

Athreya Lahiri Chapter 3 Solutions

Arthur Chen

October 7, 2025

Problem 3.1

Let $\phi : (a, b) \rightarrow \mathbb{R}$ be convex. Show the following.

We will constantly use the following convexity formula: if ϕ is convex and $a < x_1 < x_2 < x_3 < b$, then

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(x_3) - \phi(x_1)}{x_3 - x_1} \leq \frac{\phi(x_3) - \phi(x_2)}{x_3 - x_2}$$

Part a

For each $x \in (a, b)$,

$$\phi'_+(x) = \lim_{y \downarrow x} \frac{\phi(y) - \phi(x)}{y - x}, \phi'_-(x) = \lim_{y \uparrow x} \frac{\phi(y) - \phi(x)}{y - x}$$

exist and are finite.

Consider ϕ'_+ first. Let $z < x$ be arbitrary and let $\{y_n\}_{n=1} \geq x$ such that $y_n \rightarrow x$. Wlog, let y_n be decreasing. Then by convexity,

$$\frac{\phi(y_1) - \phi(x)}{y_1 - x} \geq \frac{\phi(y_2) - \phi(x)}{y_2 - x} \geq \dots$$

which is bounded below by $\frac{\phi(x) - \phi(z)}{x - z}$. Thus $\frac{\phi(y_n) - \phi(x)}{y_n - x}$ is a monotonically decreasing sequence bounded below, thus it has a limit. Since this sequence was arbitrary, the limit and thus ϕ'_+ exists. The same applies in reverse for ϕ'_- .

Problem 3.3

Prove the following.

Part a

Let $a_1 \dots a_k$ be real and $p_1 \dots p_k$ be positive numbers such that $\sum_{i=1}^k p_i = 1$. Then

$$\sum_{i=1}^k p_i \exp(a_i) \geq \exp\left(\sum_{i=1}^k p_i a_i\right)$$

Let P be the probability measure on \mathbb{R} that assigns probability p_i to point a_i and apply Jensen's inequality with $\phi(x) = e^x$.

Part b

Let $b_1 \dots b_k$ be nonnegative numbers and $p_1 \dots p_k$ be as in Part a. Then

$$\sum_{i=1}^k p_i b_i \geq \prod_{i=1}^k b_i^{p_i}$$

Furthermore, equality holds iff $b_1 = b_2 = \dots = b_k$.

Let $a_i = \log b_i$ and apply Part a. For the iff, since the exponential function is strictly convex, inequality holds iff $f(\omega)$ is a constant, which in this context means that all the b_i s are equal.

Part c

For any $a, b \in \mathbb{R}$ and $1 \leq p < \infty$,

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$$

Let $f(x) = x$, $\phi(x) = |x|^p$, which is convex on the range of p , and let P be the probability measure with $1/2$ probability on $\{a, b\}$. Thus by Jensen's inequality,

$$\phi\left(\int x dP\right) = \frac{1}{2^p} |a + b|^p \leq \int |x|^p dP = \frac{1}{2}(|a|^p + |b|^p)$$

which implies the desired result.

Problem 3.12

Part b

Prove that for $p \in (0, 1)$, $\int |f + g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$.

Building off of equation 2.2, we have that

$$\left(\frac{|x|}{|x| + |y|}\right)^p + \left(\frac{|y|}{|x| + |y|}\right)^p \geq \frac{|x|}{|x| + |y|} + \frac{|y|}{|x| + |y|} = 1$$

implies

$$|x + y|^p \leq (|x| + |y|)^p \leq |x|^p + |y|^p$$

and integrating pointwise gives the result.