Practice problems for Linear and Quadratic functions
Consider the beta distribution

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

of which $\alpha = \beta = 2$ and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$ (or n as a positive integer) and $n! = n(n-1)(n-2)\ldots(2)(1)$. Answer the following questions

1. What is the Y-axis intersection(s) of $f(x;2,2)$? How many are there?

2. How many X-axis intersections of $f(x;2,2)$? Is there any?

3. (Continuation of 2) What is/are the solution(s) of $f(x;2,2) = 0$? Do they exist?

4. Which of the following lines will intersect with the $f(x;2,2)$? If intersection exists, create the linear $y = ax$ (or affine, $y = ax + b$) function for each intersection point to the point $T(0,1)$

   (a) $y = 3x - 1$
   (b) $y = -2$
   (c) $x = 3$
   (d) $-2y + x = 0$
   (e) $y = 3$
   (f) $y = 0.2x + 9$

5. (Challenging) As the function $f(x;\alpha,\beta)$ describes the probability density function, it does means that the area under the graph $f(x;2,2)$ over the X-axis is equal to 1 unit (from the axiom of probability). Indicates the point $P(a,b)$ such that it will make the area under the graph $f(x;2,2)$ over the X-axis equal to 0.5 unit if we calculate the area starting from the intersection point of the graph with Y-axis and the point $P(a,b)$.

6. (Even more challenging) Let say we consider $h(x) = 4f(x;2,2)$, what would be the under the graph $h(x)$ over the X-axis from its Y-axis intersection to its maximum? [Hint: What is the shape of $f(x;2,2)$? What is the definition of that shape? What if we make a rectangle touching $f(x;2,2)$ on its corners within that shape?]

7. (A lot more challenging) Let say we consider $k(x) = h(x) + 2$ ($h(x)$ from question 6), what would be the area under the graph $k(x)$ over the X-axis from its Y-axis intersection to its maximum? [Hint: What if we make a rectangle within that shape, but this time wider?]

8. (Advanced) If we cut the area under the graph $h(x)$ (from problem 6) over the X-axis with $y = 6x$, what would be the area over the graph $y = 6x$ and under the graph $k(x)$ from Y-axis intersection of $h(x)$ to its maximum? [Hint: What is the shape of $y = 6x$ from Y-axis intersection of $k(x)$ to some line like $x = 2$?]

[Hint: $\Gamma(2) = 1$, $\Gamma(4) = 6$]

Brief solution $f(x;2,2) = (x)(1-x)\Gamma(4)/(\Gamma(2)\Gamma(2)) = 6x - 6x^2$

1. $(0,0)$, only one.
2. two points: $(0,0)$ and $(1,0)$
3. see 2.
4. In standard form
   (a) $-(6+\sqrt{33})x + y = 1$
   $(\sqrt{33}-6)x + y = 1$
   (b) $-(9+3\sqrt{21})x + 2y = 2,$
   $18x + (3+\sqrt{21})y = 3+\sqrt{21}$
   (c) no intersection
   (d) $x = 0$
   $13x + 22y = 22$
   (e) no intersection
   (f) no intersection
5. As parabola a symmetric shape, its vertex $P(0.5, 1.5)$ would cut the area into half.
6. 2, As parabola is similar shape no matter how you scale or translate it. It means that the area also scale with the same factor also. As the vertex is scaled 4 times, it means that the area also scales 4 times. (Can be proven using the area of similar rectangles like suggested on hint part)
7. 3, With the same reason with 5. The parabola is shifted upward for 2 units and thus, create the rectangle with the area 2.
8. 1.25, the area under the line $y = 6x$ from $x = 0$ to $x = 0.5$ is right triangle with the area 0.75. We can ideally subtract the area of the triangle (yellow) with the area of the function (green) itself to find the target area (blue).