Title

Alexander Uhlmann*
Advisor: Prof. Dr. Peter Hintz

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Abstract

((TODO))

*auhlmann@ethz.ch

Chapter 1

Introduction

1.1 Main Results

((Introduce Notation etc.))

Definition 1.1.1 (Suitable). We call $p^- \ll p^+ \in M$ *suitable* if p^+ has no past cut points in $\mathcal{L}^-_{p^+} \cap J^+(p^-)$ and p^- has no future cut points in $\mathcal{L}^+_{p^-} \cap J^-(p^+)$.

Theorem 1.1.2 (Interior Reconstruction). *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

Now let $V_j \subset J(p_j^-, p_j^+)^{\circ}$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Theorem 1.1.3 (Boundary Reconstruction). *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

Now let $V_j \subset J(p_j^-, p_j^+) \setminus p_j^+$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Chapter 2

Geometric Preliminaries

2.1 Null Conjugate Points

((TODO Connection to cut points)) ((Leave this here?))

Definition 2.1.1 (Null Conjugate Point)**.** Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. We then call $p = \gamma_{q,w}(b)$ a *null conjugate point* if there exists a nontrivial variation $\mathbf{x} : [0, b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma_{q,w}$ through null geodesics such that $\mathbf{x}_v(b, 0) = 0$.

We have the following useful characterization:

Proposition 2.1.2. *Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. Then $p = \gamma_{q,w}(b)$ is a null conjugate point if and only if $\exp_q : L_q M \to M$ is singular at bw, i.e. if there exists a nonzero $\xi \in T_{bw}(L_q M)$ such that $d \exp_q(\xi) = 0$.*

Proof. We begin by proving the backwards direction and to that end assume that there exist a nonzero $\xi \in T_{bw}(L_q M)$ such that $d \exp_q(\xi) = 0$. By the construction of the tangent space there thus exists a non-constant path $\xi : (-\varepsilon, \varepsilon) \to L_q M$ with $\xi(0) = bw$. This allows us to construct the variation $\mathbf{x}(u, v) = \exp_q(\frac{u}{b}\xi(v))$ which has $\mathbf{x}(t, 0) = \gamma_{q,w}(t)$ and is a variation through null geodesics. Finally we have $\mathbf{x}_v(b, 0) = d \exp_q(\xi) = 0$ by the chain rule.

For the other direction we first note that by definition $\mathbf{x}(u, v) = \exp_q(u\mathbf{x}_u(0, v))$ and $\mathbf{x}_u(0, v) \in L_q M$ as $\mathbf{x}$ is a variation through *null* geodesics. Now again by the chain rule we have $0 = \mathbf{x}_v(b, 0) = d \exp_q|_{bw} \circ \frac{d}{dv}(bx_u(0, v))|_{v=0}$. But since $\xi := \frac{d}{dv}(bx_u(0, v))|_{v=0} \in T_{bw}(L_q M)$ we are done. $\square$

Null conjugate points are also conformal invariants:

Proposition 2.1.3. *Let $\Phi : (M, g) \to (N, h)$ be a conformal diffeomorphism and $\gamma : [0, b] \to M$ a null geodesic. Then $\gamma(b)$ is a null conjugate point of γ if and only if $\Psi(\gamma(b))$ is a null conjugate point of $\Psi \circ \gamma$.*

Proof. ((Cite relevant prop)) Because of the symmetry of the situation we only need to prove one direction and suppose that $\gamma(b)$ is a null conjugate point of γ. We thus have a variation $\mathbf{x}$ of γ through null geodesics. But since Φ maps null geodesics to null geodesics, $\Phi \circ \mathbf{x}$ is a variation of $\Phi \circ \gamma$ through null geodesics in N, which implies that $\Phi(\gamma(b))$ is a null conjugate point of $\Phi \circ \gamma$. □

2.2 Geometry of the Light Cone Observations

Remark 2.2.1 (Data). In the following we will use an equivalent formulation to Theorems 1.1.2 and 1.1.3: Namely we will show that if $(M, g), K, V, p^+, p^-$ are as in Theorem 1.1.2 resp. 1.1.3, then given the *data*

(1) The smooth manifold K,

(2) the conformal class of $g|_K$ and

(3) the set of light cone observations $\mathcal{P}_K(V)$

we can construct a space $\widehat{V}$ which is conformally equivalent to V. Notably in Theorems 1.1.2 and 1.1.3, the assumptions assure that for both $(M_i, g_i), K_i, V_i, p_i^+, p_i^-$ we have the same data. Therefore the reconstruction will yield the same $\widehat{V}$ which will then be conformally equivalent to both V_1 and V_2. This in turn implies that V_1 and V_2 are conformally equivalent.

In light of this we will from here on restrict ourselves to only one manifold with $(M, g), K, V, p^+, p^-$ and show how given the *data* we can construct $\widehat{V}$.

2.2.1 Parameterization of Observations

Lemma 2.2.2. *We have:*

(1) $K = J(p^-, p^+) \setminus I^-(p^+)$,

(2) *There exists a surjective smooth map* $\Theta : S^{n-1} \times [0, 1] \to K$ *such that the curves* $\mu_a := t \mapsto \Theta(a, t), a \in S^n$ *are null geodesics,*

$$\Theta(S^{n-1} \times \{1\}) = \{p^+\}, \quad R := \Theta(S^{n-1} \times \{0\}) = K \setminus I^+(p^-)$$

and $\Theta : S^{n-1} \times [0, 1) \to K \setminus p^+$ *is a diffeomorphism.*

(3) $\mathcal{L}^-_{p^+} \cap V = \emptyset$ *and* $\mathcal{L}^-_{p_0} \cap V = J^-(p_0) \cap V = \emptyset \quad \forall p_0 \in R.$

Proof. (1) We first rewrite $J(p^-,p^+)\setminus I^-(p^+) = (J^-(p^+)\setminus I^-(p^+))\cap J^+(p^-)$ and immediately get $(J^-(p^+)\setminus I^-(p^+))\cap J^+(p^-)\subset \mathcal{L}^-_{p^+}\cap J^+(p^-) = K$ as $J^-(p^+)\setminus I^-(p^+)\subset \mathcal{L}^-_{p^+}$. For the other inclusion we note that by assumption for $p\in K$ we have $\tau(p,p^+)=0$ and $p\in\mathcal{L}^-_{p^-}$. This implies $p\in J^-(p^+)\setminus I^-(p^+)$. Furthermore $p\in K$ also implies $p\in J^+(p^-)$. Putting this together we get $p\in (J^-(p^+)\setminus I^-(p^+))\cap J^+(p^-)$ proving the equality.

For (2) we first note that $p^-\notin K$ because $p^+\gg p^-$ implies $\tau(p^-,p^+)>0$ which would make p^- a cut point were it in K, violating our assumption. Thus also $p^-\notin\mathcal{L}^-(p^+)$. This implies that $\mathcal{L}^-_{p^+}$ and $\mathcal{L}^+_{p^-}$ are transversal.

Next we note that the exponential map

$$\exp_{p^+}: L^-_{p^+}M\simeq S^{n-1}\times\mathbb{R}_+\to\mathcal{L}^-_{p^+}$$

is smooth and surjective.

We now aim to construct a smooth surjecive map $\theta: S^{n-1}\times[0,1]\to\exp^{-1}_{p^+}(K)$ which is a diffeomorphism on $S^{n-1}\times[0,1)$. To that end we look at the set of *unit null directions*

$$CL^-_{p^+}M := \{v\in L^-_{p^+}M \mid \|v\|_{g^+}=1\}\simeq S^{n-1}$$

for some riemannian metric g^+ on M. By ((Leavescompact)) for a given null direction $v\in CL^-_{p^+}M$ there exists a $s_v>0$ such that $\gamma_{p^+,v}(s_v)\in J^+(p^-)$ but $\gamma_{p^+,v}(s')\notin J^+(p^-)$ for all $s'>s_v$. Furthermore for any $s'\le s_\mu$ we have $\gamma_{p^+,v}(s')\in J^+(p^-)$ because we can append the lightlike path from p^- to $\gamma_{p^+,v}(s_v)\in J^+(p^-)$ to $\gamma_{p^+,v}|_{[s',s_v]}$ and get a lightlike path from p^- to $\gamma_{p^+,v}(s')$. We also have $\gamma_{p^+,v}(t_v)\notin I^+(p^-)$ because $I^+(p^-)$ is open which would imply the existence of a $t'>t_v$ such that $\gamma_{p^+,v}(t')\in I^+(p^-)\subset J^+(p^-)$ violating the maximality of t_v. Finally, because $\exp_{p^+}$ is transverse to $\mathcal{L}^+_{p^-}$, $exp^-_{p^+}1(\mathcal{L}^+_{p^-})$ is a smooth submanifold of $L^-_{p^+}M$, by lemma A.0.1. This implies that the map that $v\mapsto t_v$ is smooth.

We now define

$$\begin{aligned}\theta: S^{n-1}\times[0,1]&\to\exp^{-1}_{p^+}(K)\\ (v,t)&\mapsto(1-t)s_v v\end{aligned}$$

where we used $CL^-_{p^+}\simeq S^{n-1}$ to identify $v\in S^{n-1}$ with the corresponding $v\in CL^-_{p^+}$. Using the results from the above paragraph it follows that θ is well-defined and has the desired properties.

We now set $\Theta := \exp_{p^+}\circ\,\theta$, which satisfies all properties in (2) and are done with this part.

Finally, for part (3) we assume there exists a $p\in\mathcal{L}^-_{p^+}\cap V$. Recall that $V\subset J(p^-,p^+)^\circ = I^+(p^-)\cap I^-(p^+)$. We thus have $p\in I^+(p^-)\subset J^+(p^-)$, which together with $p\in\mathcal{L}^-_{p^+}$ implies $p\in K$. But now we have $p\in I^-(p^+)$ and $p\in K$, a contradiction to (1).

Now we assume that there exists a $p_0 \in R$ and $p \in J^-(p_0) \cap V$. Because $V \subset I^+(p^-)$ there exists a timelike path from p^- to p. Because $p \in J^-(p_0)$ as well we can construct a timelike path ((REF)) from p^- to p_0 implying $p_0 \in I^+(p^-)$. But because $p \in R = K \setminus I^+(p^-)$ this is a contradiction. $\mathcal{L}^-_{p_0} \subset J^-(p_0)$ yields the second equality. $\square$

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will often treat K itself as a submanifold when it is clear that we are working away from the boundary. This is often the case as by (3) no null geodesic originating from the interior of $J(p^-, p^+)$ can reach p^+ or R, i.e. the boundary of K.

Furthermore by the properties of Θ we have

$$\mu_a([0,1]) \cap \mu_{a'}([0,1]) = \{p^+\} \text{ for } a \neq a' \in S^{n-1} \text{ and} \tag{2.1}$$

$$\bigcup_{a \in S^{n-1}} \mu_a([0,1]) = K \tag{2.2}$$

And finally we can see that we can construct the map Θ and thus the geodesics μ_a using only the data outlined in remark 2.2.1, because we know K and $g|_K$ determines all null geodesics on K

2.2.2 Geometry of Light Observation Sets

Lemma 2.2.3. *For any $q \in V$ the restriction of the exponential map to null vectors* $\exp_q : L^+_q M \to M$ *is* transverse *to K, i.e. for all $w \in L^+_q M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_pK$.*

Proof. In order to achieve a contradiction we assume that there exists a $q \in V$ and a $w \in L^+_q M$ such that with $v := \gamma_{q,w}(1) \in L_pK$. Since K is generated by backwards null geodesics originating at p^+ there exists a $u \in L^-_{p^+}M$ such that there exists a $t \in \mathbb{R}_+$ with $\gamma_{p^+,u}(t) = p, \gamma'_{p^+,u}(t) = -v$. We can thus obtain an unbroken past-pointing null geodesic from p^+ to q by connecting $\gamma_{p^+,u}$ and $\gamma_{p,-v}$. But this implies that $q \in \mathcal{L}^-_{p^+}$ which is a contradiction to 2.2.2(3).

We now prove that this implies that $\exp_q : L^+_q M \to M$ is transverse to K. We need to show that for every $w \in L^+_q M$ with $\exp_q(w) = p \in K$ we have

$$\operatorname{im}(d\exp_q|_w) \oplus T_pK = T_pM.$$

As T_pK is a null hypersurface we only need to prove that $\operatorname{im}(d\exp_q|_w)$ contains a null vector which is not a multiple of the null vector $v \in T_pK$ generating $T_pK = v^\perp$. But by the properties of the exponential map, $\operatorname{im}(d\exp_q|_w)$ contains $v' = \gamma'_{q,w}(1) \in T_pM$. And since we just proved that $v' \notin T_pK$, $v + v'$ must be a timelike vector and $\operatorname{im}(d\exp_q|_w) \oplus T_pK = T_pM$, as desired. $\square$

This lemma closely resembles lemma 2.5 in [HU17] with only minor adjustments to adapt it to our case. It is reproduced here for the sake of completeness. This lemma will allow us to reconstruct the direction of incoming light rays at point in $\mathcal{P}_K(q)$ which will locally correspond to the spacelike hypersurface.

Lemma 2.2.4 (Direction Reconstruction)**.** *Let $p \in K$ then there exists a bijection Φ between the space $\mathcal{S}$ of spacelike hyperplanes $S \subset T_pK$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+V \subset T_pM$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto T_pK \cap V^\perp \in \mathcal{S}$.*

Moreover there exists a bijection between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_pM$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \ni S \mapsto S \oplus \operatorname{span}\Phi(S) \in \mathcal{N}$.

Proof. Let $p \in K$, and $S \subset T_pK$ be a spacelike hyperplane. The orthogonal complement $S^\perp \subset T_pM$ then is a two-dimensional lorentzian subspace. Hence there exist four light rays which are multiples of the vectors $V, -V, W, -W$ in $S^\perp$, where we WLOG assume that V and W are future-pointing. Since $T_pK = v^\perp$ for some future-pointing null vector $v \in T_pK$, we have $v \in S^\perp$ and can WLOG assume $\mathbb{R}_+W = \mathbb{R}_+v$, i.e. $\mathbb{R}_+W$ is the ray pointing along the null hypersurface K. This leaves $\mathbb{R}_+V$ as the unique future-pointing outward null ray which is perpendicular to S, and we can thus set $\Phi(S) = \mathbb{R}_+V$.

For to prove Φ is a bijection, we let $0 \neq V \in T_pM$ be an outward future-pointing null vector. In particular this means that $V \notin T_pK$. Thus $S = V^\perp \cap T_pK$ is a spacelike hyperplane in T_pK which satisfies $S = \Phi^{-1}(V)$.

For the last claim we note that the map $\mathcal{N} \ni N \mapsto N^\perp \cap L_p^+M \in \mathcal{V}$ maps a null hypersurface N to the unique ray along a future-pointing outward null generator of N. The inverse of this map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto V^\perp \in \mathcal{N}$. Composition of these maps with Φ yields the desired bijection $\mathcal{N} \to \mathcal{S}$. □

Lemma 2.2.5. *For $q \in V$ and $w \in L_q^+M$ there exists exactly one $t_w \in (0, \infty)$ such that $\gamma_{q,w}(t_w) \in K$.*

Proof. Let $q \in V$ and $w \in L_q^+M$, by ((Leavescompact)) any geodesic starting in the compact set $J(p^-, p^+)$ must eventually leave it, intersecting the boundary. As K is the future boundary of $J(p^-, p^+)$ there exists at least one $t_w \in (0, \infty)$ with $p = \gamma_{q,w}(t_w) \in K$. We now show $\gamma_{q,w}(t') \notin K$ for any other $t' \neq t_w$.

First let us consider the case $t' < t_w$. We can then append $\gamma_{q,w}|_{[t', t_w]}$ to the path $\mu_a|_{[s,1]}$, whre $a \in S^{n-1}, s \in [0, 1]$ such that $\mu_a(s) = p$, to get a broken lightlike path from $\gamma_{q,w}(t')$ to p^+. The fact that this path must be broken follows from the transversality proven in the previous lemma. But the existence of this broken path implies $\tau(\gamma_{q,w}(t'), p^+) > 0$ and thus $\gamma_{q,w}(t') \in I^-(p^+)$. But as $K = J(p^-, p^+) \backslash I^-(p^+)$ we have $\gamma_{q,w}(t') \notin K$

Conversely we now assume $t' > t_w$. Again by the transversality of $\gamma_{q,w}$ to K we get that for $t'-t_w > \varepsilon > 0$ small enough we have $\gamma_{q,w}(t_w+\varepsilon) \notin J(p^-, p^+) = J^+(p^-)\cap J^-(p^+)$ because K is the future boundary of $J(p^-, p^+)$. As any point on $\gamma_{q,w}$ is in $J^+(p^-)$ we must have have $\gamma_{q,w}(t_w+\varepsilon) \notin J^-(p^+)$, i.e. there exists no lightlike path from $\gamma_{q,w}(t_w+\varepsilon)$ to p^+. But if $\gamma_{q,w}(t') \in J^-(p^+)$ there exists a path σ from $\gamma_{q,w}(t')$ to p^+ and we could construct a lightlike path from $\gamma_{q,w}(t_w+\varepsilon)$ to p^+ by appending $\gamma_{q,w}|_{[t_w+\varepsilon,t']}$ to σ, a contradiction. We thus have $\gamma_{q,w}(t') \notin J^-(p^+) \supset J(p^-, p^+) \supset K$, completing the proof. $\square$

Definition 2.2.6 (Observation Preimage)**.** For any $q \in V$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* $L_q^K M$ to be the preimage of K under the exponential map restricted to $L_q^+ M$, i.e.

$$L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K) \subset L_q^+ M$$

Lemma 2.2.7. *For any $q \in V$, the observation preimage $L_q^K M$ is a $n-1$-dimensional submanifold of $L_q^+ M$.*

Furthermore, for any $w \in L_q^K M$ there exist a relatively open neighborhood $O_w \subset L_q^K M$ such that $\exp_q : O_w \to U_w := \exp_q(O_w) \subset \mathcal{P}_K(q)$ is a diffeomorphism.

Proof. By lemma **??**, $\exp_q : L_q^+ M \to M$ is transverse to K (here we treat $L_q^+ M$ and K as submanifolds, because by lemma 2.2.2(3) we can disregard the boundary points). Thus by the preimage lemma A.0.1 $L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K)$ is a $n-1$-dimensional submanifold of $L_q^+ M$.

For the second part let $w \in L_q^K M$, since $p := \exp_q(w) \in K$ and we assumed that such a p cannot be a null conjugate point, we know that $\exp_q : L_q^+ M \to M$ has an invertible differential at w. Thus, by the implicit function theorem, there exists an open neighborhood $O'_w \subset L_q^+ M$ of w such that $\exp_q : O'_w \to \exp_q(O'_w)$ is a diffeomorphism. If we then restrict $\exp_q$ to $O_w := O'_w \cap L_q^K M$ the map is still a diffeomorphism as desired. $\square$

Note that by the invariance of domain theorem U_w is an open submanifold of $\mathcal{P}_K(q)$

Corollary 2.2.8. *The map*

$$\begin{aligned} S^{n-1} \simeq CL_q^+ M &\to L_q^K M \\ w &\mapsto t_w w \end{aligned}$$

where t_w is as in 2.2.5, is a diffeomorphism.

Proof. This result follows immediately from lemma 2.2.5 together with the fact that since K is (away from its boundary) a smooth submanifold, the map $w \mapsto t_w$ is smooth. $\square$

Lemma 2.2.9. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $w_1, \dots, w_N \in L_q^K M$ such that $\exp_q(w_i) = p$. Furthermore for O_{w_i} as in the previous lemma such that $\exp_q : O_{w_i} \to U_{w_i}$ is a diffeomorphism, there exists an open neighborhood $U \subset \mathcal{P}_K(q)$ of p such that*

$$\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^{N} O_{w_i}$$

Proof. Note that the previous corollary immediately yields that $L_q^K M$ is compact. Let $q \in V$, $p \in \mathcal{P}$. We first remark that, by the previous lemma, for any $w \in \exp_q^{-1}(p) \cap L_q^K M$ there exist open neighborhoods $w \in O_w \subset L_q^K M$ and $p \in U_w = \exp_q(O_w) \subset \mathcal{P}_K(q)$ making $\exp_q : O_w \to U_w$ a diffeomorphism.

To show that there can only be finitely many $w \in L_q^K M$ with $\exp_q(w) = p$ we let

$$C := \exp_q^{-1}(p) \cap L_q^K M.$$

As M is hausdorff, p is closed and because $\exp_q$ is continuous, so is C. Now $C \subset L_q^K M$ is a closed subset of a compact space, making C itself compact as well. Now the family $\{O_w \mid w \in \exp_q^{-1}(p) \cap L_q^K M\}$ is an open cover of C. But because C is compact there must exist a finite subcover such that

$$C \subset O := \bigcup_{i=1}^{N} O_{w_i}.$$

We can now make some observations: By definition, for any $w \in L_q^K M \setminus C$ we have $\exp_q(w) \neq p$. And as $\exp_q$ is a diffeomorphism on O_{w_i} for all $i = 1, \dots, N$, it must be injective and we get $\exp_q^{-1}(p) \cap O_{w_i} = \{w_i\}$. We thus have

$$\exp_q^{-1}(p) \cap O = \{w_1, \dots, w_N\}.$$

Furthermore, as $C \subset O$ for any $p \in L_q^K M \setminus O \subset L_q^K M \setminus C$ we still have $\exp_q(w) \neq p$. In other words:

$$\exp_q^{-1}(p) \cap L_q^K M \setminus O = \emptyset.$$

Putting these two observations together we get

$$\exp_q^{-1}(p) \cap L_q^K M = \{w_1, \dots, w_N\},$$

as desired.

To show the second part we denote

$$L^\times := L_q^K M \setminus O \quad \text{and have } L^\times \cap \exp_q^{-1}(p) = \emptyset.$$

Note that $L^\times$ is a closed and thus compact subset of L_q^K. We then endow M with an arbitrary metric d compatible with its topology. This lets us define the continuous function

$$\begin{aligned} g : L^\times &\to \mathbb{R} \\ w &\mapsto d(\exp_q(w), p). \end{aligned}$$

Because $L^\times \cap \exp_q^{-1}(p) = \emptyset$ we have $g(w) > 0$ for all $w \in L^\times$. But now, as $L^\times$ is compact there exists a $\varepsilon > 0$ such that $g(w) = d(\exp_q(w), p) > \varepsilon$ for all $w \in L^\times$. We can now choose

$$U := B_\varepsilon(p) \cap \mathcal{P}_K(q)$$

and get an open neighborhood of p in $\mathcal{P}_K(q)$ with $\exp^{-1}(U) \cap L^\times = \emptyset$. But this means

$$\exp_q^{-1}(U) \cap L_q^K M = O = \bigcup_{i=1}^{N} O_{w_i}$$

completing the proof.

□

We can immediately put these lemmas to use and prove this proposition characterizing the light observation set.

Proposition 2.2.10. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$. There exists an open neighborhood $p \in U \subset \mathcal{P}_K(q)$, a positive integer N and N pairwise transversal, spacelike, codimension 1 submanifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ and $p \in \mathcal{U}_i$ for $i = 1, \ldots, N$.*

Proof. Let $q \in V$ and $p \in \mathcal{P}_K(q)$. By the previous lemma we know that there can only be finitely many $w_1, \ldots, w_n \in L_q^K M$ with $\exp_q(w_i) = p$.

By lemma 2.2.7, for each w_i there exists a neighborhood $O_{w_i} \subset L_q^K M$ of w_i such that $\exp_q : O_{w_i} \to U_{w_i} := \exp_q(O_{w_i})$ is a diffeomorphism. Thus $U_{w_i} \subset \mathcal{P}_K(q)$ is a codimension 1 submanifold of K and we have $\bigcup_{i=1}^N U_{w_i} \subset \mathcal{P}_K(q)$.

Now we use the second part of the previous lemma to obtain an open neighborhood $U \subset \mathcal{P}_K(q)$ of p, such that $\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$. Thus any point $p \in \mathcal{P}_K(q) \cap U$ is contained in some $\mathcal{V}_i$ and we have $\bigcup_{i=1}^N U_{w_i} \supset \mathcal{P}_K(q) \cap U$. We then define

$$\mathcal{U}_i := U \cap U_{w_i}$$

and have

$$\bigcup_{i=1}^N \mathcal{U}_i = \mathcal{P}_K(q) \cap U$$

as desired. Furthermore, because U is an open neighborhood of p, $\mathcal{U}_i$ is still a codimension 1 submanifold of K and $p \in \mathcal{U}_i$.

We show that $\mathcal{U}_i$ is spacelike. To that end let $p \in \mathcal{U}_i$. Note that we have $\mathcal{U}_i \subset K$ and $\mathcal{U}_i \subset U'_{w_i} = \exp_q(O'_{w_i})$, where $w_i \in O'_{w_i} \subset L_q^+ M$ is an open neighborhood of w_i in $L_q^+ M$ such that on O'_{w_i}, exp_q is a diffeomorphism onto its image. Both K and U'_{w_i} are null hypersurfaces around p but by proposition **??** they are transversal and thus cannot be generated by the same null rays. Thus $T_p\mathcal{U}_i = T_pK \cap T_pU'_{w_i}$ can only contain spacelike vectors.

Finally to prove that they are transversal at p, we assume by contradiction that there exist $i \neq j$ such that $T_p\mathcal{U}_i = T_p\mathcal{U}_j$. But by lemma 2.2.4 this would imply that $v_i = c * v_j$ for a $c \in \mathbb{R}_+$, where $v_i = \gamma'(1)_{q,w_i}$. This would imply $w_i = w_j$, a contradiction. □

Definition 2.2.11 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{U} \subset M$ of p such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a $n-1$ dimensional submanifold of M.

Note that $p \in \mathcal{P}_K(q)$ is regular if and only if $N = 1$ for p in the previous proposition.

Corollary 2.2.12. *The subset of regular points, $\mathcal{P}_K^{reg}(q) \subset \mathcal{P}_K(q)$ is open and dense in $\mathcal{P}_K(q)$.*

Proof. The fact that it is open follows immediately from the definition: Let $p \in \mathcal{P}_K(q)$ be regular. There thus exists an open neighborhood $p \in \mathcal{U} \subset M$ such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a submanifold. But now for every point $p' \in \mathcal{U} \cap \mathcal{P}_K(q)$, $\mathcal{U}$ also makes p' a regular point making $\mathcal{U} \cap \mathcal{P}_K(q)$ an open neighborhood of regular points of p. Thus every regular point has an open neighborhood of regular points making the set of regular points itself open.

To prove the set of regular points is dense in $\mathcal{P}_K(q)$ we to show that for every point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $U' \subset \mathcal{P}_K(q)$ contains a regular point. By the previous proposition, for U' small enough we have $\mathcal{P}_K(q) \cap U' = \bigcup_{i=1}^N \mathcal{U}_i$, where $\mathcal{U}_i$ are pairwise transversal. This means their intersection is of lower dimension and

$$\mathcal{U}_i \setminus \bigcup_{j \neq i} \mathcal{U}_j \quad \text{is open and nonempty for every } i = 1, \dots N.$$

((Give name and close to p)) and we can find a $p' \in \mathcal{V}_i$ for some $i \in 1, \dots, N$ such that $p' \notin \mathcal{V}_j$ for $j \neq i$. Thus we can find an open neighborhood $\mathcal{O}'$ around p' such that $\mathcal{O}' \cap \mathcal{P}_K(q) \subset \mathcal{V}_i$ which means p' is a regular point, as desired. □

2.3 Observation Time Functions

Definition 2.3.1 (Observation Time Function)**.** For $a \in S^{n-1}$ the *observation time function* is defined as

$$\begin{aligned} f_a : J(p^-, p^+) &\to [0, 1] \\ q &\mapsto \inf(\{s \in [0, 1] \mid \mu_a(s) \in J^+(q)\} \cup \{1\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) := \mu_a(f_a(q)) \in M$ be the earliest point where μ_a sees light from q.

Lemma 2.3.2. *Let $a \in S^{n-1}$ and $q \in V$. Then*

(1) *It holds that $f_a(q) \in (0, 1)$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[0, 1]$ and strictly increasing on $[f_a(q), 1]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p, q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L_q^+ M$ and $t \in [0, \rho(q, v)]$ such that $p = \gamma_{q,v}(t)$.*

Proof. Let $a \in \mathcal{A}$ and $q \in V$.

We begin by showing (1): Because $q \in V \subset J(p^-, p^+)^{\circ} = I^+(p^-) \cap I^-(p^+)$ we have $q \in I^-(p^+)$ and conversely $p^+ \in I^+(q)$. By ((REF)) we know that $I^+(q)$ is open and thus it forms an open neighborhood of p^+. But as μ_a is a continuous path with $\mu_a(1) = p^+$ there must exist a $t < 1$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. Hence we have $f_a(q) < 1$.

To show $f_a(q) > 0$ we assume $f_a(q) = 0$ to achieve a contradiction. We thus have $0 = \inf\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\}$. This means that there exists a convergent sequence $t_n \searrow 0$ as $n \to \infty$ such that $\mu_a(t_n) \in J^+(q)$ for all n. Because μ_a is continuous and $J^+(q)$ closed we have $p_0 := \mu_a(0) \in J^+(q)$. But $p_0 = \mu_a(0) \in R$ by 2.2.2(2). Hence we get $p_0 \in J^+(q) \cap R$ for $q \in V$, which is a contradiction to 2.2.2(3).

To show (2) we proceed as follows: By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all t_n we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from q to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such t.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[0,1]$ we first note that it is the composition of two continuous functions. Monotony then follows from the reverse triangle inequality together with the fact that μ_a is a null path.

Finally to show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 1]$ we let $f_a \leq t_1 < t_2 \leq 1$. Now by ((REF)) there exists a causal geodesic $\gamma_1 : [0,1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect γ_1 to $\mu_a|_{[t_1,t_2]}$ we get a path γ_2 connecting q to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as μ_a is a null geodesic. Next we argue that γ_2 must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If γ_1 is timelike this observation is trivial as μ_a is lightlike. If however, γ_1 is lightlike (which is only the case if $t_1 = f_a(q)$), this fact follows from the transversality of light cone observations as noted in proposition **??**. This means that γ_2 is a broken causal geodesic, which by ((REF)) implies that there exists a strictly longer timelike path γ_3 connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Next to prove (3): To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q,p) = 0$. But now, by ((REF)) there exists a null geodesic from q to p which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q,p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0,1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q,p) = 0$.

The other equivalence follows from the definition of $\mathcal{P}_K(q)$ together with the definition of cut points. $\square$

By (3) of the above lemma, for any $q \in V$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.3.1 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-1,1] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2.3}$$

Proposition 2.3.3. *The function* $f : V \times S^{n-1} \to [0,1]$; $(q,a) \mapsto f_a(q)$ *is continuous.*

Proof. We want to show that for every convergent sequence $(q_n, a_n) \to (q_0, a_0) \in V \times S^{n-1}$ we have $t_n := f_{a_n}(q_n) \to f_{a_0}(q_0) =: t_0$ as $n \to \infty$. Because the sequence t_n lives in $[0,1]$ it suffices to show that for every convergent subsequence $t_j = f_{a_j}(q_j) \to t'$ we have $t' = t_0$. Note that still $(q_j, a_j) \to (q_0, a_0)$ because they are the subsequence of a convergent sequence. The points of earliest observation converge:

$$\mathcal{E}_{a_j}(q_i) = \mu_{a_j}(f_{a_j}(q_j)) = \mu_{a_j}(t_j) = \Theta(a_j, t_j) \to \Theta(a_0, t') = \mu_{a_0}(t') = p'$$

because $(a_j, t_j) \to (a_0, t')$ and Θ is continuous. The first key observation is that because $q_j \to q_0$ and $J^+(q_i) \ni \mathcal{E}_{a_j}(q_j) \to p'$ ((REF)) implies $p' \in J^+(q_0)$.

Furthermore we have

$$0 = \tau(q_j, \mathcal{E}_{a_j}(q_j)) = \tau(q_j, \Theta(a_j, t_j)) \to \tau(q_0, \Theta(a_0, t')) = \tau(q_0, p') = 0$$

because τ and Φ are continuous.

We can now combine these observations and get: $p' \in \mathcal{L}^+_{q_0}$ because $p' \in J^+(q_0)$ and $\tau(q_0, p') = 0$ imply that there exist a null geodesic from q_0 to p'. $p' \in \mathcal{P}_K(q_0)$ because $p' \in \mu_{a_0}([0,1]) \subset K$ and $p' \in \mathcal{L}^+_{q_0}$. But now lemma 2.3.2(3) yields that $p' = \mathcal{E}_{a_0}(q_0)$ and we get

$$\mu_{a_0}(t') = p' = \mathcal{E}_{a_0}(q_0) = \mu_{a_0}(f_{a_0}(q_0)) = \mu_{a_0}(t_0).$$

Because μ_a is injective we get $t' = t_0$, as desired. Hence every convergent subsequence of t_n goes to t_0 which, by compactness of $[0,1]$, implies that also $f_{a_n}(q_n) = t_n \to t_0 = f_{a_0}(q_0)$, proving that f is continuous.

□

Proposition 2.3.4. *If $q_n \to q_0 \in V$ as $n \to \infty$ and we denote $F_q : S^{n-1} \to \mathbb{R}; a \mapsto f_a(q)$. Then $F_{q_n} \to F_{q_0}$ uniformly over S^{n-1} as $n \to \infty$.*

Proof. Let $q_n \to q_0 \in V$ be a convergent sequence. We can endow M with an arbitrary metric d, which is compatible with the topology. Then there exists an $\varepsilon > 0$ and a $N \in \mathbb{N}$ such that $q_n \in \overline{B_\varepsilon(q_0)}$ for all $n \geq N$. After discarding the first N points of the sequence we may assume that $q_n \in \overline{B_\varepsilon(q_0)}$ $\forall n$.

By the previous proposition

$$f : (\overline{B_\varepsilon(q_0)}, d) \times (S^{n-1}, d_{S^{n-1}}) \to ([0,1], d_{[0,1]})$$

is a continuous function from and to compact spaces. Now we can apply lemma A.0.2 to find that $F_{q_n} \to F_{q_0}$ uniformly.

□

2.3.1 Set of earliest observations

Definition 2.3.5 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned}\mathcal{D}_K(q) = \{(p,v) \in L^+M \mid &(p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L^+_q M, 0 \leq t \leq \rho(q,w)\}, \\ \mathcal{D}^{reg}_K(q) = \{(p,v) \in L^+M \mid &(p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L^+_q M, 0 < t < \rho(q,w)\},\end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}^{reg}_K(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_K(q) = \pi(\mathcal{D}_K(q))$ and $\mathcal{E}^{reg}_K(q) = \pi(\mathcal{D}^{reg}_K(q))$, where $\pi : TM \to M$ is the canonical projection. We say that $\mathcal{E}_K(q)$ is the set of earliest observations and $\mathcal{E}^{reg}_K(q)$ is the set of earliest regular observations of q in K. We denote the collection of earliest observation sets by $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$.

Note that $\mathcal{E}_K(q) = \{\mathcal{E}_a(q) \mid a \in S^{n-1}\}$.

Proposition 2.3.6. *For any $q \in V$ it holds that*

(1) *Let $T = \{p \in \mathcal{L}^+_q \mid \tau(q,p) = 0\}$ then*

$$\mathcal{E}_K(q) = \mathcal{P}_K(q) \cap T \quad \text{and} \quad \mathcal{E}^{reg}_K(q) = \mathcal{P}^{reg}_K(q) \cap T,$$

(2) *$\mathcal{E}^{reg}_K(q)$ is an open subset of $\mathcal{P}^{reg}_K(q)$, and is thus also a $n-1$-dimensional spacelike submanifold of K,*

(3) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points*

(4) *$\overline{\mathcal{E}_K^{reg}(q)}$ is open and dense in $\mathcal{E}_K(q)$,*

(5) *$\mathcal{D}_K^{reg}(q)$ is a nonempty open n-dimensional submanifold of $\overrightarrow{K} := \pi^{-1}(K)$.*

Proof. Let $q \in V$. We first look at a useful relation of the exponential map to cut points: We define $\mathcal{V} := \{w \in L_q^+M \mid \rho(q,w) > 1\}$. By B.5.6, $\rho(q,w)$ is lower semicontinuous and $\mathcal{V}$ is thus open. Furthermore by the definition of cut points, $\mathcal{V}$ is star-shaped around $0 \in L_q^+M$. Because by B.5.5 cut points are exactly the points where $\exp_q$ first fails to be a diffeomorphism, $\exp_q : \mathcal{V} \to \mathcal{W} := \exp_q(\mathcal{V})$ is a diffeomorphism. Furthermore by the invarance of domain theorem we get that $\mathcal{W} \subset \mathcal{L}_q^+$ is relatively open. Note that this also implies that for any $p \in \mathcal{W}$, there exists a $p \in U \subset M$ open such that $p \in \mathcal{L}_q^+ \cap U$ is a n-dimensional submanifold of M.

We can now move on to proving (1): $p \in \mathcal{E}_K(q) \iff p \in \mathcal{P}_K(q) \cap T$ follows immediately lemma 2.3.2(3).

Let $p \in \mathcal{E}_K^{reg}(q)$. By definition this implies that $p \in \mathcal{W}$ and we get an $p \in U \subset M$ open such that $p \in \mathcal{L}_q^+ \cap U$ is a dimension n submanifold. Now, around p, K is also a dimension n submanifold, transversal to $\mathcal{L}_q^+$ and thus $K \cap \mathcal{L}_q^+ \cap U = \mathcal{P}_K(q) \cap U$ is a dimension $n-1$ submanifold around p. Thus p is a regular point, i.e. $p \in \mathcal{P}_K^{reg}(q)$. $\tau(q,p) = 0$ follows immediately from the fact that $\rho(q,w) > 1$, proving the first direction.

To show the reverse direction we assume $p \in \mathcal{P}_K^{reg}(q)$ with $\tau(q,p) = 0$. Because $p \in \mathcal{P}_K^{reg}(q)$ by definition 2.2.11 there exists exactly one $w \in L_q^K M$ such that $\exp_q(w) = p$. From $\tau(q,w) = 0$ we get $\rho(q,p) \geq 1$. Now if $\rho(q,w) = 1$, p would be a cut point. By theorem B.5.5 this would mean that either $p \in K$ is a conjugate point to q or there exists a $w \neq w' \in L_q^K M$ with $\exp_q(w') = p$. The first option is impossible because in the statement of theorem 1.1.2 we assumed that no $q \in V$ can have a conjugate point on K. The second option is also impossible because we assumed p to be a regular point in $\mathcal{P}_K(q)$. We thus must have $\rho(q,w) > 1$, implying $p \in \mathcal{E}_K^{reg}(q)$.

We now move on to (2): To prove that $\mathcal{E}_K^{reg}(q)$ is open in $\mathcal{P}_K^{reg}(q)$ we claim that $\mathcal{E}_K^{reg}(q) = \mathcal{P}_K^{reg}(q) \cap \mathcal{W}$. To that end we first note that $\mathcal{E}_K^{reg}(q) \subset \mathcal{W} \subset T$. Recall that by (1) we have $\mathcal{E}_K^{reg}(q) = \mathcal{P}_K^{reg}(q) \cap T$. Applying $\cap\mathcal{W}$ to both sides yields

$$\mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q) \cap \mathcal{W} = \mathcal{P}_K^{reg}(q) \cap T \cap \mathcal{W} = \mathcal{P}_K^{reg}(q) \cap \mathcal{W}$$

as desired.

Proposition 2.2.10 implies that $\mathcal{P}_K^{reg}(q)$ is a $n-1$ dimensional spacelike submanifold of M. Because $\mathcal{W} \subset \mathcal{L}_q^+$ is open and $\mathcal{P}_K^{reg}(q) \subset \mathcal{L}_q^+$, $\mathcal{E}_K^{reg}(q)$ is a relatively open subset of $\mathcal{P}_K^{reg}(q)$, as desired. This also means that $\mathcal{E}_K^{reg}(q)$ itself is a open subset of a $n-1$-dimensional spacelike submanifold of M as well.

We can now tackle (3): Let $p \in \mathcal{E}_K(q)$ be a cut point, then by proposition 2.2.10, there exists an open neighborhood $p \in U \subset M$ and N codimension 1 pairwise transversal manifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$. Because $\tau(q,p) = 0$ and the manifolds are pairwise transversal and intersect at p ((SEE FIGURE)), $\mathcal{E}_K(q)$ must have a sharp edge at p meaning it cannot be a submanifold. For the other direction we assume that $p \in \mathcal{E}_K(q)$ is not a cut point. Then, by definition we have $p \in \mathcal{E}_K(q)$ which is a submanifold.

Moving on to (4), the fact fact that $\mathcal{E}_K^{reg}(q)$ is dense in $\mathcal{E}_K(q)$ follows by an argument which is analogous to the one used in the proof of corollary 2.2.12. To show that it is relatively open in $\mathcal{E}_K(q)$ we use that $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K(q) \cap \mathcal{W}$ with $\mathcal{W}$ open in $\mathcal{L}_q^+$.

Finally the proof of (5) is analogous to (2) with the difference in submanifold dimension originating from the face that for any $(p,v) \in \mathcal{D}_K^{reg}(q)$ we also have $(p,cv) \in \mathcal{D}_K^{reg}(q)$ for all $c \in \mathbb{R}_+$ (explain more). □

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

Proposition 2.3.7. *Let $q \in V$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

Proof. For the left inclusion assume $p \in \mathcal{E}_K(q)$, i.e. there exists an $a \in S^{n-1}$ such that $\mathcal{E}_a(q) = p$. Then lemma 2.3.2(3) immediately yields, $p \in \mathcal{P}_K(q)$ and $\tau(q,p) = 0$. Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. Because $\mathcal{P}_K(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p,q) > 0$, a contradiction.

For the other direction we assume we have $p = \mu_a(t) \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 2.3.2(3) we only need to prove that $\tau(p,q) = 0$. Suppose that $\tau(p,q) > 0$. Now since $\tau(p,q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since μ_a is timelike, which is a contradiction. □

Thus $\mathcal{E}_K(q)$ truly deserves to be called the "set of earliest observations".

2.3.2 Observation Reconstruction

Proposition 2.3.8. *Given the data outlined in remark 2.2.1 we can uniquely determine $\mathcal{E}_K(q)$ and $\mathcal{E}_K^{reg}(q)$, as well as $\mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Proof. What we want to show is that given K, the conformal class of $g|_K$ and the set $\{\mathcal{P}_K(q) \mid q \in V\}$ we can reconstruct the sets stated above. Note that as described in 2.2.2 ((Move to own remark?)) this data allows us to construct $\Theta : S^{n-1} \times [0,1] \to K$ and μ_a.

We first show that for a given $\mathcal{P}_K(q)$ we can determine $\mathcal{E}_K(q)$: By equation 2.3, for any $a \in S^{n-1}$ we can determine $f_a(q)$ and thus $\mathcal{E}_a(q) = \mu_a(f_a(q))$ using only $\mathcal{P}_K(q)$. We can then construct $\mathcal{E}_K(q) = \bigcup_{a \in S^{n-1}} \mathcal{E}_a(q)$. Furthermore, by proposition 2.3.6, $\mathcal{E}_K^{reg}(q)$ contains exactly the points $p \in \mathcal{E}_K(q)$ where $\mathcal{E}_K(q)$ is locally a submanifold of M and thus K. But because we know K we can determine all points where this is the case and reconstruct $\mathcal{E}_K^{reg}(q)$.

To reconstruct the direction set we first note that by lemma 2.2.10 for any $p \in \mathcal{P}_K(q)$ such that $exp_q^{-1}(p) = \{w_1, \dots, w_N\} \subset L_q^K M$, we have $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ where $p \in U \subset M$ open and $p \in \mathcal{U}_i$ are pairwise transversal spacelike hypersurfaces of K. For each w_i we let $v_i = \gamma'_{q,w_i}(1)$ be the outbound velocity vector of the null geodesic which starts at q with velocity w_i, once it hits K. To find $\mathcal{D}_K(q)$ we must reconstruct all such v_i.

To that end, note that we have $T_p\mathcal{P}_K(q) = \bigcup_{i=1}^N T_p\mathcal{U}_i$ where $T_p\mathcal{U}_i$ are spacelike hyperplanes. For each such hypersurface, using lemma 2.2.4 we can then find the outward pointing orthogonal null ray $\mathbb{R}_+ v_i$ which must contain the outbound velocity vector v_i at p. Thus for any $p \in \mathcal{P}_K(q)$ we can reconstruct $\mathbb{R}_+ v_i$ for all geodesics γ_{q,w_i} from q to p.

Now by definition for any $p \in \mathcal{E}_K(q)$, we have

$$\mathcal{D}_p := \pi^{-1}(p) \cap \mathcal{D}_K(q) = \{(p, \mathbb{R}_+ v_1), \dots, (p, \mathbb{R}_+ v_N)\}$$

where $\pi : TM \to M$ is the canonical projection. As we saw for any $p \in \mathcal{E}_K(q) \subset \mathcal{P}_K(q)$ we can reconstruct $\mathcal{D}_p$ which allows us to reconstruct $\mathcal{D}_K(q) = \bigcup_{p \in \mathcal{E}_K(q)} \mathcal{D}_p$.

Finally we can reconstruct $\mathcal{D}_K^{reg}(q)$ by using $\mathcal{D}_K^{reg}(q) = \pi^{-1}(\mathcal{E}_K^{reg}(q)) \cap \mathcal{D}_K(q)$. $\square$

Note that we can adapt this proof to show that $\mathcal{E}_K(q)$ uniquely determines $\mathcal{E}_K^{reg}(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.

Proposition 2.3.9. *Let $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$. Then $q = q'$.*

Proof. We assume by contradiction that $q, q' \in V$ such that $\mathcal{E}_K(q) = \mathcal{E}_K(q')$ and $q \neq q'$ Let $p_1, p_2 \in \mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ with $p_1 \neq p_2$. Because p_1 and p_2 cannot be cut points there must exist unique $w_1, w_2 \in L_q^K M$ and $w'_1, w'_2 \in L_{q'}^K M$ such that $\gamma_{q,w_i}(1) = p_i$ and $\gamma_{q',w'_i}(1) = p_i$. Because $\mathcal{E}_K^{reg}(q) = \mathcal{E}_K^{reg}(q')$ we can use lemma 2.2.4 to show that

$$v_i = \gamma'_{q,w_i}(1) = c_i \gamma_{q',w'_i}(1) = c_i v'_i$$

for some $c_i > 0$.

Now $\gamma_{p_i,-v_i}$ are two past-pointing null geodesics going from p_i through q and q'. Hence there either exists a null geodesic from q to q' or from q' to q. We will WLOG assume $q' \in J^+(q)$. Now there must exist $t_1, t_2 \in (0,1)$ such that $\gamma_{q,w_i}(t_i) = q'$. But this would make q' a cut point of q which is impossible as we assumed $p_i \in \mathcal{E}_K^{reg}(q)$. $\square$

Chapter 3

Interior Reconstruction

3.1 Construction of the topology

We aim to reconstruct the topological and differential data of V. To that end we define the following functions.

For $q \in V$ we define the function $F_q : S^{n-1} \to \mathbb{R}$ by $a \mapsto f_a(q)$. We can then define the function

$$\begin{aligned} \mathcal{F} : V &\to (C(S^{n-1}), d_\infty) \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in V$ to the function $F_q : S^{n-1} \to \mathbb{R}$. $(C(S^{n-1}), d_\infty)$ is the space of continuous functions from S^{n-1} to $\mathbb{R}$, together with the metric $d_\infty(f, g) = \max_{a \in S^{n-1}} |f(a) - g(a)|$.

The following proposition establishes that the canonical topological structure on $\mathcal{F}(V)$, i.e. the topology obtained by taking the subspace topology wrt. the topology induced by d_∞ on $C(S^{n-1})$, is the same as the pushforward under $\mathcal{F}$ of the topology on V, making $\mathcal{F}$ a homeomorphism. ((Explain that data allows us to determine d_∞))

Lemma 3.1.1. *The map $\mathcal{F} : V \to \widehat{V} := \mathcal{F}(V)$ is a well-defined continuous and injective.*

Proof. First of all $\mathcal{F} : V \to (C(S^{n-1}), d_\infty)$ is well-defined by proposition 2.3.3, i.e. for any $q \in V$, $F_q = f(q, \cdot)$ is continuous on S^{n-1}.

Because the topology induced on $C(S^{n-1})$ by d_∞ is uniform convergence, proposition 2.3.4 implies that $\mathcal{F}$ is continuous.

Finally injectivity follows from the fact that for any $q, q' \in V$ we have $\mathcal{F}(q) = \mathcal{F}(q') \implies F_q = F_{q'} \implies \mathcal{E}_K(q) = \mathcal{E}_K(q')$ which implies $q = q'$ by proposition 2.3.9. $\square$

However there is still some work required to show that also $\mathcal{F}^{-1}$ is continous on $\widehat{V}$:

Proposition 3.1.2. *Let $(q_n)_{n=1}^{\infty}$ be a sequence in V and $q_0 \in V$ such that $F_{q_n} \to F_{q_0}$ uniformly then $q_n \to q_0$ as $n \to \infty$.*

Proof. Let $(q_n)_{n=1}^{\infty}, q_0 \in V$ such that $F_{q_n} \to F_{q_0}$ uniformly. Note that because $\mathcal{F}$ is injective the choice of q_n for $F_{q_n} = \mathcal{F}(q_n)$ is unambigous.

Let $t_0 = \max_{a \in S^{n-1}} F_{q_0}(a)$, because $q_0 \in V$, lemma 2.2.2(3) implies that $t_0 < 1$. Hence there exists an $\varepsilon > 0$ such that $t' := t_0 + \varepsilon < 1$. And because F_{q_n} to F_{q_0} uniformly there exists a $N \in \mathbb{N}$ such that $t_n := \max_{a \in S^{n-1}} F_{q_n}(a) < t'$ for all $n \geq N$.

We now define

$$C_{t'} := \{q \in J(p^-, p^+) \mid \max_{a \in S^{n-1}} F_q(q) \leq t'\}$$

and after removing the first N elements of $(q_n)_{n=1}^{\infty}$ we may assume $q_n, q_0 \in C_{t'}$.

We now show that $C_{t'} \subset J(p^-, p^+)$ is closed: Let $p_n \to p_0 \in J(p^-, p^+)$ with $p_n \in C_{t'}$. We must have $p_0 \in J(p^-, p^+)^{\circ}$ because otherwise $\max_{a \in S^{n-1}} F_{p_n}(a) \to 1 > t'$. Because p_n, p_0 does not necessarily lie in V we but we still want to use all the machinery we built up so far we now look at $V_0 := J(p^-, p^+)^{\circ}$. We can do this because all results were stated in terms of any arbitrary $V \subset J(p^-, p^+)^{\circ}$ which includes the case $V_0 = J(p^-, p^+)^{\circ}$. We then apply proposition 2.3.4 to V_0, to get $F_{p_n} \to F_{p_0}$ uniformly. This in turn implies $\max_{a \in S^{n-1}} F_{p_n} \to \max_{a \in S^{n-1}} F_{p_0}$ and because $p_n \in C_{t'}$ we have $\max_{a \in S^{n-1}} F_{p_n} \leq t'$ for all n and hence also $\max_{a \in S^{n-1}} F_{p_0} \leq t'$. This proves $p_0 \in C_{t'}$ making $C_{t'}$ closed.

Now $C_{t'}$ is a closed subset of a compact space and thus itself compact. Again using V_0 and the previous lemma we get that $\mathcal{F} : V_0 \to \mathcal{F}(V_0)$ is well-defined continuous and injective. We can then restrict $\mathcal{F}$ to $C_{t'}$ and all these properties are preserved. But now $\mathcal{F} : C_{t'} \to \mathcal{F}(C_{t'})$ is a continuous, injective map from a compact space to a hausdorff space, making it a homeomorphism.

Because $q_n, q_0 \in C_{t'}$ for all n, we have $F_{q_n} = \mathcal{F}(q_n) \to F_{q_0} = \mathcal{F}(q_0) \in \mathcal{F}(C_{t'})$. Using that $\mathcal{F}^{-1}$ continuous on $\mathcal{F}(C_{t'})$ we get

$$q_n = \mathcal{F}^{-1}(F_{q_n}) \to \mathcal{F}^{-1}(F_{q_0}) = q_0,$$

as desired. □

And we get:

Corollary 3.1.3. *$\mathcal{F} : V \to \widehat{V}$ is a homeomorphism.*

3.2 Smooth Reconstruction

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

3.2.1 Preparation

Definition 3.2.1 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_K^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w \in L_q^K M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q,w) > 1$. Existence follows from lemma 2.3.2 while uniqueness follows from the fact that $p \in \mathcal{E}_K^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Omega : \mathcal{Z} &\mapsto L^K V \\ (q,p) &\mapsto (q,w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0,w_0) \subset TM$ be a ε-neighborhood of (q_0,w_0) with respect to the Sasaki-metric induced on TM by g^+.

Lemma 3.2.2. *((Move to appendix?))The function*

$$\begin{aligned} T_+ : L^+ J(p^-,p^+) &\to \mathbb{R} \\ (q,w) &\mapsto \sup\{t \geq 0 \mid \gamma_{q,w}(t) \in J^-(p^+)\} \end{aligned}$$

is finite and upper semicontinuous.

Proof. Finiteness follows from lemma B.2.2. We now want to show that T_+ is upper semicontinuous. To that end let $(q_n,w_n) \to (q_0,w_0) \in L^+J(p^-,p^+)$, we want to show that $\limsup_{n\to\infty} T_+(q_n,w_n) \leq T_+(q_0,w_0)$: Let $\varepsilon > 0$ and set $t_0 = T_+(q_0,w_0)$. Then by definition we have $\gamma_{q_0,w_0}(t_0) \in M \setminus J^-(p^+)$. Because $\gamma_{q_n,w_n}(t_0) \to \gamma_{q_0,w_0}(t_0)$ and $M \setminus J^-(p^+)$ open, there exists a $N \in \mathbb{N}$ such that $\gamma_{q_n,w_n}(t_0) \in M \setminus J^-(p^+)$ for all $n \geq N$. Note that if $\gamma_{q_n,w_n}(t_0) \notin J^-(p^+)$ then for any $t' \geq t_0$ we also have $\gamma_{q_n,w_n}(t') \notin J^-(p^+)$ because otherwise we could obtain a lightlike path from $\gamma_{q_n,w_n}(t_0)$ to p^+, a contradiction. Thus, by definition $T_+(q_n,w_n) \leq t_0$ and $\limsup_{n\to\infty} T_+(q_n,w_n) \leq t_0 = T_+(q_0,w_0) + \varepsilon$. Finally because $\varepsilon > 0$ was arbitrary we get $\limsup_{n\to\infty} T_+(q_n,w_n) \leq T_+(q_0,w_0)$ as desired. □

Lemma 3.2.3. *Let $(q_0,p_0) \in \mathcal{Z}$ and $(q_0,w_0) = \Omega(q_0,p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0,w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0,w_0) \to \mathcal{U}_\varepsilon(q_0,p_0) := X(\mathcal{W}_\varepsilon(q_0,w_0))$. When ε is small enough, Ω coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0,p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $2n$-dimensional manifold and the map $\Omega : \mathcal{Z} \to L^K M$ is smooth.

Proof. Because $p_0 \in \mathcal{E}_K^{reg}(q)$ we have $\rho(q_0, w_0) > 1$. Because ρ is lower semicontinuous, for $\varepsilon > 0$ small enough we have $\rho(q', w') >$ for all $(q', w') \in \mathcal{W}_\varepsilon(q_0, w_0) \subset TV$. Thus $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) = X(\mathcal{W}_\varepsilon(q_0, w_0))$ is a diffeomorphism with $\mathcal{U}_\varepsilon(q_0, p_0)$ open in $M \times M$ by the invariance of of domain theorem.

Next we aim to show that $\Omega : \mathcal{Z} \to L^K V$ is continuous at $(q_0, p_0) \in \mathcal{Z}$. We proceed by assuming there exists a sequence $(q_n, p_n) \in \mathcal{Z}$ converging to (q_0, p_0) such that $\Theta(q_n, p_n) = (q_n, w_n) \in L^+V$ does not converge to $\Theta(q_0, p_0) = (q_0, w_0)$.

First of all we aim to show that the sequence (q_n, w_n) is bounded and thus has a convergent subsequence: Because $q_n \to q_0$ we only need to show that w_n is bounded. To that end we introduce an arbitrary riemannian metric consistent with the topology on M and can write $w_n = t_n \overline{w_n}$ where $\|\overline{w_n}\|_{g^+} = 1$. To show that t_n is bounded we first define

$$C := \{(q, w) \in L^+M \mid q \in J(p^-, p^+) \text{ and } \|w\|_{g^+} = 1\}$$

and C is compact and because T_+ is upper semicontinuous on C, there exists a $c_0 > 0$ such that $T_+(q, w) \leq c_0$ for all $(q, w) \in C$. Recall that we have $\gamma_{q_n, \overline{w_n}}(t_n) = \exp_{q_n}(w_n) = p_n \in K \subset J(p^-, p^+)$. Together with $(q_n, \overline{w_n}) \in C$ this yields

$$\|w_n\|_{g^+} = t_n \|\overline{w_n}\|_{g^+} = t_n \leq T_+(q_n, \overline{w_n}) < c_0,$$

proving $(q_n, w_n) \in L^K V$ is bounded.

We can thus obtain a convergent subsequence $(q_k, w_k) = \Theta(q_k, p_k) \to (q_0, w')$ with $w' \neq w_0$. Since the exponential map is continuous, we would have

$$\exp_{q_n}(w') = \lim_{n\to\infty} \exp_{q_n}(w_n) = \lim_{n\to\infty} p_n = p_0 = \exp_{q_n}(w_0).$$

with $w' \neq w_0$. But since $p_0 \in \mathcal{E}_K^{reg}(q)$ cannot be a cut point this is a contradiction and $\Omega : \mathcal{Z} \to L^K V$ must be continuous.

Next we use the fact that Ω is continuous and get $\Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0)) \subset \mathcal{Z}$ is open. We can thus find a $\varepsilon_1 \in (0, \varepsilon)$ such that for the open ball $\mathcal{U}_{\varepsilon_1}(q_0, w_0) \subset M$ we have

$$\mathcal{Y}_{\varepsilon_1} := \mathcal{U}_{\varepsilon_1}(q_0, w_0) \cap \mathcal{Z} \subset \Omega^{-1}(\mathcal{W}_\varepsilon(q_0, w_0))$$

implying $\Omega(\mathcal{Y}_{\varepsilon_1}) \subset \mathcal{W}_\varepsilon(q_0, w_0)$. Then for $(q, p) \in \mathcal{Y}_{\varepsilon_1}$ and $(q, w) = \Omega(q, p) \in \mathcal{W}_\varepsilon(p_0, w_0)$ we have $\exp_q(w) = p$. Hence $X(\Omega(q, p)) = (q, p)$. But now since $(q, p) \in \mathcal{U}_\varepsilon(p_0, q_0)$ we can apply X^{-1} to both sides and get $\Omega(q, p) = X^{-1}(q, p)$. Thus on $\mathcal{Y}_{\varepsilon_1}$ the function $\Omega : \mathcal{Y}_{\varepsilon_1} \to TM$ coincides with the smooth function $X^{-1} : \mathcal{Y}_{\varepsilon_1} \to TM$, which implies that Ω is smooth with full rank differential on $\mathcal{Y}_{\varepsilon_1}$ as well.

Now since $(q_0, p_0) \in \mathcal{Z}$ was arbitrary we get that $\Theta : \mathcal{Z} \to L^+V$ is smooth everywhere, injective and locally diffeomorphic with full rank. Thus $\mathcal{Z}$ diffeomorphic to an open subset of $L^K V$. This makes it a manifold with dimension $(n+1) + (n-1) = 2n$. $\square$

Proposition 3.2.4. *Let $q \in V$ and $(q_0, p_j) \in \mathcal{Z}, j = 0, \dots, n$ and $w_j \in L^K_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \dots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=0}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

Proof. First we need some setup: Let $(q_0, p_0) \in \mathcal{Z}$ and $w_0 \in L^+_{q_0}M$ such that $\gamma_{q_0,w_0}(1) = p_0$. Furthermore let $\varepsilon > 0$ be small enough such that the map $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0)$ is a diffeomorphism (see the previous lemma). We will denote this inverse by $X^{-1}(q, p) = (q, w(q, p))$ and write $\mathcal{W} = \mathcal{W}_\varepsilon(q_0, w_0), \mathcal{U} = \mathcal{U}_\varepsilon(q_0, p_0)$.

We associate with any $(q, p) \in \mathcal{U}$ the energy $E(q, p) = E(\gamma_{q,w(q,p)}([0, 1]))$ of the geodesic segment connecting q to p. The energy of a piecewise smooth curve $\alpha : [0, l] \to M$ is defined as

$$E(\alpha) = \frac{1}{2} \int_0^l g(\alpha'(t), \alpha'(t)) dt.$$

Note that the sign of $E(\alpha)$ depends on the causal nature of $\gamma_{q,w(q,p)}$. In particular $E(q, p) = 0$ if and only if $w(q, p)$ is light-like. Moreover, as X^{-1} is smooth on $\mathcal{U}$, so is $E(p, q)$.

We now return to consider $(q_0, p_0) \in \mathcal{Z}$ and let $a \in S^{n-1}$ be such that $p_0 \in \mu_a$. Then $p_0 = \mu_a(s_0)$ with $s_0 = f_a(q_0)$ as $p_0 \in \mathcal{E}^{reg}_K(q_0)$ and $s_0 \in (0, 1)$ by lemma 2.3.2(1).

Let $V_0 \subset V$ be an open neighborhood of q_0 and $t_1, t_2 \in (0, 1), t_1 < s_0 < t_2$, such that $V_0 \times \mu_a([t_1, t_2]) \subset \mathcal{U}$, which exist because $\mathcal{U}$ is open. Then for any $q \in V_0, s \in (t_1, t_2)$ the function $\mathbf{E}_a(q, s) := E(q, \mu_a(s))$ is well defined and smooth.

We want to use first variation formula for $\mathbf{E}_a(q, s)$ ((E Reference)) to calculate $\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0}$ and $\nabla_q \mathbf{E}_a(q, s_0)|_{q=q_0}$.

For the first part we define the variation $\mathbf{x}(t, s) = \gamma_{q_0,w(s)}(t), t \in [0, 1]$ where $w(s) := w(q_0, \mu_a(s + s_0)), s \in [t_1 - s_0, t_2 - s_0]$. Note that $\mathbf{x}(t, 0) = \gamma_{q_0,w_0}(t)$. We then get

$$\left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} = E'_{\mathbf{x}}(0) = g(V, \gamma'_{q_0,w_0})\big|_0^1$$

since γ_{q_0,w_0} is a geodesic and $\mathbf{x}$ has no breaks. If we now further notice that $V(0) = 0$ as $\mathbf{x}(0, s) = q_0$ for all $s \in [t_1, t_2]$ and $V(1) = \mu'_a(s_0) = \mu'_a(f_a(q_0))$ as

$\mathbf{x}(1,s) = \mu_a(s+s_0)$ we can conclude

$$\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(V(1),\gamma'_{q_0,w_0}(1)) - g(V(0),\gamma'_{q_0,w_0}(0))$$
$$= g(\mu'_a(f_a(q_0)),\gamma'_{q_0,w_0}(1))$$

For the second part we will introduce coordinates $\mathbf{q} = (q_0,\dots,q_n)$ around q_0. Then the gradient can be written as

$$\nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} \partial_j.$$

To calculate $\left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0}$ we now introduce variations $\mathbf{x}_i(t,s) = \gamma_{q(s),w(s)}(t)$ where $w(s) := w(q(s),\mu_a(s_0))$ and $q(s) := q^{-1}(q_0(q_0),\dots,q_i(q_0)+s,\dots q_n(q_0))$ is obtained by increasing the i-th coordinate by s. Note that these variations all have $\mathbf{x}_i(t,0) = \gamma_{q_0,w_0}(t)$, $\mathbf{x}_i(1,s) = \mu_a(s_0)$ thus $V_{\mathbf{x}_i}(1) = 0$ and $V_{\mathbf{x}_i}(0) = \frac{\partial}{\partial s}\mathbf{x}_i(0,s)|_{s=0} = \partial_i$. After again applying proposition ((E REF))

$$\left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} = E'_{\mathbf{x}_i}(0) = -g(V(0),\gamma'_{q_0,w_0}(0)) = -g(\partial_i,w_0).$$

Combining this with coordinate representation of the gradient we get

$$\nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = g^{ij} \left.\frac{\partial \mathbf{E}_a(q,s_0)}{\partial q_i}\right|_{q=q_0} \partial_j = -g^{ij}(g_{\alpha\beta}\partial_i^\alpha w_0^\beta)\partial_j$$
$$= -g^{ij}g_{i\beta}w_0^\beta\partial_j = -\delta^j_\beta w_0^\beta \partial_j$$
$$= -w_0^j\partial_j = -w_0.$$

We thus managed to calculate what we wanted and can summarize as

$$\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(v,\mu'_a(f_a(q_0))), \quad \nabla_q \mathbf{E}_a(q,s_0)|_{q=q_0} = -w_0 \tag{3.1}$$

where $w_0 = w(q_0,p_0)$ and $v = \gamma'_{q_0,w_0}(1)$. Since $\mu'_a(f_a(q_0))$ and v are both future-pointing null vectors, which by lemma 2.2.3 must be transversal we have $\left.\frac{\partial \mathbf{E}_a(q_0,s)}{\partial s}\right|_{s=s_0} = g(v,\mu'_a(f_a(q))) < 0$.

We can now use the implicit function theorem on $V_0 \times [t_1,t_2]$ with equation $E_a(q,s) = 0$ and single solution $E_a(q_0,s_0) = 0$. This yields an open neighborhood $V_a \subset V_0$ and a smooth function $q \mapsto s_a(q)$ such that $E_a(q,s_a(q)) = 0$ for all $q \in V_a$. Now $E_a(q,s_a(q)) = E(q,\mu_a(s_a(q))) = 0$, implies $\mu_a(s_a(q)) \in \mathcal{P}_K(q)$. This together with $(q, s_a(q)) \in \mathcal{U}$ implies that $\mu_a(s_a(q)) \in \mathcal{E}_K^{reg}(q)$ and thus $s_a(q) = f_a(q)$ on V_a. Hence we have $\nabla f_a(q)|_{q=q_0} = \nabla s_a(q)|_{q=q_0}$ and from equation 3.1 together with the implicit function theorem it follows that

$$\nabla f_a(q)|_{q=q_0} = \frac{1}{c(q_0, a)} w_0, \quad c(q_0, a) = \left.\frac{\partial \mathbf{E}_a(q_0, s)}{\partial s}\right|_{s=s_0} < 0, \tag{3.2}$$

where $p_0 = \mu_a(s_0) = \mathcal{E}_a(q_0), s_0 = f_a(q_0)$ and $w_0 = w(q_0, p_0)$.

Next we choose $p_0, \ldots, p_n \in \mathcal{E}_K^{reg}(q_0)$ and let $w_0, \ldots, w_n \in L_{q_0}^K M$ such that $p_i = \gamma_{q_0, w_i}(1)$, i.e. $w_i = w(q_0, p_i)$. We assume that $w_0, \ldots, w_n$ are linearly independent. Moreover let $a_j \in S^{n-1}$ such that $p_i \in \mu_{a_j}$ and $\overrightarrow{a} = (a_j)_{j=1}^n$. Finally we denote by $q \mapsto s_{a_j}(q) = f_{a_j}(q)$ the above constructed smooth functions which are defined on some neighborhoods $V_{a_j} \subset V$ of q_0.

Let $V_{\overrightarrow{a}} = \bigcap_{j=1}^n V_{a_j}$ and consider the map

$$\begin{aligned} \mathbf{f}_{\overrightarrow{a}} : V_{\overrightarrow{a}} &\to \mathbb{R}^n \\ q &\mapsto (f_{a_1}(q), \ldots, f_{a_n}(q)). \end{aligned}$$

Because all of its components are smooth, $\mathbf{f}_{\overrightarrow{a}}$ itself is smooth as well. By equation 3.2 each component has gradient $\nabla f_{a_j}(q)\big|_{q=q_0} = \frac{1}{c(q_0, a_j)} w_i$ with $c(q_0, a_j) \neq 0$. Since we assumed that $w_0, \ldots, w_n$ be independent, $\mathbf{f}_{\overrightarrow{a}}$ is non-degenerate at q_0 and thus defines a smooth coordinate system in some neighborhood V_1 of q_0. $\square$

3.2.2 Construction of smooth coordinates

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

Definition 3.2.5 (Observation Coordinates). Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=0}^n \subset (S^{n-1})^{n+1}$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_K^{reg}(q)$ for all $j = 0, \ldots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in (S^{n-1})^{n+1}$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will often denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

Proposition 3.2.6. *Let $\widehat{q} \in \widehat{V}$ then the following holds:*

(1) *Given the data from 2.2.1 we can determine all C^0-observation coordinates around $\widehat{q}$,*

(2) *there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$ and*

(3) *given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data 2.2.1, allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.*

Proof. We begin with some setup: Let $q \in V$. We say that $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ are *associated* with respect to q if $p \in \mu_a$, i.e. $p = \mathcal{E}_a(q)$.

To prove part (1), we let $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. We want to show that for any choice of observers $\overrightarrow{a} = (a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ we can determine if they form C^0-observation coordinates. First of all we need to check whether the associated $p_j = \mathcal{E}_{a_j}(q)$ are regular points, i.e. $p_j \in \mathcal{E}_K^{reg}(q)$. But as $\widehat{q} = \mathcal{F}(q) = F_q$ we can recover $\mathcal{E}_K q = \bigcup_{a \in S^{n-1}} \mu_a(F_q(a))$ and also the associated points $p_j = \mu_{a_j}(F_q(a_j))$. By proposition 2.3.8 this allows us to determine $\mathcal{E}_K^{reg}(q)$ and for all p_j we can then simply check whether they lie in $\mathcal{E}_K^{reg}(q)$.

We now need to check whether there exists an open neighborhood W of $\widehat{q}$ such that the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is injective. By definition we have

$$\mathbf{s}_{\overrightarrow{a}}(\widehat{q}) = (\widehat{q}(a_1), \dots, \widehat{q}(a_n)) = (F_q(a_0), \dots, F_q(a_n))$$

which means that the data allows us to fully determine $\mathbf{s}_{\overrightarrow{a}}$ on $\widehat{V}$. But since by corollary 3.1.3, the data allows us do construct the topology on $\widehat{V}$ we can determine whether there exists an open neighborhood W of $\widehat{q}$ such that $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is injective and thus open by the invariance of domain theorem.

To show (2) we let again $\widehat{q} \in \widehat{V}$ with $\widehat{q} = \mathcal{F}(q)$. Let $(a_j)_{j=0}^n \in (S^{n-1})^{n+1}$ such that the associated $p_j \in \mathcal{E}_K^{reg}(q)$ and the vectors $\{w_j = w(q, p_j) \mid j = 0, \dots n\}$ are linearly independent. We can find such a set of linearly independent vectors because by proposition 2.3.6 $\mathcal{E}_K^{reg}(q)$ is an open subset of $\mathcal{E}_K(q)$. Now by proposition 3.2.4 the observation time functions $\mathbf{f}_{\overrightarrow{a}}$ define smooth coordinates on a neighborhood V_1 of q. Thus $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F}$ are smooth local coordinates as well making $(\mathbf{s}_{\overrightarrow{a}}, \mathcal{F}(V_1))$ C^∞-observation coordinates.

Moving on to part (3): We begin by proving that the set of points in $(\mathcal{E}_K^{reg}(q))^{n+1}$ which yield C^∞-observation coordinates is open and dense in $(\mathcal{E}_K^{reg}(q))^{n+1}$. We consider $p \in \mathcal{E}_K^{reg}(q)$ and $a \in S^{n-1}$ which are associated. Let

$$K(q) = \{(w_j)_{j=0}^n \mid w_j \in L_q^K M, \rho(q, w_j) > 1, \gamma_{q,w_j}(1) \in K\}$$

and define on $K(q)$ the map

$$\begin{aligned} H : K(q) &\to K^{n+1} \\ (w_j)_{j=0}^n &\mapsto (\gamma_{q,w_j}(1))_{j=0}^n. \end{aligned}$$

We will denote $p_j = \gamma_{q,w_j}(1) = \exp_q(w_j)$. Then by definition $p_j \in \mathcal{E}_K^{reg}(q)$ and $w_j = \Omega(q, p_j)$. As ρ is lower semi-continuous, we see that $K(q) \subset (L_q^K M)^n$ is open by an analogous argument to the one in the proof of 2.3.6. As the exponential map is continuous, H is also continuous. Furthermore as $\Omega : \mathcal{Z} \to L^+V$ is continuous and injective, we can construct a continuous inverse to H, making $H : K(q) \to H(K(q)) = (\mathcal{E}_K^{reg}(q))^{n+1}$ a homeomorphism. We will denote $Y(q) \coloneqq (\varepsilon_U^{reg}(q))^{n+1}$. Note that for all $\widehat{q} \in \widehat{V}$, the data 2.2.1 determine $\mathcal{E}_K^{reg}(q)$ and thus also the set $Y(q) \subset K^n$, where $q = \mathcal{F}^{-1}(\widehat{q})$.

Let us now consider the set

$$K_0(q) = \{(w_j)_{j=1}^n \in K(q) \mid w_1, \ldots, w_n \text{ are linearly independent}\}.$$

As linear independence is an open and non-degenerate property $K_0(q)$ is open and dense in $K(q)$. Since H is a homeomorphism, $Y_0(q) = H(K_0(q))$ is open and dense in $Y(q)$ as well.

We can now prove the final part of the proposition: Recall that given C^0-observation coordinates around $\widehat{q}$, we want to determine if they are also C^∞-observation coordinates $\widehat{q}$. To that end, let $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ be C^0-observation coordinates around $\widehat{q} \in W_{\overrightarrow{a}}$ with $q = \mathcal{F}^{-1}(\widehat{q})$. By definition we have $p_j \in \mathcal{E}_K^{reg}(q)$ where $p_j = \mathcal{E}_{a_j}(q)$ are associated with a_j and hence $(p_j)_{j=0}^n \subset Y(q)$. In the case where $(p_j)_{j=0}^n \in Y_0(q)$, by proposition 3.2.4, q has a neighborhood $V_1 \subset M$ on which the function $\mathbf{f}_{\overrightarrow{a}} : V_1 \to \mathbb{R}^n$ gives smooth local coordinates. Thus, after possibly restricting $W_{\overrightarrow{a}}$, $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$. We then let $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$, $\overrightarrow{b} \in (S^{n-1})^{n+1}$ be different C^0-observation coordinates around $\widehat{q}$ and let $(\widetilde{p}_j)_{j=0}^n \in Y(q)$ be such that $\widetilde{p}_j$ is associated to b_j. Since all smooth coordinates must be compatible, then $(\widetilde{p}_j)_{j=0}^n \in Y_0(q)$ if and only if

$$\begin{aligned} &\text{The function } \mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1} \text{ is smooth at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ and the Jacobian determinant} \\ &\qquad\qquad \det(D(\mathbf{s}_{\overrightarrow{b}} \circ \mathbf{s}_{\overrightarrow{a}}^{-1})) \text{ at } \mathbf{s}_{\overrightarrow{a}}(\widehat{q}) \text{ is non-zero.} \end{aligned} \tag{3.3}$$

Here the "only if"-direction follows from the fact that the nondegeneracy of the Jacobian ensures that the linear independence of the spanning vectors is preserved.

For some $\overrightarrow{p} = (p_j)_{j=0}^n \in Y(q)$ with $\overrightarrow{a}$ associated we define $\mathcal{X}_{\overrightarrow{p}} \subset Y(q)$ to be the set of $(\widetilde{p}_j)_{j=0}^n \in Y(q)$, such that for the associated $\overrightarrow{b}$ there exists $W_{\overrightarrow{b}}$ such that $(W_{\overrightarrow{b}}, \mathbf{s}_{\overrightarrow{b}})$ are C^0-coordinates around $\widehat{q}$ and condition 3.3 is satisfied.

If $\overrightarrow{p} \in Y_0(q)$ we see that $Y_0(q) \subset \mathcal{X}_{\overrightarrow{p}}$. On the other hand $\overrightarrow{p} \notin Y_0(q)$ we have $Y_0(q) \cap \mathcal{X}_{\overrightarrow{p}} = \emptyset$. Since the set $Y_0(q)$ is open and dense in $Y(q)$, we see that

$\overrightarrow{p} \in Y_0(q)$ if and only if the interior of $\mathcal{X}_{\overrightarrow{p}}$ is dense subset of $Y(q)$. Since the data 2.2.1 is sufficient to determine $Y(q)$ and $\mathcal{X}_{\overrightarrow{p}}$, we can determine whether $\overrightarrow{p} \in Y_0(q)$ or not. And since, by proposition 3.2.4, the C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q} = \mathcal{F}(q)$ are C^∞-observation coordinates if and only if $\overrightarrow{p} \in Y_0(q)$, where $\overrightarrow{p}$ are associated to $\overrightarrow{a}$ wrt. q, we can determine all C^0-observation coordinates around $\widehat{q}$ which are also C^∞-observation coordinates. $\square$

Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric on U determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

Lemma 3.2.7. *The data given in 2.2.1 allows us to determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Proof. Let $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ be C^∞-observation coordinates on $\widehat{V}$ and $\widehat{q} \in W_{\overrightarrow{a}}$. We begin by constructing a time orientation on $\widehat{V}$: Let $a_1, a_2 \in \overrightarrow{a}$ and $p_1, p_2 \in U$ be associated wrt. the point $q = \mathcal{F}^{-1}(\widehat{q})$, i.e. $p_i = \mathcal{E}_{a_i}(q)$. Because $\mathbf{f}_{\overrightarrow{a}} = \mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F}$ are smooth coordinates we have that the vectors $w(q, p_1)$ and $w(q, p_2)$ pointing from q to p_i must be non-parallel. Therefore, by equation 3.2 we see that the gradient vectors $\nabla f_{a_i}(q)$ are non-parallel, lightlike and past-pointing. Thus the co-vectors $-ds_{a_1}|_{\widehat{q}}$ and $-ds_{a_2}|_{\widehat{q}}$ are non-parallel lightlike and future-pointing. This follows from the fact that $\mathcal{F}$ is an isometry and the co-vector df_a is the image of ∇f_a under the canonical isomophism. Moreover because the data allows us to fully determine s_{a_1} and s_{a_2} on $\widehat{V}$ (see previous proof) we can also determine ds_{a_1} resp. ds_{a_2}.

The co-vector field $X = (-ds_{a_1}) + (-ds_{a_2})$ is timelike and future-pointing and forms a local time-orientation on $W_{\overrightarrow{a}}$. Using bump functions and a partition of unity we can then obtain a time-orientation on the whole of $\widehat{V}$ since all orientations agree where they overlap.

Now we turn our attention to the construction of a metric G which is conformal to $\widehat{g}$: Let again $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ be C^∞-observation coordinates on $\widehat{V}$ with $\widehat{q}_0 \in W_{\overrightarrow{a}}$ and $q_0 \in V$ such that $\widehat{q}_0 = \mathcal{F}(q_0)$. As in the previous proof, using the data given in 2.2.1 and the function $\widehat{q}_0 = F_{q_0}$ we can determine $\mathcal{E}_K(q_0), \mathcal{E}_K^{reg}(q_0), \mathcal{D}_K(q_0)$ and $\mathcal{D}_K^{reg}(q_0)$ by 2.3.8.

We then fix the point $\widehat{q}_0 = \mathcal{F}(q_0)$ and the tuple $(p, v) \in \mathcal{D}_K^{reg}(q_0)$. Let $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{p,v}((-\widehat{t}, 0]) \subset M$ is defined and has no cut point. For $q \in V$, we have that $q \in \gamma_{p,v}((-\widehat{t}, 0))$ if and only if $(p, v) \in \mathcal{D}_K^{reg}(q)$. Hence for a fixed $(p, v) \in \mathcal{D}_K^{reg}(q_0)$ the data allows us to whether some $\widehat{q} \in W_{\overrightarrow{a}}$ has $q = \mathcal{F}^{-1}(\widehat{q}) \in \gamma_{p,v}((-\widehat{t}, 0))$ by checking if $(p, v) \in \mathcal{D}^{reg}(q)$. This allows us to determine

$$\beta = \{\widehat{q} \in W_{\overrightarrow{d}} \mid \widehat{q} = \mathcal{F}(q), \mathcal{D}_K^{reg}(q) \ni (p,v)\} = \mathcal{F}(\gamma_{p,v}((-\widehat{t},0))) \cap W_{\overrightarrow{d}}.$$

Therefore, on $W_{\overrightarrow{d}} \subset \widehat{V}$ we can find the image, under the map $\mathcal{F}$, of the light-like geodesic segment $\gamma_{p,v}((-\widehat{t},0)) \cap \mathcal{F}^{-1}(W_{\overrightarrow{d}})$ that contains $q_0 = \gamma_{p,v}(-t_1)$. Let $\alpha(s)$, $s \in (-s_0, s_0)$ be a smooth path on $W_{\overrightarrow{d}}$ such that $\partial_s \alpha(s)$ is never zero, $\alpha((-s_0, s_0)) \subset \beta$ and $\alpha(0) = \widehat{q}_0$. Such a smooth path can, for example be obtained by endowing $\widehat{V}$ with some arbitrary Riemannian metric and parameterizing by arc-length. Then $\widehat{w} = \partial_s \alpha(s)|_{s=0} \in T_{\widehat{q}_0}\widehat{V}$ has the form $\widehat{w} = c\mathcal{F}(\gamma'_{p,v}(-t_1))$ where $c \neq 0$.

Since we can do the above construction for all points $(p,v) \in \mathcal{D}_U^{reg}(q_0)$, we can determine in the tangent space $T_{\widehat{q}_0}\widehat{V}$ the set

$$\Gamma = \mathcal{F}_*(\{cw \in L_{q_0}M \mid \exp_{q_0}(w) \in \mathcal{E}_K^{reg}(q_0), c \in \mathbb{R} \setminus \{0\}\})$$

which is an open, non-empty subset of the light cone at $\widehat{q}_0$ wrt. the metric $\widehat{g}$. But now, since the light cone is determined by a quadratic equation in the tangent space, having an open set Γ determines the whole light cone. By repeating this construction for all points $\widehat{q} \in \widehat{V}$, we can uniquely determine $L\widehat{V}$. Using proposition B.6.3 we can then determine the conformal class of the tensor $\widehat{g} = \mathcal{F}_* g$ in the manifold $\widehat{V}$.

The above shows that the data 2.2.1 determine the conformal class of the metric tensor $\widehat{g}$. And in particular we can construct a metric G on $\widehat{V}$ that is conformal to $\widehat{g}$ and satisfies $G(X,X) = -1$. □

Proof. We have now gone through all the steps necessary to reconstruct the conformal, differential and topological data of V, we will not tie this all together to give a detailed account of the actual reconstruction and prove theorem **??**.

Assume we are given data 2.2.1, that is, the smooth manifold U, the conformal class of $g|_U$, the family of smooth and timelike future-pointing paths $\mu_a, a \in \mathcal{A}$ and the collection of light observation sets $\mathcal{P}_U(V) = \{\mathcal{P}_U(q) \mid q \in V\}$. Note that for a given $\mathcal{P}_U(q)$, we cannot, a priory, determine to which $q \in V$ it belongs.

To then construct $\widehat{V}$ which is conformally diffeomorphic to V we follow these steps:

- As $f_a(q) = \min\{s \in [-1,1] \mid \mu_a(s) \in \mathcal{P}_U(q)$ we can determine $\varepsilon_U(V) = \{\varepsilon_U(q) \mid q \in V\}$ from $\mathcal{P}_U(V)$.

- Proposition **??** then allows us to determine $\mathcal{D}_U^{reg}(q)$, $\mathcal{D}_U(q)$ and $\varepsilon_U^{reg}(q)$ for a given $\varepsilon_U(q) \in \varepsilon_U(V)$. We can thus construct $\mathcal{D}_U^{reg}(V)$, $\mathcal{D}_U(V)$ and $\varepsilon_U^{reg}(V)$.

- We define the function

$$
\begin{aligned}
\mathcal{F} : V &\to \mathcal{F}(V) = \widehat{V} \subset \mathbb{R}^{\mathcal{A}} \\
q &\mapsto \widehat{q} = F_q = (a \mapsto f_a(q)).
\end{aligned}
$$

  For a given $\varepsilon_U(q)$ we can construct $\widehat{q}$ by $\widehat{q}(a) = f_a(q) = s$ such that $\mu_a(s) \in \varepsilon_U(q)$. This allows us to construct the following map

$$
\begin{aligned}
\widetilde{\mathcal{F}} : V &\to \widehat{V} \\
\varepsilon_U(q) &\mapsto \widehat{q}.
\end{aligned}
$$

  Note that by the same procedure we can define the map

$$
\begin{aligned}
\widetilde{f_a} : \varepsilon_U(V) &\to \mathbb{R} \\
\varepsilon_U(q) &\mapsto f_a(q).
\end{aligned}
$$

  Furthermore if we endow $\widehat{V}$ with the product topology and let $\pi_a : \widehat{V} \to \mathbb{R}$ with $\widehat{q} \mapsto \widehat{q}(a)$ then the following diagrams commute for all $a \in \mathcal{A}$:

$$
\begin{tikzcd}
V \arrow[d, "q \mapsto \varepsilon_U(q)"'] \arrow[dr, "\mathcal{F}"] & \\
\varepsilon_U(V) \arrow[r, "\widetilde{\mathcal{F}}"'] & \widehat{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{V} \arrow[d, "\pi_a"] & \\
V \arrow[ur, "\mathcal{F}"] \arrow[r, "f_a"'] & \mathbb{R} & \varepsilon_U(V) \arrow[l, "\widetilde{f_a}"] \arrow[ul, "\widetilde{\mathcal{F}}"']
\end{tikzcd}
$$

  Note that in the above diagrams, the data 2.2.1 only allows us to construct the sets $\varepsilon_U(V)$ and $\widehat{V}$ and the maps $\widetilde{F}$, $\widetilde{f_a}$ and π_a.

  We thus can construct the space $\widehat{V}$ and endow it with the product topology since both $\mathcal{A}$ and $\mathbb{R}$ are metric spaces.

- We can then reconstruct the topological structure of V, since lemma **??** shows that $\mathcal{F} : V\widehat{V}$ is a homeomorphism and we know the topology of $\widehat{V}$.

- For a given point $\widehat{q} \in \widehat{V}$ we can use proposition 3.2.6 and the date 2.2.1 to determine all C^0-observation coordinates around $\widehat{q}$. We can then determine for each of these coordinates if they are also C^∞-observation coordinates, and find at least one such coordinates since existence is guaranteed. We then repeat that step for each $\widehat{q} \in \widehat{V}$ to find smooth coordinates on $\widehat{V}$. Since these coordinates make $\mathcal{F}$ a diffeomorphism this allows us to recover the differential structure of V.

- Finally we use lemma 3.2.7 to construct a metric G and time-orientation X on $\widehat{V}$ which is conformal to $\widehat{g} = \mathcal{F}_* g$ and makes $\mathcal{F}$ causal. $\mathcal{F} : (V, g|_U) \to (\widehat{V}, G)$ is thus a causal diffeomorphism and we can reconstruct the causal and conformal structure of V.

Given the data 2.2.1, we were able to reconstruct the conformal, differential and topological structure of V and since the data is invariant under conformal diffeomorphism, the first consequence of theorem **??** follows.

For the final part of theorem **??**, by **??**(1) we can identify all $q \in U \cap V$ using data 2.2.1 which uniquely determines the map. $\square$

Chapter 4

Boundary Reconstruction

4.1 Setting

In this section we will examine how we can extend our reconstruction result to the case where the observed set V is no longer contained within the interior of $J(p^-, p^+)$ but is now allowed to extend up to the boundary. In other words we want to recover the conformal structure of $J(p^-, p^+)$ from light cone observations made on the future null boundary $K = J(p^-, p^+) \setminus I^-(p^+)$.

This setting is complicated by the fact that as $q \in J(p^-, p^+)$ approaches the boundary, the light observation $\mathcal{P}_K(q)$ get increasingly warped and is degenerate if q is in the boundary.

Theorem 4.1.1 (Boundary Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exist a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let $V_j \subset J(p_j^-, p_j^+)$. We assume that no null geodesic starting in V_j has a conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a homeomorphism $\Phi : V_1 \to V_2$.((conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality))

Remark 4.1.2. ((Move this to intro)) We will again prove the previous theorem by working on just one globally hyperbolic Lorentzian manifold (M, g) with $p^\pm, V, K$ as above. We show that given the following data we can reconstruct the topology of V:

(1) The smooth manifold (with edge) K,

(2) the conformal class of $g|_K$,

(3) the set $\mathcal{P}_K(V)$.

4.2 Preliminaries

To extend the reconstruction up to the edge of $J(p^-, p^+)$ we need to introduce some new concepts.

Definition 4.2.1 (Unique minimum domain)**.** We define the *unique minimum domain* $D \subset J(p^-, p^+)$ to be

$$D := \{q \in J(p^-, p^+) \mid F_q \text{ has a unique minimum}\}. \tag{4.1}$$

Appendix A

Technical Lemmas

Lemma A.0.1 (Transverse Map)**.** *Let $f : M \to N$ be a smooth map transverse to the submanifold $L \subset N$ of codimension k and $f^{-1}(L)$ is nonempty. Then $f^{-1}(L)$ is a codimension k submanifold of M.*

Lemma A.0.2. *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be compact metric spaces. Let $f : X \times Y \to Z$ be a continuous functions and denote $f_x : Y \to Z; y \mapsto f_x(y) := f(x, y)$ for $x \in X$. Let $x_n \to x_0 \in X$ as $n \to \infty$ be a convergent sequence.*

Then $f_{x_n} \to f_{x_0}$ uniformly as $n \to \infty$.

Proof. Let $x_n \to x_0 \in X$ be a convergent sequence. We want to show that for any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon.$$

To that end let $\varepsilon > 0$. Then because X and Y are compact, we can use the Heine-Cantor theorem to get a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \wedge d_Y(y_1, y_2) < \delta \implies d_Z(f_{x_1}(y_1), f_{x_2}(x_2)) < \varepsilon.$$

Now if $N \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta \ \forall n \geq N$ and $y \in Y$ arbitrary we have $d_X(x_n, x_0) < \delta \wedge d_Y(y, y) < \delta$ which implies $d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$. Because $y \in Y$ was arbitrary we also get $max_{y \in Y} d_Z(f_{x_n}(y), f_{x_0}(y)) < \varepsilon$ and the proof is complete. □

Lemma A.0.3. *Let A be a first-countable topological space and $P : A \to \{false, true\}$ a property defined for all points $a \in A$. Suppose now that for any converging sequence $a_n \to a_0 \in A$ there exists a $N \in N$ such that $P(a_n)$ is true for all $n \geq N$.*

Then there exists an open neighborhood $O \in A$ of a_0 such that $P(a)$ is true for all $a \in O$.

Appendix B

Causality and Global Hyperbolicity

B.1 Causal Relations

In this first section we will establish which points in a Lorentzian manifold can be connected by timelike or lightlike paths under which circumstances.

We will take (M, g) to be a time-oriented Lorentzian manifold. First we will set up some basic causality structure:

Definition B.1.1. We write

1. $p \ll q$ if $p \neq q$ and there exist a future-pointing timelike curve from p to q,
2. $p < q$ if $p \neq q$ and there exist a future-pointing causal curve from p to q,
3. $p \leq q$ if $p = q$ or $p < q$.

We then define the *chronological future* and *causal future* of a point $p \in M$ as

$$\begin{aligned} I^+(p) &:= \{q \in M \mid p \ll q\} \\ J^+(p) &:= \{q \in M \mid p \leq q\}. \end{aligned}$$

We can extend these definitions to arbitrary sets by setting $I^+(A) := \bigcup_{p \in A} I^+(p)$ and $J^+(A)$ analogously.

Note that in the Minkowski case $\mathbb{R}^n_1$ the set $I^+(p)$ is open and $J^+(p) = \overline{I^+(p)}$ is closed. Furthermore $I^+(p)$ resp. $J^+(p)$ is the set of all $q \in R^n_1$ such that $\overrightarrow{pq}$ is timelike resp. causal. We will see that under sufficient conditions the first of the above facts also hold in the general case.

Corollary B.1.2. *If $x \ll y$ and $y \leq z$ or $x \leq y$ and $y \ll z$, then $x \ll z$.*

Proof. This follows immediately from proposition **??** □

Let $\mathcal{U} \subset M$ be an open set. Then the *intrinsic* causality relations in $\mathcal{U}$ imply the ones in M. In particular, if we denote by $I^+(A,\mathcal{U})$ the chronological future in $\mathcal{U}$ of the set $A \subset \mathcal{U}$, we have that $I^+(A,\mathcal{U}) \subset I^+(A) \cap \mathcal{U}$.

With this in mind we will now consider the case of a convex set $\mathcal{C}$:

Lemma B.1.3. *Let $\mathcal{C}$ be a convex open set in M, then*

(1) *For $p \neq q$ in $\mathcal{C}$, $q \in J^+(p,\mathcal{C}) \iff \overrightarrow{pq}$ is future-pointing causal.*

(2) *$I^+(p,\mathcal{C})$ is open in $\mathcal{C}$ (hence also in M).*

(3) *$J^+(p,\mathcal{C})$ is the closure in $\mathcal{C}$ of $I^+(p,\mathcal{C})$.*

(4) *The relation $\leq$ is closed on $\mathcal{C}$, i.e. if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{C}$ then $q_n \in J^+(p_n,\mathcal{C})$ for all n implies $q \in J^+(p,\mathcal{C})$.*

(5) *A causal curve α contained in a compact $K \subset \mathcal{C}$ is continuously extendable.*

Proof. Properties (1-3) follow from the fact that the convex open set $\mathcal{C}$ is via the exponential map everywhere diffeomorphic to the tangent space $T_pM \simeq \mathbb{R}^n_1$ and thus the properties of the minkovski space also apply here.

To prove (4) we first note that by (1) we have that $q_n \in J^+(p_n,\mathcal{C})$ implies $\overrightarrow{p_nq_n}$ is future-pointing causal. Now by **??** $(p_n,q_n) \mapsto \overrightarrow{p_nq_n}$ is continuous and thus $\overrightarrow{pq}$ is also future-pointing causal. Fact (4) then follows from again applying property (1).

To prove (5) we suppose that the domain of α is $[0,B)$ where $B < \infty$. As K is compact there exist a sequence $s_i \to B$ such that $\alpha(s_i)$ converges to a point $p \in K$. We must now prove that for any sequence $t_i \to B$ such that $\alpha(t_i) \to q$ we have $p = q$. Assume by contradiction that $p \neq q$. By possibly taking subsequences we can achieve that $s_i \leq t_i \leq s_{i+1}$. Then since α is causal we get $\alpha(s_i) \leq \alpha(t_i) \leq \alpha(s_{i+1})$ and thus $\alpha(t_i) \in J^+(\alpha(s_i),\mathcal{C})$ and $\alpha(s_{i+1}) \in J^+(\alpha(t_i),\mathcal{C})$. By (4) we now have $q \in J^+(p,\mathcal{C})$ and $p \in J^+(q,\mathcal{C})$ which by (1) implies that $\overrightarrow{pq}$ is at the same time, future and past pointing, a contradiction. □

(2) can be generalized:

Lemma B.1.4. *The relation $\ll$ is open; that is if $p \ll q$ there exist neighborhoods $\mathcal{U},\mathcal{V}$ of p and q respectively such that for any $p' \in \mathcal{U}$ and $q' \in \mathcal{V}$ we still have $p \ll q$.*

Proof. Let σ be a timelike curve from p to q. Let $\mathcal{C}$ be a convex open neighborhood of q and q^- a point on σ which comes before q and still lies in $\mathcal{C}$. Then $I^+(q^-,\mathcal{C})$ is also an open neighborhood of q. If we proceed analogously for p with p^+ and $\mathcal{C}'$. Then we get that $I^-(p^+,\mathcal{C}')$ and $I^+(q^-,\mathcal{C})$ are the neighborhoods we were looking for. □

Note that this lemma implies that $I^+(A)$ is open for any set A.

We can now further develop the topology of causality:

Lemma B.1.5. *For $A \subset M$ we have that:*

(1) $\operatorname{int} J^+(A) = I^+(A)$

(2) $J^+(A) \subset \overline{I^+(A)}$ *with equality iff* $J^+(A)$ *is closed.*

Proof. To prove (1) we first note that $I^+(A)$ is open as remarked above. Also $I^+(A) \subset J^+(A)$ by definition. Now if $q \in \operatorname{int} J^+(A)$, then for a convex neighborhood $\mathcal{C}$ of q, $I^-(q, \mathcal{C})$ contains a point of $J^+(A)$. Hence $q \in I^+J^+(A) = I^+(A)$.

Now to prove part (2): The equality assertion is clear, as $I^+(A) \subset J^+(A)$. Note that is suffices to consider only the case where $A = \{p\}$, since the general case then follows from

$$\bigcup_{p \in A} J^+(p) \subset \bigcup_{p \in A} \overline{I^+(p)} \subset \overline{\bigcup_{p \in A} I^+(p)}.$$

Let us thus consider the case of $\overline{I^+(p)}$. Clearly $p \in \overline{I^+(p)}$. Thus we only need to consider $p < q$. Let σ be a causal path from p to q. Let $\mathcal{C}$ be a convex neighborhood of q and q^- a point lying on γ in $\mathcal{C}$. Now by lemma B.1.3, $q^- \in J^+(p)$ and $I^+(J^+(p)) = I^+(p)$ we have

$$q \in J^+(q^-, \mathcal{C}) = \overline{I^+(q^-, \mathcal{C})} \subset \overline{I^+(J^+(p))} = \overline{I^+(p)}.$$

□

B.2 Causality Conditions

Definition B.2.1 (Strong Causality Condition)**.** We say that the *strong causality condition* holds at $p \in M$ if for any given neighborhood $\mathcal{U}$ of p there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that any causal curve with endpoints in $\mathcal{V}$ lies entirely within $\mathcal{U}$.

Intuitively this condition states that any causal curve which starts arbitrarily close to p and leaves some fixed neighborhood cannot return arbitrarily close to p. In particular this rules out closed causal loops.

The following lemma is in line with this intuition:

Lemma B.2.2. *Suppose the strong causality condition holds on a compact subset K of M. If α is a future-inextendable causal curve that starts in K, then α eventually permanently leaves K. That is, there exists a $s > 0$ such that $\alpha(t) \notin K$ for all $t \geq s$.*

Proof. Assume that the conclusion is false. Thus if the domain of α is $[0, B)$ for $B \leq \infty$, by the compactness of K, there exists a sequence $s_i \to B$ such that $\alpha(s_i) \to p \in K$. Since α has no future endpoint there must be some other sequence $t_j \to B$ such that $\alpha(t_j)$ does not converge to p. After taking further subsequences we can assume that some neighborhood $\mathcal{U}$ of p contains no $\alpha(t_j)$ and the sequences are alternating, i.e. $s_1 < t_1 < s_2 < t_2 < s_3 < \dots$. But now the curves $\alpha|_{[s_k, s_{k+1}}$ always leave the neighborhood $\mathcal{U}$ but return arbitrarily close and thus violated the strong causality condition. □

Under these conditions there exists a very useful lemma for constructing geodesics joining some $p < q$.

Lemma B.2.3. *Suppose the strong causality condition holds on a compact subset $K \subset M$. Let (α_n) be a sequence of future-pointing causal curve segments in K such that $\alpha_n(0) \to p$ and $\alpha_n(1) \to q \neq p$. Then there exists a future-pointing causal broken geodesic γ from p to q and a subsequence (α_m) of (α_n) such that $\lim_{m\to\infty} L(\alpha_m) \leq L(\gamma)$.*

This lemma is proven by leveraging the existence of quasi-limits together with the fact that given the strong causality condition, future inextendable curves must eventually leave a compact set K permanently. This proof can be found in detail in [One83, Lemma 14.14].

B.3 Time Separation Function

There is a natural way to generalize the notion of the separation of points $p \leq q$ in $\mathbb{R}^n_1$ to an arbitrary Lorentzian manifold M.

Definition B.3.1 (Time Separation)**.** Let $p, q \in M$, we define the *time separation* $\tau(p, q)$ from p to q as

$$\tau(p, q) \coloneqq \sup\{L(\alpha) \mid \alpha \text{ is a future-pointing causal curve segment from } p \text{ to } q\}.$$

We have $\tau(p, q) = \infty$ if the length is unbounded and $\tau(p, q) = 0$ if the separation is spacelike, i.e. $q \notin J^+(p)$. Note that for any causal path α the function $s \mapsto \tau(\alpha(0), \alpha(s))$ is monotonously increasing.

Lemma B.3.2. (1) *$\tau(p, q) > 0$ iff $p \ll q$.*

(2) *Reverse triangle inequality: If $p \leq q \leq r$, then $\tau(p, q) + \tau(q, r) \leq \tau(p, r)$.*

Proof. (1) If $\tau(p,q) > 0$ there exists a future-pointing causal curve α from p to q with $L(\alpha) > 0$. Thus α cannot be a null pregeodesic. By proposition **??** there now exists a timelike curve from p to q. The converse follows immediately from the definition.

(2) If there are future-pointing causal curves from p to q and q to r we can pick causal curves α from p to q and β from q to r such that, for an arbitrarily small $\delta > 0$

$$L(\alpha) \geq \tau(p,q) - \delta/2, \quad L(\beta) \geq \tau(q,r) - \delta/2.$$

We then have

$$\tau(p,r) \geq L(\alpha + \beta) = L(\alpha) + L(\beta) \geq \tau(p,q) + \tau(q,r) - \delta$$

for any $\delta > 0$, as required. If there is no future-pointing causal path from WLOG p to q then $\tau(p,q) = 0$ and the result follows immediately. □

Lemma B.3.3. *The time separation function $\tau : M \times M \to [0,\infty]$ is lower semicontinuous.*

Proof. If $\tau(p,q) = 0$ there is nothing to prove. Suppose $q \in I^+(p)$ and $0 < \tau(p,q) < \infty$.

Given $\delta > 0$ we must find neighborhoods $\mathcal{U}, \mathcal{V}$ such that for all $p' \in \mathcal{U}, q' \in \mathcal{V}$ we have $\tau(p',q') > \tau(p,q) - \delta$.

Let α be a timelike curve from p to q with $L(\alpha) > \tau(p,q) - \delta/3$. Let $\mathcal{C}$ be a convex neighborhood of q and q^- on α and in $\mathcal{C}$. Since in convex neighborhoods the map $q' \mapsto L(\sigma_{q^-q'})$, where $\sigma_{q^-q'}$ is the radial geodesic, is continuous there exists a neighborhood $\mathcal{V}$ of q such that for all $q' \in \mathcal{V}$ we have $L(\sigma_{q^-q'}) > L(\sigma_{q^-q}) - \delta/3$.

By analogous argument we get that there exists a p^+ and neighborhood $\mathcal{U}$ of p such that for all $p' \in \mathcal{U}$ we have $L(\sigma_{p'p^+}) > L(\sigma_{pp^+}) - \delta/3$.

Putting this together and using the fact that $L(\sigma_{q^-q}) \geq L(\alpha|_{[q^-,q]})$, resp $L(\sigma_{pp^+}) \geq L(\alpha|_{[p,p^+]})$ we have

$$\begin{aligned}
\tau(p',q') &\geq L(\sigma_{p'p^+}) + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q'}) \\
&> L(\sigma_{pp^+}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\sigma_{q^-q}) - \delta/3 \\
&\geq L(\alpha|_{[p,p^+]}) - \delta/3 + L(\alpha|_{[p^+,q^-]}) + L(\alpha|_{[q^-,q]}) - \delta/3 \\
&= L(\alpha) - 2\delta/3 > \tau(p,q) - \delta
\end{aligned}$$

as required. □

B.4 Globally Hyperbolic Manifolds

It is convenient to define

$$J(p,q) := J^+(p) \cap J^-(q)$$

Note that any future-pointing causal path from p to q must be contained in $J(p,q)$.

We can now give a powerful condition as to when the supremal path of $\tau(p,q)$ is actually achieved:

Proposition B.4.1. *For $p < q$, if the set $J(p,q)$ is compact and the strong causality condition holds on it, then there is a causal geodesic from p to q of length $\tau(p,q)$.*

Proof. Let (α_n) be a sequence of future-pointing curve segments from p to q whose lengths converge to $\tau(p,q)$ (the existence of such a sequence is guaranteed as $\tau(p,q)$ is the supremum of such curves). These curves are all in $J(p,q)$ which is compact. Hence, by lemma B.2.3, there exists a broken causal geodesic γ with

$$\tau(p,q) = \lim_{n\to\infty} L(\alpha_n) \le L(\gamma) \le \tau(p,q).$$

But now, if γ were to have any actual breaks, by corollary **??** there would exist a longer curve, which is a contradiction. □

Note that this implies in particular that $\tau(p,q)$ is always finite if $J(p,q)$ is compact.

This motivates the following definitions:

Definition B.4.2 (Globally Hyperbolic)**.** A subset $\mathcal{H} \subset M$ is called *globally hyperbolic* if (1) the strong causality conditions holds and (2) for all $p, q \in \mathcal{H}$ with $p < q$, $J(p,q)$ is compact.

Definition B.4.3. Let $\gamma : [0,T]$ be a causal geodesic from $p = \gamma(0)$ to $q = \gamma(T)$. We call γ *maximal* if we have $L(\gamma) = \tau(p,q)$ and hence $L(\gamma|_{[0,t]}) = \tau(p,\gamma(t))$ for all $0 \le t \le T$.

Lemma B.4.4. *If $\mathcal{U}$ is globally hyperbolic open set, then the time separation function $\tau : \mathcal{U} \times \mathcal{U} \to [0,\infty)$ is continuous.*

Proof. We know from a previous lemma that τ is always lower semicontinuous. Suppose, for contradiction, that is is not upper semicontinuous at (p,q), i.e. there exists a number $\delta > 0$ and sequences $p_n \to p$ and $q_n \to q$ such that $\tau(p_n,q_n) \ge \tau(p,q) + \delta$ for all n.

Since $\tau(p_n,q_n) > 0$, there exists a causal curve α_n from p_n to q_n such that $L(\alpha_n) > \tau(p_n,q_n) - 1/n$. Because $\mathcal{U}$ is open it contains also the slightly earlier resp. later points $p^- \ll p$, $q^+ \gg q$. As $I^+(p^-)$ resp. $I^-(q^+)$ are open neighborhoods of p resp. q, p_n and q_n are eventually contained in them and we can WLOG assume that they always are. It follows that the curves α_n are all contained in the compact set $J(p^-,q^+)$. Now we can apply lemma B.2.3 to obtain a broken geodesic γ from $p = \lim p_n$ to $q = \lim q_n$ with

$$L(\gamma) \ge \lim_{n\to\infty} L(\alpha_n) \ge \lim_{n\to\infty} \tau(p_n,q_n) \ge \tau(p,q) + \delta.$$

But since δ itself is a curve from p to q this is a contradiction. □

Lemma B.4.5. *If $\mathcal{U} \subset M$ is a globally hyperbolic open set, then the causality relation $\leq$ is closed on $\mathcal{U}$.*

Proof. We again have to show that if $p_n \to p$ and $q_n \to q$ with all points in $\mathcal{U}$ and $p_n \leq q_n$ for all n, then also $p \leq q$.

If $p = q$ the result follows immediately. We can thus assume $p \neq q$ and $p_n < q_q$ for all n. Let α_n then be a causal curve from p_n to q_n. As in the preceding proof, all α are in $J(p^-, q^+)$ and by lemma $B.2.3$, there exists a causal curve γ from p to q. This implies $p < q$. $\square$

Remark B.4.6. We can now summarize the results from this section for the case where (M, g) is a globally hyperbolic Lorentzian manifold:

For any $p \in M$, $I^{\pm}(p)$ is open and $J^{\pm}(p)$ is closed with $\operatorname{int} J^{\pm}(p) = I^{\pm}(p)$ and $\overline{I^{\pm}(p)} = J^{\pm}(p)$.

For the time separation function we can say the following:

(1) $\tau(p, q) > 0$ iff $p \ll q$.

(2) $\tau(x, y)$ satisfies the *reverse triangle inequality*:

$$\tau(x, y) + \tau(y, z) \leq \tau(x, z) \quad \text{for } x \leq y \leq z.$$

(3) $(x, y) \mapsto \tau(x, y)$ is continuous in $M \times M$.

(4) For $x < y$ there exists a causal geodesic γ from x to y such that $L(\gamma) = \tau(x, y)$.

B.5 Light Cones

Definition B.5.1 (Light Cones)**.** Let

$$L_pM := \{v \in T_pM \setminus \{0\} \mid \langle v, v\rangle = 0\}$$

be the set of null vectors at $p \in M$. We can split L_pM into L_p^+M and L_p^-M the future- and past-pointing null vectors. Furthermore we can define the bundle $LV := \bigcup_{p \in V} L_pV \subset TM$.

We now define the *future light cone* of $p \in M$ to be

$$\mathcal{L}_p^+ := \exp_p(L_p^+M) \cup \{p\}.$$

$\mathcal{L}_p^-$ is defined analogously.

Note that for $p \in M$ we have $\mathcal{L}^+{}_p \subset J^+(p)$ and $\mathcal{L}^+{}_p \supset J^+(p) \setminus I^+(p)$ if M is globally hyperbolic.

B.5.1 Null Cut Points

To better understand the behavior of null geodesics we will introduce so called *cut points* which intuitively are the points where a null geodesic stops being maximal. Such cut points are the product of curvature as in the minkovski case there are none.

For $(p, v) \in TM$ with $v \neq 0$ let $\mathcal{T}(x, v) \in (0, \infty]$ be the maximal value for which $\gamma_v : [0, \mathcal{T}(x, v))$ is defined.

Definition B.5.2 (Cut Locus Function and Cut Points)**.** For $(p, v) \in L^+M$ we define the *cut locus function*

$$\rho(p, v) \coloneqq \sup\{s \in [0, \mathcal{T}(p, v)) \mid \tau(x, \gamma_v(s)) = 0\}.$$

The points $x_1 = \gamma_v(t_1), x_2 = \gamma_v(t_2), t_1 < t_2 \in [0, t_0]$ are called *cut points* on $\gamma_v([0, t_0])$ if $t_2 - t_1 = \rho(x_1, v_1)$ for $v_1 = \gamma_v'(t_1)$. In particular, the point $p(x, v) = \gamma_v(s)|_{s=\rho(x,v)}$, if it exists, is called the *first cut point* on the geodesic γ_v.

Lemma B.5.3. *Let $p < q \in M$. Suppose there are two distinct future-pointed null geodesics $\alpha : [0, a) \to M, \beta : [0, b) \to M$ from $p = \alpha(0) = \beta(0)$ through $q = \alpha(1) = \beta(1)$. Then both geodesics have a cut point in* $[0, 1]$*, i.e. q comes on or after the first cut point.*

Proof. We will show that for any $s \in (1, a)$ we have $\tau(p, \alpha(s)) > 0$ since this implies that α must have a cut point at or before 1. Let $\gamma = \beta|_{[0,1]} + \alpha|_{(1,a)}$ be the broken null geodesic obtained by traveling from p to q on β and then continuing on α. Thus for any $s \in (1, a)$, $\gamma|_{[0,s]}$ is a broken null geodesic and by proposition **??** there exists a timelike curve from p to $\gamma(s) = \alpha(s)$ which implies $\tau(p, \alpha(t)) > 0$ as required.

The proof for β follows analogously. □

Lemma B.5.4. *Let now (M, g) be globally hyperbolic, and let $p < q \in M$ with $\tau(p, q) = 0$. Assume that $p_n \to p$ and $q_n \to q$ with $p_n \leq q_n$. Let γ_n be maximal geodesics joining p_n to q_n with initial direction v_n. Then the set (v_n) has a limit w and γ_w is a maximal null geodesic from p to q.*

Proof. As in the proof of lemma B.4.4 there exist $p^- \ll p$ $q^+ \gg q$ such that p_n, q_n, γ_n all lie in $J(p^-, q^+)$ which is compact. By lemma B.2.3 there exists a future-pointing broken geodesic λ which is the quasi-limit of γ_n (see [One83, Def. 14.7]). Thus there exists a convex neighborhood $\mathcal{C}$ of p and a sequence s_n such that $\lim_{n\to\infty} x_n \coloneqq \gamma_n(s_n) \to x = \lambda(s) \in \mathcal{C}$ and $\gamma_n|_{[0,s_n]} \in \mathcal{C}$. Note that since γ_n is a maximal geodesic we have that $\gamma_n|_{[0,s_n]}$ is the unique radial geodesic from p_n to x_n

and we have $v_n = \gamma_n'(0) = \overrightarrow{p_n x_n}$. Now by lemma B.1.3 $(p', q') \to \overrightarrow{p'q'}$ is continuous and we thus have that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \overrightarrow{p_n x_n} = \overrightarrow{px} =: w.$$

By construction, see [One83, Lemma 14.14], $\lambda|_{[0,s]}$ is the radial geodesic in $\mathcal{C}$ from p to x and thus also $\lambda'(0) = \overrightarrow{px} = w$.

It remains to show that λ is an actual unbroken geodesic. But since $L(\lambda) \leq \tau(p, q) = 0$ it follows from proposition **??** that λ must be smooth null geodesic.

Thus also $\lambda = \gamma_w$ since λ is a geodesic with initial velocity w. □

Theorem B.5.5 (Cut Point Characterization). *Let (M, g) be globally hyperbolic. Then for $(x, p) \in L^+M$, $p(x, v)$ is either the first conjugate point on γ_v or the first point on γ_v where there exists another null geodesic γ_w from x to $p(x, v)$ where $v \neq cw$.*

Proof. Let $q = p(x, v) = \gamma_v(t)$ be the first cut point on the null geodesic γ_v. Let furthermore $t_n \to t$ be a monotonously decreasing sequence such that $\gamma_v(t_n)$ is well defined for all n. Now since M is globally hyperbolic there exist maximal geodesics γ_n from p to $q_n := \gamma_v(t_n)$. Note that since $q = \gamma_v(t)$ is the first cut point of γ_v we have $\tau(p, \gamma_v(t_n)) > 0$ for all n. But since γ_v is a null geodesic, it has zero length and cannot be maximal up until any of the t_n. Thus γ_n cannot equal γ_v and in particular $v_n := \gamma_n'(0) \neq v$ for all n. We can apply the previous lemma to obtain a geodesic γ_w and a null vector w such that $v_n \to w$ and γ_w is a maximal geodesic from p to q.

Now we can distinguish to cases: If $v \neq w$ there exist two distinct maximal geodesics, namely γ_v and γ_w joining p and q.

If however, $v = w$ we can view γ_n as a variation of γ_v through geodesics starting at p which additionally satisfy that the limiting variation at q is zero (since the q_n converge to q). q is thus a conjugate point of γ_v. □

Proposition B.5.6. *For (M, g) globally hyperbolic, $\rho(p, v)$ is lower semicontinuous.*

Proof. It suffices to prove that if $(p_n, v_n) \to (p, v)$ in TM and $\rho(p_n, v_n) \to A$ in $\mathbb{R} \cup \{\infty\}$, then $\rho(p, v) \leq A$. If $A = \infty$ there is nothing to prove we will thus assume that $A < \infty$. We further assume $\rho(p, v) > A$ to derive a contradiction.

We can choose a $\delta > 0$ such that $A + \delta < \rho(p, v)$ and $q := \gamma_v(A + \delta)$ exists. We define $b_n = \rho(p_n, v_n) + \delta$ and can force for n large enough $b_n < \rho(p, v)$ and $\gamma_n := \gamma_{v_n}$ defined past b_n. We then denote $q_n = \gamma_n(b_n)$.

Since $b_n > \rho(p_n, v_n)$, γ_n cannot be maximal from p_n to q_n. Now, since M is globally hyperbolic, by B.4.1 we can find maximal null geodesics σ_n from p_n to q_n with initial velocity w_n. By B.5.4 $w_n \to w$ with γ_w a maximal null geodesic from p to q.

Since q cannot be conjugate point (because this would make it a cut point) we cannot have $w_n \to w = v$. Thus we must have $w \neq v$, but this implies that there are two distinct maximal geodesics from p to q, namely γ_v and γ_w, thus $q = \gamma : v(A+\delta)$ must be a cut point of γ_v. This implies that $\rho(p, v) \leq A+\delta$, which is a contradiction since we assumed $A + \delta < \rho(p, v)$. □

B.6 Conformal Structure

Definition B.6.1 (Conformal Diffeomorphism)**.** A map $\Psi : (M_1, g_1) \to (M_2, g_2)$ is called a *conformal diffeomorphism* or *homothety* if $\Psi : M_1 \to M_2$ is a diffeomorphism and $\Psi^* g_2 = e^{2\Omega} g_1$ where $\Omega \in C(M_1)$ and nowhere zero.

We further say that $\Psi : V_1 \to V_2$ *preserves causality* if $x < y$ implies $\Psi(x) < \Psi(y)$.

It can be calculated that the connections D on M_1 and $\widetilde{D}$ on M_2 are related by the following equation:

$$\widetilde{D}_{\Psi_* X} \Psi_* Y = f_* D_X Y + X(\Omega)\Psi_* Y + Y(\Omega)\Psi \tag{B.1}$$

Proposition B.6.2. *$\gamma : I \to M_1$ is a null geodesic if, and only if $\sigma \coloneqq \Psi \circ \gamma$ is also a null geodesic.*

Proof. By the symmetry of the situation (i.e. Ψ^{-1} is also a conformal diffeomorphism) is suffices to show only one direction. Suppose now $\gamma : I \to M_1$ is a null geodesic on M_1 and $\sigma = \Psi \circ \gamma$. By the previous equation we have

$$\widetilde{D}_{\sigma'} \sigma'(t) = 2\gamma'(t)(\Omega)\sigma'(t).$$

We can now reparameterize σ such that $2\gamma'(t)(\Omega)$ is always zero and σ is a null geodesic as desired. □

The following proposition asserts that the conformal data of a metric can be reconstructed from knowledge of the null cones:

Proposition B.6.3. *Let M be a smooth manifold of dimension $n \geq 3$ with Lorentzian metrics g and h. Suppose that for any $v \in TM$ we have $g(v, v) = 0$ iff $h(v, v) = 0$. Then there exists a smooth nowhere zero function $\Omega \in C(M)$ such that $g = e^{2\Omega} h$.*

Proof. The proof follows from the fact that the nullcones are given by systems of quadratic equations and some linear algebra. It can be found in more detailed form at [Bee81, Theorem 2.3] □

We can see that even the cut locus is conserved under conformal transformation:

Proposition B.6.4. *Let $\gamma : [0, a) \to (M_1, g_1)$ be a null geodesic with first cut point $q = \gamma(t_0)$. Then $q' = \Psi(q)$ is the first null cut point of $p' = \Psi(p)$ along the null pregeodesic $\Psi \circ \gamma$.*

Proof. We can WLOG (since Ψ either causal or anti-causal and the proof of the anti-causal case is analogous) assume that Ψ is causal and γ is future-pointing. $\Psi \circ \gamma$ is thus also a future-pointed pre-geodesic which can be reparameterized to a null geodesic σ with $p' = \sigma(0)$ and $q' = \sigma(t_1)$. We will denote by τ_j the time separation function on M_j.

We first show that $\tau_2(p', \sigma(t)) = 0$ for $t \in [0, t_1]$, i.e. that q', if it is a cut point, is indeed the first cut point. To obtain a contradiction we assume that there exists a $t \in [0, t_1]$ with $\tau_2(p', \sigma(t)) > 0$. We my thus find a future-pointing causal curve β from p' to $\sigma(t)$ with $L_{g_2}(\beta) > 0$. Now $\Psi^{-1} \circ \beta$ is a future-directed causal curve in M_1 from p to $\Psi^{-1}(\sigma(t))$ with $L_{g_1}(\Psi^{-1} \circ \beta) > 0$. But since $t \leq t_1$ we have $\Psi^{-1}(\sigma(t) = \gamma(t_2)$ with $t_2 \in [0, t_0]$ and thus $\tau_1(p, \gamma(t_2)) > 0$. This would mean that γ has a cut point at t_2, before t_0 which is a contradiction.

We will now show that $\tau_2(q', \sigma(t)) > 0$ for any $t > t_1$, as this would make $q' = \sigma(t_1)$ a future cut point of p along σ as required. Let thus $t > t_1$. There exists a $t_2 > t_0$ such that $\Psi^{-1}(\sigma(t)) = \gamma(t_2)$. Now since $\gamma(t_2)$ lies past the first cut point of γ, we have $\tau_1(p, \gamma(t_2)) > 0$ and there exists a future-pointing causal curve α in M_1 with $L_{g_1}(\alpha) > 0$. Now $\Psi \circ \alpha$ is also a future-pointing causal curve from p' to $\sigma(t)$ with $L_{g_2}(\Psi \circ \alpha) > 0$ and thus $\tau_2(p', \sigma(t)) \geq L_{g_2}(\Psi \circ \alpha) > 0$ as required. □

B.7 Short Cut Argument

Theorem B.7.1. *Let (M, g) be globally hyperbolic and $p < q$ in M, then there exists a future-pointed null geodesic $\gamma : [0, a) \to M$ from $p = \gamma(0)$ to $q = \gamma(t_0)$ and we have $\tau(p, q) = 0$ if and only if γ has no cut points in $[0, t_0)$.*

Proof. The existence of γ is assured by proposition B.4.1. Now suppose we have $\tau(p, q) > 0$ by the continuity of τ there must be a cut point $\gamma(t)$ before q, i.e. $t < t_0$. Suppose on the other hand that γ has a cut point $\gamma(t)$ with $t < t_0$. Then by the definition of cut points we must have $\tau(p, q) > 0$ as $t < t_0$. □

We can apply this theorem to the case of a path from p to q which is the union of the future pointing light-like pregeodesics $\gamma_{p,v}([0, t_0])$ and $\gamma_{x_1,w}([0, t_1])$ where $x_1 = \gamma_{p,v}(t_0), q = \gamma_{x_1,w}(t_1)$. Let $\zeta = \gamma'_{p,v}(t_0)$. If there are no $c > 0$ such that $\zeta = cw$ or equivalently, the union of these two paths is not also a light-like pregeodesic, then we have $\tau(p, q) > 0$. By B.4.6, this implies that there exists a time-like geodesic from p to q and thus $\tau(p, q) > 0$. This is called a *short-cut argument.*

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