Athreya Lahiri Chapter 2 Solutions

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Problem 2.1

Prove de Morgan's laws. Let $\Omega_i, i = 1, 2$ be two nonempty sets, and let $T : \Omega_1 \to \Omega_2$ be a map. For any collection $\{A_\alpha : \alpha \in I\}$ of subsets of Ω_2, prove that

$$T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} T^{-1}(A_\alpha)$$
$$T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$$
$$\left(T^{-1}(A)\right)^C = T^{-1}(A^C)$$

For the first,

$$\begin{aligned} T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) &= \{B \in \Omega_1 : T(B) \in \bigcup_{\alpha \in I} A_\alpha\} \\ &= \bigcup_{\alpha \in I} \{B \in \Omega_1 : T(B) \in A_\alpha\} \\ &= \bigcup_{\alpha \in I} T^{-1}(A_\alpha) \end{aligned}$$

with a similar argument holding for the second. For the third,

$$\left(T^{-1}(A)\right)^C = \{B \in \Omega_1 : T(b) \in A\}^C = \{B \in \Omega_1 : T(b) \in A^C\} = T^{-1}(A^C)$$

Problem 2.3

Let $f, g : \Omega \to \mathbb{R}$ be $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable. Set

$$h(\omega) = \frac{f(\omega)}{g(\omega)} \mathbb{1}(g(\omega) \neq 0)$$

Verify that h is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Directly from the definition, we have that for $a \in \mathbb{R}$,

$$\begin{aligned}
h^{-1}((-\infty, a]) &= \{\omega : \frac{f(\omega)}{g(\omega)} \mathbb{1}(g(\omega) \neq 0) \leq a\} \\
&= \{\omega : f(\omega) \leq ag(\omega), g(\omega) > 0\} \cup \{\omega : f(\omega) \geq ag(\omega), g(\omega) < 0\} \\
&= \{\omega : f(\omega) - ag(\omega) \leq 0, g(\omega) > 0\} \cup \{\omega : f(\omega) - ag(\omega) \geq 0, g(\omega) < 0\}
\end{aligned}$$

$f(\omega) - ag(\omega)$ and $g(\omega)$ are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable functions, so for the left and right sets, the conditions individually define sets in $\mathcal{F}$ so their intersection is in $\mathcal{F}$. Thus $h(\omega)$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Problem 2.6

Let $X_i, i = 1, 2, 3$ be random variables on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Consider the equation (with $t \in \mathbb{R}$)

$$X_1(\omega)t^2 + X_w(\omega)t + X_3(\omega) = 0$$

Part a

Show that $A := \{\omega \in \Omega : \text{The above equation has two distinct roots}\} \in \mathcal{F}$.

The condition for $\omega \in A$ is equivalent to

$$\{\omega : X_1(\omega) \neq 0\} \cap \{\omega : X_2^2(\omega) - 4X_1(\omega)X_3(\omega) > 0\}$$

because this indicates that the polynomial is second-order and its discriminant is positive. $X_1(\omega)$ and $X_2^2(\omega) = 4X_1(\omega)X_3(\omega)$ are random variables on probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ and the sets $(0, \infty)$ and $(-\infty, 0) \cup (0, \infty)$ are Borel sets in $\mathbb{R}$, so both of the above sets are in F, and their intersection is thus in F.

Part b

Let $T_1(\omega)$ and $T_2(\omega)$ denote the two roots of the above equation on A. Let

$$f_i(\omega) = \begin{cases} T_i(\omega) & \omega \in A \\ 0 & \omega \in A^C \end{cases}$$

Show that (f_1, f_2) is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}^2) \rangle$-measurable.

Lemma 1 *Let f be a non-negative $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable function. Then $\sqrt{f}$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.*

Proof: *For all $a \geq 0$,*

$$(\sqrt{f(\omega)})^{-1}((-\infty, a]) = \{\omega : 0 \leq \sqrt{f(\omega)} \leq a\} = \{\omega : 0 \leq f(\omega) \leq a^2\} \in F$$

$\square$

By the quadratic formula (arbitrarily letting $i = 1$ be the positive root), we have that

$$f_1(\omega) = \frac{-X_2(\omega) + \sqrt{X_2^2(\omega) - 4X_1(\omega)X_3(\omega)}}{2X_1(\omega)} \mathbb{1}_A$$

and f_2 is the negative root. By Problem 2.3, since the numerator and denominator are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, $A \ inF$, and the restriction in A prevents the denominator from being zero, $f_i, i = 1, 2$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, and the Cartesian product of $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable functions is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Problem 2.7

Let $M := ((X_{ij})), 1 \leq i, j \leq k$ be a random matrix of random variables X_{ij} defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$.

Part a

Show that $det(M)$ and $tr(M)$ are both $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

The trace is trivial; the diagonal entries of M are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable and so is their sum. The determinant follows by induction on the size of M. When $k = 1$, the determinant is the random variable X_{11}, which by assumption is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable. Assuming that the determinant of a size $k-1$ matrix of random variables in $(\Omega, \mathcal{F}, \boldsymbol{P})$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, by the Laplace expansion, we can rewrite the determinant of the size k matrix as the sum of random variables multiplied by the determinant of size $k-1$ matrices, and this sum is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Part b

Show that the largest eigenvalue of $M'M$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

Using the hint, I will note that the largest eigenvalue is equal to

$$\sup_x \frac{x'M'Mx}{x'x} \mathbb{1}_{x \neq 0}$$

The numerator and denominator are the sums of products of random variables in $(\Omega, \mathcal{F}, \boldsymbol{P})$ and the denominator is restricted from zero, so the internal function is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable, and the supremum of a measurable function is measurable.

Problem 2.8

Let $f : \mathbb{R} \to \mathbb{R}$. Let $\overline{f}(x) = \inf_{\delta>0} \sup_{|y-x|<\delta} f(y)$ and $\underline{f}(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y), x \in \mathbb{R}$.

Part a

Show that for any $t \in \mathbb{R}$,

$$\{x : \overline{f}(x) < t\}$$

is open and hence $\overline{f}$ is $\langle \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\rangle$-measurable.

Denote the set above as A. By the definition of $\overline{f}$ and the properties of infimum, for each $x \in A$, there exists δ_0 such that

$$\sup_{|y-x|<\delta_0} f(y) < t$$

Thus for all x' such that $|y - x'| < \delta_0$, there exists an open ball centered at x' such that $B(x', r) \subset B(x, \delta_0)$, and thus

$$\sup_{y\in B(x',r)} f(y) \leq \sup_{y\in B(x,\delta_0)} f(y) < t$$

implies

$$\inf_{r\to 0} \sup_{y\in B(x',r)} f(y) < t$$

and thus $x' \in A$, which implies that $B(x, \delta_0) \subset A$. Thus A is open. A similar argument holds in reverse for $\underline{f}$ using $\{x : \underline{f}(x) > t\}$

Part b

Show that for any $t > 0$,

$$\{x : \overline{f}(x) - \underline{f}(x) < t\} = \bigcup_{r\in\mathbb{Q}} \{x : \overline{f}(x) < t + r, \underline{f}(x) > r\}$$

and hence is open.

Denote the set on the left A. A can be built up as a union of smaller sets. For $r \in \mathbb{R}$, consider $A_r = \{x : \overline{f}(x) < r + t, \underline{f}(x) = r\}$. It's clear that this captures all x in the inclusion condition of A for a given value of $\underline{f}(x) = r$. By taking the union over all $r \in \mathbb{R}$, we get the inclusion condition of A for all possible $x \in \mathbb{R}$ and thus the union equals A. Thus

$$A = \bigcup_{r\in\mathbb{R}} \{x : \overline{f}(x) < r + t, \underline{f}(x) = r\}$$

Since we're taking the union over all $r \in \mathbb{R}$ and our inclusion criteria is a strictly less than sign, we can replace the $\underline{f}(x) = r$ conditions with $\underline{f}(x) > r$, such that $r \in \mathbb{Q}$ instead of $\mathbb{R}$.

To see this, let $A'_r = \{x : \overline{f}(x) < r + t, \underline{f}(x) > r\}$. If $x \in A_r$, then $\overline{f}(x) < r + t, \underline{f}(x) = r$. Since rationals are dense in the reals and the inequality is strict, there exists $r' \in \mathbb{Q}$ such that $\overline{f}(x) < r' + t, \underline{f}(x) > r'$, and thus $x \in A'_{r'}$. Thus

$$A = \bigcup_{r \in \mathbb{Q}} \{x : \overline{f}(x) < r + t, \underline{f}(x) > r\}$$

But by Part a, each of the subsets is the finite intersection of two open sets and is thus open, and the countable union of open sets is open. Thus A is open, as desired.

Part c

Show that f is continuous at $x_0 \in \mathbb{R}$ iff $\overline{f}(x_0) = \underline{f}(x_0)$.

For the forward, for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$. Thus

$$\sup_{|y-x_0|<\delta} f(y) - \inf_{|y-x_0|<\delta} f(y) < \epsilon$$

which implies $\overline{f}(x_0) - \underline{f}(x_0) = 0$. For the reverse, we have that for all $\delta > 0$, $\sup_{|y-x_0|<\delta} f(y) \geq \overline{f}(x_0)$ and $\sup_{|y-x_0|<\delta} f(y) \leq f(x_0)$, combined with $\overline{f}(x_0) = \underline{f}(x_0)$, imply that $\overline{f}(x_0) = \underline{f}(x_0) = f(x_0)$. Thus for all $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$\sup_{|y-x_0|<\delta_1} f(y) - f(x_0) < \frac{\epsilon}{2}$$
$$f(x_0) - \inf_{|y-x_0|<\delta_2} f(y) < \frac{\epsilon}{2}$$

Thus for all $y : |y - x_0| < \min(\delta_1, \delta_2)$,

$$f(y) - f(x_0) < \epsilon$$

as desired.

Part d

Show that the set $C_f = \{x : f(\cdot) \text{ is continuous at } x\}$ is a G_δ set, i.e. the intersection of a countable number of open sets, and hence C_f is a Borel set.

Letting A_n be the set in Part b with $t = \frac{1}{n}$, $\bigcup_{n=1}^{\infty} A_n = \{x : x : \overline{f}(x) - \underline{f}(x) = 0\}$, which by Part c is the set of x such that $f(x)$ is continuous. By Part b, this is the countable union of open sets.

Problem 2.15

Consider the probability space $\big((0,1), \mathcal{B}((0,1)), m\big)$, where m is the Lebesgue measure.

Part a

Let Y_1 be the random variable $Y_1(x) = \sin(2\pi x)$ for $x \in (0,1)$. Find the cdf of Y_1.

Looking at the graph of $\sin(2\pi x)$, for $y \in (0,1)$, we have that $\sin^{-1} y > 0$, thus

$$\begin{aligned} \boldsymbol{P}(Y_1 \le y) &= mY_1^{-1}\big((-\infty, y]\big) \\ &= m\left(\left(0, \frac{\sin^{-1} y}{2\pi}\right) \bigcup \left(1/2 - \frac{\sin^{-1} y}{2\pi}, 1\right)\right) \\ &= \frac{1}{2} + \frac{\sin^{-1} y}{\pi} \end{aligned}$$

Similarly, for $y \in (-1,0)$, we have that $\sin^{-1} y < 0$, thus

$$\boldsymbol{P}(Y_1 \le y) = m\left(\frac{1}{2} - \frac{\sin^{-1} y}{2\pi}, 1 + \frac{\sin^{-1} y}{2\pi}\right) = \frac{1}{2} + \frac{\sin^{-1} y}{\pi}$$

Thus

$$\boldsymbol{P}(Y_1 \le y) = \begin{cases} 0 & y \le -1 \\ \frac{1}{2} + \frac{\sin^{-1} y}{\pi} & y \in (-1,1) \\ 1 & y \ge 1 \end{cases}$$

Part b

Let Y_2 be the random variable $Y_2(x) = \log x$ for $x \in (0,1)$. Find the cdf of Y_2.

We have that for $y < 0$,

$$\boldsymbol{P}(Y_2 \le y) = mY_2^{-1}\big((-\infty, y]\big) = m\big((0, e^y)\big) = e^y$$

Thus

$$\boldsymbol{P}(Y_2 \le y) = \begin{cases} e^y & y < 0 \\ 1 & y \ge 0 \end{cases}$$

Part c

Let $F : \mathbb{R} \to \mathbb{R}$ be a cdf. For $x \in (0,1)$, let

$$\begin{aligned} F_1^{-1}(x) &= \inf\{y : y \in \mathbb{R}, F(y) \ge x\} \\ F_2^{-1}(x) &= \sup\{y : y \in \mathbb{R}, F(y) \le x\} \end{aligned}$$

Let Z_i be the random variable defined by

$$Z_i = F_i^{-1}(x), x \in (0,1), i = 1, 2$$

Subpart i

IN PROGRESS

Find the cdf of $Z_i, i = 1, 2$.

We begin with a characterization of F_1^{-1} and F_2^{-1}.

Lemma 2 *Let $A_x = \{y : y \in \mathbb{R}, F(y) = x\}$. Then A is either the empty set, a singleton, or an interval.*

Proof: *F is right-continuous and nondecreasing. If there is no y such that $F(y) = x$, then F has a jump discontinuity that jumps from below y to above y, since otherwise by the intermediate value theorem F would achieve the value x. If A_x is a nonempty nonsingleton, then there exist multiple y such that $F(y) = x$. This must occur when F is flat, and since F is nondecreasing, this can only happen on a connected interval.* □

Lemma 3 *$F_i^{-1}(x)$ can be broken up into cases. When $F(y)$ is flat, letting y_1 and y_2 be the left and right endpoints of the interval, $F_1^{-1}(x) = y_1$ and $F_2^{-1}(x) = y_2$. When $F(y)$ has a jump discontinuity at y that jumps from x_1 to x_2, $x : x \in [x_1, x_2] \to F_i^{-1}(x) = y$. Otherwise, F is invertible at y and $F_i^{-1}(x) = F^{-1}(F(y)) = y$.*

Lemma 4 *For any $x \in (0,1), t \in \mathbb{R}, F(t) \geq x \Leftrightarrow F_1^{-1}(x) \leq t$.*

Proof: *For the forward, assume that $F(t) \geq x$. Because F is nondecreasing, the infimum of y such that $F(y) \geq x$ must be less than or equal to t. For the reverse, assume that $F_1^{-1}(x) \leq t$. Because F is nondecreasing and right continuous, the sets $F(y) \geq x$ are closed intervals, and thus their infimum lies within the set - specifically, at the left endpoint, which is $F_1^{-1}(x)$. Thus for $t \geq F_1^{-1}(x)$, we know that $F(t) \geq F(F_1^{-1}(x)) = x$, since $F_1^{-1}(x)$ is the left endpoint and this is where the sets achieve their minimum.* □

Thus for Z_1,

$$\boldsymbol{P}(Z_1 \leq z) = mZ_1^{-1}\big((-\infty, z]\big) = m\big([0, F(z)]\big) = F(z)$$

where we used the lemma substituting z for t.

Problem 2.16

Part a

Let $(\Omega, \mathcal{F}_1, \mu)$ be a σ-finite measure space. Let $T : \Omega \to \mathbb{R}$ be $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$ measureable. Show by counterexample that the induced measure μT^{-1} need not be σ-finite.

Let $(\Omega, \mathcal{F}_1, \mu)$ be the Lebesgue measure on the Borel sets of $\mathbb{R}$, which is obviously σ-finite. Let $T(x) = 0$. T is continuous and thus $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$ measurable. However, the induced measure $\mu T^{-1}(A)$ equals infinity if $x \in A$, zero otherwise. Thus there is no collection of sets with finite measure such that their union is $\mathbb{R}$, and thus μT^{-1} is not σ-finite.

Part b

Let $(\Omega_i, \mathcal{F}_i)$ be measurable spaces for $i = 1, 2$ and let $T : \Omega_1 \to \Omega_2$ be $\langle \mathcal{F}_1, \mathcal{F}_2\rangle$-measurable. Show that any measure μ on $(\Omega_1, \mathcal{F}_1)$ is σ-finite if μT^{-1} is σ-finite on $(\Omega_2, \mathcal{F}_2)$.

By assumption of σ-finiteness of $(\Omega_2, \mathcal{F}_2)$, there is a countable collection of sets $\{A_n\}_{n\geq 1} \subset \mathcal{F}_2$ such that $\bigcup_{n\geq 1} A_n = \Omega_2$ and $\mu T^{-1}(A_n) < \infty$ for all n. By the first assumption,

$$\Omega_1 = T^{-1}(\Omega_2) = T^{-1}\left(\bigcup_{n\geq 1} A_n\right) = \bigcup_{n\geq 1} T^{-1}(A_n)$$

By the second assumption, $\mu(T^{-1}(A_n)) < \infty$ for all n. Thus the sets $\{T^{-1}(A_n)\}_{n\geq 1}$ show that μ is σ-finite.

Problem 2.20

Use Corollary 2.3.5 to show that for any collection $\{a_{ij} : i, j \in \mathbb{N}\}$ of nonnegative numbers,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

Let $(\mathbb{N}, P(\mathbb{N}))$ with the counting measure be a measure space, and let $h_n : \mathbb{N} \to \mathbb{R}$ be the function/sequence $h_i(j) = a_{ij}$ for all $i, j \in \mathbb{R}$. Then

$$\sum_{i=1}^{\infty} \int h_i(j) d\mu = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right)$$
$$\int \sum_{i=1}^{\infty} h_i(j) d\mu = \int \left(\sum_{i=1}^{\infty} a_{ij}\right) d\mu = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

and by Corollary 2.3.5, since $h_i(j)$ is nonnegative and measurable, the two expressions equal each other.

Problem 2.26

Establish Theorem 2.5.1. Suppose that $\mu(\Omega) < \infty$. Then $f_n \to f$ a.e. (μ) implies that $f_n \to^m f$.

I will use the hint. Let

$$D = \{\omega : f_j(\omega) \nrightarrow f(\omega)\}$$
$$A_{jr} = \{\omega : |f_j(\omega) - f(\omega)| > 1/r\}$$
$$B_{nr} = \bigcup_{j \geq n} A_{jr}$$
$$C_r = \bigcap_{n \geq 1} B_{nr}$$

I claim that $D = \bigcup_{r \geq 1} C_r$. A_{jr} is the set of ω such that for a given j and r, $f_j(\omega) - f(\omega)| > 1/r$. B_{nr} is the set of ω such that for a given n and r, for all $j \geq n$, $f_j(\omega) - f(\omega) > 1/r$. C_r is the set such that for all $n \geq 1$, $f_j(\omega) - f(\omega) > 1/r$. It's clear from the definition that the union of these sets is the set of all ω such that $f_n(\omega)$ does not converge to $f(\omega)$, which is the definition of D.

Since f_n converges to f a.e. (μ), $\mu(D) = 0$. Thus $\mu(D) = \mu(\bigcup_{r \geq 1} C_r)$ implies that for all r,

$$\mu(C_r) = 0$$

Since B_{nr} monotonically decrease to C_r, m.c.f.a. implies that

$$0 = \mu(C_r) = \mu(\bigcap_{n \geq 1} B_{nr}) = \lim_{n \to \infty} \mu(B_{nr})$$

Since this holds for all r, the definition of B_{nr} implies that B_{nr} is the set of ω such that for a given r, $|f_j(\omega) - f(\omega)| > 1/r$ for some $j \geq n$. Since we know that this set's measure goes to zero as $n \to \infty$, looking at the definition of convergence in measure, we have that $f_n \to^m f$.

Problem 2.27

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing and let $\frac{\phi(x)}{x} \uparrow \infty$ as $x \uparrow \infty$. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a subset of $L^1(\Omega, \mathcal{F}, \mu)$. Show that if $\sup_{\lambda \in \Lambda} \int \phi(|f_\lambda|) d\mu < \infty$, then $\{f_\lambda : \lambda \in \Lambda\}$ is uniformly integrable.

Consider $\sup_{\lambda \in \Lambda} \int_{|f_\lambda| > t} \phi(|f_\lambda|) d\mu$. For arbitrary $M > 0$ large, let t be large enough such that for all $|f_\lambda| > t$, $\frac{\phi(|f_\lambda|)}{|f_\lambda|} \geq M$. Then

$$\sup_{\lambda \in \Lambda} \int_{|f_\lambda| > t} \phi(|f_\lambda|) d\mu \geq \sup_{\lambda \in \Lambda} \int_{|f_\lambda| > t} M |f_\lambda| d\mu$$

which implies

$$0 \leq \sup_{\lambda \in \Lambda} \int_{|f_\lambda|>t} |f_\lambda| d\mu \leq \frac{1}{M} \sup_{\lambda \in \Lambda} \int_{|f_\lambda|>t} \phi(|f_\lambda|) d\mu$$

We know that $\sup_{\lambda \in \Lambda} \int \phi(|f_\lambda|) d\mu$ is finite, so $\sup_{\lambda \in \Lambda} \int_{|f_\lambda|>t} \phi(|f_\lambda|) d\mu$ is finite, so as $M \to \infty$, $t \to \infty$ and the right side goes to zero, implying that the middle goes to zero. This is the definition of uniform integrability.

Problem 2.28

Let μ be the Lebesgue measure on $([-1,1], \mathcal{B}([-1,1]))$. For $n \geq 1$, define $f_n(x) = nI_{(0,n^{-1})}(x) - nI_{(-n^{-1},0)}(x)$ and $f(x) = 0$ for $x \in [-1,1]$. Show that $f_n \to f$ a.e. (μ) and $\int f_n d\mu \to \int f d\mu$ but $\{f_n\}_{n \geq 1}$ is not uniformly integrable.

It's clear that f_n converges pointwise to f and $\int f_n d\mu = 0 = \int f d\mu$ for all $n \geq 1$, so the first two conditions hold. However, $\int_{|f_n|>t} |f_n| d\mu$ always equals 2 for some n, and since $|f_n|$ has arbitrarily large magnitude, we have that

$$\sup_{n \in \mathbb{N}} \int_{|f_n|>t} |f_n| d\mu = 2$$

which does not go to zero as $t \to \infty$. Thus $\{f_n\}_{n \geq 1}$ are not uniformly integrable.

Problem 2.29

Let $\{f_n : n \geq 1\} \cup \{f\} \subset L^1(\Omega, \mathcal{F}, \mu)$.

Part a

Show that $\int |f_n - f| d\mu \to 0$ iff $f_n \to^m f$ and $\int |f_n| d\mu \to \int |f| d\mu$.

For the forward, we know from Theorem 2.5.3 that $f_n \to^{L^1} f$ implies $f_n \to^m f$. We also have that

$$\int |f_n| d\mu = \int |f_n - f + f| d\mu \leq \int |f_n - f| d\mu + \int |f| d\mu$$

where by assumption $\int |f_n - f| d\mu \to 0$, so $\lim_{n\to\infty} \int |f_n| d\mu \leq \int |f| d\mu$. A similar argument in reverse shows that $\lim_{n\to\infty} \geq \int |f| d\mu$.

For the reverse, we use the extended dominated convergence theorem. By Problem 2.37, the results hold even if the two sequences f_n and g_n converge in measure, not a.e. (μ). Let $g_n = |f_n|$. $f_n \to^m f$ holds by assumption, and $|f_n| \to^m |f|$ holds because $||f_n| - |f|| \leq |f_n - f|$ pointwise, so for any $\epsilon > 0$ $\mu(\{||f_n| - |f|| > \epsilon\}) \leq \mu(\{|f_n - f| > \epsilon\})$. Thus $\mu(\{|f_n - f| > \epsilon\}) \to 0$ implies that $\mu(\{||f_n| - |f|| > \epsilon\}) \to 0$. We similarly have by assumption that $g_n, g \in L^1(\Omega, \mathcal{F}, \mu)$ and $\int |g_n| d\mu \to \int |g| d\mu$. Thus by the extended dominated convergence theorem, $\int |f_n - f| d\mu \to 0$, as desired.

Part b

Show that if $\mu(\Omega) < \infty$ then the above two conditions are equivalent to $f_n \to^m f$ and $\{f_n\}$ is uniformly integrable.

For the forward, we want to show that $\nu_{|f_n|}$ is uniformly absolutely continuous. We know that $\sup_n \int |f_n| d\mu < \infty$ because $\int |f_n| d\mu \to \int |f| d\mu < \infty$. Specifically, for all $\epsilon > 0$, eventually for all $n > n_0$ $\int |f_n| d\mu < \int |f| d\mu + \epsilon < \infty$, and the supremum of the finite $n \leq n_0$ must be finite because each f_n is integrable. Thus if we can show that $\nu_{|f_n|}$ is absolutely continuous, Theorem 2.5.9 implies that $\{f_n\}$ is uniformly integrable.

Fix $\epsilon > 0$. Because each f_n is integrable, each $\nu_{|f_n|}$ is absolutely continuous, so there exist $\{\delta_n\}$ such that for all n, $\mu(A) < \delta_n \Rightarrow \int_A |f_n| d\mu < \frac{\epsilon}{3}$. For each f_n, we have

$$\int_A |f_n| d\mu \leq \int_A |f_n - f| d\mu + \int_A |f| d\mu$$

We know that because f is integrable, there exists $\delta_f > 0$ such that $\int_A |f| d\mu < \frac{\epsilon}{3}$. Similarly, because $\int_A |f_n - f| d\mu \leq \int |f_n - f| d\mu \to 0$, we know that there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ implies that $\int |f_n - f| d\mu < \frac{\epsilon}{3}$.

Thus letting $\delta = \min(\delta_1, \ldots \delta_{n_0}, \delta_f)$, if $\mu(A) < \delta$, then $\int_A |f_n| d\mu < \epsilon$ for all n, as desired.

For the reverse, the proof of Theorem 2.5.10 applies, since the only use of $f_n \to f$ a.e. (μ) was in applying the DCT, which still applies when convergence a.e. is replaced with convergence in measure.

Problem 2.30

For $n \geq 1$, let $f_n(x) = n^{-1/2} I_{(0,n)}(x), x \in \mathbb{R}$, and let $f(x) = 0, x \in \mathbb{R}$. Let m denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that $f_n \to f$ a.e. (m) and $\{f_n\}_{n \geq 1}$ is uniformly integrable, but $\int f_n dm \nrightarrow \int f dm$.

f_n converges pointwise to f, and thus converges to f a.e.. Similarly, for $t = 1$, $I_{|f_n| > 1}(x)$ is always zero, so the supremum condition is fulfilled and $\{f_n\}_{n \geq 1}$ is uniformly integrable. However, $\int f_n dm \to \infty$, which does not equal $\int f dm = 0$.