ as well as ${}_{\widetilde{g}}T_a$ depend smoothly on $\widetilde{g}$ and its first and second derivatives. Because we have ${}_g\rho(p^+, a) > {}_gT_a$ for all $a \in S^{n-1}$ there exists a $\varepsilon_4 > 0$ such that ${}_{\widetilde{g}}\rho(p^+, a) > {}_{\widetilde{g}}T_a$ for all $a \in S^{n-1}$ and we have proved that p^-, p^+ are suitable.

Note that because $\widetilde{g} = g$ outside of V, we still have $V \subset {}_{\widetilde{g}}J(p^-, p^+)^{\mathrm{o}}$. We can then see that p^-, p^+, V are still suitable with respect to $\widetilde{g}$ and some $\varepsilon_5 > 0$ after noting that ${}_{\widetilde{g}}L^K\overline{V}$ is compact and ${}_g\exp$ has no conjugate points in ${}_gL^K\overline{V}$ by assumption. □

Corollary 5.1.2. *Let (M, g) be a globally hyperbolic manifold with $p^-, p^+ \in M$* suitable *and such that no geodesic starting at p^- has a cut point in $\mathcal{L}^+ \cap J^-(p^+)$. Furthermore let $V \in J(p^-, p^+)^{\mathrm{o}}$ such for any $q \in \overline{V}$, no null geodesic starting at q has a conjugate point in K.*

If we vary the metric g slightly to $\widetilde{g} := g + h$ such that

$$|h_{ij}|_q| < \varepsilon, \quad |h_{ij,\alpha}|_q| < \varepsilon, \quad |h_{ij,\alpha,\beta}|_q| < \varepsilon \quad \textit{for } i, j, \alpha, \beta \in \{1, \ldots, 1+n\}$$

$q \in J(p^-, p^+)^{\mathrm{o}}$ and $\widetilde{g}$ is smooth and has $h = 0$ for all $q \in M \setminus J(p^-, p^+)^{\mathrm{o}}$. Then if $\varepsilon > 0$ is small enough, $(M, \widetilde{g})$ is globally hyperbolic and p^-, p^+, V are still suitable. *Furthermore we have ${}_gJ(p^-, p^+) = {}_{\widetilde{g}}J(p^-, p^+)$ and ${}_gK = {}_{\widetilde{g}}K$.*

Proof. The proof follows from the observation that the fact that $p^\pm$ have no cut points in $\mathcal{L}^\mp_{p^\pm} \cap J^\pm(p^\mp)$ implies ${}_g\exp_{p^\pm} = {}_{\widetilde{g}}\exp_{p^\pm}$ together with an analogous argument to the previous lemma. □

Example 5.1.3. Because Minkowski space $(\mathbb{R}^{1+n}, g_M := -dt^2 + \sum dx^2)$ has no cut points. We can pick any p^-, p^+, V *suitable* such that $V \in J(p^-, p^+)^{\mathrm{o}}$ using the previous lemma we can see that for small deviatations from g_M with support on V, theorem 1.1.2 still applies and we can reconstruct V from the light cone observations on K.

However this example is somewhat limited in scope because such a deviation cannot be physical. To get a physical example we will use the reconstruction result on the Einstein universe:

5.2 Einstein Universe

Definition 5.2.1 (Einstein Universe)**.** Let $(\mathbb{R}, -dt^2)$ be the real line with negatively definite metric $-dt^2$ and (S^n, h) the n-sphere with the canonical Riemannian metric. The $1 + n$ dimensional *Einstein universe* is then defined as the product $(\mathbb{R} \times S^n, -ds^2 \oplus h)$

Remark 5.2.2. We can parameterize S^n by an angle $\alpha \in [0,\pi]$ and a point $\omega \in S^{n-1}$ via the map

$$\begin{aligned} S : [0,\pi] \times S^{n-1} &\to S^n \\ (\alpha, \omega) &\mapsto (\cos\alpha, \sin\alpha\omega) \end{aligned}$$

If for a $X \in S^n$ we write $X = (X_0, \overrightarrow{X}), X_0 \in \mathbb{R}, \overrightarrow{X} \in \mathbb{R}^n$. We can invert S by

$$\alpha = \arccos X_0, \quad \omega = \frac{\overrightarrow{X}}{\|\overrightarrow{X}\|}.$$

S is surjective and smooth but we have

$$(1, 0, \dots, 0) = S(0, \omega) \quad \text{and} \quad (-1, 0, \dots, 0) = S(\pi, \omega) \quad \text{for all } \omega \in S^{n-1},$$

which means S fails to be injective if $\alpha = [0,\pi]$. Nontheless for every $X \in S^n$, α is well defined.

We define

$$\begin{aligned} M^M :=&\{(T, X) \in \mathbb{R} \times S^n \mid T \in (-\pi,\pi), \alpha < \pi - |T|\} \text{ and} \\ \mathcal{J}^+ :=&\{(T, X) \in \mathbb{R} \times S^n \mid T \in (0,\pi), \alpha = \pi - T\} \\ \mathcal{J}^- :=&\{(T, X) \in \mathbb{R} \times S^n \mid T \in (\pi, 0), \alpha = \pi + T\} \\ i^\pm := \{T = \pm\pi, \alpha = 0\} \end{aligned}$$

((Explanation, $\mathcal{J}$ are null infinities and essentially K))

We can now construct our conformal embedding:

Proposition 5.2.3. *Let $(\mathbb{R} \times S^n, g)$ be the $1+n$ dimensional Einstein universe and $(\mathbb{R}^{1+n}, h = dt^2 - dx_n dx^n)$ the $1+n$ dimensional Minkovski space. Then the map*

$$\Psi : M^M \to \mathbb{R}^{1+n} \tag{5.1}$$

$$(T, X) \mapsto \frac{1}{\cos T + X_0}(\sin T, \overrightarrow{X}) \tag{5.2}$$

is a conformal diffeomorphism from M^M to the whole Minkovski space.

Proof. ((Look at Friedrich)) $\square$

Example 5.2.4. vary EU a bit use cor to reconstr show that

Appendix A

Technical Lemmas

Lemma A.0.1 (Transverse Map)**.** *Let $f : M \to N$ be a smooth map transverse to the submanifold $L \subset N$ of codimension k and $f^{-1}(L)$ is nonempty. Then $f^{-1}(L)$ is a codimension k submanifold of M.*

Lemma A.0.2. *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces with X, Y compact. Let $f : X \times Y \to Z$ be a continuous functions and denote $f_x : Y \to Z; y \mapsto f_x(y) := f(x, y)$ for $x \in X$. Let $x_n \to x_0 \in X$ as $n \to \infty$ be a convergent sequence.*

Then $f_{x_n} \to f_{x_0}$ uniformly as $n \to \infty$.

Proof. Let $x_n \to x_0 \in X$ be a convergent sequence. We want to show that for any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon.$$

To that end let $\varepsilon > 0$. Then because X and Y are compact, we can use the Heine-Cantor theorem to get a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \wedge d_Y(y_1, y_2) < \delta \implies d_Z(f_{x_1}(y_1), f_{x_2}(x_2)) < \varepsilon.$$

Now if $N \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta \ \forall n \geq N$ and $y \in Y$ arbitrary we have $d_X(x_n, x_0) < \delta \wedge d_Y(y, y) < \delta$ which implies $d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$. Because $y \in Y$ was arbitrary we also get $max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$ and the proof is complete. □

Lemma A.0.3. *Let A be a first-countable topological space and $P : A \to \{false, true\}$ a property defined for all points $a \in A$. Suppose now that for any converging sequence $a_n \to a_0 \in A$ there exists a $N \in N$ such that $P(a_n)$ is true for all $n \geq N$.*

Then there exists an open neighborhood $O \in A$ of a_0 such that $P(a)$ is true for all $a \in O$.

Appendix B

Causality and Global Hyperbolicity

B.1 Causal Relations

In this first section we will establish which points in a Lorentzian manifold can be connected by timelike or lightlike paths under which circumstances.

We will take (M, g) to be a time-oriented Lorentzian manifold. First we will set up some basic causality structure:

Definition B.1.1. We write

1. $p \ll q$ if $p \neq q$ and there exist a future-pointing timelike curve from p to q,
2. $p < q$ if $p \neq q$ and there exist a future-pointing causal curve from p to q,
3. $p \leq q$ if $p = q$ or $p < q$.

We then define the *chronological future* and *causal future* of a point $p \in M$ as

$$I^+(p) := \{q \in M \mid p \ll q\}$$
$$J^+(p) := \{q \in M \mid p \leq q\}.$$

We can extend these definitions to arbitrary sets by setting $I^+(A) := \bigcup_{p \in A} I^+(p)$ and $J^+(A)$ analogously.

Note that in the Minkowski case $\mathbb{R}^n_1$ the set $I^+(p)$ is open and $J^+(p) = \overline{I^+(p)}$ is closed. Furthermore $I^+(p)$ resp. $J^+(p)$ is the set of all $q \in R^n_1$ such that $\overrightarrow{pq}$ is timelike resp. causal. We will see that under sufficient conditions the first of the above facts also hold in the general case.

Corollary B.1.2. *If $x \ll y$ and $y \leq z$ or $x \leq y$ and $y \ll z$, then $x \ll z$.*

Proof. This follows immediately from proposition **??** $\square$

Let $\mathcal{U} \subset M$ be an open set. Then the *intrinsic* causality relations in $\mathcal{U}$ imply the ones in M. In particular, if we denote by $I^+(A,\mathcal{U})$ the chronological future in $\mathcal{U}$ of the set $A \subset \mathcal{U}$, we have that $I^+(A,\mathcal{U}) \subset I^+(A) \cap \mathcal{U}$.

With this in mind we will now consider the case of a convex set $\mathcal{C}$:

Lemma B.1.3. *Let $\mathcal{C}$ be a convex open set in M, then*

(1) *For $p \neq q$ in $\mathcal{C}$, $q \in J^+(p,\mathcal{C}) \iff \overrightarrow{pq}$ is future-pointing causal.*

(2) *$I^+(p,\mathcal{C})$ is open in $\mathcal{C}$ (hence also in M).*

(3) *$J^+(p,\mathcal{C})$ is the closure in $\mathcal{C}$ of $I^+(p,\mathcal{C})$.*

(4) *The relation $\leq$ is closed on $\mathcal{C}$, i.e. if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{C}$ then $q_n \in J^+(p_n,\mathcal{C})$ for all n implies $q \in J^+(p,\mathcal{C})$.*

(5) *A causal curve α contained in a compact $K \subset \mathcal{C}$ is continuously extendable.*

Proof. Properties (1-3) follow from the fact that the convex open set $\mathcal{C}$ is via the exponential map everywhere diffeomorphic to the tangent space $T_pM \simeq \mathbb{R}^n_1$ and thus the properties of the minkovski space also apply here.

To prove (4) we first note that by (1) we have that $q_n \in J^+(p_n,\mathcal{C})$ implies $\overrightarrow{p_nq_n}$ is future-pointing causal. Now by **??** $(p_n, q_n) \mapsto \overrightarrow{p_nq_n}$ is continuous and thus $\overrightarrow{pq}$ is also future-pointing causal. Fact (4) then follows from again applying property (1).

To prove (5) we suppose that the domain of α is $[0, B)$ where $B < \infty$. As K is compact there exist a sequence $s_i \to B$ such that $\alpha(s_i)$ converges to a point $p \in K$. We must now prove that for any sequence $t_i \to B$ such that $\alpha(t_i) \to q$ we have $p = q$. Assume by contradiction that $p \neq q$. By possibly taking subsequences we can achieve that $s_i \leq t_i \leq s_{i+1}$. Then since α is causal we get $\alpha(s_i) \leq \alpha(t_i) \leq \alpha(s_{i+1})$ and thus $\alpha(t_i) \in J^+(\alpha(s_i),\mathcal{C})$ and $\alpha(s_{i+1}) \in J^+(\alpha(t_i),\mathcal{C})$. By (4) we now have $q \in J^+(p,\mathcal{C})$ and $p \in J^+(q,\mathcal{C})$ which by (1) implies that $\overrightarrow{pq}$ is at the same time, future and past pointing, a contradiction. $\square$

(2) can be generalized:

Lemma B.1.4. *The relation $\ll$ is open; that is if $p \ll q$ there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of p and q respectively such that for any $p' \in \mathcal{U}$ and $q' \in \mathcal{V}$ we still have $p \ll q$.*

Proof. Let σ be a timelike curve from p to q. Let $\mathcal{C}$ be a convex open neighborhood of q and q^- a point on σ which comes before q and still lies in $\mathcal{C}$. Then $I^+(q^-,\mathcal{C})$ is also an open neighborhood of q. If we proceed analogously for p with p^+ and $\mathcal{C}'$. Then we get that $I^-(p^+,\mathcal{C}')$ and $I^+(q^-,\mathcal{C})$ are the neighborhoods we were looking for. $\square$

Note that this lemma implies that $I^+(A)$ is open for any set A.

We can now further develop the topology of causality:

Lemma B.1.5. *For $A \subset M$ we have that:*

(1) $\operatorname{int} J^+(A) = I^+(A)$

(2) $J^+(A) \subset \overline{I^+(A)}$ *with equality iff* $J^+(A)$ *is closed.*

Proof. To prove (1) we first note that $I^+(A)$ is open as remarked above. Also $I^+(A) \subset J^+(A)$ by definition. Now if $q \in \operatorname{int} J^+(A)$, then for a convex neighborhood $\mathcal{C}$ of q, $I^-(q, \mathcal{C})$ contains a point of $J^+(A)$. Hence $q \in I^+J^+(A) = I^+(A)$.

Now to prove part (2): The equality assertion is clear, as $I^+(A) \subset J^+(A)$. Note that is suffices to consider only the case where $A = \{p\}$, since the general case then follows from

$$\bigcup_{p\in A} J^+(p) \subset \bigcup_{p\in A} \overline{I^+(p)} \subset \overline{\bigcup_{p\in A} I^+(p)}.$$

Let us thus consider the case of $\overline{I^+(p)}$. Clearly $p \in \overline{I^+(p)}$. Thus we only need to consider $p < q$. Let σ be a causal path from p to q. Let $\mathcal{C}$ be a convex neighborhood of q and q^- a point lying on γ in $\mathcal{C}$. Now by lemma B.1.3, $q^- \in J^+(p)$ and $I^+(J^+(p)) = I^+(p)$ we have

$$q \in J^+(q^-, \mathcal{C}) = \overline{I^+(q^-, \mathcal{C})} \subset \overline{I^+(J^+(p))} = \overline{I^+(p)}.$$

□

B.2 Causality Conditions

Definition B.2.1 (Strong Causality Condition)**.** We say that the *strong causality condition* holds at $p \in M$ if for any given neighborhood $\mathcal{U}$ of p there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that any causal curve with endpoints in $\mathcal{V}$ lies entirely within $\mathcal{U}$.

Intuitively this condition states that any causal curve which starts arbitrarily close to p and leaves some fixed neighborhood cannot return arbitrarily close to p. In particular this rules out closed causal loops.

The following lemma is in line with this intuition:

Lemma B.2.2. *Suppose the strong causality condition holds on a compact subset K of M. If α is a future-inextendable causal curve that starts in K, then α eventually permanently leaves K. That is, there exists a $s > 0$ such that $\alpha(t) \notin K$ for all $t \geq s$.*

Proof. Assume that the conclusion is false. Thus if the domain of α is $[0, B)$ for $B \leq \infty$, by the compactness of K, there exists a sequence $s_i \to B$ such that $\alpha(s_i) \to p \in K$. Since α has no future endpoint there must be some other sequence $t_j \to B$ such that $\alpha(t_j)$ does not converge to p. After taking further subsequences we can assume that some neighborhood $\mathcal{U}$ of p contains no $\alpha(t_j)$ and the sequences are alternating, i.e. $s_1 < t_1 < s_2 < t_2 < s_3 < \dots$. But now the curves $\alpha|_{[s_k, s_{k+1}}$ always leave the neighborhood $\mathcal{U}$ but return arbitrarily close and thus violated the strong causality condition. $\square$

Under these conditions there exists a very useful lemma for constructing geodesics joining some $p < q$.

Lemma B.2.3. *Suppose the strong causality condition holds on a compact subset $K \subset M$. Let (α_n) be a sequence of future-pointing causal curve segments in K such that $\alpha_n(0) \to p$ and $\alpha_n(1) \to q \neq p$. Then there exists a future-pointing causal broken geodesic γ from p to q and a subsequence (α_m) of (α_n) such that $\lim_{m\to\infty} L(\alpha_m) \leq L(\gamma)$.*

This lemma is proven by leveraging the existence of quasi-limits together with the fact that given the strong causality condition, future inextendable curves must eventually leave a compact set K permanently. This proof can be found in detail in [**oneill**].

B.3 Time Separation Function

There is a natural way to generalize the notion of the separation of points $p \leq q$ in $\mathbb{R}^n_1$ to an arbitrary Lorentzian manifold M.

Definition B.3.1 (Time Separation)**.** Let $p, q \in M$, we define the *time separation* $\tau(p, q)$ from p to q as

$$\tau(p, q) \coloneqq \sup\{L(\alpha) \mid \alpha \text{ is a future-pointing causal curve segment from } p \text{ to } q\}.$$

We have $\tau(p, q) = \infty$ if the length is unbounded and $\tau(p, q) = 0$ if the separation is spacelike, i.e. $q \notin J^+(p)$. Note that for any causal path α the function $s \mapsto \tau(\alpha(0), \alpha(s))$ is monotonously increasing.

Lemma B.3.2. (1) *$\tau(p, q) > 0$ iff $p \ll q$.*

(2) *Reverse triangle inequality: If $p \leq q \leq r$, then $\tau(p, q) + \tau(q, r) \leq \tau(p, r)$.*

Proof. (1) If $\tau(p,q) > 0$ there exists a future-pointing causal curve α from p to q with $L(\alpha) > 0$. Thus α cannot be a null pregeodesic. By proposition **??** there now exists a timelike curve from p to q. The converse follows immediately from the definition.

(2) If there are future-pointing causal curves from p to q and q to r we can pick causal curves α from p to q and β from q to r such that, for an arbitrarily small $\delta > 0$

$$L(\alpha) \geq \tau(p,q) - \delta/2, \quad L(\beta) \geq \tau(q,r) - \delta/2.$$

We then have

$$\tau(p,r) \geq L(\alpha + \beta) = L(\alpha) + L(\beta) \geq \tau(p,q) + \tau(q,r) - \delta$$

for any $\delta > 0$, as required. If there is no future-pointing causal path from WLOG p to q then $\tau(p,q) = 0$ and the result follows immediately. $\square$

Lemma B.3.3. *The time separation function $\tau : M \times M \to [0,\infty]$ is lower semicontinuous.*

Proof. If $\tau(p,q) = 0$ there is nothing to prove. Suppose $q \in I^+(p)$ and $0 < \tau(p,q) < \infty$.

Given $\delta > 0$ we must find neighborhoods $\mathcal{U}, \mathcal{V}$ such that for all $p' \in \mathcal{U}, q' \in \mathcal{V}$ we have $\tau(p',q') > \tau(p,q) - \delta$.

Let α be a timelike curve from p to q with $L(\alpha) > \tau(p,q) - \delta/3$. Let $\mathcal{C}$ be a convex neighborhood of q and q^- on α and in $\mathcal{C}$. Since in convex neighborhoods the map $q' \mapsto L(\sigma_{q^-q'})$, where $\sigma_{q^-q'}$ is the radial geodesic, is continuous there exists a neighborhood $\mathcal{V}$ of q such that for all $q' \in \mathcal{V}$ we have $L(\sigma_{q^-q'}) > L(\sigma_{q^-q}) - \delta/3$.

By analogous argument we get that there exists a p^+ and neighborhood $\mathcal{U}$ of p such that for all $p' \in \mathcal{U}$ we have $L(\sigma_{p'p^+}) > L(\sigma_{pp^+}) - \delta/3$.

Putting this together and using the fact that $L(\sigma_{q^-q}) \geq L(\alpha|_{[q^-,q]})$, resp $L(\sigma_{pp^+}) \geq L(\alpha|_{[p,p^+]})$ we have

$$\begin{aligned}
\tau(p',q') &\geq L(\sigma_{p'p^+}) + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q'}) \\
&> L(\sigma_{pp^+}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q}) - \delta/3 \\
&\geq L(\alpha|_{[p,p^+]}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\alpha|_{[q^-,q]}) - \delta/3 \\
&= L(\alpha) - 2\delta/3 > \tau(p,q) - \delta
\end{aligned}$$

as required. $\square$

B.4 Globally Hyperbolic Manifolds

It is convenient to define

$$J(p,q) := J^+(p) \cap J^-(q)$$

Note that any future-pointing causal path from p to q must be contained in $J(p, q)$.

We can now give a powerful condition as to when the supremal path of $\tau(p, q)$ is actually achieved:

Proposition B.4.1. *For $p < q$, if the set $J(p, q)$ is compact and the strong causality condition holds on it, then there is a causal geodesic from p to q of length $\tau(p, q)$.*

Proof. Let (α_n) be a sequence of future-pointing curve segments from p to q whose lengths converge to $\tau(p, q)$ (the existence of such a sequence is guaranteed as $\tau(p, q)$ is the supremum of such curves). These curves are all in $J(p, q)$ which is compact. Hence, by lemma B.2.3, there exists a broken causal geodesic γ with

$$\tau(p, q) = \lim_{n \to \infty} L(\alpha_n) \leq L(\gamma) \leq \tau(p, q).$$

But now, if γ were to have any actual breaks, by corollary **??** there would exist a longer curve, which is a contradiction. □

Note that this implies in particular that $\tau(p, q)$ is always finite if $J(p, q)$ is compact.

This motivates the following definitions:

Definition B.4.2 (Globally Hyperbolic)**.** A subset $\mathcal{H} \subset M$ is called *globally hyperbolic* if (1) the strong causality conditions holds and (2) for all $p, q \in \mathcal{H}$ with $p < q$, $J(p, q)$ is compact.

Definition B.4.3. Let $\gamma : [0, T]$ be a causal geodesic from $p = \gamma(0)$ to $q = \gamma(T)$. We call γ *maximal* if we have $L(\gamma) = \tau(p, q)$ and hence $L(\gamma|_{[0,t]}) = \tau(p, \gamma(t))$ for all $0 \leq t \leq T$.

Lemma B.4.4. *If $\mathcal{U}$ is globally hyperbolic open set, then the time separation function $\tau : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ is continuous.*

Proof. We know from a previous lemma that τ is always lower semicontinuous. Suppose, for contradiction, that is is not upper semicontinuous at (p, q), i.e. there exists a number $\delta > 0$ and sequences $p_n \to p$ and $q_n \to q$ such that $\tau(p_n, q_n) \geq \tau(p, q) + \delta$ for all n.

Since $\tau(p_n, q_n) > 0$, there exists a causal curve α_n from p_n to q_n such that $L(\alpha_n) > \tau(p_n, q_n) - 1/n$. Because $\mathcal{U}$ is open it contains also the slightly earlier resp. later points $p^- \ll p$, $q^+ \gg q$. As $I^+(p^-)$ resp. $I^-(q^+)$ are open neighborhoods of p resp. q, p_n and q_n are eventually contained in them and we can WLOG assume that they always are. It follows that the curves α_n are all contained in the compact set $J(p^-, q^+)$. Now we can apply lemma B.2.3 to obtain a broken geodesic γ from $p = \lim p_n$ to $q = \lim q_n$ with

$$L(\gamma) \geq \lim_{n \to \infty} L(\alpha_n) \geq \lim_{n \to \infty} \tau(p_n, q_n) \geq \tau(p, q) + \delta.$$

But since δ itself is a curve from p to q this is a contradiction. □

Lemma B.4.5. *If $\mathcal{U} \subset M$ is a globally hyperbolic open set, then the causality relation $\leq$ is closed on $\mathcal{U}$.*

Proof. We again have to show that if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{U}$ and $p_n \leq q_n$ for all n, then also $p \leq q$.

If $p = q$ the result follows immediately. We can thus assume $p \neq q$ and $p_n < q_q$ for all n. Let α_n then be a causal curve from p_n to q_n. As in the preceding proof, all α are in $J(p^-, q^+)$ and by lemma $B.2.3$, there exists a causal curve γ from p to q. This implies $p < q$. $\square$

Remark B.4.6. We can now summarize the results from this section for the case where (M, g) is a globally hyperbolic Lorentzian manifold:

For any $p \in M$, $I^{\pm}(p)$ is open and $J^{\pm}(p)$ is closed with $\operatorname{int} J^{\pm}(p) = I^{\pm}(p)$ and $\overline{I^{\pm}(p)} = J^{\pm}(p)$.

For the time separation function we can say the following:

(1) $\tau(p, q) > 0$ iff $p \ll q$.

(2) $\tau(x, y)$ satisfies the *reverse triangle inequality*:

$$\tau(x, y) + \tau(y, z) \leq \tau(x, z) \quad \text{for } x \leq y \leq z.$$

(3) $(x, y) \mapsto \tau(x, y)$ is continuous in $M \times M$.

(4) For $x < y$ there exists a causal geodesic γ from x to y such that $L(\gamma) = \tau(x, y)$.

B.5 Light Cones

Definition B.5.1 (Light Cones)**.** Let

$$L_pM := \{v \in T_pM \setminus \{0\} \mid \langle v, v\rangle = 0\}$$

be the set of null vectors at $p \in M$. We can split L_pM into L_p^+M and L_p^-M the future- and past-pointing null vectors. Furthermore we can define the bundle $LV := \bigcup_{p \in V} L_pV \subset TM$.

We now define the *future light cone* of $p \in M$ to be

$$\mathcal{L}_p^+ := \exp_p(L_p^+M) \cup \{p\}.$$

$\mathcal{L}_p^-$ is defined analogously.

Note that for $p \in M$ we have $\mathcal{L}^+{}_p \subset J^+(p)$ and $\mathcal{L}^+{}_p \supset J^+(p) \setminus I^+(p)$ if M is globally hyperbolic.

B.5.1 Null Cut Points

To better understand the behavior of null geodesics we will introduce so called *cut points* which intuitively are the points where a null geodesic stops being maximal. Such cut points are the product of curvature as in the minkovski case there are none.

For $(p, v) \in TM$ with $v \neq 0$ let $\mathcal{T}(x, v) \in (0, \infty]$ be the maximal value for which $\gamma_v : [0, \mathcal{T}(x, v))$ is defined.

Definition B.5.2 (Cut Locus Function and Cut Points)**.** For $(p, v) \in L^+M$ we define the *cut locus function*

$$\rho(p, v) \coloneqq \sup\{s \in [0, \mathcal{T}(p, v)) \mid \tau(x, \gamma_v(s)) = 0\}.$$

The points $x_1 = \gamma_v(t_1), x_2 = \gamma_v(t_2), t_1 < t_2 \in [0, t_0]$ are called *cut points* on $\gamma_v([0, t_0])$ if $t_2 - t_1 = \rho(x_1, v_1)$ for $v_1 = \gamma_v'(t_1)$. In particular, the point $p(x, v) = \gamma_v(s)|_{s=\rho(x,v)}$, if it exists, is called the *first cut point* on the geodesic γ_v.

Lemma B.5.3. *Let $p < q \in M$. Suppose there are two distinct future-pointed null geodesics $\alpha : [0, a) \to M, \beta : [0, b) \to M$ from $p = \alpha(0) = \beta(0)$ through $q = \alpha(1) = \beta(1)$. Then both geodesics have a cut point in $[0, 1]$, i.e. q comes on or after the first cut point.*

Proof. We will show that for any $s \in (1, a)$ we have $\tau(p, \alpha(s)) > 0$ since this implies that α must have a cut point at or before 1. Let $\gamma = \beta|_{[0,1]} + \alpha|_{(1,a)}$ be the broken null geodesic obtained by traveling from p to q on β and then continuing on α. Thus for any $s \in (1, a)$, $\gamma|_{[0,s]}$ is a broken null geodesic and by proposition **??** there exists a timelike curve from p to $\gamma(s) = \alpha(s)$ which implies $\tau(p, \alpha(t)) > 0$ as required.

The proof for β follows analogously. $\square$

Lemma B.5.4. *Let now (M, g) be globally hyperbolic, and let $p < q \in M$ with $\tau(p, q) = 0$. Assume that $p_n \to p$ and $q_n \to q$ with $p_n \leq q_n$. Let γ_n be maximal geodesics joining p_n to q_n with initial direction v_n. Then the set (v_n) has a limit w and γ_w is a maximal null geodesic from p to q.*

Proof. As in the proof of lemma B.4.4 there exist $p^- \ll p$ $q^+ \gg q$ such that p_n, q_n, γ_n all lie in $J(p^-, q^+)$ which is compact. By lemma B.2.3 there exists a future-pointing broken geodesic λ which is the quasi-limit of γ_n (see [**oneill**]). Thus there exists a convex neighborhood $\mathcal{C}$ of p and a sequence s_n such that $\lim_{n\to\infty} x_n \coloneqq \gamma_n(s_n) \to x = \lambda(s) \in \mathcal{C}$ and $\gamma_n|_{[0,s_n]} \in \mathcal{C}$. Note that since γ_n is a maximal geodesic we have that $\gamma_n|_{[0,s_n]}$ is the unique radial geodesic from p_n to x_n

and we have $v_n = \gamma_n'(0) = \overrightarrow{p_n x_n}$. Now by lemma B.1.3 $(p', q') \to \overrightarrow{p'q'}$ is continuous and we thus have that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \overrightarrow{p_n x_n} = \overrightarrow{px} =: w.$$

By construction, see [**oneill**], $\lambda|_{[0,s]}$ is the radial geodesic in $\mathcal{C}$ from p to x and thus also $\lambda'(0) = \overrightarrow{px} = w$.

It remains to show that λ is an actual unbroken geodesic. But since $L(\lambda) \leq \tau(p, q) = 0$ it follows from proposition **??** that λ must be smooth null geodesic.

Thus also $\lambda = \gamma_w$ since λ is a geodesic with initial velocity w. □

Theorem B.5.5 (Cut Point Characterization). *Let (M, g) be globally hyperbolic. Then for $(x, p) \in L^+M$, $p(x, v)$ is either the first conjugate point on γ_v or the first point on γ_v where there exists another null geodesic γ_w from x to $p(x, v)$ where $v \neq cw$.*

Proof. Let $q = p(x, v) = \gamma_v(t)$ be the first cut point on the null geodesic γ_v. Let furthermore $t_n \to t$ be a monotonously decreasing sequence such that $\gamma_v(t_n)$ is well defined for all n. Now since M is globally hyperbolic there exist maximal geodesics γ_n from p to $q_n := \gamma_v(t_n)$. Note that since $q = \gamma_v(t)$ is the first cut point of γ_v we have $\tau(p, \gamma_v(t_n)) > 0$ for all n. But since γ_v is a null geodesic, it has zero length and cannot be maximal up until any of the t_n. Thus γ_n cannot equal γ_v and in particular $v_n := \gamma_n'(0) \neq v$ for all n. We can apply the previous lemma to obtain a geodesic γ_w and a null vector w such that $v_n \to w$ and γ_w is a maximal geodesic from p to q.

Now we can distinguish to cases: If $v \neq w$ there exist two distinct maximal geodesics, namely γ_v and γ_w joining p and q.

If however, $v = w$ we can view γ_n as a variation of γ_v through geodesics starting at p which additionally satisfy that the limiting variation at q is zero (since the q_n converge to q). q is thus a conjugate point of γ_v. □

Proposition B.5.6. *For (M, g) globally hyperbolic, $\rho(p, v)$ is lower semicontinuous.*

Proof. It suffices to prove that if $(p_n, v_n) \to (p, v)$ in TM and $\rho(p_n, v_n) \to A$ in $\mathbb{R} \cup \{\infty\}$, then $\rho(p, v) \leq A$. If $A = \infty$ there is nothing to prove we will thus assume that $A < \infty$. We further assume $\rho(p, v) > A$ to derive a contradiction.

We can choose a $\delta > 0$ such that $A + \delta < \rho(p, v)$ and $q := \gamma_v(A + \delta)$ exists. We define $b_n = \rho(p_n, v_n) + \delta$ and can force for n large enough $b_n < \rho(p, v)$ and $\gamma_n := \gamma_{v_n}$ defined past b_n. We then denote $q_n = \gamma_n(b_n)$.

Since $b_n > \rho(p_n, v_n)$, γ_n cannot be maximal from p_n to q_n. Now, since M is globally hyperbolic, by B.4.1 we can find maximal null geodesics σ_n from p_n to q_n with initial velocity w_n. By B.5.4 $w_n \to w$ with γ_w a maximal null geodesic from p to q.

Since q cannot be conjugate point (because this would make it a cut point) we cannot have $w_n \to w = v$. Thus we must have $w \neq v$, but this implies that there are two distinct maximal geodesics from p to q, namely γ_v and γ_w, thus $q = \gamma : v(A+\delta)$ must be a cut point of γ_v. This implies that $\rho(p, v) \leq A+\delta$, which is a contradiction since we assumed $A + \delta < \rho(p, v)$. □

B.6 Conformal Structure

Definition B.6.1 (Conformal Diffeomorphism)**.** A map $\Psi : (M_1, g_1) \to (M_2, g_2)$ is called a *conformal diffeomorphism* or *homothety* if $\Psi : M_1 \to M_2$ is a diffeomorphism and $\Psi^* g_2 = e^{2\Omega} g_1$ where $\Omega \in C(M_1)$ and nowhere zero.

We further say that $\Psi : V_1 \to V_2$ *preserves causality* if $x < y$ implies $\Psi(x) < \Psi(y)$.

It can be calculated that the connections D on M_1 and $\widetilde{D}$ on M_2 are related by the following equation:

$$\widetilde{D}_{\Psi_* X} \Psi_* Y = f_* D_X Y + X(\Omega)\Psi_* Y + Y(\Omega)\Psi \qquad \text{(B.1)}$$

Proposition B.6.2. *$\gamma : I \to M_1$ is a null geodesic if, and only if $\sigma \coloneqq \Psi \circ \gamma$ is also a null geodesic.*

Proof. By the symmetry of the situation (i.e. Ψ^{-1} is also a conformal diffeomorphism) is suffices to show only one direction. Suppose now $\gamma : I \to M_1$ is a null geodesic on M_1 and $\sigma = \Psi \circ \gamma$. By the previous equation we have

$$\widetilde{D}_{\sigma'} \sigma'(t) = 2\gamma'(t)(\Omega)\sigma'(t).$$

We can now reparameterize σ such that $2\gamma'(t)(\Omega)$ is always zero and σ is a null geodesic as desired. □

The following proposition asserts that the conformal data of a metric can be reconstructed from knowledge of the null cones:

Proposition B.6.3. *Let M be a smooth manifold of dimension $n \geq 3$ with Lorentzian metrics g and h. Suppose that for any $v \in TM$ we have $g(v, v) = 0$ iff $h(v, v) = 0$. Then there exists a smooth nowhere zero function $\Omega \in C(M)$ such that $g = e^{2\Omega} h$.*

Proof. The proof follows from the fact that the nullcones are given by systems of quadratic equations and some linear algebra. It can be found in more detailed form at [**beem**] □

We can see that even the cut locus is conserved under conformal transformation:

Proposition B.6.4. *Let $\gamma : [0, a) \to (M_1, g_1)$ be a null geodesic with first cut point $q = \gamma(t_0)$. Then $q' = \Psi(q)$ is the first null cut point of $p' = \Psi(p)$ along the null pregeodesic $\Psi \circ \gamma$.*

Proof. We can WLOG (since Ψ either causal or anti-causal and the proof of the anti-causal case is analogous) assume that Ψ is causal and γ is future-pointing. $\Psi \circ \gamma$ is thus also a future-pointed pre-geodesic which can be reparameterized to a null geodesic σ with $p' = \sigma(0)$ and $q' = \sigma(t_1)$. We will denote by τ_j the time separation function on M_j.

We first show that $\tau_2(p', \sigma(t)) = 0$ for $t \in [0, t_1]$, i.e. that q', if it is a cut point, is indeed the first cut point. To obtain a contradiction we assume that there exists a $t \in [0, t_1]$ with $\tau_2(p', \sigma(t)) > 0$. We my thus find a future-pointing causal curve β from p' to $\sigma(t)$ with $L_{g_2}(\beta) > 0$. Now $\Psi^{-1} \circ \beta$ is a future-directed causal curve in M_1 from p to $\Psi^{-1}(\sigma(t))$ with $L_{g_1}(\Psi^{-1} \circ \beta) > 0$. But since $t \leq t_1$ we have $\Psi^{-1}(\sigma(t) = \gamma(t_2)$ with $t_2 \in [0, t_0]$ and thus $\tau_1(p, \gamma(t_2)) > 0$. This would mean that γ has a cut point at t_2, before t_0 which is a contradiction.

We will now show that $\tau_2(q', \sigma(t)) > 0$ for any $t > t_1$, as this would make $q' = \sigma(t_1)$ a future cut point of p along σ as required. Let thus $t > t_1$. There exists a $t_2 > t_0$ such that $\Psi^{-1}(\sigma(t)) = \gamma(t_2)$. Now since $\gamma(t_2)$ lies past the first cut point of γ, we have $\tau_1(p, \gamma(t_2)) > 0$ and there exists a future-pointing causal curve α in M_1 with $L_{g_1}(\alpha) > 0$. Now $\Psi \circ \alpha$ is also a future-pointing causal curve from p' to $\sigma(t)$ with $L_{g_2}(\Psi \circ \alpha) > 0$ and thus $\tau_2(p', \sigma(t)) \geq L_{g_2}(\Psi \circ \alpha) > 0$ as required. $\square$

B.7 Short Cut Argument

Theorem B.7.1. *Let (M, g) be globally hyperbolic and $p < q$ in M, then there exists a future-pointed null geodesic $\gamma : [0, a) \to M$ from $p = \gamma(0)$ to $q = \gamma(t_0)$ and we have $\tau(p, q) = 0$ if and only if γ has no cut points in $[0, t_0)$.*

Proof. The existence of γ is assured by proposition B.4.1. Now suppose we have $\tau(p, q) > 0$ by the continuity of τ there must be a cut point $\gamma(t)$ before q, i.e. $t < t_0$. Suppose on the other hand that γ has a cut point $\gamma(t)$ with $t < t_0$. Then by the definition of cut points we must have $\tau(p, q) > 0$ as $t < t_0$. $\square$

We can apply this theorem to the case of a path from p to q which is the union of the future pointing light-like pregeodesics $\gamma_{p,v}([0, t_0])$ and $\gamma_{x_1,w}([0, t_1])$ where $x_1 = \gamma_{p,v}(t_0), q = \gamma_{x_1,w}(t_1)$. Let $\zeta = \gamma'_{p,v}(t_0)$. If there are no $c > 0$ such that $\zeta = cw$ or equivalently, the union of these two paths is not also a light-like pregeodesic, then we have $\tau(p, q) > 0$. By B.4.6, this implies that there exists a time-like geodesic from p to q and thus $\tau(p, q) > 0$. This is called a *short-cut argument.*