

Athreya Lahiri Chapter 1 Solutions

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Problem 1.19

Let Ω be a nonempty set and let $C \subset \mathcal{P}(\Omega)$ be a semialgebra. Let

$$\mathcal{A}(C) = \{A : A = \bigcup_{i=1}^k B_i : B_i \in C, i = 1, 2 \dots k, k \in \mathbb{N}\}$$

Part a

Show that $\mathcal{A}(C)$ is the smallest algebra containing C .

Lemma 1 $\mathcal{A}(C)$ is an algebra.

Proof: For $\Omega \in \mathcal{A}(C)$, let $A \in C$ be arbitrary. Because C is a semialgebra, $A^C = \bigcap_{i=1}^k B_i, B_i \in C$, so $A^C \in C$. Thus $A \cap A^C = \Omega \in \mathcal{A}(C)$.

For closure under compliments, $A \in \mathcal{A}(C)$ implies by definition that $A = \bigcup_{i=1}^k B_i, B_i \in C$, and taking compliments $A^C = \bigcap_{i=1}^k B_i^C$. Because $B_i \in C$ and C is a semialgebra, each B_i^C is the finite union of $C_j \in C$. Thus

$$A^C = \bigcap_{i=1}^k \bigcup_{j=1}^{l_i} C_j$$

Distributing the intersection, we get the union of a finite number of pairwise intersections, e.g. $A^C = (C_{11} \cap C_{12}) \cup (C_{11} \cap C_{13}) \cup \dots$. All of the pairwise intersections are in C , and thus their union is in $\mathcal{A}(C)$. Thus $A^C \in \mathcal{A}(C)$.

Closure under finite union is immediate. \square

Showing that $\mathcal{A}(C)$ is the smallest algebra containing C is equivalent to showing that if B is an algebra such that $C \subset B$, then $\mathcal{A}(C) \subset B$. But this is almost immediate. Let $M \in \mathcal{A}(C)$. By definition of $\mathcal{A}(C)$, $M = \bigcap_{i=1}^k M_k, M_k \in C, k \in \mathbb{N}$. Since B is an algebra contain C , $M \in B$. Thus $\mathcal{A}(C) \subset B$, as desired.

Part b

Show that $\sigma\langle C \rangle = \sigma\langle \mathcal{A}(C) \rangle$.

Trivially, $C \subset \mathcal{A}(C)$ implies $\sigma\langle C \rangle \subset \sigma\langle \mathcal{A}(C) \rangle$. For the reverse, let $M \in \sigma\langle \mathcal{A}(C) \rangle$. Then M is the countable union of elements in $\mathcal{A}(C)$, and thus it is the countable union of elements in C and thus in $\sigma\langle C \rangle$.

Problem 1.20

Let C be a semialgebra Ω with μ a measure on C . Let μ^* be the outer measure induced by μ^* , defined as

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n \geq 1} \subset C, A \subset \bigcup_{n \geq 1} A_n \right\}$$

Show that μ^* satisfies the following three properties making it an outer measure:

- Non-negativity: $\mu^*(\emptyset) = 0$
- Monotonicity: $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
- Countable sub-additivity: For any $\{A_n\}_{n \geq 1} \subset P(\Omega)$, $\mu^*\left(\bigcup_{n \geq 1} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Lemma 2 For a (non-empty) semi-algebra C , $\emptyset \in C$.

Proof: Let $A \in C$. Then $A^C = \bigcup_{i=1}^k B_i$, k finite, $B_i \in C$. Then $A \cap B_i = \emptyset$, and since C is closed to complements, $\emptyset \in C$. \square

For non-negativity, $\emptyset \in C$ and $\mu(\emptyset) = 0$ imply $\mu^*(\emptyset) = 0$. For monotonicity, consider any cover of B with sets in C . Then that cover also covers A . Since $\mu^*(A)$ is the infimum of the sum of the measures of sets in C that cover A , monotonicity follows immediately.

For countable sub-additivity, if the sum on the right is infinite, then the result is immediate. Otherwise, $\mu^*(A_n) < \infty$ for all n and the outer measures of A_n decrease enough so that the sum is (absolutely) convergent.

Let $0 < \epsilon < \infty$. Since $\mu^*(A_n)$ is the infimum over covers of A_n with elements in C , there exist $\{A_{nj}\}_{j \geq 1} \subset C$ such that

$$\mu^*(A_n) \leq \sum_{j=1}^{\infty} \mu(A_{nj}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$$

The union of covers over the A_n , $\bigcup_{n \geq 1} \bigcup_{j \geq 1} A_{nj}$, form a cover for $\bigcup_{n \geq 1} A_n$. Thus

$$\mu^* \left(\bigcup_{n \geq 1} A_n \right) \leq \mu \left(\bigcup_{n=1} \bigcup_{j=1} A_{nj} \right) \leq \sum_{n=1} \sum_{j=1} \mu(A_{nj}) \leq \sum_{n=1} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$

which proves the desired result.