Title

Alexander Uhlmann*
Advisor: Prof. Dr. Peter Hintz

July 18, 2022

Abstract

((TODO))

*auhlmann@ethz.ch

Chapter 1

Introduction

1.1 Main Results

((Introduce Notation etc.))

Definition 1.1.1 (Suitable)**.** We call $p^- \ll p^+ \in M$ *suitable* if p^+ has no past cut points in $\mathcal{L}^-_{p^+} \cap J^+(p^-)$ and p^- has no future cut points in $\mathcal{L}^+_{p^-} \cap J^-(p^+)$.

Theorem 1.1.2 (Interior Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

Now let $V_j \subset J(p_j^-, p_j^+)^{\circ}$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Theorem 1.1.3 (Boundary Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$* suitable *in M_j we denote $K_j = \mathcal{L}^-_{p_j^+} \cap J^+(p_j^-)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$.*

Now let $V_j \subset J(p_j^-, p_j^+) \setminus p_j^+$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Chapter 2

Geometric Preliminaries

2.1 Null Conjugate Points

((Leave this here?))

Definition 2.1.1 (Null Conjugate Point)**.** Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. We then call $p = \gamma_{q,w}(b)$ a *null conjugate point* if there exists a nontrivial variation $\mathbf{x} : [0, b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma_{q,w}$ through null geodesics such that $\mathbf{x}_v(b, 0) = 0$.

We have the following useful characterization:

Proposition 2.1.2. *Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. Then $p = \gamma_{q,w}(b)$ is a null conjugate point if and only if $\exp_q : L_qM \to M$ is singular at bw, i.e. if there exists a nonzero $\xi \in T_{bw}(L_qM)$ such that $d\exp_q(\xi) = 0$.*

Proof. We begin by proving the backwards direction and to that end assume that there exist a nonzero $\xi \in T_{bw}(L_qM)$ such that $d\exp_q(\xi) = 0$. By the construction of the tangent space there thus exists a non-constant path $\xi : (-\varepsilon, \varepsilon) \to L_qM$ with $\xi(0) = bw$. This allows us to construct the variation $\mathbf{x}(u, v) = \exp_q(\frac{u}{b}\xi(v))$ which has $\mathbf{x}(t, 0) = \gamma_{q,w}(t)$ and is a variation through null geodesics. Finally we have $\mathbf{x}_v(b, 0) = d\exp_q(\xi) = 0$ by the chain rule.

For the other direction we first note that by definition $\mathbf{x}(u, v) = \exp_q(u\mathbf{x}_u(0, v))$ and $\mathbf{x}_u(0, v) \in L_qM$ as $\mathbf{x}$ is a variation through *null* geodesics. Now again by the chain rule we have $0 = \mathbf{x}_v(b, 0) = d\exp_q|_{bw} \circ \frac{d}{dv}(bx_u(0, v))|_{v=0}$. But since $\xi := \frac{d}{dv}(bx_u(0, v))|_{v=0} \in T_{bw}(L_qM)$ we are done. □

Null conjugate points are also conformal invariants:

Proposition 2.1.3. *Let $\Phi : (M, g) \to (N, h)$ be a conformal diffeomorphism and $\gamma : [0, b] \to M$ a null geodesic. Then $\gamma(b)$ is a null conjugate point of γ if and only if $\Psi(\gamma(b))$ is a null conjugate point of $\Psi \circ \gamma$.*

Proof. ((Cite relevant prop)) Because of the symmetry of the situation we only need to prove one direction and suppose that $\gamma(b)$ is a null conjugate point of γ. We thus have a variation $\mathbf{x}$ of γ through null geodesics. But since Φ maps null geodesics to null geodesics, $\Phi \circ \mathbf{x}$ is a variation of $\Phi \circ \gamma$ through null geodesics in N, which implies that $\Phi(\gamma(b))$ is a null conjugate point of $\Phi \circ \gamma$. $\square$

2.2 Geometry of the Light Cone Observations

From now on $(M, g), K, V, p^+, p^-$ be as in theorem 1.1.2 (we suppress the indices to simplify notation). ((More explanation))

2.2.1 Parameterization of Observations

Lemma 2.2.1. *We have:*

(1) $K = J(p^-, p^+) \setminus I^-(p^+)$,

(2) *There exists a surjective smooth map* $\Theta : S^{n-1} \times [0, 1] \to K$ *such that the curves* $\mu_a := t \mapsto \Theta(a, t), a \in S^n$ *are null geodesics,*

$$\Theta(S^{n-1} \times \{1\}) = \{p^+\}, \quad R := \Theta(S^{n-1} \times \{0\}) = K \setminus I^+(p^-)$$

and $\Theta : S^{n-1} \times [0, 1) \to K \setminus p^+$ *is a diffeomorphism.*

(3) $\mathcal{L}^-_{p^+} \cap V = \emptyset$ *and* $\mathcal{L}^-_{p_0} \cap V = J^-(p_0) \cap V = \emptyset \quad \forall p_0 \in R$.

Proof. (1) We first rewrite $J(p^-, p^+) \setminus I^-(p^+) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ and immediately get $(J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-) \subset \mathcal{L}^-_{p^+} \cap J^+(p^-) = K$ as $J^-(p^+) \setminus I^-(p^+) \subset \mathcal{L}^-_{p^+}$. For the other inclusion we note that by assumption for $p \in K$ we have $\tau(p, p^+) = 0$ and $p \in \mathcal{L}^-_{p^-}$. This implies $p \in J^-(p^+) \setminus I^-(p^+)$. Furthermore $p \in K$ also implies $p \in J^+(p^-)$. Putting this together we get $p \in (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$ proving the equality.

For (2) we first note that $p^- \notin K$ because $p^+ \gg p^-$ implies $\tau(p^-, p^+) > 0$ which would make p^- a cut point were it in K, violating our assumption. Thus also $p^- \notin \mathcal{L}^-(p^+)$. This implies that $\mathcal{L}^-_{p^+}$ and $\mathcal{L}^+_{p^-}$ are transversal.

Next we note that the exponential map

$$\exp_{p^+} : L^-_{p^+}M \simeq S^{n-1} \times \mathbb{R}_+ \to \mathcal{L}^-_{p^+}$$

is smooth and surjective.

We now aim to construct a smooth surjecive map $\theta : S^{n-1} \times [0,1] \to \exp_{p^+}^{-1}(K)$ which is a diffeomorphism on $S^{n-1} \times [0,1)$. To that end we look at the set of *unit null directions*

$$CL^-_{p^+}M := \{v \in L^-_{p^+}M \mid \|v\|_{g^+} = 1\} \simeq S^{n-1}$$

for some riemannian metric g^+ on M. By ((Leavescompact)) for a given null direction $v \in CL^-_{p^+}M$ there exists a $s_v > 0$ such that $\gamma_{p^+,v}(s_v) \in J^+(p^-)$ but $\gamma_{p^+,v}(s') \notin J^+(p^-)$ for all $s' > s_v$. Furthermore for any $s' \leq s_\mu$ we have $\gamma_{p^+,v}(s') \in J^+(p^-)$ because we can append the lightlike path from p^- to $\gamma_{p^+,v}(s_v) \in J^+(p^-)$ to $\gamma_{p^+,v}|_{[s',s_v]}$ and get a lightlike path from p^- to $\gamma_{p^+,v}(s')$. We also have $\gamma_{p^+,v}(t_v) \notin I^+(p^-)$ because $I^+(p^-)$ is open which would imply the existence of a $t' > t_v$ such that $\gamma_{p^+,v}(t') \in I^+(p^-) \subset J^+(p^-)$ violating the maximality of t_v. Finally, because $\exp_{p^+}$ is transverse to $\mathcal{L}^+_{p^-}$, $exp^-_{p^+}1(\mathcal{L}^+_{p^-})$ is a smooth submanifold of $L^-_{p^+}M$, by lemma A.0.1. This implies that the map that $v \mapsto t_v$ is smooth.

We now define

$$\begin{aligned} \theta : S^{n-1} \times [0,1] &\to \exp_{p^+}^{-1}(K) \\ (v,t) &\mapsto (1-t)s_v v \end{aligned}$$

where we used $CL^-_{p^+} \simeq S^{n-1}$ to identify $v \in S^{n-1}$ with the corresponding $v \in CL^-_{p^+}$. Using the results from the above paragraph it follows that θ is well-defined and has the desired properties.

We now set $\Theta := \exp_{p^+} \circ \theta$, which satisfies all properties in (2) and are done with this part.

Finally, for part (3) we assume there exists a $p \in \mathcal{L}^-_{p^+} \cap V$. Recall that $V \subset J(p^-,p^+)^\circ = I^+(p^-) \cap I^-(p^+)$. We thus have $p \in I^+(p^-) \subset J^+(p^-)$, which together with $p \in \mathcal{L}^-_{p^+}$ implies $p \in K$. But now we have $p \in I^-(p^+)$ and $p \in K$, a contradiction to (1).

Now we assume that there exists a $p_0 \in R$ and $p \in J^-(p_0) \cap V$. Because $V \subset I^+(p^-)$ there exists a timelike path from p^- to p. Because $p \in J^-(p_0)$ as well we can construct a timelike path ((REF)) from p^- to p_0 implying $p_0 \in I^+(p^-)$. But because $p \in R = K \setminus I^+(p^-)$ this is a contradiction. $\mathcal{L}^-_{p_0} \subset J^-(p_0)$ yields the second equality.

□

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will often treat K itself as a submanifold when it is clear that we are working away from the boundary. This is often the case as by (3) no null geodesic originating from the interior of $J(p^-,p^+)$ can reach p^+ or R, i.e. the boundary of K.

Furthermore by the properties of Θ we have

$$\mu_a([0,1]) \cap \mu_{a'}([0,1]) = \{p^+\} \text{ for } a \neq a' \in S^{n-1} \text{ and} \tag{2.1}$$

$$\bigcup_{a \in S^{n-1}} \mu_a([0,1]) = K \tag{2.2}$$

2.2.2 Geometry of Light Observation Sets

Lemma 2.2.2. *For any $q \in V$ the restriction of the exponential map to null vectors $\exp_q : L_q^+ M \to M$ is* transverse *to K, i.e. for all $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_p K$.*

Proof. In order to achieve a contradiction we assume that there exists a $q \in V$ and a $w \in L_q^+ M$ such that with $v := \gamma_{q,w}(1) \in L_p K$. Since K is generated by backwards null geodesics originating at p^+ there exists a $u \in L_{p^+}^- M$ such that there exists a $t \in \mathbb{R}_+$ with $\gamma_{p^+,u}(t) = p, \gamma'_{p^+,u}(t) = -v$. We can thus obtain an unbroken past-pointing null geodesic from p^+ to q by connecting $\gamma_{p^+,u}$ and $\gamma_{p,-v}$. But this implies that $q \in \mathcal{L}_{p^+}^-$ which is a contradiction to 2.2.1(3).

We now prove that this implies that $\exp_q : L_q^+ M \to M$ is transverse to K. We need to show that for every $w \in L_q^+ M$ with $\exp_q(w) = p \in K$ we have

$$\operatorname{im}(d \exp_q|_w) \oplus T_p K = T_p M.$$

As $T_p K$ is a null hypersurface we only need to prove that $\operatorname{im}(d \exp_q|_w)$ contains a null vector which is not a multiple of the null vector $v \in T_p K$ generating $T_p K = v^\perp$. But by the properties of the exponential map, $\operatorname{im}(d \exp_q|_w)$ contains $v' = \gamma'_{q,w}(1) \in T_p M$. And since we just proved that $v' \notin T_p K$, $v + v'$ must be a timelike vector and $\operatorname{im}(d \exp_q|_w) \oplus T_p K = T_p M$, as desired. □

This lemma closely resembles lemma 2.5 in [HU17] with only minor adjustments to adapt it to our case. It is reproduced here for the sake of completeness. This lemma will allow us to reconstruct the direction of incoming light rays at point in $\mathcal{P}_K(q)$ which will locally correspond to the spacelike hypersurface.

Lemma 2.2.3 (Direction Reconstruction)**.** *Let $p \in K$ then there exists a bijection Φ between the space $\mathcal{S}$ of spacelike hyperplanes $S \subset T_p K$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+ V \subset T_p M$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+ V \mapsto T_p K \cap V^\perp \in \mathcal{S}$.*

Moreover there exists a bijection between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_p M$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \ni S \mapsto S \oplus \operatorname{span} \Phi(S) \in \mathcal{N}$.

Proof. Let $p \in K$, and $S \subset T_pK$ be a spacelike hyperplane. The orthogonal complement $S^\perp \subset T_pM$ then is a two-dimensional lorentzian subspace. Hence there exist four light rays which are multiples of the vectors $V, -V, W, -W$ in $S^\perp$, where we WLOG assume that V and W are future-pointing. Since $T_pK = v^\perp$ for some future-pointing null vector $v \in T_pK$, we have $v \in S^\perp$ and can WLOG assume $\mathbb{R}_+W = \mathbb{R}_+v$, i.e. $\mathbb{R}_+W$ is the ray pointing along the null hypersurface K. This leaves $\mathbb{R}_+V$ as the unique future-pointing outward null ray which is perpendicular to S, and we can thus set $\Phi(S) = \mathbb{R}_+V$.

For to prove Φ is a bijection, we let $0 \neq V \in T_pM$ be an outward future-pointing null vector. In particular this means that $V \notin T_pK$. Thus $S = V^\perp \cap T_pK$ is a spacelike hyperplane in T_pK which satisfies $S = \Phi^{-1}(V)$.

For the last claim we note that the map $\mathcal{N} \ni N \mapsto N^\perp \cap L_p^+M \in \mathcal{V}$ maps a null hypersurface N to the unique ray along a future-pointing outward null generator of N. The inverse of this map is given by $\mathcal{V} \ni \mathbb{R}_+V \mapsto V^\perp \in \mathcal{N}$. Composition of these maps with Φ yields the desired bijection $\mathcal{N} \to \mathcal{S}$. □

Lemma 2.2.4. *For $q \in V$ and $w \in L_q^+M$ there exists exactly one $t_w \in (0, \infty)$ such that $\gamma_{q,w}(t_w) \in K$.*

Proof. Let $q \in V$ and $w \in L_q^+M$, by ((Leavescompact)) any geodesic starting in the compact set $J(p^-, p^+)$ must eventually leave it, intersecting the boundary. As K is the future boundary of $J(p^-, p^+)$ there exists at least one $t_w \in (0, \infty)$ with $p = \gamma_{q,w}(t_w) \in K$. We now show $\gamma_{q,w}(t') \notin K$ for any other $t' \neq t_w$.

First let us consider the case $t' < t_w$. We can then append $\gamma_{q,w}|_{[t',t_w]}$ to the path $\mu_a|_{[s,1]}$, whre $a \in S^{n-1}, s \in [0,1]$ such that $\mu_a(s) = p$, to get a broken lightlike path from $\gamma_{q,w}(t')$ to p^+. The fact that this path must be broken follows from the transversality proven in the previous lemma. But the existence of this broken path implies $\tau(\gamma_{q,w}(t'), p^+) > 0$ and thus $\gamma_{q,w}(t') \in I^-(p^+)$. But as $K = J(p^-, p^+) \backslash I^-(p^+)$ we have $\gamma_{q,w}(t') \notin K$

Conversely we now assume $t' > t_w$. Again by the transversality of $\gamma_{q,w}$ to K we get that for $t' - t_w > \varepsilon > 0$ small enough we have $\gamma_{q,w}(t_w + \varepsilon) \notin J(p^-, p^+) = J^+(p^-) \cap J^-(p^+)$ because K is the future boundary of $J(p^-, p^+)$. As any point on $\gamma_{q,w}$ is in $J^+(p^-)$ we must have have $\gamma_{q,w}(t_w + \varepsilon) \notin J^-(p^+)$, i.e. there exists no lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+. But if $\gamma_{q,w}(t') \in J^-(p^+)$ there exists a path σ from $\gamma_{q,w}(t')$ to p^+ and we could construct a lightlike path from $\gamma_{q,w}(t_w + \varepsilon)$ to p^+ by appending $\gamma_{q,w}|_{[t_w+\varepsilon,t']}$ to σ, a contradiction. We thus have $\gamma_{q,w}(t') \notin J^-(p^+) \supset J(p^-, p^+) \supset K$, completing the proof. □

Definition 2.2.5 (Observation Preimage)**.** For any $q \in V$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* L_q^KM to be the preimage of K under the exponential map restricted to L_q^+M, i.e.

$$L_q^KM := (\exp_q|_{L_q^+M})^{-1}(K) \subset L_q^+M$$

Lemma 2.2.6. *For any $q \in V$, the observation preimage $L_q^K M$ is a $n-1$-dimensional submanifold of $L_q^+ M$.*

Furthermore, for any $w \in L_q^K M$ there exist a relatively open neighborhood $O_w \subset L_q^K M$ such that $\exp_q : O_w \to U_w := \exp_q(O_w) \subset \mathcal{P}_K(q)$ is a diffeomorphism.

Proof. By lemma 2.2.2, $\exp_q : L_q^+ M \to M$ is transverse to K (here we treat $L_q^+ M$ and K as submanifolds, because by lemma 2.2.1(3) we can disregard the boundary points). Thus by the preimage lemma A.0.1 $L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K)$ is a $n-1$-dimensional submanifold of $L_q^+ M$.

For the second part let $w \in L_q^K M$, since $p := \exp_q(w) \in K$ and we assumed that such a p cannot be a null conjugate point, we know that $\exp_q : L_q^+ M \to M$ has an invertible differential at w. Thus, by the implicit function theorem, there exists an open neighborhood $O'_w \subset L_q^+ M$ of w such that $\exp_q : O'_w \to \exp_q(O'_w)$ is a diffeomorphism. If we then restrict $\exp_q$ to $O_w := O'_w \cap L_q^K M$ the map is still a diffeomorphism as desired. □

Note that by the invariance of domain theorem U_w is an open submanifold of $\mathcal{P}_K(q)$

Corollary 2.2.7. *The map*

$$\begin{aligned} S^{n-1} \simeq CL_q^+ M &\to L_q^K M \\ w &\mapsto t_w w \end{aligned}$$

where t_w is as in 2.2.4, is a diffeomorphism.

Proof. This result follows immediately from lemma 2.2.4 together with the fact that since K is (away from its boundary) a smooth submanifold, the map $w \mapsto t_w$ is smooth. □

Lemma 2.2.8. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $w_1, \ldots, w_N \in L_q^K M$ such that $\exp_q(w_i) = p$. Furthermore for O_{w_i} as in the previous lemma such that $\exp_q : O_{w_i} \to U_{w_i}$ is a diffeomorphism, there exists an open neighborhood $U \subset \mathcal{P}_K(q)$ of p such that*

$$\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$$

Proof. Note that the previous corollary immediately yields that $L_q^K M$ is compact. Let $q \in V$, $p \in \mathcal{P}$. We first remark that, by the previous lemma, for any $w \in \exp_q^{-1}(p) \cap L_q^K M$ there exist open neighborhoods $w \in O_w \subset L_q^K M$ and $p \in U_w = \exp_q(O_w) \subset \mathcal{P}_K(q)$ making $\exp_q : O_w \to U_w$ a diffeomorphism.

To show that there can only be finitely many $w \in L_q^K M$ with $\exp_q(w) = p$ we let

$$C := \exp_q^{-1}(p) \cap L_q^K M.$$

As M is hausdorff, p is closed and because $\exp_q$ is continuous, so is C. Now $C \subset L_q^K M$ is a closed subset of a compact space, making C itself compact as well. Now the family $\{O_w \mid w \in \exp_q^{-1}(p) \cap L_q^K M\}$ is an open cover of C. But because C is compact there must exist a finite subcover such that

$$C \subset O := \bigcup_{i=1}^{N} O_{w_i}.$$

We can now make some observations: By definition, for any $w \in L_q^K M \setminus C$ we have $\exp_q(w) \neq p$. And as $\exp_q$ is a diffeomorphism on O_{w_i} for all $i = 1, \dots, N$, it must be injective and we get $\exp_q^{-1}(p) \cap O_{w_i} = \{w_i\}$. We thus have

$$\exp_q^{-1}(p) \cap O = \{w_1, \dots, w_N\}.$$

Furthermore, as $C \subset O$ for any $p \in L_q^K M \setminus O \subset L_q^K M \setminus C$ we still have $\exp_q(w) \neq p$. In other words:

$$\exp_q^{-1}(p) \cap L_q^K M \setminus O = \emptyset.$$

Putting these two observations together we get

$$\exp_q^{-1}(p) \cap L_q^K M = \{w_1, \dots, w_N\},$$

as desired.

To show the second part we denote

$$L^\times := L_q^K M \setminus O \quad \text{ and have } L^\times \cap \exp_q^{-1}(p) = \emptyset.$$

Note that $L^\times$ is a closed and thus compact subset of L_q^K. We then endow M with an arbitrary metric d compatible with its topology. This lets us define the continuous function

$$\begin{aligned} g : L^\times &\to \mathbb{R} \\ w &\mapsto d(\exp_q(w), p). \end{aligned}$$

Because $L^\times \cap \exp_q^{-1}(p) = \emptyset$ we have $g(w) > 0$ for all $w \in L^\times$. But now, as $L^\times$ is compact there exists a $\varepsilon > 0$ such that $g(w) = d(\exp_q(w), p) > \varepsilon$ for all $w \in L^\times$. We can now choose

$$U := B_\varepsilon(p) \cap \mathcal{P}_K(q)$$

and get an open neighborhood of p in $\mathcal{P}_K(q)$ with $\exp^{-1}(U) \cap L^\times = \emptyset$. But this means

$$\exp_q^{-1}(U) \cap L_q^K M = O = \bigcup_{i=1}^{N} O_{w_i}$$

completing the proof.

□

We can immediately put these lemmas to use and prove this proposition characterizing the light observation set.

Proposition 2.2.9. *Let $q \in V$ and $p \in \mathcal{P}_K(q)$. There exists an open neighborhood $p \in U \subset \mathcal{P}_K(q)$, a positive integer N and N pairwise transversal, spacelike, codimension 1 submanifolds $\mathcal{U}_i \subset K$ such that $\mathcal{P}_K(q) \cap U = \bigcup_{i=1}^N \mathcal{U}_i$ and $p \in \mathcal{U}_i$ for $i = 1, \dots, N$.*

Proof. Let $q \in V$ and $p \in \mathcal{P}_K(q)$. By the previous lemma we know that there can only be finitely many $w_1, \dots, w_n \in L_q^K M$ with $\exp_q(w_i) = p$.

By lemma 2.2.6, for each w_i there exists a neighborhood $O_{w_i} \subset L_q^K M$ of w_i such that $\exp_q : O_{w_i} \to U_{w_i} := \exp_q(O_{w_i})$ is a diffeomorphism. Thus $U_{w_i} \subset \mathcal{P}_K(q)$ is a codimension 1 submanifold of K and we have $\bigcup_{i=1}^N U_{w_i} \subset \mathcal{P}_K(q)$.

Now we use the second part of the previous lemma to obtain an open neighborhood $U \subset \mathcal{P}_K(q)$ of p, such that $\exp_q^{-1}(U) \cap L_q^K M \subset \bigcup_{i=1}^N O_{w_i}$. Thus any point $p \in \mathcal{P}_K(q) \cap U$ is contained in some $\mathcal{V}_i$ and we have $\bigcup_{i=1}^N U_{w_i} \supset \mathcal{P}_K(q) \cap U$. We then define

$$\mathcal{U}_i := U \cap U_{w_i}$$

and have

$$\bigcup_{i=1}^N \mathcal{U}_i = \mathcal{P}_K(q) \cap U$$

as desired. Furthermore, because U is an open neighborhood of p, $\mathcal{U}_i$ is still a codimension 1 submanifold of K and $p \in \mathcal{U}_i$.

We show that $\mathcal{U}_i$ is spacelike. To that end let $p \in \mathcal{U}_i$. Note that we have $\mathcal{U}_i \subset K$ and $\mathcal{U}_i \subset U'_{w_i} = \exp_q(O'_{w_i})$, where $w_i \in O'_{w_i} \subset L_q^+ M$ is an open neighborhood of w_i in $L_q^+ M$ such that on O'_{w_i}, exp_q is a diffeomorphism onto its image. Both K and U'_{w_i} are null hypersurfaces around p but by proposition 2.2.2 they are transversal and thus cannot be generated by the same null rays. Thus $T_p\mathcal{U}_i = T_pK \cap T_pU'_{w_i}$ can only contain spacelike vectors.

Finally to prove that they are transversal at p, we assume by contradiction that there exist $i \neq j$ such that $T_p\mathcal{U}_i = T_p\mathcal{U}_j$. But by lemma 2.2.3 this would imply that $v_i = c * v_j$ for a $c \in \mathbb{R}_+$, where $v_i = \gamma'(1)_{q,w_i}$. This would imply $w_i = w_j$, a contradiction. □

Definition 2.2.10 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{U} \subset M$ of p such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a submanifold.

Note that $p \in \mathcal{P}_K(q)$ if and only if $N = 1$ for p in the previous proposition.

Corollary 2.2.11. *The subset of regular points is open and dense in $\mathcal{P}_K(q)$.*

Proof. The fact that it is open follows immediately from the definition: Let $p \in \mathcal{P}_K(q)$ be regular. There thus exists an open neighborhood $p \in \mathcal{U} \subset M$ such that $\mathcal{U} \cap \mathcal{P}_K(q)$ is a submanifold. But now for every point $p' \in \mathcal{U} \cap \mathcal{P}_K(q)$, $\mathcal{U}$ also makes p' a regular point making $\mathcal{U} \cap \mathcal{P}_K(q)$ an open neighborhood of regular points of p. Thus every regular point has an open neighborhood of regular points making the set of regular points itself open.

To prove the set of regular points is dense in $\mathcal{P}_K(q)$ we to show that for every point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $U' \subset \mathcal{P}_K(q)$ contains a regular point. By the previous proposition, for U' small enough we have $\mathcal{P}_K(q) \cap U' = \bigcup_{i=1}^N \mathcal{U}_i$, where $\mathcal{U}_i$ are pairwise transversal. This means their intersection is of lower dimension and

$$\mathcal{U}_i \setminus \bigcup_{j \neq i} \mathcal{U}_j \quad \text{is open and nonempty for every } i = 1, \dots N.$$

((Give name and close to p)) and we can find a $p' \in \mathcal{V}_i$ for some $i \in 1, \dots, N$ such that $p' \notin \mathcal{V}_j$ for $j \neq i$. Thus we can find an open neighborhood $\mathcal{O}'$ around p' such that $\mathcal{O}' \cap \mathcal{P}_K(q) \subset \mathcal{V}_i$ which means p' is a regular point, as desired. □

2.2.3 Observation Time Functions

Definition 2.2.12 (Observation Time Function)**.** For $a \in S^{n-1}$ the *observation time function* is defined as

$$\begin{aligned} f_a : J(p^-, p^+) &\to [0,1] \\ q &\mapsto \inf(\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\} \cup \{1\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) \coloneqq \mu_a(f_a(q)) \in M$ be the earliest point where μ_a sees light from q.

Lemma 2.2.13. *Let $a \in S^{n-1}$ and $q \in V$. Then*

(1) *It holds that $f_a(q) \in (0,1)$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[0,1]$ and strictly increasing on $[f_a(q), 1]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p,q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L_q^+ M$ and $t \in [0, \rho(q,v)]$ such that $p = \gamma_{q,v}(t)$.*

Proof. Let $a \in \mathcal{A}$ and $q \in V$.

We begin by showing (1): Because $q \in V \subset J(p^-, p^+)^\circ = I^+(p^-) \cap I^-(p^+)$ we have $q \in I^-(p^+)$ and conversely $p^+ \in I^+(q)$. By ((REF)) we know that $I^+(q)$ is open and thus it forms an open neighborhood of p^+. But as μ_a is a continuous path with $\mu_a(1) = p^+$ there must exist a $t < 1$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. Hence we have $f_a(q) < 1$.

To show $f_a(q) > 0$ we assume $f_a(q) = 0$ to achieve a contradiction. We thus have $0 = \inf\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\}$. There thus exists a convergent sequence $s_n \to 0$ as $n \to \infty$ such that $\mu_a(s_n) \in J^+(q)$ for all n. But because μ_a is continuous and $J^+(q)$ closed we have $\mu_a(0) \in J^+(q)$. This implies that there exists a lightlike future-pointing geodesic from q to

By lemma 2.2.1(3) we have that $\mu_a(t_-) \notin J^+(q)$ and $\mu_a(t_+) \in J^+(q)$. The second part immediately yields $f_a(q) \leq t_+$ as $f_a(q)$ is the infimum over all observation times. For the first part we assume by contradiction that there were to exist a $t_{-2} < t_-$ with $\mu_a(t_{-2}) \in J^+(q)$. This allows us to construct a causal path from q to $\mu_a(t_-)$ by joining the causal path from $q \to \mu_a(t_{-2})$ and the null geodesic μ_a from t_{-2} to t_-. Since this would imply that $\mu_a(t_-) \in J^+(q)$ this is a contradiction and $f_a(q)$ must be bigger than t_- proving (1).

(2) By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all t_n we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from q to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such t.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[0, 1]$ we first note that it is the composition of two continuous functions. Non-decreasing then follows from the reverse triangle inequality together with the fact that μ_a is a null path.

Finally to show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 1]$ we let $f_a \leq t_1 < t_2 \leq 1$. Now by ((REF)) there exists a causal geodesic $\gamma_1 : [0, 1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect γ_1 to $\mu_a|_{[t_1,t_2]}$ we get a path γ_2 connecting q to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as μ_a is a null geodesic. Next we argue that γ_2 must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If γ_1 is timelike this observation is trivial as μ_a is lightlike. If however, γ_1 is lightlike (which is exactly the case if $t_1 = f_a(1)$), this fact follows from the transversality of light cone observations as noted in proposition 2.2.2. This means that γ_2 is a broken causal geodesic, which by ((REF)) implies that there exists a strictly longer timelike path γ_3 connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Next to prove (3): To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q, p) = 0$. But now, by ((REF)) there exists a null geodesic from q to p which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q, p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0, 1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q, p) = 0$.

The other equivalence follows the definition of $\mathcal{P}_K(q)$ together with the definition of cut points. $\square$

By (3) of the above lemma, for any $q \in \overline{V}$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.2.12 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-1,1] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2.3}$$

Lemma 2.2.14. *Let $a \in S^{n-1}$. Then the function $q \mapsto f_a(q)$ is continuous on V.*

Proof. Let $q_i \to q$ in $\overline{V}$, let $t_i = f_a(q_i)$ and $t = f_a(q)$. Since τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(q_j, \mu_a(t+\varepsilon)) = \tau(q, \mu_a(t+\varepsilon)) > 0$. Thus for i big enough we have $\tau(q_i, \mu_a(t+\varepsilon)) > 0$. But by (3) this implies that a must have observed q_i before $t+\varepsilon$ i.e. $f_a(q_i) < t+\varepsilon = f_a(q)+\varepsilon$. As ε was arbitrary we get $\limsup_{j\to\infty} t_j \le t$.

We assume now that $\liminf_{i\to\infty} t_i = t' < t$. Let (t_i) be a convergent subsequence such that $f_a(q_i) = t_i \to t' < f_a(q)$. Now by the continuity of τ and μ_a we have

$$0 = \tau(q_i, \mu_a(f_a(q_i))) \to \tau(q, \mu_a(t')).$$

Furthermore by ((REF)) $\mu(s_i) \in J^+(q_i)$ for all i implies $\mu(s') \in J^+(q)$. But now we have $\mu(s') \in \mathcal{P}_K(q)$ and $\tau(q, \mu_a(s')) = 0$ which by (3) implies that $\mu_a(s') = \mathcal{E}_a(q) = \mu_a(f_a(q))$. But this is a contradiction as $s' < f_a(q)$. ((More in-detail?)) $\square$

Lemma 2.2.15. *Let $q \in V$. Then the function $a \mapsto f_a(q)$ is continuous on S^{n-1}.*

Corollary 2.2.16. *The map $f : V \times S^{n-1} \to \mathbb{R}; (q,a) \mapsto f_a(q)$ is continuous.*

Proposition 2.2.17. *If $q_n \to q_0 \in V$ as $n \to \infty$ and we denote $F_q : S^{n-1} \to \mathbb{R}; a \mapsto f_a(q)$. Then $F_{q_n} \to F_{q_0}$ uniformly over S^{n-1} as $n \to \infty$.*

Proposition 2.2.18. *Let $q, q' \in V$ such that $F_q = F_{q'}$. Then $q = q'$.*

Proposition 2.2.19. *Let $(q_n)_{n=1}^\infty$ be a sequence in V and $q_0 \in V$ such that $F_{q_n} \to F_{q_0}$ uniformly then $q_n \to q_0$ as $n \to \infty$.*

2.2.4 Set of earliest observations

Definition 2.2.20 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned}\mathcal{D}_K(q) = \{(p,v) \in L^+M \mid & (p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } p \in K, w \in L^+_q M, 0 \le t \le \rho(q,w)\}, \\ \mathcal{D}_K^{reg}(q) = \{(p,v) \in L^+M \mid & (p,v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } p \in K, w \in L^+_q M, 0 < t < \rho(q,w)\},\end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}_K^{reg}(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_K(q) = \pi(\mathcal{D}_K(q))$ and $\mathcal{E}_K^{reg}(q) = \pi(\mathcal{D}_K^{reg}(q))$, where $\pi : TM \to M$ is the canonical projection. We say that $\mathcal{E}_K(q)$ is the set of earliest observations and $\mathcal{E}_K^{reg}(q)$ is the set of earliest regular observations of q in K. We denote the collection of earliest observation sets by $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$.

Note that $\mathcal{E}_K(q) = \{\mathcal{E}_a(q) \mid a \in S^{n-1}\}$.

Proposition 2.2.21. *For any $q \in V$ it holds that*

(1) *$\mathcal{E}_K^{reg}(q)$ is a $n-1$-dimensional nonempty spacelike submanifold of K which is open relative to $\mathcal{P}_K(q)$ and has $\overline{\mathcal{E}_K^{reg}(q)} = \mathcal{E}_K(q)$ and,*

(2) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points,*

(3) *$\mathcal{D}_K^{reg}$ is a nonempty open n-dimensional submanifold of $\overrightarrow{K} := \pi^{-1}(K)$.*

Proof. We begin by proving (1):

Let p

$\square$

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

Proposition 2.2.22. *For any $q \in \overline{V}$, $\mathcal{E}reg_K(q) \subset K$ and $\mathcal{D}_K^{reg}(q) \subset TU$ are smooth submanifolds of dimension $n-1$ ((D has dim n)).*

Proof. ((TODO))

We will focus our attention to the case of $\mathcal{E}reg_U(q)$ as the argument for $\mathcal{D}_U^{reg}(q)$ is analogous Note first that $\mathcal{E}reg_U(q)$ can be rewritten as

$$\{\exp_q(w) \mid w \in L_q^+M \text{ with } 1 < \rho(q,w)\}.$$

Next by lower semi-continuity of ρ we get that $R = \{w \in L_q^+M \mid 1 < \rho(q,w)$ is an open set and thus a dimension $(n-1)$ submanifold (this is because L_q^+M itself is of dimension $(n-1)$). But since $\rho(q,w)$ describes where $\exp_q$ first fails to be a diffeomorphism we get that the surjection $\exp_p : R \to \mathcal{E}reg_U(q)$ is a diffeomorphism. Thus, since R was a manifold of dimension $(n-1)$, $\mathcal{E}reg_U(q)$ is also a manifold and has the required dimension. $\square$

Finally in this section we will prove

Proposition 2.2.23. *Let $q \in V$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

Proof. ((Still True?)) For the left inclusion assume $p \in \mathcal{E}_U(q)$, i.e. there exists an $a \in \mathcal{A}$ such that $\mathcal{E}_a(q) = p$. Then lemma 2.2.13(3) immediately yields, $p \in \mathcal{P}_U(q)$ and $\tau(q,p) = 0$. Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. By as $\mathcal{P}_U(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p,q) > 0$, a contradiction.

For the other direction we assume we have $p \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 2.2.13(3) we only need to prove that $\tau(p,q) = 0$. Suppose that $\tau(p,q) > 0$. By equation **??** there exists an $a \in \mathcal{A}$ and a $s \in [-1,1]$ such that $\mu_a(s) = p$. Now since $\tau(p,q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since μ_a is timelike, which is a contradiction. □

Thus $\mathcal{E}_K(q)$ truly deserves to be called the "set of earliest observations".

Proposition 2.2.24. *((Given data)) The light observations $\mathcal{P}_K(q)$ uniquely determines the light* direction *observation set $\mathcal{D}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$.*

Proof. 2nd part: from formula

1st part: from lemma + only finite nonconj cut points + we can parameterize $\mathcal{P}_K(q)$ by a spacelike submanifold of the forwards lightcone □

Proposition 2.2.25. *((Given data)) Given the light direction observation set $\mathcal{P}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$, we can determine the sets $\mathcal{E}_K^{reg}(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Proof. ((Take $\pi^-1(\mathcal{E}_K(q)) \cap \mathcal{C}_K(q)$ for $\mathcal{D}_U(q)$, then remove all cut points (in this case points with equal p but different v) in $\mathcal{D}_U(q)$ to obtain $\mathcal{D}_K^{reg}(q)$ and project again)) □

Chapter 3

Interior Reconstruction

3.1 Topology Reconstruction

Lemma 3.1.1. *Thing with dir set reconstruction Also intersection is spacelike somewhere*

Construction of V as a topological manifold

((Intro))

Next we aim to reconstruct the topological and differential data of V. To that end we define the following functions.

For $q \in V$ we define the function $F_q : S^{n-1} \to \mathbb{R}$ by $a \mapsto f_a(q)$. We can then define the function

$$\begin{aligned} \mathcal{F} : V &\to (C(S^{n-1}), d_\infty) \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in V$ to the function $F_q : S^{n-1} \to \mathbb{R}$. $(C(S^{n-1}), d_\infty)$ is the space of continuous functions from S^{n-1} to $\mathbb{R}$, together with the metric $d_\infty(f, g) = \max_{a \in S^{n-1}} |f(a) - g(a)|$.

The following proposition establishes that the canonical topological structure on $\mathcal{F}(V)$, i.e. the topology obtained by taking the subspace topology wrt. the topology induced by d_∞ on $C(S^{n-1})$, is the same as the pushforward under $\mathcal{F}$ of the topology on V, making $\mathcal{F}$ a homeomorphism.

Lemma 3.1.2. *The map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

Proof. ((Works the same, use direction set reconstruction)) $\square$

3.2 Smooth Reconstruction

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

3.2.1 Preliminaries

Definition 3.2.1 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_U^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q,w) > 1$. Existence follows from lemma 2.2.13 while uniqueness follows from the fact that $p \in \mathcal{E}_U^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Omega : \mathcal{Z} &\mapsto L^+V \\ (q,p) &\mapsto (q,w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0,w_0) \subset TM$ be a ε-neighborhood of (q_0,w_0) with respect to the Sasaki-metric induced on TM by g^+.

Lemma 3.2.2. *((Move to appendix?))The function*

$$\begin{aligned} T_+ : L^+J(p^-,p^+) &\to \mathbb{R} \\ (q,w) &\mapsto \sup\{t \geq 0 \mid \gamma_{q,w}(t) \in J^-(p^+)\} \end{aligned}$$

is finite and upper semicontinuous.

Corollary 3.2.3. *Let (q_n,w_n) be a sequence in $L^+J(p^-,p^+)$ such that $X(q_n,w_n) = (q_n,p_n) \to (q_0,p_0)$ as $n \to \infty$ then $\|w_n\|_{g^+}$ is bounded for any riemannian metric on M.*

Lemma 3.2.4. *Let $(q_0,p_0) \in \mathcal{Z}$ and $(q_0,w_0) = \Omega(q_0,p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0,w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0,w_0) \to \mathcal{U}_\varepsilon(q_0,p_0) := X(\mathcal{W}_\varepsilon(q_0,w_0))$. When ε is small enough, Ω coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0,p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and the map $\Omega : \mathcal{Z} \to L^+M$ is smooth.

Proof. ((Works the same with minor adjustments?)) □

((Explain what we're doing now))

Proposition 3.2.5. *Let $q \in V$ and $(q_0, p_j) \in \mathcal{Z}, j = 1, \dots, n$ and $w_j \in L^+_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \dots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=1}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

Proof. ((Works almost the same, maybe clarify implicit function theorem stuff)) □

3.2.2 Reconstruction

Definition 3.2.6 (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=1}^n \subset \mathcal{A}^n$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_U^{reg}(q)$ for all $j = 1, \dots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, the above $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in \mathcal{A}^n$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will sometimes denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

Proposition 3.2.7. *Let $\widehat{q} \in \widehat{V}$. Then there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$.*

Furthermore, given the data from **??** *we can determine all C^0-observation coordinates around $\widehat{q}$.*

Finally given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data **??***, allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.*

Proof. ((Works the same way)) □

Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric on U determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

Lemma 3.2.8. *The data given in* **??** *determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Proof. ((Works the same)) $\square$

Chapter 4

Boundary Reconstruction

4.1 Setting

In this section we will examine how we can extend our reconstruction result to the case where the observed set V is no longer contained within the interior of $J(p^-, p^+)$ but is now allowed to extend up to the boundary. In other words we want to recover the conformal structure of $J(p^-, p^+)$ from light cone observations made on the future null boundary $K = J(p^-, p^+) \setminus I^-(p^+)$.

This setting is complicated by the fact that as $q \in J(p^-, p^+)$ approaches the boundary, the light observation $\mathcal{P}_K(q)$ get increasingly warped and is degenerate if q is in the boundary.

Theorem 4.1.1 (Boundary Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exist a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let $V_j \subset J(p_j^-, p_j^+)$. We assume that no null geodesic starting in V_j has a conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a homeomorphism $\Phi : V_1 \to V_2$.((conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality))

Remark 4.1.2. ((Move this to intro)) We will again prove the previous theorem by working on just one globally hyperbolic Lorentzian manifold (M, g) with $p^\pm, V, K$ as above. We show that given the following data we can reconstruct the topology of V:

(1) The smooth manifold (with edge) K,

(2) the conformal class of $g|_K$,

(3) the set $\mathcal{P}_K(V)$.

4.2 Preliminaries

To extend the reconstruction up to the edge of $J(p^-, p^+)$ we need to introduce some new concepts.

Definition 4.2.1 (Unique minimum domain)**.** We define the *unique minimum domain* $D \subset J(p^-, p^+)$ to be

$$D := \{q \in J(p^-, p^+) \mid F_q \text{ has a unique minimum}\}. \tag{4.1}$$

Lemma 4.2.2. *Let A be a first-countable topological space and $P : A \to \{false, true\}$ a property defined for all points $a \in A$. Suppose now that for any converging sequence $a_n \to a_0 \in A$ there exists a $N \in N$ such that $P(a_n)$ is true for all $n \geq N$.*

Then there exists an open neighborhood $O \in A$ of a_0 such that $P(a)$ is true for all $a \in O$.

Lemma 4.2.3. *Let $(A, d_A), (B, d_B), (C, d_C)$ be metric spaces with A, B compact. Let $f : A \times B \to C$ be a continuous functions and denote $f_a : B \to C; b \mapsto f_a(b) := f(a, b)$ for $a \in A$. Let $a_n \to a_0 \in A$ as $n \to \infty$ be a convergent sequence.*

Then $f_{a_n} \to f_{a_0}$ uniformly as $n \to \infty$.

Appendix A

Technical Lemmas

Lemma A.0.1 (Transverse Map). *Let $f : M \to N$ be a smooth map transverse to the submanifold $L \subset N$ of codimension k and $f^{-1}(L)$ is nonempty. Then $f^{-1}(L)$ is a codimension k submanifold of M.*

Bibliography

[Bee81] Easley Beem Ehrlich. *Global Lorenzian Geometry*. Pure and Applied Mathematics, 1981.

[Hal08] Rodney G Halburd. “Partial Differential Equations in General Relativity”. In: *Journal of Physics A: Mathematical and Theoretical* 41.47 (2008), p. 479003.

[HU17] Peter Hintz and Gunther Uhlmann. “Reconstruction of Lorentzian Manifolds from Boundary Light Observation Sets”. In: *International Mathematics Research Notices* 2019.22 (Feb. 2017), pp. 6949–6987. ISSN: 1073-7928. DOI: 10.1093/imrn/rnx320. eprint: https://academic.oup.com/imrn/article-pdf/2019/22/6949/30945542/rnx320.pdf. URL: https://doi.org/10.1093/imrn/rnx320.

[KLU17] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*. 2017. arXiv: 1405.3386 [math.DG].

[One83] Barrett O’neill. *Semi-Riemannian geometry with applications to relativity*. Academic press, 1983.