Athreya Lahiri Chapter 1 Solutions

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Problem 1.19

Let Ω be a nonempty set and let $C \subset \mathcal{P}(\Omega)$ be a semialgebra. Let

$$\mathcal{A}(C) = \{A : A = \bigcup_{i=1}^{k} B_i : B_i \in C, i = 1, 2 \ldots k, k \in \mathbb{N}\}$$

Part a

Show that $\mathcal{A}(C)$ is the smallest algebra containing C.

Lemma 1 *$\mathcal{A}(C)$ is an algebra.*

Proof: *For $\Omega \in \mathcal{A}(C)$, let $A \in C$ be arbitrary. Because C is a semialgebra, $A^C = \bigcap_{i=1}^{k} B_i, B_i \in C$, so $A^C \in C$. Thus $A \cap A^C = \Omega \in \mathcal{A}(C)$.*

For closure under compliments, $A \in \mathcal{A}(C)$ implies by definition that $A = \bigcup_{i=1}^{k} B_i, B_i \in C$, and taking compliments $A^C = \bigcap_{i=1}^{k} B_i^C$. Because $B_i \in C$ and C is a semialgebra, each B_i^C is the finite union of $C_j \in C$. Thus

$$A^C = \bigcap_{i=1}^{k} \bigcup_{j=1}^{l_i} C_j$$

Distributing the intersection, we get the union of a large number of pairwise intersections, e.g. $A^C = (C_{11} \cap C_{12}) \cup (C_{11} \cap C_{13}) \cup \ldots$. All of the pairwise intersections are in C, and thus their union is in $\mathcal{A}(C)$. Thus $A^C \in \mathcal{A}(C)$.

Closure under finite union is immediate. $\square$

Showing that $\mathcal{A}(C)$ is the smallest algebra containing C is equivalent to showing that if B is an algebra such that $C \subset B$, then $\mathcal{A}(C) \subset B$. But this is almost immediate. Let $M \in \mathcal{A}(C)$. By definition of $\mathcal{A}(C)$, $M = \bigcap_{i=1}^{k} M_k, M_k \in C, k \in \mathbb{N}$. Since B is an algebra contain C, $M \in B$. Thus $\mathcal{A}(C) \subset B$, as desired.

Part b

Show that $\sigma\langle C\rangle = \sigma(\langle \mathcal{A}(C)\rangle$.

Trivially, $C \subset \mathcal{A}(C)$ implies $\sigma\langle C\rangle \subset \sigma\langle \mathcal{A}(C)\rangle$. For the reverse, let $M \in \sigma\langle \mathcal{A}(C)\rangle$. Then M is the countable union of elements in $\mathcal{A}(C)$, and thus it is the countable union of elements in C and thus in $\sigma\langle C\rangle$.