Notes Masters Thesis

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May 17, 2022

Abstract

Notes

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1 Null Conjugate Points

Definition 1.1 (Null Conjugate Point)**.** Let $\gamma_{q,w} : [0,b] \to M$ be a null geodesic. We then call $p = \gamma_{q,w}(b)$ a *null conjugate point* if there exists a nontrivial variation $\mathbf{x} : [0,b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma_{q,w}$ through null geodesics such that $\mathbf{x}_v(b,0) = 0$.

We have the following useful characterization:

Proposition 1.2. *Let $\gamma_{q,w} : [0,b] \to M$ be a null geodesic. Then $p = \gamma_{q,w}(b)$ is a null conjugate point if and only if $\exp_q : L_qM \to M$ is singular at bw, i.e. if there exists a nonzero $\xi \in T_{bw}(L_qM)$ such that $d\exp_q(\xi) = 0$.*

Proof. We begin by proving the backwards direction and to that end assume that there exist a nonzero $\xi \in T_{bw}(L_qM)$ such that $d\exp_q(\xi) = 0$. By the construction of the tangent space there thus exists a non-constant path $\xi : (-\varepsilon, \varepsilon) \to L_qM$ with $\xi(0) = bw$. This allows us to construct the variation $\mathbf{x}(u,v) = \exp_q(\frac{u}{b}\xi(v))$ which has $\mathbf{x}(t,0) = \gamma_{q,w}(t)$ and is a variation through null geodesics. Finally we have $\mathbf{x}_v(b,0) = d\exp_q(\xi) = 0$ by the chain rule.

For the other direction we first note that by definition $\mathbf{x}(u,v) = \exp_q(u\mathbf{x}_u(0,v))$ and $\mathbf{x}_u(0,v) \in L_qM$ as $\mathbf{x}$ is a variation through *null* geodesics. Now again by the chain rule we have $0 = \mathbf{x}_v(b,0) = d\exp_q|_{bw} \circ \frac{d}{dv}(bx_u(0,v))|_{v=0}$. But since $\xi := \frac{d}{dv}(bx_u(0,v))|_{v=0} \in T_{bw}(L_qM)$ we are done. □

Null conjugate points are also conformal invariants:

Proposition 1.3. *Let $\Phi : (M,g) \to (N,h)$ be a conformal diffeomorphism and $\gamma : [0,b] \to M$ a null geodesic. Then $\gamma(b)$ is a null conjugate point of γ if and only if $\Psi(\gamma(b))$ is a null conjugate point of $\Psi \circ \gamma$.*

Proof. ((Cite relevant prop)) Because of the symmetry of the situation we only need to prove one direction and suppose that $\gamma(b)$ is a null conjugate point of γ. We thus have a variation $\mathbf{x}$ of γ through null geodesics. But since Φ maps null geodesics to null geodesics, $\Phi \circ \mathbf{x}$ is a variation of $\Phi \circ \gamma$ through null geodesics in N, which implies that $\Phi(\gamma(b))$ is a null conjugate point of $\Phi \circ \gamma$. □

2 Reconstruction from null cone observations

Theorem 2.1 (Baby Case)**.** *Let $(M_j, g_j), j = 1,2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exist a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let V_j be open sets such that $\overline{V_j} \subset \operatorname{int} J(p_j^-, p_j^+)$ is compact. We assume that no null geodesic starting in V_j has a conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

2.1 Preliminary Constructions

((Intro))

Lemma 2.2. *Let $(M, g), K, V, p^+, p^-$ be as in the statement of theorem 2.1 (we suppress the indices to simplify notation) then the following holds:*

(1) $(J^-(p^+) \setminus I^-(p^+)) \cap K = \mathcal{L}^-_{p^+} \cap K$ *and thus* $K = \mathcal{L}^-_{p^+} \cap J^+(p^-)$.

(2) *There exists a surjective smooth map $\Psi : S^n \times [0, 1] \to K$ such that the curves $t \mapsto \Psi(v, t), v \in S^n$ are null geodesics and*

$$\Psi(S^n \times \{1\}) = \{p^+\}, \quad \Psi(S^n \times \{0\}) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$$

(3) *There exist $0 < t_- < t_+ < 1$ such that the restriction $\Psi|_{S^n \times [t_-, t_+]}$ is a diffeomorphism onto its image and that for all $v \in S^n$, we have*

$$\Psi(v, t_-) \notin \bigcup_{q \in \overline{V}} J^+(q), \quad \Psi(v, t_+) \in \bigcap_{q \in \overline{V}} J^+(q).$$

Proof. ((Overhaul)) As we have no cut point in $J(p_1^-, p_1^+)$, the exponential map at p_1^+ is a diffeomorphism onto $J(p_1^-, p_1^+)$. Thus the preimage $\exp^{-1}_{p_1^+}(R)$ of the smooth submanifold

$$R = (J^-(p_j^+) \setminus I^-(p_j^+)) \cap (J^+(p_j^-) \setminus I^+(p_j^-)) = \mathcal{L}^-_{p_1^+} \cap \mathcal{L}^+_{p_1^-}$$

is a smooth submanifold of $L^-_{p_1^+} M$. We then let $\mathcal{A} = R$ and denote by $\mu_a(s) = \gamma_{p_1^+, a}(1 - s)$ for $a \in R$. It is then easily checked that this parameterization satisfies all requirements and we are done. □

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will often treat K itself as a submanifold when it is clear that we are working away from the boundary. This is almost always the case since any light cone originating in $\overline{V}$ will intersect K in $\Psi(S^n \times (t_-, t_+)) \subset K$.

The next proposition allows us to endow K with a number of "laboratory frames" we will use to conveniently describe the light cone observations on K.

Proposition 2.3 (Laboratory Frames)**.** *Let $(M_j, g_j), K_j, V_j, p_j^+, p_j^-, \Phi$ be as in the statement of theorem 2.1 Then there exists a family of future pointing, null geodesics $\mu_a^{(1)} : [0,1] \to K_1$ indexed by $a \in \mathcal{A}$ where $\mathcal{A}$ is a metric space. Furthermore we can require the map $[0,1] \times \mathcal{A} \to K_1; (s,a) \mapsto \mu_a^{(1)}(s)$ to be open ((almost, needed?)) and continuous. If we then take $\mu_a^{(2)} := \Phi(\mu_a^{(1)})$ we can achieve*

$$K_j = \bigcup_{a \in \mathcal{A}} \mu_a^{(j)}([0,1]). \tag{1}$$

Proof. ((TODO)) $\square$

Remark 2.4. To simplify notation we will continue with the construction on just one Lorentzian manifold (M, g) of dimension $1 + n$ and assume that we are given the following data to construct the required conformal diffeomorphism ((explain better)) from theorem 2.1.

1. A the quasi-manifold K,
2. the conformal class of $g|_K$ (but not only restricted to tangent vectors in K ((i think??))),
3. the paths $\mu_a : [0,1] \to K, a \in \mathcal{A}$,
4. the set $\mathcal{P}_K(V)$ where V is open and $\overline{V} \subset \operatorname{int} J(p^-, p^+)$ is compact.

Note that these data are invariant under conformal diffeomorphism, and any map we construct from it will thus also be invariant. We also remark that $\overline{V} \subset \operatorname{int} J(p^-, p^+)$ implies that $q \notin K$ for any $q \in \overline{V}$.

2.1.1 Geometry of Light Observation Sets

Lemma 2.5. *For any $q \in \overline{V}$ the restriction of the exponential map to null vectors $\exp_q : L_q^+ M \to M$ is* transverse *to K, i.e. for all $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p \in K$ we have $\gamma'_{q,w}(1) \notin T_p K$.*

Proof. We first establish that $\mathcal{L}_{p^+}^- \cap \overline{V} = \emptyset$. By remark 2.4 we have $\overline{V} \cap K = \emptyset$. By lemma 2.2(1) we know that $K = \mathcal{L}_{p^+}^- \cap J^+(p^-)$. This means any intersection of $\overline{V}$ and $\mathcal{L}_{p^+}^-$ must occur in $\mathcal{L}_{p^+}^- \setminus J^+(p^-)$. But since $\overline{V}$ lies entirely within $J^+(p^-)$ this is also impossible and $\mathcal{L}_{p^+}^- \cap \overline{V}$ must be empty.

In order to achieve a contradiction we now assume that there exists a $q \in \overline{V}$ and a $w \in L_q^+ M$ such that with $v := \gamma_{q,w}(1) \in L_p K$. Since K is generated by backwards null geodesics originating at p^+ there exists a $u \in L_{p^+}^- M$ such that there exists a $t \in \mathbb{R}_+$ with $\gamma_{p^+,u}(t) = p, \gamma'_{p^+,u}(t) = -v$. We can thus obtain an unbroken past-pointing null geodesic from p^+ to q by connecting $\gamma_{p^+,u}$ and $\gamma_{p,-v}$. But this implies that $q \in \mathcal{L}^-_{p^+}$ which is a contradiction to our previous fact.

Finally we prove that this implies that $\exp_q : L^+_q M \to M$ is transverse to K, i.e. we need to prove that for every $w \in L^+_q M$ with $\exp_q(w) = p \in K$ we have

$$\operatorname{im}(d\exp_q|_w) \oplus T_pK = T_pM.$$

As T_pK is a null hypersurface we only need to prove that $\operatorname{im}(d\exp_q|_w)$ contains a null vector which is not a multiple of the null vector $v \in T_pK$ generating $T_pK = v^\perp$. But by the properties of the exponential map, $\operatorname{im}(d\exp_q|_w)$ contains $v' = \gamma'_{q,w}(1) \in T_pM$. And since we just proved that $v' \notin T_pK$, $v + v'$ must be a timelike vector and $\operatorname{im}(d\exp_q|_w) \oplus T_pK = T_pM$, as desired. □

Lemma 2.6. *For $q \in \overline{V}$ and $w \in L^+_q M$ there exists exactly one $t \in (0,\infty)$ such that $\gamma_{q,w}(t) \in K$.*

Proof. Let $q \in \overline{V}$ and $w \in L^+_q M$, by ((Leavescompact)) any geodesic starting in the compact set $J(p^-, p^+)$ must eventually leave it, intersecting the boundary. Thus there exists at least one $t \in (0,\infty)$ with $p = \gamma_{q,w}(t) \in K$. ((Do compactness argument?)) We WLOG assume that t is the smallest such value. Now, by the previous lemma, we have $\gamma'_{q,w}(t) \notin T_pK$. ((But now we are outside $J^-(p^+)$)) □

Lemma 2.7 (Direction Reconstruction)**.** *((Iso = Bijection)) Let $p \in K$ then there exists an isomorphism Φ between the space $\mathcal{S}$ of spacelike hyperplanes $S \subset T_pK$ and the space $\mathcal{V}$ of rays $\mathbb{R}_+ V \subset T_pM$ along future-directed outward facing null vectors, given by the mapping $S \in \mathcal{S}$ to the unique future-directed outward pointing null ray $\Phi(S)$ contained in $S^\perp$. The inverse map is given by $\mathcal{V} \ni \mathbb{R}_+ V \mapsto T_pK \cap V^\perp \in \mathcal{S}$.*

Moreover there exists an isomorphism between $\mathcal{S}$ and the space $\mathcal{N}$ of linear null hypersurfaces $N \subset T_pM$ which contain a future-directed outward pointing null vector given by $\mathcal{S} \mapsto S \oplus \operatorname{span}\Phi(S) \in \mathcal{N}$.

Proof. Let $p \in K$, and $S \subset T_pK$ be a spacelike hyperplane. The orthogonal complement $S^\perp \subset T_pM$ then is a two-dimensional lorentzian subspace. There thus exist four light rays $V, -V, W, -W$ in $S^\perp$. Since $T_pK = v^\perp$ for some future-pointing null vector $v \in T_pK$, we have $v \in S^\perp$ and can WLOG assume $V = v$. This leaves W as the unique future-pointing outward null ray which is perpendicular to S, and we can thus set $\Phi(S) = W$.

For the other we let $0 \neq V \in T_pM$ be an outward future-pointing null vector. In particular this means that $V \notin T_pK$. Thus $S = V^\perp \cap T_pK$ is a spacelike hyperplane in T_pK which satisfies $S = \Phi^{-1}(V)$. ((... do isomorphism fun?))

For the final claim we note that ((... Why iso?)) □

Definition 2.8 (Observation Preimage)**.** For any $q \in \overline{V}$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* $L_q^K M$ to be the preimage of K under the exponential map restricted to $L_q^+ M$, i.e.

$$L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K) \subset L_q^+ M$$

Lemma 2.9. *For any $q \in \overline{V}$, the observation preimage $L_q^K M$ is a $n-1$-dimensional submanifold of $T_p M$.*

Furthermore, for any $w \in L_q^K M$ there exist a relatively open neighborhood $\mathcal{W} \subset L_q^K M$ such that $\exp_q : \mathcal{W} \to \exp_q(\mathcal{W}) \subset \mathcal{P}_K(q)$ is a diffeomorphism.

Proof. By lemma 2.5, $\exp_q : L_q^+ M \to M$ is transverse to K (here we treat $L_q^+ M$ and K as submanifolds as the points where they fail to be submanifolds can be removed without impacting the proof). Thus by the preimage lemma $L_q^K M := (\exp_q|_{L_q^+ M})^{-1}(K)$ is a $n-1$-dimensional submanifold of $L_q^+ M$.

For the second part let $w \in L_q^K M$, since $p := \exp_q(w) \in K$ and we assumed that such a p cannot be a null conjugate point, we know that $\exp_q : L_q M \to M$ has an invertible differential at w. Thus, by the implicit function theorem, there exists an open neighborhood $\mathcal{W}' \subset L_q M$ of w such that $\exp_q : \mathcal{W}' \to \exp_q(\mathcal{W}')$ is a diffeomorphism. If we then restrict $\exp_q$ to $\mathcal{W} := \mathcal{W}' \cap L_q^K M$ the map is still a diffeomorphism as desired. □

Lemma 2.10. *Let $q \in \overline{V}$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $v_1, \ldots, v_n \in L_q^K M$ such that $\exp_q(v_i) = p$. Furthermore for any neighborhood $W \subset L_q^K M$ of $v_1, \ldots, v_n$, there exists a neighborhood $V \subset \mathcal{P}_K(q)$ of p such that $\exp_q^{-1}(V) \cap L_q^+ M \subset W$.*

Proof. ((Overhaul))

Let $q \in \overline{V}$ and $v \in L_q^+ M$ such that $p = \exp_q(v)$. Since we required that $p \in K$ cannot be a null conjugate point of q, $\exp_q$ must be a local diffeomorphism around v. This means that there exist open sets $v \in \mathcal{O}_v \subset L_q M, p \in \mathcal{U}_v \subset M$ such that $\exp_q : \mathcal{O}_v \to \mathcal{U}_v$ is a diffeomorphism. But this means that there cannot exist another $v' \in \mathcal{O}_v$ with $\exp_q(v') = p$. We now restrict ourselves only to null *directions* at q i.e. the quotient $L_q^+ M/\mathbb{R}_+ \simeq S^{n-1}$. Since any null vector v with $\exp_q(v) = p$ has an open neighborhood where no other vector can have this property, the set of null *directions* in S^{n-1} which hit p is discrete and thus finite because S^{n-1} is compact. Because we only have finitely many null directions which hit p, $\pi^{-1}(p) \cap \overrightarrow{\mathcal{P}_K}(q)$ can only have finitely many elements, as desired. □

We can immediately put these lemmas to use and prove this proposition characterizing the light observation set.

Proposition 2.11. *Let $q \in \overline{V}$ and $p \in \mathcal{P}_K(q)$. There exists a neighborhood $\mathcal{O}$ of p, a positive integer N and N pairwise transversal codimension 1 submanifolds of $\mathcal{V}_i \subset K$ such that $\mathcal{P}_K(q) \cap \mathcal{O} = \bigcup_{i=1}^N \mathcal{V}_i$.*

Proof. Let $q \in \overline{V}$ and $p \in \mathcal{P}_K(q)$. By the previous lemma we know that there can only be finitely many $w_1, \dots, w_n \in L_q^K M$ with $\exp_q(w_i) = p$.

By lemma 2.9, for each w_i there exists a neighborhood $\mathcal{W}_i \subset L_q^K M$ of w_i such that $\exp_q : \mathcal{W}_i \to \mathcal{V}_i := \exp_q(\mathcal{W}_i)$ is a diffeomorphism. Thus $\mathcal{V}_i \subset \mathcal{P}_K(q)$ is a submanifold of K and we have $\bigcup_{i=1}^N \mathcal{V}_i \subset \mathcal{P}_K(q)$.

Now we use the second part of the previous lemma to obtain an open neighborhood $\mathcal{O} \subset \mathcal{P}_K(q)$ of p, such that $\exp_q^{-1}(\mathcal{O}) \cap L_q^+ M \subset \bigcup_{i=1}^N \mathcal{W}_i$. Thus any point $p \in \mathcal{P}_K(q) \cap \mathcal{O}$ is contained in some $\mathcal{V}_i$ and we have $\bigcup_{i=1}^N \mathcal{V}_i \supset \mathcal{P}_K(q) \cap \mathcal{O}$. After possibly shrinking some $\mathcal{W}_i$ we get equality.

The fact that each $\mathcal{V}_i$ is spacelike follows as it can be written as the intersection of two transversal null hypersurfaces, $\mathcal{L}_q^+$ and K.

Finally to prove that they are transversal at p, we assume by contradiction that there exist $i \neq j$ such that $T_p\mathcal{V}_i = T_p\mathcal{V}_j$. But by lemma 2.7 this would imply that $v_i = c * v_j$ for a $c \in \mathbb{R}_+$, where $v_i = \gamma'(1)_{q,w_i}$. Thus we would have $w_i = w_j$, a contradiction. ☐

Definition 2.12 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{O} \subset M$ of p such that $\mathcal{O} \cap \mathcal{P}_K(q)$ is a submanifold.

Corollary 2.13. *The subset of regular points is dense in $\mathcal{P}_K(q)$.*

Proof. It suffices to show that for every cut point $p \in \mathcal{P}_K(q)$, every relatively open neighborhood $\mathcal{O} \subset \mathcal{P}_K(q)$ contains a regular point. By the previous proposition, for $\mathcal{O}$ small enough we have $\mathcal{P}_K(q) \cap \mathcal{O} = \bigcup_{i=1}^N \mathcal{V}_i$, where $\mathcal{V}_i$ are pairwise transversal. This means their intersection is of lower dimension and we can find a $p' \in \mathcal{V}_i$ for some $i \in 1, \dots, N$ such that $p' \notin \mathcal{V}_j$ for $j \neq i$. Thus we can find an open neighborhood $\mathcal{O}'$ around p' such that $\mathcal{O}' \cap \mathcal{P}_K(q) \subset \mathcal{V}_i$ which means p' is a regular point, as desired. ☐

2.1.2 Observation Time Functions

Definition 2.14 (Observation Time Function)**.** For $a \in \mathcal{A}$ the *observation time function* is defined as

$$\begin{aligned} f_a : \overline{V} &\to [0,1] \\ q &\mapsto \inf(\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\} \cup \{1\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) := \mu_a(f_a(q)) \in M$ be the earliest point where μ_a sees light from q.

Lemma 2.15. *Let $a \in \mathcal{A}$ and $q \in \overline{V}$. Then*

(1) *It holds that $t_- \leq f_a(q) \leq t_+$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[0,1]$ and strictly increasing on $[f_a(q), 1]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p,q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L_q^+M$ and $t \in [0, \rho(q,v)]$ such that $p = \gamma_{q,v}(t)$.*

(4) *The function $q \mapsto f_a(q)$ is continuous on $\overline{V}$.*

Proof. Let $a \in \mathcal{A}$ and $q \in \overline{V}$.

We begin by showing (1): By lemma 2.2(3) we have that $\mu_a(t_-) \notin J^+(q)$ and $\mu_a(t_+) \in J^+(q)$. The second part immediately yields $f_a(q) \leq t_+$ as $f_a(q)$ is the infimum over all observation times. For the first part we assume by contradiction that there were to exist a $t_{-2} < t_-$ with $\mu_a(t_{-2}) \in J^+(q)$. This allows us to construct a causal path from q to $\mu_a(t_-)$ by joining the causal path from $q \to \mu_a(t_{-2})$ and the null geodesic μ_a from t_{-2} to t_-. Since this would imply that $\mu_a(t_-) \in J^+(q)$ this is a contradiction and $f_a(q)$ must be bigger than t_- proving (1).

(2) By the definition of the infimum we can find a sequence $t_n \searrow f_a(q)$ such that for all t_n we have $\mu_a(t_n) \in J^+(q)$. Now since $t \mapsto \mu_a(t)$ is continuous we have that $\mu_a(t_n) \to \mu_a(f_a(q)) = \mathcal{E}_a(q)$. Since $J^+(q)$ is closed this yields $\mathcal{E}_a(q) \in J^+(q)$.

For the second part we assume by contradiction that $\tau(q, \mathcal{E}_a(q)) > 0$. Since this means that a timelike path from q to $\mathcal{E}_a(q)$ exists we have $\mathcal{E}_a(q) \in I^+(q)$. Then, since $I^+(q)$ is open we can find a $t < f_a(q)$ such that $\mu_a(t) \in I^+(q) \subset J^+(q)$. This is a contradiction since $f_a(q)$ is the infimum over such t.

To show that $s \mapsto \tau(q, \mu_a(s))$ is continuous and non-decreasing on $[0,1]$ we first note that it is the composition of two continuous functions. Non-decreasing then follows from the reverse triangle inequality together with the fact that μ_a is a null path.

Finally to show that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing in $[f_a(q), 1]$ we let $f_a \leq t_1 < t_2 \leq 1$. Now by ((REF)) there exists a causal geodesic $\gamma_1 : [0,1] \to M$ with $\gamma_1(0) = q$ and $\gamma_1(1) = \mu_a(t_1)$ such that $L(\gamma_1) = \tau(p, \mu_a(t_1))$. If we then connect γ_1 to $\mu_a|_{[t_1,t_2]}$ we get a path γ_2 connecting q to $\mu_a(t_2)$ which has length $L(\gamma_2) = L(\gamma_1)$ as μ_a is a null geodesic. Next we argue that γ_2 must have a break at the connecting point, i.e. $\gamma_1'(1) \neq c\mu_a'(t_1)$ for any $c \in \mathbb{R}_+$. If γ_1 is timelike this observation is trivial as μ_a is lightlike. If however, γ_1 is lightlike (which is exactly the case if $t_1 = f_a(1)$), this fact follows from the transversality of light cone observations as noted in proposition 2.5. This means that γ_2 is a broken causal geodesic, which by ((REF)) implies that there exists a strictly longer timelike path γ_3 connecting the endpoints and we get

$$\tau(q, \mu_a(t_2)) \geq L(\gamma_3) > L(\gamma_2) = L(\gamma_1) = \tau(q, \mu_a(t_1)).$$

Next to prove (3): To prove the fist direction we assume that $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$. Now by (2) we have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = \tau(q, p) = 0$. But now, by ((REF)) there exists a null geodesic from q to p which means $p \in \mathcal{P}_K(q)$.

For the other direction we let $p \in \mathcal{P}_K(q)$ with $\tau(q, p) = 0$. Now let $a \in \mathcal{A}$ such that $p = \mu_a(t)$ for some $t \in [0, 1]$. We then assume by contradiction that $\mathcal{E}_a(q) \neq p$, i.e. $f_a(q) < t$. But by (2) we have that $s \mapsto \tau(q, \mu_a(s))$ is strictly increasing after $f_a(q)$ which is in contradiction with $\tau(q, p) = 0$.

The other equivalence follows the definition of $\mathcal{P}_K(q)$ together with the definition of cut points.

Finally we prove (4): Let $q_i \to q$ in $\overline{V}$, let $t_i = f_a(q_i)$ and $t = f_a(q)$. Since τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(q_j, \mu_a(t + \varepsilon)) = \tau(q, \mu_a(t + \varepsilon)) > 0$. Thus for j big enough we have $\tau(q_i, \mu_a(t + \varepsilon) > 0$. But by (3) this implies that a must have observed q_i before $t+\varepsilon$ i.e. $f_a(q_i) < t+\varepsilon = f_a(q)+\varepsilon$. As ε was arbitrary we get $\limsup_{j\to\infty} t_j \leq t$.

We assume now that $\liminf_{j\to\infty} t_j = t' < t$. Let (t_i) be a convergent subsequence such that $f_a(q_i) = t_i \to t' < f_a(q)$. Now by the continuity of τ and μ_a we have

$$0 = \tau(q_i, \mu_a(f_a(q_i))) \to \tau(q, \mu_a(t')).$$

Furthermore by ((REF)) $\mu(s_i) \in J^+(q_i)$ for all i implies $\mu(s') \in J^+(q)$. But now we have $\mu(s') \in \mathcal{P}_K(q)$ and $\tau(q, \mu_a(s')) = 0$ which by (3) implies that $\mu_a(s') = \mathcal{E}_a(q) = \mu_a(f_a(q))$. But this is a contradiction as $s' < f_a(q)$. ((More in-detail?)) $\square$

By (3) of the above lemma, for any $q \in \overline{V}$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.14 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-1, 1] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2}$$

2.1.3 Set of earliest observations

Definition 2.16 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned} \mathcal{D}_K(q) = \{(p, v) \in L^+K \mid & (p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } w \in L^+_q M, 0 \leq t \leq \rho(q, w)\}, \\ \mathcal{D}^{reg}_K(q) = \{(p, v) \in L^+K \mid & (p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ & \text{where } w \in L^+_q M, 0 < t < \rho(q, w)\}, \end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}_K^{reg}(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_U(q) = \pi(\mathcal{D}_U(q))$ and $\mathcal{E}_U^{reg}(q) = \pi(\mathcal{D}_U^{reg}(q))$, where $\pi : TU \to U$ is the canonical projection. We say that $\mathcal{E}_U(q)$ is the set of earliest observations and $\mathcal{E}reg_U(q)$ is the set of earliest regular observations of q in U. We denote the collection of earliest observation sets by $\mathcal{E}_U(V) = \{\mathcal{E}_U(q) \mid q \in V\}$.

Note that $\mathcal{E}_U(q) = \{\mathcal{E}_a(q) \mid a \in \mathcal{A}\}$.

Proposition 2.17. *For any $q \in \overline{V}$ it holds that*

(1) *$\mathcal{E}_K^{reg}(q)$ is a $n-1$-dimensional nonempty spacelike submanifold of K which is open relative to $\mathcal{P}_K(q)$ and has $\overline{\mathcal{E}_K^{reg}(q)} = \mathcal{E}_K(q)$ and,*

(2) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points,*

(3) *$\mathcal{D}_K^{reg}$ is a nonempty submanifold of $\overrightarrow{K} := \pi^{-1}(K)$ ((...)) which is open*

Proof. We begin by proving (1):

Let p

$\square$

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

Proposition 2.18. *For any $q \in \overline{V}$, $\mathcal{E}reg_K(q) \subset K$ and $\mathcal{D}_K^{reg}(q) \subset TU$ are smooth submanifolds of dimension $n-1$ ((D has dim n)).*

Proof. ((TODO))

We will focus our attention to the case of $\mathcal{E}reg_U(q)$ as the argument for $\mathcal{D}_U^{reg}(q)$ is analogous Note first that $\mathcal{E}reg_U(q)$ can be rewritten as

$$\{\exp_q(w) \mid w \in L_q^+M \text{ with } 1 < \rho(q,w)\}.$$

Next by lower semi-continuity of ρ we get that $R = \{w \in L_q^+M \mid 1 < \rho(q,w)$ is an open set and thus a dimension $(n-1)$ submanifold (this is because L_q^+M itself is of dimension $(n-1)$). But since $\rho(q,w)$ describes where $\exp_q$ first fails to be a diffeomorphism we get that the surjection $\exp_p : R \to \mathcal{E}reg_U(q)$ is a diffeomorphism. Thus, since R was a manifold of dimension $(n-1)$, $\mathcal{E}reg_U(q)$ is also a manifold and has the required dimension. $\square$

Finally in this section we will prove

Proposition 2.19. *Let $q \in \overline{V}$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

Proof. ((Still True?)) For the left inclusion assume $p \in \mathcal{E}_U(q)$, i.e. there exists an $a \in \mathcal{A}$ such that $\mathcal{E}_a(q) = p$. Then lemma 2.15(3) immediately yields, $p \in \mathcal{P}_U(q)$ and $\tau(q,p) = 0$. Now suppose there were a $p' \in \mathcal{P}_U(q)$ with $p' \ll p$. By as $\mathcal{P}_U(q) \subset J^+(q)$ we have $q \leq p'$, then as $p' \ll p$ we get $q \ll p$. But this would imply $\tau(p,q) > 0$, a contradiction.

For the other direction we assume we have $p \in \mathcal{P}_U(q)$ such that there are no $p' \in \mathcal{P}_U(q)$ such that $p' \ll p$. Again by lemma 2.15(3) we only need to prove that $\tau(p,q) = 0$. Suppose that $\tau(p,q) > 0$. By equation 1 there exists an $a \in \mathcal{A}$ and a $s \in [-1,1]$ such that $\mu_a(s) = p$. Now since $\tau(p,q) > 0$, we must have $s > f_a(q)$. But then $\mathcal{E}_a(q) = \mu_a(f_a(q)) \ll \mu_a(s)$, since μ_a is timelike, which is a contradiction. □

Thus $\mathcal{E}_U(q)$ truly deserves to be called the "set of earliest observations".

2.2 Constructive Solution of the Inverse Problem

((Intro))

2.2.1 Reconstruction ...

Lemma 2.20. *Thing with dir set reconstruction Also intersection is spacelike somewhere*

Proposition 2.21. *((Given data)) The light observations $\mathcal{P}_K(q)$ uniquely determines the light* direction *observation set $\mathcal{C}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$.*

Proof. 2nd part: from formula

1st part: from lemma + only finite nonconj cut points + we can parameterize $\mathcal{P}_K(q)$ by a spacelike submanifold of the forwards lightcone □

Proposition 2.22. *((Given data)) Given the light direction observation set $\mathcal{P}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$, we can determine the sets $\mathcal{E}reg_K(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Proof. ((Take $\pi^-1(\mathcal{E}_K(q)) \cap \mathcal{C}_K(q)$ for $\mathcal{D}_U(q)$, then remove all cut points (in this case points with equal p but different v) in $\mathcal{D}_U(q)$ to obtain $\mathcal{D}_K^{reg}(q)$ and project again)) □

2.2.2 Construction of V as a topological manifold

((Intro))

Next we aim to reconstruct the topological and differential data of V. To that end we define the following functions.

For $q \in \overline{V}$ we define the function $F_q : \mathcal{A} \to \mathbb{R}$ by $a \mapsto f_a(q)$. We can then define the function

$$\begin{aligned} \mathcal{F} : \overline{V} &\to \mathbb{R}^{\mathcal{A}} \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in \overline{V}$ to the function $F_q : \mathcal{A} \to \mathbb{R}$. We endow the set $\mathbb{R}^{\mathcal{A}} = \{f : \mathcal{A} \to \mathbb{R}\}$ with the product topology.

((...))

We begin by establishing the topological structure:

Lemma 2.23. *((V or $\overline{V}$?)) The map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

Proof. ((Works the same, use direction set reconstruction)) □

2.2.3 Construction of V as a smooth manifold

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

Definition 2.24 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_U^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q,w) > 1$. Existence follows from lemma 2.15 while uniqueness follows from the fact that $p \in \mathcal{E}_U^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Theta : \mathcal{Z} &\mapsto L^+ V \\ (q,p) &\mapsto (q,w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0, w_0) \subset TM$ be a ε-neighborhood of (q_0, w_0) with respect to the Sasaki-metric induced on TM by g^+.

Lemma 2.25. *Let $(q_0, p_0) \in \mathcal{Z}$ and $(q_0, w_0) = \Theta(q_0, p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0, w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0, w_0) \to \mathcal{U}_\varepsilon(q_0, p_0) := X(\mathcal{W}_\varepsilon(q_0, w_0))$. When ε is small enough, Θ coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0, p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and the map $\Theta : \mathcal{Z} \to L^+ M$ is smooth.

Proof. ((Works the same with minor adjustments?)) □

((Explain what we're doing now))

Proposition 2.26. *Let $q \in \overline{V}$ and $(q_0, p_j) \in \mathcal{Z}, j = 1, \dots, n$ and $w_j \in L^+_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \dots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=1}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

Proof. ((Works almost the same, maybe clarify implicit function theorem stuff)) □

Definition 2.27 (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=1}^n \subset \mathcal{A}^n$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_U^{reg}(q)$ for all $j = 1, \dots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, the above $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in \mathcal{A}^n$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will sometimes denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

Proposition 2.28. *Let $\widehat{q} \in \widehat{V}$. Then there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$.*

Furthermore, given the data from 2.4 we can determine all C^0-observation coordinates around $\widehat{q}$.

Finally given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data 2.4, allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.

Proof. ((Works the same way)) □

2.2.4 Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric on U determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

Lemma 2.29. *The data given in 2.4 determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Proof. ((Works the same)) □

3 Applications

Definition 3.1 (Einstein Universe)**.** Let $(\mathbb{R}, -dt^2)$ be the real line with negatively definite metric $-dt^2$ and (S^n, h) the n-sphere with the canonical Riemannian metric. The $1+n$ dimensional *Einstein universe* is then defined as the product $(\mathbb{R} \times S^n, -ds^2 \oplus h)$

Remark 3.2. We can parameterize S^n by an angle $\alpha \in (0, \pi)$ and a point $\omega \in S^{n-1}$ via the map

$$\begin{aligned} S : (0, \pi) \times S^{n-1} &\to S^n \\ (\alpha, \omega) &\mapsto (\cos\alpha, \sin\alpha\omega) \end{aligned}$$

If for a $X \in S^n$ we write $X = (X_0, \overrightarrow{X}), X_0 \in \mathbb{R}, \overrightarrow{X} \in \mathbb{R}^n$. We can invert S by

$$\alpha = \arccos X_0, \quad \omega = \frac{\overrightarrow{X}}{\|\overrightarrow{X}\|}.$$

Note that S has the irregular points $(\pm 1, 0 \ldots, 0)$

We can now construct our conformal embedding:

Proposition 3.3. *Let $(\mathbb{R} \times S^n, g)$ be the $1+n$ dimensional Einstein universe and $(\mathbb{R}^{1+n}, h = dt^2 - dx_n dx^n)$ the $1+n$ dimensional Minkovski space. Then the map*

$$\Psi : \mathbb{R} \times S^n \to \mathbb{R}^{1+n} \tag{3}$$

$$(T, X) \mapsto \frac{1}{\cos T + X_0}(\sin T, \overrightarrow{X}) \tag{4}$$

is a conformal diffeomorphism from a suitable subset $U \subset \mathbb{R} \times S^n$ to the whole Minkovski space.

Proof. □

4 Stability Results

((Overhaul, include globally hyperbolic stability)) We aim to show the reconstruction result in a simplified case and first establish that small deviations from the minkovsky metric on a compact set introduce no conjugate points:

Proposition 4.1. *Let $K \subset \mathbb{R}^n_1$ be a compact subset of the $1+n$-dimensional minkovsky space with metric $g = -dt^2 + \sum_{i=1}^n dx_i^2$. And let $\widetilde{g}$ be a slightly disturbed metric $\widetilde{g} = g + \varepsilon h$ where $\varepsilon > 0$ and h is another metric.*

Then we can choose $\varepsilon > 0$ small enough such that under the disturbed metric $\widetilde{g}$ no causal geodesic starting in K has a conjugate point in K.

Proof. We begin by defining the set $H = \{(p,v) \in TK \mid \exp_p(v) \in K\}$. Note that for $\varepsilon > 0$ small enough this set is compact as well. This is because in the minkovsky case, every geodesic starting in K leaves K in a finite time which depends continuously on the starting point and initial direction. This property still holds for $\widetilde{g}$ if ε is small enough and thus H is compact.

Recall that ((REF GEOD)) for a geodesic $\gamma_{p,v}$ starting at p with initial velocity v, $q = \gamma_{p,v}(b)$ is a conjugate point of $p = \gamma_{p,v}(0)$ if and only if the differential of the exponential map $d\exp_p$ is singular at bv. We we then define

$$\varepsilon_{p,v} = \frac{1}{2}\sup\{\varepsilon' > 0 \mid d\exp\}$$

((Bla bla bla, do it with open cover instead)) By compactness of H we can achieve that exp is never singular on it which means that no geodesic starting in K has a conjugate point in K. $\square$

((Is global hyperbolicity needed?)) Note that this proof can be directly generalized to show that if K is a compact subset of a globally hyperbolic manifold (M,g) and every causal geodesic starting in K has no conjugate point in K, then also for a slightly perturbed $(M, g+\varepsilon h)$, causal geodesics starting in K have no conjugate points in K. ((Expand conj point to cut points?))

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