Athreya Lahiri Chapter 3 Solutions

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Problem 3.1

Let $\phi : (a, b) \to \mathbb{R}$ be convex. Show the following.

We will constantly use the following convexity formula: if ϕ is convex and $a < x_1 < x_2 < x_3 < b$, then

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(x_3) - \phi(x_1)}{x_3 - x_1} \leq \frac{\phi(x_3) - \phi(x_2)}{x_3 - x_2}$$

Part a

For each $x \in (a, b)$,

$$\phi'_+(x) = \lim_{y \downarrow x} \frac{\phi(y) - \phi(x)}{y - x}, \phi'_-(x) = \lim_{y \uparrow x} \frac{\phi(y) - \phi(x)}{y - x}$$

exist and are finite.

Consider ϕ'_+ first. Let $z < x$ be arbitrary and let $\{y_n\}_{n=1} \geq x$ such that $y_n \to x$. Wlog, let y_n be decreasing. Then by convexity,

$$\frac{\phi(y_1) - \phi(x)}{y_1 - x} \geq \frac{\phi(y_2) - \phi(x)}{y_2 - x} \geq \dots$$

which is bounded below by $\frac{\phi(x)-\phi(z)}{x-z}$. Thus $\frac{\phi(y_n)-\phi(x)}{y_n-x}$ is a monotonically decreasing sequence bounded below, thus it has a limit. Since this sequence was arbitrary, the limit and thus ϕ'_+ exists. The same applies in reverse for $\phi'+$.

Problem 3.3

Prove the following.

Part a

Let $a_1 \dots a_k$ be real and $p_1 \dots p_k$ be positive numbers such that $\sum_{i=1}^k p_i = 1$. Then

$$\sum_{i=1}^{k} p_i \exp(a_i) \geq \exp\left(\sum_{i=1}^{k} p_i a_i\right)$$

Let P be the probability measure on $\mathbb{R}$ that assigns probability p_i to point a_i and apply Jensen's inequality with $\phi(x) = e^x$.

Part b

Let $b_1 \dots b_k$ be nonnegative numbers and $p_1 \dots p_k$ be as in Part a. Then

$$\sum_{i=1}^{k} p_i b_i \geq \prod_{i=1}^{k} b_i^{p_i}$$

Furthermore, equality holds iff $b_1 = b_2 = \cdots = b_k$.

Let $a_i = \log b_i$ and apply Part a. For the iff, since the exponential function is strictly convex, inequality holds iff $f(\omega)$ is a constant, which in this context means that all the b_is are equal.

Part c

For any $a, b \in \mathbb{R}$ and $1 \leq p < \infty$,

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$$

Let $f(x) = x$, $\phi(x) = |x|^p$, which is convex on the range of p, and let P be the probability measure with 1/2 probability on $\{a, b\}$. Thus by Jensen's inequality,

$$\phi\left(\int x dP\right) = \frac{1}{2^p}|a + b|^p \leq \int |x|^p dP = \frac{1}{2}(|a|^p + |b|^p)$$

which implies the desired result.

Problem 3.12

Part b

Prove that for $p \in (0, 1)$, $\int |f + g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$.

Building off of equation 2.2, we have that

$$\left(\frac{|x|}{|x| + |y|}\right)^p + \left(\frac{|y|}{|x| + |y|}\right)^p \geq \frac{|x|}{|x| + |y|} + \frac{|y|}{|x| + |y|} = 1$$

implies

$$|x + y|^p \leq (|x| + |y|)^p \leq |x|^p + |y|^p$$

and integrating pointwise gives the result.

Problem 3.14

Show that $(L^\infty(\mu), d_\infty)$ is a complete metric space.

Using the hint, let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $L^\infty(\mu)$. For each $k \geq 1$, let f_{n_k} be a subsequence such that $||f_{n_{k+1}} - f_{n_k}||_\infty < 2^{-k}$. By the definition of L^∞, for all $f \in L^\infty$, the set $\{\omega : |f(\omega)| > ||f||_\infty\}$ has measure zero.

Let

$$A = \bigcap_{k=1}^{\infty} \{\omega : |f_{n_{k+1}}(\omega) - f_{n_k}(\omega)| \leq ||f_{n_{k+1}} - f_{n_k}||_\infty\}$$

A^C has measure zero because it is the countable union of zero sets.

Thus, for $\omega \in A$ and for all $k \geq 1$, $|f_{n_{k+1}}(\omega) - f_{n_k}(\omega)| \leq ||f_{n_{k+1}} - f_{n_k}||_\infty < 2^{-k}$. Thus for $\omega \in A$, $\{f_{n_k}(\omega)\}_{k\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, which converges to a point we denote $f(\omega)$. For $\omega \in A^C$, let $f(\omega) = 0$. Then $\lim_{k\to\infty} f_{n_k} = f$ a.e. (μ), and the rest of the proof follows as in the proof of Theorem 3.2.2, which proves completeness for L^p, $p \in (0, \infty)$.

Problem 3.18

Let X be a nonnegative random variable.

Part a

Show that $EX \log X \geq (EX)(E \log X)$.

Log is concave, so by Jensen's inequality, $\log(EX) \geq E \log X$. Consider $\phi(x) = x \log x$. Its second derivative is $1/x$, which is positive over the range of x, so $x \log x$ is convex. Thus by Jensen's inequality, $EX \log(EX) \leq E(X \log X)$. Putting it all together,

$$E(X \log X) \geq EX \log(EX) \geq (EX)(E \log X)$$

as desired.

Part b

Show that

$$\sqrt{1 + (EX)^2} \leq E(\sqrt{1 + X^2}) \leq 1 + EX$$

For the first inequality, consider $\phi(x) = \sqrt{1 + x^2}$. Its second derivative is $(1 + x^2)^{-3/2}$, which is positive, so $\phi(x)$ is convex and the first inequality follows by Jensen's inequality. The second inequality follows by the triangle inequality on $L^{1/2}$ with the functions $f = 1, g = x^2$, which was proven directly in Problem 3.12.