Problem 1

Determine if the following are injective (one to one), surjective (onto), or total.

- $f \colon \mathbb{R} \to \mathbb{R}$
  $f(x) = \sin x$
  IS total because $\forall x \in \mathbb{R}, \sin x \in \mathbb{R}$
  NOT injective because $\sin 0 \equiv \sin 2\pi$
  NOT surjective because $x \in \mathbb{R}, \sin x \neq 5$

- $f \colon \mathbb{R} \to \mathbb{R}$
  $f(x) = \sqrt{x}$
  NOT total because $\sqrt{-1} \notin \mathbb{R}$
  IS injective because if $\sqrt{x} = \sqrt{y}$ then $x = y$
  NOT surjective because $x \in \mathbb{R}, \sqrt{x} \neq -$

- $f \colon \mathbb{N} \to \mathbb{R}^+$
  $f(x) = \sqrt{x}$
  IS total because $\forall x \in \mathbb{N}, \sqrt{x} \in \mathbb{R}^+$
  IS injective because every root is unique
  NOT surjective because $x \in \mathbb{R}, f(x) \neq 2.5$

- $f \colon \mathbb{R}^+ \to \mathbb{N}$
  $f(x) = \sqrt{x}$
  NOT total because $\sqrt{3} \notin \mathbb{N}$
  IS injective because if $\sqrt{x} = \sqrt{y}$ then $x = y$
  IS surjective because $\forall x \in \mathbb{N}, x^2 \in \mathbb{R}$

- $f \colon \mathbb{R} \to \mathbb{R}^+$
  $f(x) = x^2$
  NOT total because $0^2 \notin \mathbb{R}$
  NOT injective because $f(2) \equiv f(-2)$
  IS surjective because $\forall x \in \mathbb{R}^+, \sqrt{(x)} \in \mathbb{R}$

- $f \colon \mathbb{R} \to \mathbb{R}$
  $f(x) = x^3$
  IS total because $\forall x \in \mathbb{R}, x^2 \in \mathbb{R}$
  IS injective because $\forall x \in \mathbb{R}, x^3$ is unique
  IS surjective because $\forall x \in \mathbb{R}, x^{\frac{1}{3}} \in \mathbb{R}$

Problem 2

Determine if the relations are reflexive, symmetric, antisymmetric, or transitive.

- $a \sim b$ if $a + b = 10$
  NOT reflexive $3 + 3 \neq 10$
  IS symmetric $a + b = 10 \equiv b + a = 10$
  NOT antisymmetric $3 + 7 = 10$ and $7 + 3 = 10$
  NOT transitive $4 + 6 = 10, 6 + 4 = 10, 4 + 4 \neq 10$

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- $a \sim b$ if a and b are both even or odd
  IS reflexive $a \sim a$ if a is even or odd
  IS symmetric $e_1 \sim e_2$ and $e_2 \sim o_1, o_1 \sim o_2$ and $o_2 \sim o_1, e \not\sim o$ and $o \not\sim e$
  NOT antisymmetric $2 \sim 4$ and $4 \sim 2$ and $2 \neq 4$
  IS transitive because $a \sim b, b \sim c, a \sim c$ meaning a, b, and c are all even or all odd

- $\frac{p}{q} \sim \frac{r}{s}$ if $q \leq s$
  IS reflexive $\frac{p}{q} \sim \frac{r}{q}$ because $q = q$
  NOT symmetric $\frac{1}{2} \sim \frac{1}{4}$ but $\frac{1}{4} \not\sim \frac{1}{2}$
  NOT antisymmetric $\frac{3}{7} \sim \frac{5}{7}$ and $\frac{5}{7} \sim \frac{3}{7}$ but $\frac{3}{7} \neq \frac{5}{7}$
  IS transitive $q \leq s, s \leq z, q \leq z$

- if s and t are strings, then $s \sim t$ if $s = reverse(t)$
  NOT reflexive, CAT $\not\sim$ CAT
  IS symmetric, every string is the reverse of its reverse
  NOT antisymmetric CAT $\neq$ TAC
  NOT transitive, it is symmetric but no reflexive so it cannot be transitive

- $a \sim b$ if $b = c * a$ for some c $a, b, c \in \mathbb{N}$
  IS reflexive, for $c = 1, a \sim a$
  NOT symmetric, for 2 4 but 4 2
  IS antisymmetric for same reason as symmetric
  IS transitive for same reason as symmetric
  For other values of c, none of these properties hold except antisymmetry. I don't think I totally understood this problem.

- $a \sim b$ if $a^b = b^a$
  IS reflexive, $a^a \equiv a^a$
  IS symmetric, $a^b = b^a \equiv b^a = a^b$
  NOT antisymmetric, $2 \sim 4$ and $4 \sim 2$ but $4 \neq 2$
  IS transitive because this is an equivalence relation (symmetric reasoning)

Problem 3

Prove that if $f: B \to C$ is injective, and $g: A \to B$ is injective, then $f \circ g$ is also injective.

1. Assume $f \circ g(a) = b$ and $f \circ g(a) = c$ (that $f \circ g$ is not injective)

2. $g(a) = d$ and only d (definition of injective)

3. Given the assumption, $f(d) = b$ and $f(d) = c$ (definition of composition) resulting that $f: B \to C$ is not injective, which is a contradiction.

Problem 4

Prove or disprove.

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- For any sets A and B, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cap \mathbb{P}(B)$

$$x \in \mathbb{P}(A) \cap \mathbb{P}(B) \qquad \text{(assumption)}$$
$$x \in \mathbb{P}(A) \wedge x \in \mathbb{P}(A) \qquad \text{(definition of intersect)}$$
$$x \subseteq A \wedge x \subseteq B \qquad \text{(definition of powerset)}$$
$$x \subseteq A \cap B \qquad \text{(definition of intersect)}$$
$$x \in \mathbb{P}(A \cap B) \qquad \text{(definition of powerset)}$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cap \mathbb{P}(B) \qquad \text{(set extensionality)}$$

- For any sets A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) \cup \mathbb{P}(B)$
  Counter example:
  $\mathbb{P}(\{1\} \cup \{2\}) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$
  $\mathbb{P}(\{1\}) \cup \mathbb{P}(\{2\}) = \{\{\}, \{1\}, \{2\}\}$

Problem 5

Prove that $A^c \cup B^c = (A \cap B)^c$

$$x \in (A \cap B)^c \qquad \text{(assumption)}$$
$$x \notin A \cap B \qquad \text{(definition of compliment)}$$
$$x \notin A \vee x \notin B \qquad \text{(definition of intersection)}$$
$$x \in A^c \vee x \in B^c \qquad \text{(definition of compliment)}$$
$$x \in A^c \cup B^c \qquad \text{(definition of logical or)}$$
$$A^c \cup B^c = (A \cap B)^c \qquad \text{(set extensionality)}$$

Problem 6

Prove there will always be an odd number of doors at the bottom of the triangle.

1. Take any series of blue and green dots starting with blue and ending with green.

2. Traverse the dots from start to finish. The first green dot encountered will create the first door. If the first green dot is the last dot, there is only one door (odd).

3. Traverse to the next blue dot. If all the remaining dots are green, there is only one door (odd). If there is another blue dot, there is two doors. Two is an even number. An even plus and even is an even and an even plus an odd is an odd. Zero plus an even is an even and zero plus an odd is an odd. This means we can ignore everything before this dot, making it the new start with zero doors.

4. Repeat steps two through four until you reach the last dot. The number of doors in the final recursive problem can only be one.

Problem 7

If $f\colon A \to B$ and $g\colon A \to B$, and $\forall x \in A, f(x) = g(x)$ then $f = g$. Show this is an equivalence relation.

- Reflexive: $\forall x \in A, xRx$ It is given that $f(x) = g(x)$

- Symmetric: $\forall x \in A, xRy \wedge yRx$ Because $f(x) = g(x)$ $x \equiv y$ meaning yRx is always true

- Transitive: Let x, y, and z be in A. Suppose xRy and yRz. Because x = y and y = z, x must be equal to z and xRz given that $f(x) = g(x)$.

Because this relation is reflexive, symmetric, and transitive, it is an equivalence relation.

Problem 8

Show that a dictionary order is a partial order.
A relation that is reflexive, antisymmetric, and transitive is a partial order.

- Reflexive: A string is always the same as itself. This means that dictionary order is reflexive.

- Antisymmetric: Take any two words a and b. If $a \leq b$ and $b \leq a$ then $a \equiv b$. This is true because if words are not identical, then one must be less than the other given the definition of dictionary order. This means that symmetry outside of equality is impossible.

- Transitive: Take any 3 words a, b, and c. If $a \leq b$ and $b \leq c$ then $a \leq c$. If a[0] $\leq$ b[0] and b[0] $\leq$ c[0] then a[0] $\leq$ c[0] following from the definition of $\leq$ and the transitivity property of order. Doing this for every letter in each word demonstrates that dictionary order also follows this property.