Problem 1

$$\text{prove } n! > 2^n \ \forall n > 4$$

$n = 5, \quad 5! = 120, \quad 2^5 = 32, \quad 120 > 32$	Base case of n = 5.
$k! > 2^k$	inductive hypothesis
$(k+1)!$	inductive case
$= (k+1)k!$	definition of factorial
$(k+1)k! > 2^k(k+1)$	because of the inductive hypothesis
$(k+1)k! > 2^k(2)$	because $k > 4, 2 < k+1$
$(k+1)k! > 2^{k+1}$	algebra

Problem 2

$$\text{prove } \sum_{i=1}^{n} 2i - 1 = n^2$$

$n = 1, \quad 2 - 1 = 1, \quad \sqrt{1}$	base case
$\displaystyle\sum_{i=1}^{k} 2i - 1 = k^2$	inductive hypothesis
$\displaystyle\sum_{i=1}^{k+1} 2i - 1$	inductive case
$= k^2 + (2(k+1) - 1)$	peel and substitute
$= k^2 + 2k + 1$	algebra
$= (k+1)^2$	factor

Problem 3

$$\text{prove } \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

$n = 1, \quad \dfrac{2}{2}^2 = 1, \quad 1^3 = 1$	base case
$\displaystyle\sum_{i=1}^{k+1} i^3 = \left(\frac{k(k+1)}{2} \right)^2$	inductive hypothesis
$\left(\dfrac{k(k+1)}{2} \right)^2 + (k+1)^3$	peel and substitute
$\left(\dfrac{(k+1)(k+2)}{2} \right)^2$	algebra

Problem 4

A Show that a tree with depth d has 2^{d-1} leaves.
 Base case tree of depth one has one leaf. $(2^{1-1} = 1)$
 $L_{d+1} = 2L_d$ because every time a layer is added to the tree every leaf gets 2 leaves added to it.
 As a result, a tree with depth 2 has 2 leaves, depth 3 has 4 leaves, depth 4 has 8 leaves, and so on.
 This doubling relationship is 2^{d-1}

B Show that a full binary tree with depth d has $2^d - 1$ nodes.
 This is easy to demonstrate given that a tree of depth d has 2^{d-1} leaves.
 The total number of nodes is the sum of all the leaves at each level of the tree.

$$\sum_{i=1}^{d} 2^{i-1} = 2^d - 1$$

$$d = 1, \quad 2^0 = 1, \quad 2^1 - 1 = 1 \qquad \text{base case}$$

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1 \qquad \text{inductive hypothesis}$$

$$\sum_{i=1}^{k+1} 2^{i-1} \qquad \text{inductive case}$$

$$= 2^k - 1 + 2^k \qquad \text{peel and substitute}$$

$$= 2^{k+1} - 1 \qquad \text{algebra}$$

Problem 5

$$\text{prove} \sum_{i=1}^{n} ia^i = \frac{a - (n+1)a^{n+1} + na^{n+2}}{(a-1)^2}$$

$$n = 1, \quad 1a^1 = a, \quad \frac{a - (n+1)a^2 + a^3}{(a-1)^2} = a \qquad \text{base case}$$

$$\sum_{i=1}^{k} ia^i = \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2} \qquad \text{inductive hypothesis}$$

$$\sum_{i=1}^{k+1} ia^i \qquad \text{inductive case}$$

$$= \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2} + (k+1)a^{k+1} \qquad \text{peel and substitute}$$

$$= \frac{a - (k+2)(a^{k+2}) + (k+1)(a^{k+3})}{(a-1)^2} \qquad \text{algebra}$$

Problem 6

prove $length(a + b) = length(a) + length(b)$ using structural induction given:

concatenation:		length:		
$+ : \mathcal{L} \times \mathcal{L} \to \mathcal{Z}$		$length : \mathcal{L} \to \mathcal{Z}$		
$Nil + b$	$= b$	$length(Nil)$	$= 0$	$rule1$
$Cons(h,t) + b$	$= Cons(h, t + b)$	$length(Cons(h,t))$	$= 1 + length(t)$	$rule2$

base case:

$length(a + Nil)$

$$= length(a) \qquad \text{rule one of concatenation}$$
$$= length(a) + 0 \qquad \text{add zero}$$
$$= length(a) + length(Nil) \qquad \text{rule one of length}$$

$length(t + b) = length(t) + length(b) \qquad\qquad$ inductive hypothesis

inductive case:

$length(Cons\ h\ t + b)$

$$= length(Cons\ h(t + b)) \qquad \text{rule two of concatenation}$$
$$= 1 + length(t + b) \qquad \text{rule two of length}$$
$$= 1 + length(t) + length(b) \qquad \text{inductive hypothesis}$$

Problem 7

  A The first lemma has issues because the set of green vegetables already includes that head of lettuce. This means it cannot be added to the set and the induction is invalid.

  B The second lemma depends on the first lemma being true to prove that all edible green things are vegetables. This invalidates it right off the start.
But let us suppose there are infinite green vegetables. The justification for this lemma requires us to sort an infinite set. This doesn't sound like a good idea. . . Sorting is an algorithm; algorithms must terminate to be correct. Sorting an infinite set will never terminate.