Chapter 3 - Matrix Algebra

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Problem Set C

Problem 3

If A is a fixed (i.e. nonrandom) matrix of size $n \times \ell$ and B is a fixed matrix of size $1 \times m$, show that $\mathrm{E}(AUB) = A\mathrm{E}(U)B$.

$AU \in \mathbb{R}^{n \times 1}$. The ith element is

$$(AU)_i = \sum_{k=1}^{\ell} a_{ik} u_k$$

and the i, jth element of $AUB \in \mathbb{R}^{n \times m}$ is

$$(AUB)_{ij} = \sum_{k=1}^{\ell} a_{ik} u_k b_j$$

The expectation of a random matrix is the expectation of each of its elements, so

$$\mathrm{E}(AUB)_{ij} = \sum_{k=1}^{\ell} a_{ik} \mathrm{E}(u_k) b_j$$

Unwinding the matrix multiplications shows that this equals $A\mathrm{E}(U)B$, as desired.

This result generalizes to arbitrary matrices. Let $A \in \mathbb{R}^{n \times d}, U \in \mathbb{R}^{d \times c}$, and $B \in \mathbb{R}^{c \times m}$. We first calculate the elements of $A\mathrm{E}(U)B$. Given

$$(UB)_{ij} = \sum_{\ell=1}^{c} U_{i\ell} B_{ellj}$$

$$\begin{aligned}(AUB)_{ij} &= \sum_{k=1}^{d} A_{ik} (UB)_{kj} \\ &= \sum_{\ell=1}^{c} \sum_{k=1}^{d} A_{ik} U_{k\ell} B_{\ell j}\end{aligned}$$

The expectation of a random matrix is the elementwise expectation of each of its elements, so

$$(E(AUB))_{ij} = E((AUB)_{ij}) = \sum_{\ell=1}^{c}\sum_{k=1}^{d} A_{ik}\mathrm{E}(U)_{k\ell}B_{\ell j}$$

and by unwinding this, we see that this equals $A\mathrm{E}(U)B$, as desired.

Problem 4

Show that $\mathrm{Cov}(AU) = A\mathrm{Cov}(U)A'$.

By definition, $\mathrm{Cov}(AU) = \mathrm{E}\left((AU - \mathrm{E}(AU))(AU - \mathrm{E}(AU))'\right)$. Looking at the $AU - \mathrm{E}(AU)$ term, by Problem 3,

$$AU - \mathrm{E}(AU) = AU - A\mathrm{E}(U) = A(U - E(U))$$

Similarly,

$$(AU - \mathrm{E}(AU))' = (U - \mathrm{E}(U))'A'$$

and so

$$\begin{aligned}
\mathrm{Cov}(AU) &= \mathrm{E}\left((AU - \mathrm{E}(AU))(AU - \mathrm{E}(AU))'\right) \\
&= \mathrm{E}\left(A(U - \mathrm{E}(U))(U - \mathrm{E}(U))'A'\right) \\
&= A\mathrm{E}\left((U - \mathrm{E}(U))(U - \mathrm{E}(U))'\right)A' = A\mathrm{Cov}(U)A'
\end{aligned}$$

where the last result follows from Problem 3 because A and A' are fixed, while $(U - \mathrm{E}(U))(U - \mathrm{E}(U))'$ is random.