CS250 homeowrk 3

name:Austen Nelson

Due: 7/16/19

1. Prove that $n^2\%3 \neq 2$ for all integers n.
   Remember $n\%3$ is the remainder of $n/3$, so it can only ever be 0, 1, or 2.

   Proof by cases:
   $x \in \mathbf{N} : 3 \mid x$ and $x = 3k : k \in \mathbf{N}$ because of definition of divides

   1. case 1: $x^2 \equiv 0 \pmod 3$

   $$\begin{aligned} x^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \end{aligned}$$

   $$3(3k^2) \equiv 0 \pmod 3 \qquad \text{case 1 is true because of definition of modulus}$$

   2. case 2: $(x+1)^2 \equiv 1 \pmod 3$

   $$\begin{aligned} (x+1)^2 &= x^2 + 2x + 1 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

   $$\begin{aligned} 3(3k^2 + 2k) \mod 3 &= 0 && \text{definition of modulus} \\ 1 &\equiv 1 \pmod 3 && \text{case 2 is true through substitution} \end{aligned}$$

   3. case 3: $(x+2)^2 \equiv 1 \pmod 3$

   $$\begin{aligned} (x+2)^2 &= x^2 + 4x + 4 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k) + 4 \end{aligned}$$

   $$\begin{aligned} 3(3k^2 + 4k) \mod 3 &= 0 && \text{definition of modulus} \\ 4 &\equiv 1 \pmod 3 && \text{case 3 is true through substitution} \end{aligned}$$

   These are all the cases we need to prove that $x^2 \mod 3 \in \{0, 1\}$. To demonstrate this:

   $$\begin{aligned} (x+3)^2 &= x^2 + 6x + 9 \\ 6x + 9 &= 3(2x + 3) \\ 3(2x+3) \mod 3 &= 0 \\ (x+3)^2 &\equiv x^2 \pmod 3 \end{aligned}$$

2. Fermat's Last theorem is a famous theorem in Math that was unproven for 200 years. The theorem says for all $n > 2, a, b, c \in \mathbf{N}. a^n + b^n \neq c^n$. Or $a^n + b^n = c^n$ has no integer solutions for n larger than 2. Use this theorem to prove that $\sqrt[n]{2}$ is irrational for n larger than 2.

$$\forall n > 2, a, b, c \in \mathbf{N}.$$
$$a^n + b^n \neq c^n$$
$$\text{assume } \sqrt[n]{2} = \frac{p}{q} : p, q \in \mathbf{N}$$
$$2 = \frac{p^n}{q^n}$$
$$2q^n = p^n$$
$$q^n + q^n = p^n \text{ which is a contradiction with Fermat's. So our assumption must be false.}$$

3. (a) Prove that there is no smallest positive rational number greater than 0.
   Assume x is the smallest positive rational number greater than 0.

   $x \in \mathbf{Q}^+, p \in \mathbf{N}, q \in \mathbf{N}.$
   $x = \frac{p}{q}$ definition of a rational number
   $\frac{p}{q} > \frac{p}{q+1}$ by incrementing q we create a smaller rational than x, which is a contradiction

   (b) Prove that for every positive real number greater than 0 there is a smaller positive rational number. (Hint: if $r < 1$ then $1/r > 1$)
   Let x be a real number between 0 and 1.
   Let y be the ceiling of 1/x.
   1/y > x.

   (c) Now Prove that there is no smallest positive real number greater than 0.
   Let x be the smallest positive real number greater than 0.
   Let y be x/2. This is a contradiction because y < x. There must be no smallest positive real number.

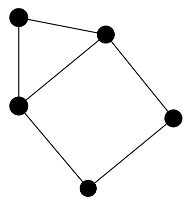
4. In a graph G we have a relation $u \sim v$ if u and v have an edge between them.
   Is this relation reflexive, symmetric, antisymmetric, transitive.

   This relation is not reflexive because a vertix is not required to have an edge to itself. For an undirected graph, it is symmetric (but not for a directed graph). An undirected graph cannot be antisymmetric (some directed graphs could possibly be antisymmetric). It is not transitive. If vertex A has an edge to B and B to C, there is a path from A to C (not an edge). A complete graph would be transitive.

5. remember a graph is a bunch of vertices connected by edges.
   A *path* is a sequence of vertices $v_1, v_2, \ldots$ where there is an edge between every vertex.
   A *cycle* is a path that starts and ends at the same vertex.
   Prove that if a graph has no cycles, then there is at most one path between any two vertices.

   Suppose there are two different paths between nodes a and b in the graph. Trace the paths simultaneously. When the paths separate and join again they create a cycle.

6. The degree of a vertex in a graph is the number of vertices it's connected to. so $deg(v) = |\{u : u \sim v\}|$ For the following graph give the degree of each vertex.



7. Prove that for the vertices $\{v_1, v_2, \ldots v_n\}$ in a graph that $deg(v_1) + deg(v_2) + \ldots deg(v_n) = 2 \cdot |E|$.
Every edge added to a graph has to connect two vertices. This means that the number of vertices will always be 2 times the total number of edges.