Athreya Lahiri Chapter 1 Solutions

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Problem 1.19

Let Ω be a nonempty set and let $C \subset \mathcal{P}(\Omega)$ be a semialgebra. Let

$$\mathcal{A}(C) = \{A : A = \bigcup_{i=1}^{k} B_i : B_i \in C, i = 1, 2 \ldots k, k \in \mathbb{N}\}$$

Part a

Show that $\mathcal{A}(C)$ is the smallest algebra containing C.

Lemma 1 *$\mathcal{A}(C)$ is an algebra.*

Proof: *For $\Omega \in \mathcal{A}(C)$, let $A \in C$ be arbitrary. Because C is a semialgebra, $A^C = \bigcap_{i=1}^{k} B_i, B_i \in C$, so $A^C \in C$. Thus $A \cap A^C = \Omega \in \mathcal{A}(C)$.*

For closure under compliments, $A \in \mathcal{A}(C)$ implies by definition that $A = \bigcup_{i=1}^{k} B_i, B_i \in C$, and taking compliments $A^C = \bigcap_{i=1}^{k} B_i^C$. Because $B_i \in C$ and C is a semialgebra, each B_i^C is the finite union of $C_j \in C$. Thus

$$A^C = \bigcap_{i=1}^{k} \bigcup_{j=1}^{l_i} C_j$$

Distributing the intersection, we get the union of a finite number of pairwise intersections, e.g. $A^C = (C_{11} \cap C_{12}) \cup (C_{11} \cap C_{13}) \cup \ldots$. All of the pairwise intersections are in C, and thus their union is in $\mathcal{A}(C)$. Thus $A^C \in \mathcal{A}(C)$.

Closure under finite union is immediate. □

Showing that $\mathcal{A}(C)$ is the smallest algebra containing C is equivalent to showing that if B is an algebra such that $C \subset B$, then $\mathcal{A}(C) \subset B$. But this is almost immediate. Let $M \in \mathcal{A}(C)$. By definition of $\mathcal{A}(C)$, $M = \bigcap_{i=1}^{k} M_k, M_k \in C, k \in \mathbb{N}$. Since B is an algebra contain C, $M \in B$. Thus $\mathcal{A}(C) \subset B$, as desired.

Part b

Show that $\sigma\langle C\rangle = \sigma\langle \mathcal{A}(C)\rangle$.

Trivially, $C \subset \mathcal{A}(C)$ implies $\sigma\langle C\rangle \subset \sigma\langle \mathcal{A}(C)\rangle$. For the reverse, let $M \in \sigma\langle \mathcal{A}(C)\rangle$. Then M is the countable union of elements in $\mathcal{A}(C)$, and thus it is the countable union of elements in C and thus in $\sigma\langle C\rangle$.

Problem 1.20

Let C be a semialgebra Ω with μ a measure on C. Let μ^* be the outer measure induced by μ^*, defined as

$$\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n\geq 1} \subset C, A \bigcup_{n\geq 1} A_n\right\}$$

Show that μ^* satisfies the following three properties making it an outer measure:

- Non-negativity: $\mu^*(\emptyset) = 0$
- Monotonicity: $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
- Countable sub-additivity: For any $\{A_n\}_{n\geq 1} \subset P(\Omega), \mu^*\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Lemma 2 *For a (non-empty) semi-algebra C, $\emptyset \in C$.*

Proof: *Let $A \in C$. Then $A^C = \bigcup_{i=1}^{k} B_i$, k finite, $B_i \in C$. Then $A \cap B_i = \emptyset$, and since C is closed to complements, $\emptyset \in C$.* □

For non-negativity, $\emptyset \in C$ and $\mu(\emptyset) = 0$ imply $\mu^*(\emptyset) = 0$. For monotonicity, consider any cover of B with sets in C. Then that cover also covers A. Since $\mu^*(A)$ is the infimum of the sum of the measures of sets in C that cover A, monotonicity follows immediately.

For countable sub-additivity, if the sum on the right is infinite, then the result is immediate. Otherwise, $\mu^*(A_n) < \infty$ for all n and the outer measures of A_n decrease enough so that the sum is (absolutely) convergent.

Let $0 < \epsilon < \infty$. Since $\mu^*(A_n)$ is the infimum over covers of A_n with elements in C, there exist $\{A_{nj}\}_{j\geq 1} \subset C$ such that

$$\mu^*(A_n) \leq \sum_{j=1}^{\infty} \mu(A_{nj}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$$

The union of covers over the A_n, $\bigcup_{n\geq 1}\bigcup_{j\geq 1} A_{nj}$, form a cover for $\bigcup_{n\geq 1} A_n$. Thus

$$\mu^*\left(\bigcup_{n\geq 1} A_n\right) \leq \mu\left(\bigcup_{n=1}\bigcup_{j=1} A_{nj}\right) \leq \sum_{n=1}\sum_{j=1}\mu(A_{nj}) \leq \sum_{n=1}\left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right)$$

which proves the desired result.

Problem 1.22

Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing. Let $(a, b], (a_n, b_n], n \in \mathbb{N}$ be intervals in $\mathbb{R}$ such that $(a, b] = \bigcup_{n\geq 1}(a_n, b_n]$ and $\{(a_n, b_n] : n \geq 1\}$ are disjoint. Let $\mu_F(\cdot)$ be

$$\mu_F((a, b]) = F(b+) - F(a+)$$
$$\mu_F((a, \infty)) = F(\infty) - F(a+)$$

Show that $\mu_F((a, b]) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n])$ and is thus countably additive with the following:

Part b

Assume wlog that $F(\cdot)$ is right continuous (which follows trivially from the definition). Show that for any $k \in \mathbb{N}$,

$$F(b) - F(a) \geq \sum_{i=1}^{k}(F(b_i) - F(a_i))$$

which implies that

$$F(b) - F(a) \geq \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

Wlog, assume that the intervals are indexed so that $a_{i+1} > a_i$. Choose k of the intervals, being sure to choose the $a_1 = a$ and a_k corresponding to the interval with right endpoint b. Since the intervals are disjoint, $b_{i-1} \leq a_i$ and since F is nondecreasing, this implies that $F(b_{i-1}) \leq F(a_i)$. Thus by telescoping,

$$\begin{aligned} F(b) - F(a) &= \sum_{i=2}^{k} F(b_i) - F(b_{i-1}) + F(b_1) - F(a) \\ &\geq \sum_{i=1}^{k} F(b_i) - F(a_i) \end{aligned}$$

for all k, which implies the desired result.

Part c

Fix $\eta > 0$. Choose $c > a$ and $d_n > b_n, n \geq 1$ such that

$$F(c) - F(a) < \eta$$
$$F(d_n) - F(b_n) < \frac{\eta}{2^n}$$

We know that $\{(a_n, d_n)\}_{n\geq 1}$ is an open cover for $[c, b]$, so by the Heine-Borel theorem, there exists a finite subcover $\{(a_i, d_i)\}_{i=1}^k$ for $[c, b]$. Wlog, index the intervals in the finite subcover such that $c \in (a_1, d_1)$ and $b \in (a_k, d_k)$. Thus letting b_k be the b_i that corresponds to d_k,

$$\begin{aligned}
F(b) - F(a) &= F(b_k) - F(a_k) + F(a_k) - \cdots + F(a_1) - F(a) \\
&\leq F(d_k) - F(a_k) + F(d_{k-1}) - F(a_{k-1}) + \cdots + F(c) - F(a) \\
&\leq \sum_{i=1}^{k} \left(F(b_i) - F(a_i) + \frac{\eta}{2^n} \right) + \eta \\
&\leq \sum_{i=1}^{k} F(b_i) - F(a_i) + 2\eta
\end{aligned}$$

which implies the desired result.

Problem 1.26

Establish the uniqueness of the Caratheodory extension with the following. Let μ be a σ-finite measure on a semialgebra C. Let ν be a measure on the measurable space $(\Omega, \sigma\langle C\rangle)$, such that $\nu = \mu$ on C. We want to prove that $\nu = \mu^*$ on $\sigma\langle C\rangle$.

Part a

Suppose that $\nu(\Omega) < \infty$. Verify that $L := \{A : A \in \sigma\langle C\rangle, \mu^*(A) = \nu(A)\}$ is a λ system and use the $\pi - \lambda$ theorem.

We first state a lemma that $\sigma\langle C\rangle \subset M_{\mu^*}$, since M_{μ^*} is a σ-algebra that contains C.

We verify the assumptions of a λ system. For $\Omega \in L$, since C is a semialgebra, we know that there are finite disjoint sets $A_1 \dots A_k \in C$ such that $\bigcup_{i=1}^k A_i = \Omega$. Thus

$$\nu(\Omega) = \nu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \nu(A_i) = \sum_{i=1}^{k} \mu^*(A_i) = \mu^*\left(\bigcup_{i=1}^{k} A_i\right) = \mu^*(\Omega)$$

where the middle equality follows because $\nu = \mu = \mu^*$ on C. Thus $\Omega \in L$.

For closure under complement, let $A, B \in L$. Then since $A, B \in \sigma\langle C\rangle$, both are measurable under ν and μ^*, so

$$\nu(B \backslash A) = \nu(B) - \nu(A) = \mu^*(B) - \mu^*(A) = \mu^*(B \backslash A)$$

and so $B \backslash A \in L$. These expressions are well-defined because $\nu(\Omega) < \infty$, so $\nu(A) < \infty$.

For closure under monotone increasing union, let $A_1, \dots \in L$, with $A_n \subset A_{n+1}$ for all n. Let $B_1 = A_1, B_n = A_n$
A_{n-1}. Since the $A_n \in L$, by the above, $B_n \in L$. It's also clear that $\bigcup_{i=1} A_i = \bigcup_{i=1} B_i$. Thus

$$\nu\left(\bigcup_{i=1} B_i\right) = \sum_{i=1} \nu(B_i) = \sum_{i=1} \mu^*(B_i) = \mu^*\left(\bigcup_{i=1} B_i\right)$$

Thus $\bigcup_{i=1} A_i \in L$, and L is a λ-system. Since C is a semialgebra and thus a π-system, the $\pi - \lambda$ theorem states that $\lambda\langle C\rangle = \sigma\langle C\rangle$, which implies that $\sigma\langle C\rangle \subset L$. But by definition, $L \subset \sigma\langle C\rangle$, and thus $L = \sigma\langle C\rangle$. Thus $\mu^*(A) = \nu(A)$ for all $A \in \sigma\langle C\rangle$, as desired.

Part b

Extend the result to the σ-finite case.

The only result that needs to be changed in Part ii. Because Ω is σ-finite on C, there exist countable sets $\{C_i\} \subset C$, not necessarily disjoint, such that $\mu(C_i) < \infty$ and $\bigcup_{i=1}^{\infty} C_i = \Omega$. Since C is a semialgebra, we can take intersections of all subsets of $\{C_i\}$ to get a countable disjoint collection of subsets in C with finite measure whose union is Ω. Thus, wlog, we can assume that $\{C_i\}$ are disjoint.

Thus

$$\begin{aligned}
\nu(B \backslash A) = \nu(B \backslash A \cap \Omega) = \nu\left(B \cap A^C \cap \bigcup_{i=1}^{\infty} C_i\right) &= \nu\left(\bigcup_{i=1}^{\infty} B \cap A^C \cap C_i\right) \\
&= \sum_{i=1}^{\infty} \nu\left(B \cap A^C \cap C_i\right) = \sum_{i=1}^{\infty} \mu^*\left(B \cap A^C \cap C_i\right) \\
&= \mu^*\left(B \cap A^C \cap \bigcup_{i=1}^{\infty} C_i\right) \\
&= \mu^*(B \backslash A)
\end{aligned}$$

where the equalities follow because $B \cap A \cap C_i \in \sigma\langle C\rangle$ and disjoint.

Problem 1.28

Let F be a discrete distribution function, i.e. F is of the form

$$F(x) = \sum_{j=1}^{\infty} a_j I(x_j \leq x), x \in \mathbb{R}$$

where $0 < a_j < \infty$, $\sum_{j\geq 1} a_j = 1$, $x_j \in \mathbb{R}$, $j \geq 1$. Show that $\mathcal{M}_{\mu_F^*} = \mathcal{P}(\mathbb{R})$.

We know that F is cadlag, so it is right-continuous and has left limits everywhere. Define $A = \{x_j\}_{j\geq 1}$ to be the set of jump points. It's clear by the definition of F that F is discontinuous on A and continuous everywhere else. By the definition of the Riemann-Stieltjes measure,

$$\mu_F^*(\{x_j\}) = F(x_j^+) - F(x_j^-) = a_j$$

for all $j \geq 1$, and $\mu_F^*(x) = 0$ everywhere else.

Lemma 3 $A \in \mathcal{M}_{\mu_F^*}$.

Proof: *All singletons are measurable, because for* $x \in \mathbb{R}$, $\{x\} = \Big((-\infty, x) \cup (x, \infty)\Big)^C$, *so* $\{x\} \in \mathcal{B} = \sigma\langle C\rangle \subset \mathcal{M}_{\mu_F^*}$. *The countable union of measurable sets is measurable.* □

Thus

$$\mu_F^*(A) = \sum_{j=1}^{\infty} \mu_F^*(\{x_j\}) = \sum_{j=1}^{\infty} a_j = 1$$

By definition, $\mu^*(\mathbb{R}) = 1$. Thus

$$\mu_F^*(A^C) = \mu_F^*(\mathbb{R}\backslash A) = \mu_F^*(\mathbb{R}) - \mu_F^*(A) = 0$$

For all $B \subset \mathbb{R}$, $B \cap A \in \mathcal{B}(\mathbb{R})$, so $B \cap A$ is measurable. μ_F^* is complete, so $\mu_F^*(A^C) = 0$ implies that $B \cap A^C$ is measurable. Thus B is the finite union of measurable sets, and is thus measurable.