Title

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Abstract

((TODO))

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Chapter 1

Introduction

1.1 Main Results

((Introduce Notation etc.))

Theorem 1.1.1 (Interior Reconstruction). *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let $V_j \subset \operatorname{int} J(p_j^-, p_j^+)$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Theorem 1.1.2 (Boundary Reconstruction). *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exists a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let $V_j \subset J(p_j^-, p_j^+) \setminus p_j^+$ be open sets. We assume that no null geodesic starting in V_j has a null conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality.

Chapter 2

Geometric Preliminaries

2.1 Null Conjugate Points

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Definition 2.1.1 (Null Conjugate Point)**.** Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. We then call $p = \gamma_{q,w}(b)$ a *null conjugate point* if there exists a nontrivial variation $\mathbf{x} : [0, b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma_{q,w}$ through null geodesics such that $\mathbf{x}_v(b, 0) = 0$.

We have the following useful characterization:

Proposition 2.1.2. *Let $\gamma_{q,w} : [0, b] \to M$ be a null geodesic. Then $p = \gamma_{q,w}(b)$ is a null conjugate point if and only if $\exp_q : L_q M \to M$ is singular at bw, i.e. if there exists a nonzero $\xi \in T_{bw}(L_q M)$ such that $d\exp_q(\xi) = 0$.*

Null conjugate points are also conformal invariants:

Proposition 2.1.3. *Let $\Phi : (M, g) \to (N, h)$ be a conformal diffeomorphism and $\gamma : [0, b] \to M$ a null geodesic. Then $\gamma(b)$ is a null conjugate point of γ if and only if $\Psi(\gamma(b))$ is a null conjugate point of $\Psi \circ \gamma$.*

2.2 Geometry of the Light Cone Observations

From now on $(M, g), K, V, p^+, p^-$ be as in theorem 1.1.1 (we suppress the indices to simplify notation). ((More explanation))

2.2.1 Parameterization of Observations

Lemma 2.2.1. *We have:*

(1) $(J^-(p^+) \setminus I^-(p^+)) \cap K = \mathcal{L}^-_{p^+} \cap K$ *and thus* $K = \mathcal{L}^-_{p^+} \cap J^+(p^-)$.

(2) *There exists a surjective smooth map* $\Theta : S^{n-1} \times [0,1] \to K$ *such that the curves* $\mu_a := t \mapsto \Theta(a,t), a \in S^n$ *are null geodesics,*

$$\Theta(S^{n-1} \times \{1\}) = \{p^+\}, \quad R := \Theta(S^{n-1} \times \{0\}) = (J^-(p^+) \setminus I^-(p^+)) \cap J^+(p^-)$$

and $\Theta : S^{n-1} \times [0,1) \to K \setminus p^+$ *is a diffeomorphism.*

(3) $\mathcal{L}^-_{p^+} \cap J(p^-,p^+)^{\mathrm{o}} = \emptyset$ *and* $\mathcal{L}^-_p \cap J(p^-,p^+)^{\mathrm{o}} = \emptyset \quad \forall p \in R.$

Note that this implies that K is a smooth n-dimensional submanifold of M at any point away from its boundary. We will often treat K itself as a submanifold when it is clear that we are working away from the boundary. This is often the case as by (3) no null geodesic originating from the interior of $J(p^-,p^+)$ can reach p^+ or R, i.e. the boundary of K.

2.2.2 Geometry of Light Observation Sets

Lemma 2.2.2. *For any* $q \in \overline{V}$ *the restriction of the exponential map to null vectors* $\exp_q : L^+_q M \to M$ *is* transverse *to* K*, i.e. for all* $w \in L^+_q M$ *such that* $\gamma_{q,w}(1) = p \in K$ *we have* $\gamma'_{q,w}(1) \notin T_p K$.

Lemma 2.2.3. *For* $q \in \overline{V}$ *and* $w \in L^+_q M$ *there exists exactly one* $t \in (0,\infty)$ *such that* $\gamma_{q,w}(t) \in K$.

Lemma 2.2.4 (Direction Reconstruction)**.** *((Iso = Bijection)) Let* $p \in K$ *then there exists an isomorphism* Φ *between the space* $\mathcal{S}$ *of spacelike hyperplanes* $S \subset T_p K$ *and the space* $\mathcal{V}$ *of rays* $\mathbb{R}_+ V \subset T_p M$ *along future-directed outward facing null vectors, given by the mapping* $S \in \mathcal{S}$ *to the unique future-directed outward pointing null ray* $\Phi(S)$ *contained in* $S^\perp$*. The inverse map is given by* $\mathcal{V} \ni \mathbb{R}_+ V \mapsto T_p K \cap V^\perp \in \mathcal{S}$.

Moreover there exists an isomorphism between $\mathcal{S}$ *and the space* $\mathcal{N}$ *of linear null hypersurfaces* $N \subset T_p M$ *which contain a future-directed outward pointing null vector given by* $\mathcal{S} \mapsto S \oplus \operatorname{span} \Phi(S) \in \mathcal{N}$.

Definition 2.2.5 (Observation Preimage)**.** For any $q \in J(p^-,p^+)^{\mathrm{o}}$ with light observation set $\mathcal{P}_K(q) \subset K$ we define the *observation preimage* $L^K_q M$ to be the preimage of K under the exponential map restricted to $L^+_q M$, i.e.

$$L^K_q M := (\exp_q|_{L^+_q M})^{-1}(K) \subset L^+_q M$$

Lemma 2.2.6. *For any* $q \in J(p^-,p^+)^{\mathrm{o}}$*, the observation preimage* $L^K_q M$ *is a* $n-1$*-dimensional submanifold of* $T_p M$.

Furthermore, for any $w \in L^K_q M$ *there exist a relatively open neighborhood* $\mathcal{W} \subset L^K_q M$ *such that* $\exp_q : \mathcal{W} \to \exp_q(\mathcal{W}) \subset \mathcal{P}_K(q)$ *is a diffeomorphism.*

Lemma 2.2.7. *Let $q \in J(p^-, p^+)^{\circ}$ and $p \in \mathcal{P}_K(q)$ then there exist only finitely many $v_1, \ldots, v_n \in L_q^K M$ such that $\exp_q(v_i) = p$. Furthermore for any neighborhood $W \subset L_q^K M$ of $v_1, \ldots, v_n$, there exists a neighborhood $U \subset \mathcal{P}_K(q)$ of p such that $\exp_q^{-1}(U) \cap L_q^+ M \subset W$.*

We can immediately put these lemmas to use and prove this proposition characterizing the light observation set.

Proposition 2.2.8. *Let $q \in J(p^-, p^+)^{\circ}$ and $p \in \mathcal{P}_K(q)$. There exists a neighborhood $\mathcal{O}$ of p, a positive integer N and N pairwise transversal, spacelike, codimension 1 submanifolds $\mathcal{V}_i \subset K$ such that $\mathcal{P}_K(q) \cap \mathcal{O} = \bigcup_{i=1}^{N} \mathcal{V}_i$ and $p \in \mathcal{V}_i$ for $i = 1, \ldots, N$.*

Definition 2.2.9 (Regular Point)**.** We call a point $p \in \mathcal{P}_K(q)$ *regular* if there exists an open neighborhood $\mathcal{O} \subset M$ of p such that $\mathcal{O} \cap \mathcal{P}_K(q)$ is a submanifold.

Corollary 2.2.10. *The subset of regular points is open and dense in $\mathcal{P}_K(q)$.*

2.2.3 Observation Time Functions

Definition 2.2.11 (Observation Time Function)**.** For $a \in S^{n-1}$ the *observation time function* is defined as

$$\begin{aligned} f_a : J(p^-, p^+) &\to [0,1] \\ q &\mapsto \inf(\{s \in [0,1] \mid \mu_a(s) \in J^+(q)\} \cup \{1\}). \end{aligned}$$

Moreover, let $\mathcal{E}_a(q) \coloneqq \mu_a(f_a(q)) \in M$ be the earliest point where μ_a sees light from q.

Lemma 2.2.12. *Let $a \in S^{n-1}$ and $q \in J(p^-, p^+)^{\circ}$. Then*

(1) *It holds that $f_a(q) \in (0, 1)$.*

(2) *We have $\mathcal{E}_a(q) \in J^+(q)$ and $\tau(q, \mathcal{E}_a(q)) = 0$. Moreover the function $s \mapsto \tau(q, \mu_a(s))$ is continuous, non-decreasing on $[0, 1]$ and strictly increasing on $[f_a(q), 1]$.*

(3) *Let $p \in K$. Then $p = \mathcal{E}_a(q)$ with some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_K(q)$ and $\tau(p, q) = 0$. Furthermore, these are equivalent to the fact that there are $v \in L_q^+ M$ and $t \in [0, \rho(q, v)]$ such that $p = \gamma_{q,v}(t)$.*

By (3) of the above lemma, for any $q \in \overline{V}$ and $a \in \mathcal{A}$ we have $\mathcal{E}_a(q) \in \mathcal{P}_K(q)$. Since $\mathcal{P}_K(q) \subset J^+(q)$, we can see using definition 2.2.11 that the set of earliest observations $\mathcal{P}_K(q)$ and the path μ_a completely determine the functions

$$f_a(q) = \min\{s \in [-1, 1] \mid \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)) \tag{2.1}$$

Lemma 2.2.13. *Let $a \in S^{n-1}$. Then the function $q \mapsto f_a(q)$ is continuous on $J(p^-, p^+)^{\mathrm{o}}$.*

Lemma 2.2.14. *Let $q \in J(p^-, p^+)^{\mathrm{o}}$. Then the function $a \mapsto f_a(q)$ is continuous on S^{n-1}.*

Corollary 2.2.15. *The map $f : J(p^-, p^+)^{\mathrm{o}} \times S^{n-1} \to \mathbb{R}; (q, a) \mapsto f_a(q)$ is continuous.*

Proposition 2.2.16. *If $q_n \to q_0 \in J(p^-, p^+)^{\mathrm{o}}$ as $n \to \infty$ and we denote $F_q : S^{n-1} \to \mathbb{R}; a \mapsto f_a(q)$. Then $F_{q_n} \to F_{q_0}$ uniformly over S^{n-1} as $n \to \infty$.*

Proposition 2.2.17. *Let $q, q' \in V$ such that $F_q = F_{q'}$. Then $q = q'$.*

Proposition 2.2.18. *Let $(q_n)_{n=1}^{\infty}$ be a sequence in V and $q_0 \in V$ such that $F_{q_n} \to F_{q_0}$ uniformly then $q_n \to q_0$ as $n \to \infty$.*

2.2.4 Set of earliest observations

Definition 2.2.19 (Set of earliest observations)**.** For $q \in \overline{V}$ we define

$$\begin{aligned}\mathcal{D}_K(q) = \{(p, v) \in L^+M \mid &(p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L_q^+M, 0 \le t \le \rho(q, w)\}, \\ \mathcal{D}_K^{reg}(q) = \{(p, v) \in L^+M \mid &(p, v) = (\gamma_{q,w}(t), \gamma'_{q,w}(t)) \\ &\text{where } p \in K, w \in L_q^+M, 0 < t < \rho(q, w)\},\end{aligned}$$

We say that $\mathcal{D}_K(q)$ is the *direction set* of q and $\mathcal{D}_K^{reg}(q)$ is the *regular direction set* of q.

Let $\mathcal{E}_K(q) = \pi(\mathcal{D}_K(q))$ and $\mathcal{E}_K^{reg}(q) = \pi(\mathcal{D}_K^{reg}(q))$, where $\pi : TM \to M$ is the canonical projection. We say that $\mathcal{E}_K(q)$ is the set of earliest observations and $\mathcal{E}_K^{reg}(q)$ is the set of earliest regular observations of q in K. We denote the collection of earliest observation sets by $\mathcal{E}_K(V) = \{\mathcal{E}_K(q) \mid q \in V\}$.

Note that $\mathcal{E}_K(q) = \{\mathcal{E}_a(q) \mid a \in S^{n-1}\}$.

Proposition 2.2.20. *For any $q \in V$ it holds that*

(1) *$\mathcal{E}_K^{reg}(q)$ is a $n-1$-dimensional nonempty spacelike submanifold of K which is open relative to $\mathcal{P}_K(q)$ and has $\overline{\mathcal{E}_K^{reg}(q)} = \mathcal{E}_K(q)$ and,*

(2) *$\mathcal{E}_K(q)$ fails to be a submanifold exactly at cut points,*

(3) *$\mathcal{D}_K^{reg}$ is a nonempty open n-dimensional submanifold of $\overrightarrow{K} := \pi^{-1}(K)$.*

Note that since $\mathcal{E}_K^{reg}(q)$ is exactly $\mathcal{E}_K(q)$ without the cut points, it is also the collection of all points where $\mathcal{E}_K(q)$ is locally a submanifold.

Proposition 2.2.21. *For any $q \in \overline{V}$, $\mathcal{E}reg_K(q) \subset K$ and $\mathcal{D}_K^{reg}(q) \subset TU$ are smooth submanifolds of dimension $n-1$ ((D has dim n)).*

Finally in this section we will prove

Proposition 2.2.22. *Let $q \in V$, then*

$$\mathcal{E}_K(q) = \{p \in \mathcal{P}_K(q) \mid \textit{there are no } p' \in \mathcal{P}_K(q) \textit{ such that } p' < p\}.$$

Thus $\mathcal{E}_K(q)$ truly deserves to be called the "set of earliest observations".

Proposition 2.2.23. *((Given data)) The light observations $\mathcal{P}_K(q)$ uniquely determines the light* direction *observation set $\mathcal{D}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$.*

Proposition 2.2.24. *((Given data)) Given the light direction observation set $\mathcal{P}_K(q)$ and the set of earliest observations $\mathcal{E}_K(q)$, we can determine the sets $\mathcal{E}_K^{reg}(q), \mathcal{D}_K(q)$ and $\mathcal{D}_K^{reg}(q)$.*

Chapter 3

Interior Reconstruction

3.1 Topology Reconstruction

Lemma 3.1.1. *Thing with dir set reconstruction Also intersection is spacelike somewhere*

Construction of V as a topological manifold

((Intro))

Next we aim to reconstruct the topological and differential data of V. To that end we define the following functions.

For $q \in V$ we define the function $F_q : S^{n-1} \to \mathbb{R}$ by $a \mapsto f_a(q)$. We can then define the function

$$\begin{aligned} \mathcal{F} : V &\to (C(S^{n-1}), d_\infty) \\ q &\mapsto F_q \end{aligned}$$

mapping a $q \in V$ to the function $F_q : S^{n-1} \to \mathbb{R}$. $(C(S^{n-1}), d_\infty)$ is the space of continuous functions from S^{n-1} to $\mathbb{R}$, together with the metric $d_\infty(f, g) = \max_{a \in S^{n-1}} |f(a) - g(a)|$.

The following proposition establishes that the canonical topological structure on $\mathcal{F}(V)$, i.e. the topology obtained by taking the subspace topology wrt. the topology induced by d_∞ on $C(S^{n-1})$, is the same as the pushforward under $\mathcal{F}$ of the topology on V, making $\mathcal{F}$ a homeomorphism.

Lemma 3.1.2. *The map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

3.2 Smooth Reconstruction

Having established the topological structure of V we next aim to establish coordinates on $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

3.2.1 Preliminaries

Definition 3.2.1 (Coordinates on V)**.** We first define

$$\mathcal{Z} = \{(q,p) \in V \times K \mid p \in \mathcal{E}_U^{reg}(q)\}.$$

Then for every $(q,p) \in \mathcal{Z}$ there is a unique $w \in L_q^+ M$ such that $\gamma_{q,w}(1) = p$ and $\rho(q,w) > 1$. Existence follows from lemma 2.2.12 while uniqueness follows from the fact that $p \in \mathcal{E}_U^{reg}(q)$ and thus cannot be a cut point. We can then define the map

$$\begin{aligned} \Omega : \mathcal{Z} &\mapsto L^+ V \\ (q,p) &\mapsto (q,w) \end{aligned}$$

Note that this map is injective. Below we will $\mathcal{W}_\varepsilon(q_0,w_0) \subset TM$ be a ε-neighborhood of (q_0,w_0) with respect to the Sasaki-metric induced on TM by g^+.

Lemma 3.2.2. *((Move to appendix?))The function*

$$\begin{aligned} T_+ : L^+ J(p^-,p^+) &\to \mathbb{R} \\ (q,w) &\mapsto \sup\{t \geq 0 \mid \gamma_{q,w}(t) \in J^-(p^+)\} \end{aligned}$$

is finite and upper semicontinuous.

Corollary 3.2.3. *Let (q_n,w_n) be a sequence in $L^+J(p^-,p^+)$ such that $X(q_n,w_n) = (q_n,p_n) \to (q_0,p_0)$ as $n \to \infty$ then $\|w_n\|_{g^+}$ is bounded for any riemannian metric on M.*

Lemma 3.2.4. *Let $(q_0,p_0) \in \mathcal{Z}$ and $(q_0,w_0) = \Omega(q_0,p_0)$. When $\varepsilon > 0$ is small enough the map*

$$\begin{aligned} X : \mathcal{W}_\varepsilon(q_0,w_0) &\to M \times M \\ (q,w) &\mapsto (q, \exp_q(w)) \end{aligned}$$

is open and defines a diffeomorphism $X : \mathcal{W}_\varepsilon(q_0,w_0) \to \mathcal{U}_\varepsilon(q_0,p_0) := X(\mathcal{W}_\varepsilon(q_0,w_0))$. When ε is small enough, Ω coincides in $\mathcal{Z} \cap \mathcal{U}_\varepsilon(q_0,p_0)$ with the inverse map of X. Moreover $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and the map $\Omega : \mathcal{Z} \to L^+M$ is smooth.

((Explain what we're doing now))

Proposition 3.2.5. *Let $q \in V$ and $(q_0, p_j) \in \mathcal{Z}, j = 1, \dots, n$ and $w_j \in L^+_{q_0}M$ such that $\gamma_{q_0,w_j}(1) = p_j$. Assume that $w_j, j = 1, \dots, n$ are linearly independent. Then, if $a_j \in A$ and $\overrightarrow{a} = (a_j)_{j=1}^n$ are such that $p_j \in \mu_{a_j}$, there is a neighborhood $V_1 \subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\overrightarrow{a}}(q) = (f_{a_j}(q))_{j=1}^n$$

define smooth coordinates on V_1. Moreover $\nabla f_{a_j}|_{q_0}$, i.e. gradient of f_{a_j} with respect to q at q_0, satisfies $\nabla f_{a_j}|_{q_0} = c_j w_j$ for some $c_j \neq 0$.

3.2.2 Reconstruction

Definition 3.2.6 (Observation Coordinates)**.** Let $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ and $\overrightarrow{a} = (a_j)_{j=1}^n \subset \mathcal{A}^n$ with $p_j = \mathcal{E}_{a_j}(q)$ such that $p_j \in \mathcal{E}_U^{reg}(q)$ for all $j = 1, \dots, n$. Let $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$ and $\mathbf{s}_{\overrightarrow{a}} = \mathbf{f}_{\overrightarrow{a}} \circ \mathcal{F}^{-1}$. Let $W \subset \widehat{V}$ be an open neighborhood of $\widehat{q}$. We say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^0-observation coordinates around $\widehat{q}$ if the map $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open and injective. Also we say that $(W, \mathbf{s}_{\overrightarrow{a}})$ are C^∞-observation coordinates around $\widehat{q}$ if $\mathbf{s}_{\overrightarrow{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are smooth local coordinates on $V \subset M$.

Note that by the invariance of domain theorem, the above $\mathbf{s}_{\overrightarrow{a}} : W \to \mathbb{R}^n$ is open if it is injective. Although for a given $\overrightarrow{a} \in \mathcal{A}^n$ there might be several sets W for which $(W, \mathbf{s}_{\overrightarrow{a}})$ form C^0-observation coordinates to clarify the notation we will sometimes denote the coordinates $(W, \mathbf{s}_{\overrightarrow{a}})$ as $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$.

We will consider $\mathcal{F}(V)$ a topological space and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V and makes $\mathcal{F}$ a diffeomorphism.

Proposition 3.2.7. *Let $\widehat{q} \in \widehat{V}$. Then there exist C^∞-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$.*

Furthermore, given the data from **??** *we can determine all C^0-observation coordinates around $\widehat{q}$.*

Finally given any C^0-observation coordinates $(W_{\overrightarrow{a}}, \mathbf{s}_{\overrightarrow{a}})$ around $\widehat{q}$, the data **??**, *allows us to determine whether they are C^∞-observation coordinates around $\widehat{q}$.*

Construction of the conformal type of the metric

We will denote by $\widehat{g} = \mathcal{F}_* g$ the metric on $\widehat{V} = \mathcal{F}$ that makes $\mathcal{F} : V \to \widehat{V}$ an isometry. Next we will show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric on U determine the conformal class of $\widehat{g}$ on $\widehat{V}$.

Lemma 3.2.8. *The data given in* **??** *determine a metric G on $\widehat{V} = \mathcal{F}(V)$ that is conformal to $\widehat{g}$ and a time orientation on $\widehat{V}$ that makes $\mathcal{F} : V \to \widehat{V}$ a causality preserving map.*

Chapter 4

Boundary Reconstruction

4.1 Setting

In this section we will examine how we can extend our reconstruction result to the case where the observed set V is no longer contained within the interior of $J(p^-, p^+)$ but is now allowed to extend up to the boundary. In other words we want to recover the conformal structure of $J(p^-, p^+)$ from light cone observations made on the future null boundary $K = J(p^-, p^+) \setminus I^-(p^+)$.

This setting is complicated by the fact that as $q \in J(p^-, p^+)$ approaches the boundary, the light observation $\mathcal{P}_K(q)$ get increasingly warped and is degenerate if q is in the boundary.

Theorem 4.1.1 (Boundary Reconstruction)**.** *Let $(M_j, g_j), j = 1, 2$ be two open globally hyperbolic, time-oriented Lorentzian manifolds. For $p_j^- \ll p_j^+$ two points in M_j we denote $K_j = J(p_j^-, p_j^+) \setminus I^-(p_j^+)$, the closed and compact backwards light cone from p_j^+ cut off at the intersection with the forwards light cone of p_j^-. We assume that there exist a conformal diffeomorphism $\Phi : K_1 \to K_2$ and that none of the past null geodesics starting at p_j^+ have a cut point in K_j.*

Now let $V_j \subset J(p_j^-, p_j^+)$. We assume that no null geodesic starting in V_j has a conjugate point on K_j.

Then, if

$$\widetilde{\Phi}(\mathcal{P}_{K_1}(V_1)) = \mathcal{P}_{K_2}(V_2)$$

there exists a homeomorphism $\Phi : V_1 \to V_2$.((conformal diffeomorphism $\Phi : V_1 \to V_2$ that preserves causality))

Remark 4.1.2. ((Move this to intro)) We will again prove the previous theorem by working on just one globally hyperbolic Lorentzian manifold (M, g) with $p^{\pm}, V, K$ as above. We show that given the following data we can reconstruct the topology of V:

(1) The smooth manifold (with edge) K,

(2) the conformal class of $g|_K$,

(3) the set $\mathcal{P}_K(V)$.

4.2 Preliminaries

To extend the reconstruction up to the edge of $J(p^-, p^+)$ we need to introduce some new concepts.

Definition 4.2.1 (Unique minimum domain)**.** We define the *unique minimum domain* $D \subset J(p^-, p^+)$ to be

$$D := \{q \in J(p^-, p^+) \mid F_q \text{ has a unique minimum}\}. \tag{4.1}$$

Lemma 4.2.2. *Let A be a first-countable topological space and $P : A \to \{false, true\}$ a property defined for all points $a \in A$. Suppose now that for any converging sequence $a_n \to a_0 \in A$ there exists a $N \in N$ such that $P(a_n)$ is true for all $n \geq N$.*

Then there exists an open neighborhood $O \in A$ of a_0 such that $P(a)$ is true for all $a \in O$.

Lemma 4.2.3. *Let $(A, d_A), (B, d_B), (C, d_C)$ be metric spaces with A, B compact. Let $f : A \times B \to C$ be a continuous functions and denote $f_a : B \to C; b \mapsto f_a(b) := f(a, b)$ for $a \in A$. Let $a_n \to a_0 \in A$ as $n \to \infty$ be a convergent sequence.*

Then $f_{a_n} \to f_{a_0}$ uniformly as $n \to \infty$.

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