

Casella Berger Chapter 2

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Problem 2.29

Part c

A nasty discrete distribution is the beta-binomial, with pmf

$$P(X = k) = \frac{a}{k+a} \frac{\binom{n}{k} \binom{a+b-1}{a}}{\binom{n+a+b-1}{k+a}}$$

where n , a , and b are integers, and $y = 0, 1, 2 \dots n$. Use factorial moments to calculate the variance of the beta-binomial.

Note that being a probability distribution implies that

$$\sum_{k=0}^n \frac{1}{k+a} \frac{\binom{n}{k}}{\binom{n+a+b-1}{k+a}} = [a \binom{a+b-1}{a}]^{-1} = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

Furthermore, expanding the binomial coefficients gives

$$\begin{aligned} \frac{1}{k+a} \frac{\binom{n}{k}}{\binom{n+a+b-1}{k+a}} &= \frac{\frac{n!}{(n-k)!k!}}{(k+a) \frac{(n+a+b-1)!}{(n+b-k-1)!(k+a)!}} \\ &= \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} \frac{n!}{(n+a+b-1)!} \end{aligned}$$

implying

$$\sum_{k=0}^n \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} = \frac{(a-1)!(b-1)!(n+a+b-1)!}{n!(a+b-1)!} \quad (1)$$

$$= (a-1)!(b-1)! \binom{n+a+b-1}{a+b-1} \quad (2)$$

further implying that the probability mass function can be rewritten as

$$P(Y = k) = \frac{n!(a+b-1)!}{(a-1)!(b-1)!(n+a+b-1)!} \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} \quad (3)$$

$$= \alpha \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} \quad (4)$$

where $\alpha := \frac{n!(a+b-1)!}{(a-1)!(b-1)!(n+a+b-1)!}$ and doesn't depend on k . We first calculate the mean, $E(X)$.

$$E(X) = \alpha \sum_{k=0}^n k \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} \quad (5)$$

$$= \alpha \sum_{k=1}^n k \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} \quad (6)$$

$$= \alpha \sum_{k=1}^n \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!(k-1)!} \quad (7)$$

where Equation 6 follows from the $k = 0$ term being zero.