Chapter 3 - Matrix Algebra

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Problem Set C

Problem 3

If A is a fixed (i.e. nonrandom) matrix of size $n \times \ell$ and B is a fixed matrix of size $1 \times m$, show that $\mathrm{E}(AUB) = A\mathrm{E}(U)B$.

$AU \in \mathbb{R}^{n \times 1}$. The ith element is

$$(AU)_i = \sum_{k=1}^{\ell} a_{ik} u_k$$

and the i, jth element of $AUB \in \mathbb{R}^{n \times m}$ is

$$(AUB)_{ij} = \sum_{k=1}^{\ell} a_{ik} u_k b_j$$

The expectation of a random matrix is the expectation of each of its elements, so

$$\mathrm{E}(AUB)_{ij} = \sum_{k=1}^{\ell} a_{ik} \mathrm{E}(u_k) b_j$$

Unwinding the matrix multiplications shows that this equals $A\mathrm{E}(U)B$, as desired.

Problem 4

Show that $\mathrm{Cov}(AU) = A\mathrm{Cov}(U)A'$.

By definition, $\mathrm{Cov}(AU) = \mathrm{E}\left((AU - \mathrm{E}(AU))(AU - \mathrm{E}(AU))'\right)$. Looking at the $AU - \mathrm{E}(AU)$ term, by Problem 3,

$$AU - \mathrm{E}(AU) = AU - A\mathrm{E}(U) = A(U - E(U))$$

Similarly,

$$(AU - \mathrm{E}(AU))' = (U - \mathrm{E}(U))'A'$$

and so

$$\begin{aligned}
\mathrm{Cov}(AU) &= \mathrm{E}\left((AU - \mathrm{E}(AU))(AU - \mathrm{E}(AU))'\right) \\
&= \mathrm{E}\left(A(U - \mathrm{E}(U))(U - \mathrm{E}(U))' A'\right) \\
&= A\mathrm{E}\left((U - \mathrm{E}(U))(U - \mathrm{E}(U))'\right) A' = A\mathrm{Cov}(U)A'
\end{aligned}$$