Athreya Lahiri Chapter 3 Solutions

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Problem 3.3

Prove the following.

Part a

Let $a_1 \dots a_k$ be real and $p_1 \dots p_k$ be positive numbers such that $\sum_{i=1}^k p_i = 1$. Then

$$\sum_{i=1}^{k} p_i \exp(a_i) \geq exp\left(\sum_{i=1}^{k} p_i a_i\right)$$

Let P be the probability measure on $\mathbb{R}$ that assigns probability p_i to point a_i and apply Jensen's inequality with $\phi(x) = e^x$.

Part b

Let $b_1 \dots b_k$ be nonnegative numbers and $p_1 \dots p_k$ be as in Part a. Then

$$\sum_{i=1}^{k} p_i b_i \geq \prod_{i=1}^{k} b_i^{p_i}$$

Furthermore, equality holds iff $b_1 = b_2 = \cdots = b_k$.

Let $a_i = \log b_i$ and apply Part a. For the iff, since the exponential function is strictly convex, inequality holds iff $f(\omega)$ is a constant, which in this context means that all the b_is are equal.

Part c

For any $a, b \in \mathbb{R}$ and $1 \leq p < \infty$,

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$$

Let $f(x) = x$, $\phi(x) = |x|^p$, which is convex on the range of p, and let P be the probability measure with $1/2$ probability on $\{a, b\}$. Thus by Jensen's inequality,

$$\phi\left(\int x dP\right) = \frac{1}{2^p}|a+b|^p \leq \int |x|^p dP = \frac{1}{2}(|a|^p + |b|^p)$$

which implies the desired result.