Athreya Lahiri Chapter 1 Supplement

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Theorem 1.1.2: The $\pi - \lambda$ Theorem

If C is a π-system, then $\lambda\langle C\rangle = \sigma\langle C\rangle$.

Proof: For the forward, every σ-algebra is a λ-system and $C \subset \sigma\langle C\rangle$ so $\lambda\langle C\rangle \subset \sigma\langle C\rangle$. Thus, it suffices to show that if C is a π-system, then $\lambda\langle C\rangle$ is a σ-algebra so that $\sigma\langle C\rangle \subset \lambda\langle C\rangle$.

Since $\lambda\langle C\rangle$ is a λ-system, it is closed under complementation and monotone increasing unions. By Proposition 1.1.1, showing that it is closed under intersection implies that it is a σ-algebra.

Let $\lambda_1(C) = \{A : A \in \lambda\langle C\rangle, A \cap B \in \lambda\langle C\rangle \text{ for all } B \in C\}$.

Lemma 1 $C \subset \lambda_1(C)$.

Proof: *Let $A \in C \subset \lambda\langle C\rangle$. Then for all $B \in C$, because C is a π-system, $(A \cap B) \in C \subset \lambda\langle C\rangle$. Thus $A \in \lambda_1(C)$.* □

Lemma 2 *$\lambda_1(C)$ is a λ-system.*

Proof: *$\Omega \in \lambda_1(C)$ because $\Omega \in \lambda\langle C\rangle$ by definition and for all $B \in C$, $(\Omega \cap B) = B \in C \subset \lambda\langle C\rangle$. Thus $\Omega \in \lambda_1(C)$.*

For closure under set compliment, let $A, X \in \lambda_1(C), X \subset A$. Then $A, X \in \lambda\langle C\rangle$, and for all $B \in C$, $A \cap B, X \cap B \in \lambda\langle C\rangle$. Then $(A \cap B)\backslash(X \cap B) = (A\backslash X) \cap B \in \lambda_1(C)$, because $\lambda\langle C\rangle$ is a λ-system so $A\backslash X \in \lambda\langle C\rangle$.

For closure under countable monotone increasing union, let $A_1, A_2 \cdots \in \lambda_1(C), A_1 \subset A_2 \subset \dots$. Then $A_n \in \lambda\langle C\rangle$ and $A_n \cap B \in \lambda\langle C\rangle$ for all $B \in C$. Let $A = \bigcup_{n=1} A_n$. Then $A \cap B = \bigcup_{n=1}(A_n \cap B)$, and by assumption $(A_n \cap B) \in \lambda\langle C\rangle$ for all n, so $\bigcup_{n=1}(A_n \cap B) \in \lambda\langle C\rangle$ for all $B \in C$. Thus $A \in \lambda_1(C)$. □

Lemma 3 $\lambda_1(C) = \lambda\langle C\rangle$.

Proof: *$\lambda_1(C)$ is a λ-system containing C, so $\lambda\langle C\rangle \subset \lambda_1(C)$. However, by the definition of $\lambda_1(C)$, $\lambda_1(C) \subset \lambda\langle C\rangle$.* □

Let $\lambda_2(C) = \{A : A \in \lambda\langle C\rangle, A \cap B \in \lambda\langle C\rangle$ for all $B \in \lambda\langle C\rangle\}$. $\lambda_2(C)$ is a λ-system for the same reasons that $\lambda_1(C)$ is - the proofs are essentially unchanged.

Lemma 4 *$C \subset \lambda_2(C)$.*

Proof: *Let $X \in C$ be arbitrary. Then $X \in \lambda\langle C\rangle = \lambda_1(C)$. Thus by the definition of $\lambda_1(C)$, for all $B \in C$, $(X \cap B) \in \lambda(C)$. Flipping this around and letting $B \in C$ be arbitrary, we see that for all $X \in \lambda\langle C\rangle$, $(B \cap X) \in \lambda(C)$. Thus $C \subset \lambda_2(C)$.* □

By the definition of $\lambda_2(C)$ and the above lemma we see that $C \subset \lambda_2(C) \subset \lambda\langle C\rangle$, and taking the λ-systems shows that $\lambda_2(C) = \lambda\langle C\rangle$. Thus from the definition of $\lambda_2(C)$, we see that $\lambda\langle C\rangle$ is closed under finite intersection. Thus $\lambda\langle C\rangle$ is a σ-algebra. □

Theorem 1.2.4: Uniqueness of Measures

Let μ_1 and μ_2 be two finite measures on a measurable space (Ω, F). Let $\mathcal{C} \subset F$ be a π-system such that $F = \sigma\langle\mathcal{C}\rangle$. If $\mu_1(C) = \mu_2(C)$ for all $C \in \mathcal{C}$ and $\mu_1(\Omega) = \mu_2(\Omega)$, then $\mu_1(A) = \mu_2(A)$ for all $A \in F$.

Proof: Let $L = \{A : A \in F, \mu_1(A) = \mu_2(A)\}$.

Lemma 5 *L is a λ-system.*

Proof: *$\Omega \in L$ follows from the assumption that $\mu_1(\Omega) = \mu_2(\Omega)$. For closure under set compliment, let $A, B \in L, A \subset B$. Then $B \backslash A$ has measure $\mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A)$ and thus is in L. For closure under countable monotone increasing union, let $A_1, A_2 \dots$ have $A_n \subset A_{n+1}$ and $\mu_1(A_n) = \mu_2(A_n)$ for all $n \in \mathbb{N}$. By mcfb of measures,*

$$\mu_1\left(\bigcup_{i\geq 1} A_i\right) = \lim_{n\to\infty} \mu_1(A_i) = \lim_{n\to\infty} \mu_2(A_i) = \mu_2\left(\bigcup_{i\geq 1} A_i\right)$$

and so $\left(\bigcup_{i\geq 1} A_i\right) \in L$. □

Since $C \subset L$, by the π-λ theorem, $F = \sigma\langle C\rangle \subset L$, and so by the definition of L, the measures are equal on F. □