Casella Berger Chapter 2

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January 19, 2022

Problem 2.29

Part c

A nasty discrete distribution is the beta-binomial, with pmf

$$P(X=k) = \frac{a}{k+a}\frac{\binom{n}{k}\binom{a+b-1}{a}}{\binom{n+a+b-1}{k+a}}$$

where n, a, and b are integers, and $y = 0, 1, 2 \ldots n$. Use factorial moments to calculate the variance of the beta-binomial.

Note that being a probability distribution implies that

$$\sum_{k=0}^{n} \frac{1}{k+a}\frac{\binom{n}{k}}{\binom{n+a+b-1}{k+a}} = [a\binom{a+b-1}{a}]^{-1} = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

Furthermore, expanding the binomial coefficients gives

$$\begin{aligned}\frac{1}{k+a}\frac{\binom{n}{k}}{\binom{n+a+b-1}{k+a}} &= \frac{\frac{n!}{(n-k)!k!}}{(k+a)\frac{(n+a+b-1)!}{(n+b-k-1)!(k+a)!}} \\ &= \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!}\frac{n!}{(n+a+b-1)!}\end{aligned}$$

implying

$$\begin{aligned}\sum_{k=0}^{n} \frac{(n+b-k-1)!(k+a-1)!}{(n-k)!k!} &= \frac{(a-1)!(b-1)!(n+a+b-1)!}{n!(a+b-1)!} &(1)\\ &= (a-1)!(b-1)!\binom{n+a+b-1}{a+b-1} &(2)\end{aligned}$$

We first calculate the mean, $E(X)$.

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{n} \frac{ka}{k+a} \frac{\binom{n}{k}\binom{a+b-1}{a}}{\binom{n+a+b-1}{k+a}} \\
&= \sum_{k=1}^{n} \frac{ka}{k+a} \frac{\binom{n}{k}\binom{a+b-1}{a}}{\binom{n+a+b-1}{k+a}}
\end{aligned}
$$

because the $k = 1$ term is zero. Rewriting the $k\binom{n}{k}$,

$$
k\binom{n}{k} = k\frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-k)!(k-1)!} = n\binom{n-1}{k-1}
$$

implies

$$
E(X) = \sum_{k=1}^{n} \frac{a}{k+a} \frac{\binom{n-1}{k-1}\binom{a+b-1}{a}}{\binom{n+a+b-1}{k+a}}
$$