CS251: Homework #3

Due on October 29, 2019 at 2:00pm

Steven Libby Section A

Austen Nelson

Problem 1

Convert the following into first order predicate calculus:

- For any two real numbers, there is a real number between them.
  $\forall x, y \in \mathbb{R}.\exists z \in \mathbb{R}.x < z < y \vee x > z > y$
- A natural number is prime if, and only if, it has no divisors other than 1 and itself.
  $\forall p \in \mathbb{N}.p \in P \iff (\nexists d \in \{x \in \mathbb{N} \mid x \neq p \wedge x \neq 1\}.d \mid p)$
- A Language is Turing Complete if it can encode Turing Machines $L = TM$. There is a compiler between any two Turing Complete languages.
  L is the set of all languages.
  $\forall x, y \in L.TuringComp(x) \wedge TuringComp(y) \rightarrow (\exists f : x \rightarrow y)$
- The set of closed balls in n dimensional space is B^n. A fixed point is a point that doesn't change after applying a function. Any continuous function on a closed ball has at least one fixed point.
  $\forall n \in \mathbb{N}.\forall f : B^n \rightarrow B^n.Continuous(f) \rightarrow (\exists b \in B^n.b \equiv f(b))$
- Every planer graph is 4-colorable.
  $\forall g \in Graphs.Planer(g) \rightarrow Colorable(g, 4)$

Problem 2

Restate the following in english:

- $\forall n, a, b \in \mathbb{N}.n \mid a \wedge n \mid b \rightarrow \exists x, y \in \mathbb{N}.n \mid ax + by$
  For every three natural numbers, if one divides the other two, there are two natural scalars that scale the dividends such that the divisor divides the sum of the scaled values.
- $\forall n \in \mathbb{N}.n > 2 \rightarrow (\nexists a, b, c \in \mathbb{N}.a^n + b^n = c^n)$
  Fermat's last theorem. There are no natural solutions that satisfy $a^n + b^n = c^n$ larger than $n = 2$ for every natural a, b, c.
- $\nexists p, q \in \mathbb{N}.\frac{p}{q}^2 \neq 2$
  The square root of two is not a rational number.
- $\exists x \in \mathbb{R}.x^2 + 1 = 0$
  There is a real number that is also imaginary.

Problem 3

Determine if the following are well formed (legal). P is a predicate with arity 3, Q is a predicate with arity 2, and f is a function with arity 2, and g is a function with arity 1. The only domain is $\mathbb{N}$.

- $\forall x.P(f(x, x), g(x), x) = x$
  No, predicate P is a true or false value and cannot be compared to x, a natural.
- $\forall x.(\exists y.Q(x, y) \wedge P(x, y, z)) \rightarrow f(x, y) = g(x)$
  Well formed. Both sides of the implication evaluate to a boolean value.

Problem 3 continued on next page. . .

- $\nexists y.y = f(y,y)$
  Well formed.
- $\forall x, y, z.P(x,y,z) \to Q(x,y,z)$
  No, Q has an arity of 2.
- $\forall z.1 + 1 = 2$
  Pretty useless but well formed.

Problem 4

Reduce the following to shortest form.

- $\nexists x \exists y.(\neg x \vee \neg y)$

$$\begin{aligned} &\forall x.\forall y \neg(\neg x \vee \neg y) && DM_\exists * 2 \\ &\forall x, y.x \wedge y && DM_\vee \end{aligned}$$

- $\forall q.((\forall x.x) \wedge (\forall y.y) \to (\forall z.q))$

$$\begin{aligned} &\forall q, z.((\forall x.x) \wedge (\forall y.y) \to q) && Cov_\forall \\ &\forall q, z.(\forall x, y.x \wedge y) \to q && Com_\wedge, \forall_\wedge * 2 \\ &\forall q, z \exists x, y.x \wedge y \to q && Cont_\forall \end{aligned}$$

- $(\forall x.x) \to (\exists y.\neg y)$

$$\begin{aligned} &\exists x.x \to (\exists y.\neg y) && Cont_\forall \\ &\exists x, y.x \to \neg y && Cov_\exists \end{aligned}$$

Problem 5

Convert the following into PNF.

- $(\forall a \exists b.a \to b) \wedge (\forall b \exists a.a \to b)$

$$\begin{aligned} &(\forall a \exists b.a \to b) \wedge (\forall c \exists d.a \to b[b \mapsto c][a \mapsto d]) && Rep_\forall, Rep_\exists \\ &\forall a, c \exists b, d.a \to b \wedge d \to c && \forall_\wedge, \exists_\wedge \end{aligned}$$

- $\forall a.P(a) \to (\exists a.R(a,a))$

$$\begin{aligned} &\forall a.P(a) \to (\exists b.R(a,a)[a \mapsto b]) && Rep_\exists \\ &\forall a \exists b.P(a) \to R(b,b) && Cov_\exists \end{aligned}$$

- $\forall x.(\forall y.x \to y) \wedge (\forall y.y \to x)$

$$\begin{aligned} &\forall x.(\forall y.x \to y) \wedge (\forall z.y \to x[y \mapsto z]) && Rep_\forall \\ &\forall x, y, z.x \to y \wedge z \to x && \forall_\wedge * 2 \end{aligned}$$

Problem 5 continued on next page...

- $\forall x.(\forall y.x \to y) \to (\forall z.z)$

$$\forall x, z \exists y.(x \to y) \to z \qquad Cont_{\forall}, Cov_{\exists}$$

- $\forall x.\neg(\forall y.y) \to \neg(\forall z.x \land z)$

$$\forall x.(\exists y.\neg y) \to (\exists z.\neg(x \land z)) \qquad DM_{\exists}$$
$$\forall x, y \exists z.\neg y \to \neg(x \land z) \qquad Cont_{\exists}, Cov_{\exists}$$