Athreya Lahiri Chapter 4 Solutions

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Problem 4.4

Let $\nu, \mu, \mu_1, \mu_2 \ldots$ be σ-finite measures on a measurable space $(\Omega, \mathcal{F})$. Prove the following.

Part a

If $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_3$, then $\mu_1 \ll \mu_3$ and

$$\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} \text{ a.e. } (\mu_3)$$

$\mu_1 \ll \mu_3$ is trivial; the zero sets of μ_3 are zero sets of μ_2 are zero sets of μ_1. Using the Radon-Nikodym derivatives,

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2} d\mu_2$$

where $\frac{d\mu_1}{d\mu_2}$ is non-negative and measurable. Thus the integral can be approximated by a series of non-decreasing simple functions,

$$\begin{aligned}
\int_A \frac{d\mu_1}{d\mu_2} d\mu_2 &= \lim_{n\to\infty} \int_A \left(\frac{d\mu_1}{d\mu_2}\right)^{(n)} d\mu_2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} \left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)} \mu_2(A_{i_n})
\end{aligned}$$

where for all n, $\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}$ is constant on A_{i_n}. Replacing μ_2 with its Radon-Nikodym derivative,

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{k_n}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\mu_2(A_{i_n}) &= \lim_{n\to\infty}\sum_{i=1}^{k_n}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\int_{A_{i_n}}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \lim_{n\to\infty}\sum_{i=1}^{k_n}\int_{A_{i_n}}\left(\frac{d\mu_1}{d\mu_2}\right)_i^{(n)}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \lim_{n\to\infty}\int_A\left(\frac{d\mu_1}{d\mu_2}\right)^{(n)}\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \int_A\frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3}d\mu_3
\end{aligned}
$$

where the second line follows because $\left(\frac{d\mu_1}{d\mu_2}\right)^{(n)}$ is constant on A_{i_n}, the third line because adding up the integrals on the partition of A gives an integral on the whole of A, and the fourth line by the MCT. Since the Radon-Nikodym derivative of μ_3 is unique a.e. (μ_3), this implies that $\frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} = \frac{d\mu_1}{d\mu_3}$ a.e. (μ_3), as desired.

Part b

Suppose that μ_1 and μ_2 are dominated by μ_3. Then for any $\alpha, \beta \geq 0$, $\alpha\mu_1 + \beta\mu_2$ is dominated by μ_3 and

$$\frac{d(\alpha\mu_1 + \beta\mu_2)}{\mu_3} = \alpha\frac{d\mu_1}{d\mu_3} + \beta\frac{d\mu_2}{d\mu_3} \text{ a.e. } (\mu_3)$$

Domination is trivial. We also have

$$
\begin{aligned}
(\alpha\mu_1 + \beta\mu_2)(A) &= \alpha\mu_1(A) + \beta\mu_2(A) \\
&= \alpha\int_A\frac{d\mu_1}{d\mu_3}d\mu_3 + \beta\int_A\frac{d\mu_2}{d\mu_3}d\mu_3 \\
&= \int_A\alpha\frac{d\mu_1}{d\mu_3} + \beta\frac{d\mu_2}{d\mu_3}d\mu_3
\end{aligned}
$$

and the result follows from the uniqueness of the Radon-Nikodym derivative a.e. (μ_3).

Part c

If $\mu \ll \nu$ and $\frac{d\mu}{d\nu} > 0$ a.e. (ν), then $\nu \ll \mu$ and

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1} \text{ a.e. } (\mu)$$

Suppose that $\mu(A) = 0$. Then by the derivative, $0 = \int_A \frac{d\mu}{d\nu} d\nu$. Since $\frac{d\mu}{d\nu}$ is strictly positive, the integral can only be zero if $\nu(A) = 0$. This shows that $\nu \ll \mu$. Using a similar argument to Part a,

$$\begin{aligned}
\nu(A) &= \int_A \frac{d\nu}{d\mu} d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n_i} \left(\frac{d\nu}{d\mu}\right)_i^{(n)} \mu(A_{i_n}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n_i} \left(\frac{d\nu}{d\mu}\right)_i^{(n)} \int_{A_{i_n}} \frac{d\mu}{d\nu} d\nu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n_i} \int_{A_{i_n}} \left(\frac{d\nu}{d\mu}\right)_i^{(n)} \frac{d\mu}{d\nu} d\nu \\
&= \lim_{n\to\infty} \int_A \left(\frac{d\nu}{d\mu}\right)^{(n)} \frac{d\mu}{d\nu} d\nu \\
&= \int_A \frac{d\nu}{d\mu} \frac{d\mu}{d\nu} d\nu
\end{aligned}$$

Since this is true for all A, this implies that $\frac{d\nu}{d\mu} = 1$ a.e. (ν), which is equivalent to $\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}$ a.e. (μ), since the zero sets of μ are zero sets of ν and vice versa.

Part d

Let $\{\mu_n\}_{n\geq1}$ be a sequence of measures and let $\{\alpha_n\}_{n\geq1}$ be a sequence of positive real numbers. Define $\mu = \sum_{n=1}^{\infty} \alpha_n \mu_n$.

Subpart i

Show that $\mu \ll \nu$ iff $\mu_n \ll \nu$ for each $n \geq 1$, and in this case

$$\frac{d\mu}{d\nu} = \sum_{n=1}^{\infty} \alpha_n \frac{d\mu_n}{d\nu} \text{ a.e. } (\nu)$$

For the forward, if A is a zero set of ν and therefore μ, then $\mu(0) = \sum_{n=1}^{\infty} \alpha_n \mu_n(A)$ iff $\mu_n(A) = 0$ for all $n \geq 1$. The reverse is trivial.

For the derivative, let $\mu_{(k)}(A)$ be the partial sum $\sum_{n=1}^{k} \alpha_n \mu_n(A)$. Then

$$\int_A \frac{d\mu_{(k)}}{d\nu} d\nu = \int_A \sum_{i=1}^{k} \alpha_n \frac{d\mu_n}{d\nu} d\nu$$

and the claim follows by the MCT.

Subpart ii

Show that $\mu \perp \nu$ iff $\mu_n \perp \nu$ for all $n \geq 1$.

The forward is the same as the proof in Subpart i. For the reverse, let A_k be sets such that $\mu_k(A_k) = 0, \nu(A_k^C) = 0$. Consider $A = \bigcap_{k=1}^{\infty} A_k$. For each μ_n, A is the subset of a zero set A^n, so $\mu_n(A) = 0$ for all $n \geq 1$ implies that $\mu(A) = 0$. Conversely, $A^C = \bigcup_{k=1}^{\infty} A_k^C$ is the countable union of zero sets in ν and so $\nu(A^C) = 0$.