Block (2/3)

The Finite Element Method for the Diffusion Equation in 1D

EE4375 - FEM For EE Applications

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1D FEM: (1/11) One-Dim Problem to Solve (1/2)

Same as Before: Boundary Value Problem

Second Order Differential Equations

- given $x \in \Omega = (0, 1)$ with left and right boundary point
- given: $f(x)$ given function and α given number
- find: $u(x)$ such that

$$\begin{aligned} -u''(x) &= f(x) \text{ for } 0 < x < 1 \text{ (differential equation on } \Omega) \\ u(x=0) &= 0 \text{ (Dirichlet boundary condition in } x = 0) \\ \frac{du}{dx}(x=1) &= \alpha \text{ (Neumann boundary condition in } x = 1) \end{aligned}$$

- differential equation not valid in $x = 0$ or $x = 1$

  boundary conditions are valid in $x = 0$ or $x = 1$

1D FEM: (1/11) One-Dim Problem to Solve (2/2)

Residual function r(x)

- definition: $r(x) = u''(x) + f(x)$
- quality of the solution:
  small (large) residual is indication of good (poor) approximation
- solve $-u''(x) = f(x)$ + b.c. equivalent to
- find $u(x)$ such that $r(x) = 0$ and $u(x)$ satisfies b.c.

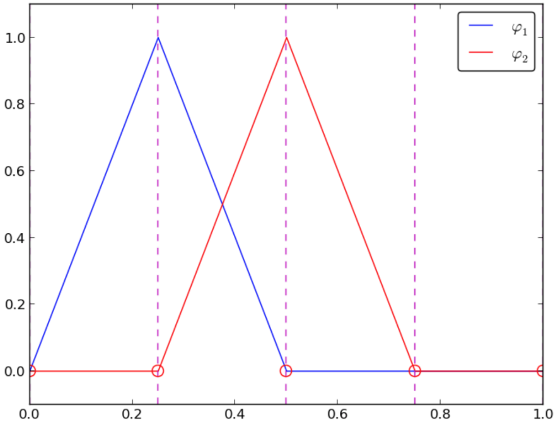
1D FEM: (2/11) Mesh and Shape Functions (1/9)

$$\text{Mesh } \Omega^h = \{x_i \mid 1 \leq i \leq N+1\}$$

- create mesh on $\Omega = (0,1)$
- $(N+1)$ points or nodes: x_i for $1 \leq i \leq N+1$
- N intervals or elements: $e_i = [x_i, x_{i+1}]$ for $1 \leq i \leq N$
- element size $h_i = x_{i+1} - x_i$
- mesh uniform in case that h_i is independent of i
  non-uniform otherwise
- Let op (Dutch) - Dhyaan (Hindi) - Zhuyi li (Chinese) - Alaintibah (Arabic):
  N elements e_i- (N+1) nodes x_i at which 2 on the boundary

1D FEM: (2/11) Mesh and Shape Functions (2/9)

Shape Functions



1D FEM: (2/11) Mesh and Shape Functions (3/9)

Definition of shape function $\phi_i(x)$

- linear Lagrange interpolation function
- for $x_{i-1} \leq x \leq x_i$ (left of x_i): $\phi_i(x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$ (positive slope)
- for $x_i \leq x \leq x_{i+1}$ (right of x_i): $\phi_i(x) = \dfrac{x_{i+1} - x}{x_{i+1} - x_i}$ (negative slope)
- for $x \leq x_{i-1}$ or $x \geq x_{i+1}$: $\phi_i(x) = 0$
- each node x_i (including 2 boundary nodes) has its $\phi_i(x)$
- mesh defines in total N+1 functions $\phi_i(x)$
- $\phi_i(x_j) = \delta_{ij}$ (see figure)

1D FEM: (2/11) Mesh and Shape Functions (4/9)

Mesh Terminology

1D Domains Ω

- two nodes x_i
- interval $e_i = [x_i, x_{i+1}]$
- interval length $h_i = x_{i+1} - x_i$

2D Domains Ω

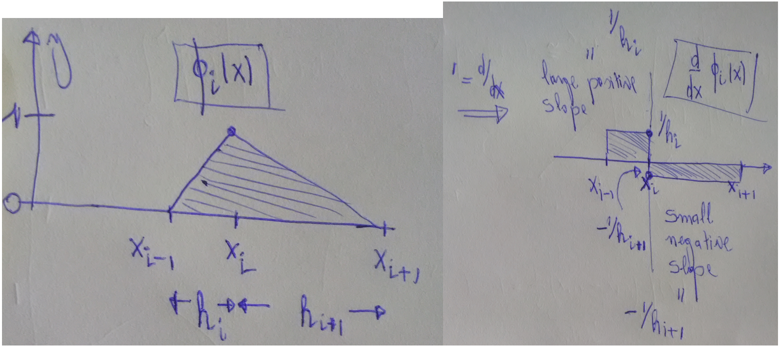
- three nodes x_i
- triangle formed by x_i
- area of triangle

Shape functions

- the mesh determines the set of shape functions
- construction carries over to non-uniform meshes
- one element $e_i = [x_i, x_{i+1}]$ ”sees” two basis functions
  - left node x_i has basis function $\phi_i(x)$
  - right node x_{i+1} has basis function $\phi_{i+1}(x)$
  - all other basis function $\phi_j(x)$ with $j \neq i$ or $j \neq i+1$ are zero on e_i

1D FEM: (2/11) Mesh and Shape Functions (6/9)

Shape function $\phi_i(x)$ (left) - derivative $\phi_i'(x) = \frac{d}{dx}\phi_i(x)$ (right)



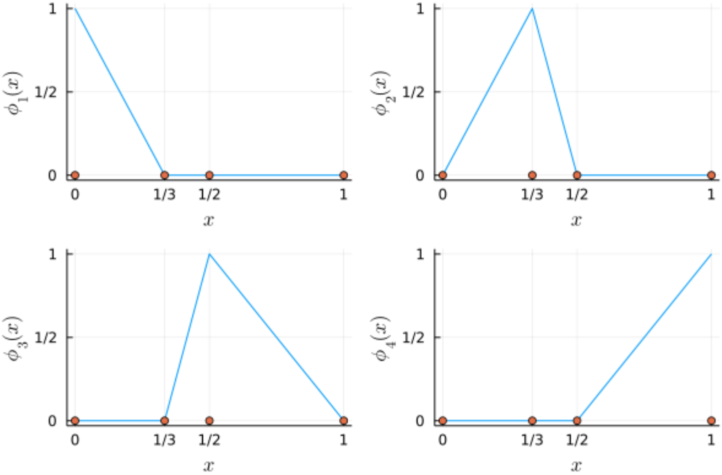
Mesh and Shape Functions (7/9)

Exercise

- assume a mesh with the four nodes $x_1 = 0$, $x_2 = 1/3$, $x_3 = 1/2$ and $x_4 = 1$;
- plot the mesh and label the nodes x_1, x_2, x_3 and x_4;
- plot the shape functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$;
- plot the first derivative of the shape functions $\frac{d\,\phi_1(x)}{dx}$, $\frac{d\,\phi_2(x)}{dx}$, $\frac{d\,\phi_3(x)}{dx}$ and $\frac{d\,\phi_4(x)}{dx}$;

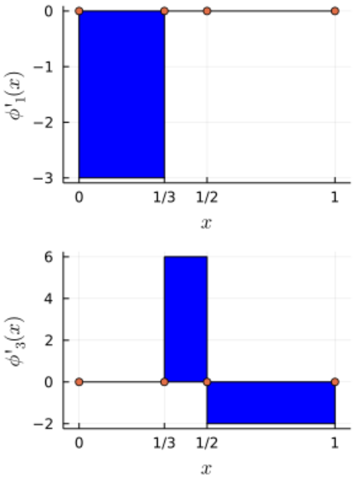
Mesh and Shape Functions (8/9)

Solution (1/2)



Mesh and Shape Functions (9/9)

Solution (1/2)



1D FEM: (3/11) Linear Combination of Shape Functions (1/5)

Linear Combination of Shape Functions

- set of functions $\{\phi_i(x) | 1 \leq i \leq N+1\}$
- linear combinations of these functions $\phi_i(x)$ can be made
- $V_0^h(\Omega)$: function space defined by all linear combinations

$$V_0^h(\Omega) = \text{span}\{\phi_1(x), \ldots, \phi_{N+1}(x)\}$$

- $u^h(x) \in V_0^h(\Omega)$: there exists coordinates $c_1, \ldots, c_{N+1}$ such that

$$u^h(x) = c_1\,\phi_1(x) + \ldots + c_{N+1}\,\phi_{N+1}(x)$$

1D FEM: (3/11) Linear Combination of Shape Functions (2/5)

Analogy with Geometry

- geometry in 2D plane: $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$
- x and y coordinates - $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ basis vectors

- geometry in 3D space: $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$
- x, y and z: coordinates
- $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$: basis vectors

1D FEM: (3/11) Linear Combination of Shape Functions (3/5)

Analogy with Fourier Analysis

- $f(x)$: given function on domain $x \in \Omega = (0, 1)$
- $f^N(x)$: approximation of $f(x)$ with all harmonics up to order N

$$f^N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(k\,\pi\,x) + b_k \sin(k\,\pi\,x)$$

- a_0, a_k and b_k for $1 \leq k \leq N$: coordinates
- 1, $\cos(k\,\pi\,x)$ and $\sin(k\,\pi\,x)$: Fourier modes or basis functions

1D FEM: (3/11) Linear Combination of Shape Functions (4/5)

Application to Finite Elements

- $u(x)$: exact solution of the boundary value problem
- $u^h(x)$: finite element approximation to $u(x)$ computed on Ω^h
- $u^h(x) \in V_0^h(\Omega) = \text{span}\{\phi_1(x), \ldots, \phi_{N+1}(x)\}$
- expansion of $u^h(x)$ as linear combination of shape function

$$u^h(x) = c_1\, \phi_1(x) + \ldots + c_{N+1}\, \phi_{N+1}(x)$$

- $c_1, \ldots, c_{N+1}$ coordinates - $\phi_1(x), \ldots, \phi_{N+1}(x)$ basis functions
- non-uniform meshes can be accommodated

1D FEM: (3/11) Linear Combination of Shape Functions (5/5)

Exercises

- assume a non-uniform mesh on Ω with 4 nodes $x = 0$, $x = 1/3$, $x = 1/2$ and $x = 1$
- plot $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$
- plot linear combinations of $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$
- assume a non-uniform mesh on Ω with (N+1) nodes
- plot $\phi_i(x) + \phi_{i+1}(x)$ on e_i

1D FEM: (4/11) Strong vs. Weak Equal to Zero (1/8)

Strong Equal to Zero (works as expected)

- assume $g(x)$ scalar function in x with domain $\Omega = (0, 1)$ (as $u(x)$ and $f(x)$ before)
- analytically:

$$g(x) = 0 \text{ if and only if for all } x \in \Omega \text{ we have that } g(x) = 0$$

$$\boxed{g(x) = 0 \text{ in continuous strong form } \Leftrightarrow \forall x \in \Omega : g(x) = 0}$$

- numerically: we only have mesh points Ω^h at disposal - enforce above in discrete nodes only

$$\boxed{g(x) = 0 \text{ in discrete strong form } \Leftrightarrow \forall x_i \in \Omega^h : g(x_i) = 0}$$

1D FEM: (4/11) Strong vs. Weak Equal to Zero (2/8)

Discussion of Strong Equal to Zero

- analytically: $g(x) = 0 \;\; \forall x \in \Omega = (0, 1)$

  no space to choose $g(x)$ for input x

- numerically: $g(x_i) = 0 \;\; \forall x_i \in \Omega^h = \{x_j | 1 \leq j \leq N + 1\}$

  lots of space to choose $g(x)$ for $x \neq x_i$

  e.g. $g(\frac{x_i + x_{i+1}}{2}) =$ whatever-you-please

1D FEM: (4/11) Strong vs. Weak Equal to Zero (3/8)

Example of Strong Equal to Zero

- choose: $g(x) = r(x) = u''(x) + f(x)$ (residual function)
- analytically:

$$g(x) = 0 + \text{bc} \Leftrightarrow -u''(x) = f(x) + \text{bc} \Leftrightarrow \text{problem to solve}$$

- numerically:

$$g(x_i) = 0 + \text{bc} \Leftrightarrow -u''(x_i) = f(x_i) + \text{bc} \Leftrightarrow \text{finite diff. method}$$

1D FEM: (4/11) Strong vs. Weak Equal to Zero (4/8)

Weak Equal to Zero (using averaging)

- first idea

$$g(x) = 0 \text{ in weak form} \Leftrightarrow \int_{\Omega} g(x)\,dx = \int_0^1 g(x)\,dx = 0$$

- strong equal zero implies weak equal zero

$$g(x) = 0 \Rightarrow \int_0^1 g(x)\,dx = 0$$

- converse is false (try e.g. $g(x) = x - 1/2$)
- hence the terminology

1D FEM: (4/11) Strong vs. Weak Equal to Zero (5/8)

Weak Equal to Zero (using weighted avering)

- towards a more proper definition

$$g(x) = 0 \text{ in weak form} \Leftrightarrow \int_{\Omega} g(x)\, v(x)\, dx = \int_0^1 g(x)\, v(x)\, dx = 0$$

- in previous slide $v(x) = 1$ was choosen
- What is $v(x)$? How many are there? How to choose them?

1D FEM: (4/11) Strong vs. Weak Equal to Zero (6/8)

Inspiration from Linear Algebra

- vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$
- unit-vector $\mathbf{i}$ i-th component equal to one, all other components equal to zero
- $\mathbf{v} = \mathbf{0} \Leftrightarrow v_i = 0 \;\; 1 \leq i \leq n$

  again $N + 1$ equations indexed by i
- $\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{v}^T \mathbf{i} =< \mathbf{v}, \mathbf{i} >= 0 \;\; 1 \leq i \leq n$
- how to bring this to finite elements?

1D FEM: (4/11) Strong vs. Weak Equal to Zero (7/8)

Applied to Finite Elements

- $N + 1$ basis functions $\phi_i(x)$ defined by the mesh Ω^h
- inner product: $< g(x), \phi_i(x) >= \int_0^1 g(x)\, \phi_i(x)\, dx$
- $< g(x), \phi_i(x) >$ coordinate of $g(x)$ along the basis function $\phi_i(x)$
- numerically: essential in remainder of the course

$$\boxed{g(x) = 0 \text{ in discrete weak form } \Leftrightarrow \forall x_i \in \Omega^h :< g(x), \phi_i(x) >= 0}$$

again $N + 1$ equations indexed by i

- analytically: see later

1D FEM: (4/11) Strong vs. Weak Equal to Zero (8/8)

Applied to Finite Elements

- choose $g(x) = r(x) = u''(x) + f(x)$
- enforce $g(x) = 0$ plus boundary conditions in discrete weak form

$$
\begin{aligned}
< g(x), \phi_i(x) > \quad &= \quad 0 \quad \text{for } 1 \leq i \leq N+1 \\
\Leftrightarrow \int_0^1 g(x)\, \phi_i(x)\, dx \quad &= \quad 0 \quad \text{for } 1 \leq i \leq N+1 \\
\Leftrightarrow \int_0^1 -u''(x)\, \phi_i(x)\, dx \quad &= \quad \int_0^1 f(x)\, \phi_i(x)\, dx \quad \text{for } 1 \leq i \leq N+1
\end{aligned}
$$

1D FEM: (5/11) Computation of the RHS Vector (1/4)

Terminology

- $f(x)$: source term or right-hand side in the differential equation
- α: value imposed on the derivative
- $\phi_i(x)$: basis function centered on $x_i \in \Omega^h$
- f_i: i-th component of the right-hand side vector $\mathbf{f}$
- relationship (see previous slide)

$$f_i = \int_0^1 f(x)\, \phi_i(x)\, dx$$

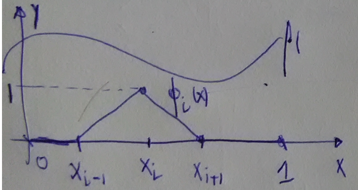
1D FEM: (5/11) Computation of the RHS Vector (2/4)

$$
\begin{aligned}
f_i &= \int_0^1 f(x)\,\phi_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} f(x)\,\phi_i(x)\,dx \quad (\text{ restrict to } e_{i-1} \cup e_i) \\
&= \int_{x_{i-1}}^{x_i} f(x)\,\phi_i(x)\,dx + \int_{x_i}^{x_{i+1}} f(x)\,\phi_i(x)\,dx \quad (\text{ seperate } e_{i-1} \text{ and } e_i) \\
&\approx \frac{h_{i-1}}{2}\,[f(x_{i-1})\,\phi_i(x_{i-1}) + f(x_i)\,\phi_i(x_i)] + \frac{h_i}{2}\,[f(x_i)\,\phi_i(x_i) + f(x_{i+1})\,\phi_i(x_{i+1})] \\
&\quad (\text{ use trapezoidal rule twice }) \\
&= \frac{h_{i-1}}{2}\,[f(x_{i-1})\,0 + f(x_i)\,1] + \frac{h_i}{2}\,[f(x_i)\,1 + f(x_{i+1})\,0)] \\
&\quad (\text{ use properties of } \phi_i(x)) \\
&= \frac{h_{i-1} + h_i}{2}\,f(x_i) \\
&= h\,f(x_i) \qquad (\text{ in case of uniform mesh })
\end{aligned}
$$

1D FEM: (5/11) Computation of the RHS Vector (3/4)

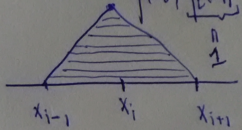
component i of the right-hand side vector

$$f[i] = \int_0^1 f(x)\,\phi_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} f(x)\,\phi_i(x)\,dx \cong \frac{x_{i+1}-x_{i-1}}{2} f(x_i) \overset{\text{uniform mesh}}{=} h\,f(x_i)$$



zoom $\Rightarrow$

$$f(x_i)\,\underbrace{\phi_i(x_i)}_{=1} = f(x_i)$$



$$\boxed{} = \frac{1}{2}\underbrace{(x_{i+1}-x_{i-1})}_{\text{width}}\,\underbrace{f(x_i)}_{\text{height}}$$

1D FEM: (5/11) Computation of the RHS Vector (4/4)

Treatment of the Dirichlet and Neumann boundary conditions

- Dirichlet boundary condition

  fix $u(x)$ - grounding:

  same as for Finite Difference method

- Neumann boundary condition

  fix $\frac{d}{dx}u(x)$ - imposed symmetry or flux:

  different from Finite Difference method - details later

1D FEM: (6/11) Mathematical Preliminaries (1/4)

Wish - Desire - Aim - Ambition?

- manipulate $-\int_0^1 u''(x)\,\phi_i(x)\,dx$
- challenge: $u(x)$ is unknown
- observe: $u''(x)$ has double prime - $\phi_i(x)$ has no prime

1D FEM: (6/11) Mathematical Preliminaries (2/4)

Derivative: Integration by Parts in one variable (1/2)

- fundamental theorem of calculus:

  assume $0 < x < 1$ or $x \in \Omega = (0, 1)$ and $F'(x) = \dfrac{d\,F(x)}{dx}$

$$\int_\Omega F'(x)\,dx = \int_0^1 F'(x)\,dx = [F(x)]_0^1 = F(1) - F(0)$$

- choose $F(x) = \frac{d\,u}{dx}\,v(x)$ (motivation on next slide)

  observe $u(x)$ has prime and $v(x)$ has no prime

- then $F'(x) = \dfrac{d}{dx}\left(\dfrac{d\,u}{dx} v(x)\right) = u''(x)\,v(x) + u'(x)\,v'(x)$

1D FEM: (6/11) Mathematical Preliminaries (3/4)

Derivative: Integration by Parts in one variable (2/2)

- from $\int_0^1 F'(x)\,dx = [F(x)]_0^1$ we obtain that

$$\int_0^1 \left[u''(x)\,v(x) + u'(x)\,v'(x)\right]\,dx = \left[u'(x)\,v(x)\right]_0^1$$

- after rearranging terms

$$\boxed{-\int_0^1 u''(x)\,v(x)\,dx = \int_0^1 u'(x)\,v'(x)\,dx - \left[u'(x)\,v(x)\right]_0^1}$$

LHS: $u(x)$ has double prime and $v(x)$ has no prime

RHS: both $u(x)$ and $v(x)$ have one prime - additional fudge term (used later)

1D FEM: (6/11) Mathematical Preliminaries (4/4)

Integration: Quadrature by Trapezoidal Rule

- trapezoidal rule: $0 \leq a < b \leq 1$

$$\int_a^b g(x)\,dx \approx \frac{b-a}{2}\,[g(a)+g(b)]$$

- Simpson rule: $0 \leq a < b \leq 1$

$$\int_a^b g(x)\,dx \approx \frac{b-a}{6}\left[g(a)+4\,g(\frac{a+b}{2})+g(b)\right]$$

- typically applied to $a = x_i$ and $b = x_{i+1}$

1D FEM: (7/11) Discrete Weak Form (1/4)

Apply Integration by Parts on Weak Formulation

- earlier we set the residual $r(x) = u''(x) + f(x)$ to zero in weak form and arrived at

$$\int_0^1 -u''(x)\,\phi_i(x)\,dx = \int_0^1 f(x)\,\phi_i(x)\,dx \quad \text{for } 1 \leq i \leq N+1$$

- apply integration by part to the LHS

$$\int_0^1 u'(x)\,\phi_i'(x)\,dx = \int_0^1 f(x)\,\phi_i(x)\,dx + [u'(x)\,\phi_i(x)]_0^1 \quad \text{for } 1 \leq i \leq N+1$$

observe: minus sign in LHS disappeared
- first order derivative on both $u(x)$ and $\phi_i(x)$

1D FEM: (7/11) Discrete Weak Form (2/4)

- boundary conditions: $u(x=0)=0$ and $u'(x=1)=\alpha$
- boundary term in the RHS of the weak form

$$\begin{aligned}[u'(x)\,\phi_i(x)]_0^1 &= u'(x=1)\,\phi_i(x=1) - u'(x=0)\,\phi_i(x=0)\\ &= \alpha\,\phi_i(x=1) - (\text{no information given})\,\phi_i(x=0)\end{aligned}$$

- Dirichlet and Neumann boundary conditions treated differently
- for now save assume that

$$[u'(x)\,\phi_i(x)]_0^1 = \alpha\,\phi_i(x=1) - 0 = \alpha\,\phi_i(x=1)$$

1D FEM: (7/11) Discrete Weak Form (3/4)

Discrete Weak Form Becomes

$$\int_0^1 u'(x)\,\phi_i'(x)\,dx = \int_0^1 f(x)\,\phi_i(x)\,dx + \alpha\,\phi_i(x=1) \quad \text{for } 1 \le i \le N+1$$

- assume $u(x)$ approximate by $u^h(x)$ where $u^h(x) = \sum_{j=1}^{N+1} c_j\,\phi_j(x)$
- thus $u'(x)$ approximate by $[u^h(x)]' = \sum_{j=1}^{N+1} c_j\,\phi_j'(x)$
- then for $1 \le i \le N+1$

$$\sum_{j=1}^{N+1} \int_0^1 \phi_j'(x)\,\phi_i'(x)\,dx\,c_j = \int_0^1 f(x)\,\phi_i(x)\,dx + \alpha\,\phi_i(x=1)$$

1D FEM: (7/11) Discrete Weak Form (4/4)

Discrete Weak Form Becomes

- for $1 \leq i \leq N+1$

$$\sum_{j=1}^{N+1} \int_0^1 \phi_i'(x)\,\phi_j'(x)\,dx\,c_j = \int_0^1 f(x)\,\phi_i(x)\,dx + \alpha\,\phi_i(x=1)$$

- can be written in the form: for $1 \leq i \leq N+1$

$$\sum_{j=1}^{N+1} A_{ij}\,c_j = f_i$$

- and thus as a $N+1$ by $N+1$ linear system

$$A\,\mathbf{c} = \mathbf{f}$$

- index i counts equations - index j counts unknowns

1D FEM: (8/11) Linear System Formulation (1/8)

Expression for Matrix and Vector Elements

- Matrix elements:

$$A_{ij} = \int_0^1 \phi_i'(x)\, \phi_j'(x)\, dx \text{ for } 1 \le i, j \le N + 1$$

- Vector elements:

$$f_i = \int_0^1 f(x)\, \phi_i(x)\, dx + \alpha\, \phi_i(x = 1) \text{ for } 1 \le i \le N + 1$$

1D FEM: (8/11) Linear System Formulation (2/8)

Properties of Matrix A

- diagonal of A is positive - $A_{ii} > 0$

$$A_{ii} = \int_0^1 \underbrace{\phi_i'(x)\,\phi_i'(x)}_{>0}\,dx > 0$$

- A is symmetric - $A = A^T$

$$A_{ij} = \int_0^1 \phi_i'(x)\,\phi_j'(x)\,dx = \int_0^1 \phi_j'(x)\,\phi_i'(x)\,dx = A_{ji}$$

- off-diagonal of A is negative

$$A_{i,i+1} = \int_0^1 \phi_i'(x)\,\phi_{i+1}'(x)\,dx = \int_{x_i}^{x_{i+1}} \underbrace{\phi_i'(x)}_{<0}\,\underbrace{\phi_{i+1}'(x)}_{>0}\,dx < 0$$

- A is tri-diagonal and thus sparse
- A many other cool properties $\Rightarrow$ fast solvers for $A\,\mathbf{c} = \mathbf{f}$ exist

1D FEM: (8/11) Linear System Formulation (3/8)

Remember that

- length of element $e_i = [x_i, x_{i+1}]$ is denoted by $h_i = x_{i+1} - x_i$
- length of element $e_{i-1} = [x_{i-1}, x_i]$ is denoted by $h_{i-1} = x_i - x_{i-1}$
- slope of $\phi_i(x)$ on e_i is equal to $\frac{-1}{h_i}$
- slope of $\phi_{i+1}(x)$ on e_i is equal to $\frac{1}{h_i}$

1D FEM: (8/11) Linear System Formulation (4/8)

Diagonal Matrix Elements A_{ii}

$$
\begin{aligned}
A_{ii} &= \int_0^1 \phi_i'(x)\,\phi_i'(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} \phi_i'(x)\,\phi_i'(x)\,dx \quad (\text{ restrict to } e_{i-1} \cup e_i) \\
&= \int_{x_{i-1}}^{x_i} \phi_i'(x)\,\phi_i'(x)\,dx + \int_{x_i}^{x_{i+1}} \phi_i'(x)\,\phi_i'(x)\,dx \quad (\text{ seperate } e_{i-1} \text{ and } e_i) \\
&= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_{i-1}}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h_i}\right)^2 dx \quad (\text{ use information on derivatives} \\
&= h_{i-1}\left(\frac{1}{h_{i-1}}\right)^2 + h_i\left(\frac{1}{h_i}\right)^2 \quad (\text{ integration of constant integrands }) \\
&= \frac{1}{h_{i-1}} + \frac{1}{h_i} \quad (\text{ in case of non-uniform mesh }) \\
&= \frac{2}{h} \quad (\text{ in case of uniform mesh for which all } h_i \text{ are equal })
\end{aligned}
$$

1D FEM: (8/11) Linear System Formulation (5/8)

Upper-Diagonal Matrix Elements

$$
\begin{aligned}
A_{i,i+1} &= \int_0^1 \phi_i'(x)\,\phi_{i+1}'(x)\,dx = \int_{x_i}^{x_{i+1}} \phi_i'(x)\,\phi_{i+1}'(x)\,dx \quad (\text{ restrict to } e_i) \\
&= \int_{x_i}^{x_{i+1}} \frac{-1}{h_i}\,\frac{1}{h_i}\,dx \quad (\text{ use information on derivatives }) \\
&= h_i\,\frac{-1}{h_i}\,\frac{1}{h_i} \quad (\text{ integration of constant integrands }) \\
&= \frac{-1}{h_i} \quad (\text{ in case of non-uniform mesh }) \\
&= \frac{-1}{h} \quad (\text{ in case of uniform mesh for which all } h_i \text{ are equal })
\end{aligned}
$$

1D FEM: (8/11) Linear System Formulation (6/8)

Summary: Matrix A and Right-Hand Vector $\mathbf{f}$

Internal mesh points x_i for $2 \leq i \leq N$

- Matrix

$$A[i,:] = \begin{pmatrix} 0 & \dots & 0 & \underbrace{-1/h_{i-1}}_{i-1} & \underbrace{1/h_{i-1} + 1/h_i}_{i} & \underbrace{-1/h_i}_{i+1} & 0 & \dots & 0 \end{pmatrix}$$

- Vector

$$f[i] = \frac{h_{i-1} + h_i}{2} f(x_i)$$

1D FEM: (8/11) Linear System Formulation (7/8)

Summary: Matrix A and Right-Hand Vector $\mathbf{f}$

Treatment of the Boundary Conditions

- Dirichlet boundary conditions: $u(x = 0) = 0$:

  modify first equation in linear system

  see finite difference method

- Neumann boundary conditions: $u'(x = 1) = \alpha$

  add term $\alpha \, \phi_i(x = 1)$ to vector $\mathbf{f}$

  see lab sessions for details

1D FEM: (8/11) Linear System Formulation (8/8)

Exercise

- assume again a non-uniform mesh on Ω with 4 nodes $x = 0$, $x = 1/3$, $x = 1/2$ and $x = 1$
- compute matrix A and vector $\mathbf{f}$

1D FEM: (9/11) Element-by-Element Construction of the Vector (1/7)

Remember that

- i-th component of the right-hand side vector $\mathbf{f}$ is given by

$$f_i = \int_{x_{i-1}}^{x_{i+1}} f(x)\,\phi_i(x)\,dx = \int_{e_{i-1}} f(x)\,\phi_i(x)\,dx + \int_{e_i} f(x)\,\phi_i(x)\,dx$$

1D FEM: (9/11) Element-by-Element Construction of the Vector (2/7)

Element e_i contributes to f_i and f_{i+1} only

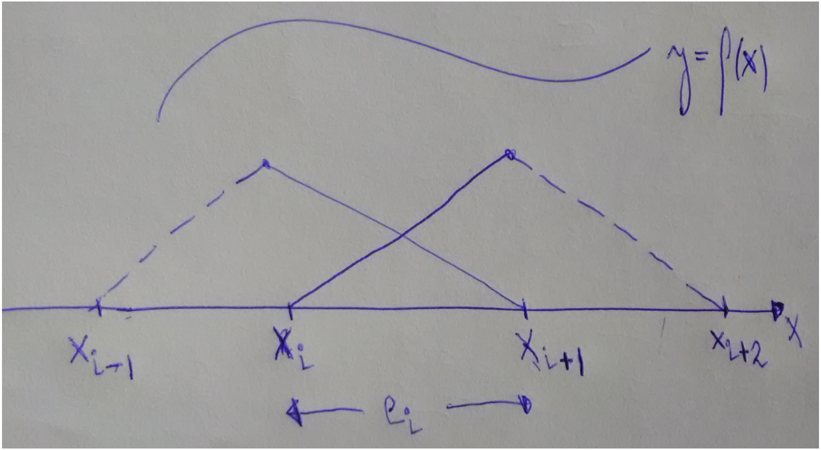
- i-th component of the right-hand side vector $\mathbf{f}$

$$f_i = \int_{e_{i-1}} f(x)\,\phi_i(x)\,dx + \int_{e_i} f(x)\,\phi_i(x)\,dx$$

- $i+1$-th component of the right-hand side vector $\mathbf{f}$

$$f_{i+1} = \int_{e_i} f(x)\,\phi_{i+1}(x)\,dx + \int_{e_{i+1}} f(x)\,\phi_{i+1}(x)\,dx$$

1D FEM: (9/11) Element-by-Element Construction of the Vector (3/7)



1D FEM: (9/11) Element-by-Element Construction of the Vector (4/7)

How does element e_i contribute to the vector $\mathbf{f}$?

- we here fix element e_i (rather than fixing component f_i)
- element $e_i = [x_i, x_{i+1}]$ sees shape functions $\phi_i(x)$ and $\phi_{i+1}(x)$
  all other basis functions are zero on the element e_i
- the element e_i therefore contributes to two component of $\mathbf{f}$
  these components are f_i and f_{i+1}
  (same indices as basis functions)
- this leads to an element-by-element construction of vector $\mathbf{f}$

1D FEM: (9/11) Element-by-Element Construction of the Vector (5/7)

How does element e_i contribute to the vector $\mathbf{f}$?

- $f_{e_i} \in \mathbb{R}^2$ contribution of element e_i to global vector $\mathbf{f}$

$$f_{e_i} = \begin{pmatrix} \int_{e_i} f(x)\,\phi_i(x)\,dx \\ \int_{e_i} f(x)\,\phi_{i+1}(x)\,dx \end{pmatrix}$$

- use trapezoidal rule of integration (see earlier)

$$f_{e_i} = \begin{pmatrix} \int_{e_i} f(x)\,\phi_i(x)\,dx \\ \int_{e_i} f(x)\,\phi_{i+1}(x)\,dx \end{pmatrix} = \frac{h_i}{2} \begin{pmatrix} f(x_i) \\ f(x_{i+1}) \end{pmatrix}$$

- given mesh Ω^h and source $f(x)$, f_{e_i} for each e_i can be computed

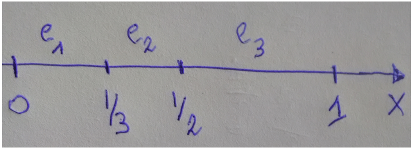
1D FEM: (9/11) Element-by-Element Construction of the Vector (6/7)

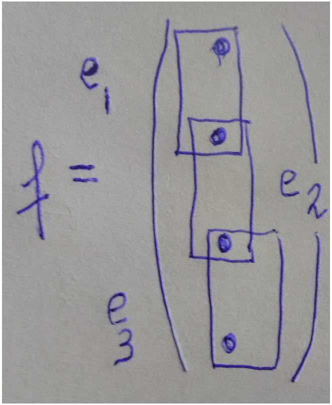
Finite Element Assembly of the Vector **f**

- loop over all of the N elements e_i in the mesh Ω^h
- on e_i compute the local element vector $f_{e_i} \in \mathbb{R}^2$
  the local element vector has two components
- add local element vector to the global vector $\mathbf{f} \in \mathbb{R}^{N+1}$
  the global vector **f** has $N + 1$ components
- $\mathbf{f} = \mathbf{f} + f_{e_i}$
  assembly requires taking the mesh connectivity into account
  connectivity here between nodes x_i and the elements e_i into account

1D FEM: (9/11) Element-by-Element Construction of the Vector (7/7)

Assembly for Three-Element Mesh





1D FEM: (10/11) Element-by-Element Construction of the Matrix (1/4)

How does element e_i contribute to the vector A?

- element $e_i = [x_i, x_{i+1}]$ sees shape functions $\phi_i(x)$ and $\phi_{i+1}(x)$
- the element e_i therefore contributes to four component of A

  these components are A_{ii}, $A_{i,i+1} = A_{i+1,i}$ and $A_{i+1,i+1}$

1D FEM: (10/11) Element-by-Element Construction of the Matrix (2/4)

How does element e_i contribute to the vector A?

- $A_{e_i} \in \mathbb{R}^{2\times 2}$ contribution of element e_i to global vector A

$$A_{e_i} = \begin{pmatrix} \int_{e_i} \phi_i'(x)\,\phi_i'(x)\,dx & \int_{e_i} \phi_i'(x)\,\phi_{i+1}'(x)\,dx \\ \int_{e_i} \phi_i'(x)\,\phi_{i+1}(x)\,dx & \int_{e_i} \phi_{i+1}'(x)\,\phi_{i+1}'(x)\,dx \end{pmatrix}$$

- use derivative of the shape functions

$$A_{e_i} = \frac{1}{h_i} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

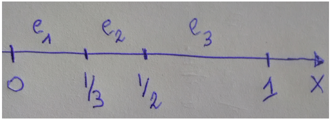
1D FEM: (9/11) Element-by-Element Construction of the Matrix (3/4)

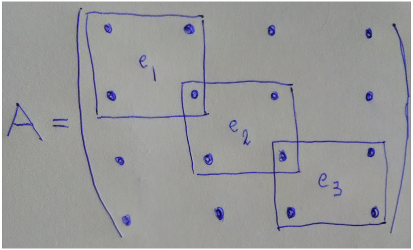
Finite Element Assembly of the Matrix A

- loop over all of the N elements e_i in the mesh Ω^h
- on e_i compute the local element matrix $A_{e_i} \in \mathbb{R}^{2 \times 2}$
  the local element matrix has two by two components
- add local element matrix to the global matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$
  the global matrix A has $N+1$ by $N+1$ components
- $A = A + A_{e_i}$
  assembly requires taking the mesh connectivity into account
  connectivity here between nodes x_i and the elements e_i into account

1D FEM: (10/11) Element-by-Element Construction of the Matrix (4/4)

Assembly for Three-Element Mesh





1D FEM: (11/11) Implementation in Julia

- see Lab Session