Chapter 1 Probability and Measure

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Problem 2

For a set $B \subset \mathbb{N} = \{1, 2, \dots\}$, define

$$\mu(B) = \lim_{n\to\infty} \frac{\#[B \cap \{1, \dots n\}]}{n}$$

when the limit exists, and let $\mathcal{A}$ denote the collection of all such sets.

Part b

If A and B are disjoint sets in $\mathcal{A}$, show that $\mu(A \cup B) = \mu(A) + \mu(B)$.

By definition,

$$\begin{aligned}
\mu(A \cup B) &= \lim_{n\to\infty} \frac{\#[(A \cup B) \cap \{1, \dots n\}]}{n} \\
&= \lim_{n\to\infty} \left[\frac{\#[A \cap \{1, \dots n\}]}{n} + \frac{\#[B \cap \{1, \dots n\}]}{n} \right] \\
&= \lim_{n\to\infty} \frac{\#[A \cap \{1, \dots n\}]}{n} + \lim_{n\to\infty} \frac{\#[B \cap \{1, \dots n\}]}{n}
\end{aligned}$$

assuming the limits are nice enough. The second line follows because A and B are disjoint.

Part c

Is μ a measure? Explain your answer.

No. A measure must satisfy countable additivity. Let B_n be the singleton set $\{n\}$, and let $A = \cup_{n=1}^{\infty} B_n = \mathbb{N}$. It's clear that $\mu(B_n) = 0$. Then

$$\mu(\cup_{n=1}^{\infty} B_n) = \mu(\mathbb{N}) = 1$$

but

$$\sum_{n=1}^{\infty} \mu(B_n) = 0$$