

# Athreya Lahiri Chapter 2 Solutions

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## Problem 2.1

Prove de Morgan's laws. Let  $\Omega_i, i = 1, 2$  be two nonempty sets, and let  $T : \Omega_1 \rightarrow \Omega_2$  be a map. For any collection  $\{A_\alpha : \alpha \in I\}$  of subsets of  $\Omega_2$ , prove that

$$\begin{aligned} T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) &= \bigcup_{\alpha \in I} T^{-1}(A_\alpha) \\ T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) &= \bigcap_{\alpha \in I} T^{-1}(A_\alpha) \\ (T^{-1}(A))^C &= T^{-1}(A^C) \end{aligned}$$

For the first,

$$\begin{aligned} T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) &= \{B \in \Omega_1 : T(B) \in \bigcup_{\alpha \in I} A_\alpha\} \\ &= \bigcup_{\alpha \in I} \{B \in \Omega_1 : T(B) \in A_\alpha\} \\ &= \bigcup_{\alpha \in I} T^{-1}(A_\alpha) \end{aligned}$$

with a similar argument holding for the second. For the third,

$$(T^{-1}(A))^C = \{B \in \Omega_1 : T(b) \in A\}^C = \{B \in \Omega_1 : T(b) \in A^C\} = T^{-1}(A^C)$$

## Problem 2.3

Let  $f, g : \Omega \rightarrow \mathbb{R}$  be  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable. Set

$$h(\omega) = \frac{f(\omega)}{g(\omega)} \mathbf{1}(g(\omega) \neq 0)$$

Verify that  $h$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

Directly from the definition, we have that for  $a \in \mathbb{R}$ ,

$$\begin{aligned} h^{-1}((-\infty, a]) &= \{\omega : \frac{f(\omega)}{g(\omega)} \mathbb{1}(g(\omega) \neq 0) \leq a\} \\ &= \{\omega : f(\omega) \leq ag(\omega), g(\omega) > 0\} \cup \{\omega : f(\omega) \geq ag(\omega), g(\omega) < 0\} \\ &= \{\omega : f(\omega) - ag(\omega) \leq 0, g(\omega) > 0\} \cup \{\omega : f(\omega) - ag(\omega) \geq 0, g(\omega) < 0\} \end{aligned}$$

$f(\omega) - ag(\omega)$  and  $g(\omega)$  are  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable functions, so for the left and right sets, the conditions individually define sets in  $\mathcal{F}$  so their intersection is in  $\mathcal{F}$ . Thus  $h(\omega)$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

## Problem 2.6

Let  $X_i, i = 1, 2, 3$  be random variables on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Consider the equation (with  $t \in \mathbb{R}$ )

$$X_1(\omega)t^2 + X_2(\omega)t + X_3(\omega) = 0$$

### Part a

Show that  $A := \{\omega \in \Omega : \text{The above equation has two distinct roots}\} \in \mathcal{F}$ .

The condition for  $\omega \in A$  is equivalent to

$$\{\omega : X_1(\omega) \neq 0\} \cap \{\omega : X_2^2(\omega) - 4X_1(\omega)X_3(\omega) > 0\}$$

because this indicates that the polynomial is second-order and its discriminant is positive.  $X_1(\omega)$  and  $X_2^2(\omega) - 4X_1(\omega)X_3(\omega)$  are random variables on probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and the sets  $(0, \infty)$  and  $(-\infty, 0) \cup (0, \infty)$  are Borel sets in  $\mathbb{R}$ , so both of the above sets are in  $\mathcal{F}$ , and their intersection is thus in  $\mathcal{F}$ .

### Part b

Let  $T_1(\omega)$  and  $T_2(\omega)$  denote the two roots of the above equation on  $A$ . Let

$$f_i(\omega) = \begin{cases} T_i(\omega) & \omega \in A \\ 0 & \omega \in A^c \end{cases}$$

Show that  $(f_1, f_2)$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}^2) \rangle$ -measurable.

**Lemma 1** *Let  $f$  be a non-negative  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable function. Then  $\sqrt{f}$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.*

**Proof:** For all  $a \geq 0$ ,

$$(\sqrt{f(\omega)})^{-1}((-\infty, a]) = \{\omega : 0 \leq \sqrt{f(\omega)} \leq a\} = \{\omega : 0 \leq f(\omega) \leq a^2\} \in F$$

□

By the quadratic formula (arbitrarily letting  $i = 1$  be the positive root), we have that

$$f_1(\omega) = \frac{-X_2(\omega) + \sqrt{X_2^2(\omega) - 4X_1(\omega)X_3(\omega)}}{2X_1(\omega)} \mathbf{1}_A$$

and  $f_2$  is the negative root. By Problem 2.3, since the numerator and denominator are  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable,  $A \in F$ , and the restriction in  $A$  prevents the denominator from being zero,  $f_i, i = 1, 2$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable, and the Cartesian product of  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable functions is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

## Problem 2.7

Let  $M := ((X_{ij})), 1 \leq i, j \leq k$  be a random matrix of random variables  $X_{ij}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ .

### Part a

Show that  $\det(M)$  and  $\text{tr}(M)$  are both  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

The trace is trivial; the diagonal entries of  $M$  are  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable and so is their sum. The determinant follows by induction on the size of  $M$ . When  $k = 1$ , the determinant is the random variable  $X_{11}$ , which by assumption is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable. Assuming that the determinant of a size  $k - 1$  matrix of random variables in  $(\Omega, \mathcal{F}, \mathbf{P})$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable, by the Laplace expansion, we can rewrite the determinant of the size  $k$  matrix as the sum of random variables multiplied by the determinant of size  $k - 1$  matrices, and this sum is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

### Part b

Show that the largest eigenvalue of  $M'M$  is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

Using the hint, I will note that the largest eigenvalue is equal to

$$\sup_x \frac{x' M' M x}{x' x} \mathbf{1}_{x \neq 0}$$

The numerator and denominator are the sums of products of random variables in  $(\Omega, \mathcal{F}, \mathbf{P})$  and the denominator is restricted from zero, so the internal function is  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ -measurable, and the supremum of a measurable function is measurable.

## Problem 2.8

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Let  $\bar{f}(x) = \inf_{\delta > 0} \sup_{|y-x| < \delta} f(y)$  and  $\underline{f}(x) = \sup_{\delta > 0} \inf_{|y-x| < \delta} f(y)$ ,  $x \in \mathbb{R}$ .

### Part a

Show that for any  $t \in \mathbb{R}$ ,

$$\{x : \bar{f}(x) < t\}$$

is open and hence  $\bar{f}$  is  $\langle \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}) \rangle$ -measurable.

Denote the set above as  $A$ . By the definition of  $\bar{f}$  and the properties of infimum, for each  $x \in A$ , there exists  $\delta_0$  such that

$$\sup_{|y-x| < \delta_0} f(y) < t$$

Thus for all  $x'$  such that  $|y - x'| < \delta_0$ , there exists an open ball centered at  $x'$  such that  $B(x', r) \subset B(x, \delta_0)$ , and thus

$$\sup_{y \in B(x', r)} f(y) \leq \sup_{y \in B(x, \delta_0)} f(y) < t$$

implies

$$\inf_{r \rightarrow 0} \sup_{y \in B(x', r)} f(y) < t$$

and thus  $x' \in A$ , which implies that  $B(x, \delta_0) \subset A$ . Thus  $A$  is open. A similar argument holds in reverse for  $\underline{f}$  using  $\{x : \underline{f}(x) > t\}$

### Part b

Show that for any  $t > 0$ ,

$$\{x : \bar{f}(x) - \underline{f}(x) < t\} = \bigcup_{r \in \mathbb{Q}} \{x : \bar{f}(x) < t + r, \underline{f}(x) > r\}$$

and hence is open.

Denote the set on the left  $A$ .  $A$  can be built up as a union of smaller sets. For  $r \in \mathbb{R}$ , consider  $A_r = \{x : \bar{f}(x) < r + t, \underline{f}(x) = r\}$ . It's clear that this captures all  $x$  in the inclusion condition of  $A$  for a given value of  $\underline{f}(x) = r$ . By taking the union over all  $r \in \mathbb{R}$ , we get the inclusion condition of  $A$  for all possible  $x \in \mathbb{R}$  and thus the union equals  $A$ . Thus

$$A = \bigcup_{r \in \mathbb{R}} \{x : \bar{f}(x) < r + t, \underline{f}(x) = r\}$$

Since we're taking the union over all  $r \in \mathbb{R}$  and our inclusion criteria is a strictly less than sign, we can replace the  $\underline{f}(x) = r$  conditions with  $\underline{f}(x) > r$ , such that  $r \in \mathbb{Q}$  instead of  $\mathbb{R}$ .

To see this, let  $A'_r = \{x : \bar{f}(x) < r + t, \underline{f}(x) > r\}$ . If  $x \in A_r$ , then  $\bar{f}(x) < r + t, \underline{f}(x) = r$ . Since rationals are dense in the reals and the inequality is strict, there exists  $r' \in \mathbb{Q}$  such that  $\bar{f}(x) < r' + t, \underline{f}(x) > r'$ , and thus  $x \in A'_{r'}$ . Thus

$$A = \bigcup_{r \in \mathbb{Q}} \{x : \bar{f}(x) < r + t, \underline{f}(x) > r\}$$

But by Part a, each of the subsets is the finite intersection of two open sets and is thus open, and the countable union of open sets is open. Thus  $A$  is open, as desired.

### Part c

Show that  $f$  is continuous at  $x_0 \in \mathbb{R}$  iff  $\bar{f}(x_0) = \underline{f}(x_0)$ .

For the forward, for arbitrary  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$ . Thus

$$\sup_{|y-x_0|<\delta} f(y) - \inf_{|y-x_0|<\delta} f(y) < \epsilon$$

which implies  $\bar{f}(x_0) - \underline{f}(x_0) = 0$ . For the reverse, we have that for all  $\delta > 0$ ,  $\sup_{|y-x_0|<\delta} f(y) \geq \bar{f}(x_0)$  and  $\sup_{|y-x_0|<\delta} f(y) \leq f(x_0)$ , combined with  $\bar{f}(x_0) = \underline{f}(x_0)$ , imply that  $\bar{f}(x_0) = \underline{f}(x_0) = f(x_0)$ . Thus for all  $\epsilon > 0$ , there exist  $\delta_1, \delta_2 > 0$  such that

$$\begin{aligned} \sup_{|y-x_0|<\delta_1} f(y) - f(x_0) &< \frac{\epsilon}{2} \\ f(x_0) - \inf_{|y-x_0|<\delta_2} f(y) &< \frac{\epsilon}{2} \end{aligned}$$

Thus for all  $y : |y - x_0| < \min(\delta_1, \delta_2)$ ,

$$f(y) - f(x_0) < \epsilon$$

as desired.

### Part d

Show that the set  $C_f = \{x : f(\cdot) \text{ is continuous at } x\}$  is a  $G_\delta$  set, i.e. the intersection of a countable number of open sets, and hence  $C_f$  is a Borel set.

Letting  $A_n$  be the set in Part b with  $t = \frac{1}{n}$ ,  $\bigcup_{n=1}^{\infty} A_n = \{x : x : \bar{f}(x) - \underline{f}(x) = 0\}$ , which by Part c is the set of  $x$  such that  $f(x)$  is continuous. By Part b, this is the countable union of open sets.

## Problem 2.15

Consider the probability space  $((0, 1), \mathcal{B}((0, 1)), m)$ , where  $m$  is the Lebesgue measure.

### Part a

Let  $Y_1$  be the random variable  $Y_1(x) = \sin(2\pi x)$  for  $x \in (0, 1)$ . Find the cdf of  $Y_1$ .

Looking at the graph of  $\sin(2\pi x)$ , for  $y \in (0, 1)$ , we have that  $\sin^{-1} y > 0$ , thus

$$\begin{aligned} P(Y_1 \leq y) &= mY_1^{-1}((-\infty, y]) \\ &= m\left(\left(0, \frac{\sin^{-1} y}{2\pi}\right) \cup \left(1/2 - \frac{\sin^{-1} y}{2\pi}, 1\right)\right) \\ &= \frac{1}{2} + \frac{\sin^{-1} y}{\pi} \end{aligned}$$

Similarly, for  $y \in (-1, 0)$ , we have that  $\sin^{-1} y < 0$ , thus

$$P(Y_1 \leq y) = m\left(\frac{1}{2} - \frac{\sin^{-1} y}{2\pi}, 1 + \frac{\sin^{-1} y}{2\pi}\right) = \frac{1}{2} + \frac{\sin^{-1} y}{\pi}$$

Thus

$$P(Y_1 \leq y) = \begin{cases} 0 & y \leq -1 \\ \frac{1}{2} + \frac{\sin^{-1} y}{\pi} & y \in (-1, 1) \\ 1 & y \geq 1 \end{cases}$$

### Part b

Let  $Y_2$  be the random variable  $Y_2(x) = \log x$  for  $x \in (0, 1)$ . Find the cdf of  $Y_2$ .

We have that for  $y < 0$ ,

$$P(Y_2 \leq y) = mY_2^{-1}((-\infty, y]) = m((0, e^y)) = e^y$$

Thus

$$P(Y_2 \leq y) = \begin{cases} e^y & y < 0 \\ 1 & y \geq 0 \end{cases}$$

### Part c

Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a cdf. For  $x \in (0, 1)$ , let

$$\begin{aligned} F_1^{-1}(x) &= \inf\{y : y \in \mathbb{R}, F(y) \geq x\} \\ F_2^{-1}(x) &= \sup\{y : y \in \mathbb{R}, F(y) \leq x\} \end{aligned}$$

Let  $Z_i$  be the random variable defined by

$$Z_i = F_i^{-1}(x), x \in (0, 1), i = 1, 2$$

### Subpart i

#### IN PROGRESS

Find the cdf of  $Z_i, i = 1, 2$ .

We begin with a characterization of  $F_1^{-1}$  and  $F_2^{-1}$ .

**Lemma 2** *Let  $A_x = \{y : y \in \mathbb{R}, F(y) = x\}$ . Then  $A$  is either the empty set, a singleton, or an interval.*

**Proof:**  *$F$  is right-continuous and nondecreasing. If there is no  $y$  such that  $F(y) = x$ , then  $F$  has a jump discontinuity that jumps from below  $y$  to above  $y$ , since otherwise by the intermediate value theorem  $F$  would achieve the value  $x$ . If  $A_x$  is a nonempty nonsingleton, then there exist multiple  $y$  such that  $F(y) = x$ . This must occur when  $F$  is flat, and since  $F$  is nondecreasing, this can only happen on a connected interval.*  $\square$

**Lemma 3**  *$F_i^{-1}(x)$  can be broken up into cases. When  $F(y)$  is flat, letting  $y_1$  and  $y_2$  be the left and right endpoints of the interval,  $F_1^{-1}(x) = y_1$  and  $F_2^{-1}(x) = y_2$ . When  $F(y)$  has a jump discontinuity at  $y$  that jumps from  $x_1$  to  $x_2$ ,  $x : x \in [x_1, x_2] \rightarrow F_i^{-1}(x) = y$ . Otherwise,  $F$  is invertible at  $y$  and  $F_i^{-1}(x) = F^{-1}(F(y)) = y$ .*

**Lemma 4** *For any  $x \in (0, 1), t \in \mathbb{R}, F(t) \geq x \Leftrightarrow F_1^{-1}(x) \leq t$ .*

**Proof:** *For the forward, assume that  $F(t) \geq x$ . Because  $F$  is nondecreasing, the infimum of  $y$  such that  $F(y) \geq x$  must be less than or equal to  $t$ . For the reverse, assume that  $F_1^{-1}(x) \leq t$ . Because  $F$  is nondecreasing and right continuous, the sets  $F(y) \geq x$  are closed intervals, and thus their infimum lies within the set - specifically, at the left endpoint, which is  $F_1^{-1}(x)$ . Thus for  $t \geq F_1^{-1}(x)$ , we know that  $F(t) \geq F(F_1^{-1}(x)) = x$ , since  $F_1^{-1}(x)$  is the left endpoint and this is where the sets achieve their minimum.*  $\square$

Thus for  $Z_1$ ,

$$P(Z_1 \leq z) = mZ_1^{-1}((-\infty, z]) = m([0, F(z)]) = F(z)$$

where we used the lemma substituting  $z$  for  $t$ .

## Problem 2.16

### Part a

Let  $(\Omega, \mathcal{F}_1, \mu)$  be a  $\sigma$ -finite measure space. Let  $T : \Omega \rightarrow \mathbb{R}$  be  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$  measurable. Show by counterexample that the induced measure  $\mu T^{-1}$  need not be  $\sigma$ -finite.

Let  $(\Omega, \mathcal{F}_1, \mu)$  be the Lebesgue measure on the Borel sets of  $\mathbb{R}$ , which is obviously  $\sigma$ -finite. Let  $T(x) = 0$ .  $T$  is continuous and thus  $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$  measurable. However, the induced measure  $\mu T^{-1}(A)$  equals infinity if  $x \in A$ , zero otherwise. Thus there is no collection of sets with finite measure such that their union is  $\mathbb{R}$ , and thus  $\mu T^{-1}$  is not  $\sigma$ -finite.

## Part b

Let  $(\Omega_i, \mathcal{F}_i)$  be measurable spaces for  $i = 1, 2$  and let  $T : \Omega_1 \rightarrow \Omega_2$  be  $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ -measurable. Show that any measure  $\mu$  on  $(\Omega_1, \mathcal{F}_1)$  is  $\sigma$ -finite if  $\mu T^{-1}$  is  $\sigma$ -finite on  $(\Omega_2, \mathcal{F}_2)$ .

By assumption of  $\sigma$ -finiteness of  $(\Omega_2, \mathcal{F}_2)$ , there is a countable collection of sets  $\{A_n\}_{n \geq 1} \subset \mathcal{F}_2$  such that  $\bigcup_{n \geq 1} A_n = \Omega_2$  and  $\mu T^{-1}(A_n) < \infty$  for all  $n$ . By the first assumption,

$$\Omega_1 = T^{-1}(\Omega_2) = T^{-1}\left(\bigcup_{n \geq 1} A_n\right) = \bigcup_{n \geq 1} T^{-1}(A_n)$$

By the second assumption,  $\mu(T^{-1}(A_n)) < \infty$  for all  $n$ . Thus the sets  $\{T^{-1}(A_n)\}_{n \geq 1}$  show that  $\mu$  is  $\sigma$ -finite.

## Problem 2.20

Use Corollary 2.3.5 to show that for any collection  $\{a_{ij} : i, j \in \mathbb{N}\}$  of nonnegative numbers,

$$\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \right)$$

Let  $(\mathbb{N}, P(\mathbb{N}))$  with the counting measure be a measure space, and let  $h_n : \mathbb{N} \rightarrow \mathbb{R}$  be the function/sequence  $h_i(j) = a_{ij}$  for all  $i, j \in \mathbb{N}$ . Then

$$\begin{aligned} \sum_{i=1}^{\infty} \int h_i(j) d\mu &= \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) \\ \int \sum_{i=1}^{\infty} h_i(j) d\mu &= \int \left( \sum_{i=1}^{\infty} a_{ij} \right) d\mu = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \right) \end{aligned}$$

and by Corollary 2.3.5, since  $h_i(j)$  is nonnegative and measurable, the two expressions equal each other.