# Athreya Lahiri Chapter 2 Solutions

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## Problem 2.1

Prove de Morgan's laws. Let $\Omega_i, i = 1, 2$ be two nonempty sets, and let $T : \Omega_1 \to \Omega_2$ be a map. For any collection $\{A_\alpha : \alpha \in I\}$ of subsets of $\Omega_2$, prove that

$$T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} T^{-1}(A_\alpha)$$
$$T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$$
$$\left(T^{-1}(A)\right)^C = T^{-1}(A^C)$$

For the first,

$$\begin{aligned} T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) &= \{B \in \Omega_1 : T(B) \in \bigcup_{\alpha \in I} A_\alpha\} \\ &= \bigcup_{\alpha \in I} \{B \in \Omega_1 : T(B) \in A_\alpha\} \\ &= \bigcup_{\alpha \in I} T^{-1}(A_\alpha) \end{aligned}$$

with a similar argument holding for the second. For the third,

$$\left(T^{-1}(A)\right)^C = \{B \in \Omega_1 : T(b) \in A\}^C = \{B \in \Omega_1 : T(b) \in A^C\} = T^{-1}(A^C)$$

## Problem 2.3

Let $f, g : \Omega \to \mathbb{R}$ be $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable. Set

$$h(\omega) = \frac{f(\omega)}{g(\omega)} \mathbb{1}(g(\omega) \neq 0)$$

Verify that $h$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.

Directly from the definition, we have that for $a \in \mathbb{R}$,

$$\begin{aligned} h^{-1}((-\infty, a]) &= \{\omega : \frac{f(\omega)}{g(\omega)} \mathbb{1}(g(\omega) \neq 0) \leq a\} \\ &= \{\omega : f(\omega) \leq ag(\omega), g(\omega) > 0\} \cup \{\omega : f(\omega) \geq ag(\omega), g(\omega) < 0\} \\ &= \{\omega : f(\omega) - ag(\omega) \leq 0, g(\omega) > 0\} \cup \{\omega : f(\omega) - ag(\omega) \geq 0, g(\omega) < 0\} \end{aligned}$$

$f(\omega) - ag(\omega)$ and $g(\omega)$ are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable functions, so for the left and right sets, the conditions individually define sets in $\mathcal{F}$ so their intersection is in $\mathcal{F}$. Thus $h(\omega)$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.

# Problem 2.6

Let $X_i, i = 1, 2, 3$ be random variables on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Consider the equation (with $t \in \mathbb{R}$)

$$X_1(\omega)t^2 + X_w(\omega)t + X_3(\omega) = 0$$

## Part a

Show that $A \coloneqq \{\omega \in \Omega : \text{The above equation has two distinct roots}\} \in \mathcal{F}$.

The condition for $\omega \in A$ is equivalent to

$$\{\omega : X_1(\omega) \neq 0\} \cap \{\omega : X_2^2(\omega) - 4X_1(\omega)X_3(\omega) > 0\}$$

because this indicates that the polynomial is second-order and its discriminant is positive. $X_1(\omega)$ and $X_2^2(\omega) = 4X_1(\omega)X_3(\omega)$ are random variables on probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ and the sets $(0, \infty)$ and $(-\infty, 0) \cup (0, \infty)$ are Borel sets in $\mathbb{R}$, so both of the above sets are in $F$, and their intersection is thus in $F$.

## Part b

Let $T_1(\omega)$ and $T_2(\omega)$ denote the two roots of the above equation on $A$. Let

$$f_i(\omega) = \begin{cases} T_i(\omega) & \omega \in A \\ 0 & \omega \in A^C \end{cases}$$

Show that $(f_1, f_2)$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}^2)\rangle$-measurable.

**Lemma 1** *Let $f$ be a non-negative $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable function. Then $\sqrt{f}$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.*

**Proof:** *For all $a \geq 0$,*

$$(\sqrt{f(\omega)})^{-1}((-\infty, a]) = \{\omega : 0 \leq \sqrt{f(\omega)} \leq a\} = \{\omega : 0 \leq f(\omega) \leq a^2\} \in F$$

$\square$

By the quadratic formula (arbitrarily letting $i = 1$ be the positive root), we have that

$$f_1(\omega) = \frac{-X_2(\omega) + \sqrt{X_2^2(\omega) - 4X_1(\omega)X_3(\omega)}}{2X_1(\omega)} \mathbb{1}_A$$

and $f_2$ is the negative root. By Problem 2.3, since the numerator and denominator are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, $A\ inF$, and the restriction in $A$ prevents the denominator from being zero, $f_i, i = 1, 2$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, and the Cartesian product of $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable functions is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

# Problem 2.7

Let $M := ((X_{ij})), 1 \leq i, j \leq k$ be a random matrix of random variables $X_{ij}$ defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$.

## Part a

Show that $det(M)$ and $tr(M)$ are both $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

The trace is trivial; the diagonal entries of $M$ are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable and so is their sum. The determinant follows by induction on the size of $M$. When $k = 1$, the determinant is the random variable $X_{11}$, which by assumption is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable. Assuming that the determinant of a size $k-1$ matrix of random variables in $(\Omega, \mathcal{F}, \boldsymbol{P})$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, by the Laplace expansion, we can rewrite the determinant of the size $k$ matrix as the sum of random variables multiplied by the determinant of size $k - 1$ matrices, and this sum is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

## Part b

Show that the largest eigenvalue of $M'M$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Using the hint, I will note that the largest eigenvalue is equal to

$$\sup_x \frac{x'M'Mx}{x'x} \mathbb{1}_{x \neq 0}$$

The numerator and denominator are the sums of products of random variables in $(\Omega, \mathcal{F}, \boldsymbol{P})$ and the denominator is restricted from zero, so the internal function is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, and the supremum of a measurable function is measurable.