Athreya Lahiri Chapter 4 Solutions

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October 8, 2025

Problem 4.4

Let $\nu, \mu, \mu_1, \mu_2 \ldots$ be σ-finite measures on a measurable space $(\Omega, \mathcal{F})$. Prove the following.

Part a

If $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_3$, then $\mu_1 \ll \mu_3$ and

$$\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} \text{ a.e. } (\mu_3)$$

$\mu_1 \ll \mu_3$ is trivial; the zero sets of μ_3 are zero sets of μ_2 are zero sets of μ_1. Using the Radon-Nikodym derivatives,

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2} d\mu_2$$

where $\frac{d\mu_1}{d\mu_2}$ is non-negative and measurable. Thus the integral can be approximated by a series of non-decreasing simple functions,

$$\begin{aligned}
\int_A \frac{d\mu_1}{d\mu_2} d\mu_2 &= \lim_{n\to\infty} \int_A \left(\frac{d\mu_1}{d\mu_2}\right)^{(n)} d\mu_2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} \left(\frac{d\mu_1}{d\mu_2}\right)^{(n)}_i \mu_2(A_{i_n})
\end{aligned}$$

Replacing μ_2 with its Radon-Nikodym derivative,

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{k_n}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\mu_2(A_{i_n}) &= \lim_{n\to\infty}\sum_{i=1}^{k_n}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\int_{A_{i_n}}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \lim_{n\to\infty}\sum_{i=1}^{k_n}\int_{A_{i_n}}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \lim_{n\to\infty}\int_A\left(\frac{d\mu_1}{d\mu_2}\right)^{(n)}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \int_A\frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3}d\mu_3
\end{aligned}
$$

where the second line follows because $\left(\frac{d\mu_1}{d\mu_2}\right)^{(n)}$ is constant on A_{i_n}, the third line because adding up the integrals on the partition of A gives an integral on the whole of A, and the fourth line by the MCT. Since the Radon-Nikodym derivative of μ_3 is unique a.e. (μ_3), this implies that $\frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} = \frac{d\mu_1}{d\mu_3}$ a.e. (μ_3), as desired.